Title: Yang-Baxter Equations, Extra-special Two-groups and Topological-like Features in Quantum Information Theory

Date: Apr 29, 2008 11:10 AM

URL: http://pirsa.org/08040065

Abstract: Recently a simple but perhaps profound connection has been observed between the unitary solutions of the Yang-Baxter Equations (YBE) and the entangled Bell states and their higher dimensional (or more-qubit) extensions, the generalized GHZ states. We have shown that this connection can be made more explicit by exploring the relation between the solutions of the YBE and the representations of the extra-special two-groups. This relationship brings certain topological-like features to quantum information theory, and makes a connection to the well-known Jones polynomials which are topological invariants of knots and links. This emerging connection may deepen our understanding, through new representations of extra-special two-groups, of quantum error correction and topological quantum computation. This work is a collaboration with Eric Rowell, Zhenghan Wang, Molin Ge, and Yong-Zhang.

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

1 / 23 100% Find

Yang-Baxter Equations, Extra-Special Two-groups, and Topological-like Features in Quantum Information Theory

Yong-Shi Wu and Yong Zhang
Department of Physics, University of Utah

(Perimeter Institute for Theoretical Physics, April 29, 2008)

with Eric C. Rowell, Zhenghan Wang, and Mo-Lin Ge

See: arXiv:0706.1761v2

We derive unitary braid representations in terms of representations of extra-special two-groups. We suggest that this provides a possible link between quantum error correcting codes and topological-like quantum computing.

Motivation: Quantum Error Correcting Codes (QECC)

Obstacle for physical realization of quantum computer: Decoherence due to noise

QECC: To protect quantum information from noise.

1.1. Shor (1995): Shor nine-qubit codes;

1.2. Calderbank, Rains Shor, Steane codes (1996): The Galois Field GF(4) orthogonal geometry approach;

CRSS (1996) suggested that the extraspecial two-groups provide "a bridge between quantum error-correcting codes in Hilbert space and orthogonal geometry".

2. Shor (1996): Fault-tolerant Quantum Computation

GHZ states are often called cat states acting as ancillas in many fault-tolerant quantum computation.

3.1. Kitaev (1997): Topological quantum error correcting codes

(Qubits: anyons; Quantum gates: unitary braid representation)

M. Freedman, A. Kitaev, M. Larsen, and Zhenghan Wang (2003): Topological Quantum Computation

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

2 / 23 100% Find

Interdisciplinary involving Math, Physics and Computer Science

Interdisciplinary topics include

Computer Science: New (Topological) Codes
Mathematics: Algebra, Topology, Category Theory, etc.
Physics: Quantum Mechanics, Quantum Field Theories, etc.
(Math-Physics:) Yang–Baxter Equation, Statistical Integrable Models etc.

The Main Theme of my presentation today: Connections Among above Three.

More concretely,

Topology: Braid Group Repres., Jones polynomial, Chern-Simons-Witten Theory
Math Physics: Yang-Baxter Equations, Integrable Statistical Models Physics:

1

Topological Phases, e.g. Quantum Charge and Spin Hall Effect

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

3 / 23 100% Find

Topological Phases, e.g. Quantum Charge and Spin Hall Effect

How is Topological Protection of Quantum Information related to Many-body Integrability?

Kitaevs two models have shown some details. Other models?

Braid Group Relation and Quantum Gates (I)

The Artin braid group $\mathcal{B}_n$ with Generators: $b_1, b_2, \cdots, b_{n-1}$.

The commutative relation: $b_i b_j = b_j b_i, |i-j| > 1$

The braid group relations (also called the braided Yang–Baxter equation (YBE)): $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$, which is symmetric under the exchange of i and $i+1$.

Braid group is the symmetry group that determines quantum statistics (Abelian and Non-abelian anyons) in 2-dim. systems.

Yong-Shi Wu, General Theory of Quantum Statistics in Two Dimensions, Phys. Rev. Lett. 52, 2103 (1984).

In quantum computing, a nontrivial unitary solution of the braid group relation plays the role of a universal quantum gate.

WTL_PI080429.pdf - Adobe Reader

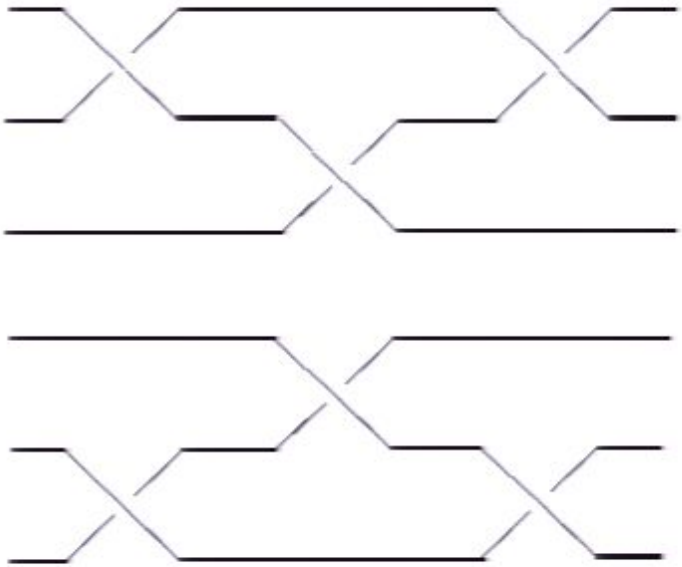
File Edit View Document Tools Window Help

5 / 23 100% Find

J.L. Brylinski and R. Brylinski (2002). A unitary braiding operator can be regarded as a universal quantum gate if it can transform a separable state to an entangling state.

Braiding Gate and Topological Quantum Computing

The braid group relation = an equation of universal quantum gates



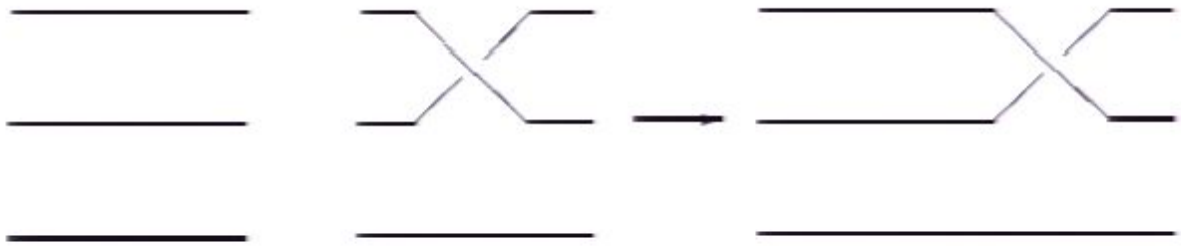
Braid configuration: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

This configuration is also for the braided YBE and the quantum YBE.

WTL_PI080429.pdf - Adobe Reader

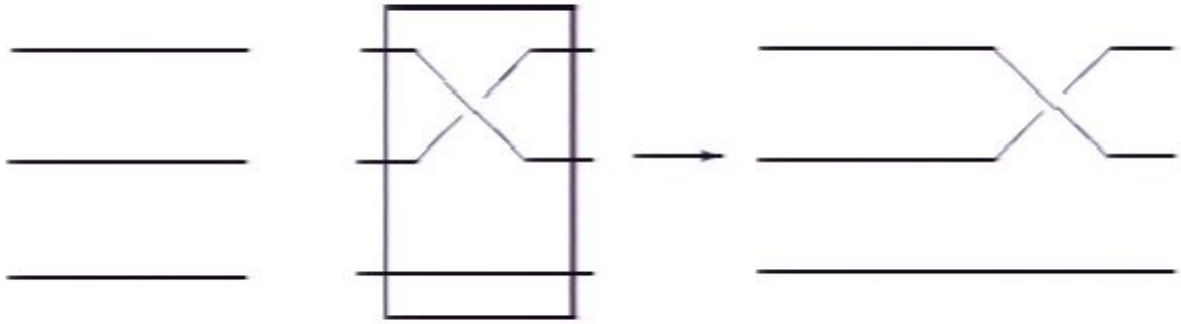
File Edit View Document Tools Window Help

4 / 23 100% Find



$b \otimes Id$

Fig. 1. A braiding operator $b \otimes Id$ in braiding configuration



$b \otimes Id$

Fig. 1. A braiding gate $b \otimes Id$ in quantum circuit

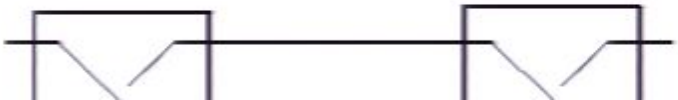
Braiding Gate and Topological Quantum Computing

The braid group relation = an equation of universal quantum gates

=

Braid configuration: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

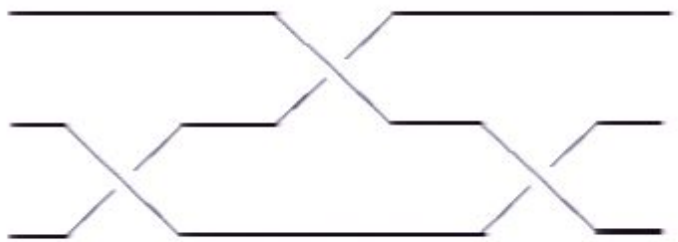
This configuration is also for the braided YBE and the quantum YBE.



Braiding Gate and Topological Quantum Computing

The braid group relation = an equation of universal quantum gates

=



Braid configuration: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

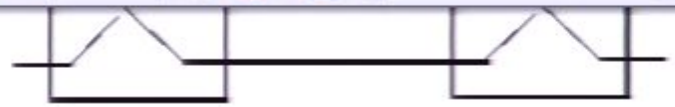
This configuration is also for the braided YBE and the quantum YBE.

=

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

7 / 23 100% Find



Quantum circuit: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

In topological quantum computing, all gates are Braiding Gates

5

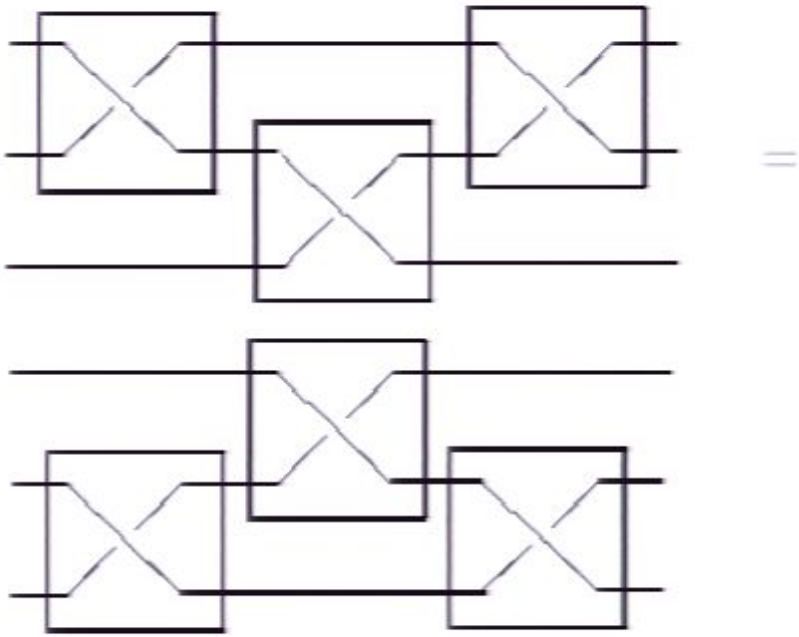
Topological(-Like) Quantum Computing (I)

In topological-like quantum computating, both unitary braiding gates and non-braiding gates are involved.

Given a solution $\check{R}(x)$ of quantum YBE, there is always an integrable model to be constructed. The QYBE with x, y spectral parameter has a similar form

Braid configuration: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

This configuration is also for the braided YBE and the quantum YBE.



Quantum circuit: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

In topological quantum computing, all gates are Braiding Gates

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

7 / 23 100% Find

Quantum circuit: $(b \otimes Id)(Id \otimes b)(b \otimes Id) = (Id \otimes b)(b \otimes Id)(Id \otimes b)$

In topological quantum computing, all gates are Braiding Gates

Topological(-Like) Quantum Computing (I)

In topological-like quantum computating, both unitary braiding gates and non-braiding gates are involved.

Given a solution $\check{R}(x)$ of quantum YBE, there is always an integrable model to be constructed. The QYBE with x, y spectral parameter has a similar form

Topological(-Like) Quantum Computing (I)

In topological-like quantum computating, both unitary braiding gates and non-braiding gates are involved.

Given a solution $\check{R}(x)$ of quantum YBE, there is always an integrable model to be constructed. The QYBE with x, y spectral parameter has a similar form

$$(\check{R}(x)\otimes Id)(Id\otimes\check{R}(xy))(\check{R}(y)\otimes Id) = (Id\otimes\check{R}(y))(\check{R}(xy)\otimes Id)(Id\otimes\check{R}(y))$$

where $\check{R}(0)$ forms a braid representation, i.e.,

$$(\check{R}(0)\otimes Id)(Id\otimes\check{R}(0))(\check{R}(0)\otimes Id) = (Id\otimes\check{R}(0))(\check{R}(0)\otimes Id)(Id\otimes\check{R}(0))$$

But $\check{R}(x)$ is often not a braid representation.

Yang–Baxterization: A procedure of deriving the solution $\check{R}(x)$ of QYBE with the asymptotic condition $\check{R}(0) = b$. QYBE like differential equation; $\check{R}(0) = b$ like initial condition; Yang–Baxterization like solving differential equation with initial condition.

Yang–Baxteization gives an example for topological-like quantum computing.

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

7 / 23 100% Find

Yang–Baxteization gives an example for topological-like quantum computing.

Unitary solutions of quantum YBE acting as universal quantum gates: A possible link among topology, YBE, integrable model and quantum computing.

Topological(-Like) Quantum Computing (II)

Argument: Topological quantum error correction codes in topological quantum computing by Kitaev (1997) is the most natural link between Quantum Error Correcting Codes and Topological Quantum Computing.

Our work: From Extraspecial Two-Groups to Unitary Braid Representation. From quantum error correction to Topological-like quantum computing.

Topological-like quantum computation involves both unitary braiding gates and non-braiding gates. researchers on topological-like quantum computation collaborating with us include:

Collaborators:

Louis. H. Kauffman (Topology): University of Illinois at Chicago
Reinhard F. Werner (Quantum Information): TU Braunschweig
Mo-Lin Ge (Yang-Baxter Equations): S.S. Chern Inst. of Math (Nankai)

Naihuan Jing (Algebra): North Carolina State University

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

8 / 23 100% Find

and non-braiding gates. researchers on topological-like quantum computation collaborating with us include:

Collaborators:

Louis. H. Kauffman (Topology): University of Illinois at Chicago
Reinhard F. Werner (Quantum Information): TU Braunschweig
Mo-Lin Ge (Yang-Baxter Equations): S.S. Chern Inst. of Math (Nankai)

Naihuan Jing (Algebra): North Carolina State University
Eric C. Rowell (Topology): Texas A&M University
Zhenghan Wang (TQC): Microsoft, Station Q
Yong Zhang (Theor Physics): University of Utah

Generalized GHZ states (I): Definition

$\mathcal{H}_2$: $|\uparrow\rangle$ and $|\downarrow\rangle$ satisfying $\sigma_z|\uparrow\rangle = |\uparrow\rangle$, $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$

Coordinate vector: $|\uparrow\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Complex field: $\mathbb{C}^2 \cong \mathcal{H}_2$

Qubit: A state vector in this two dimensional Hilbert space $\mathcal{H}_2$

N-qubit: $(\mathbb{C}^2)^{\otimes N}$, orthonormal tensor product basis $|\Phi_k\rangle$, $1 \leq k \leq 2^N$,

$$|\Phi_k\rangle = |m_1, \cdots, m_N\rangle, \quad m_1, \cdots, m_N = \uparrow \text{ or } \downarrow .$$

which has a converse counterpart

$$|\Phi_{\bar{k}}\rangle = |\bar{m}_1, \cdots, \bar{m}_N\rangle, \quad \bar{m}_i = -m_i,$$

Greenberger-Horne-Zeilinger (1989) States (GHZ states):

In terms of $|\Phi_l\rangle$ and $|\Phi_{\bar{l}}\rangle$, $1 \leq l \leq N$, the Hilbert space $\mathcal{H}_{2^N}$ is spanned by the 2^N orthonormal GHZ states $|\Psi_l\rangle$ of N qubits,

Qubit: A state vector in this two dimensional Hilbert space $\mathcal{H}_2$

N-qubit: $(\mathbb{C}^2)^{\otimes N}$, orthonormal tensor product basis $|\Phi_k\rangle$, $1 \leq k \leq 2^N$,

$$|\Phi_k\rangle = |m_1, \cdots, m_N\rangle, \quad m_1, \cdots, m_N = \uparrow \text{ or } \downarrow .$$

which has a converse counterpart

$$|\Phi_{\bar{k}}\rangle = |\bar{m}_1, \cdots, \bar{m}_N\rangle, \quad \bar{m}_i = -m_i,$$

Greenberger-Horne-Zeilinger (1989) States (GHZ states):

In terms of $|\Phi_l\rangle$ and $|\Phi_{\bar{l}}\rangle$, $1 \leq l \leq N$, the Hilbert space $\mathcal{H}_{2^N}$ is spanned by the 2^N orthonormal GHZ states $|\Psi_l\rangle$ of N qubits,

$$|\Psi_l\rangle \equiv \frac{1}{\sqrt{2}}(|\Phi_l\rangle + |\Phi_{\bar{l}}\rangle), \quad |\Psi_{\bar{l}}\rangle \equiv \frac{1}{\sqrt{2}}(|\Phi_l\rangle - |\Phi_{\bar{l}}\rangle).$$

GHZ states can be classified into two classes:

1. GHZ states of an even number of qubits, example, Bell states of 2-qubit;
2. GHZ states of an odd number of qubits, example, GHZ states of 3-qubit;

GHZ States (II): Bell States

Bell states (two-qubit case): orthonormal maximally entangled basis

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

1. Bell states are originally called Einstein-Podolsky-Rosen (EPR) states or EPR-Bohm states.

2. Bell Theorem, Bell Inequalities and Bell states: the incompatibility between quantum theory and classical deterministic local models is expressed in the form of inequalities (the Bell inequalities) among various statistical correlations.

3. Bell states are assumed to be maximally entangled states by various known entanglement measures in quantum information.

GHZ States (II): Bell States

Bell states (two-qubit case): orthonormal maximally entangled basis

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + |\Downarrow\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle)$$

1. Bell states are originally called Einstein-Podolsky-Rosen (EPR) states or EPR-Bohm states.

2. Bell Theorem, Bell Inequalities and Bell states: the incompatibility between quantum theory and classical deterministic local models is expressed in the form of inequalities (the Bell inequalities) among various statistical correlations.

3. Bell states are assumed to be maximally entangled states by various known entanglement measures in quantum information.

4. Bell states as maximally entanglement sources have been widely exploited in quantum information theory, example, quantum key distribution (Ekert, 1991), quantum teleportation (BBCJPW, 1993), etc.

GHZ States (III): Three Qubits

GHZ states of three qubits: orthonormal maximally entangled basis

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle), \quad |\Psi_8\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle), \quad |\Psi_7\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle), \quad |\Psi_6\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle)$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle), \quad |\Psi_5\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle)$$

1. The GHZ states are multipartite generalization of the bipartite maximally entangled Bell states.

2. GHZ Theorem and GHZ states: the incompatibility between quantum theory and classical deterministic local models is expressed in the form of equalities (not inequalities) and can be tested by perfect correlations instead of statistical correlations.

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

11 / 23 100% Find

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle), \quad |\Psi_6\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle)$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle), \quad |\Psi_5\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle)$$

1. The GHZ states are multipartite generalization of the bipartite maximally entangled Bell states.

2. GHZ Theorem and GHZ states: the incompatibility between quantum theory and classical deterministic local models is expressed in the form of equalities (not inequalities) and can be tested by perfect correlations instead of statistical correlations.

3. GHZ states are assumed to be maximally entangled states by various known entanglement measures in quantum information.

4. GHZ states as multipartite maximally entanglement sources have been widely exploited in multipartite quantum information theory.

Bell matrix (I): Unitary Basis Transformation Matrix

Orthonormal tensor product basis, Hilbert space, $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$|\Phi_1\rangle = |\uparrow\uparrow\rangle, |\Phi_2\rangle = |\uparrow\downarrow\rangle, |\Phi_3\rangle = |\downarrow\uparrow\rangle, |\Phi_4\rangle = |\downarrow\downarrow\rangle$$

Bell states, orthonormal maximally entangled basis, $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow \frac{1}{2}\rangle)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

Unitary basis transformation matrix: $B = (|\Psi_4\rangle, |\Psi_2\rangle, -|\Psi_3\rangle, |\Psi_1\rangle)$

$$B = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

This B-matrix, called the Bell matrix, forms a unitary braid representation,

A quantum gate, equivalent to the $CNOT$ gate by single qubit transformations, is a universal quantum gate.

Kauffman and Lomonaco (2004): The Bell matrix is a universal quantum gate by verifying that $CNOT = (\alpha \otimes \beta) \cdot B \cdot (\gamma \otimes \delta)$.

The $CNOT$ gate is a universal two-qubit quantum gate and it is a quantum generalization of classical NOT gate

$$CNOT|YY\rangle = |YY\rangle, \quad CNOT|YN\rangle = |YN\rangle,$$
$$CNOT|NY\rangle = |NN\rangle, \quad CNOT|NN\rangle = |NY\rangle$$

$$\alpha = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \quad \beta = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$
$$\gamma = -\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

which are local unitary transformations of single qubit.

Bell matrix: 1) unitary basis transformation matrix; 2) unitary braid representation; 3) universal quantum gate.

Bell matrix: 1) quantum mechanics; 2) topology; 3) topological-like quantum

WTL_PI080429.pdf - Adobe Reader

File Edit View Document Tools Window Help

13 / 23 100% Find

Bell matrix: 1) unitary basis transformation matrix; 2) unitary braid representation; 3) universal quantum gate.

Bell matrix: 1) quantum mechanics; 2) topology; 3) topological-like quantum computing.

12

Bell matrix (III): Almost-Complex Structure

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

where $M^2 = -1$ and $M^\dagger = -M$.

The Bell matrix has exponential formulation: $B = \frac{1}{\sqrt{2}}(\mathbb{1}_4 + M) = e^{\frac{\pi}{4}M}$;

In differential geometry, the almost-complex structure usually is denoted by the symbol J, but in quantum mechanics J usually represents the total angular momentum $\vec{J} = \vec{L} + \vec{S}$.

Define $M_i = \mathbb{1}_2^{\otimes i-1} \otimes M \otimes \mathbb{1}_2^{\otimes n-i-2}$, $1 \leq i \leq n-1$ satisfying

$$M_i^2 = -\mathbb{1}_2, \qquad M_i M_j = M_j M_i, \quad \text{if } |i-j| > 1,$$

$$M_i M_{i+1} = -M_{i+1} M_i, \quad \text{for all } 1 \leq i \leq (n-2).$$

The proof for the Bell matrix satisfying the braid group representation is greatly simplified if we use the above algebraic relations.

$$b_i b_{i+1} b_i = 2(M_i + M_{i+1}) + (M_i M_{i+1} + M_{i+1} M_i)$$

M: Anti-Hermitian almost-complex structure

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

where $M^2 = -1$ and $M^\dagger = -M$.

The Bell matrix has exponential formulation: $B = \frac{1}{\sqrt{2}}(\mathbb{1}_4 + M) = e^{\frac{\pi}{4}M}$;

In differential geometry, the almost-complex structure usually is denoted by the symbol J, but in quantum mechanics J usually represents the total angular momentum $\vec{J} = \vec{L} + \vec{S}$.

Define $M_i = \mathbb{1}_2^{\otimes i-1} \otimes M \otimes \mathbb{1}_2^{\otimes n-i-2}$, $1 \leq i \leq n-1$ satisfying

$$M_i^2 = -\mathbb{1}_2, \qquad M_i M_j = M_j M_i, \quad \text{if } |i-j| > 1,$$

$$M_i M_{i+1} = -M_{i+1} M_i, \quad \text{for all } 1 \leq i \leq (n-2).$$

The proof for the Bell matrix satisfying the braid group representation is greatly simplified if we use the above algebraic relations.

the symbol J, but in quantum mechanics J usually represents the total angular momentum $\vec{J} = \vec{L} + \vec{S}$.

Define $M_i = \mathbb{1}_2^{\otimes i-1} \otimes M \otimes \mathbb{1}_2^{\otimes n-i-2}$, $1 \leq i \leq n-1$ satisfying

$$M_i^2 = -\mathbb{1}_2, \qquad M_i M_j = M_j M_i, \quad \text{if } |i-j| > 1,$$

$$M_i M_{i+1} = -M_{i+1} M_i, \quad \text{for all } 1 \leq i \leq (n-2).$$

The proof for the Bell matrix satisfying the braid group representation is greatly simplified if we use the above algebraic relations.

$$b_i b_{i+1} b_i = 2(M_i + M_{i+1}) + (M_i M_{i+1} + M_{i+1} M_i)$$

which is symmetric under $i \leftrightarrow i+1$, i.e., $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$

A finite group G of order 2^{m+1} is an **extraspecial 2-group** if the center $Z(G)$ and the commutator subgroup G' coincide and are isomorphic to $\mathbb{Z}_2$ as well as $G/Z(G) \cong (\mathbb{Z}_2)^m$.

The abstract group $\mathbf{E}_m$ generated by $e_1, \ldots, e_m$ satisfying

$$e_i^2 = -\mathbb{1}, \; e_i e_j = e_j e_i, \; |i-j| \geq 2, \quad e_{i+1} e_i = -e_i e_{i+1}, \; 1 \leq i, j \leq m-1,$$

where $-\mathbb{1}$ is an order 2 central element.

The commutator subgroup of $\mathbf{E}_m$ is $\{\pm\mathbb{1}\}$ due to its (anti-)commutation relations, and hence it is immediate that $\mathbf{E}_{2k}$ is an extraspecial 2-group.

When m is odd, $\mathbf{E}_m$ does not fit the technical definition of extraspecial 2-group as the center will have order 4. However, since $\mathbf{E}_{m-1} \subset \mathbf{E}_m \subset \mathbf{E}_{m+1}$ we obtain an extraspecial 2-group from $\mathbf{E}_m$ by adding or removing a generator, so we will call $\mathbf{E}_{2k+1}$ a *nearly* extraspecial 2-group.

1. Almost-complex structure provides an anti-Hermitian representation for extraspecial two-groups or nearly extraspecial two-groups.

2. Almost-complex structure provides a way of deriving the unitary braid representation by $B = e^{\frac{\pi}{4}M}$.

Extraspecial Two-Groups

A finite group G of order 2^{m+1} is an **extraspecial 2-group** if the center $Z(G)$ and the commutator subgroup G' coincide and are isomorphic to $\mathbb{Z}_2$ as well as $G/Z(G) \cong (\mathbb{Z}_2)^m$.

The abstract group $\mathbf{E}_m$ generated by $e_1, \ldots, e_m$ satisfying

$$e_i^2 = -\mathbb{1}, \quad e_i e_j = e_j e_i, \quad |i-j| \geq 2, \quad e_{i+1} e_i = -e_i e_{i+1}, \quad 1 \leq i, j \leq m-1,$$

where $-\mathbb{1}$ is an order 2 central element.

The commutator subgroup of $\mathbf{E}_m$ is $\{\pm\mathbb{1}\}$ due to its (anti-)commutation relations, and hence it is immediate that $\mathbf{E}_{2k}$ is an extraspecial 2-group.

When m is odd, $\mathbf{E}_m$ does not fit the technical definition of extraspecial 2-group as the center will have order 4. However, since $\mathbf{E}_{m-1} \subset \mathbf{E}_m \subset \mathbf{E}_{m+1}$ we obtain an extraspecial 2-group from $\mathbf{E}_m$ by adding or removing a generator, so we will call $\mathbf{E}_{2k+1}$ a *nearly* extraspecial 2-group.

Theorem 1. Let $\{T_1, \ldots, T_{n-1}\}$ be the images of the generators $\{e_1, \ldots, e_{n-1}\}$ of $\mathbf{E}_{n-1}$ under a representation of $\mathbf{E}_{n-1}$ such that:

$$T_i^2 = -Id, \qquad T_iT_j = T_jT_i, \quad \text{if } |i-j| > 1,$$

$$T_iT_{i+1} = -T_{i+1}T_i, \quad \text{for all } 1 \leq i \leq (n-2).$$

Then

(a) The set of $\{\check{R}_1, \ldots, \check{R}_{n-1}\}$ defined by $\check{R}_i = \frac{1}{\sqrt{2}}(Id + T_i)$ gives a representation of $\mathcal{B}_n$ by $b_i \to \check{R}_i$.

(b) If in addition the T_i are anti-Hermitian (i.e. $T_i = -T_i^\dagger$), the $\mathcal{B}_n$ representation is unitary.

Proof (a): $\sqrt{2}\check{R}_i\check{R}_{i+1}\check{R}_i = 2T_{i+1} + 2T_i + T_iT_{i+1} + T_{i+1}T_i$ which is symmetric under $i \leftrightarrow i+1$.

Proof (b): $T^\dagger = -T \Rightarrow \check{R}^\dagger\check{R} = \mathbb{1}$.

From extraspecial two-groups to unitary braid representation
+ From unitary braid representation to GHZ states
= From extraspecial two-groups to GHZ states
= From quantum error correction to topological-like quantum computing

Theorem 1: From Extraspecial 2-Groups To Unitary BGR

Theorem 1. Let $\{T_1, \ldots, T_{n-1}\}$ be the images of the generators $\{e_1, \ldots, e_{n-1}\}$ of $\mathbf{E}_{n-1}$ under a representation of $\mathbf{E}_{n-1}$ such that:

$$T_i^2 = -Id, \qquad T_iT_j = T_jT_i, \quad \text{if } |i-j| > 1,$$

$$T_iT_{i+1} = -T_{i+1}T_i, \quad \text{for all } 1 \leq i \leq (n-2).$$

Then

(a) The set of $\{\check{R}_1, \ldots, \check{R}_{n-1}\}$ defined by $\check{R}_i = \frac{1}{\sqrt{2}}(Id + T_i)$ gives a representation of $\mathcal{B}_n$ by $b_i \to \check{R}_i$.

(b) If in addition the T_i are anti-Hermitian (i.e. $T_i = -T_i^\dagger$), the $\mathcal{B}_n$ representation is unitary.

Proof (a): $\sqrt{2}\check{R}_i\check{R}_{i+1}\check{R}_i = 2T_{i+1} + 2T_i + T_iT_{i+1} + T_{i+1}T_i$ which is symmetric under $i \leftrightarrow i+1$.

if, the deformation parameters q_{ij} in M^{JJ} satisfy the three constraints:

$$q_{ij}q_{\bar{i}\,\bar{j}} = 1, \quad q_{ij}q_{\bar{i}j} = q_{jl}q_{j\bar{l}}, \quad \overset{*}{q}_{ij}q_{ij} = 1.$$

Comment 1. Factorization of the almost-complex structure.

$$M^{JJ} = M_{2k} \otimes P_{2k}, \;\; M_{2k} = \sum_{i=-J}^{J} \epsilon(i) q_i |i\rangle\langle\bar{i}|, \;\; P_{2k} = \sum_{j=-J}^{J} q_j |j\rangle\langle\bar{j}|.$$

Comment 2. GHZ states of $2n$-qubit.

$$k = 2^{n-1}, 2k \times 2k = 2^n \times 2^n, \quad B_{2^{2n}\otimes 2^{2n}} = e^{\frac{\pi}{4}M_{2n}\otimes P_{2n}}$$

Comment 3. Rescaling transformation. $|i'\rangle = q_i^{\frac{1}{2}}|i\rangle, \quad \langle j'| = \langle j|(q_j^*)^{\frac{1}{2}},$

$$M_{2k} = \sum_{i=-J}^{J} \epsilon(i)|i'\rangle\langle\bar{i}'|, \quad P_{2k} = \sum_{i=-J}^{J} |i'\rangle\langle\bar{i}'|.$$

Representation with deformation parameters is the same as the representation without deformation parameters. But these unimodular deformation parameters can have an interpretation with geometric phase, for example, the Berry phase.

Class (2): The almost-complex structure, $M_{2^N} \equiv M_2 \otimes P_{2^{N-1}} = \sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}$, a generalization of **Class (1)** by taking $M_{2k_1} \otimes P_{2k_2}$, $k_1 = 1, k_2 = 2^{N-2}$, N even or odd.

Theorem 3. Define a map $\phi_m^{(2)}$ into the set of unitary matrices $\mathrm{U}(2^{N+k(m-1)})$ on the generators of $\mathbf{E}_m$ by

$$\phi_m^{(2)}(e_i) = \mathbb{1}_{2^k}^{\otimes i-1} \otimes (\sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}) \otimes \mathbb{1}_{2^k}^{\otimes m-i}, \quad i = 1, \cdots, m.$$

Then $\phi_m^{(2)}$ defines an (anti-Hermitian) representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N-1$.

Example: GHZ states of three qubits. The Bell matrix B_8 in terms of the almost-complex structure

$$M_8 = \sqrt{-1}\sigma_y \otimes \sigma_x^{\otimes 2}$$

having the form

$$B_8 = e^{i\frac{\pi}{4}\sigma_y \otimes \sigma_x^{\otimes 2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 3: GHZ states of odd number of qubits

Class (2): The almost-complex structure, $M_{2^N} \equiv M_2 \otimes P_{2^{N-1}} = \sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}$, a generalization of **Class (1)** by taking $M_{2k_1} \otimes P_{2k_2}$, $k_1 = 1, k_2 = 2^{N-2}$, N even or odd.

Theorem 3. Define a map $\phi_m^{(2)}$ into the set of unitary matrices $\mathrm{U}(2^{N+k(m-1)})$ on the generators of $\mathbf{E}_m$ by

$$\phi_m^{(2)}(e_i) = \mathbb{1}_{2^k}^{\otimes i-1} \otimes (\sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}) \otimes \mathbb{1}_{2^k}^{\otimes m-i}, \quad i = 1, \cdots, m.$$

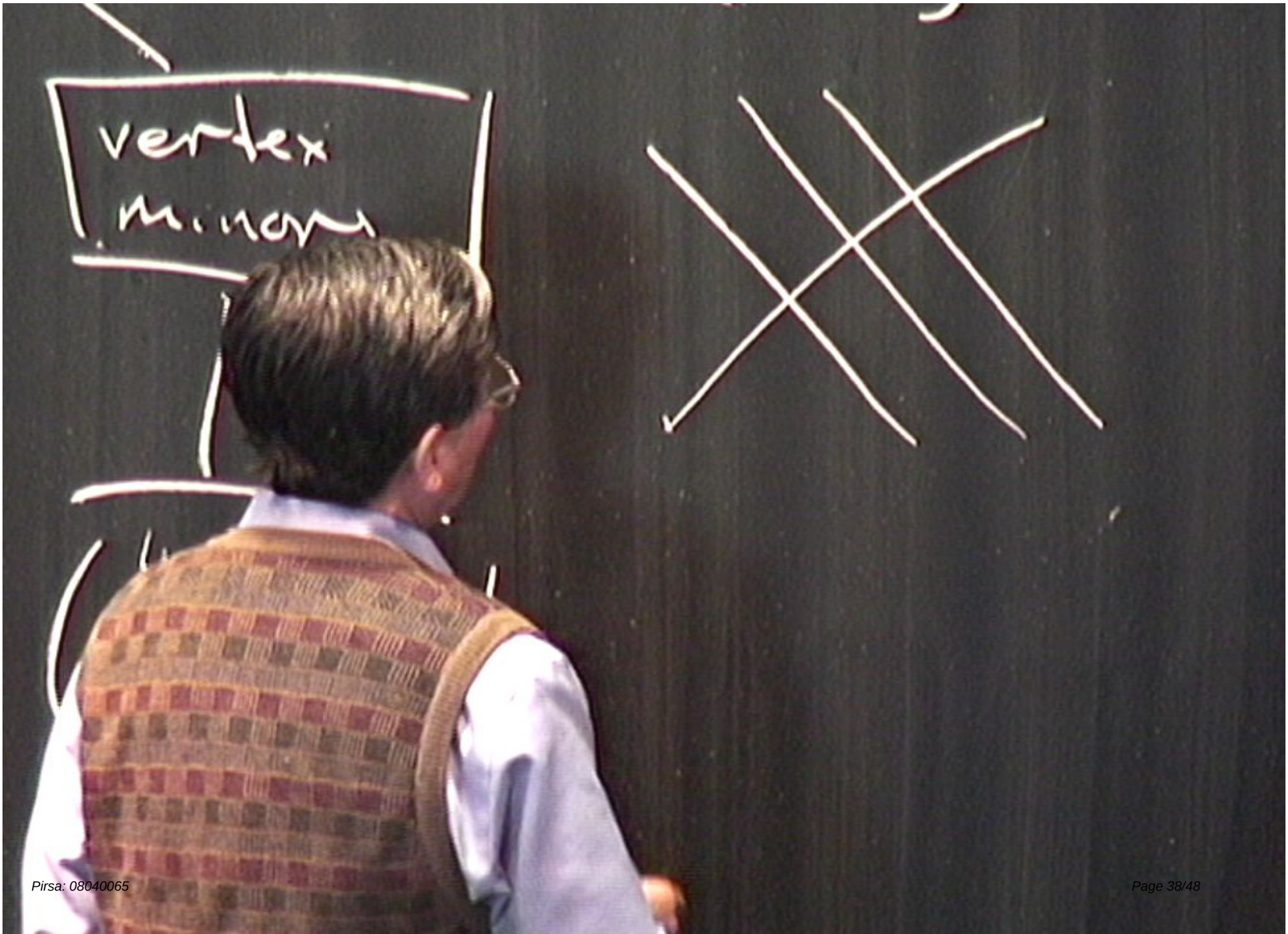
Then $\phi_m^{(2)}$ defines an (anti-Hermitian) representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N-1$.

Example: GHZ states of three qubits. The Bell matrix B_8 in terms of the almost-complex structure

$$M_8 = \sqrt{-1}\sigma_y \otimes \sigma_x^{\otimes 2}$$

having the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$


vertex
minor

Theorem 3: GHZ states of odd number of qubits

Class (2): The almost-complex structure, $M_{2^N} \equiv M_2 \otimes P_{2^{N-1}} = \sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}$, a generalization of **Class (1)** by taking $M_{2k_1} \otimes P_{2k_2}$, $k_1 = 1$, $k_2 = 2^{N-2}$, N even or odd.

Theorem 3. Define a map $\phi_m^{(2)}$ into the set of unitary matrices $\mathrm{U}(2^{N+k(m-1)})$ on the generators of $\mathbf{E}_m$ by

$$\phi_m^{(2)}(e_i) = \mathbb{1}_{2^k}^{\otimes i-1} \otimes (\sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N-1}}) \otimes \mathbb{1}_{2^k}^{\otimes m-i}, \quad i = 1, \cdots, m.$$

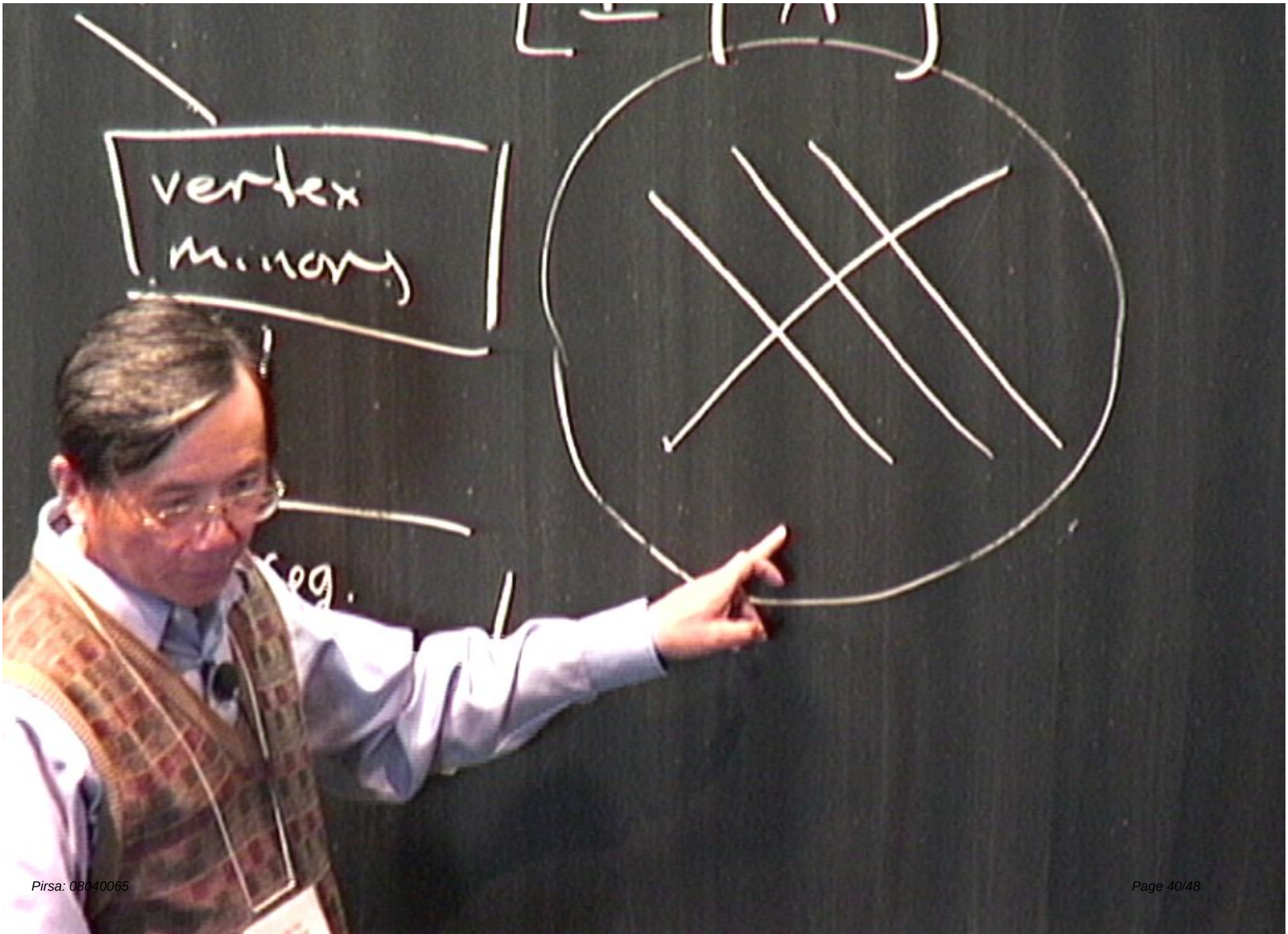
Then $\phi_m^{(2)}$ defines an (anti-Hermitian) representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N-1$.

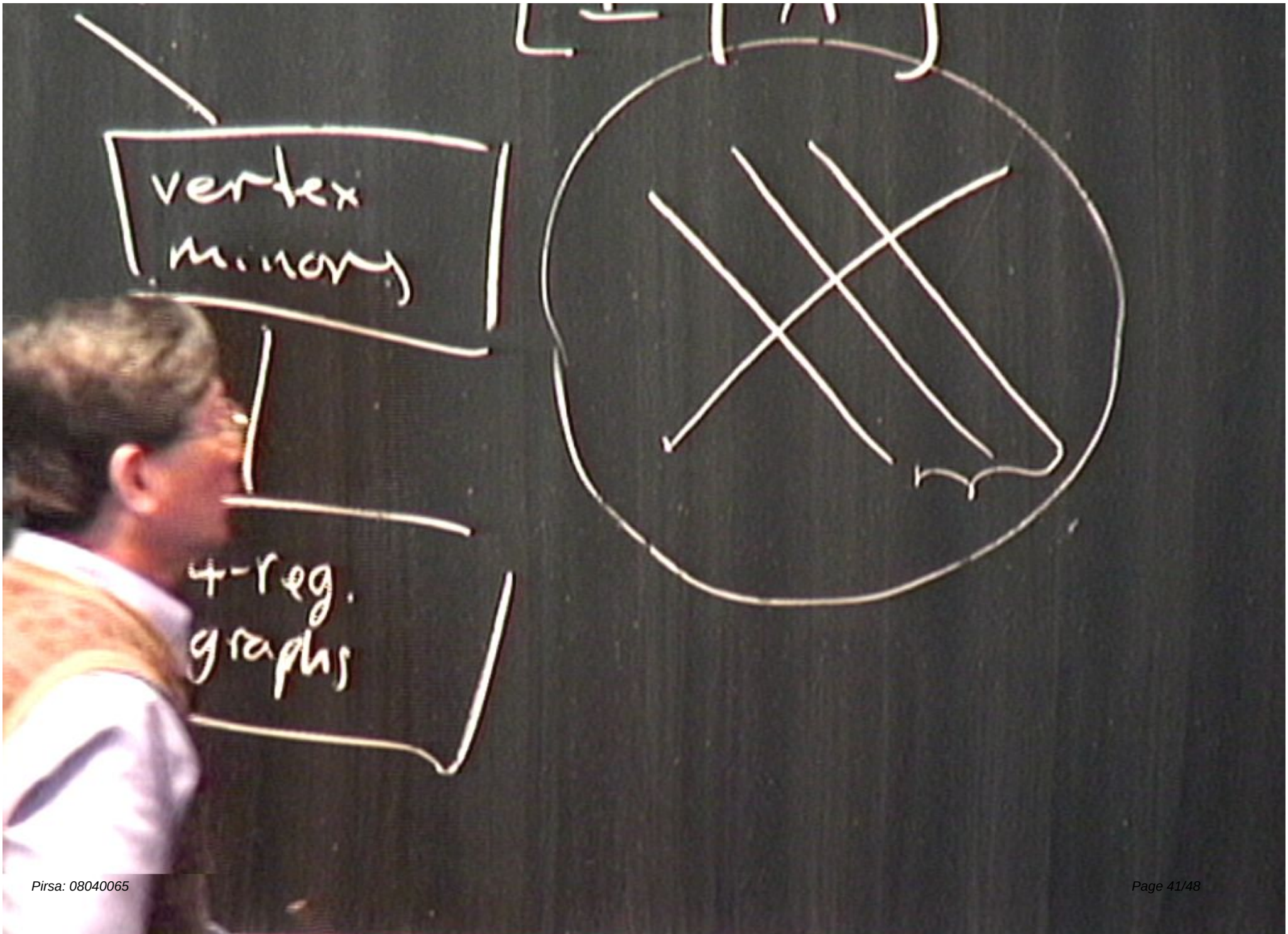
Example: GHZ states of three qubits. The Bell matrix B_8 in terms of the almost-complex structure

$$M_8 = \sqrt{-1}\sigma_y \otimes \sigma_x^{\otimes 2}$$

having the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$


vertex
memory


vertex
memory
4-reg.
graphs

Theorem 3. Define a map $\phi_m^{(2)}$ into the set of unitary matrices $\mathrm{U}(2^{N+k(m-1)})$ on the generators of $\mathbf{E}_m$ by

$$\phi_m^{(2)}(e_i) = \mathbb{1}_{2^k}^{\otimes i-1} \otimes (\sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N}-1}) \otimes \mathbb{1}_{2^k}^{\otimes m-i}, \quad i = 1, \cdots, m.$$

Then $\phi_m^{(2)}$ defines an (anti-Hermitian) representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N-1$.

Example: GHZ states of three qubits. The Bell matrix B_8 in terms of the almost-complex structure

$$M_8 = \sqrt{-1}\sigma_y \otimes \sigma_x^{\otimes 2}$$

having the form

$$B_8 = e^{i\frac{\pi}{4}\sigma_y \otimes \sigma_x^{\otimes 2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which denotes a unitary basis transformation matrix from the product basis to the

The generalized Yang–Baxter equation

Theorem 3. Define a map $\phi_m^{(2)}$ into the set of unitary matrices $\mathrm{U}(2^{N+k(m-1)})$ on the generators of $\mathbf{E}_m$ by

$$\phi_m^{(2)}(e_i) = \mathbb{1}_{2^k}^{\otimes i-1} \otimes (\sqrt{-1}\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{x}}^{\otimes \mathbf{N}-1}) \otimes \mathbb{1}_{2^k}^{\otimes m-i}, \quad i = 1, \cdots, m.$$

Then $\phi_m^{(2)}$ defines an (anti-Hermitian) representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N-1$.

Comment: The square of the dimension of the identity operator is allowed to be different to the dimension of the almost-complex structure. We define the generalized Yang–Baxter equation in the way.

$$(\check{R} \otimes \mathbb{1}_l)(\mathbb{1}_l \otimes \check{R})(\check{R} \otimes \mathbb{1}_l) = (\mathbb{1}_l \otimes \check{R})(\check{R} \otimes \mathbb{1}_l)(\mathbb{1}_l \otimes \check{R}),$$

where $l = p^k$ $(2 \leq p \in \mathbb{N})$ and $\check{R}$ is an invertible $d' \otimes d'$ matrix with $d' = p^N$. Both sides act on the tensor product $(\mathbb{C}^p)^{\otimes(k+N)}$.

At $k = 1$, $N = 2$, the generalized YBE is the same as the ordinary YBE.

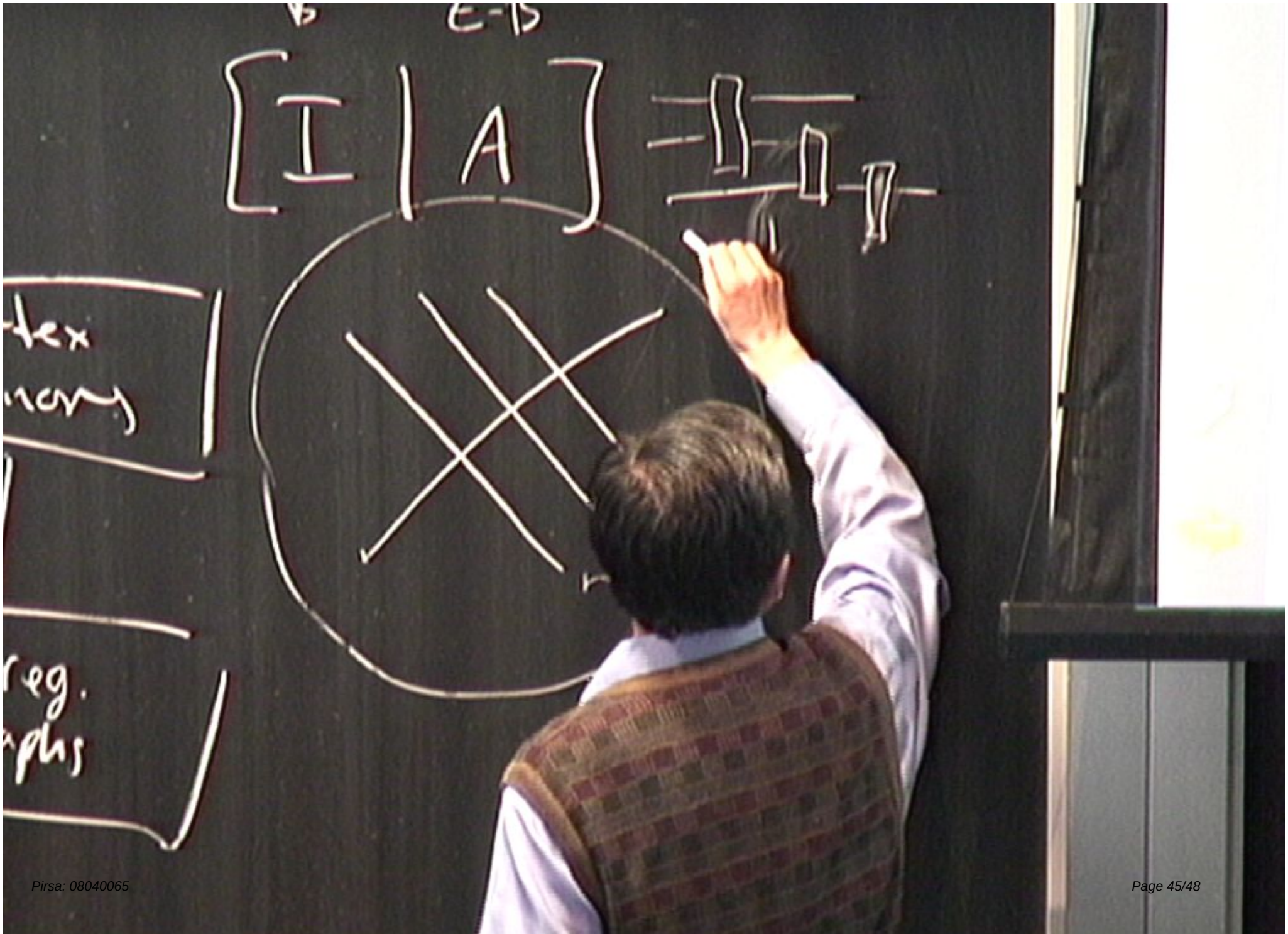
Jones Polynomials

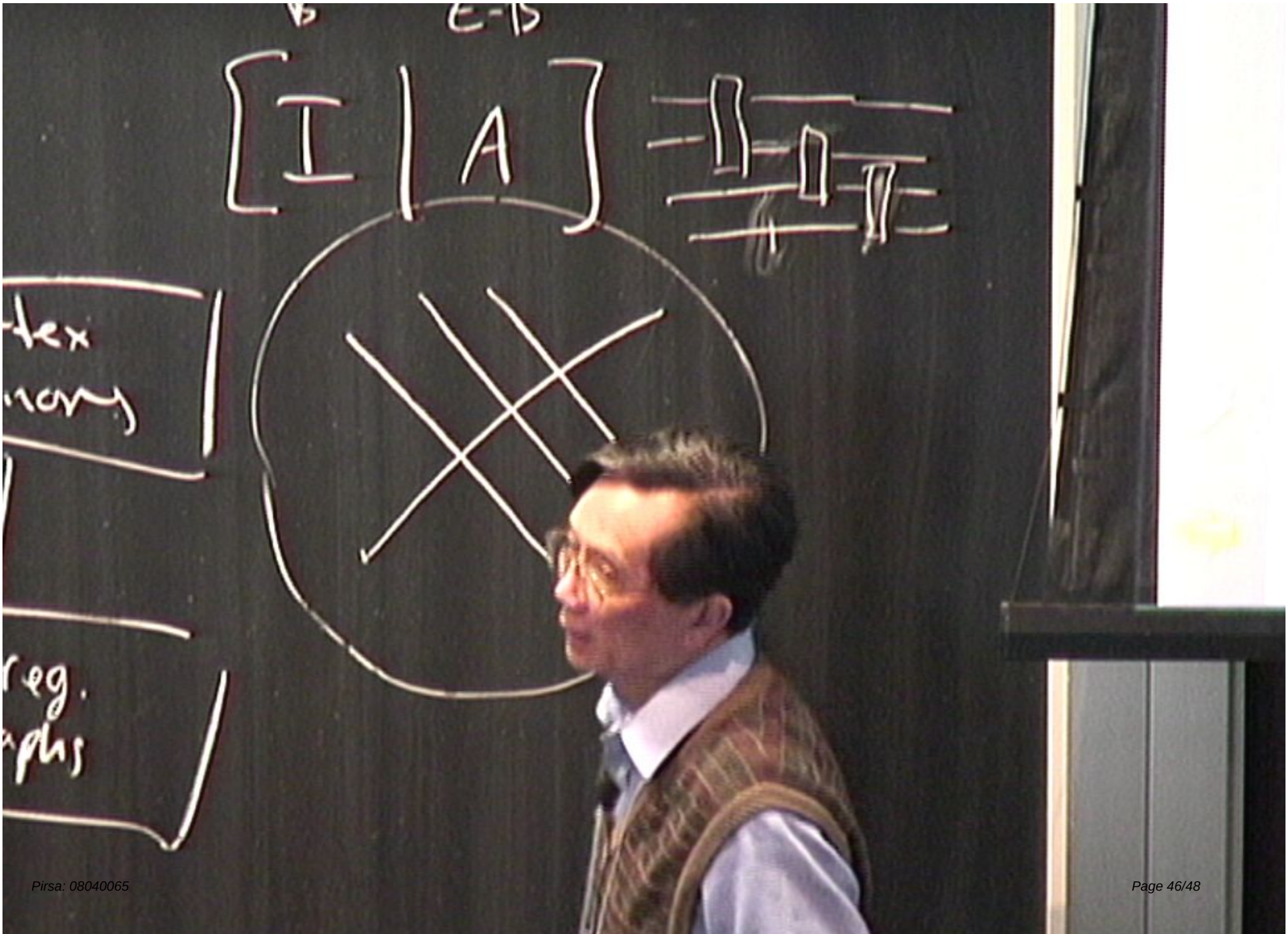
A math remark is that our braid group repres. obtained through the extra-special two-groups can be used to construct link polynomials. It turns out that the link polynomial is essentially the Jones polynomials at the 4th root of unity, i.e. at $q = \sqrt{-1}$ for either the usual braids or for the color braids (i.e. different strands may have different dimensions.)

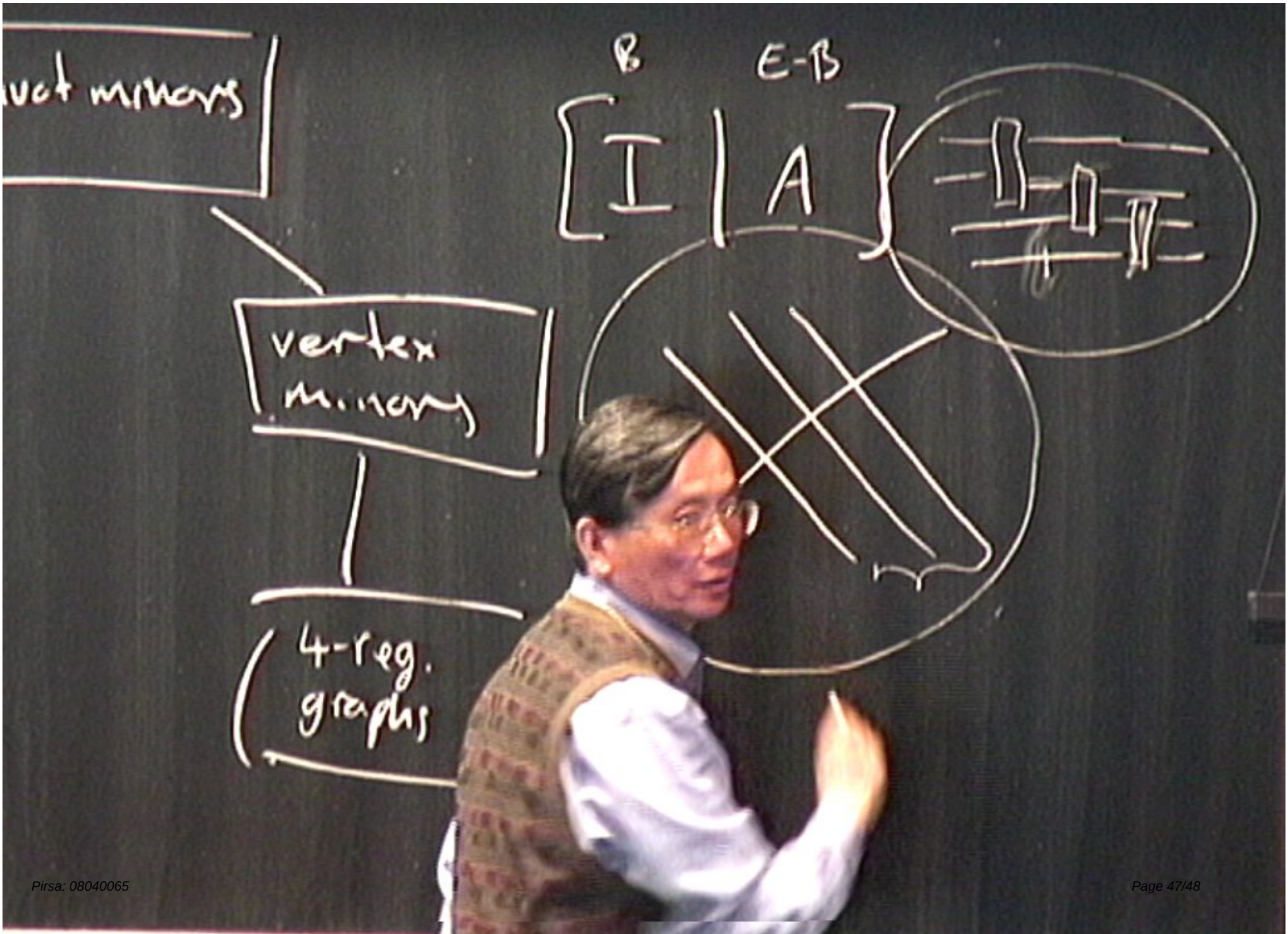
Proposition: (Maximally) entangled states, such as Bell states, GHZ states or the mutli-qubit GHZ states that we constructed are all related to Jones polynomial or its natural generalizations at the 4th root of unity, if we consider the states are obtained by quantum gates acting on computational basis (separable states).

Connection to the graph theory or quantum circuits Consider graphs (or a special class of quantum circuits) obtained by 2-qubit gates acting on the nearest neighboring qubits. When the gates are the Bell gates, there is a "topological" invariant associated with this graph, i.e. the Jones polynomial at the 4th root of unity. "Topological" here means the equivalence relations modulo changes represented by the braid Yang-Baxter relations.

(This paragraph was written last night, inspired by yesterday's lectures.)


[I | A]


[I | A]


uct minors
B
E-B
I
A
vertex minors
4-reg. graphs

Jones Polynomials

A math remark is that our braid group repres. obtained through the extra-special two-groups can be used to construct link polynomials. It turns out that the link polynomial is essentially the Jones polynomials at the 4th root of unity, i.e. at $q = \sqrt{-1}$ for either the usual braids or for the color braids (i.e. different strands may have different dimensions.)

Proposition: (Maximally) entangled states, such as Bell states, GHZ states or the mutli-qubit GHZ states that we constructed are all related to Jones polynomial or its natural generalizations at the 4th root of unity, if we consider the states are obtained by quantum gates acting on computational basis (separable states).

Connection to the graph theory or quantum circuits Consider graphs (or a special class of quantum circuits) obtained by 2-qubit gates acting on the nearest neighboring qubits. When the gates are the Bell gates, there is a "topological" invariant associated with this graph, i.e. the Jones polynomial at the 4th root of unity. "Topological" here means the equivalence relations modulo changes represented by the braid Yang-Baxter relations.

(This paragraph was written last night, inspired by yesterday's lectures.)