Title: A unifying view of graph theory in quantum field theory

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Abstract: A fundamental theorem of quantum field theory states that the generating functionals of connected graphs and one-particle irreducible graphs are related by Legendre transformation. An equivalent statement is that the tree level Feynman graphs yield the solution to the classical equations of motion. Existing proofs of either fact are either lengthy or are short but less rigorous. Here we give a short transparent rigorous proof. On the practical level, our methods could help make the calculation of Feynman graphs more efficient. On the conceptual level, our methods yield a new, unifying view of the structure of perturbative quantum field theory, and they reveal the fundamental role played by the Euler characteristic of graphs. This is joint work with D.M. Jackson (UW) and A. Morales (MIT)

A unifying view of graph theory in quantum field theory

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Joint work with D.M. Jackson (UW) and A. Morales (MIT)

QUANTUM INFORMATION AND GRAPH THEORY

Perimeter Institute, April 28 - May 2, 2008

Overview

Quantum Field Theory (QFT)

- QFT is generalization of QM: includes relativity and particle creation/annihilation processes.
- Quantum Information to be generalized to QFT: useful even for Quantum Gravity.
- Main solution method for QFT: Feynman graphs.

New results

- Combinatorial proof of a key theorem on Feynman graphs.
- Reduction to a basic homological statement. This yields generalizations of the theorem.

New insights

- "Path integral of graphs" is more robust (thus more fundamental?) than "path integral of fields".
- Beautiful and unifying duality of Feynman graphs and Feynman rules.

Quantum Field Theory

- There exit a dozen or so of <u>species</u> of fundamental particles in nature.

  Examples:

  - Electrons, quarks, photons, neutrinos, Z particles …
  - Each has its own "wave function" $\Phi(x)$, $q(x)$, $A(x)$, … (neglecting fermion and gauge issues)

    In QFT, they are called <u>fields</u>.

- Interpretation, very roughly:

  - Large field amplitude means large probability amplitude for a particle.

    (Caution: in relativistic case, antiparticles spoil that picture)
  - Fields are not normalized => one field can describe many particles.
  - The norm can change => particle creation/annihilation processes.

- What is the dynamics of the fields?

Quantum Field Theory

- Fields do not obey equations of motion, but pursue, virtually, all evolutions. This is the principle of 2^{nd} quantization.

  - The probability amplitude for the fields to undergo a particular evolution is:

$$\textbf{Probabilityamplitude}[\Phi, q, A\ldots] = N^{-1}\ e^{iS[\phi, q, A\ldots]}$$

    The action, $S(\Phi.q,A)$, is a scalar polynomial in $\Phi.q,A$.

  - Normalization: $$N = \int e^{iS[\phi, q, A\ldots]}\, D\phi Dq DA\ldots$$

  - Formal stationary phase analysis => evolutions near extremal action most likely.

  - Thus, in limiting case:
    - have approximately Schroedinger (or Dirac) evolution
    - but with quantum fluctuations.

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    "Path Integral"

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Quantum Field Theory

- Key predictions: correlation functions
  - E.g., correlation of field amplitudes at two points x,x′ in spacetime:

$$G^{(2)}(\phi(x),\phi(x')) = N^{-1}\int \phi(x)\ \phi(x')\ e^{iS[\phi,q,A,...]}D\phi DqDA...$$

- Meaning (roughly):
  - Elevated field amplitudes (in wave packets) describe particles.
  - Thus, from this two-point correlation function for field amplitudes, the propagation of particles between x and x′ can be predicted.
- More generally:

$$G^{(3)}(\phi(x),\phi(x'),q(x'')) = N^{-1}\int \phi(x)\ \phi(x')\ q(x'')\ e^{iS[\phi,q,A,...]}D\phi DqDA...$$

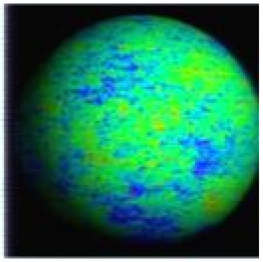
  - n-point functions describe the propagation <u>and interaction</u> of particles

Aspects of quantum information

One can show (Feynman):

n-point functions are generally nonzero almost everywhere, even in vacuum and even when the events x, x', x'' etc are chosen to be at equal time.

- Because of causality, this correlation is not due to propagation or interaction.
  - This correlation is due to entanglement of the vacuum state.
- More generally:
  - Enlarge the n points $x, x', x'', \ldots$ to n finite-size regions of space.
  - Consider the n regions of space as n systems (amplitudes are degrees of freedom).
  - Even the ground state (vacuum) of the combined system is generally entangled.
- Ramifications:
  - Connection to Unruh and Hawking effects, and holography.
  - Central to one of the greatest scientific ideas of past 50 years, Cosmic inflation:
    - Origin of all inhomogeneities is primordial equal-time field quantum fluctuations
    - The entanglement-predicted nontrivial correlation functions are so far being confirmed by the measurements of the cosmic microwave background.



Calculation of n-point functions

- An important special case:
  - Assume that the action, S, is an even polynomial in a field as, e.g., in cosmic inflation.
  - Then, all odd n-partite vacuum entanglement vanishes, e.g.:

$$G^{(3)}(\phi(x),\phi(x'),\phi(x'')) = N^{-1}\int \phi(x)\ \phi(x')\ \phi(x'')\ e^{iS[\phi,q,A,...]}D\phi Dq DA... \ = \ 0$$

  - **Note:**

    This prediction of "gaussian fluctuations" of cosmic inflation is currently under very close scrutiny (also: recent conference at PI)

  - In general, however, the path integral is analytically ill defined!
  - **Crucial trick (Schwinger, Feynman et al):**
    - Suitably pull field products in front of the path integral.
    - Remaining path integral cancels normalization factor path integral, N.
    - Price to pay: Evaluation of n-point function requires Feynman graphs.

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Towards Feynman graphs

- **Example:**

  A single field action for an M-dimensional function space (i.e., use UV and IR cutoff):

$$S[\phi] = \frac{1}{2} Q_{i,j}\ \phi_i\ \phi_j\ +\ \frac{1}{3!} V_{a,b,c}\, \phi_a\, \phi_b\, \phi_c$$

- Following Schwinger, introduce an auxiliary field, and define Z[J]:

$$Z[J] = \int_{R^n} e^{iS[\phi]+i\phi_a J_a}\ \prod_{m=1}^{M} d\phi_m$$

- <u>**Notice:**</u> This path integral is formally a Fourier transform !

- Define: $W[J] = -i\log(Z[J])$

- The n-point functions are then:

$$G^{(n)}(a_1,\dots,a_n) = Z[0]^{-1}\,(-i)^n\,\frac{\partial^n}{\partial J_{a_1}\dots\partial J_{a_n}}\,Z[J]\Big|_{at\ J=0} = (-i)^n\,\frac{\partial^n}{\partial J_{a_1}\dots\partial J_{a_n}}\,W[J]\Big|_{at\ J=0}$$

Towards Feynman graphs

- Now we can pull the fields in front of the path integral.

$$Z[J] = \int_{R^n} e^{iS[\phi]+i\phi_a J_a} \prod_{m=1}^{M} d\phi_m \qquad \text{with} \qquad S[\phi] = \frac{1}{2} Q_{i,j}\, \phi_i\, \phi_j \; + \; \frac{1}{3!} V_{a,b,c}\, \phi_a \phi_b \phi_c$$

becomes:

$$Z[J] = e^{\frac{i}{3!}(-i)^3 V_{a,b,c} \partial_{J_a} \partial_{J_b} \partial_{J_c}} \int e^{\frac{i}{2} Q_{i,j}\, \phi_i \phi_j \, + \, i\phi_r J_r} \prod_m d\phi_m$$

- Complete squares, carry out "gaussian" integration =>

$$Z[J] = e^{\frac{i}{3!}(-i)^3 V_{a,b,c} \partial_{J_a} \partial_{J_b} \partial_{J_c}} \; e^{-\frac{i}{2} Q^{-1}_{i,j} J_i J_j} \times const.$$

In the calculation of n-point functions, the constant drops out, as desired.

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Feynman graphs

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- Thus, defining these Feynman rules
  - edge = $i\,Q^{-1}_{i,j}$
  - 3-vertex = $i\,V_{a,b,c}$
  - 1-vertex = $i\,J_a$ (also called end-vertex)
- we have: Z[J] = generating functional of all Feynman graphs.
- Therefore, from Z[J] = -i exp(W[J]),

  we have W[J] = generating functional of all connected Feynman graphs.
  - This is combinatorially easy to see.
  - This general role of the logarithm is well known in combinatorics.

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Feynman graphs: in quantum and in classical theory

- Setup of quantum theory:
  - Definition:

    Problem $S[\Phi]$ ---Fourier transform--> Quantum solution $W[J]$

  - Theorem 1:

    $W[J]$ can be calculated through:

    $W[J] = \Sigma$ (all connected raphs)

- Setup of classical theory:
  - Definition:

    Problem $S[\Phi]$ ---Legendre transform--> Classical solution $T[J]$

  - Theorem 2:

    $T[J]$ can be calculated through:

    $T[J] = \Sigma$ (all tree graphs)

Feynman graphs: in quantum and in classical theory

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- Setup of classical theory:
  - Definition:

    Legendre: $T[J] := S[\Phi] - \langle J,\Phi\rangle$ with $dS/d\Phi = J$

    It follows: $\Phi = -dT/dJ$, which is the classical solution

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  - Theorem 2: nontrivial – we'll revisit it here

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Significance of Theorem 2

- Define a new generating functional, Γ:

  $W[J]$ <---Legendre transform---> $\Gamma[\phi]$

- Then, $\Gamma[\phi]$ is a quantum effective action:
  - Solve problem given by Γ classically => obtain full quantum solution $W[J]$
  - Concretely:
    - Read off new Feynman rules from Γ. Then:
    - $W[J] = \Sigma$ (all connected graphs of old F. rules) $= \Sigma$ (all trees of new F. rules)
  - Important in QFT, e.g., for:
    - Regularization and renormalization
    - Spontaneous symmetry breaking

The effective action, concretely

Recall:

W[J] = Σ (all connected graphs of old F. rules) = Σ (all trees of new F. rules)

- **Definition:**

  1-particle irreducible graphs (1PI graphs) := graphs that cannot be disconnected by cutting one edge.

- **Observation:**

  Any connected graph is, uniquely, a tree whose vertices are maximal 1PI graphs.

- **Conclusion:**
  - The new Feynman rules of Γ have 1PI graphs with n ends as n-vertices
  - Γ is the sum (i.e., generating functional) of all 1PI graphs

- **Practical significance:**
  - The 1PI graphs are building blocks for connected graphs.
  - Once the 1PI are renormalized (hard in QFT), the connected graph is put back together without further loops, i.e. without further renormalization.

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New results

- **Recall theorem 2:**

  $T[J] := \text{Legendre transform}(S[\Phi]) = \Sigma\,(\text{all tree graphs})$

- **We proved theorem 2, again. Why?**
  - Existing proofs un-necessarily use the Dyson Schwinger equations of motion, require tedious induction (Jona Lasigno) or employ not quite rigorous limit taking (Weinberg).
- **Advantages of our proof:**
  - Purely combinatorial
  - No analytic assumptions: S,T elements in ring of power series
  - Short, transparent
- **Proof strategy:**
  - Reduce Theorem 2 to the basic homological statement about the connectivity of tree graphs:

    $$T = S - \langle J,\Phi\rangle \qquad \text{(Legendre transform)}$$
    $$1 = V - E \qquad \text{(Euler characteristic)}$$

  - To this end, we use this key observation:

Ramifications

- Can now generalize theorem 2 for general Euler characteristic:

  $2 - 2g = V - E + F$

  - generalize decomposition of graphs into n-particle irreducible subgraphs
  - of interest for practical calculations

- **Can now generalize Legendre and Fourier transform for realistic S,T,Z,W,Γ:**
  - We showed that Theorem 2 holds even when S and T, or Γ and W are non-convergent power series (as is generally the case, except perhaps S)
  - **New definition of the Legendre transform**, without analytic assumptions:
    - Consider power series R as an action, read off the Feynman rules.
    - Define L := Legendre(R) as the sum (generating functional) of the tree graphs.
    - Notice: we also proved the involution property combinatorially.
  - **Similarly, define (exponentiated) Fourier transform** through graphs:
    - Define F := Fourier(R) as the sum (generating functional) of all connected graphs.
    - **Conjecture:** Feynman graph theoretically defined Fourier transform is also involutive

Ramifications

- These algebraic Fourier and Legendre transforms (and exponentiation) can replace the usual analytic definitions.
- In particular, there is a well-defined alternative to the ill-defined usual path integral:

$$Z[J] = \int e^{iS[\phi]+i\int \phi J\,dx}\, D\phi$$

  - It is merely the analytically-defined Fourier transform
  - For realistic actions, it is ill defined in any case.
  - **But:**
    The path integral represents the principle that classical evolution is of extremal action while quantum motion explores all evolutions.

- Does this principle have an alternative representation too? Yes.
  - Evolution is given by the sum over all graphs:
    - Classical: shortest path (which is always a tree)
    - Quantum: all paths (including loops)
  - A unified picture emerges for 1st and 2nd quantization:
    - The edge, i.e., the propagator is a sum of paths of 1st quantization.
    - **Thus, evolution is sum over all paths, including graphs.**

Outlook

- Recall:
  - Legendre transform is involutive. Thus, duality:
    - Problem S ---Legendre transform---> Solution T
    - "Problem" T ---Legendre transform---> "Solution" S
  - In terms of graphs:
    - T = Σ (all trees of F. rules of S)
    - S = Σ (all trees of F. rules of T)
    - Thus: sum of trees of trees = sum of Feynman rules !
  - Conjecture: Similar duality should exist for Fourier transform.
    (not shown algebraically yet):
    - Sum of all connected Feynman graphs from sum of all Feynman graphs = sum of Feynman rules
- Questions:
  - How general is this duality phenomenon in graph theory, beyond Feynman graphs?
  - Does this duality of "Problem" versus "Solution" possess a physical interpretation?

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