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Cosmological Perturbations and Structure Formation via

STOCHASTIC GRAVITY

B. L. Hu

Maryland Center for Fundamental Physics

University of Maryland, College Park

- Canadian Atlantic GR meeting, *Univ. New Brunswick, May 11, 2008*
- CITA *seminar Univ. Toronto April 18, 2008.*
- KITP- UCSB *Conference on Nonequilibirum Phenomena in Cosmology and Particle Physics (Feb 25-29, 2008)*

Reporting on work of Roura and Verdaguer..*Some slides courtesy E. Verdaguer*

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OUTLINE

1. Semiclassical gravity (SCG): SC Einstein Eqn: $\langle T_{mn} \rangle$

2. Stochastic gravity: Einstein-Langevin Equation

Noise Kernel: $\langle T_{mn} T_{rs} \rangle$

3. Influence functional; Stochastic Effective Action.

4. One of many areas of Application:

Primordial cosmological perturbations

- *Gives result equivalent at* linear order *to usual method of quantizing metric and inflaton perturbations*
- *But can treat quadratic order perturbations needed in R^2 trace anomaly driven (Starobinsky) inflation* .

A (very modest) Bottom-Up route:

General Relativity + Quantum Field Theory →
Quantum Field Theory in Curved Spacetime →
Semiclassical Gravity → Stochastic Gravity

1970s, 1980s

Semiclassical Gravity

▷ Quantum Fields in Curved classical Space : Test Field approx.

- Vacuum state (Fulling 71)
- Particle creation : Cosmological Spacetimes; Black Holes : Hawking Radiation (1974)
- Regularization of $\langle T_{\mu\nu} \rangle$

$$G_{\mu\nu}(g_{\alpha\beta}) = 8\pi G \langle T_{\mu\nu}(\phi) \rangle_0$$

Expectation value wrt vacuum

▷ Backreaction :

Spacetime not fixed, but determined by field and vice versa

- Self-consistency

SCG is a Mean Field Theory [mean field theory]

SEMICLASSICAL GRAVITY

Semiclassical Einstein eq. for classical geometry, metric function g

$$G_{ab}[g] = \kappa \langle \hat{T}_{ab}[g] \rangle_{ren}$$

$$\kappa = 8\pi G = 8\pi / m_P^2$$

Klein-Gordon eq. for quantum matter field ^φ

$$(\nabla_g^2 - m^2 - \xi R)\hat{\phi} = 0$$

- Solve for g and ^φ self-consistently.
- Backreaction problem is at the heart of semiclassical gravity.

SEMICLASSICAL EINSTEIN EQUATION

Renormalization introduces quadratic tensors

$$G_{ab}[g] + \Lambda g_{ab} - \alpha A_{ab}[g] - \beta B_{ab}[g] = \kappa \langle \hat{T}_{ab}[g] \rangle_{ren}$$

where

$$A^{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g}\, C_{cdef} C^{cdef}$$

$$B^{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g}\, R^2$$

$$T^{ab} = \nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} (\nabla^c \phi \nabla_c \phi + m^2 \phi^2) + \xi (g^{ab} \nabla^c \nabla_c - \nabla^a \nabla^b + G^{ab}) \phi^2$$

LIMITS OF SEMICLASSICAL GRAVITY

- Below Planck energy: measurements of time and length intervals

$$\Delta t >> t_P, \qquad \Delta \ell >> \ell_P$$

- Quantum fluctuations of stress tensor small:

$$\langle \hat{T}^2 \rangle - \langle \hat{T} \rangle^2 \approx 0$$

We want to extend semiclassical Einstein equations to account for fluctuations of $\hat{T}_{ab}$ consistently.

Semiclassical Gravity

Semiclassical Einstein Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle\hat{T}_{\mu\nu}\rangle_q \quad + \kappa\,(T_{\mu\nu})_c$$

$\tilde{G}_{\mu\nu}$ is the Einstein tensor (plus covariant terms associated with the renormalization of the quantum field)

$\kappa = 8\pi G_N$ and G_N is Newton's constant

Free massive scalar field

$$(\Box - m^2 - \xi R)\hat{\phi} = 0.$$

$\hat{T}_{\mu\nu}$ is the stress-energy tensor operator

$\langle\,\rangle_q$ denotes the expectation value

Stochastic Gravity

Einstein- Langevin Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa \left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu} \right)$$

$T^{c}_{\mu\nu}$ is due to classical matter or fields

$$T^{\rm qs}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\rm q} + T^{\rm s}_{\mu\nu}$$

$T^{\rm qs}_{\mu\nu}$ is a new stochastic term
related to the quantum fluctuations of $T_{\mu\nu}$

How could a quantum field give rise to a stochastic source?

via Influence functional (Feynman-Vernon 1963):

- We will assume linear perturbation of semiclassical solution

$g_{ab} + h_{ab}$ But stochastic gravity is NOT restricted to linear perturbations

- Einstein-Langevin equation: $G_{g+h} = \kappa(\langle \hat{T} \rangle_{g+h} + \xi)$

$$G^{(1)}_{ab}[g+h] = \kappa \langle \hat{T}^{(1)}_{ab}[g+h] \rangle_{ren} + \kappa \xi_{ab}[g]$$

$$(\nabla^2_{g+h} - m^2 - \xi R)\hat{\phi} = 0$$

NOISE KERNEL

- Exp Value of 2-point correlations of stress tensor: bitensor
- Noise kernel measures quantum flucts of stress tensor

$$N_{abcd}(x,y) = \frac{1}{2}\langle\{\hat{t}_{ab}(x), \hat{t}_{cd}(y)\}\rangle \qquad \hat{t}_{ab} \equiv \hat{T}_{ab} - \langle \hat{T}_{ab} \rangle \hat{I}$$

It can be represented by (shown via influence functional to be equivalent to) a classical stochastic tensor source $\xi_{ab}[g]$

$$\langle \xi_{ab} \rangle_s = 0 \qquad \langle \xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

- **Symmetric, traceless** (for conformal field), **divergenceless**

Noise associated with the fluctuations of a quantum field

- The noise kernel is real and positive semi-definite as a consequence of stress energy tensor being self-adjoint

the ultraviolet behaviour of $\langle \hat{T}_{ab}(x)\hat{T}_{cd}(y)\rangle$ is the same as that of $\langle \hat{T}_{ab}(x)\rangle\langle \hat{T}_{cd}(y)\rangle$.

- Classical Gaussian stochastic tensor field:

$$\langle \xi_{ab}[g; x)\rangle_s = 0, \quad \langle \xi_{ab}[g; x)\xi_{cd}[g; y)\rangle_s = N_{abcd}[g; x, y),$$

$\langle \cdots \rangle_s$

denotes statistical average wrt this noise distribution

Classical Stochastic Field
assoc.with a Quantum Field

- Stochastic tensor is covariantly conserved in the background spacetime (which is a solution of the semiclassical Einstein equation).

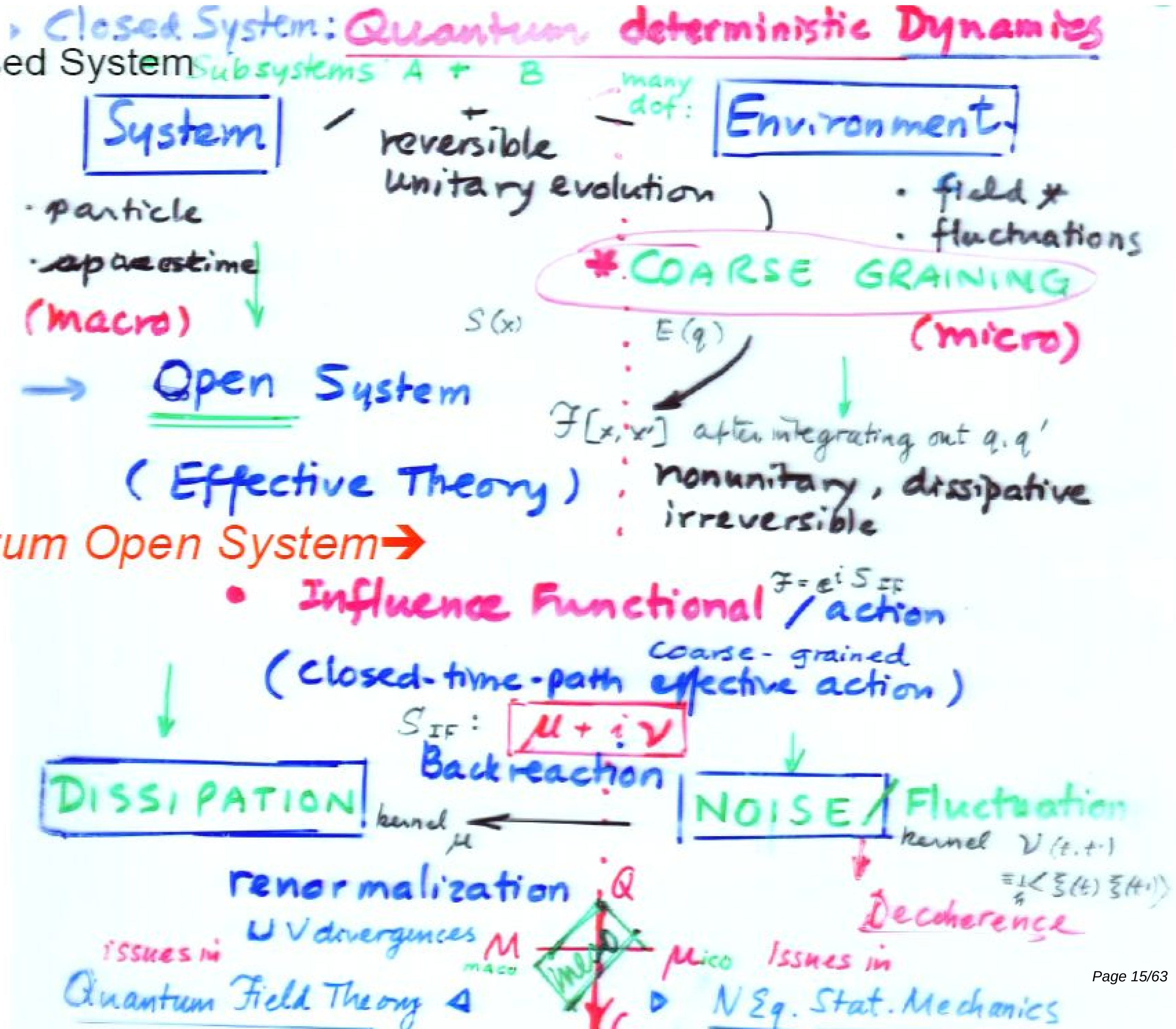
$$\nabla^a \xi_{ab}[g; x) = 0.$$

- For a conformal field ξ_{ab} is traceless:

$$g^{ab}\xi_{ab}[g; x) = 0;$$

Thus there is no stochastic correction to the trace anomaly

Q. Closed System



Quantum Open System➔

Quantum Open System

- Influence Functional / action $\mathcal{F} = e^{iS_{IF}}$

(Closed-time-path coarse-grained effective action)

S_{IF}: $\mu + i\nu$

Backreaction

DISSIPATION kernel μ ← NOISE / Fluctuation kernel $\nu(t,t')$ $= \frac{1}{\hbar}\langle \xi(t)\xi(t')\rangle$

Decoherence

renormalization Q

UV divergences M (macro) meso μico

issues in Quantum Field Theory ◄ C ► Issues in NEq. Stat. Mechanics

Quantum → Classical via (environment-induced) decoherence

↓

Master Eqn., Fokker-Planck Eqn.

Langevin Eqn. Classical Stochastic Dynamics

e.g. $$M \frac{d^2}{ds^2} X_c(s) + 2M \int_0^s ds' \, \gamma(s-s') \frac{d}{ds'} X_c(s') + M\omega_0^2 X_c(s) = F_\xi(s)$$ Noise

C M of classical paths; $\frac{d\gamma}{\cdot} \sim \mu$ Dissipation; Ω^2; ξ for linear cplg

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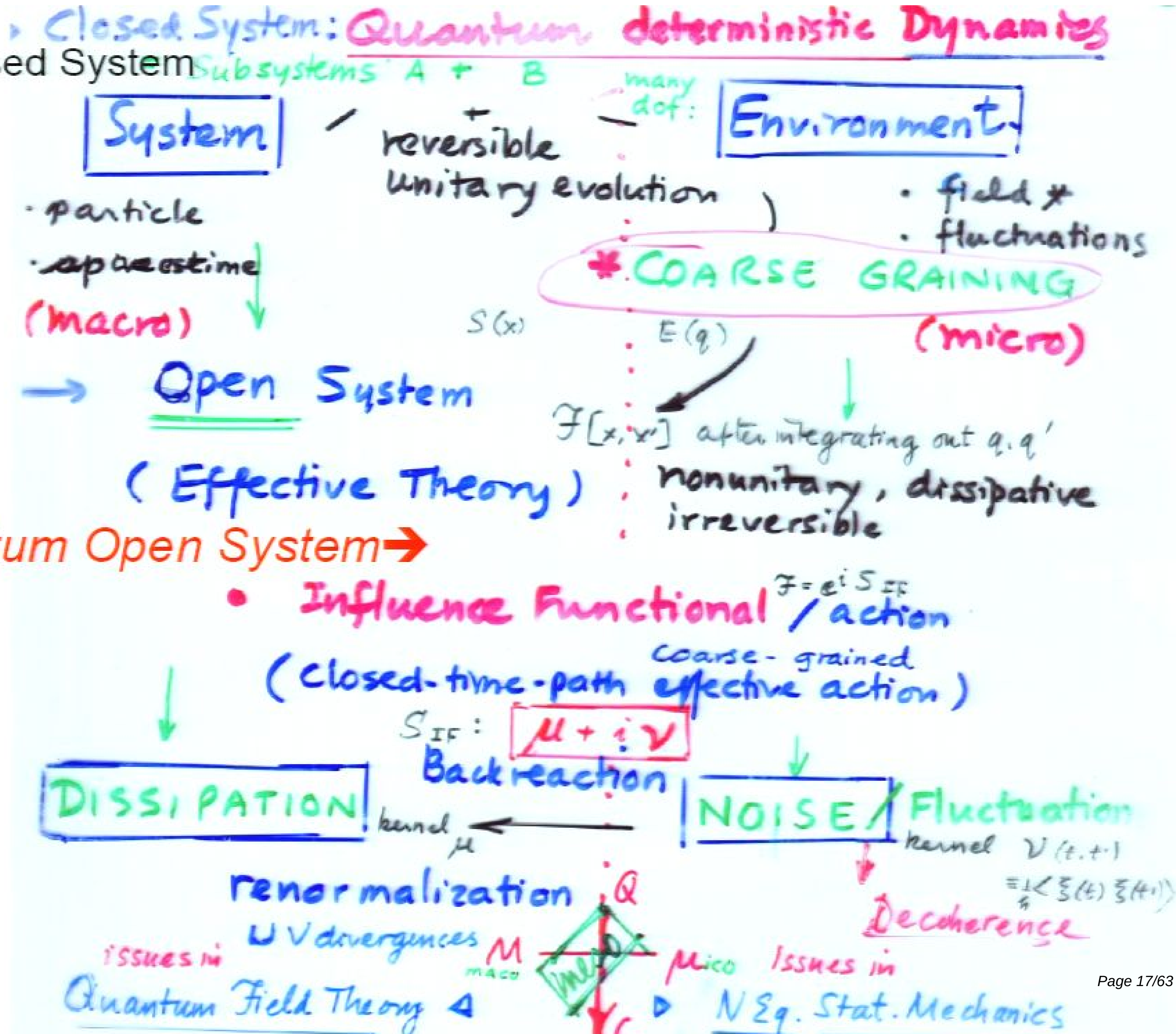

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X_c: CM of classical paths; γ: $\frac{d\gamma}{ds} = \mu$ Dissipation; $M\omega_0^2$: Ω^2; F_ξ: Noise, ξ for linear cplg.

Quantum Open System

losed System: **Density Matrix** $\hat{\rho}(t) = \mathcal{J}(t, t_i)\hat{\rho}(t_i).$

$\mathcal{J}(x, \mathbf{q}, x', \mathbf{q}', t \mid x_i, \mathbf{q}_i, x'_i, \mathbf{q}'_i, t_i)$ is the (**unitary**)

volutionary operator of the system from initial time t_I to time t.

PEN SYSTEM: System (s) interacting with an Environmen e) or Bath (b): Integrate out (coarse-graining) the bath dof renders ne system open. Its evolution is described by the Reduced Density Matrix

$$\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}')\delta(\mathbf{q} - \mathbf{q}')$$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i \; \mathcal{J}_r(x, x', t \mid x_i, x'_i, t_i) \; \rho_r(x_i, x'_i, t_i \;).$$

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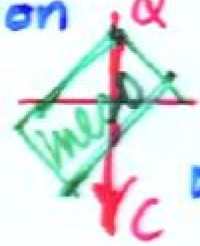

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CM of classical paths; $\frac{d\gamma}{ds} = \mu$ Dissipation; Ω^2; Noise; ξ for linear cplg.

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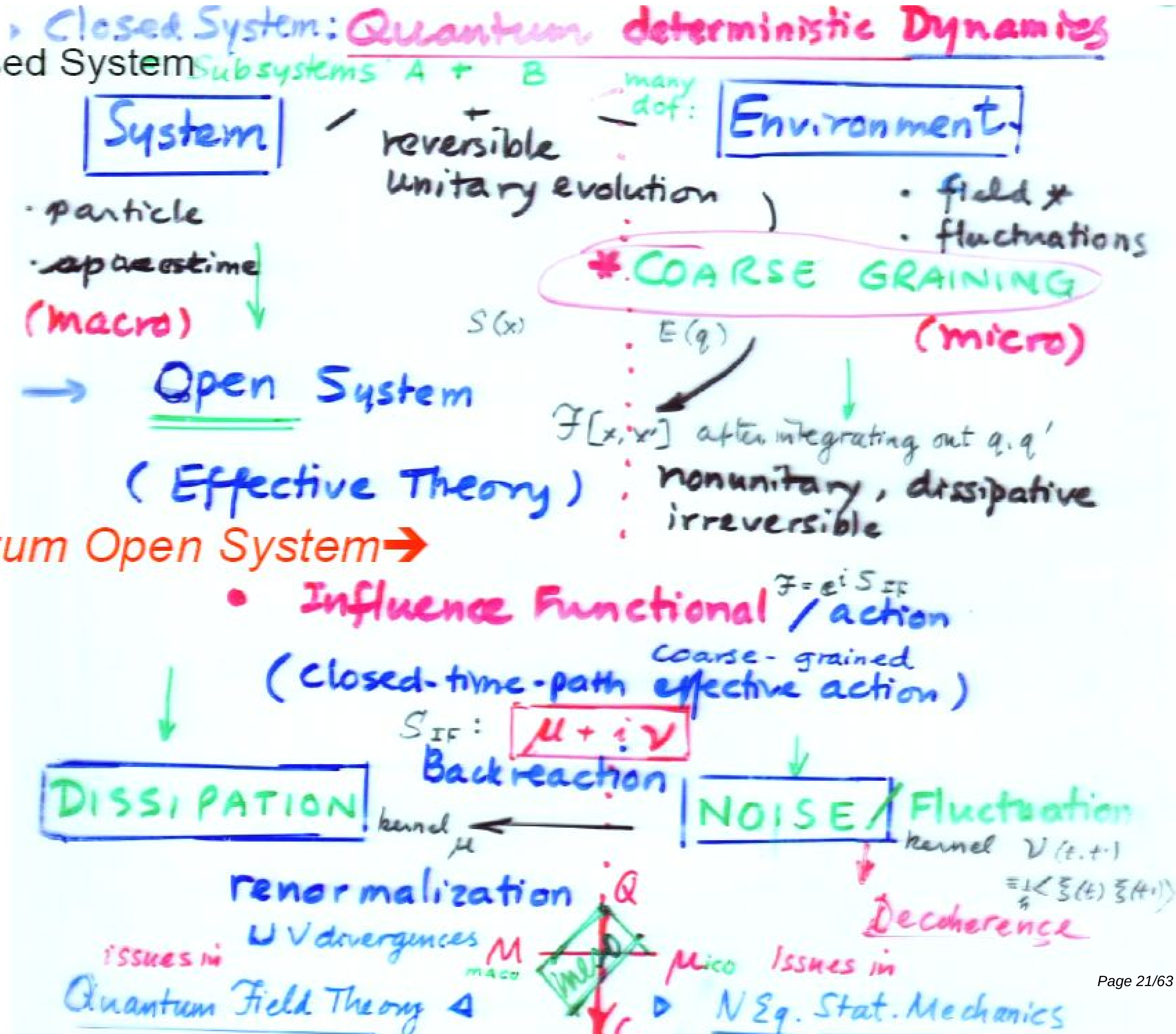

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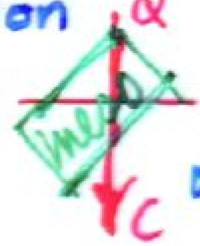

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$$\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}')\delta(\mathbf{q} - \mathbf{q}')$$

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Influence Functional

ıssume **factorizable condition** between the system (s) and the bath (b) ıitially

$$\hat{\rho}(t = t_i) = \hat{\rho}_s(t_i) \times \hat{\rho}_b(t_i),$$

:volutionary operator for the reduced density matrix is

$$J_r(x_f, x'_f, t \mid x_i, x'_i, t_i) = \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx' \, \exp\left(\frac{i}{\hbar}\left\{S[x] - S[x']\right\}\right) \mathcal{F}[x, x'$$

ıfluence
'unctional

$$\mathcal{F}[x, x'] = \int_{-\infty}^{+\infty} d\mathbf{q}_f \int_{-\infty}^{+\infty} d\mathbf{q}_i \int_{-\infty}^{+\infty} d\mathbf{q}'_i \int_{\mathbf{q}_i}^{\mathbf{q}_f} D\mathbf{q} \int_{\mathbf{q}'_i}^{\mathbf{q}_f} D\mathbf{q}'$$

$$\exp\left(\frac{i}{\hbar}\left\{S_b[\mathbf{q}] + S_{\text{int}}[x, \mathbf{q}] - S_b[\mathbf{q}'] - S_{\text{int}}[x', \mathbf{q}']\right\}\right) \times \rho_b(\mathbf{q}_i, \mathbf{q}'_i, t_i)$$

ıfluence Action

$$= \exp\left(\frac{i}{\hbar}\delta\mathcal{A}[x, x']\right)$$

Quantum Brownian Motion II

System (S): quantum oscillator *with time dependent natural frequency*
Environment (E) : n-quantum oscillators
with time-dependent natural frequencies = Scalar Field
Coupling: c_n F (x) q_n.

$$S[x, \mathbf{q}] = S[x] + S_E[\mathbf{q}] + S_{\text{int}}[x, \mathbf{q}]$$

$$= \int_0^t ds \left[\frac{1}{2} M(s)[\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2] \right.$$

$$\left. + \sum_n \left\{ \frac{1}{2} m_n(s)[\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2] + \sum_n \Big(-c_n(s)F(x)q_n \Big) \right] \right.$$

Influence functional for a Paramp

$$\mathcal{F}[x,x'] = \exp\Big\{-\frac{i}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\mu(s,s')\Big[F(x(s')) + F(x'(s'))\Big]$$
$$-\frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\nu(s,s')\Big[F(x(s')) - F(x'(s'))\Big]\Big\}$$

$$\Sigma(s) = \frac{1}{2}\Big(F(x(s)) + F(x'(s))\Big),$$

$$\Delta(s) = F(x(s)) - F(x'(s)),$$

Dissipation μ and Noise ν Kernels

Δ (s) μ (s,s') Σ (s')

$$\mathcal{F}[x,x'] = \exp\Big\{\frac{i}{\hbar}\int_{t_i}^{t} ds\Delta(s)\langle\xi(s)\rangle - \frac{1}{\hbar^2}\int_{t_i}^{t} ds \int_{t_i}^{s} ds'\Delta(s)\Delta(s')C_2(s,s')$$

$$\langle\bar{\xi}(t)\bar{\xi}(t')\rangle = C_2(s,s') \equiv \hbar\nu(s,s')$$

Langevin Equation:::

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - 2\frac{\partial F(x)}{\partial x}\int_{t_i}^{t}\mu(t,s)F(x(s))ds = -\frac{\partial F(x)}{\partial x}\bar{\xi}(t)$$

Noise and Dissipation Kernels

Equation of Motion for the amplitude function of a Parametric Oscillator

$b_n = 0$ and $m = 1$ $\quad \kappa_n = m_n(t_i)\omega_n(t_i)$ $\qquad \ddot{X}_n + \omega_n^2(t) X_n = 0,$

$$\mu(s,s') = \frac{i}{2}\int_0^\infty d\omega I(\omega,s,s')\Big[X_\omega^*(s)X_\omega(s') - X_\omega(s)X_\omega^*(s')\Big],$$

$$\nu(s,s') = \frac{1}{2}\int_0^\infty d\omega I(\omega,s,s')\coth\left(\frac{\hbar\omega(t_i)}{2k_BT}\right)\Big[\cosh 2r(\omega)\Big[X_\omega^*(s)X_\omega(s') + X_\omega(s)X_\omega^*(s')\Big]$$

$$-\sinh 2r(\omega)\Big[e^{-2i\phi(\omega)}X_\omega^*(s)X_\omega^*(s') + e^{2i\phi(\omega)}X_\omega(s)X_\omega(s')\Big]\Big].$$

$$I(\omega,s,s') = \sum_n \delta(\omega-\omega_n)\frac{c_n(s)c_n(s')}{2\kappa_n}$$

Spectral Density Function

$I(\omega) \sim \omega^n$ $\qquad$ n=1: Ohmic, $\quad$ n>1 Supra Ohmic; $\quad$ n<1 Subohmic

Squeezed and Rotation parameters: $\hat{\rho}_b(t_i) = \prod_n \hat{S}_n(r(n),\phi(n))\hat{\rho}_{\rm th}\hat{S}_n^\dagger(r(n),\phi(n)$

e.g., for an initial squeezed thermal bath

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$$-\frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\nu(s,s')\Big[F(x(s')) - F(x'(s'))\Big]\Big\}$$

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$\Delta(s)\,\mu(s,s')\,\Sigma(s')$

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Stochastic Equations

Master Equation: (Non- Markovian)

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}_r(t) = [\hat{H}_{\mathbf{ren}}, \hat{\rho}] + iD_{pp}[\hat{x}, [\hat{x}, \hat{\rho}]] + iD_{xx}[\hat{p}, [\hat{p}, \hat{\rho}]]$$

$$+ iD_{xp}[\hat{x}, [\hat{p}, \hat{\rho}]] + iD_{px}[\hat{p}, [\hat{x}, \hat{\rho}]] + \Gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}],$$

$$\hat{H}_{\mathbf{ren}} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{\mathbf{ren}}(t)\hat{x}^2.$$

Wigner Function:

$$F_W(\Sigma, p, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip\Delta/\hbar}\left\langle \Sigma - \frac{\Delta}{2}\middle|\hat{\rho}\middle|\Sigma + \frac{\Delta}{2}\right\rangle d\Delta,$$

Fokker-Planck or Wigner Equation: (Non-Markovian)

$$\frac{\partial}{\partial t}F_W(\Sigma, p, t) = \Big[-\frac{p}{M(t)}\frac{\partial}{\partial \Sigma} + \frac{1}{2}M(t)\Omega^2_{\mathbf{ren}}(t)\ \Sigma\frac{\partial}{\partial p} + \Gamma(t)\frac{\partial}{\partial p}p - 2D_{pp}(t)\frac{\partial^2}{\partial p^2}$$

$$-\hbar D_{xx}(t)\frac{\partial^2}{\partial \Sigma^2} + 2\Big(D_{xp}(t) + D_{px}(t)\Big)\frac{\partial^2}{\partial \Sigma \partial p}\Big]F_W(\Sigma, p, t).$$

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$$\mathcal{F}[x,x'] = \exp\Big\{ -\frac{i}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\mu(s,s')\Big[F(x(s')) + F(x'(s'))\Big]$$

$$-\frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Big[F(x(s)) - F(x'(s))\Big]\nu(s,s')\Big[F(x(s')) - F(x'(s'))\Big]\Big\}$$

$$\Sigma(s) = \frac{1}{2}\Big(F(x(s)) + F(x'(s))\Big),$$

$$\Delta(s) = F(x(s)) - F(x'(s)),$$

Dissipation μ and Noise ν Kernels

$$\Delta(s)\,\mu(s,s')\,\Sigma(s')$$

$$\mathcal{F}[x,x'] = \exp\Big\{ \frac{i}{\hbar}\int_{t_i}^{t} ds\Delta(s)\langle\xi(s)\rangle - \frac{1}{\hbar^2}\int_{t_i}^{t} ds \int_{t_i}^{s} ds'\Delta(s)\Delta(s')C_2(s,s')$$

$$\langle\bar{\xi}(t)\bar{\xi}(t')\rangle = C_2(s,s') \equiv \hbar\nu(s,s')$$

Langevin Equation:::

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - 2\frac{\partial F(x)}{\partial x}\int_{t_i}^{t}\mu(t,s)F(x(s))ds = -\frac{\partial F(x)}{\partial x}\bar{\xi}(t)$$

Quantum Brownian Motion II

System (S): quantum oscillator *with time dependent natural frequency*
Environment (E) : n-quantum oscillators
with time-dependent natural frequencies = Scalar Field
Coupling: c_n F (x) q_n.

$$S[x, \mathbf{q}] = S[x] + S_E[\mathbf{q}] + S_{\text{int}}[x, \mathbf{q}]$$

$$= \int_0^t ds \left[\frac{1}{2} M(s)[\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2] \right.$$

$$\left. + \sum_n \left\{ \frac{1}{2} m_n(s)[\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2] + \sum_n \left(-c_n(s)F(x)q_n \right) \right. \right]$$

Influence Functional

ıssume **factorizable condition** between the system (s) and the bath (b) ıitially

$$\hat{\rho}(t = t_i) = \hat{\rho}_s(t_i) \times \hat{\rho}_b(t_i),$$

:volutionary operator for the reduced density matrix is

$$\mathcal{J}_r(x_f, x'_f, t \mid x_i, x'_i, t_i) = \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx' \exp\left(\frac{i}{\hbar}\left\{S[x] - S[x']\right\}\right) \mathcal{F}[x, x'$$

ıfluence 'unctional

$$\mathcal{F}[x, x'] = \int_{-\infty}^{+\infty} d\mathbf{q}_f \int_{-\infty}^{+\infty} d\mathbf{q}_i \int_{-\infty}^{+\infty} d\mathbf{q}'_i \int_{\mathbf{q}_i}^{\mathbf{q}_f} D\mathbf{q} \int_{\mathbf{q}'_i}^{\mathbf{q}_f} D\mathbf{q}'$$

$$\exp\left(\frac{i}{\hbar}\left\{S_b[\mathbf{q}] + S_{\rm int}[x, \mathbf{q}] - S_b[\mathbf{q}'] - S_{\rm int}[x', \mathbf{q}']\right\}\right) \times \rho_b(\mathbf{q}_i, \mathbf{q}'_i, t_i)$$

ıfluence Action

$$= \exp\left(\frac{i}{\hbar}\delta\mathcal{A}[x, x']\right)$$

Quantum Open System

Closed System: **Density Matrix** $\hat{\rho}(t) = \mathcal{J}(t, t_i)\hat{\rho}(t_i).$

$\mathcal{J}(x, \mathbf{q}, x', \mathbf{q}', t \mid x_i, \mathbf{q}_i, x'_i, \mathbf{q}'_i, t_i)$ is the (**unitary**) **evolutionary operator** of the system from initial time t_I to time t.

OPEN SYSTEM: System (s) interacting with an Environment (e) or Bath (b): Integrate out (coarse-graining) the bath dof renders the system open. Its evolution is described by the Reduced Density Matrix

$$\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}')\delta(\mathbf{q} - \mathbf{q}')$$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i \; \mathcal{J}_r(x, x', t \mid x_i, x'_i, t_i) \; \rho_r(x_i, x'_i, t_i \,).$$

Quantum Brownian Motion II

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$$= \int_0^t ds \left[\frac{1}{2} M(s)[\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2] \right.$$

$$\left. + \sum_n \left\{ \frac{1}{2} m_n(s)[\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2] + \sum_n \left(-c_n(s)F(x)q_n \right) \right] \right.$$

Stochastic Equations

laster Equation: (Non- Markovian)

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}_r(t) = [\hat{H}_{\mathrm{ren}}, \hat{\rho}] + iD_{pp}[\hat{x}, [\hat{x}, \hat{\rho}]] + iD_{xx}[\hat{p}, [\hat{p}, \hat{\rho}]]$$

$$+ iD_{xp}[\hat{x}, [\hat{p}, \hat{\rho}]] + iD_{px}[\hat{p}, [\hat{x}, \hat{\rho}]] + \Gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}],$$

$$\hat{H}_{\mathrm{ren}} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{\mathrm{ren}}(t)\hat{x}^2.$$

Wigner Function:

$$F_W(\Sigma, p, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip\Delta/\hbar}\left\langle \Sigma - \frac{\Delta}{2}\middle|\hat{\rho}\middle|\Sigma + \frac{\Delta}{2}\right\rangle d\Delta,$$

okker-Planck or Wigner Equation: (Non-Markovian)

$$\frac{\partial}{\partial t}F_W(\Sigma, p, t) = \left[-\frac{p}{M(t)}\frac{\partial}{\partial\Sigma} + \frac{1}{2}M(t)\Omega_{\mathrm{ren}}^2(t)\ \Sigma\frac{\partial}{\partial p} + \Gamma(t)\frac{\partial}{\partial p}p - 2D_{pp}(t)\frac{\partial^2}{\partial p^2}\right.$$
$$\left. -\hbar D_{xx}(t)\frac{\partial^2}{\partial\Sigma^2} + 2\Big(D_{xp}(t) + D_{px}(t)\Big)\frac{\partial^2}{\partial\Sigma\partial p}\right]F_W(\Sigma, p, t).$$

INFLUENCE FUNCTIONAL

- Open quantum system *(Feynman-Vernon 63)*

system — h_{ab} ⇔ ϕ_j — **environment**

$$F_{IF} \equiv e^{iS_{IF}} = \int D[\phi_+] D[\phi_-] \exp(S_m[\phi_+, g^+] - S_m[\phi_-, g^-])$$

$$S_{IF}(g+h^{\pm}) = \frac{1}{2}\int \langle \hat{T}_x \rangle [h_x] - \iint [h_x] H_{xy} \{h_y\} + \frac{i}{8}\iint [h_x] N_{xy} [h_y]$$

$$[h] \equiv h^+ - h^- \qquad \{h\} \equiv (h^+ + h^-)/2$$

(x,y denotes ab.cd)

$$H'_{xy} = \frac{1}{4}\,\mathrm{Im}\langle T^*(\hat{T}_x \hat{T}_y)\rangle - \frac{i}{8}\langle [\hat{T}_x, \hat{T}_y] \rangle$$

$$\langle T^{(1)}_{ab}[g+h]\rangle_{ren} = -2\int d^4y \sqrt{-g}\, H_{abcd}(x,y) h^{cd}(y)$$

Stochastic Equations

Master Equation: (Non- Markovian)

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}_r(t) = [\hat{H}_{\rm ren},\hat{\rho}] + iD_{pp}[\hat{x},[\hat{x},\hat{\rho}]] + iD_{xx}[\hat{p},[\hat{p},\hat{\rho}]]$$

$$+ iD_{xp}[\hat{x},[\hat{p},\hat{\rho}]] + iD_{px}[\hat{p},[\hat{x},\hat{\rho}]] + \Gamma[\hat{x},\{\hat{p},\hat{\rho}\}],$$

$$\hat{H}_{\rm ren} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x}+\hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{\rm ren}(t)\hat{x}^2.$$

Wigner Function:

$$F_W(\Sigma,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip\Delta/\hbar}\left\langle \Sigma-\frac{\Delta}{2}\middle|\hat{\rho}\middle|\Sigma+\frac{\Delta}{2}\right\rangle d\Delta,$$

Fokker-Planck or Wigner Equation: (Non-Markovian)

$$\frac{\partial}{\partial t}F_W(\Sigma,p,t) = \Big[-\frac{p}{M(t)}\frac{\partial}{\partial\Sigma} + \frac{1}{2}M(t)\Omega^2_{\rm ren}(t)\;\Sigma\frac{\partial}{\partial p} + \Gamma(t)\frac{\partial}{\partial p}p - 2D_{pp}(t)\frac{\partial^2}{\partial p^2}$$

$$-\hbar D_{xx}(t)\frac{\partial^2}{\partial\Sigma^2} + 2\Big(D_{xp}(t)+D_{px}(t)\Big)\frac{\partial^2}{\partial\Sigma\partial p}\Big]F_W(\Sigma,p,t).$$

INFLUENCE FUNCTIONAL

- Open quantum system *(Feynman-Vernon 63)*

system $h_{ab} \Leftrightarrow \phi_j$ **environment**

$$F_{IF} \equiv e^{iS_{IF}} = \int D[\phi_+] D[\phi_-] \exp(S_m[\phi_+, g^+] - S_m[\phi_-, g^-])$$

$$S_{IF}(g + h^{\pm}) = \frac{1}{2}\int \langle \hat{T}_x \rangle [h_x] - \iint [h_x] H_{xy} \{h_y\} + \frac{i}{8}\iint [h_x] N_{xy} [h_y]$$

$$[h] \equiv h^+ - h^- \qquad \{h\} \equiv (h^+ + h^-)/2$$

(x,y denotes ab.cd)

$$H'_{xy} = \frac{1}{4}\mathrm{Im}\langle T^*(\hat{T}_x \hat{T}_y)\rangle - \frac{i}{8}\langle [\hat{T}_x, \hat{T}_y]\rangle$$

$$\langle T^{(1)}_{ab}[g + h]\rangle_{ren} = -2\int d^4y \sqrt{-g}\, H_{abcd}(x, y) h^{cd}(y)$$

INFLUENCE FUNCTIONAL

- Closed Time Path effective action at tree level in metric pert.

$$\Gamma^{(0)}_{CTP}\left[h^+,h^-\right]=S_g\left[h^+\right]-S_g\left[h^-\right]+S_{IF}\left[h^+,h^-\right]+O\left(h^3\right)$$

S_g is EH action plus quadratic terms.

- Integral identity (Feynman Vernon 1963):

$$e^{-\operatorname{Im}S_{IF}}\equiv\exp\left(-\frac{1}{8}\iint\left[h_x\right]N_{xy}\left[h_y\right]\right)\propto\int D\xi\exp\left(-\frac{1}{2}\iint\xi_x N^{-1}_{xy}\xi_y+\frac{i}{2}\int\xi_z\left[h_z\right]\right)$$

- Probability distribution functional of a classical stochastic field $\xi_{ab}(x)$

$$P[\xi]\propto e^{-\frac{1}{2}\iint\xi N^{-1}\xi}$$

$$e^{iS_{IF}[h^+,h^-]}=\int D\xi P[\xi]e^{i\left(\operatorname{Re}S_{IF}+\frac{1}{2}\int\xi[h]\right)}\equiv\left\langle e^{i\left(\operatorname{Re}S_{IF}+\frac{1}{2}\int\xi[h]\right)}\right\rangle_S$$

STOCHASTIC EFFECTIVE ACTION

- Define a stochastic effective action:

$$\Gamma_{stc}\left[h^{+},h^{-};\xi\right]=S_{g}\left[h^{+}\right]-S_{g}\left[h^{-}\right]+\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi_{z}\left[h_{z}\right]$$

- field equation from: $\left.\dfrac{\delta\Gamma_{stc}}{\delta h^{+}}\right|_{h^{\pm}=h}=0$

➡ the Einstein-Langevin equation

$$G_{ab}^{(1)}[g+h]=\kappa\langle\hat{T}_{ab}^{(1)}[g+h]\rangle_{ren}+\kappa\xi_{ab}[g]$$

SOLUTIONS OF EINSTEIN-LANGEVIN EQUATIONS

- These stochastic equations determine the correlations

$$h_{ab}(x) = h^0_{ab}(x) + \kappa \int d^4x' \sqrt{-g}\, G^{ret}_{abcd}(x,x')\xi^{cd}(x')$$

$$\langle h_{ab}(x) h_{cd}(y)\rangle_s = \langle h^0_{ab}(x) h^0_{cd}(y)\rangle_s + \kappa^2 \iint G^{ret}_{abef}(x,x') N^{efgh}(x',y') G^{ret}_{ghcd}(y',y)$$

Intrinsic fluctuations + **Induced fluctuations**

(flucts in the initial state) (due to matter field flucts)

- Stochastic metric correlations is equivalent to quantum metric correlations in 1/N: (Calzetta, Roura, Verdaguer)

$$\frac{1}{2}\langle\left\{\hat{h}_{ab}(x), \hat{h}_{cd}(y)\right\}\rangle = \langle h_{ab}(x) h_{cd}(y)\rangle_s$$

Applications of Stochastic Gravity: Fluctuations & Back-reaction Problems

1. Validity of Semiclassical Gravity — Hu, Roura Verdaguer (PRD04)

- Stability of solutions to SC Einstein Eqn with contributions of fluctuations – Einstein-Langevin Eqn
- Stochastic Gravity as next-to-leading-order 1/N limit. (Roura Verdaguer 03, Hartle-Horowitz 80)

2. Vacuum Fluctuations of Quantum Fields & Induced effects on Spacetime Dynamics:

- Negative energy density, quantum interest (Ford Roman)
- Re-examine Classical theorems in GR: Energy Dominance Condn with effects of quantum fluctuations

Stochastic Semiclassical Gravity

Stochastic Semiclassical Gravity : 1990s - 2000

Einstein-Langevin equation (more details later

$$G_{\mu\nu}(g_{\alpha\beta}) = 8\pi G \langle T_{\mu\nu} \rangle_q + T^{stoch}_{\mu\nu}$$

noise term from fluctuations of quantum fields

- Backreaction: dissipation

$T^{st}_{\mu\nu} = 2\tau_{\mu\nu}$ classical stochastic source

- Decoherence by quantum field leads to quantum → emergence of classical spacetime

noise

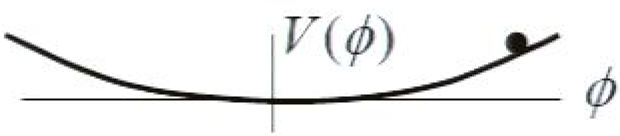
Key ingredient: NOISE, FLUCTUATIONS

➔ PHASE TRANSITION, STRUCTURE FORMATION, ENTROPY GENERATION

Stochasticity: nonequilibrium statistical mechanics

STOCHASTIC GRAVITY AND PRIMORDIAL COSMOLOGICAL PERTURBATIONS

- Quantum fluctuations of inflaton are seeds for structure formation
- Simplest chaotic inflationary model (Linde): Massive minimally coupled inflaton field, initially at average value larger than Planck scale



$$\dot{\phi}^2 \ll V(\phi) \qquad m_P \lesssim \phi_0$$

$$L(\phi) = \frac{1}{2}\partial_a \phi \partial^a \phi + \frac{1}{2} m^2 \phi^2$$

- Background inflaton field and FRW metric

$$\phi(\eta) = \langle \hat{\phi} \rangle \qquad ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij} dx^i dx^j)$$

PERTURBATIONS

- Inflaton and scalar metric perturbations

$$\hat{\phi}(x) = \phi(\eta) + \hat{\varphi}(x) \qquad \langle \hat{\varphi} \rangle_g = 0$$

$$ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j]$$

- Einstein-Langevin equations

$$G^{(0)}_{ab}[g] - \kappa\langle \hat{T}^{(0)}_{ab}[g]\rangle + G^{(1)}_{ab}[h] - \kappa\langle \hat{T}^{(1)}_{ab}[h]\rangle = \kappa\xi_{ab}[g]$$

zeroth order metric g is assumed to be quasi- de Sitter.

$$\hat{T}_{ab} = \tilde{\nabla}_a \hat{\phi} \tilde{\nabla}_b \hat{\phi} - \frac{1}{2}\tilde{g}_{ab}(\tilde{\nabla}_c \hat{\phi} \tilde{\nabla}^c \hat{\phi} + m^2 \hat{\phi}^2) \qquad \tilde{g} = g + h$$

$$\langle \xi(x)\xi(y)\rangle_s = \frac{1}{2}\langle\{\hat{t}(x), \hat{t}(y)\}\rangle[g] \qquad \hat{t} \equiv \hat{T} - \langle \hat{T}\rangle$$

STRESS TENSOR CORRELATIONS

$$\langle \hat{T}[g+h] \rangle = \langle \hat{T}[g+h] \rangle_{\phi\phi} + \langle \hat{T}[g+h] \rangle_{\phi\varphi} + \langle \hat{T}[g+h] \rangle_{\varphi\varphi}$$

$$\langle \{\hat{t},\hat{t}\} \rangle [g] = \langle \{\hat{t},\hat{t}\} \rangle [g]_{\phi^2\varphi^2} + \langle \{\hat{t},\hat{t}\} \rangle [g]_{\varphi^2\varphi^2} \equiv \langle \xi^{(1)}\xi^{(1)} \rangle_s + \langle \xi^{(2)}\xi^{(2)} \rangle_s$$

- assume Gaussian state: $\langle \hat{\varphi} \rangle = 0 \quad \langle \hat{\varphi}\hat{\varphi}\hat{\varphi} \rangle = 0$
- two independent stochastic sources: $\xi^{(1)}, \xi^{(2)}$
  independently conserved

- Including only the first (linear) term: $\langle \cdots \rangle_{\phi^2\varphi^2}$

we will show that the stochastic gravity
formulation gives equivalent results as the traditional
quantized metric and scalar field perturbations

E-L eqn for linear perturbations

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}_0^0\rangle_\Phi + \xi_0^0\right) = 3\mathcal{H}(\mathcal{H}\Phi + \Psi') - \nabla^2\Psi,$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}_0^i\rangle_\Phi + \xi_0^i\right) = \partial_i(\Psi' + \mathcal{H}\Phi),$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}_i^j\rangle_\Phi + \xi_i^j\right) = \Big[\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' + \frac{1}{2}\nabla^2 D\Big]\delta_i^j - \frac{1}{2}\delta^{jk}\partial_k\partial_i D.$$

where $\mathcal{H} = a'(\eta)/a(\eta)$, $D = \Phi - \Psi$, $\nabla^2 = \delta^{ij}\partial_i\partial_i$

- Since $\langle \hat{T}_{ij} \rangle = 0, (i \neq j)$ ➡ $\xi_{ij} = 0, (i \neq j)$

➡ metric perturbations $\Phi = \Psi$

- Fourier transf. of 0i-component: (neglecting non-local term):

$$2k_i(H\Phi_k + \Phi'_k) = \kappa \xi_{k(0i)} \qquad H \equiv \frac{a'(\eta)}{a(\eta)}$$

- Retarded propagator for Φ_k

$$G^k_{ret}(\eta, \eta') = \frac{\kappa}{2k_i}\left(\theta(\eta - \eta') \frac{a(\eta)}{a(\eta')} + f(\eta, \eta') \right)$$

With $\Psi = \Phi$ we get for the ii component of E-L eqn:

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}_i^{\ i}\rangle_\Phi + \xi_i^i\right) = \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + 3\mathcal{H}\Phi' + \Phi''.$$

Two unknowns 1. scalar metric perturbations $\Phi(x)$

2. $\langle\hat{\varphi}\rangle_\Phi$ the expectation value of the quantum operator for the inflaton perturbations on the spacetime with the perturbed metric, $\langle\hat{\varphi}[g+h]\rangle$

These three equations reduce to two because of the Bianchi Identity, which holds here since the averaged and stochastic sources in the EL eqn are separately conserved.

the one hand, the conservation of $\langle\delta\mathcal{T}_{ab}\rangle_\Phi$ is equivalent to the Klein-Gordon equation for the expectation value $\langle\hat{\varphi}\rangle_\Phi$, which is completely analogous to Eq. (36):

$$\langle\hat{\varphi}\rangle''_\Phi + 2\mathcal{H}\langle\hat{\varphi}\rangle'_\Phi - \nabla^2\langle\hat{\varphi}\rangle_\Phi + m^2a^2\langle\hat{\varphi}\rangle_\Phi - 4\phi'\Phi' + 2m^2a^2\phi\Phi = 0. \quad (41)$$

On the other hand, the conservation of the stochastic source is a consequence of the conservation of the noise kernel, which in turn relies on the fact that the quantum operator for the inflaton perturbations $\hat{\varphi}[g]$ satisfies the Klein-Gordon equation on the background spacetime. $\left(\nabla_a\nabla^a - m^2\right)\hat{\varphi}(x) = 0$.

Equivalence with Quantum approach:

Can show that EL eqn reduces to (Roura and Verdaguer 2007)

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi = 0.$$

Same as the conventional approach via quantized linear perturbations, e.g., Eq. (6.48) of

V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

Comments:

1. In Fourier space nonlocal terms in the integro-differential equation in the spatial sector simplify to products. Non-locality in time in this equation disappears due to an exact cancellation of the different contributions from $\langle \delta \hat{\mathcal{T}}_a^{\ b} \rangle_\Phi$
2. Seems like there is no dependence on the stochastic source. But the solutions to ELeqn should also satisfy the constraint eqn at the initial time in addition to the dynamical eqn. The initial conditions for $\Phi_k(\eta_0)$ and $\Phi'_k(\eta_0)$ have dependence on the stochastic source.

With $\Psi = \Phi$ we get for the ii component of E-L eqn:

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}_i^{\ i}\rangle_\Phi + \xi_i^i\right) = \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + 3\mathcal{H}\Phi' + \Phi''.$$

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CORRELATIONS FOR METRIC PERTURBATIONS

- Solutions of E-L equation:

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s = (2\pi)^2\delta(\vec{k}+\vec{k}')\iint G^k_{ret}\langle \xi_k\xi_{k'}\rangle_s G^{k'}_{ret}$$

$$\langle \xi_k\xi_{k'}\rangle_s \equiv \frac{1}{2}\langle\{\hat{t}_k,\hat{t}_{k'}\}\rangle$$

$$\langle\{\hat{t}^k_{0i}(\eta_1),\hat{t}^{-k}_{0i}(\eta_2)\}\rangle = k_ik_i\phi'(\eta_1)\phi'(\eta_2)\langle\{\hat{\varphi}_k(\eta_1),\hat{\varphi}_{-k}(\eta_2)\}\rangle$$

$$\langle\{\hat{\varphi}_k(\eta_1),\hat{\varphi}_{-k}(\eta_2)\}\rangle = G^{(1)}_k(\eta_1,\eta_2)$$

is the Hadamard function for free scalar field on de Sitter, in Euclidean vacuum

$$a(\eta) = -\frac{1}{H\eta}; -\infty < \eta < 0$$

METRIC PERTURBATION CORRELATIONS

Computing $G_k^{(1)}$ perturbatively in m / m_P

assuming slow roll $\dot{\phi}(t) \simeq -m_P^2 (m / m_P)$

taking (rather insensitive to initial conds.) $\eta_0 \to -\infty$

$$\langle \Phi_k(\eta) \Phi_{k'}(\eta') \rangle_s \simeq 8\pi^2 \left(\frac{m}{m_P} \right)^2 k^{-3} (2\pi)^3 \delta(\vec{k} + \vec{k}') \cos[k(\eta - \eta')]$$

- Harrison-Zel'dovich scale inv. spectrum large scales $k\eta \leq 1$
- Amplitude of CMB anisotropies ➡ $\frac{m}{m_P} \sim 10^{-6}$
- Agreement with linear perturbations approach (Mukhanov 92)
- Stochastic gravity can go beyond linear app. in inflaton flucts. (Weinberg 05) and deal with Starobinsky (tr anomaly) inflation

Summary: Main Features

1. Semiclassical gravity depends on e.v. of q. stress tensor

S.G fails when flucts. of quantum stress tensor are large

2. Stochastic gravity incorporates these fluctuations (at Gaussian level) through the noise kernel acting as source for the Einstein-Langevin equation

3. Stochastic two-point metric correlations agree with quantum two-point metric correlations to order 1/N in large N expansion

4. Cosmological Perturbation and Structure Formation:
Agreement with linear perturbations approach (e.g.,Mukhanov 92)
•But can go beyond linear order in inflaton fluctuations necessary for trace-anomaly driven inflations (e.g., Starobinsky 1980

Stochastic Gravity program

- *(since 1994)* *E. Calzetta (Buenos Aires),*
  B. L. Hu, A. Matacz, N.G. Phillips, S. Sinha (Maryland)
  A. Campos, R. Martin, A. Roura, Enric Verdaguer (Barcelona);

- *Current work:*
  with *A. Roura (Los Alamos), Enric Verdaguer (Barcelona)*
  - cosmological perturbations work // by Urakawa and Maeda (Waseda)

- ***Review:***

B. L. Hu and E. Verdaguer, "Stochastic gravity: Theory and Applications", in Living Reviews in Relativity 7 (2004) 3.
[update in arXiv:0802.0658]

- Since $\langle \hat{T}_{ij} \rangle = 0, (i \neq j)$ ➡ $\xi_{ij} = 0, (i \neq j)$

➡ metric perturbations $\Phi = \Psi$

- Fourier transf. of 0i-component: (neglecting non-local term):

$$2k_i(H\Phi_k + \Phi'_k) = \kappa \xi_{k(0i)} \qquad H \equiv \frac{a'(\eta)}{a(\eta)}$$

- Retarded propagator for Φ_k

$$G_{ret}^k(\eta, \eta') = \frac{\kappa}{2k_i}\left(\theta(\eta - \eta') \frac{a(\eta)}{a(\eta')} + f(\eta, \eta') \right)$$

Stochastic Semiclassical Gravity

Stochastic Semiclassical Gravity : 1990s - 2000 noise-average

Einstein-Langevin equation (more details later

$$G_{\mu\nu}(g_{\alpha\beta}) = 8\pi G \langle T_{\mu\nu} \rangle_q + T^{stoch}_{\mu\nu}$$

noise term from fluctuations of quantum fields

- Backreaction: dissipation

$T^{st}_{\mu\nu} = 2\xi_{\mu\nu}$ classical stochastic source

- Decoherence by quantum field leads to quantum → emergence of classical spacetime

noise

Key ingredient: NOISE, FLUCTUATIONS

➔ PHASE TRANSITION, STRUCTURE FORMATION, ENTROPY GENERATION

Stochasticity: nonequilibrium statistical mechanics

3. Black Hole Horizon Fluctuations & Backreaction (Hu, Raval, Sinha 98, 03)

- many speculations on the magnitude of such fluctuations but no quantitative calculations yet.
- Stochastic gravity is the theory for such inquiries

Roura Hu (06, 07)

4. Structure Formation from grav. perturbations (Roura & Verdaguer 03)

- particularly useful for trace anomaly-induced inflations.

Roura & Verdaguer (99, 07), Urakawa and Maeda (07)

5. A platform towards Quantum Gravity — defined as a theory of the micro-scopic structure of ST

NOT quantizing GR

Hu 03

SOLUTIONS OF EINSTEIN-LANGEVIN EQUATIONS

- These stochastic equations determine the correlations

$$h_{ab}(x) = h^0_{ab}(x) + \kappa \int d^4x' \sqrt{-g} G^{ret}_{abcd}(x,x') \xi^{cd}(x')$$

$$\langle h_{ab}(x) h_{cd}(y) \rangle_s = \langle h^0_{ab}(x) h^0_{cd}(y) \rangle_s + \kappa^2 \iint G^{ret}_{abef}(x,x') N^{efgh}(x',y') G^{ret}_{ghcd}(y',y)$$

Intrinsic fluctuations + **Induced fluctuations**

(flucts in the initial state) (due to matter field flucts)

- Stochastic metric correlations is equivalent to quantum metric correlations in 1/N: (Calzetta, Roura, Verdaguer)

$$\frac{1}{2} \langle \left\{ \hat{h}_{ab}(x), \hat{h}_{cd}(y) \right\} \rangle = \langle h_{ab}(x) h_{cd}(y) \rangle_s$$