Title: Trans-Planckian physics and renormalization

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Abstract: We analyze the trans-Planckian problem and its formulation in the context of cosmology, black-hole physics, and analogue models of gravity. In particular, we discuss the phenomenological approach to the trans-Planckian problem based on modified, locally Lorentz-breaking, dispersion relations (MDR). The main question is whether MDR leave an detectable imprint on macroscopic physics. In the framework of the semi-classical theory of gravity, this question can be unambiguously answered only through a rigorous formulation of quantum field theory on curved space with MDR. In this context, we propose a momentum-space analysis of the Green\'s function, which will hopefully lead to the correct renormalization of the stress tensor.

Trans-Planckian Physics and Renormalization

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Outline

1. Introduction
2. Modified Dispersion Relations
3. Momentum Space Representation of $G(x, x')$
4. Analogue Models
5. Conclusions

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WHAT HAPPENS AT THE PLANCK SCALE AND BEYOND?

Theoretical models:

- Loop Quantum Gravity
- String Theory
- Non-commutative geometry
- κ-Poincaré algebra

Facts:

- Lorentz group is non-compact
- Lorentz invariance implies ultraviolet divergences
- Analogue models of gravity

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ARE TP EFFECTS VISIBLE?

- l_P is very small, can we ignore trans-Planckian physics?
- There are at least two phenomena in which physics at the Planck scale can be crucial also in the macroscopic world: early/inflationary cosmology and near-horizon black hole physics.
- In both cases, modes with Planck-size wavelength are stretched (red-shifted) by a very large factor, and can affect macroscopic physics.

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EXAMPLE: BLACK HOLES

- Black holes emit radiation with thermal spectrum. The modes involved satisfy

$$\Box\phi = 0$$

- Outgoing null geodesics (characteristics) are given by

$$x = x_0\, e^{k(t-t_0)}\ , \qquad k = 1/4M$$

- Frequency measured by a free-falling observer is

$$\Omega \sim \omega(1 - r_h/r)^{-1}\ , \qquad r_h = 2M$$

- This property is universal
- Can we trust such a boundless growth?

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- Divergence comes from $\omega^2 = p^2$
- What happens if

$$\omega^2 = p^2 + p^4/p_c^2 \quad \text{or} \quad G(x, x') \sim \frac{1}{(x - x')^2 - \lambda_p^2}$$

- Hawking radiation is robust [Unruh, 1981]
- Unruh's effect is robust [Navarro-Salas et al 2007, Rinaldi 2008]
- Does this settle the problem? No, these results say that Hawking radiation/Unruh effect are low-energy effects
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- Some people (e.g. Starobinski) claims that TP effects are largely suppressed.
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$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{\mathrm{TP}}^{\mathrm{ren}}$$

where $\langle T_{\mu\nu} \rangle_{\mathrm{TP}}^{\mathrm{ren}}$ encodes also TP effects

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Modified Dispersion Relations
Lorentz Invariant Case

UV cut-off

- There exits a UV cut-off at $l_p = \sqrt{G\hbar/c^3}$.
- If l_p is a "zero-point length", any process with $E \gg E_p$ will be suppressed.
- [Padmanabhan, 1998] proposed a duality in the path integral.

$$G_F(x,x') = \sum_{\text{paths}} e^{-m\sigma(x,x')(1+l_p^2/\sigma(x,x')^2)}$$

- It follows that

$$\langle \phi^2 \rangle \sim \frac{1}{\sigma^2(x,x') - l_p^2}$$

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- A Lorentz-breaking modified dispersion relation appears in momentum space as [Jacobson et al, 2000]

$$\omega^2 = m^2 + |\vec{k}|^2 + F(|\vec{k}|^2)$$

- $F(|\vec{k}|^2) = a_1|\vec{k}|^4 + a_2|\vec{k}|^6 + \dots$
- The sign of a_i determines whether the modes are super or sub - luminal
- We assume that rotational invariance is preserved.
- In coordinate space, this corresponds to

$$\left[\Box - m^2 - \mathcal{F}(\hat{\nabla}^2)\right]\phi = 0$$

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- This has the form of a unit time-like vector u^μ

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- Any metric can then be written as

$$ds^2 = -(u_\mu dx^\mu)^2 + q_{\mu\nu} dx^\mu dx^\nu \ , \quad q^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

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- MDR for a scalar field coupled to the metric arise with

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- If applied to the inflaton, the extra term does not give inflation
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- also it does not solve the horizon problem: in the superluminal case, modes travel much faster than light but they traverse a super-planckian region where semi-classical approximation does not hold

Preferred frames

The main question is: what happens to $\langle T_{\mu\nu} \rangle_{\rm ren}$ in the semi-classical Einstein equations?

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{\rm ren}$$

Formally

$$\langle T_{\mu\nu} \rangle_{\rm ren} = T^{\rm modes}_{\mu\nu}(\Psi, \nabla\Psi, g_{\mu\nu}, u^\mu, \nabla_\nu u^\mu) - T^{\rm div}_{\mu\nu}$$

Our task is to find $T^{\rm div}_{\mu\nu}$ in the general case. Some result is already available.

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Consider the 2-point function in 2-D flat space [Rinaldi, 2007]

$$\Box\phi(x,t) - \epsilon^2\partial_x^4\phi(x,t) = 0\ , \quad \epsilon^2 > 0$$

In momentum space, the massless propagator reads

$$G(p) = \frac{1}{p_\mu p^\mu + \epsilon^2 p^4} = \frac{1}{\omega_p^2 - p_0^2}$$

The Wightman functions are given by

$$G^\pm(x^\mu, x'^\mu) = \frac{1}{2\pi}\int_\Lambda^{+\infty} \frac{\cos(p\Delta x)}{\sqrt{p^2+\epsilon^2 p^4}}\, e^{\left(\mp i\Delta t\sqrt{p^2+\epsilon^2 p^4}\right)} dp$$

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$$\begin{aligned}\langle\phi^2(x^\mu,x'^\mu)\rangle &= G^+ + G^- = \\ &- \sqrt{\frac{\Delta t}{2\epsilon\pi}}\left[\cos\left(\frac{\Delta x^2}{4\epsilon\Delta t}\right)-\sin\left(\frac{\Delta x^2}{4\epsilon\Delta t}\right)\right]+ \\ &- \frac{\Delta x}{2\epsilon}\left[C\left(\frac{\Delta x}{\sqrt{2\pi\epsilon\Delta t}}\right)+S\left(\frac{\Delta x}{\sqrt{2\pi\epsilon\Delta t}}\right)\right]\end{aligned}$$

where S and C are Fresnel integrals.

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- In the coincidence limit $\Delta t = 0$, $\Delta x = 0$ this function is finite, unlike in the Lorentz-invariant case:

$$\langle \phi^2(x^\mu, x'^\mu) \rangle^{\rm LI} \sim -\frac{1}{2} \ln |\Delta x^2 - \Delta t^2|$$

- The stress tensor $\langle T_{\mu\nu} \rangle$ is still divergent as it depends on derivatives of $\langle \phi^2(x^\mu, x'^\mu) \rangle$.
- What happens in curved space?

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- In Cosmology, we identify u^μ with the velocity of the comoving observer.
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$$\mathcal{L} = R - 2\Lambda - b_2(\nabla_\mu u^\mu)^2 - b_3 R_{\mu\nu} u^\mu u^\nu - \lambda(g_{\mu\nu} u^\mu u^\nu + 1)$$

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Cosmology

In FLRW cosmology, the scalar field depends on time only. If one uses the conformal metric

$$ds^2 = C(\tau)[-d\tau^2 + \delta_{ij}dx^i dx^j]$$

the modes equation for the scaled field $\chi(\tau) = \sqrt{C}\phi$ reads

$$\partial_\tau^2\chi + [(\xi - \xi_n)RC + \omega_k^2]\chi = 0$$

with $\omega_k^2 = k^2 + C[m^2 + \epsilon^2 F(C^{-1/2}|\vec{k}|^2)]$.

- The modes equation is still a second order differential equation: adiabatic WKB regularization still works.

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Cosmology

- For flat FLRW metrics, we can renormalize in the usual way, by absorbing divergences in the bare G_N and Λ:

$$\langle T_{\mu\nu}\rangle^{(\text{ren})} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle^{(0)} - \langle T_{\mu\nu}\rangle^{(2)} - \langle T_{\mu\nu}\rangle^{(4)}$$

- The last term is finite but necessary to recover the trace anomaly in the relativistic limit $\langle T^{\mu}{}_{\mu}\rangle = -1/240\pi^2\alpha^4$
- No terms like $R_{\mu\nu}u^{\mu}u^{\nu}$ appears. The reason is the high degree of symmetry.
- Renormalization is achieved in the usual way by redefining the cosmological constant and Newton's constant [Mazzitelli et al, 2007]
- What happens for less symmetric cases?

Cosmology

- Consider a Bianchi I metric

$$ds^2 = -C(\eta)d\eta^2 + \sum_{i=1}^{3} C_i(\eta)dx_i^2, \qquad C = C_1 C_2 C_3$$

Then, the second adiabatic order correction to the 2-point function is

$$\langle\phi^2\rangle^{(2)} \sim \frac{\alpha}{6}R + \beta(K^2 + 2K_{\mu\nu}K^{\mu\nu}) \qquad K_{\mu\nu} = \nabla_\mu u_\nu$$

- These new terms are related to the extrinsic curvature of the hypersurface orthogonal to u^μ. $\langle T_{\mu\nu}\rangle^{(\text{ren})}$ has similar structure.
- To renormalize, we need to redefine also b_2 and b_3, together with the cosmological constant and Newton's constant.

Stationary Backgrounds

- For stationary backgrounds (e.g. black holes) we have $\phi = \phi(x^j)$. Consider the simplest case: $F(|\vec{k}|^2) = |\vec{k}|^4$.
- The Green's functions $\mathcal{G}$ satisfy the equation

$$(\Box - m^2 - \epsilon^2 \hat{\nabla}^4)\mathcal{G}(x, x') = -g^{-1/2}\delta(x - x')$$

- The differential equation is now of fourth order. WKB methods do not apply. deWitt-Schwinger expansion does not work!
- The conjecture is that the counter-terms must contain all possible combinations of $R_{\mu\nu}$, $g_{\mu\nu}$, and u^μ [Mazzitelli et al, 2007]

Momentum Space Representation of Green's Functions

Riemann coordinates expansion

- Complicated Green's functions can be simplified in momentum space: Bunch & Parker expansion.
- We need a local orthonormal frame.
- Given two close points, we can always create a local orthonormal coordinate system: Riemann Normal Coordinates (RNC).
- In RNC, the metric can be locally written as

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^{\alpha} y^{\beta} + \dots$$

The Fourier transform makes sense locally

$$G(x, x') = \int \frac{d^N k}{(2\pi)^N} \, e^{i g_{\mu\nu} k^{\mu} y^{\nu}} \, \tilde{G}(k^{\mu})$$

Riemann coordinates expansion

- One expand the $\Box$ operator in RNC and Fourier transform.
- The equation can be solved iteratively. For example, the B&P expansion to second order of the Green's functions is

$$\tilde{G} = \tilde{G}_0 + \frac{1}{6} R \tilde{G}_0^2$$

where $\tilde{G}_0 = (k^\mu k_\mu + m^2)^{-1}$ is the flat-space propagator.
- Can we do the same when there is a preferred direction and MDR? Yes.... but it is horribly complicated!

Ultra-static metric case

- We have tried for ultra-static metrics and for a $\hat{\nabla}^4$ dispersion:

$$ds^2 = -d\tau^2 + q_{ij}(x^l)dx^i dx^j \ , \quad (\Box + m^2 - \epsilon^2\hat{\nabla}^4)\phi = 0$$

- Dimensional reduction $\mathcal{G}(x,x') = \int \frac{d\omega}{2\pi}\, e^{i\omega(\tau-\tau')}\, G(x^j, x'^j, \omega)$
- At fourth order, we find via B&P:

$$\tilde{G} = \tilde{G}_0 - \frac{1}{6}\hat{R}D\tilde{G}_0 - \frac{i}{12}\hat{R}_{;j}\tilde{\partial}^j D\tilde{G}_0 + \left(\frac{1}{72}\hat{R}^2 - \frac{1}{3}\hat{H}\right)D^2\tilde{G}_0 +$$

$$- \frac{1}{3}\hat{H}_{ij}\tilde{\partial}^i\tilde{\partial}^j D\tilde{G}_0 \quad D = \partial/\partial(k^i k_i)$$

$$\hat{H}_{ij} = -\frac{1}{30}\hat{R}^p{}_i\hat{R}_{pj} + \frac{1}{60}\hat{R}^p{}_i{}^q{}_j\hat{R}_{pq} + \frac{1}{60}\hat{R}^{pql}{}_i\hat{R}_{pqlj} + \frac{3}{40}\hat{R}_{;ij} + \frac{1}{40}\hat{R}_{ij;p}{}^p$$

Ultra-static deWitt-Schwinger expansion

- In coordinate space we have

$$\mathcal{G}(x^\mu, x'^\mu) = \int \frac{d^{n+1}k}{(2\pi)^{n+1}} e^{ik^\mu y_\mu} \left[1 - f_1 D + f_2 D^2\right] \tilde{G}_0$$

- Define $\quad \tilde{G}_0 = i\int_0^\infty ds\, e^{-is(k^2+\epsilon^2 k^4+m^2-\omega^2)}$
- Find

$$\begin{aligned} G(y) &= i\int_0^\infty ds\, e^{-is(m^2-\omega^2)}\Big[1 + (is)(1-2\epsilon^2\partial^2)f_1 + \\ &+ 2(is)^2\epsilon^2 f_2 + (is)^2(1-2\epsilon^2\partial^2)^2 f_2\Big] I_\epsilon(y,s) \end{aligned}$$

- Where

$$I_\epsilon(y,s) = \frac{e^{\frac{iy^2}{4s}}}{(4is\pi)^{n/2}} \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!}\left(\frac{i\epsilon^2}{16s}\right)^\lambda \mathcal{H}_{[4\lambda]}\left(\frac{\vec{y}}{\sqrt{4is}}\right) \quad \text{[Rinaldi, 2007]}$$

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Fermi coordinates expansion

- Can we compute these terms for a general metric?
- When there is a preferred direction, it is better to use the Fermi Normal Coordinates (FNC):

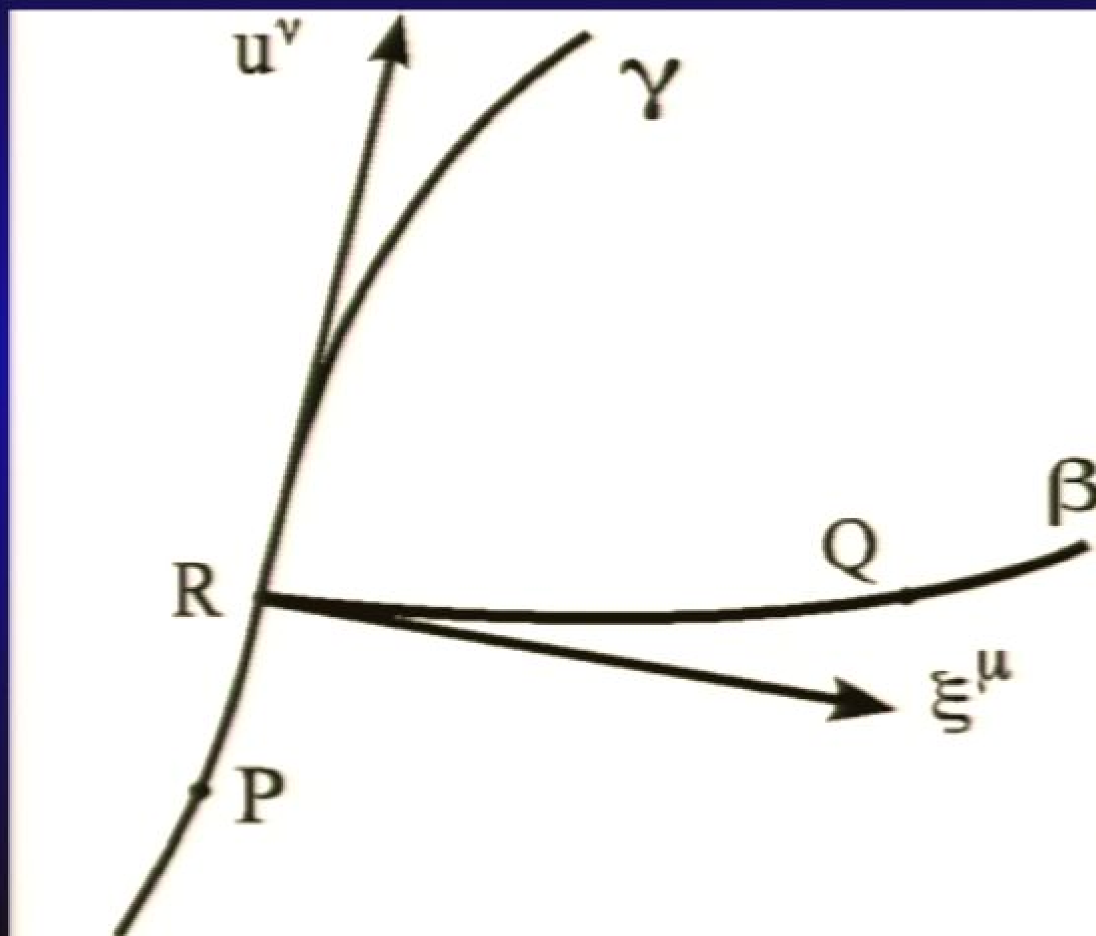
$$ds^2 = g_{00}\,dt^2 + 2g_{0i}\,dt dx^i + g_{ij}\,dx^i dx^j$$

where

$$g_{00} = -1 - R_{0c0d}(t)x^c x^d \quad g_{ab} = \delta_{ab} - \frac{1}{3}R_{acbd}(t)x^c x^d$$
$$g_{0a} = -\frac{2}{3}R_{0cad}(t)x^c x^d \quad u^A = \delta_0^A - \frac{1}{2}R^A{}_{ij0}\,x^i x^j$$

the Riemann tensor components are calculated on the integral curve of u^μ.

Fermi coordinates expansion



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Fermi coordinates expansion

The modified Klein-Gordon equation is

$$(\Box - m^2)\phi + F\left[\hat{\nabla}^2\right]\phi = 0$$

with $\hat{\nabla}^2\phi = q^{\alpha}{}_{\mu}\nabla_{\alpha}(q^{\mu}{}_{\beta}\nabla^{\beta}\phi)$. But:

$$\Box\phi = \hat{\nabla}^2\phi - u^{\alpha}u^{\beta}\nabla_{\alpha}\nabla_{\beta}\phi - K\,u^{\alpha}\partial_{\alpha}\phi$$

where $K = q^{\mu\nu}\nabla_{\mu}u_{\nu}$ is the trace of the extrinsic curvature. If, in flat space, the Fourier transform of the operator F is an analytic function of k^2, we have

$$\sum_{n=1}^{\infty}\alpha_{2n}\hat{\nabla}^{2n}\phi - u^{\alpha}u^{\beta}\nabla_{\alpha}\nabla_{\beta}\phi - K\,u^{\alpha}\partial_{\alpha}\phi - m^2\phi = 0$$

Fermi coordinates expansion

- For Lorentz-invariant dispersions, $n = 1$ and $\alpha_2 = 1$.
- The shifted Green's functions must satisfy

$$g^{1/4} \sum_{n=1}^{\infty} \alpha_{2n} \hat{\nabla}^{2n}(g^{-1/4}\bar{G}) - g^{1/4} u^A u^B \nabla_A \nabla_B (g^{-1/4}\bar{G}) +$$

$$-g^{1/4} K\, u^A \partial_A (g^{-1/4}\bar{G}) - m^2 \bar{G} = -\delta(x^a)\delta(\tau)$$

where

$$G(x, x') = g(x)^{-1/4} \bar{G}(x, x') g(x')^{-1/4}$$

Fermi coordinates expansion

- Expand the equation in FNC up to second order
- Fourier transform

$$\bar{G}(x, x') = \int \frac{d^N k}{(2\pi)^N} \, e^{i g_{AB} k^A x^B} \, \tilde{G}(k_0^2, k^2)$$

- Impose rotational invariance and solve perturbatively to find...

Fermi coordinates expansion

$$\begin{aligned} \tilde{G}_2 &= -\frac{1}{2}\delta^{ab} H_{ab} D\tilde{G}_0 + Q^0{}_b\, k_0\, k^b D(D - 3D_0)\,\tilde{G}_0 \\ &+ Q^a{}_a \left[(D + D_0)\,\tilde{G}_0 - k_0^2 D D_0\, \tilde{G}_0\right] + \\ &+ Q_{ab}\, k^a\, k^b \left[D(D + D_0)\,\tilde{G}_0 - 4\, k_0^2\, \tilde{G}_0 D^2\, \tilde{G}_0\right] \end{aligned}$$

$$\tilde{G}_0 = \left(m^2 - S - k_0^2\right)^{-1}$$

$$S = \sum_{n=1}^{\infty} \alpha_{2n} (-1)^n k^{2n}\ , \quad D_0 S = \frac{\partial S}{\partial k_0^2}\ , \quad DS = \frac{\partial S}{\partial k^2}$$

Fermi coordinates expansion

- The geometrical coefficients are

$$Q^0{}_b = -\frac{1}{6} R^0{}_b \;, \quad Q^a{}_b = R^{0a}{}_{0b} \;, \quad H_{cd} = R^0{}_{c0d} + \frac{1}{3} R^a{}_{cad}$$

- In the ultra-static case, we recover the previous results.
- The coefficients $Q^A{}_b$ and H_{ab} are the same ones found in Bianchi I cosmology via adiabatic regularization.
- They can be written as $R_{\mu\nu} u^\mu u^\nu$, using

$$R_{abcd} = R_{\alpha\beta\gamma\delta}\, e^\alpha{}_a\, e^\beta{}_b\, e^\gamma{}_c\, e^\delta{}_d$$

- Conjecture verified [Rinaldi, 2008]

Fermi coordinates expansion

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Fermi coordinates expansion

- The geometrical coefficients are

$$Q^0{}_b = -\frac{1}{6} R^0{}_b \ , \quad Q^a{}_b = R^{0a}{}_{0b} \ , \quad H_{cd} = R^0{}_{c0d} + \frac{1}{3} R^a{}_{cad}$$

- In the ultra-static case, we recover the previous results.
- The coefficients $Q^A{}_b$ and H_{ab} are the same ones found in Bianchi I cosmology via adiabatic regularization.
- They can be written as $R_{\mu\nu} u^\mu u^\nu$, using

$$R_{abcd} = R_{\alpha\beta\gamma\delta} \, e^\alpha{}_a \, e^\beta{}_b \, e^\gamma{}_c \, e^\delta{}_d$$

- Conjecture verified [Rinaldi, 2008]

Fermi coordinates expansion

- In the Lorentz-invariant case

$$S = -k^2\ , \quad D\tilde{G}_0 = -D_0\tilde{G}_0 = -\tilde{G}_0^2\ , \quad \tilde{G}_0 = (k^2 - k_0^2 + m^2)^{-1}$$

$$\tilde{G}_2^{(\mathrm{rel})} = \frac{1}{2} H\tilde{G}_0^2 + 2Q^a{}_a\, k_0^2\,\tilde{G}_0^3 + 8\,Q_{0b}\, k^0 k^b\, \tilde{G}_0^3 - 8\,Q_{ab}\, k^a p^b k_0^2\,\tilde{G}_0^4$$

different from the usual Bunch and Parker expansion!

$$\tilde{G}_2^{(\mathrm{rel})} = \frac{1}{6} R\tilde{G}_0^2$$

- The reason is that in FNC we expand around a flat metric and along a geodesics curve, not a point as in RNC!
- However, the divergent parts are the same.

Analogue Models

Analogue models

Let us consider a fluid which is

- Irrotational: $\vec{v} = \vec{\nabla}\psi$
- Homentropic: the pressure is a function of the density ρ only
- No external forces

$$S = -\int d^4x \left[\rho\partial_t\psi + \frac{1}{2}\rho(\vec{\nabla}\psi)^2 + u(\rho)\right]$$

where the energy density is given by $\mu = du/d\rho$ and t is the Newtonian (i.e. laboratory) time.

Analogue models

- We now look at fluctuations

$$\rho = \rho_0 + \rho_1, \qquad \psi = \psi_0 + \psi_1$$

- Then $S = S_0 + S_2$ as linear terms drop out (they describe mean fluctuations).
- The dynamics of the perturbations ρ_1 and ψ_1 is then described by

$$S_2 = -\int d^4x \left[\frac{1}{2}\rho_0(\vec{\nabla}\psi_1)^2 - \frac{\rho_0}{2c^2}(\dot{\psi}_1 + \vec{v}\cdot\vec{\nabla}\psi_1)^2\right]$$

$$c = \rho \frac{d\mu}{d\rho}\Big|_{\rho_0} = \text{speed of sound}$$

Analogue models

- Variation of S_2 w.r.t. ψ_1 (eq. for ρ_1 inserted) gives

$$
\begin{aligned}
- \quad & \partial_t \left[\frac{\rho_0}{2c^2} (\dot{\psi}_1 + \vec{v}_0 \cdot \vec{\nabla} \psi_1) \right] + \\
+ \quad & \vec{\nabla} \cdot \left\{ \vec{v}_0 \left[-\frac{\rho_0}{c^2} (\dot{\psi}_1 + \vec{v}_0 \cdot \vec{\nabla} \psi_1) \right] + \rho_0 \vec{\nabla} \psi_1 \right\} = 0
\end{aligned}
$$

- This equation can be written as

$$\partial_\mu (f^{\mu\nu} \partial_\nu \psi_1) = 0$$

with

$$f^{\mu\nu} = \frac{\rho_0}{c^2} \begin{pmatrix} -1 & -v_0^i \\ -v_0^i & c^2 \delta_{ij} - v_0^i v_0^j \end{pmatrix}$$

Analogue models

- By defining the matrix $g^{\mu\nu}$ such that $f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, the equation becomes

$$\Box\psi_1 = 0$$

i.e. the fluctuations ψ_1 propagates as a massless scalar field on a curved backround!

- The curvature is locally given by the background fluid velocity v_0 and the local speed of sound c.
- Also the action can be re-written as

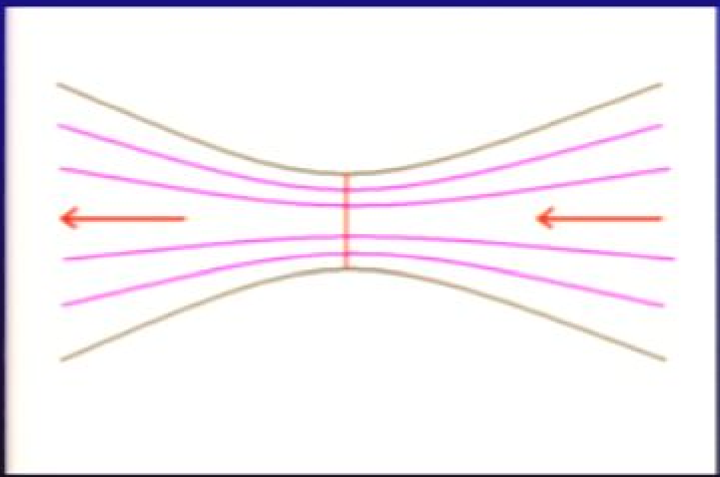
$$S_2 = -\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_1$$

Analogue models

- Inverting $g^{\mu\nu}$ one can write the acoustic metric

$$ds^2 = \frac{\rho_0}{c}\left[-(c^2 - \vec{v}_0^2)\,dt^2 - 2\delta_{ij}v_0^i\,dx^j\,dt + \delta_{ij}\,dx^i\,dx^j\right]$$

which is the Painlevé form! One can actually construct also an acoustic black hole in two dimensions with a simple nozzle



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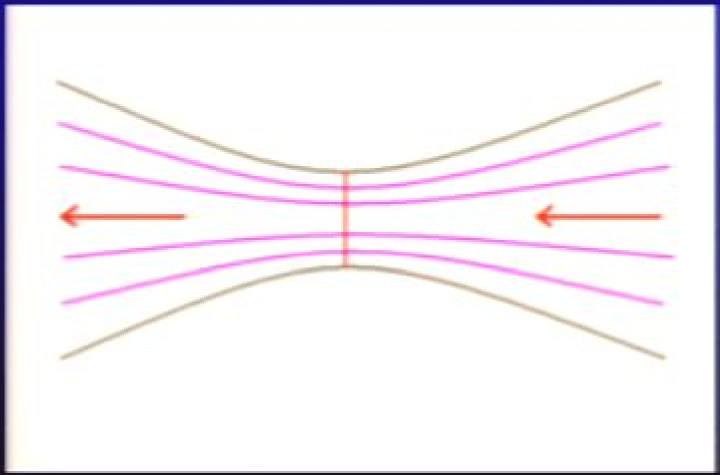

Analogue models

- Such devices can be built in a lab using Bose-Einstein condensates.
- Gross-Pitaevskii equation

$$i\hbar\partial_t \hat{\Psi} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + k(a)\hat{\Psi}^\dagger\hat{\Psi} \right) \hat{\Psi}$$

- Madelung representation: $\hat{\Psi} = \sqrt{\hat{n}}\, e^{i\hat{\theta}/\hbar}$ with $\vec{v} = \vec{\nabla}\theta/m$
- Fluctuation: $\hat{n} = n + \hat{n}_1$ and $\hat{\theta} = \theta + \hat{\theta}_1$
- We find: $\Box\hat{\theta}_1 = 0$
- With $ds^2 = \frac{n}{mc}\left[-c^2dt^2 + (d\vec{x} - \vec{v}dt)^2\right]$
- This result holds for long wavelength: $\omega = k$
- More accurate calculations for short wavelength : $\omega \sim k^2$

Conclusions

The work done so far is necessary to find the renormalized energy-momentum tensor $\langle T_{\mu\nu}\rangle_{\rm ren}$ with the point-splitting technique:

$$\langle T_{\mu\nu}\rangle_{\rm ren} \sim \lim_{y\to 0}\int \frac{d^N k}{(2\pi)^N}\, e^{ik^\mu y_\mu}\left[\tilde{T}_{\mu\nu}^{(\rm modes)} - \tilde{T}_{\mu\nu}^{(\rm MDR)}\right]$$

With this expression, we can test many predictions in semi-classical gravity and analogue models.

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Fermi coordinates expansion

$$\begin{aligned}\tilde{G}_2 &= -\frac{1}{2}\delta^{ab} H_{ab} D\tilde{G}_0 + Q^0{}_b\, k_0\, k^b D(D - 3D_0)\tilde{G}_0 \\ &+ Q^a{}_a \left[(D + D_0)\tilde{G}_0 - k_0^2 D D_0 \tilde{G}_0\right] + \\ &+ Q_{ab}\, k^a\, k^b \left[D(D + D_0)\tilde{G}_0 - 4\, k_0^2 \tilde{G}_0 D^2 \tilde{G}_0\right]\end{aligned}$$

$$\tilde{G}_0 = \left(m^2 - S - k_0^2\right)^{-1}$$

$$S = \sum_{n=1}^{\infty} \alpha_{2n}(-1)^n k^{2n}\ , \quad D_0 S = \frac{\partial S}{\partial k_0^2}\ , \quad DS = \frac{\partial S}{\partial k^2}$$

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