Title: Affine Quantum Gravity: A Different View of a Difficult Problem

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Abstract: For quantum gravity, the requirement of metric positivity suggests the use of noncanonical, affine kinematical field operators. In view of gravity\'s set of open classical first class constraints, quantization before reduction is appropriate, leading to affine commutation relations and affine coherent states. The anomaly in the quantized constraints may be accommodated within the projection operator approach, which treats first and second class quantum constraints in an equal fashion. Functional integral representations are derived for expressions both with and without constraint imposition. As with all coherent state formulations, close contact between the classical and quantum theories is maintained throughout. Perturbative nonrenormalizability is understood as a partial hard-core behavior of the interaction, which as soluble models suggest, leads to a perturbative formulation, not about the traditional free theory, but rather about a suitable pseudofree theory that properly incorporates the essence of the hard core.

Horrible Hurdles

- Signature of metric
- Irregular constraints
- Anomalies
- Nonrenormalizability
- Absence of time
- Your favorite choice ...

Go for it!

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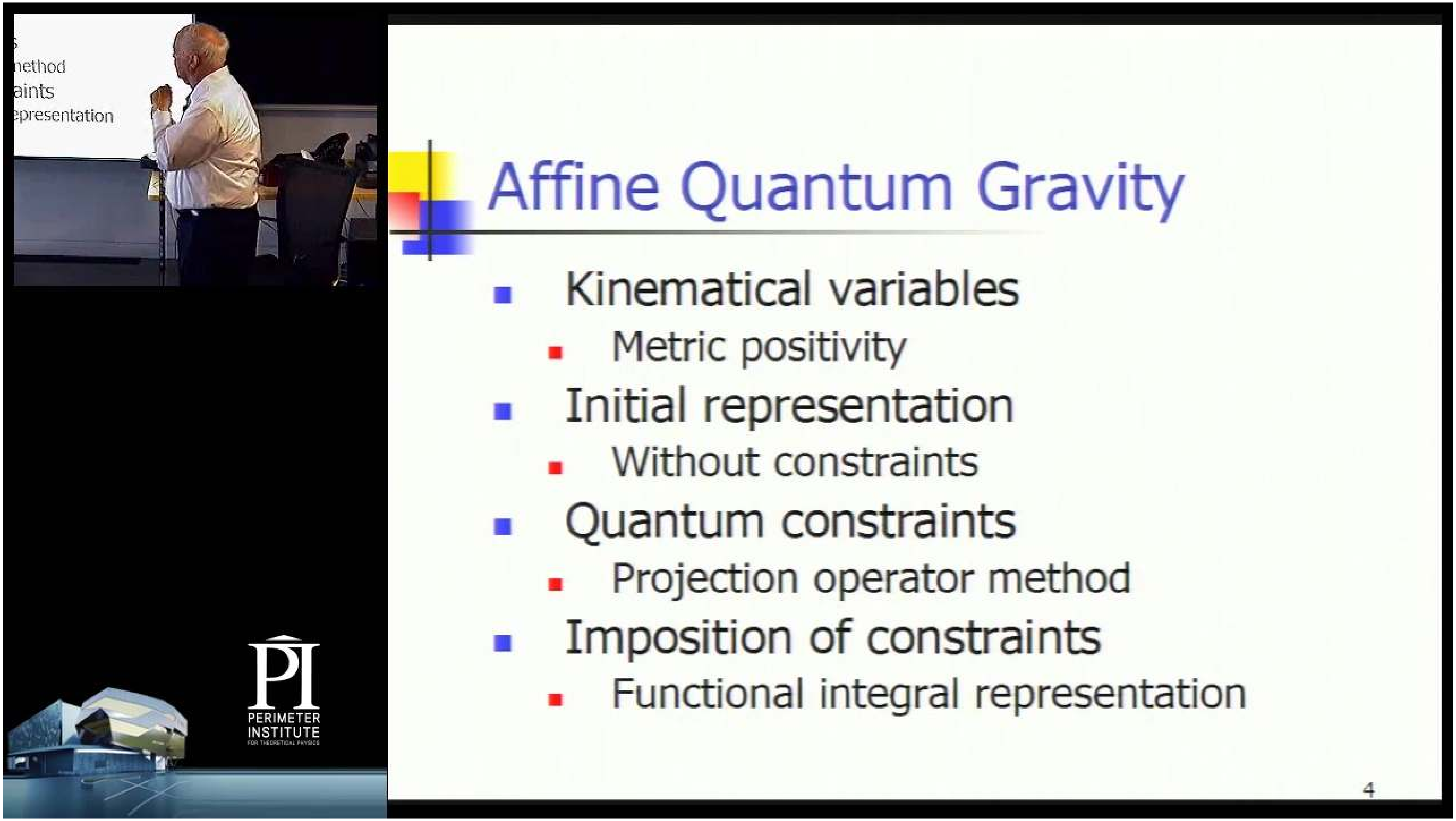
Affine Quantum Gravity
Kinematical variables
Metric positivity
Initial representation
Without constraints
Quantum constraints
Projection operator method
Imposition of constraints
Functional integral representation
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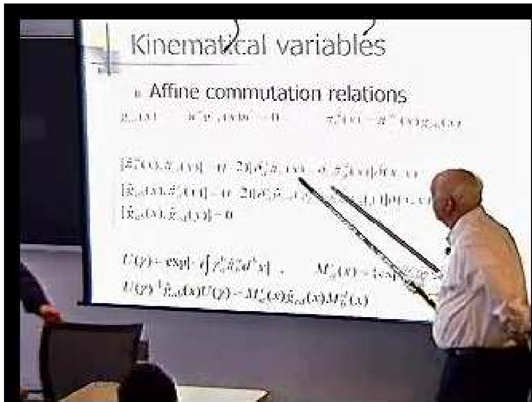

Kinematical variables

- Affine commutation relations

$$g_{ab}(x) \quad , \quad u^a g_{ab}(x) u^b > 0 \quad , \qquad \pi^a_b(x) = \pi^{ac}(x) g_{cb}(x)$$

$$[\hat{\pi}^a_b(x), \hat{\pi}^c_d(y)] = (i/2)[\delta^a_d \hat{\pi}^c_b(x) - \delta^c_b \hat{\pi}^a_d(x)]\delta(x,y)$$
$$[\hat{g}_{ab}(x), \hat{\pi}^c_d(y)] = (i/2)[\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x)]\delta(x,y)$$
$$[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0$$

$$U(\gamma) = \exp[-i\int \gamma^b_a \hat{\pi}^a_b d^3 y] \quad , \qquad M^c_a(x) = \{\exp[\gamma(x)/2]\}^c_a$$
$$U(\gamma)^{-1} \hat{g}_{ab}(x) U(\gamma) = M^c_a(x) \hat{g}_{cd}(x) M^d_b(x)$$

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Initial representation (1)

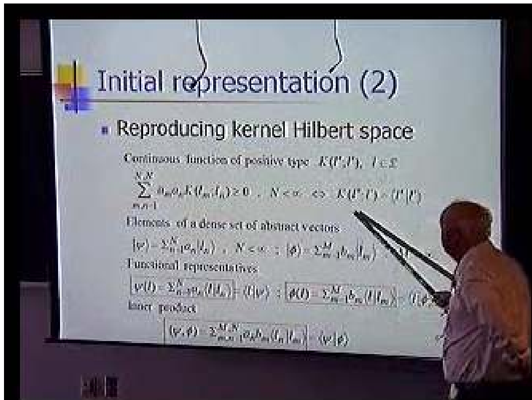
- Affine coherent states

$$|\pi, g\rangle = \exp[i\int \pi^{cd}\hat{g}_{cd}\ d^3y]\ \exp[-i\int \gamma_a^b \hat{\pi}_b^a\ d^3y]\,|\eta\rangle$$

$$\langle\eta|\hat{g}_{cd}(x)|\eta\rangle = \tilde{g}_{cd}(x)\quad,\qquad \langle\eta|\hat{\pi}_b^a(x)|\eta\rangle = 0$$

$$\langle\pi'', g''|\pi', g'\rangle = \exp\Big\{-2\int b(x)\ d^3x \times \ \ln\Big(\frac{\det\{(g''^{ab}+g'^{ab})/2 + ib(x)^{-1}(\pi''^{ab}-\pi'^{ab})/2\}}{[\det\{g''^{ab}\}\det\{g'^{ab}\}]^{1/2}}\Big)\Big\}$$

$$g_{ab}(x) \equiv M_a^c(x)\tilde{g}_{cd}(x)M_b^d(x)\quad,\qquad M_a^c(x) \equiv \{\exp[\gamma(x)/2]\}_a^c$$



Initial representation (2)

- Reproducing kernel Hilbert space

Continuous function of positive type $K(l'';l')$, $l \in \mathfrak{L}$

$$\sum_{m,n=1}^{N,N} \bar{a}_m a_n K(l_m;l_n) \geq 0 \ , \ N < \infty \quad \Leftrightarrow \quad K(l'';l') = \langle l'' | l' \rangle$$

Elements of a dense set of abstract vectors

$$|\psi\rangle = \Sigma_{n=1}^{N} a_n |l_n\rangle \ , \ N < \infty \ ; \ |\phi\rangle = \Sigma_{m=1}^{M} b_m |l_m\rangle \ , \ M < \infty$$

Functional representatives

$$\boxed{\psi(l) = \Sigma_{n=1}^{N} a_n \langle l | l_n \rangle} = \langle l | \psi \rangle \ ; \ \boxed{\phi(l) = \Sigma_{m=1}^{M} b_m \langle l | l_m \rangle} = \langle l | \phi \rangle$$

Inner product

$$\boxed{(\psi, \phi) = \Sigma_{m,n=1}^{M,N} \bar{a}_n b_m \langle l_n | l_m \rangle} = \langle \psi | \phi \rangle$$

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Quantum constraints (1)

- Enforcing the constraints

$\varphi_\alpha(p,q)=0$, $\quad\quad \sum_{\alpha=1}^{A} \varphi_\alpha(p,q)^2=0$

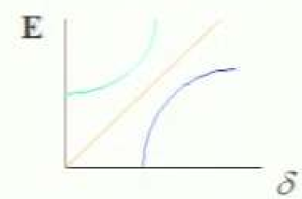
- Dirac : $\Phi_\alpha(P,Q)\,|\,\psi_{\text{phys}}\rangle=0$, (limited use)
- POM : $\mathbf{E}=\mathbf{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2)$, $\mathsf{H}_{\text{phys}}=\mathbf{E}\mathsf{H}$

(1) $\mathbf{E}(J_1^2+J_2^2+J_3^2 \le \hbar^2/2)$

(2) $\mathbf{E}(P^2+Q^2 \le \hbar)$

(3) $\mathbf{E}(Q^2 \le \delta^2)=\mathbf{E}(-\delta < Q < \delta)$

$\lim_{\delta\to 0}\langle p'',q''|\mathbf{E}|p',q'\rangle/\langle\eta|\mathbf{E}|\eta\rangle=\langle\langle p'',q''\,|\,p',q'\rangle\rangle$



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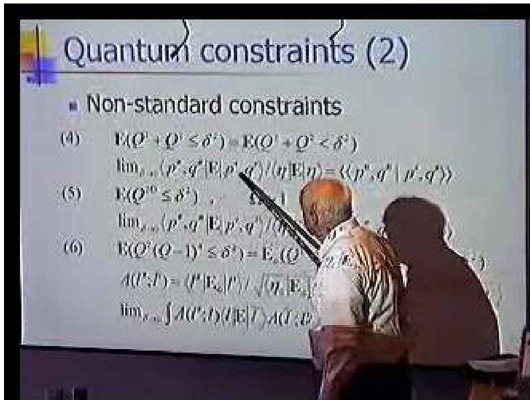

Quantum constraints (2)

- Non-standard constraints

(4) $\mathbf{E}(Q^2+Q^2 \le \delta^2) = \mathbf{E}(Q^2+Q^2 < \delta^2)$

$\lim_{\delta\to 0} \langle p'',q''|\mathbf{E}|p',q'\rangle / \langle \eta|\mathbf{E}|\eta\rangle = \langle\langle p'',q'' \,|\, p',q'\rangle\rangle$

(5) $\mathbf{E}(Q^{2\Omega} \le \delta^2)$, $\Omega > 1$

$\lim_{\delta\to 0} \langle p'',q''|\mathbf{E}|p',q'\rangle / \langle \eta|\mathbf{E}|\eta\rangle = \langle\langle p'',q'' \,|\, p',q'\rangle\rangle$

(6) $\mathbf{E}(Q^2(Q-1)^4 \le \delta^2) = \mathbf{E}_0(Q^2 \le \delta^2) + \mathbf{E}_1((Q-1)^4 \le \delta^2)$

$A(l'';l') = \langle l''|\mathbf{E}_0|l'\rangle / \sqrt{\langle \eta_0|\mathbf{E}_0|\eta_0\rangle} + \langle l''|\mathbf{E}_1|l'\rangle / \sqrt{\langle \eta_1|\mathbf{E}_1|\eta_1\rangle}$

$\lim_{\delta\to 0} \int A(l'';l)\langle l|\mathbf{E}|\tilde{l}\rangle A(\tilde{l};l') \; dl\,d\tilde{l} = \langle\langle l'' \,|\, l'\rangle\rangle$

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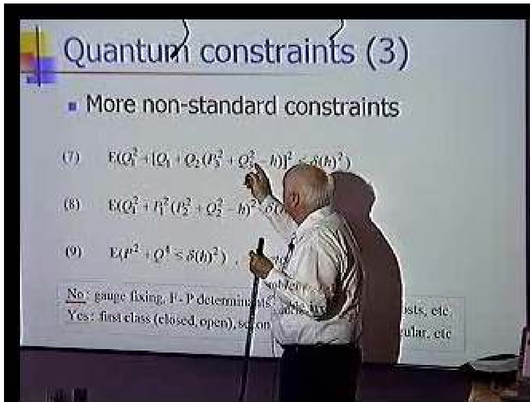

Quantum constraints (3)

- More non-standard constraints

(7) $\mathbf{E}(Q_1^2 + [Q_1 + Q_2(P_3^2 + Q_3^2 - \hbar)]^2 \leq \delta(\hbar)^2)$

(8) $\mathbf{E}(Q_1^2 + P_1^2(P_2^2 + Q_2^2 - \hbar)^2 \leq \delta(\hbar)^2)$

(9) $\mathbf{E}(P^2 + Q^4 \leq \delta(\hbar)^2)$, *etc.*

No : gauge fixing, F - P determinants, Gribov problems, ghosts, etc.
Yes : first class (closed, open), second class, reducible, irregular, etc.

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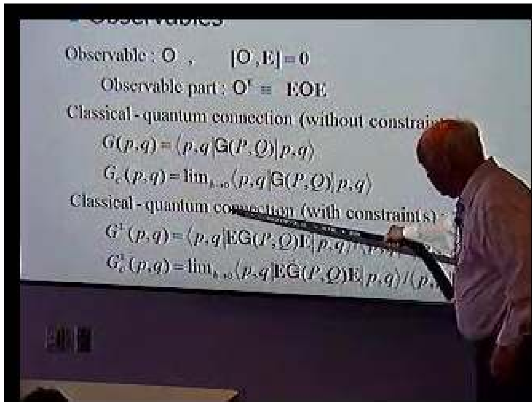

Quantum constraints (4)

- Observables

Observable : O , $[\mathsf{O}, \mathbf{E}] = 0$

Observable part : $\mathsf{O}^{\mathrm{E}} \equiv \mathbf{E}\mathsf{O}\mathbf{E}$

Classical - quantum connection (without constraints) :

$$G(p,q) = \langle p,q|\mathsf{G}(P,Q)|p,q\rangle$$

$$G_c(p,q) = \lim_{\hbar\to 0}\langle p,q|\mathsf{G}(P,Q)|p,q\rangle$$

Classical - quantum connection (with constraints) :

$$G^{\mathrm{E}}(p,q) = \langle p,q|\mathbf{E}\mathsf{G}(P,Q)\mathbf{E}|p,q\rangle / \langle p,q|\mathbf{E}|p,q\rangle$$

$$G_c^{\mathrm{E}}(p,q) = \lim_{\hbar\to 0}\langle p,q|\mathbf{E}\mathsf{G}(P,Q)\mathbf{E}|p,q\rangle / \langle p,q|\mathbf{E}|p,q\rangle$$

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Quantum constraints (5)

- Infinitely many constraints

$$\varphi_m(p,q) = p_m = 0 \;\;,\;\; m = 1,2,3,\ldots \;\;,\;\; \mathbf{E} = \mathbf{E}(\Sigma_{m=1}^{\infty} 2^{-m} P_m^2 \le \delta^2)$$

$$\langle\langle p'', q'' \mid p', q' \rangle\rangle = \lim_{\delta \to 0} \langle p'', q'' | \mathbf{E} | p', q' \rangle / \langle \eta | \mathbf{E} | \eta \rangle$$

$$\varphi_m(p,q) = p_m = 0 \;\;,\;\; \tilde{\varphi}_m(p,q) = q_m = 0 \;\;,\;\; m = 1,2,3,\ldots$$

$$\mathbf{E} = \mathbf{E}(\Sigma_{m=1}^{\infty} 2^{-m} (P_m^2 + Q_m^2) \le \hbar) = \Pi_{m=1}^{\infty} |0_m\rangle\langle 0_m|$$

$$H(x) = 0 \;\;,\;\; \hat{H}(x) \;\;,\;\; \hat{H}_m = \int u_m(x)\, \hat{H}(x) d^3x$$

$$\mathbf{E} = \mathbf{E}(\Sigma_{m=1}^{\infty} 2^{-m} \hat{H}_m^2 \le \delta(\hbar)^2)$$

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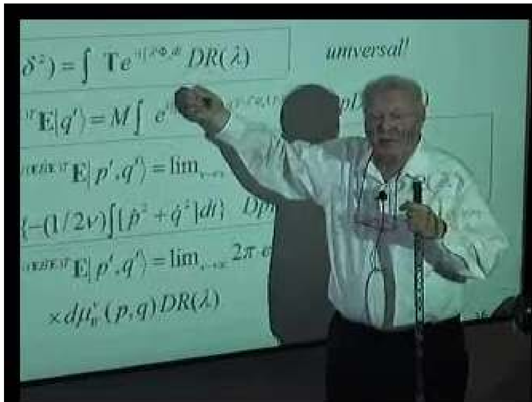

Imposition of Constraints

- Path integral

$$\mathbf{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta^2) = \int \mathbf{T} e^{-i\int \lambda^\alpha \Phi_\alpha dt} DR(\lambda) \qquad \textit{universal!}$$

$$\langle q'' | \mathbf{E} e^{-i(\mathbf{E}\hat{H}\mathbf{E})T} \mathbf{E} | q' \rangle = M \int e^{i\int [p\dot{q} - H(p,q) - \lambda^a \varphi_a(p,q)] dt} \; DpDqDR(\lambda)$$

$$\langle p'', q'' | \mathbf{E} e^{-i(\mathbf{E}\hat{H}\mathbf{E})T} \mathbf{E} | p', q' \rangle = \lim_{\nu \to \infty} \widetilde{M}_\nu \int e^{i\int [p\dot{q} - H(p,q) - \lambda^a \varphi_a(p,q)] dt}$$
$$\times \exp\{-(1/2\nu) \int [\dot{p}^2 + \dot{q}^2] dt\} \; DpDqDR(\lambda)$$

$$\langle p'', q'' | \mathbf{E} e^{-i(\mathbf{E}\hat{H}\mathbf{E})T} \mathbf{E} | p', q' \rangle = \lim_{\nu \to \infty} 2\pi \, e^{\nu T/2} \int e^{i\int [p\,dq - H(p,q)\,dt - \lambda^a \varphi_a(p,q)\,dt]}$$
$$\times d\mu_W^\nu(p,q) \, DR(\lambda)$$

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Imposition of Constraints (2)

- Functional integral

The Heart of the Matter

$$\pi'', g''|\mathbf{E}|\pi', g'\rangle = \lim_{\nu\to\infty} M_\nu \int \exp\{-i\int[g_{ab}\dot{\pi}^{ab} + N^a H_a + NH]dtd^3x\}$$
$$\times\exp\{-(1/2\nu)\int[b^{-1}(x)g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]dtd^3x\}$$
$$\times[\prod_{t,x}\prod_{a\geq b} d\pi^{ab}(t,x)dg_{ab}(t,x)]\;\; DR(N^a, N)$$

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Further Issues

- Nonrenormalizability?
  - Hard-core interpretation
- Appearance of Time?
  - Soluble examples
- Removal of Cutoffs?
  - Choose proper representation
  - Reduce δ as necessary

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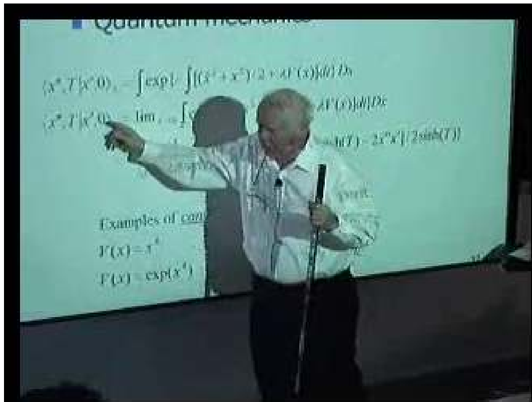
Imposition of Constraints (2)

- Functional integral

The Heart of the Matter

$$\langle \pi'', g'' | \mathbf{E} | \pi', g' \rangle = \lim_{\nu \to \infty} M_\nu \int \exp\{-i \int [g_{ab} \dot{\pi}^{ab} + N^a H_a + NH] dt d^3x\}$$
$$\times \exp\{-(1/2\nu) \int [b^{-1}(x) g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}] dt d^3x\}$$
$$\times [\textstyle\prod_{t,x} \prod_{a \geq b} d\pi^{ab}(t,x) dg_{ab}(t,x)] \quad DR(N^a, N)$$



Perturbation Theory 101

- Quantum mechanics

$$\langle x'',T|x',0\rangle_\lambda = \int \exp\{-\int[(\dot{x}^2+x^2)/2+\lambda V(x)]dt\}\,Dx$$

$$\langle x'',T|x',0\rangle_0 = \lim_{\lambda\to 0}\int \exp\{-\int[(\dot{x}^2+x^2)/2+\lambda V(x)]dt\}Dx$$

$$= \frac{1}{\sqrt{2\pi\sinh(T)}}\exp\{-[(x''^2+x'^2)\cosh(T)-2x''x']/2\sinh(T)\}$$

Examples of *continuous* perturbations

$V(x)=x^4$, $a_0+\lambda a_1+\lambda^2 a_2+\ldots$

$V(x)=\exp(x^4)$, $a_0+\lambda\infty+\ldots$; *etc.*

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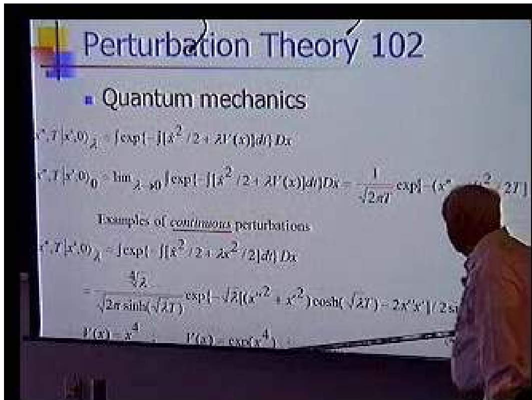

Perturbation Theory 102

- Quantum mechanics

$$''\!,T\,|x',0\rangle_{\lambda} = \int \exp\{-\int[\dot{x}^2/2 + \lambda V(x)]dt\}\, Dx$$

$$''\!,T\,|x',0\rangle_{0} = \lim_{\lambda\to 0} \int \exp\{-\int[\dot{x}^2/2 + \lambda V(x)]dt\} Dx = \frac{1}{\sqrt{2\pi T}} \exp[-(x''-x')^2/2T]$$

Examples of *continuous* perturbations

$$''\!,T\,|x',0\rangle_{\lambda} = \int \exp\{-\int[\dot{x}^2/2 + \lambda x^2/2]dt\}\, Dx$$

$$= \frac{\sqrt[4]{\lambda}}{\sqrt{2\pi \sinh(\sqrt{\lambda}T)}} \exp\{-\sqrt{\lambda}[(x''^2 + x'^2)\cosh(\sqrt{\lambda}T) - 2x''x']/2\sinh(\sqrt{\lambda}T)\}$$

$V(x) = x^4$; $V(x) = \exp(x^4)$; *etc.*

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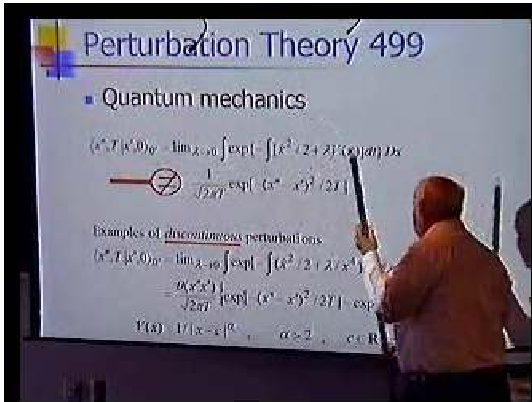

Perturbation Theory 499

- Quantum mechanics

$$\langle x'',T|x',0\rangle_{0'} = \lim_{\lambda\to 0}\int \exp\{-\int[\dot{x}^2/2+\lambda V(x)]dt\}\,Dx$$

$$\neq \frac{1}{\sqrt{2\pi T}}\exp[-(x''-x')^2/2T]$$

Examples of *discontinuous* perturbati ons

$$\langle x'',T|x',0\rangle_{0'} = \lim_{\lambda\to 0}\int \exp[-\int(\dot{x}^2/2+\lambda/x^4)dt]\,Dx$$

$$=\frac{\theta(x''x')}{\sqrt{2\pi T}}\{\exp[-(x''-x')^2/2T]-\exp[-(x''+x')^2/2T]\}$$

$$V(x)=1/|x-c|^{\alpha}\ ,\quad \alpha>2\ ,\quad c\in\mathbf{R}\ ;\quad etc.$$

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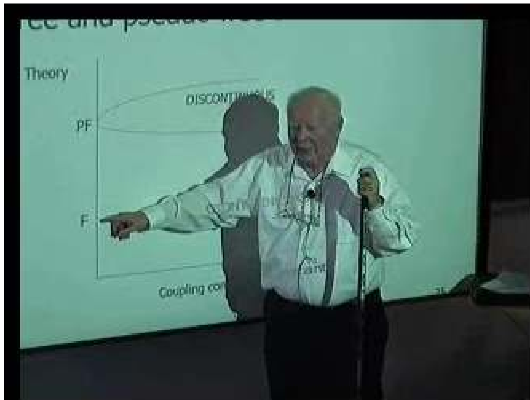
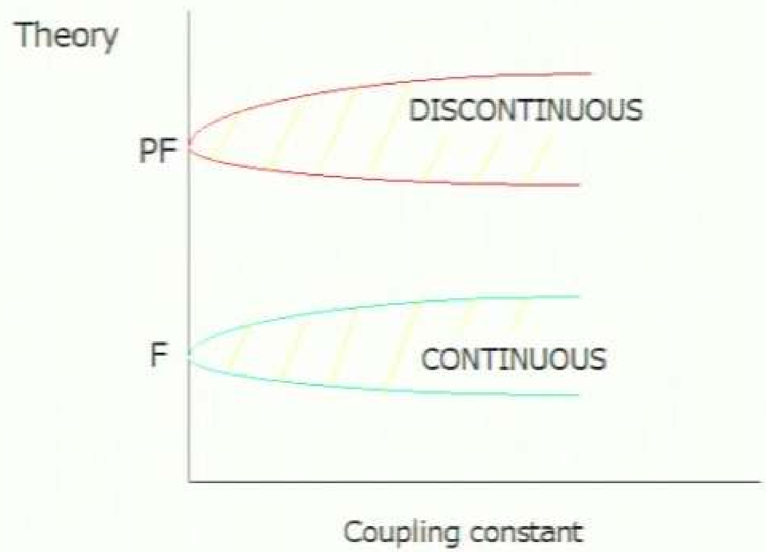


Hard-core interpretation (1)

- Free and pseudo-free theories



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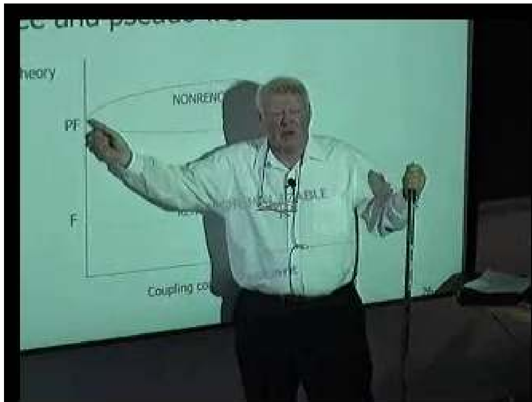
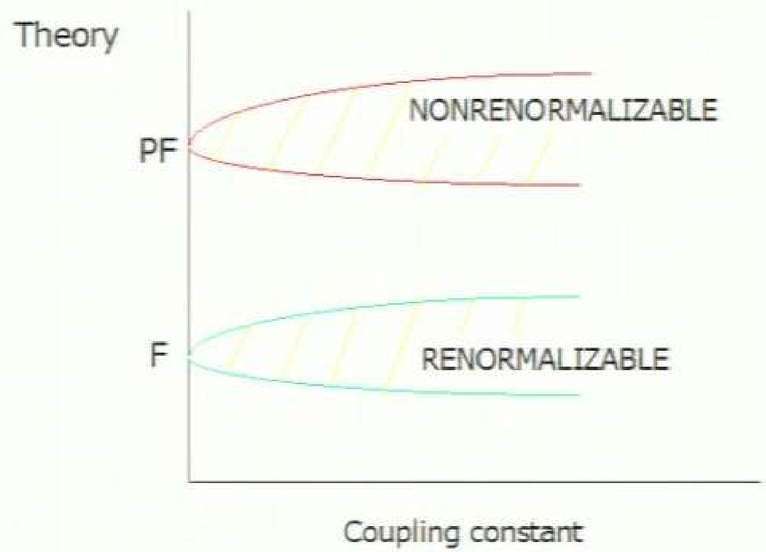


Hard-core interpretation (2)

- Free and pseudo-free theories



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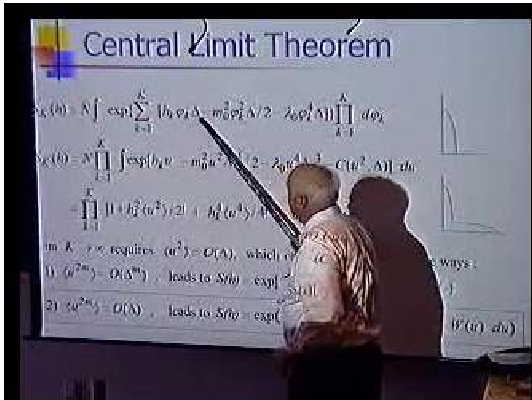
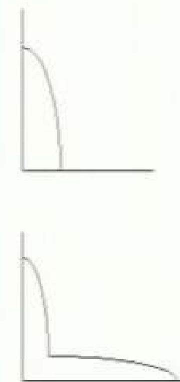


Central Limit Theorem

$$_K(h) = \tilde{N}\int \exp\{\sum_{k=1}^{K} [h_k\varphi_k\Delta - m_0^2\varphi_k^2\Delta/2 - \lambda_0\varphi_k^4\Delta]\}\prod_{k=1}^{K} d\varphi_k$$

$$_K(h) = N\prod_{k=1}^{K} \int\exp[h_k u \; - m_0^2 u^2\Delta^{-1}/2 - \lambda_0 u^4\Delta^{-3} - C(u^2,\Delta)] \; du$$

$$= \prod_{k=1}^{K} [1 + h_k^2\langle u^2\rangle/2! \; + \; h_k^4\langle u^4\rangle/4! \; + \; h_k^6\langle u^6\rangle/6! \; + \; \cdots \;]$$



m $K\to\infty$ requires $\langle u^2\rangle = O(\Delta)$, which can be done in TWO basic ways :

) $\langle u^{2m}\rangle = O(\Delta^m)$, leads to $S(h) = \exp[\; A\int d^n x \; h(x)^2]$ $(C.L.T.)$

2) $\langle u^{2m}\rangle = O(\Delta)$, leads to $S(h) = \exp(\; \int d^n x \int \{\cosh[uh(x)] \; -1\} \; W(u) \; du)$

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Hard-core interaction (3)

- Unconventional model analysis

$$S(h) = N\int \exp\{\int [h\varphi - m_0^2\varphi^2/2 - \lambda_0\varphi^4]d^n x\}D\varphi$$

$$= \exp(b\int d^n x \int [\cosh(hu) - 1]\exp[-bm^2u^2 - b^3\lambda u^4]du/|u|)$$

$$S(h) = N\int \exp(\int \{h\varphi - [\dot\varphi^2 + m_0^2\varphi^2]/2 - \lambda_0\varphi^4]dt d^{n-1}x)D\varphi$$

$$= \exp\{b\int d^{n-1}x \int c(v)\mathbf{T}[\cosh(\int hv(t)dt) - 1]c(v)dv\}$$

$$v(t) = \exp(\hat{\mathrm{h}}t)v\exp(-\hat{\mathrm{h}}t) \quad , \qquad \hat{\mathrm{h}}c(v) = 0 \quad , \quad 0.5 \le \gamma < 1.5$$

$$\hat{\mathrm{h}} = [-\partial^2/\partial v^2 + b^2m^2v^2 + \gamma(\gamma+1)/v^2]/2b + \lambda b^3 v^4 - k$$

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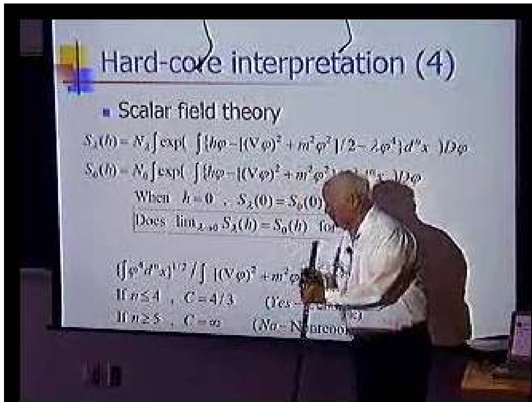

Hard-core interpretation (4)

- Scalar field theory

$$S_\lambda(h) = N_\lambda \int \exp\left(\int \{h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2 - \lambda\varphi^4\} d^n x \right) D\varphi$$

$$S_0(h) = N_0 \int \exp\left(\int \{h\varphi - [(\nabla\varphi)^2 + m^2\varphi^2]/2\} d^n x \right) D\varphi$$

When $h = 0$, $S_\lambda(0) = S_0(0) = 1$

Does $\lim_{\lambda \to 0} S_\lambda(h) = S_0(h)$ for all h?

$$\{\int \varphi^4 d^n x\}^{1/2} / \int [(\nabla\varphi)^2 + m^2\varphi^2] d^n x \le C$$

If $n \le 4$, $C = 4/3$ (*Yes* – Renormalizable)

If $n \ge 5$, $C = \infty$ (*No* – Nonrenormalizable)

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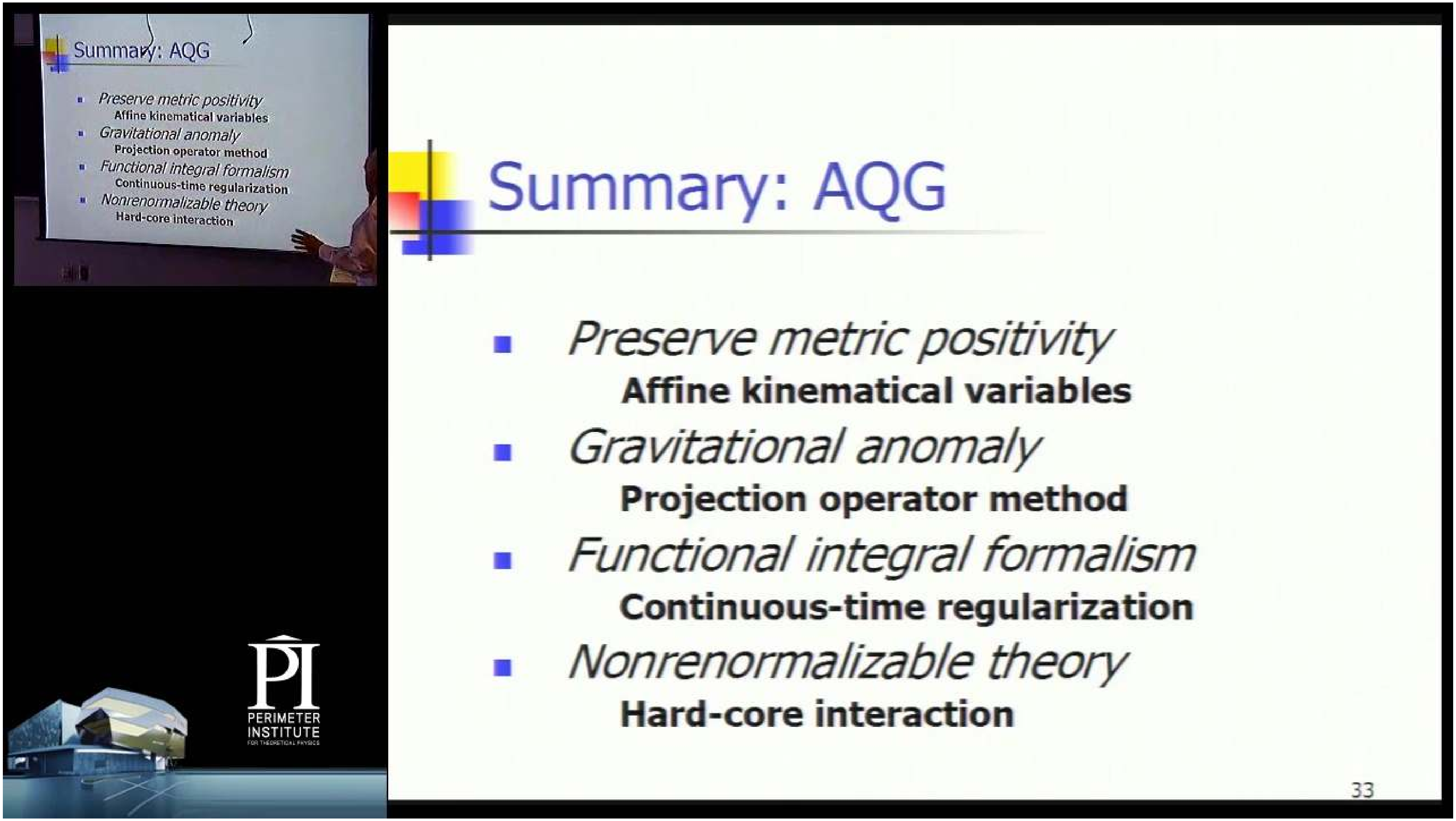
Summary: AQG
Preserve metric positivity
Affine kinematical variables
Gravitational anomaly
Projection operator method
Functional integral formalism
Continuous-time regularization
Nonrenormalizable theory
Hard-core interaction
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Imposition of Constraints (2)

- Functional integral

The Heart of the Matter

$$\langle\pi'', g''|\mathbf{E}|\pi', g'\rangle = \lim_{\nu\to\infty} M_\nu \int \exp\{-i\int[g_{ab}\dot{\pi}^{ab} + N^a H_a + NH]dtd^3x\}$$
$$\times \exp\{-(1/2\nu)\int[b^{-1}(x) g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x) g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]dtd^3x\}$$
$$\times[\textstyle\prod_{t,x}\prod_{a\geq b} d\pi^{ab}(t,x)dg_{ab}(t,x)] \quad DR(N^a,N)$$

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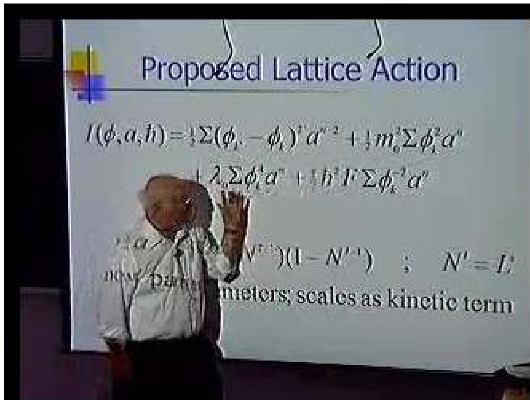
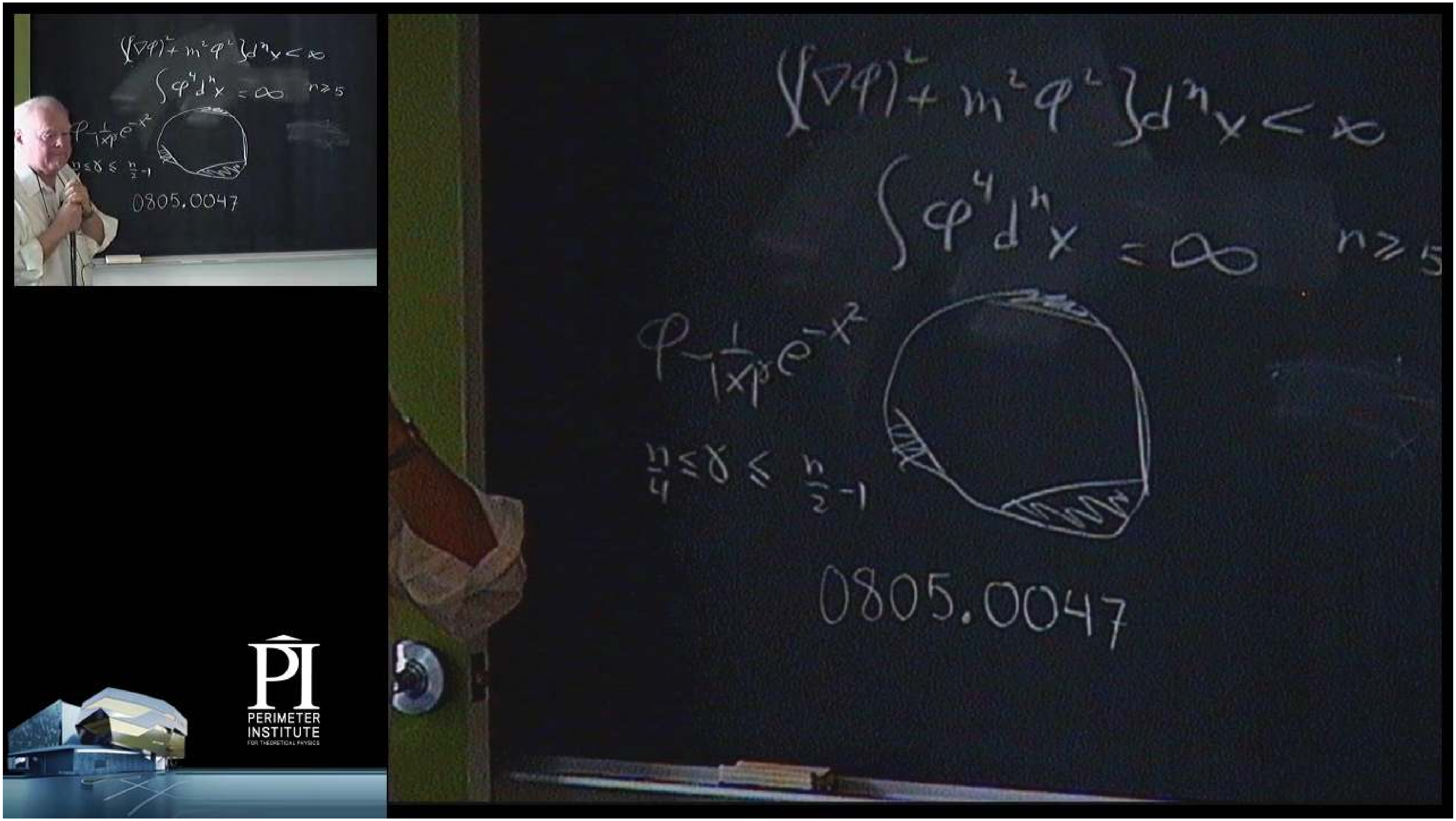


Proposed Lattice Action

$$I(\phi, a, \hbar) = \tfrac{1}{2}\Sigma(\phi_{k'} - \phi_k)^2 a^{n-2} + \tfrac{1}{2} m_0^2 \Sigma \phi_k^2 a^n + \lambda_0 \Sigma \phi_k^4 a^n + \tfrac{1}{2}\hbar^2 F \Sigma \phi_k^{-2} a^n$$

$$F = \tfrac{1}{4} a^{-2s}(3 - N'^{-1})(1 - N'^{-1}) \quad ; \quad N' = L^s$$

No new parameters; scales as kinetic term



∫{(∇φ)² + m²φ²}dⁿx < ∞
∫φ⁴dⁿx = ∞ n ≥ 5
0805.0047
PERIMETER
INSTITUTE