Title: On Spectral Triples in Quantum Gravity

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Abstract: This talk is concerned with the existence of spectral triples in quantum gravity. I will review the construction of a spectral triple over a functional space of connections. Here, the *-algebra is generated by holonomy loops and the Dirac type operator has the form of a global functional derivation operator. The spectral triple encodes the Poisson structure of General Relativity when formulated in terms of Ashtekars variables. Finally I will argue that the Hamiltonian of General Relativity may emerge from the construction via the requirement that inner automorphisms vanish on the vacuum sector.

On Spectral Triples in Quantum Gravity

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Collaboration with Johannes Aastrup and Ryszard Nest

Perimeter Institute 29.05.08

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- **Motivation.** Noncommutative Geometry: The Standard Model + Gravity — single gravitational formulation. Quantization?

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- **Our Project.** To formulate a model which combines elements of NCG and QFT. Inspiration: LQG.

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  - $\mathcal{B}_\Delta$ is an algebra of loops (functions on $\mathcal{A}$)
  - D_Δ is a Dirac type operator (functional derivation)
  - $\mathcal{H}_\Delta$ is a separable Hilbert space of states over $\mathcal{A}$.

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• Technical: The construction is based on a projective, countable system of graphs.

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- D_Δ^2 as a Hamiltonian?

loncommutative Geometry

A **Spectral Triple** is a collection (B, D, H):
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- Commutative algbra B — Riemannian spin-geometry [Rennie, Varilly].

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The Standard Model of Particle Physics (SM):
[Connes, Lott, Chamseddine, Marcolli, ...]

- $B = C^\infty(M) \otimes B_F$, *almost commutative algebra*
  $B_F = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
- $D = D_M \otimes 1 + \gamma_5 \otimes D_F$, D_F is the Yukawa coupling matrix
- $H =$ fermionic content of SM

Spectral action:

$$I = \langle \psi | \tilde{D} | \psi \rangle + Tr\left(\varphi\left(\frac{\tilde{D}^2}{\Lambda^2} \right) \right)$$

classical action of the Standard Model coupled to Gravity.

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Here

$$\tilde{D} = D + A + JAJ^{-1}$$

where A is a noncommutative 1-form generated by inner automorphisms

$$A = \sum a_i [b_i, D]$$

- generates entire bosonic sector (including Higgs).

Main point

Formulation of the Standard Model coupled to General Relativity as a single gravitational theory. The Standard Model emerges from a very simple modification of space-time geometry:

$$C^\infty(M) \to C^\infty(M) \otimes B_F$$

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- this would presumably involve Quantum Gravity.

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to construct a model which combines Noncommutative Geometry with elements of Quantum Gravity.

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"A striking aspect of this approach to geometry of $\bar{\mathcal{A}}/\mathcal{G}$ is that its general spirit is the same as that of non-commutative geometry and quantum groups: even though there is no underlying differential manifold, geometrical notions can be developed by exploiting the properties of the *algebra* of functions."

[Ashtekar, Lewandowski, 1996]

Outlook/Open questions

- The construction is based on a countable system of embedded graphs (lattices, simplicial complexes). The construction is essentially combinatorial.
- The structure of the Hamiltonian of GR emerges from a condition which restricts the triple to a sector where inner automorphisms play no role (turning off interactions — vacuum).
- The triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ depends on a set $\{a_i\}$ of scaling parameters. This resembles a regularization scheme.
- Connes distance formula: distances between "geometries".

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- Then (heuristically)

$$Tr\{D_\Delta . W\} + \{W . W\}\Big|_{\text{one vertex}} \overset{\text{classical}}{\rightsquigarrow} \epsilon^{ij}{}_k F^k_{\mu\nu} E^\mu_j E^\nu_k + \mathcal{O}\left(\alpha^2\right)$$

(we set $\alpha = a_n$ for $n \to \infty$)

- This has the form of the Hamilton constraint of GR.

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Graphs

The algebra of holonomy loops is described via the inductive

- Shift focus from connections to holonomies and flux variables

$$h_L(A) = \mathrm{Hol}(L, A)$$

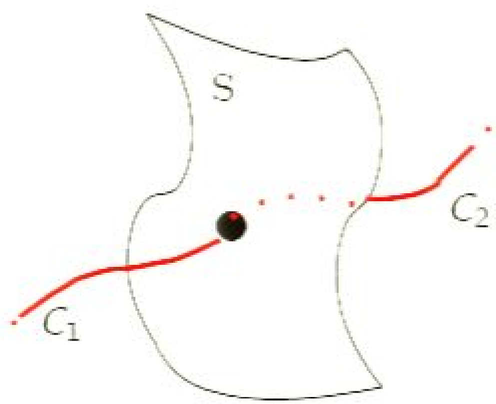
  L loop on Σ

$$F_S^a(E) = \int_S \epsilon_{mnp} E^{ma} dx^n dx^p$$

  S surface in Σ.

- Poisson brackets

$$\{F_S^a(E), h_C(A)\} = \pm h_{C_1}(A) \tau^a h_{C_2}(A)$$



  τ^a generator of $su(2)$,
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Inspiration

Loop Quantum Gravity (algebra, mathematical techniques, ideas)

Loop Quantum Gravity

- Foliation: $M = \mathbb{R} \times \Sigma$
- Ashtekar variables

$$
\begin{array}{ll}
A^i_j & SU(2)\text{-connection on } \Sigma. \\
E^i_j = |\det e|^{\frac{1}{2}} e^i_j & e^i_j \text{ orthonormal frame field.}
\end{array}
$$

- Poisson brackets

$$\{A^i_j(x), E^k_l(y)\} = \delta^i_l \delta^k_j \delta(x-y)$$

$+$ constraints (diffeomorphism, Hamilton, Gauss)

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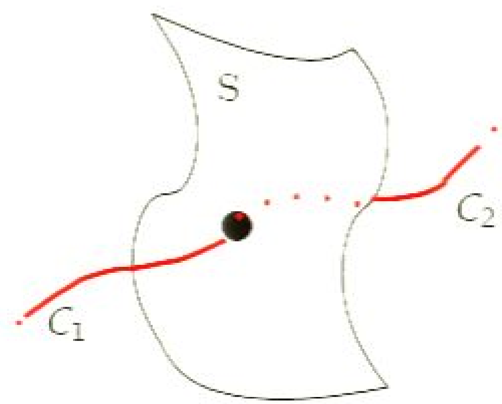

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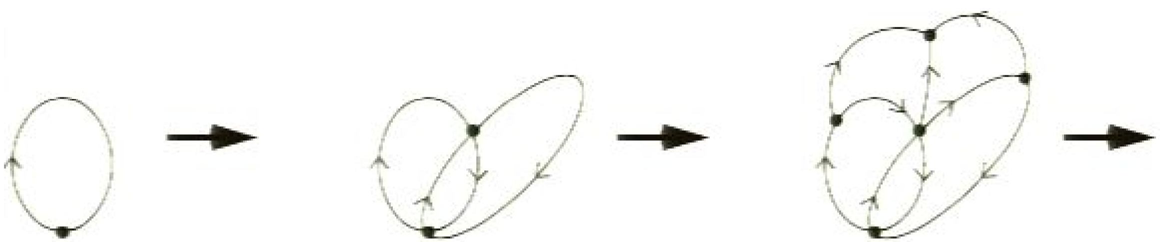
Graphs

The algebra of holonomy loops is described via the inductive system of all finite, piece-wise analytic graphs



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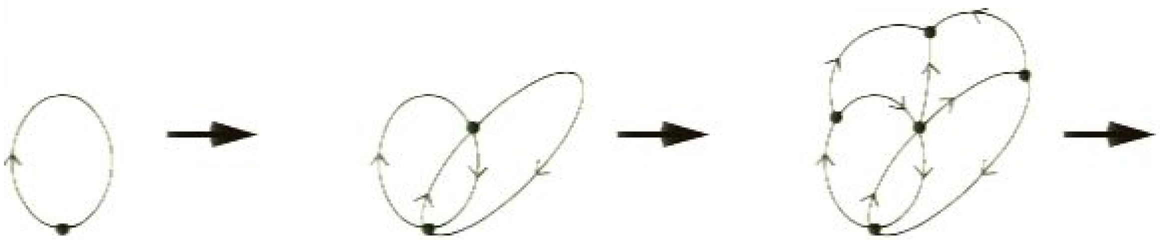

- Let $\mathcal{A}$ be the space of smooth connections with gauge group G. Denote by $\mathcal{A}_\Gamma$ the restriction of $\mathcal{A}$ to a finite graph Γ. Seen from Γ a connection $\nabla \in \mathcal{A}$ can be seen as a point in the space G^n

$$\nabla = (g_1, \ldots, g_n) \in G^{n(\Gamma)} \simeq \mathcal{A}_\Gamma$$

where $n(\Gamma)$ is the number of edges ϵ_i in Γ and where $g_i = Hol(\nabla, \epsilon_i)$ is the holonomy of ∇ along ϵ_i.

- Projective system of coarse grained spaces of connections:

$$\begin{array}{ccccccccc}
\ldots \leftarrow & \mathcal{A}_{\Gamma} & \leftarrow & \mathcal{A}_{\Gamma'} & \leftarrow & \mathcal{A}_{\Gamma''} & \leftarrow \ldots \\
 & \wr\| & & \wr\| & & \wr\| & \\
\ldots \leftarrow & G^{n(\Gamma)} & \leftarrow & G^{n(\Gamma')} & \leftarrow & G^{n(\Gamma'')} & \leftarrow \ldots
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  with structure maps

$$P_{\Gamma\Gamma'} : G^{n(\Gamma')} \leftarrow G^{n(\Gamma)}$$

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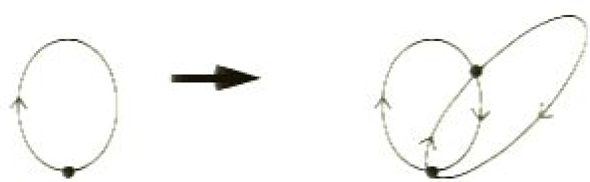
- **Example:**

$$P : G^4 \to G$$
$$(g_1, g_2, g_3, g_4) \mapsto g_1 \cdot g_3$$



- **Result:**

$$\mathcal{A} \hookleftarrow \varprojlim \mathcal{A}_\Gamma =: \overline{\mathcal{A}}^a$$

[Ashtekar, Lewandowski]

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- **Result:**

$$\mathcal{A} \hookrightarrow \varprojlim \mathcal{A}_\Gamma =: \overline{\mathcal{A}}^a$$

[Ashtekar, Lewandowski]

- **Point:** The space of connections is densely imbedded in a pro-manifold $\overline{\mathcal{A}}^a$ — Ashtekar-Lewandowski measure, Hilbert space structure ...

Our Project

- **Aim:** To construct a spectral triple that involves an algebra of loops, i.e. functions on $\mathcal{A}$:

$$L : \nabla \mapsto \text{Hol}(\nabla, L) \in M_n(\mathbb{C})$$

where the interaction between Dirac type operator and the loop algebra reproduces the Poisson structure of GR.

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- **Strategy:** Exploit the pro-manifold structure of $\mathcal{A}$

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  1. Construct a spectral triple $(\mathcal{B}, D, \mathcal{H})_\Gamma$ at the level of each finite graph Γ. Since

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     this is easy (Haar measure, Dirac operator etc.)

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  2. Ensure compatibility with the structure maps
  $$P_{\Gamma_n \Gamma_m} : \mathcal{A}_{\Gamma_n} \to \mathcal{A}_{\Gamma_m} \, .$$
  for all structures (Hilbert space, algebra, Dirac operator)

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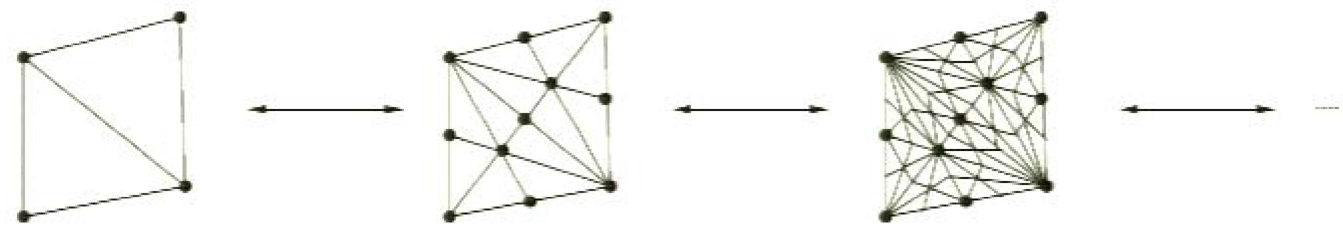

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Our Project

- **Aim:** To construct a spectral triple that involves an algebra of loops, i.e. functions on $\mathcal{A}$:

$$L : \nabla \to \mathrm{Hol}(\nabla, L) \in M_n(\mathbb{C})$$

where the interaction between Dirac type operator and the loop algebra reproduces the Poisson structure of GR.

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  1. Construct a spectral triple $(\mathcal{B}, D, \mathcal{H})_\Gamma$ at the level of each finite graph Γ. Since
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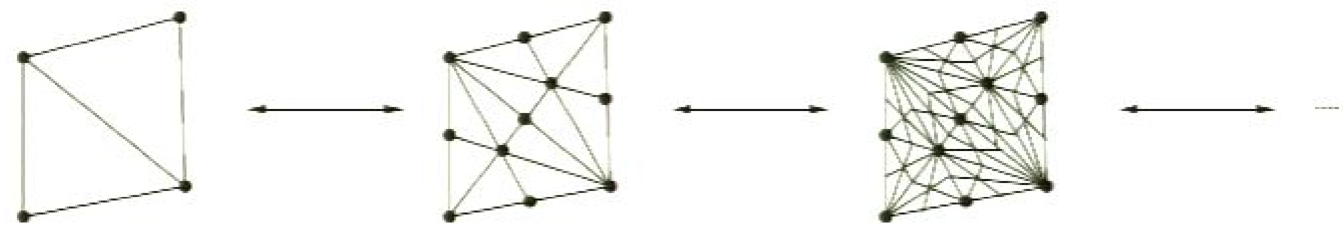

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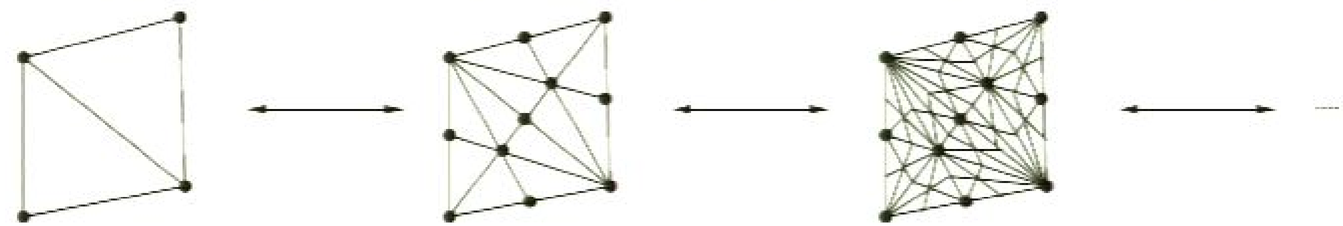
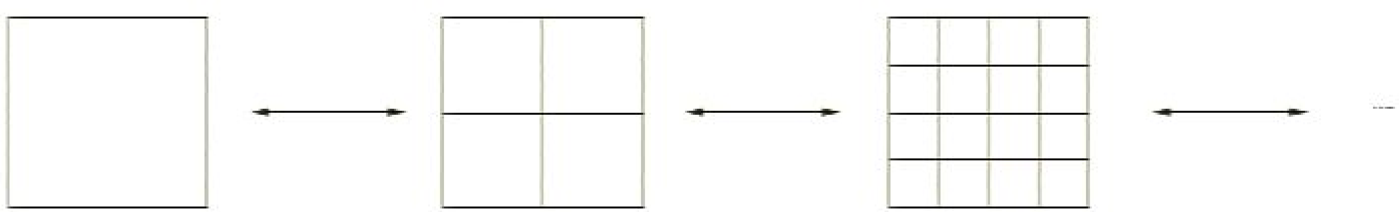


  - Here: take a projective system of cubic lattices.



The construction

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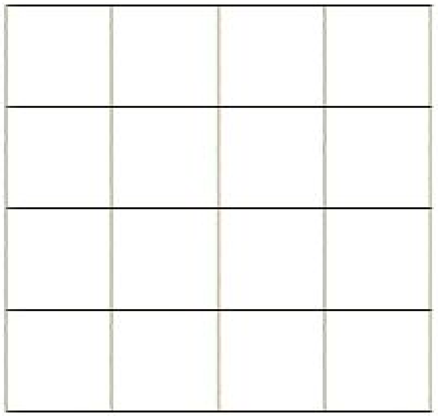
- Let Γ be a finite d-dim lattice with edges $\{\epsilon_i\}$ and vertices $\{v_i\}$ with

$$\epsilon_j : \{0. 1\} — \{v_i\}$$



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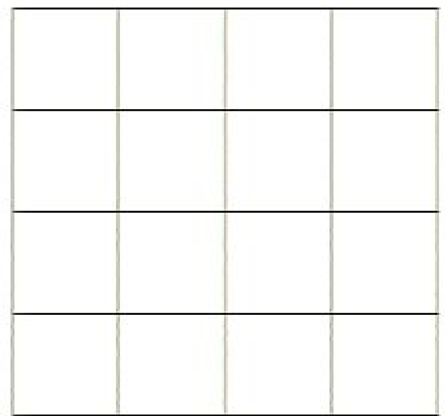

- Assign to each edge ϵ_i a group element $g_i \in G$.

$$\nabla : \epsilon_i \to g_i$$

G is a compact Lie-group. The space of such maps is denoted $\mathcal{A}_\Gamma$. Notice again:

$$\mathcal{A}_\Gamma \simeq G^n \quad \text{because} \quad \mathcal{A}_\Gamma \ni \nabla \to (\nabla(\epsilon_1), \ldots, \nabla(\epsilon_n)) \in G^n$$

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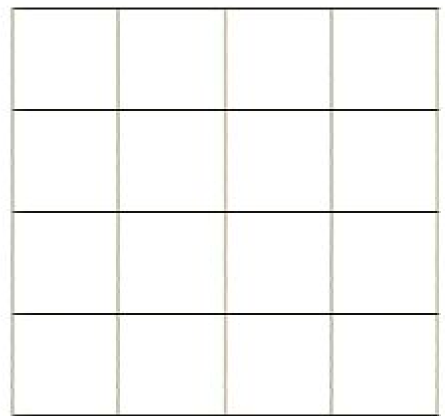

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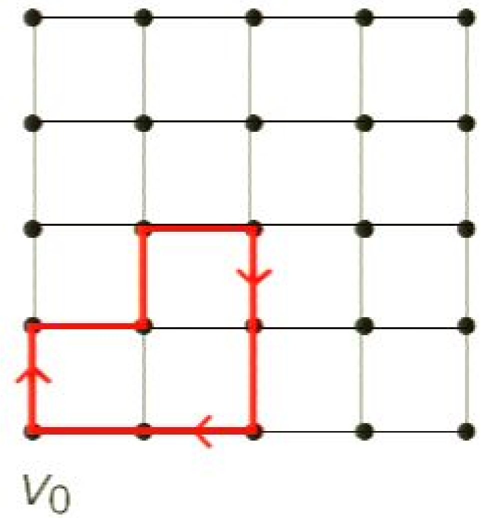
- First step: Construct a spectral triple over $\mathcal{A}_\Gamma$.

- **Algebra:** A loop L is a finite sequence of edges $L = \{\epsilon_{i_1} . \epsilon_{i_2} . \dots . \epsilon_{i_n}\}$ running in Γ (choose basepoint v_0). Discard trivial backtracking.



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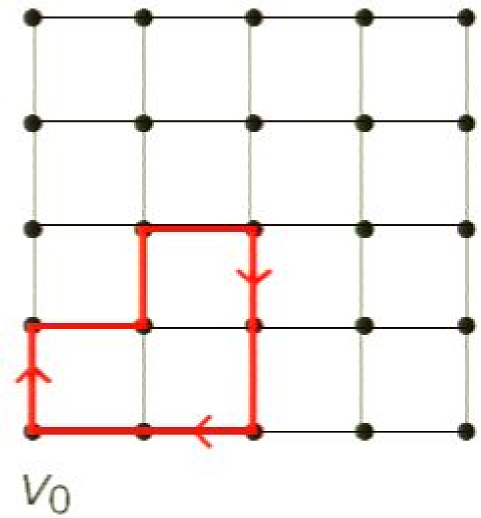

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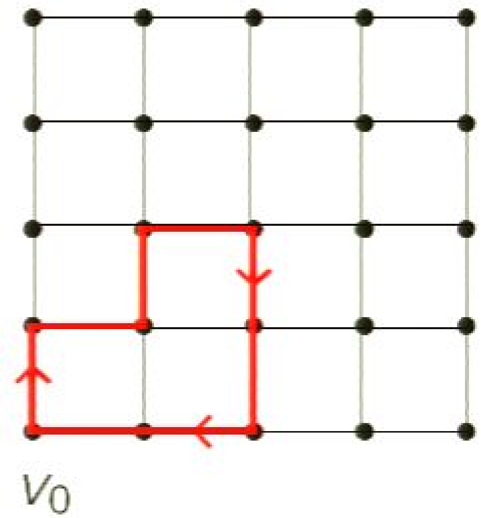

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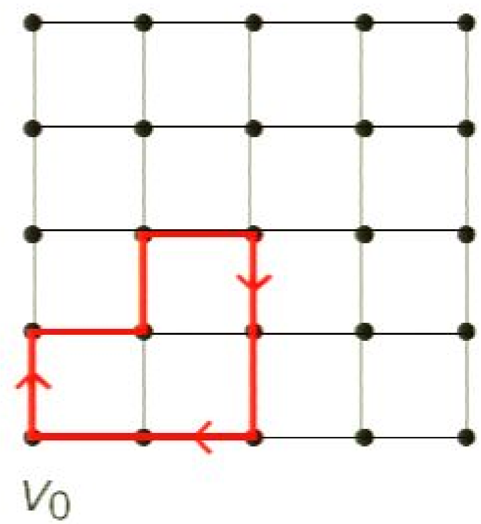

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    with $\quad \epsilon^*_j(\tau) = \epsilon_j(1-\tau)\ , \quad \tau \in \{0,1\}$
  - Consider formal, finite series of loops

$$a = \sum_i a_i L_i \ , \quad a_i \in \mathbb{C}$$

- The product between two elements a and b is defined

$$a \circ b = \sum_{i,j} (a_i \cdot b_j) L_i \circ L_j$$

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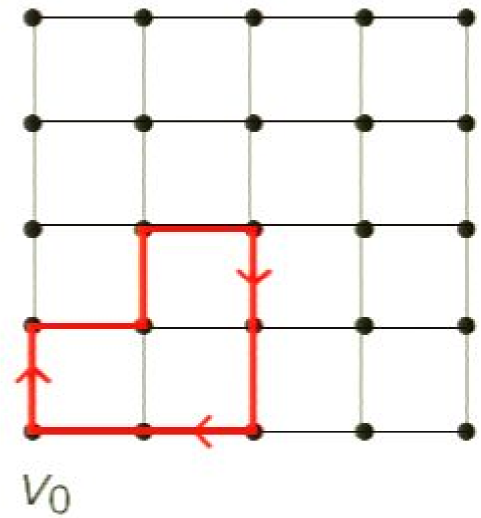

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- These elements have a natural norm

$$||a|| = \sup_{\nabla \in \mathcal{A}_\Gamma} || \sum a_i \nabla(L_i) ||_G$$

where the norm on the rhs is the matrix norm in G. The closure of the $\star$-algebra of loops with respect to this norm is a C^*-algebra. We denote this loop algebra by $\mathcal{B}$.

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Discussion

- **Hilbert space:** There is the (somewhat) natural Hilbert space

$$\mathcal{H} = L^2(G^n, Cl(T^*G^n) \otimes M_l(\mathbb{C}))$$

involving the Clifford bundle over G^n (l size of rep. of G). L^2 is with respect to the Haar measure on G^n.

  - The loop algebra $\mathcal{B}$ has a natural representation on $\mathcal{H}$

$$f_L \cdot \psi(\nabla) = (1 \otimes \nabla(L)) \cdot \psi(\nabla), \quad \psi \in \mathcal{H}$$

where the first factor acts on the Clifford-part of the Hilbert space and the second factor acts by matrix multiplication on the matrix part of the Hilbert space

$$L = \{\epsilon_1, \epsilon_4, \epsilon_6^*, \epsilon_3^*\}$$

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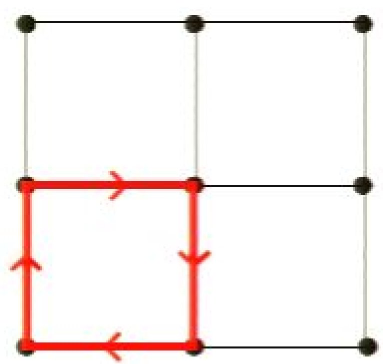

V_0

- **Dirac operator:** We choose the Dirac operator D on G^n

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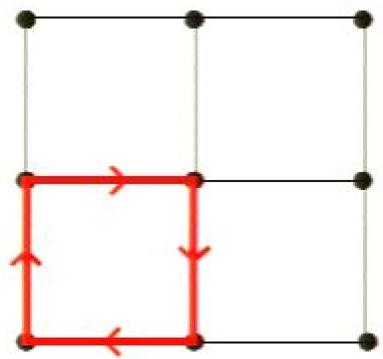

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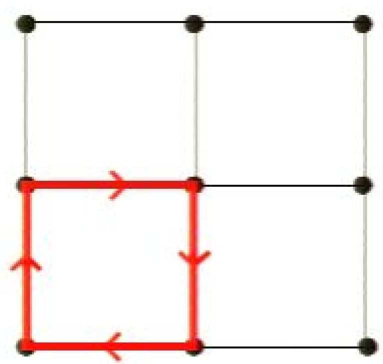

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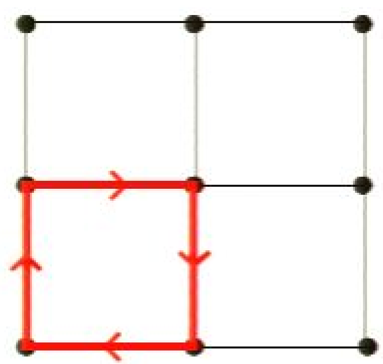

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on the level of the simplicial complex Γ.

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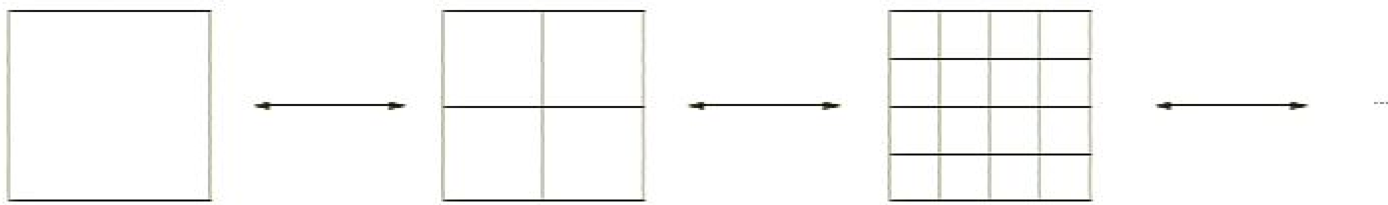
- Second step: is to allow the complexity of the simplicial complex to grow infinitely while keeping control of the spectral triple.

$\Rightarrow$ More refined version of the functional space $\mathcal{A}_\Gamma$.

- Consider a system of nested, lattices

$$\Gamma_0 - \Gamma_1 - \Gamma_2 - \dots$$

with Γ_i a subdivision of Γ_{i-1}



Γ_0 Γ_1 Γ_2

On the level of the associated manifolds $\mathcal{A}_{\Gamma_i}$ this gives rise to projections

$$G^{n_0} \xleftarrow{P_{10}} G^{n_1} \xleftarrow{P_{21}} G^{n_2} \xleftarrow{P_{32}} G^{n_3} \xleftarrow{P_{43}} \dots$$

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- Consider next a corresponding system of spectral triples

$$(\mathcal{B}.\, D.\, \mathcal{H})_{\Gamma_0} - (\mathcal{B}.\, D.\, \mathcal{H})_{\Gamma_1} - (\mathcal{B}.\, D.\, \mathcal{H})_{\Gamma_2} - \dots$$

with the additional condition that the spectral triples are compatible with the projections/embeddings between them.

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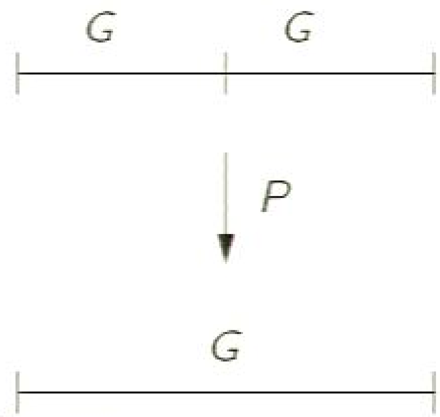
  with the additional condition that the spectral triples are compatible with the projections/embeddings between them.
- For the Hilbert space compatibility is easily obtained (weighted inner product) and compatibility for the algebra is clear.
  To obtain compatibility for the Dirac operator we need to work a little:

- It all boils down to study the simple case

$$P : G^2 \to G \,, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$



corresponding to the compatibility condition

$$P^*(D_1 v)(g_1, g_2) = D_2(P^* v)(g_1, g_2) \,, \quad v \in L^2(G, Cl(T^*G))$$

where D_1 is a Dirac operator on G using the Levi-Civita connection and D_2 is a Dirac operator on G^2 to be constructed. D_2 has the form

$$D_2 = D_{||} + aD_{\perp} \,, \quad a \in \mathbb{R}$$

where $D_{||}$ probes the embedded G and $D_{\perp}$ its orthogonal complement. a is a free parameter.

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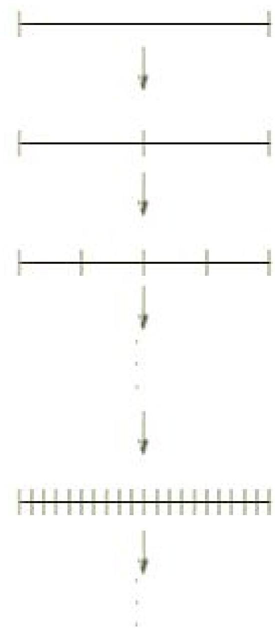

- After repeated subdivisions this gives rise to an infinite series of free parameters $\{a_i\}$.
- By solving the $G^2 - G$ problem repeatedly we end up with a Dirac-like operator on the level of Γ_i

$$D = \sum_k a_k \bar{\mathcal{E}}_k \nabla_{\mathcal{E}_k}$$

where $\mathcal{E}_k$ is an orthonormal set of covectors over G^n and $\bar{\mathcal{E}}_k$ the corresponding element in $Cl(T^*G^n)$. (exact form is complicated).

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- Result: For a compact Lie-group G the triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ is a semi-finite* spectral triple:
  - D_Δ's resolvent $(1 + D_\Delta^2)^{-1}$ is compact (wrt. trace) and
  - the commutator $[D_\Delta, a]$ is bounded

  Provided the sequence $\{a_i\}$ approaches ∞ sufficiently fast. For $G = U(1)$ we find

$$a_n = 2^n b_n \ , \quad \lim_{n \to \infty} b_n = \infty$$

**semi-finite: everything works up to a certain symmetry group with a trace.*

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Spaces of connections

- Denote

$$\overline{\mathcal{A}}^{\Delta} := \varprojlim_{\mathcal{K}} \mathcal{A}_{\mathcal{K}}$$

  or roughly:

$$G^{n_1} - G^{n_2} - \ldots - G^{\infty} \sim \overline{\mathcal{A}}^{\Delta}$$

- $\overline{\mathcal{A}}^{\Delta}$ is a space of generalised connections. To see this map the graphs $\{\Gamma_i\}$ into a manifold $\mathcal{M}$

$$h : \Gamma_i - \Gamma_i \in \mathcal{M}$$

- Denote by $\mathcal{A}$ the space of smooth G-connections. There is a natural map

$$\chi : \mathcal{A} \to \overline{\mathcal{A}}^{\Delta}\,, \quad \chi(\nabla)(\epsilon_i) = Hol(\nabla, \epsilon_i)$$

where $Hol(\nabla, \epsilon_i)$ is the holonomy of ∇ along ϵ_i (now in $\mathcal{M}$).

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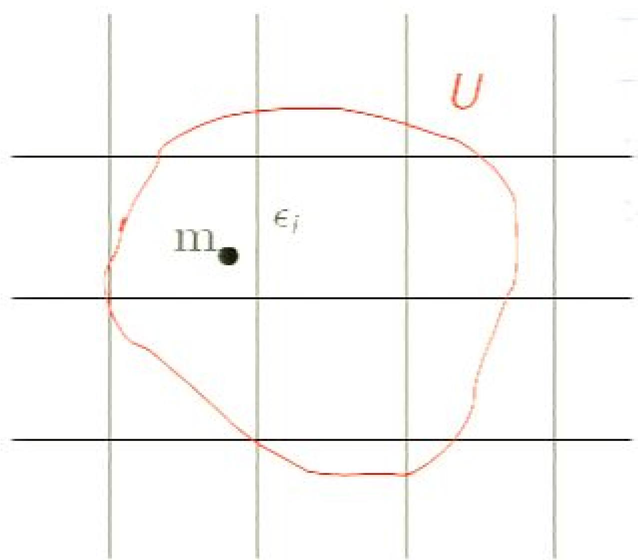
- **Result:** χ is an embedding

$$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}^{\Delta}$$

Argument: given $\nabla_1, \nabla_2 \in \mathcal{A}$ they will differ in a point $m \in \mathcal{M}$ and in a neighbourhood U of m. Choose a small edge ϵ_i in a graphs Γ_i so that $\epsilon_i \in U$. Thus

$$Hol(\nabla_1, \epsilon_i) \neq Hol(\nabla_2, \epsilon_i)$$



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- Thus: $\overline{\mathcal{A}}^{\Delta}$ contains all smooth connections. This implies:
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$$D_{\Delta} \sim \frac{\delta}{\delta \nabla}$$

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$$D_\Delta \sim \sum_x \overline{\nabla(x)} \cdot \frac{\delta}{\delta \nabla(x)}$$

where $\overline{\nabla(x)}$ represents a degree of freedom in each point.

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- **Remark:** The Dirac-type operator D_{Δ} is gauge invariant.

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$$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}^a , \quad \mathcal{A} \hookrightarrow \overline{\mathcal{A}}^\triangle$$

- The *difference* between these completions is their corresponding symmetry groups:
    - In LQG: Analytic diffeomorphisms
    - Here: discrete diffeomorphisms which preserve the graph structure: $\mathrm{Diff}(\triangle)$.

- We observe the following:

$\mathcal{A}$:
- action of diff($\mathcal{M}$)
- no Hilbert space structure
- no Dirac-like operator

$\overline{\mathcal{A}}^a$:
- action of (analytic) diff($\mathcal{M}$)
- Hilbert space structure (non-separable)
- no Dirac-like operator

$\overline{\mathcal{A}}^\Delta$:
- no action of diff($\mathcal{M}$) (few discrete)
- Hilbert space structure (separable)
- Dirac-like operator

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  - the amount of remaining diffeomorphisms
  - the *separability* of the corresponding Hilbert space
  - the existence of a Dirac-like operator.
- It appears that the use of a restricted system of graphs (simplicial complexes or cubic lattices) correspond to a kind of (partly) gauge fixing of the diffeomorphism group.

- Alternative interpretation: Notice that a cubic graph Γ is also a piecewise analytic graph. Thus:

$$L^2(\overline{\mathcal{A}}^{\Delta}) \xrightarrow{\iota} L^2(\overline{\mathcal{A}}^{a}).$$

In LQG there is the Hilbert space $\mathcal{H}_{diff}$ of (spatial) *diffeomorphism invariant states.* A surjection:

$$L^2(\overline{\mathcal{A}}^{a}) \xrightarrow{q} \mathcal{H}_{diff}$$

We therefore get a map

$$L^2(\overline{\mathcal{A}}^{\Delta}) \xrightarrow{\Xi} \mathcal{H}_{diff}$$

- We find the diagram:

$$\begin{array}{ccc} & & L^2(\overline{\mathcal{A}}^a) \\ & \overset{\iota}{\nearrow} & \downarrow q \\ L^2(\overline{\mathcal{A}}^\Delta) & \xrightarrow{\Xi} & \mathcal{H}_{diff} \end{array}$$

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- This means that $\mathcal{H}_\Delta$ is directly related to the Hilbert space of (spatial) diffeomorphism invariant states known from LQG (here we set $G = SU(2)$).

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  - so we should view a loop in $\mathcal{B}_\Delta$ as an equivalence class of loops, up to diffeomorphisms.

The Poisson Structure

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- First, for a single group element g corresponding to the $i'th$ copy of G in G^n we find

$$[D_\Delta, g] = \frac{1}{n}\sum_k \left(\pm g \mathfrak{E}_k \right) \cdot \bar{\mathcal{E}}_k \qquad (a_i \equiv 1)$$

where $\bar{\mathcal{E}}_k \in Cl(T^*G^n)$ and $\mathfrak{E}$ 'twisted' generator of the Lie algebra $\mathfrak{g}$.

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- Next, the commutator between D and the loop L is

$$[D_\Delta, f_L] = [D, g_{i_1}]g_{i_2}\dots g_{i_k} + g_{i_1}[D, g_{i_2}]\dots g_{i_k} + \dots$$

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- **In short**: the action of D is to insert generators $\mathfrak{E}_k$ at each vertex in the loop. Ex: $f_L \sim g_1 g_2 g_3$

$$[D_\Delta, f_L] \sim \dots g_1\mathfrak{E}^1 g_2 g_3 + \dots g_1 g_2 \mathfrak{E}^2 g_3 + \dots g_1 g_2 g_3 \mathfrak{E}^3$$

- **Quantization:** assume the operators $\mathbf{F}^a_S, \mathbf{C}$ exist and satisfy

$$\Rightarrow \quad [\mathbf{F}^a_S, \mathbf{C}] = \pm \mathbf{C}_1 \tau^a \mathbf{C}_2$$

where $\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2$ and where S intersects C in $C_1 \bigcap C_2$.

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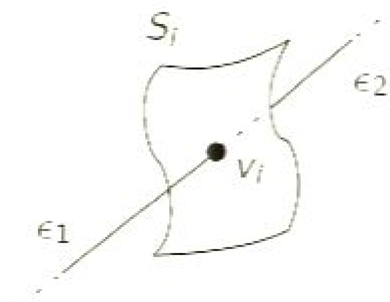
- Consider curves restricted to a lattice Γ_i and surfaces S_i which intersects loops only at vertices.



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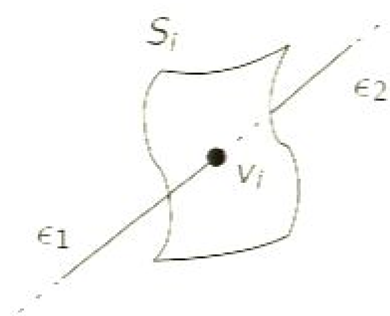

- Expand the twisted generators (ref $\mathcal{E}^i_j$) $\mathfrak{E}^i_j = b^i_{jk} \tau^k$ and define the new operators

$$\mathbf{F}^i_j = \sum_a b^i_{jk} \mathbf{F}^k_{S_i}$$

- Then the operator

$$\Delta_{\Delta} = \frac{1}{n}\sum_{k} \bar{\mathcal{E}}_k \cdot \left(\mathbf{F}_{\mathbf{i}}^{1} \pm \mathbf{F}_{\mathbf{i}}^{2} \pm \mathbf{F}_{\mathbf{i}}^{3} \pm \ldots\right)$$

  satisfy the algebra

$$[\Delta_{\Delta}. \mathbf{C}_j] = \frac{1}{n}\sum_{i} \left(\pm \mathbf{C}_j \mathfrak{E}_i^j\right) \cdot \bar{\mathcal{E}}_i$$

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- This is exactly the commutator between the Dirac operator D_{Δ} and a line segment ϵ_i.

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- In the limit we obtain a representation of the Poisson brackets of General Relativity. This representation is based on a more restrictive choice of graphs than is the representation used in LQG.
- Recall that the Hilbert space $\mathcal{H}_\triangle$ corresponds to a partial solution to the (spatial) diffeomorphism constraint.

  Thus, we can think of our construction as a quantization scheme which deals first with the constraints (partially) and next with the actual quantization (not a Dirac-type quantization).

Area Operators

- In LQG the *area operators* play an important role

$$\mathbf{A}(S) = \sum_n \sqrt{\mathbf{F}^i_{S_n} \mathbf{F}^j_{S_n} \delta_{ij}} \ .$$

where $S = \bigcup_n S_n$.

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where $S = \bigcup_n S_n$. We find that

$$D^2_\Delta = \sum_v \ldots \mathbf{A}^2(S_v) \sim \int_{\mathcal{M}} [d\mathrm{Vol}] \mathbf{A}^2(x)$$

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$$D_\Delta^2 = \sum_v \ldots \mathbf{A}^2(S_v) \sim \int_{\mathcal{M}} [d\text{Vol}] \mathbf{A}^2(x)$$

where $\mathbf{A}(x)$ is a kind of area density operator.

- The spectral action has the form of a Feynman integral

$$Tr \exp(-s\sqrt{(D_\Delta)^2}) \sim \int_{\overline{\mathcal{A}}^\Delta} [d\nabla] \exp\left(-s\sqrt{(D_\Delta)^2}\right) \delta_{\overline{\nabla}}(\nabla)$$

where D_Δ^2 plays the role of an action or an energy.

The Hamiltonian

- The algebra $\mathcal{B}_\Delta$ will, due to its noncommutativity, contain inner automorphisms of the form

$$\alpha_u(b) = ubu^*, \; b \in \mathcal{B}_\Delta$$

where u is a unitary element of $\mathcal{B}_\Delta$, $u^*u = uu^* = 1$.

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- Inner automorphisms generate fluctuations of the Dirac type operator D_Δ

$$\tilde{D}_\Delta = D_\Delta + W$$

  where $W = W^*$ has the general form

$$W = \sum_{ij} n_{ij} b_i [b_j, D_\Delta] \ , \quad b_i, b_j \in \mathcal{B}_\Delta \ , \ n_{ij} \in \mathbb{R}$$

  W is, in the terminology of noncommutative geometry, a one-form.

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(In Connes formulation of the Standard Model the entire bosonic sector is generated by inner fluctuations around D)

- Gravity has no inner automorhisms. Therefore, let us consider a sector of $\mathcal{H}_\Delta$ which is not affected by these fluctuations. We consider the operator constraint

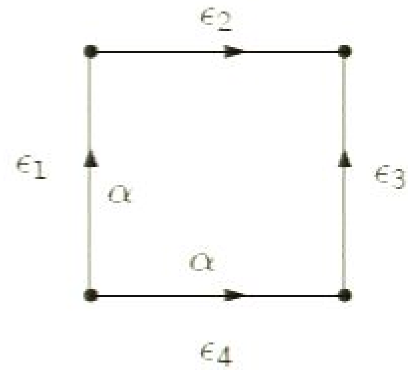
$$Tr\left(D_\Delta^2 - \tilde{D}_\Delta^2\right)\Psi = 0\,. \quad \Psi \in \mathcal{H}_\Delta$$

  (Tr is the matrix trace) which implies

$$Tr\left(\{D_\Delta . W\} + \{W . W\}\right)\Psi = 0\,.$$

Let us find a *local, classical* interpretation of this expression. Consider a fluctuation

$$W = L_1[D_\Delta, L_2]$$



$$L = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3^{-1} \cdot \epsilon_4^{-1}$$

and write

$$D_\Delta \sim \sum \mathcal{E}_j^i E_i^j\,, \quad L \sim 1 + \alpha^2 F_{\mu\nu} ds + \mathcal{O}(\alpha^4)$$

where $E_i^j, F_{\mu\nu}^i$ are the classical fields.

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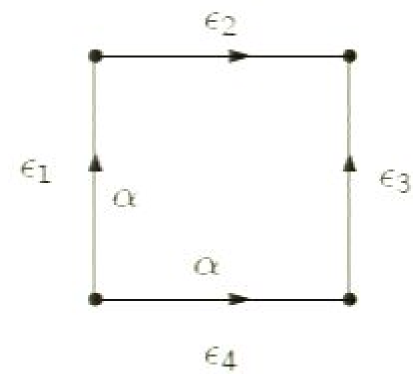

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$$W = L_1[D_\Delta, L_2]$$

and write

$$D_\Delta \sim \sum \mathcal{E}_j^i E_i^j \ , \quad L \sim 1 + \alpha^2 F_{\mu\nu} ds + \mathcal{O}(\alpha^4)$$

where $E_i^j, F_{\mu\nu}^i$ are the classical fields.

The Hamiltonian

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- Then (heuristically)

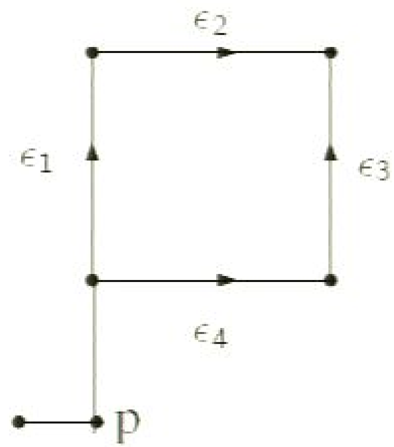
$$Tr\{D_\Delta . W\} + \{W . W\}\Big|_{\text{one vertex}} \overset{\text{classical}}{\rightsquigarrow} \epsilon^{ij}{}_k F^k_{\mu\nu} E^\mu_j E^\nu_k + \mathcal{O}\left(\alpha^2\right)$$

(we set $\alpha = a_n$ for $n \to \infty$)

- This has the form of the Hamilton constraint of GR.

- Consider next the sum over all small loops of area α^2

$$L = p \cdot \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3^{-1} \cdot \epsilon_4^{-1} \cdot p^{-1}$$



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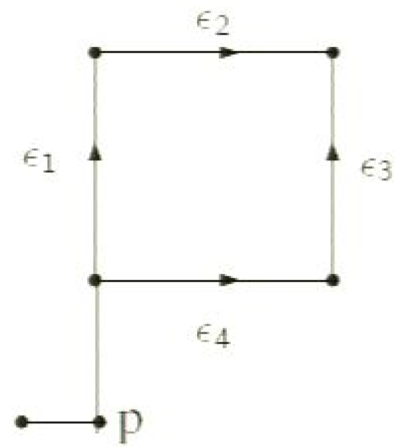

- Then

$$Tr\Big(\{D_\Delta, W\}+\{W, W\}\Big) \overset{\text{classical}}{\rightsquigarrow} \int_M d^3x N \epsilon^{ij}{}_k F^k_{\mu\nu i} E^\mu_j E^\nu_k + \mathcal{O}(\alpha^2)$$

where $N(x)$ is the weighting of each small loop.

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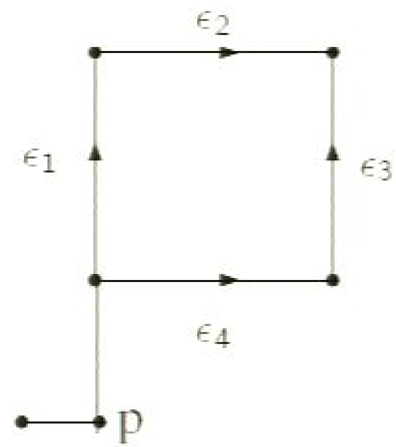

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where $N(x)$ is the weighting of each small loop.

- This has the form of the Hamiltonian of GR. N plays the role of the lapse field.

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- Warning: This is not rigorous.

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  A) Additional structure is needed for this to work (Real structure, extended Dirac operator, extended action of the algebra...)

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- Warning: This is not rigorous.

  A) Additional structure is needed for this to work (Real structure, extended Dirac operator, extended action of the algebra...)

  B) A rigorous formulation of a classical limit ...

- However, the general structure is clear.

Remarks:

- The expression $D_{\Delta}^2 - \tilde{D}_{\Delta}^2$ for a Hamilton constraint is canonical.

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Remarks:

- The expression $D_\Delta^2 - \tilde{D}_\Delta^2$ for a Hamilton constraint is canonical.
- The expression is free of ordering ambiguities and requires no regularization.

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Remarks:

- The expression $D_\Delta^2 - \tilde{D}_\Delta^2$ for a Hamilton constraint is canonical.
- The expression is free of ordering ambiguities and requires no regularization.
- The 'full' system has inner automorphisms = additional degrees of freedom.

  - It is only when one 'turns off' the inner automorphisms that we obtain the structure of the Hamilton constraint.

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  as the Hamiltonian. The advantage of the latter is that it is positive.
- Time evolution operator

$$\mathcal{U}(t) = \exp\left(\mathrm{i}t\tilde{\mathrm{D}}_\Delta^2\right)$$

The sequence $\{a_i\}$

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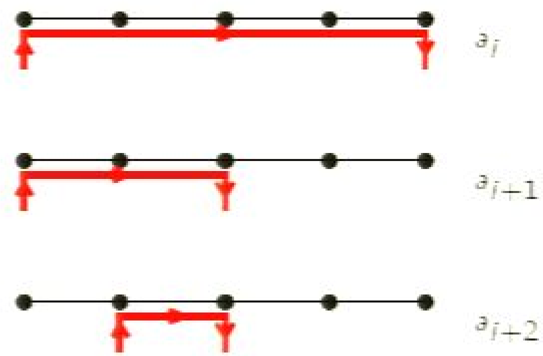
- **The role of the parameters** $\{a_i\}$ is to set a scale. A 'coarse grained' loop corresponds to small a's. A 'refined' loop corresponds to large a's.



- Natural choice:

$$a_n = 2^{-n}$$

However, exactly here D_Δ fails to be spectral — infinities

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- Natural choice:

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- Try instead

$$a_n = (2+\epsilon)^{-n}$$

  and take the limit $\epsilon \to 0$. regularization.

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- **The Goal** is to obtain a classical limit characterized by an almost commutative algebra

$$B = C^\infty(\mathcal{M}) \otimes B_F$$

Connes Distance Formula

- **Connes distance formula:** Given a spectral triple $(\mathcal{A}, D, \mathcal{H})$ over a manifold $\mathcal{M}$ the distance formula reads

$$d(\xi_x, \xi_y) = \sup_{a \in \mathcal{A}} \left\{ |\xi_x(a) - \xi_y(a)| \,\big|\, \|[D, a]\| \leq 1 \right\}$$

  where ξ_x, ξ_y are homomorphisms $\mathcal{A} \to \mathbb{C}$. This can be generalized to noncommutative spaces/algebras.
- **Question:** What about Connes distance formula for the spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$? A distance between field configurations? Yes.
- If two geometries differ on a large scale, then the distance $d(\nabla_1, \nabla_2)$ between their Levi-Civita connections will be 'large' (difference weighted with small a's - large distance)
- If they differ only on short scales, then the distance will be 'small' (difference weighted with large a's - small distance).

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- We have constructed a semi-finite spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ where:

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  - $\mathcal{B}_\Delta$ is an algebra of (holonomy) loops.

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- We have constructed a semi-finite spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ where:
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  - D_Δ resembles a global functional derivation operator.

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  - $\mathcal{B}_\Delta$ is an algebra of (holonomy) loops.
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    - D_Δ resembles a global functional derivation operator.
    - $\mathcal{H}_\Delta$ corresponds (up to a descrete symmetry group) to the Hilbert space of (spatial) diffeomorphism invariant states.
    - The interaction between $\mathcal{B}_\Delta$ and D_Δ encodes the Poisson structure of GR.
    - D_Δ is gauge invariant (Gauss constraint).

Outlook/Open questions

- The construction is based on a countable system of embedded graphs (lattices, simplicial complexes). The construction is essentially combinatorial.

Outlook/Open questions

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- The triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ depends on a set $\{a_i\}$ of scaling parameters. This resembles a regularization scheme.

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- The triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ depends on a set $\{a_i\}$ of scaling parameters. This resembles a regularization scheme.
- Connes distance formula: distances between "geometries".

Outlook/Open questions

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Conclussion

- Exact formulation of the Hamiltonian.
- The spectral action. It resembles a Feynman integral - what exactly is it?
- Computations of the inner fluctuations. What kind of degrees of freedom do they represent?
- Formulation of a classical limit.
- Noncompact structure group?
- ...