Title: Statistical Mechanical Models and Topological Color Codes.

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Abstract: We find that the overlapping of a topological quantum color code state, representing a quantum memory, with a factorized state of qubits can be written as the partition function of a 3-body classical Ising model on triangular or Union Jack lattices. This mapping allows us to test that different computational capabilities of color codes correspond to qualitatively different universality classes of their associated classical spin models. By generalizing these statistical mechanical models for arbitrary inhomogeneous and complex couplings, it is possible to study a measurement-based quantum computation with a color code state and we find that their classical simulatability remains an open problem. We complement the meaurement-based computation with the construction of a cluster state that yields the topological color code and this also gives the possibility to represent statistical models with external magnetic fields. Joint work with M.A. Martin-Delgado.

Topological Color Codes
&
Statistical Mechanical Models

"Topological Quantum Distillation", Phys. Rev. Lett. 97 180501 (2006)

"Topological Computation without Braiding", Phys. Rev. Lett. 98, 160502 (2007)

"Exact Topological Quantum Order in D=3 and Beyond", Phys. Rev. B 75, 075103 (2007)

"Optimal Resources for Topological Stabilizer Codes", Phys. Rev. A 76, 012305 (2007)

"Statistical Mechanical Models and Topological Color Codes", Phys. Rev. A

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Outline

- **Topological codes:** From stabilizers to color
- **Classical 2D Ising models:** 2-body and 3-body
- **Connection:** Partition function = product state overlap
- **An open problem:** Simulability of color codes in measurement-based quantum computation (MQC)
- **Graph states:** How to get color codes

Topological Stabilizer Codes

Topological Stabilizer Codes

- In quantum mechanics, it is frecuently useful to make **emphasis on operators rather than on states**.
- In the case of quantum information, we have the **stabilizer formalism**[1], where a subspace V is described by a set of conditions given by a group S of Pauli operators, i.e., tensor products of Pauli matrices.

$$|\psi\rangle \in V \qquad \Longleftrightarrow \qquad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Example: Bell states

$$|0\rangle|0\rangle + |1\rangle|1\rangle \qquad \mathcal{S} = (XX, ZZ); \qquad |0\rangle|0\rangle - |1\rangle|1\rangle \qquad \mathcal{S} = (-XX, ZZ);$$

$$|0\rangle|1\rangle + |1\rangle|0\rangle \qquad \mathcal{S} = (XX, -ZZ); \quad |0\rangle|1\rangle - |1\rangle|0\rangle \qquad \mathcal{S} = (-XX, -ZZ).$$

- Stabilizer **quantum error-correction codes** are such subspaces. They are particularly useful because errors in states amount to violations of the stabilizer conditions:

$$s|\psi\rangle = -|\psi\rangle$$

- The normalizer N is a subgroup of the pauli group such that

$$E \in \mathcal{N} \qquad \Longleftrightarrow \qquad \forall\, s \in \mathcal{S} \quad [E, s] = 0$$

The elements of N-S, **undetectable errors,** are important because they give the distance of the code. Indeed, they are the Pauli operators for encoded qubits.

[1] D. Gottesman 95

Topological Stabilizer Codes

- For a TSC we mean a code in which:

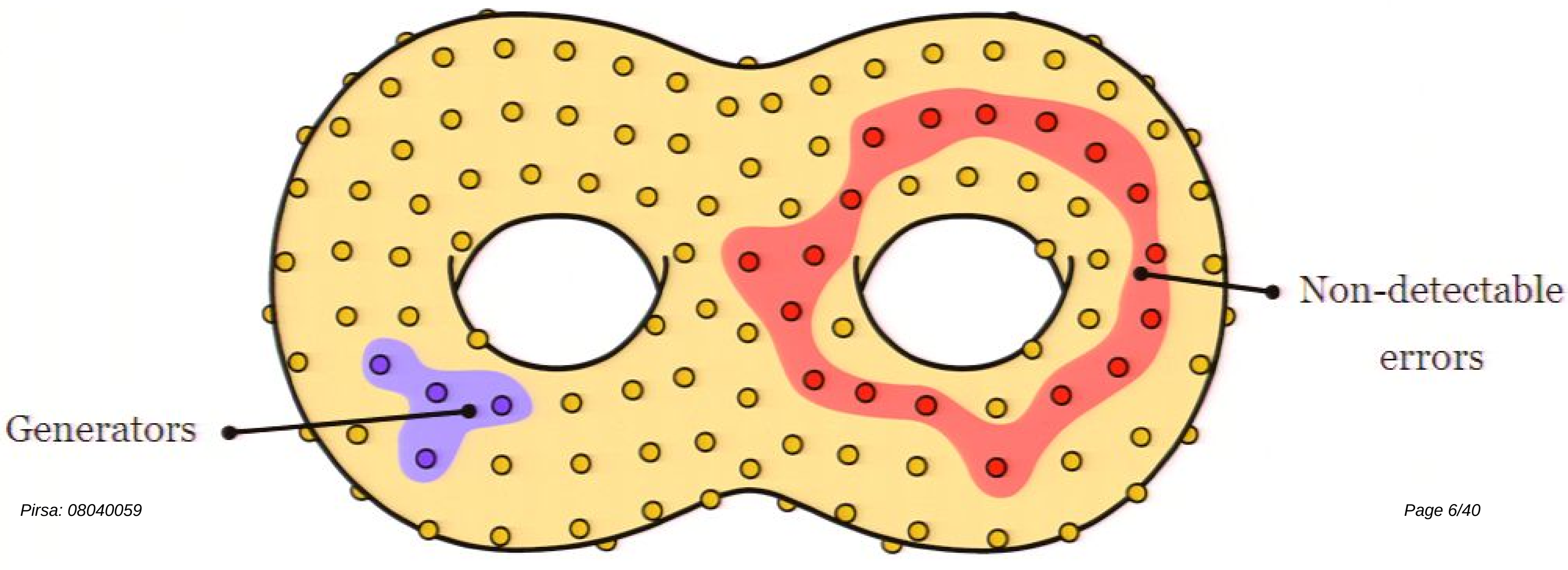
  a) the generators of the stabilizer are **local** and

  b) undetectable errors have a **global** (topological) nature.

- Usually we consider TSCs in which

  a) qubits are placed on a surface,

  b) the stabilizer S is composed of **boundaries** and its normalizer N of **cycles**, so that undetectable errors are related to cycles which are not boundaries (homology...).

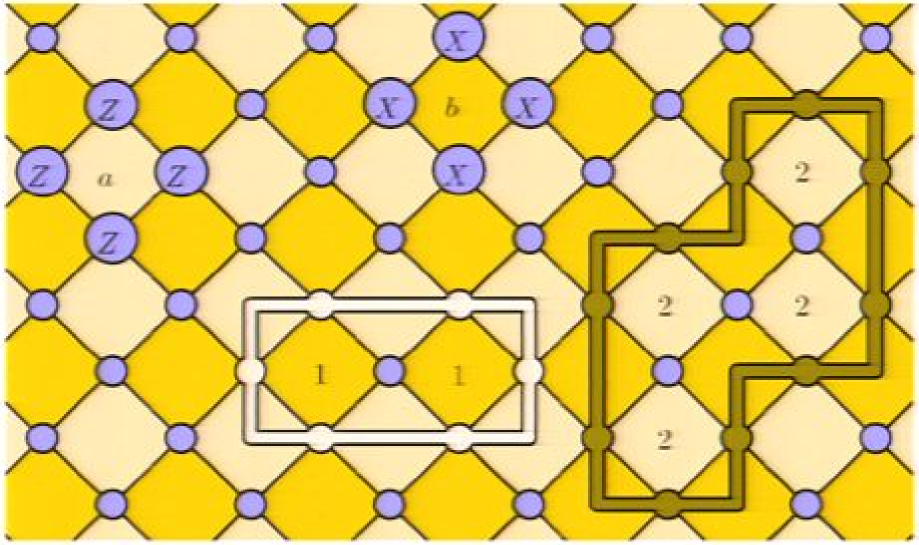


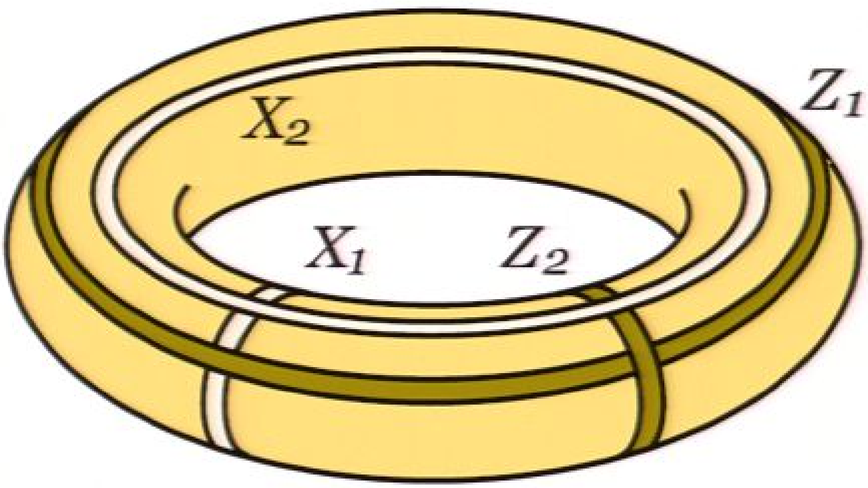
Topological Stabilizer Codes

- To construct a surface code (Kitaev '07, aka toric code), one starts from a **4-valent** lattice with **2-colorable** faces.
- Each **vertex** corresponds to a **qubit**.
- The generators of the stabilizer are light and dark **plaquette operators**:

$$Z_a := Z_1 Z_2 Z_3 Z_4$$

$$X_b := X_5 X_6 X_7 X_8$$

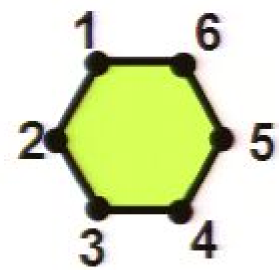




- Dark (light) **string operators** are products of Z-s (X-s).
- Plaquette operators generate the stabilizer: boundary string operators.
- Closed strings compose its normalizer.
- **Crossing** dark and light strings operators **anticommute**.
- **Encoded X-s and Z-s** can be chosen from those closed strings which are not boundaries.

Topological Stabilizer Codes

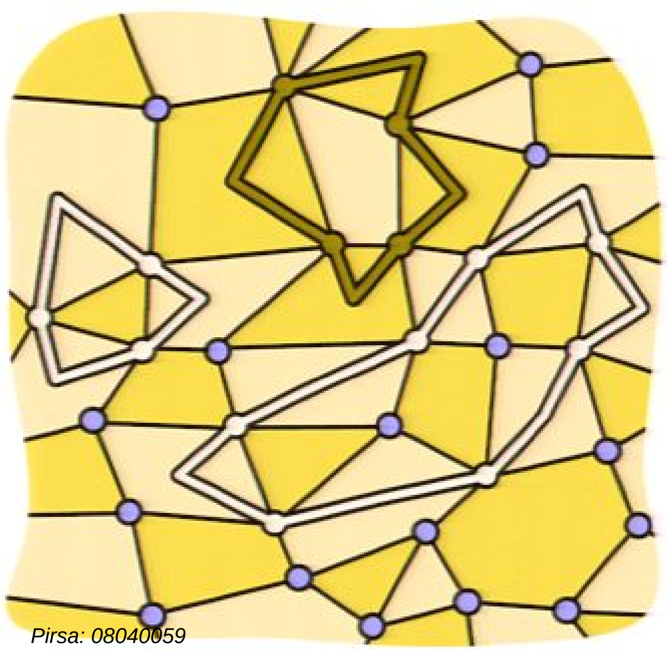
- Color codes are obtained from **trivalent** lattices with **3-colorable** faces.
- Faces are classified in red, green and blue.
- Each **vertex** corresponds to a **qubit.**



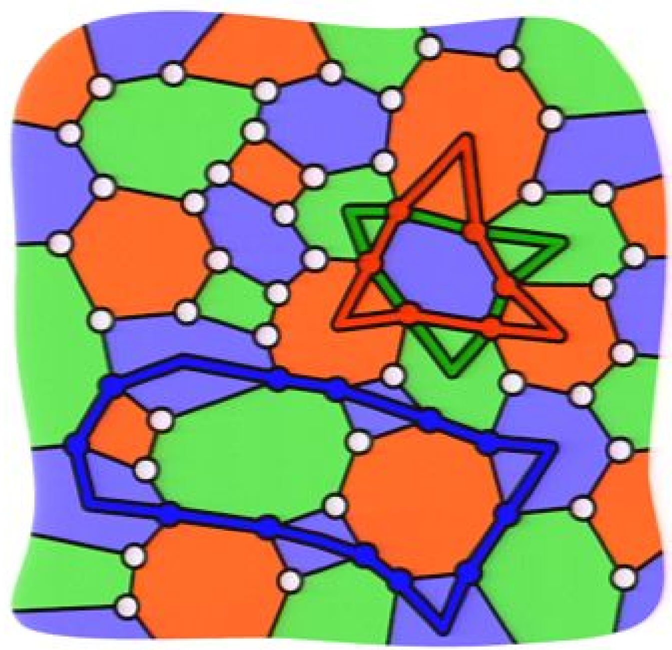
$$X_f = X_1 X_2 X_3 X_4 X_5 X_6$$

$$Z_f = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6$$

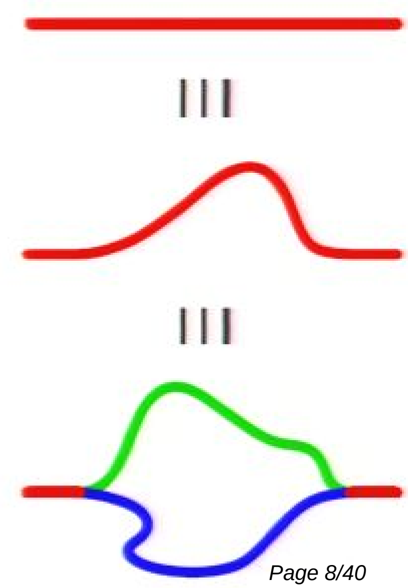
- The generators of the stabilizer are X and Z **plaquette operators**, for all plaquettes.
- As plaquettes, strings come in three colors.
- Strings not only can be deformed. A new feature appears: **branching** points.



Surface codes

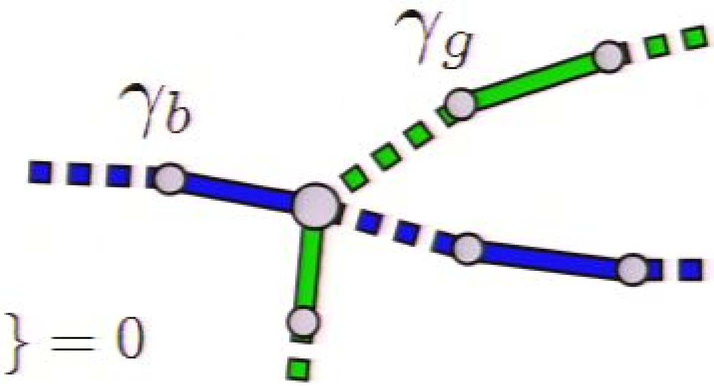


Color codes



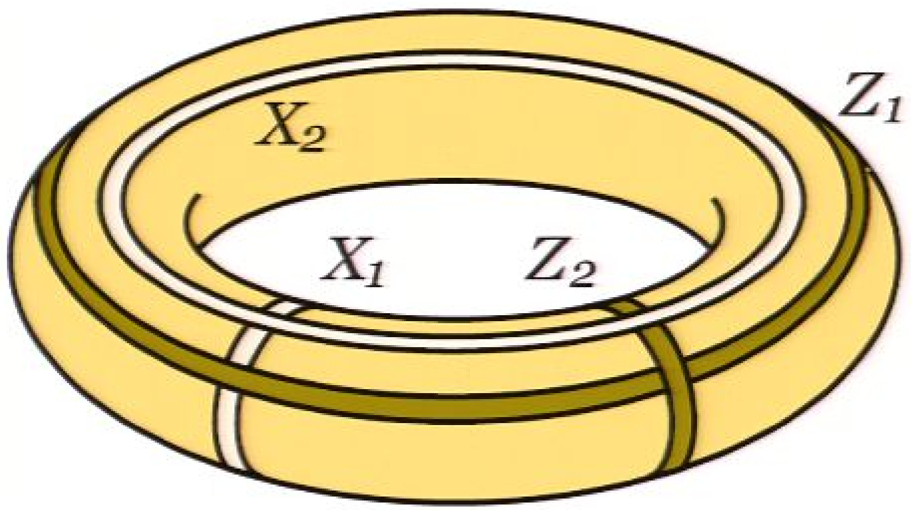
Topological Stabilizer Codes

- For each colored string γ, there are a **pair** of **string operators**, X_γ and Z_γ products of Xs or Zs along γ.
- Two string operators **anticommute** when they have **different color and type** and **cross** an odd number of times.

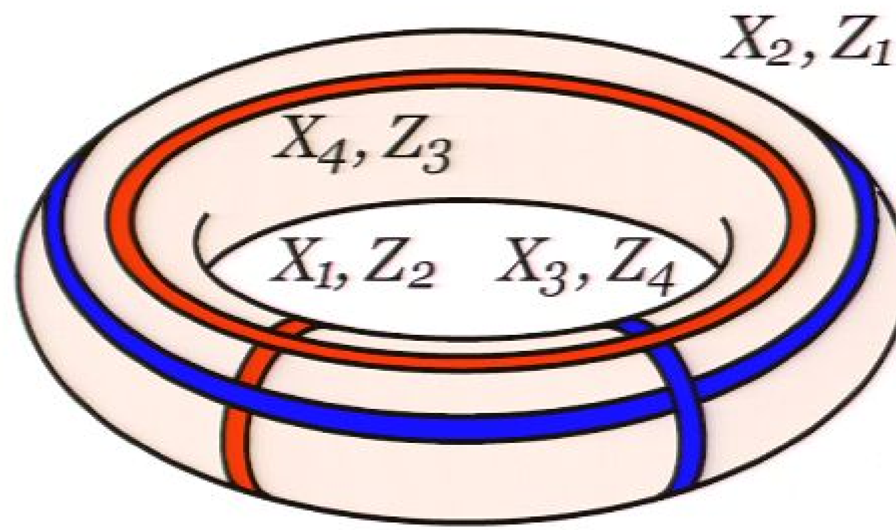


$$\{X_{\gamma_b}, Z_{\gamma_g}\} = 0$$

- As in surface codes, encoded X and Z operators can be chosen from closed string operators which are not boundaries.
- The number of **encoded qubits** is **twice** as in a surface code:



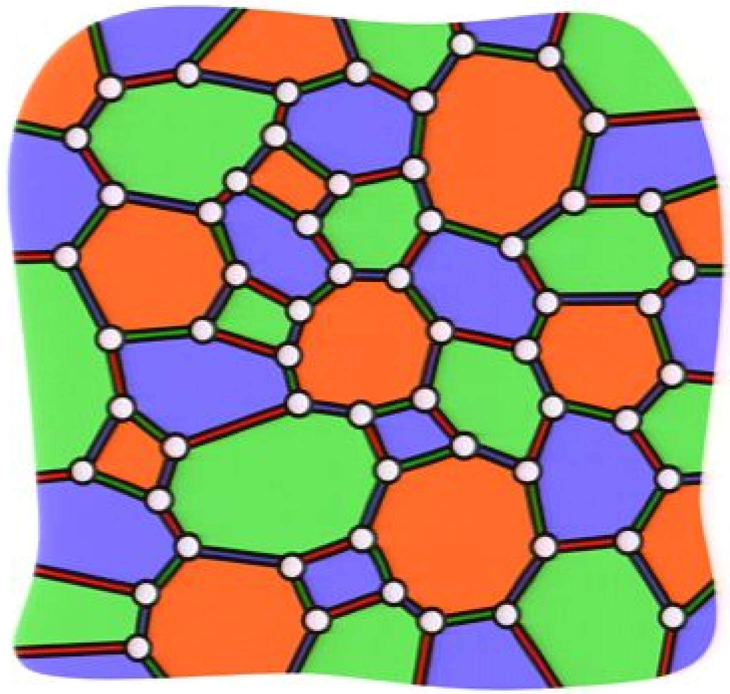
Surface code: 2 qubits



Color code: 4 qubits

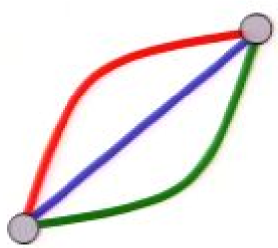
Topological Stabilizer Codes

- Color codes can be generalized to higher spatial dimensions D.
- First we have to generalize our 2D lattice. Note that **edges can be colored** in accordance with faces, so that at each vertex there are 3 links meeting, one of each color.
- In fact, the **whole structure** of the lattice is contained in its **colored graph**: faces can be reconstructed from edge coloring.



- Examples:

S



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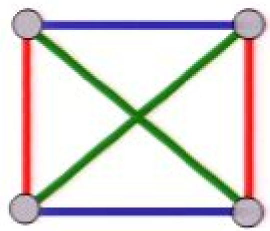

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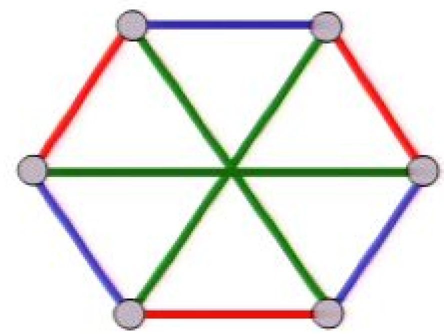
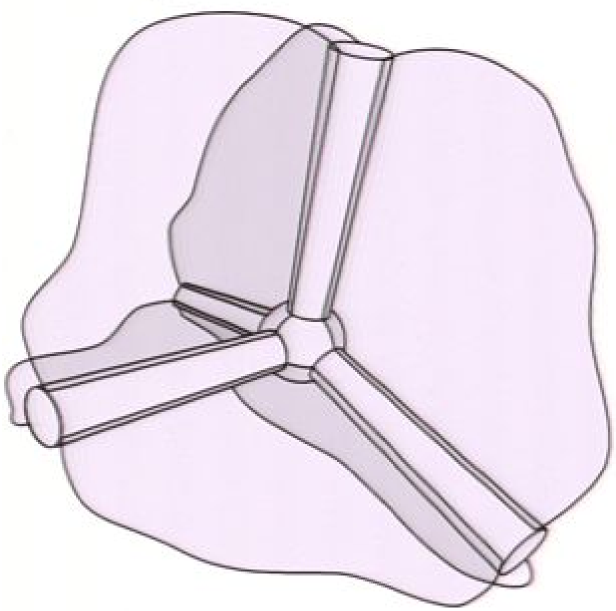
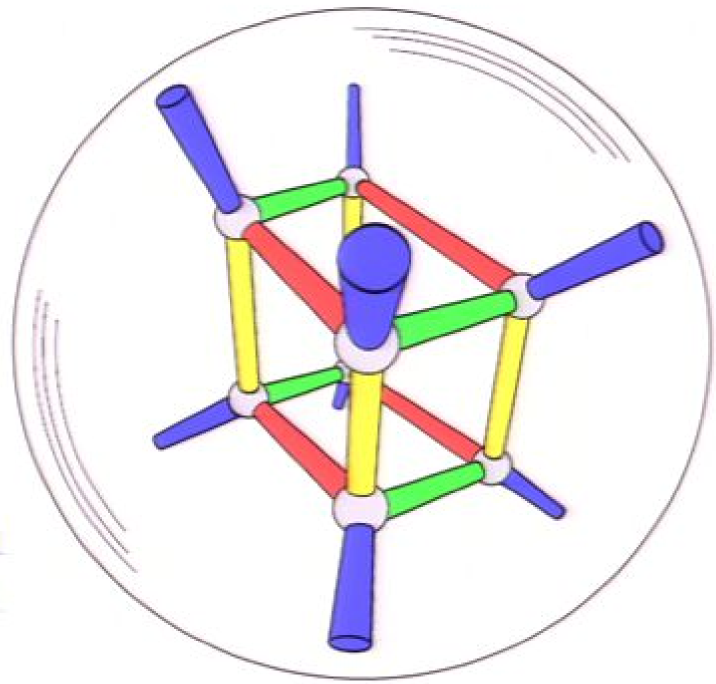


Topological Stabilizer Codes

- In dimension D, we consider graphs with D+1 edges meeting at each vertex, of D+1 different colors.
- Such graphs, with certain additional properties, give rise to D-manifolds. We call the resulting colored lattices D-colexes (for color complex).
- Of particular interest is the case D=3:



The neigborhood of a vertex.

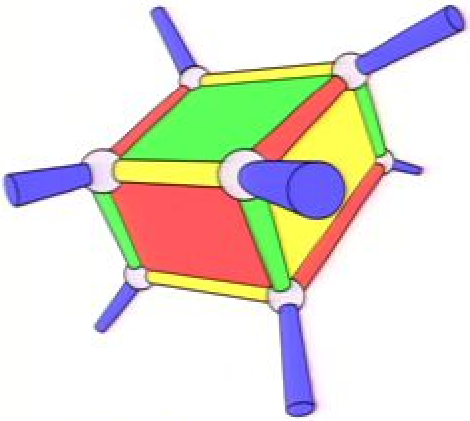


The simplest 3-colex in projective space.

- It is remarkable that **the whole topological structure of a *D*-manifold can be encoded in a colored graph**. For example, the orientability of the surface maps to the bicolorability of the graph and connected sum maps to a very simple graph manipulation. In addition, a D-colex can always be obtained from an arbitrary lattice.

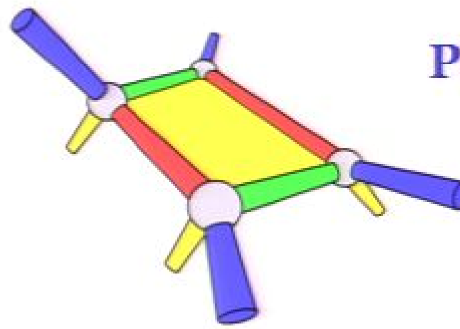
Topological Stabilizer Codes

- To build **3D color codes**, put one qubit per vertex and choose as stabilizers:



Cell operators

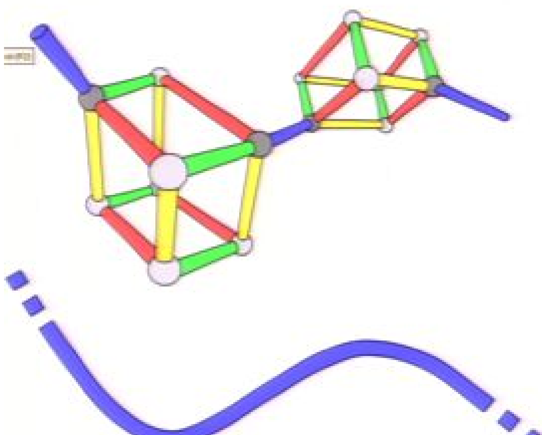
$$X_c = \bigotimes_{i=1}^{8} X_i$$



Plaquette operators

$$Z_f = \bigotimes_{i=1}^{4} Z_i$$

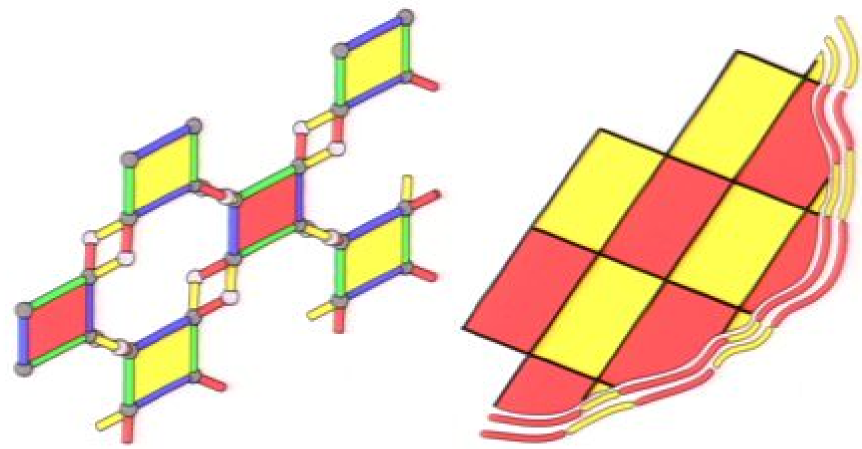
- **Strings** are constructed as in 2-D, but now come in **four colors**.
- The new feature are **membranes**. They come in **six color** combinations and, as strings, have **branching** properties.



$$Z_S = \bigotimes_{\text{string}} Z_i$$

b-string

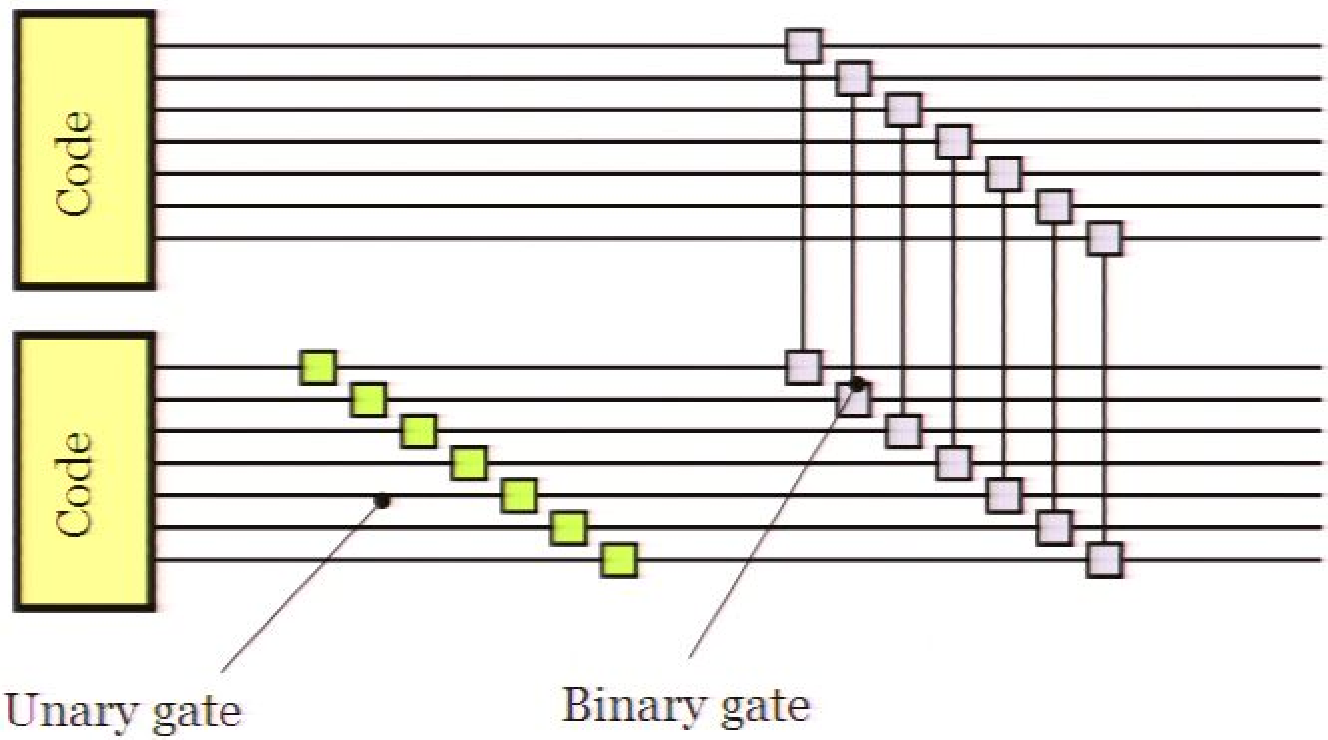
$$X_M = \bigotimes_{\text{membrane}} X_i$$



ry-membrane

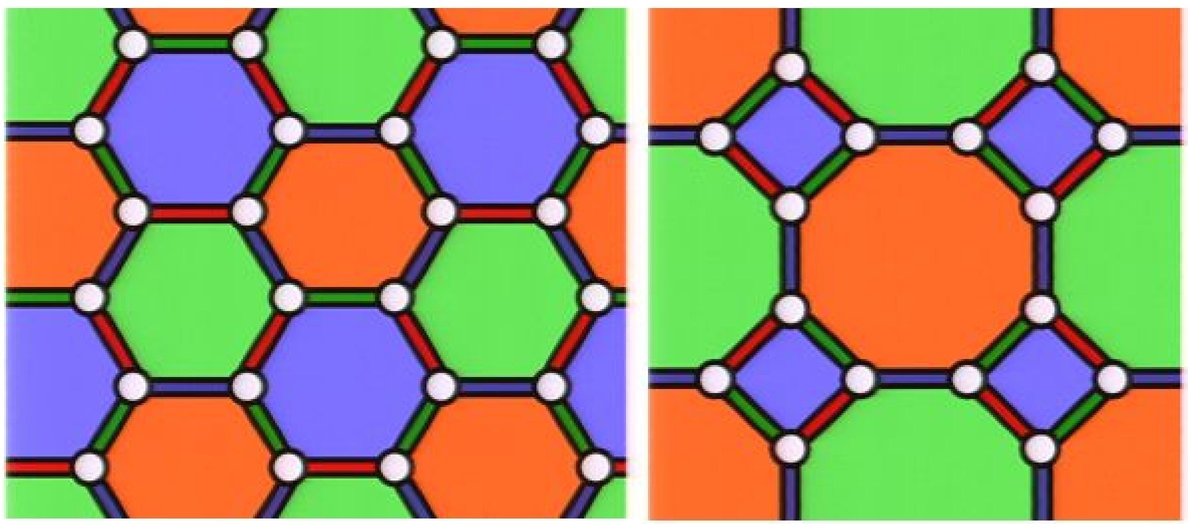
Topological Stabilizer Codes

- The reason to introduce color codes is that they have **transversality properties** not present in surface codes.
- Transversal gates have great importance in **fault tolerant quantum computation**. They can be visualized as follows in terms of quantum circuits:



Topological Stabilizer Codes

- The transversality properties of color codes **depend on the properties of the lattice**. In particular, only if all faces have $4n$ vertices can the whole **Clifford group** of operations be implemented transversally.
- This group is enough for a number of important **quantum information tasks**, such as quantum teleportation or entanglement distillation.



- Suitable 3-colexes give rise to 3D color codes that allow the **transversal implementation** of the same gates as quantum Reed-Muller codes, which is enough for **universal quantum computation**.

Topological Stabilizer Codes

- A physical system showing **topological quantum order** can be related to every TSC:

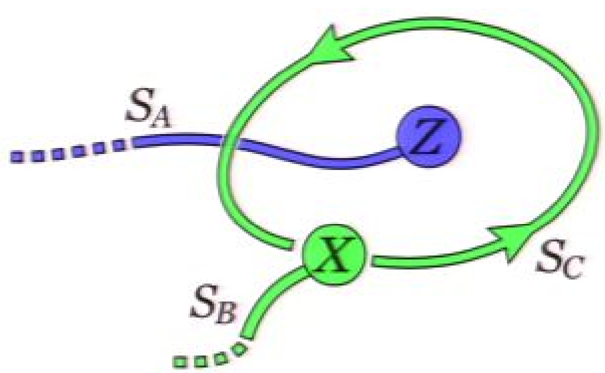
$$H = -\sum_{O \in \mathcal{S}'} O \qquad\qquad \mathcal{S}' = \text{Set of local generators of } \mathcal{S}$$

Stabilizer $\longrightarrow$ Hamiltonian

Code $\longrightarrow$ Ground state

Errors $\longrightarrow$ Excitations

- There exists a **ground state degeneracy with a topological origin**.
- Both for surface codes and *2D* color codes, **the excitations are abelian anyons,** because monodromy operations can give global phases.



Topological Stabilizer Codes

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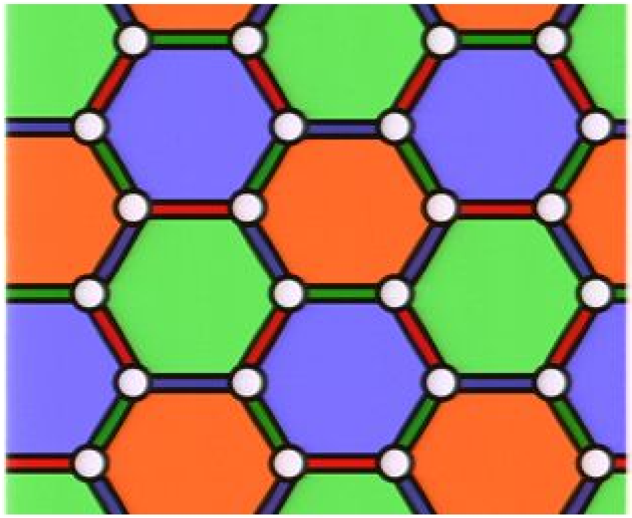
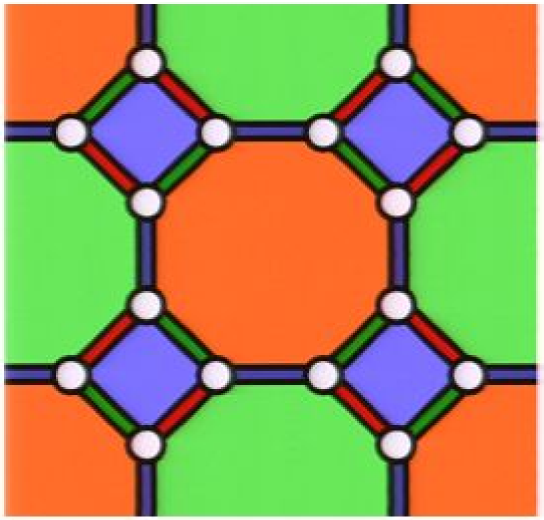
- This group is enough for a number of important **quantum information tasks**, such as quantum teleportation or entanglement distillation.

The honeycomb lattice is not suitable...





...but the 4-8 lattice is OK

- Suitable 3-colexes give rise to 3D color codes that allow the **transversal implementation** of the same gates as quantum Reed-Muller codes, which is enough for **universal quantum computation**.

Topological Stabilizer Codes

- A physical system showing **topological quantum order** can be related to every TSC:

$$H = -\sum_{O \in \mathcal{S}'} O \qquad\qquad \mathcal{S}' = \text{Set of local generators of } \mathcal{S}$$

$$\begin{aligned} \text{Stabilizer} &\longrightarrow \text{Hamiltonian} \\ \text{Code} &\longrightarrow \text{Ground state} \\ \text{Errors} &\longrightarrow \text{Excitations} \end{aligned}$$

- There exists a **ground state degeneracy with a topological origin**.
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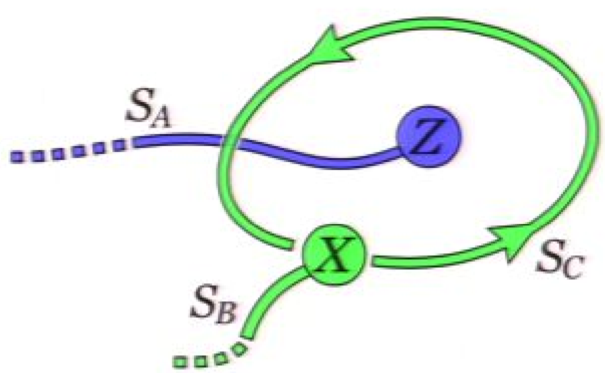
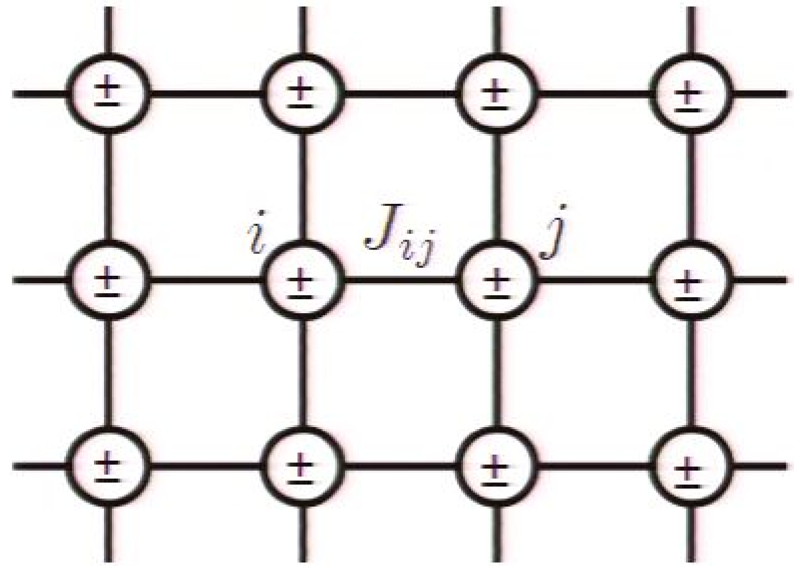


Classical Ising Models

Classical Ising Models

- Given any 2D lattice, place a classical spin variable $\sigma = \pm 1$ at each vertex. The **2-body Ising model** is given by the Hamiltonian



$$H = -\sum_{<i,j>} J_{ij}\sigma_i\sigma_j$$

- This Hamiltonian has a ***Z*$_2$ symmetry**, namely

$$\sigma_i \to -\sigma_i$$

- The thermal equilibrium is described by the **partition function**

$$Z(\beta, \mathbf{J}) := \sum_{\boldsymbol{\sigma}} e^{-\beta H} = \sum_{\boldsymbol{\sigma}} \prod_{<i,j>} e^{\beta J_{ij}\sigma_i\sigma_j}$$

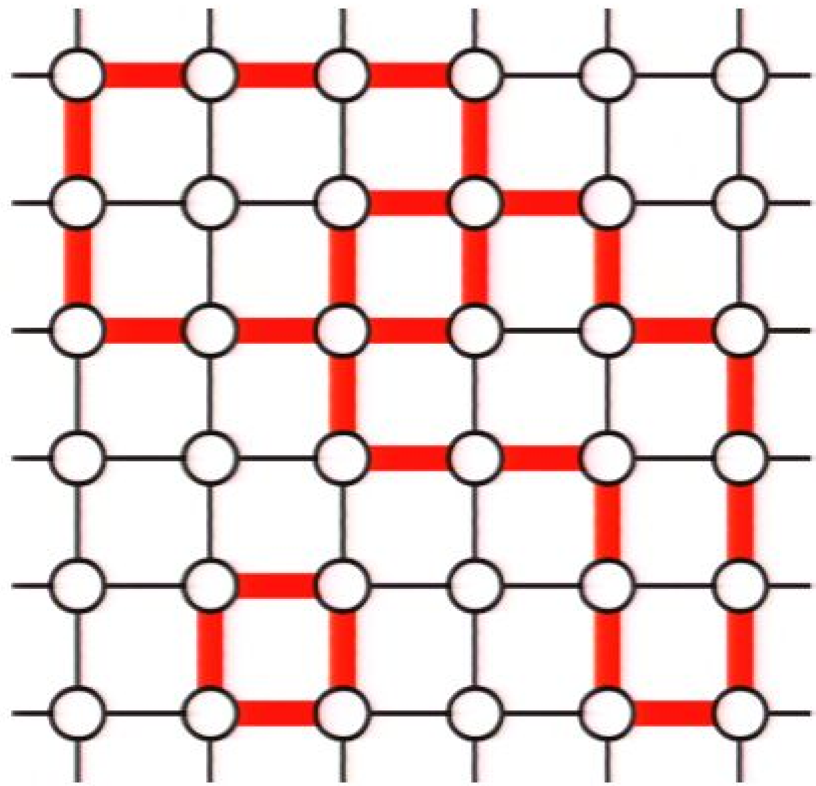
Classical Ising Models

- The **high temperature expansion** of the partition function is

$$Z(\beta, \mathbf{J}) = 2^N \prod_{<i,j>} \cosh(\beta J_{ij}) \sum_{\gamma | \partial\gamma = 0} u(\gamma)$$

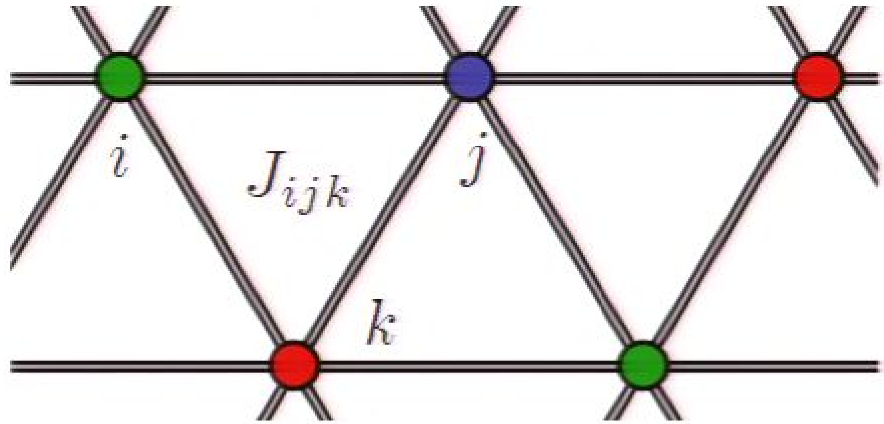
where γ runs over **closed 1-chains** of the lattice (collections of loops) and

$$u(\gamma) = \prod_{<i,j> \in \gamma} \tanh(\beta J_{ij}).$$



Classical Ising Models

- To construct a **3-body Ising model,** we place a classical spin at each vertex of a lattice with 3-colorable vertices and triangular faces. The Hamiltonian is



$$H = -\sum_{\langle i,j,k\rangle} J_{ijk}\sigma_i\sigma_j\sigma_k$$

- This Hamiltonian has a ***Z_2xZ_2* symmetry**. If we label sites with their color *c=r,g,b*

$$\sigma_i^r \to +\sigma_i^r \qquad \sigma_i^r \to -\sigma_i^r$$
$$\sigma_i^g \to -\sigma_i^g \qquad \sigma_i^g \to -\sigma_i^g$$
$$\sigma_i^b \to -\sigma_i^b \qquad \sigma_i^b \to +\sigma_i^b$$

	r	g	b
$J>0$	+	+	+
	+	-	-
	-	+	-
	-	-	+
$J<0$	-	-	-
	-	+	+
	+	-	+
	+	+	-

- For uniform J, there is no frustration. The ground state is **fourfold degenerate.**

Classical Ising Models

- Before we can write down a high temperature expansion, we need to describe the **chains of triangles** that appear on it. These are formal sums of triangles

$$\delta = \sum_{<i,j,k>} \delta_{ijk} \Delta_{ijk}, \quad \delta_{ijk} = 0, 1$$

- We can introduce a linear **boundary operator** from 0-chains to triangle chains and then back to 0-chains. Let $\Delta(v)$ be the set of triangles incident at a vertex v. We set

$$\partial v = \sum_{\Delta_{ijk} \in \Delta(v)} \Delta_{ijk}, \qquad \partial \Delta_{ijk} = v_i + v_j + v_k.$$

- With this choice, we have

$$\partial^2 = 0$$

so that we can define a **triangle homology group**, which is related to the 1-chain homology group by

$$H_T = H_1 \times H_1$$

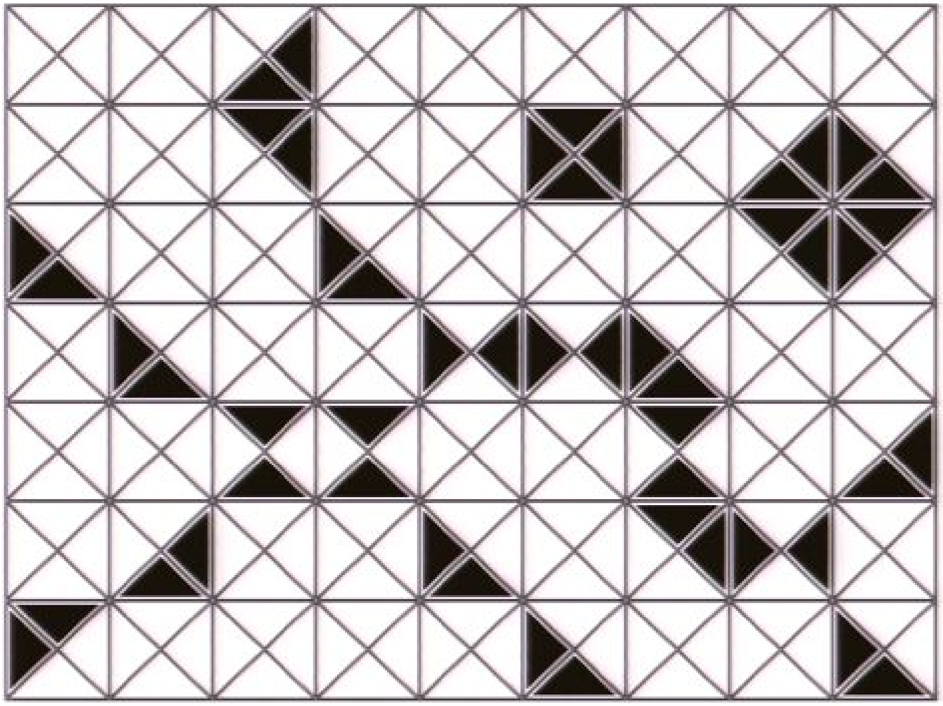
Classical Ising Models

- The **high temperature expansion of the partition function** is

$$Z(\beta, \mathbf{J}) = \sum_{\boldsymbol{\sigma}} \prod_{<i,j,k>} e^{\beta J_{ijk}\sigma_i\sigma_j\sigma_k} = 2^N \prod_{<i,j,k>} \cosh(\beta J_{ijk}) \sum_{\delta|\partial\delta=0} u(\delta),$$

where δ runs over **closed triangle chains** and

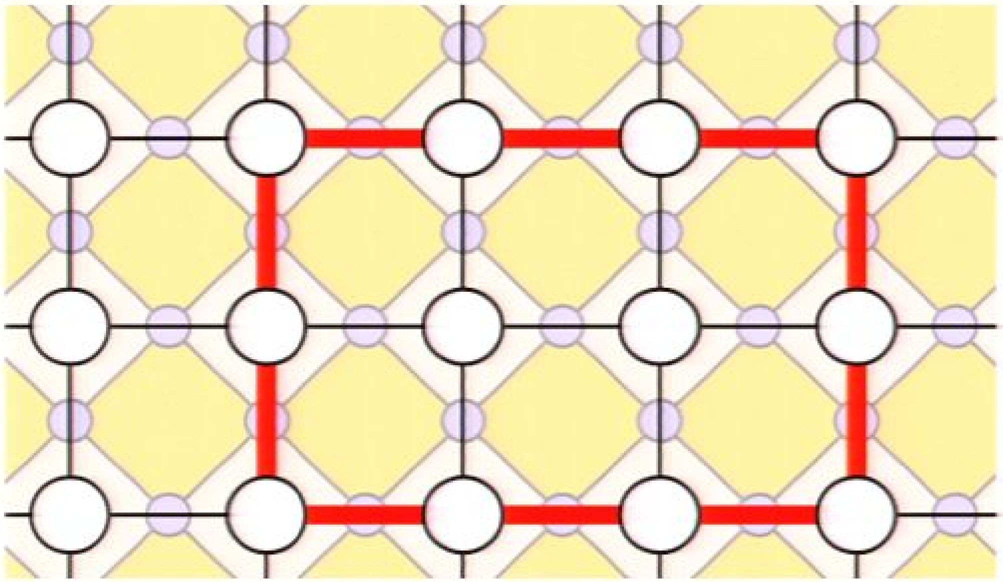
$$u(\delta) = \prod_{<i,j,k>\in\delta} \tanh(\beta J_{ijk}).$$



Connection

Connection

- There exists **a connection between classical 2-body Ising models and surface codes**[1]. We can relate the high-temperature expansion and the surface code:



$$\gamma = \sum_{<i,j>} \gamma_{ij} e_{ij}, \quad \gamma_{ij} = 0, 1$$

$$\gamma \longrightarrow |\gamma\rangle = \bigotimes_{<i,j>} |\gamma_{ij}\rangle$$

$$|\Psi_{sc}\rangle = \sum_{\gamma | \partial\gamma = 0} |\gamma\rangle$$

- Then the partition function is equal to the overlap of the surface code with a suitable product state:

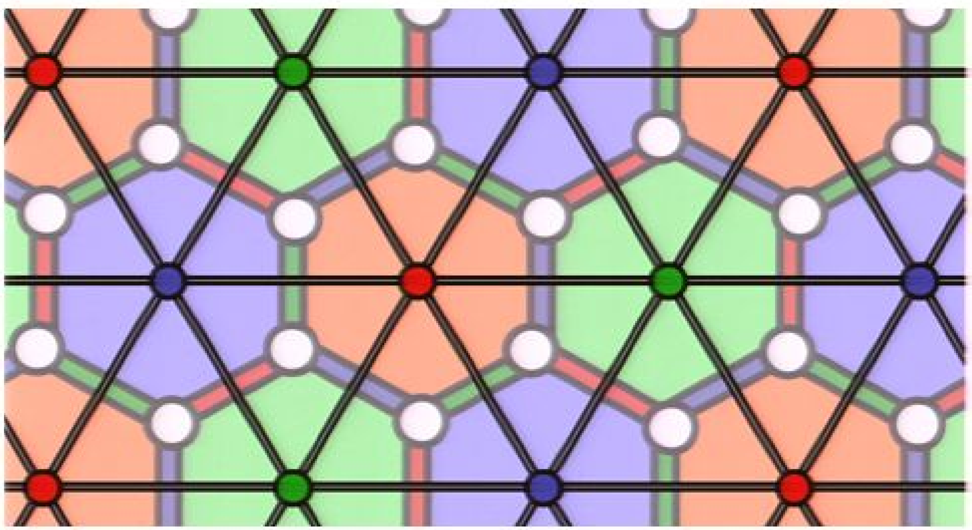
$$|\Phi\rangle := \bigotimes_{<i,j>} [\cosh(\beta J_{ij})|0\rangle + \sinh(\beta J_{ij})|1\rangle], \qquad Z(\beta, \mathbf{J}) = 2^N \langle \Psi_{sc} | \Phi \rangle$$

[1] S. Bravyi and R. Raussendorf, 07

M. Van de Neest, W. Dür and H.J. Briegel, 07

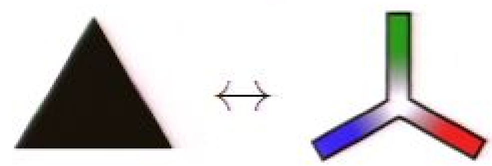
Connection

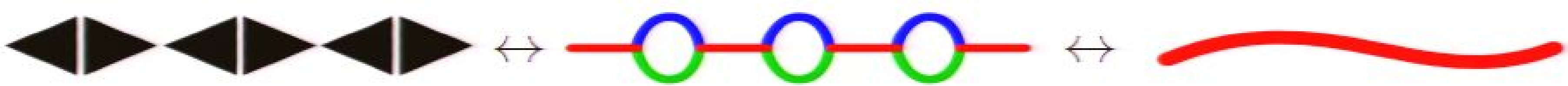
- Similarly, we can relate **classical 3-body Ising models with color codes**. Triangle chains can be mapped to suitable states:



$$\delta \longrightarrow |\delta\rangle = \bigotimes_{<i,j,k>} |\delta_{ijk}\rangle$$

$$|\Psi_{cc}\rangle = \sum_{\delta\,|\,\partial\delta=0} |\delta\rangle$$





- Again, the partition function is equal to the overlap of the color code with a suitable state:

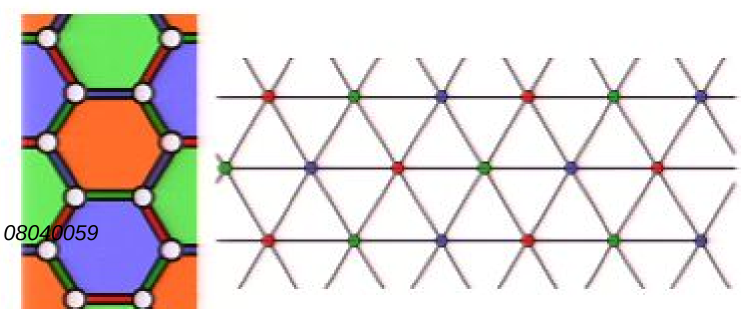
$$|\Phi\rangle := \bigotimes_{<i,j,k>} [\cosh(\beta J_{ijk})|0\rangle + \sinh(\beta J_{ijk})|1\rangle], \qquad Z(\beta, \mathbf{J}) = 2^N \langle \Psi_{cc}|\Phi\rangle$$

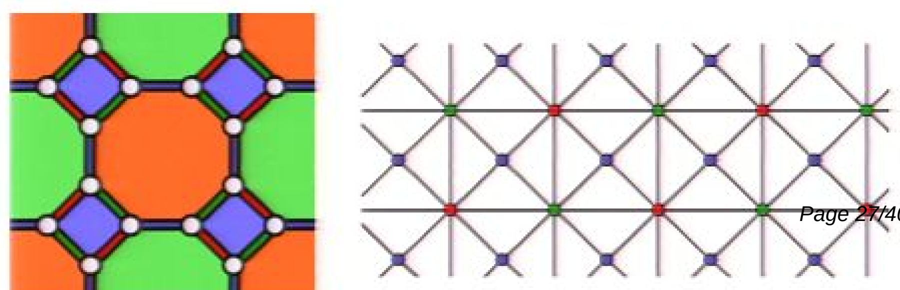
Connection

- We can compare the results for surface and color codes:
- **Interaction:** For surface codes we are led to a 2-body Ising model, but for color codes to a 3-body Ising model, which is qualitatively different.
- **Symmetry:** The symmetry of the classical model is the same as the gauge symmetry of the quantum model, $\boldsymbol{Z_2}$ for surface codes and $\boldsymbol{Z_2 x Z_2}$ for color codes. Indeed, the domain wall types of the classical systems map to the string types of the quantum ones.
- **Self-duality:** Regarding the duality between the high-T and low-T expansions, for uniform J, the 2-body model in the square lattice and the 3-body model in the triangular and Union-Jack lattices are self-dual, with critical temperature:

$$\sinh 2K_c = 1, \quad K_c := \beta_c J_c = 0.447$$

- **Universality classes:** For uniform J, the 3-body models on the triangular and Union Jack lattice are solvable under certain circumstances. This allows to check their criticality properties. It turns out that their universality classes are different. At the same time, the honeycomb and 4-8 lattices have **different q-computational capabilities**.





MQC & Color Codes

- In a measurement based quantum computer (MQC), processing is carried out via a sequence of **single-qubit measurements** on an initialized entangled quantum state.
- It turns out[1] that MQC with surface codes can be **classically simulated** as long as at every step of the computation the sets of measured and unmeasured qubits are **simply connected.**
- This results relies on the fact that the probability of obtaining a given result on a series of single qubit measurements can be written as a **partition function with complex non-uniform couplings** of a classical 2-body Ising model. Such partition functions can be calculated because there exists a mapping to a **dimer covering problem**, which in turn can be solved efficiently through the Pfaffian method in polynomial time.
- What about **color codes**? In this case we find out that it is not possible to derive a conclussive answer on the classical simulability of MQC on color codes from the connection with 3-body Ising models. **It remains an open problem.**
- The reason is that the dimer problem technique does not apply for them. Instead, there exists a mapping to a **site coloring coloring problem**. Unfortunately, this can only be solved under very restrictive conditions. In addition, this is done using the Bethe ansatz, which poses its own problems.

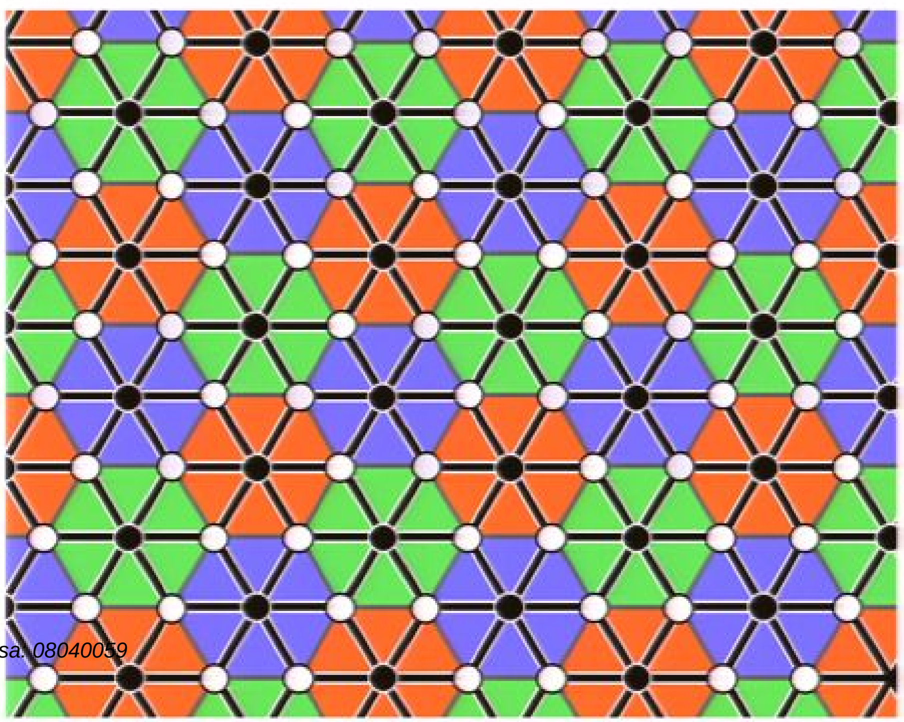
[1] S. Bravyi and R. Raussendorf, 07

Color Codes & Graph States

- Given a **bipartite graph** with a qubit at each vertex, a **graph state** is defined as a stabilizer state. Color the vertices in black and white, so that we have two sets V_b and V_w. If $N(v)$ is the set containing both v and its neighbors, the stabilizer group is

$$\mathcal{S} = (\{\, Z_{N(v)} \,|\, v \in V_b \,\} \cup \{\, X_{N(v)} \,|\, v \in V_w \,\})$$

- Given a 2-colex, we can construct a suitable bipartite graph such that from its graph state the **color code can be recovered** by measuring its black vertices in the Z basis. If the result at vertex v is x_v, the unnormalized final state is



$$|\psi(x)\rangle = \sum_{\delta \,|\, \partial\delta = x} |\delta\rangle$$

$$x := \sum_v x_v v, \quad x_v = 0, 1$$

- Indeed, the graph state can be written

$$|\kappa\rangle = \sum_x (|x\rangle \otimes |\psi(x)\rangle)$$

Color Codes & Graph States

- Such graph states can be related to suitable 3-body Ising models. In particular, we have to add a magnetic field

$$H = -\sum_{<i,j,k>} J_{ijk}\sigma_i\sigma_j\sigma_k - \sum_i h_i\sigma_i.$$

- The partition function for the new Hamiltonian is equal to the overlap of the graph state with a suitable product state:

$$Z(\beta, \mathbf{J}, \mathbf{h}) = 2^N \langle \kappa | \Phi \rangle$$

$$|\Phi\rangle := \bigotimes_i [\cosh(\beta h_i)|0\rangle + \sinh(\beta h_i)|1\rangle] \bigotimes_{<i,j,k>} [\cosh(\beta J_{ijk})|0\rangle + \sinh(\beta J_{ijk})|1\rangle]$$

Conclusions

- Color codes are constructed from D-colexes.
- They offer transversality properties not present in surface codes.
- A 2D color code is directly related to the hight-T expansion of the partition function of a classical 3-body Ising model.
- The question of the simulability of MQC with color codes remains open.
- Color codes can be obtained from suitable graph states.

- Simulable, simulatable… simulative? (courtesy of Phys. Rev. staff)

Topological Stabilizer Codes

- A physical system showing **topological quantum order** can be related to every TSC:

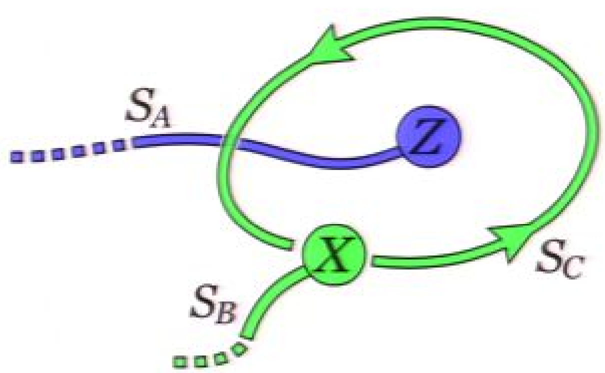
$$H = -\sum_{O \in \mathcal{S}'} O \qquad\qquad \mathcal{S}' = \text{Set of local generators of } \mathcal{S}$$

$$\text{Stabilizer} \longrightarrow \text{Hamiltonian}$$

$$\text{Code} \longrightarrow \text{Ground state}$$

$$\text{Errors} \longrightarrow \text{Excitations}$$

- There exists a **ground state degeneracy with a topological origin**.
- Both for surface codes and *2D* color codes, **the excitations are abelian anyons,** because monodromy operations can give global phases.



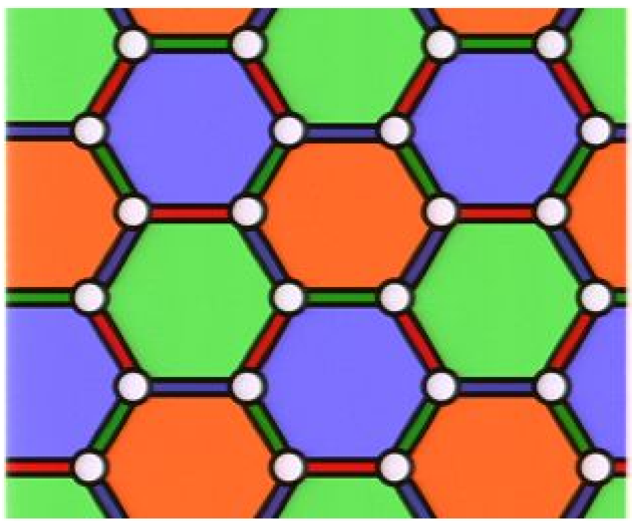
Classical Ising Models

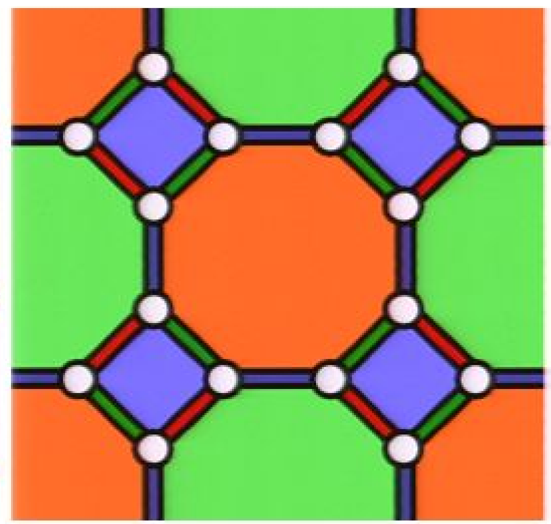
Classical Ising Models

Topological Stabilizer Codes

- The transversality properties of color codes **depend on the properties of the lattice**. In particular, only if all faces have $4n$ vertices can the whole **Clifford group** of operations be implemented transversally.
- This group is enough for a number of important **quantum information tasks**, such as quantum teleportation or entanglement distillation.

The honeycomb lattice is not suitable...





...but the 4-8 lattice is OK

- Suitable 3-colexes give rise to 3D color codes that allow the **transversal implementation** of the same gates as quantum Reed-Muller codes, which is enough for **universal quantum computation**.

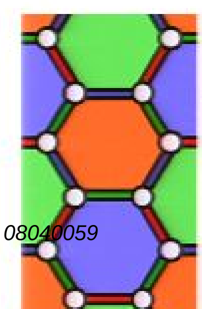
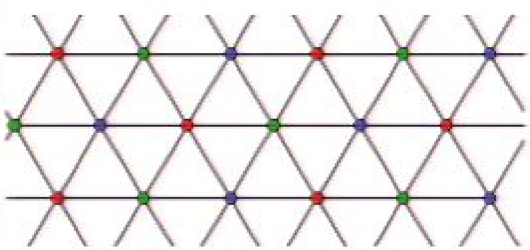
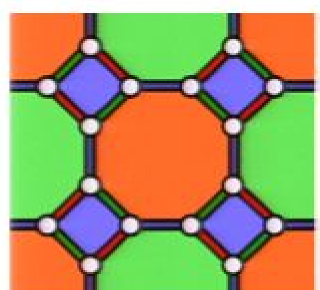
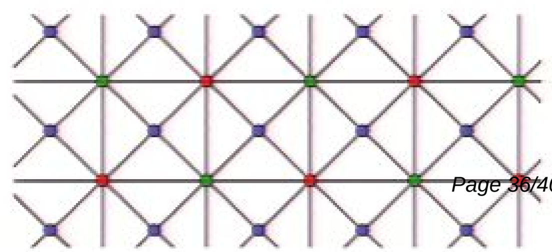
Connection

Connection

- We can compare the results for surface and color codes:
- **Interaction:** For surface codes we are led to a 2-body Ising model, but for color codes to a 3-body Ising model, which is qualitatively different.
- **Symmetry:** The symmetry of the classical model is the same as the gauge symmetry of the quantum model, $\boldsymbol{Z_2}$ for surface codes and $\boldsymbol{Z_2 x Z_2}$ for color codes. Indeed, the domain wall types of the classical systems map to the string types of the quantum ones.
- **Self-duality:** Regarding the duality between the high-T and low-T expansions, for uniform J, the 2-body model in the square lattice and the 3-body model in the triangular and Union-Jack lattices are self-dual, with critical temperature:

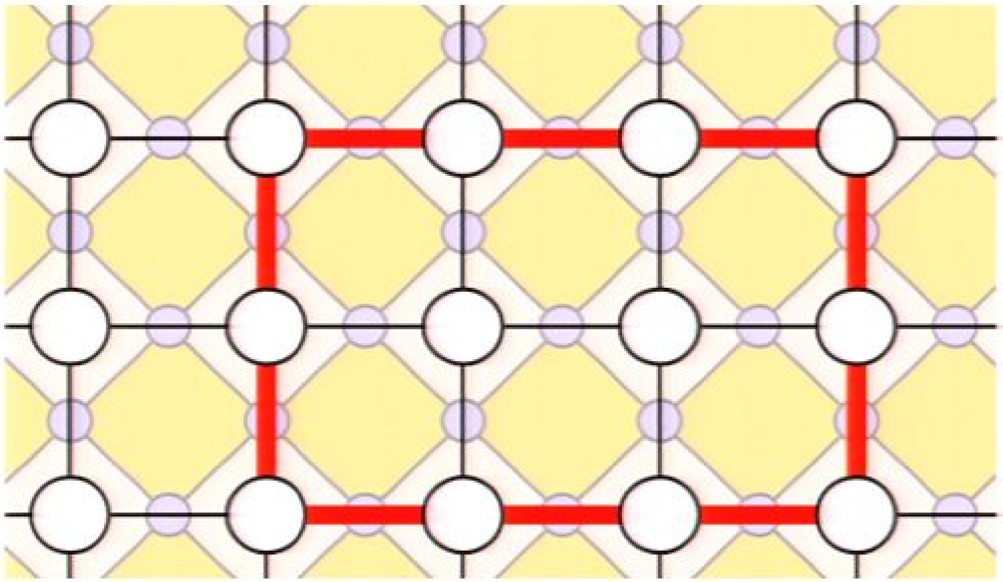
$$\sinh 2K_c = 1, \qquad K_c := \beta_c J_c = 0.447$$

- **Universality classes:** For uniform J, the 3-body models on the triangular and Union Jack lattice are solvable under certain circumstances. This allows to check their criticality properties. It turns out that their universality classes are different. At the same time, the honeycomb and 4-8 lattices have **different q-computational capabilities**.

Connection

- There exists **a connection between classical 2-body Ising models and surface codes**[1]. We can relate the high-temperature expansion and the surface code:



$$\gamma = \sum_{<i,j>} \gamma_{ij} e_{ij}, \quad \gamma_{ij} = 0, 1$$

$$\gamma \longrightarrow |\gamma\rangle = \bigotimes_{<i,j>} |\gamma_{ij}\rangle$$

$$|\Psi_{sc}\rangle = \sum_{\gamma|\partial\gamma=0} |\gamma\rangle$$

- Then the partition function is equal to the overlap of the surface code with a suitable product state:

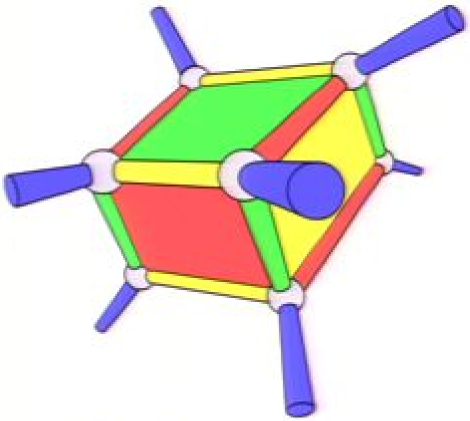
$$|\Phi\rangle := \bigotimes_{<i,j>} [\cosh(\beta J_{ij})|0\rangle + \sinh(\beta J_{ij})|1\rangle], \qquad Z(\beta, \mathbf{J}) = 2^N \langle\Psi_{sc}|\Phi\rangle$$

[1] S. Bravyi and R. Raussendorf, 07

M. Van de Neest, W. Dür and H.J. Briegel, 07

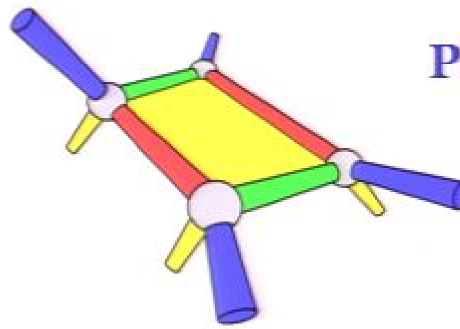
Topological Stabilizer Codes

- To build **3D color codes**, put one qubit per vertex and choose as stabilizers:



Cell operators

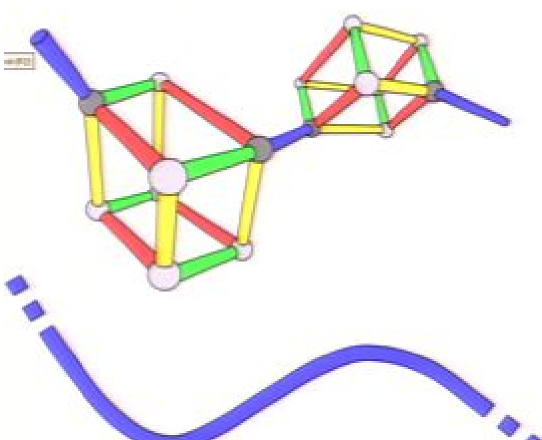
$$X_c = \bigotimes_{i=1}^{8} X_i$$



Plaquette operators

$$Z_f = \bigotimes_{i=1}^{4} Z_i$$

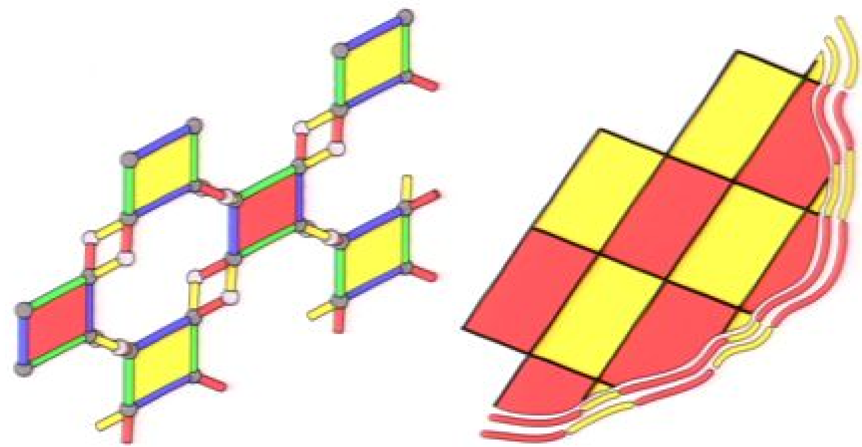
- **Strings** are constructed as in 2-D, but now come in **four colors**.
- The new feature are **membranes**. They come in **six color** combinations and, as strings, have **branching** properties.



$$Z_S = \bigotimes_{\text{string}} Z_i$$

b-string

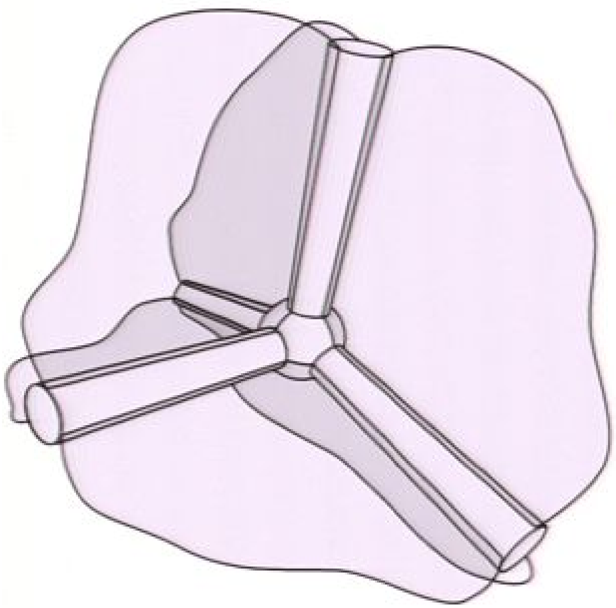
$$X_M = \bigotimes_{\text{membrane}} X_i$$



ry-membrane

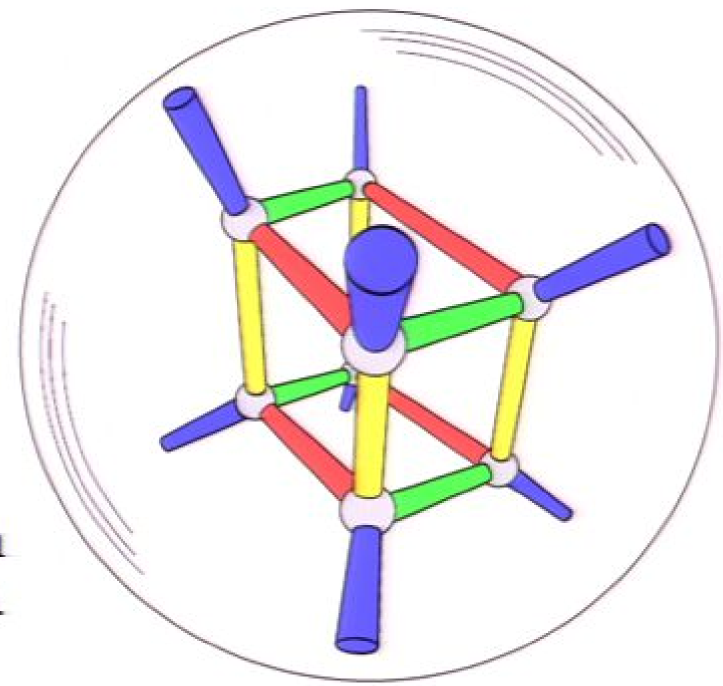
Topological Stabilizer Codes

- In dimension D, we consider graphs with $D+1$ edges meeting at each vertex, of $D+1$ different colors.
- Such graphs, with certain additional properties, give rise to D-manifolds. We call the resulting colored lattices D-colexes (for color complex).
- Of particular interest is the case D=3:



The neigborhood of a vertex.

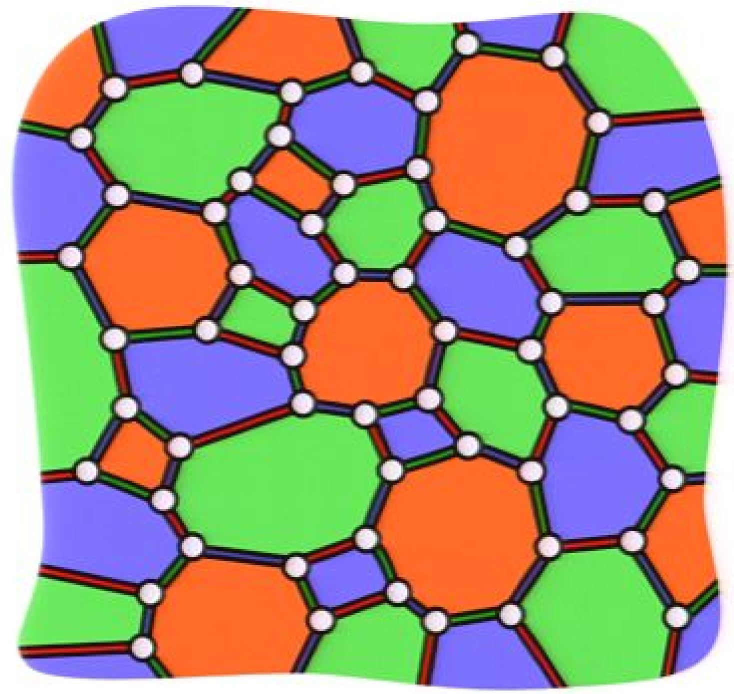
The simplest 3-colex in projective space.



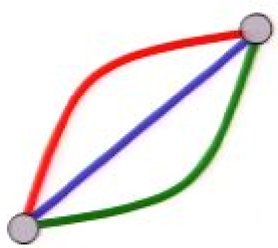
- It is remarkable that **the whole topological structure of a D-manifold can be encoded in a colored graph**. For example, the orientability of the surface maps to the bicolorability of the graph and connected sum maps to a very simple graph manipulation. In addition, a D-colex can always be obtained from an arbitrary lattice.

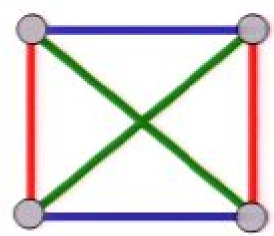
Topological Stabilizer Codes

- Color codes can be generalized to higher spatial dimensions D.
- First we have to generalize our 2D lattice. Note that **edges can be colored** in accordance with faces, so that at each vertex there are 3 links meeting, one of each color.
- In fact, the **whole structure** of the lattice is contained in its **colored graph**: faces can be reconstructed from edge coloring.



- Examples:

S 

P 

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