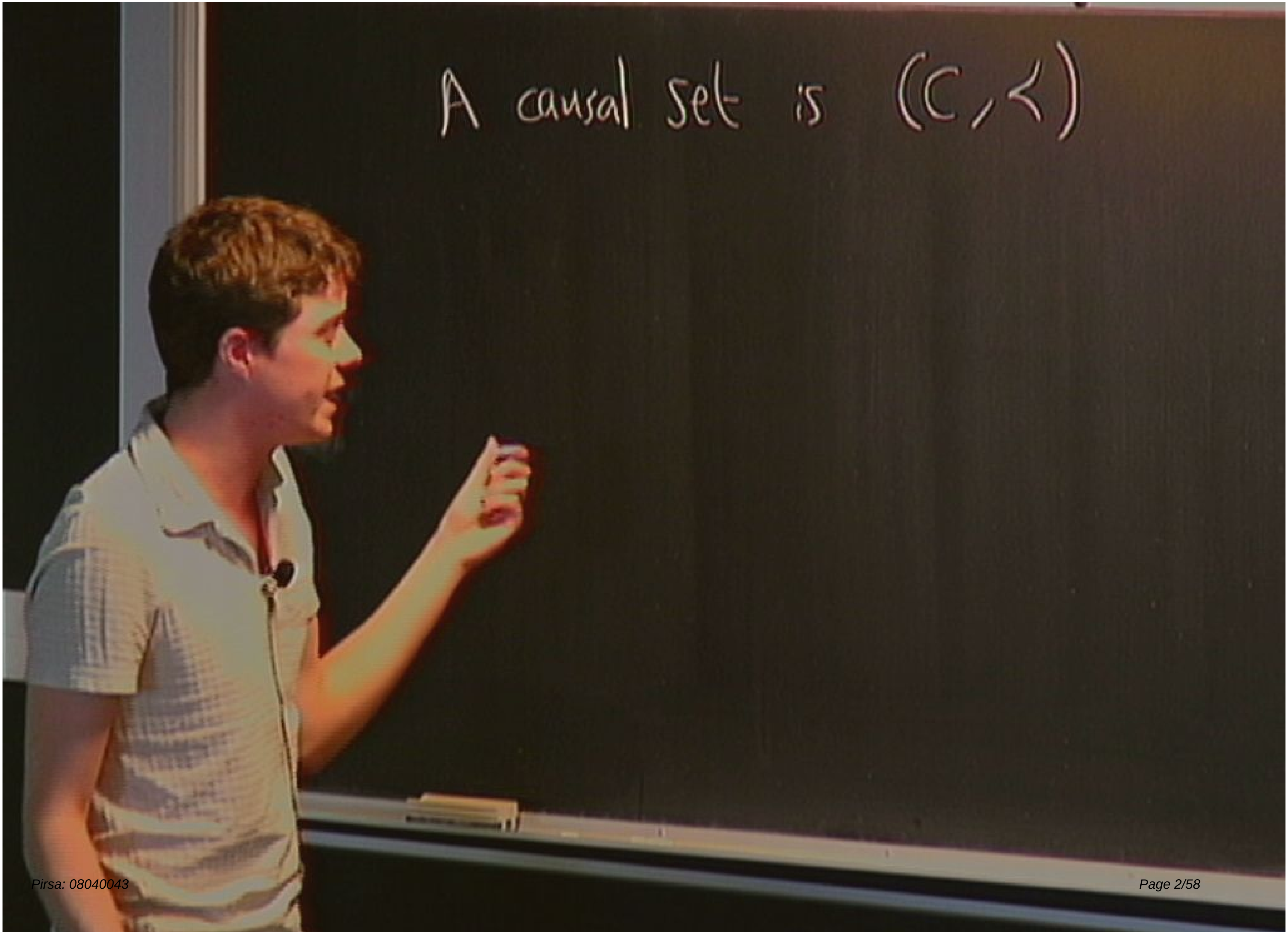
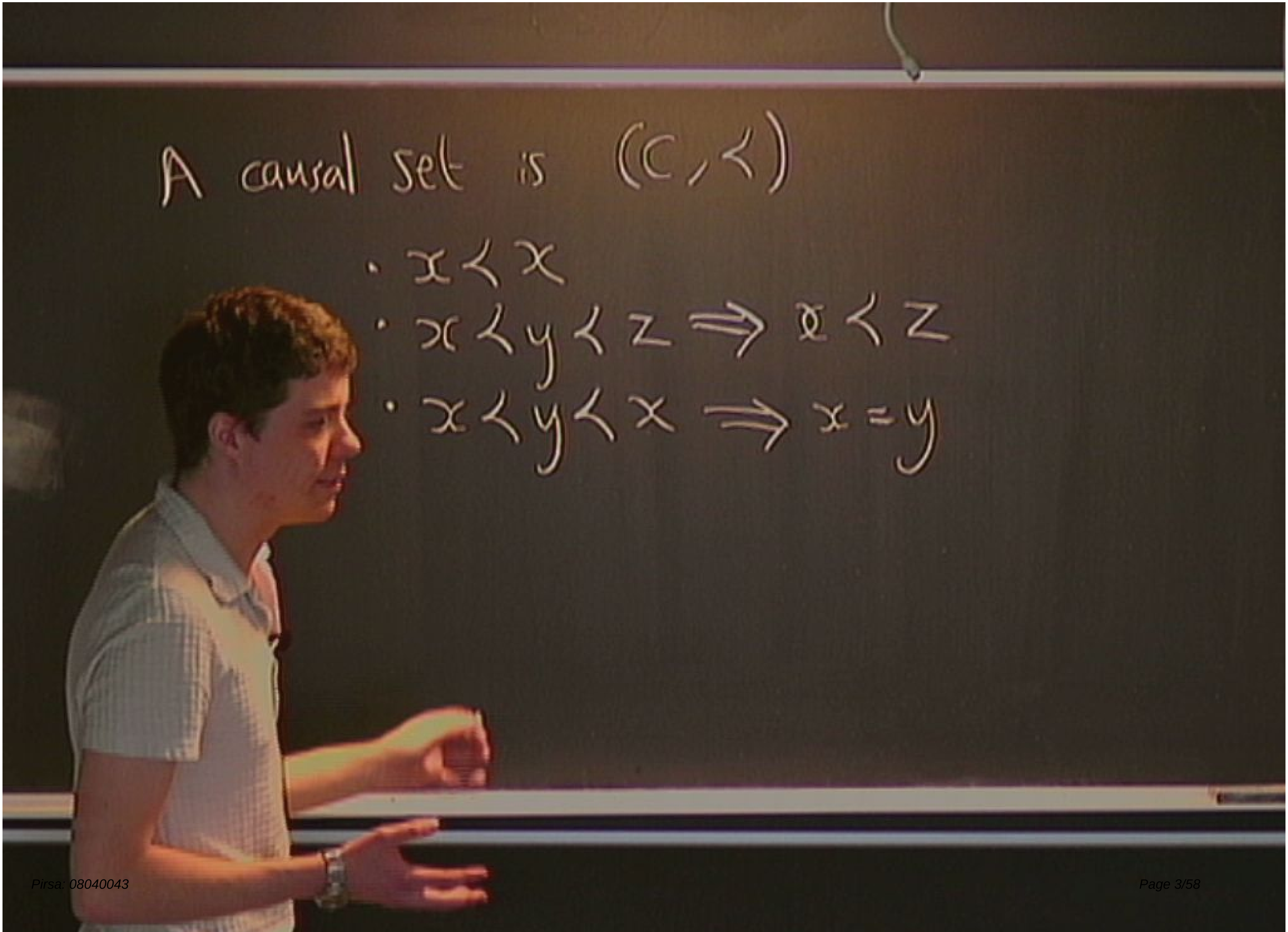
Title: Particle Propagators from Discrete Spacetime

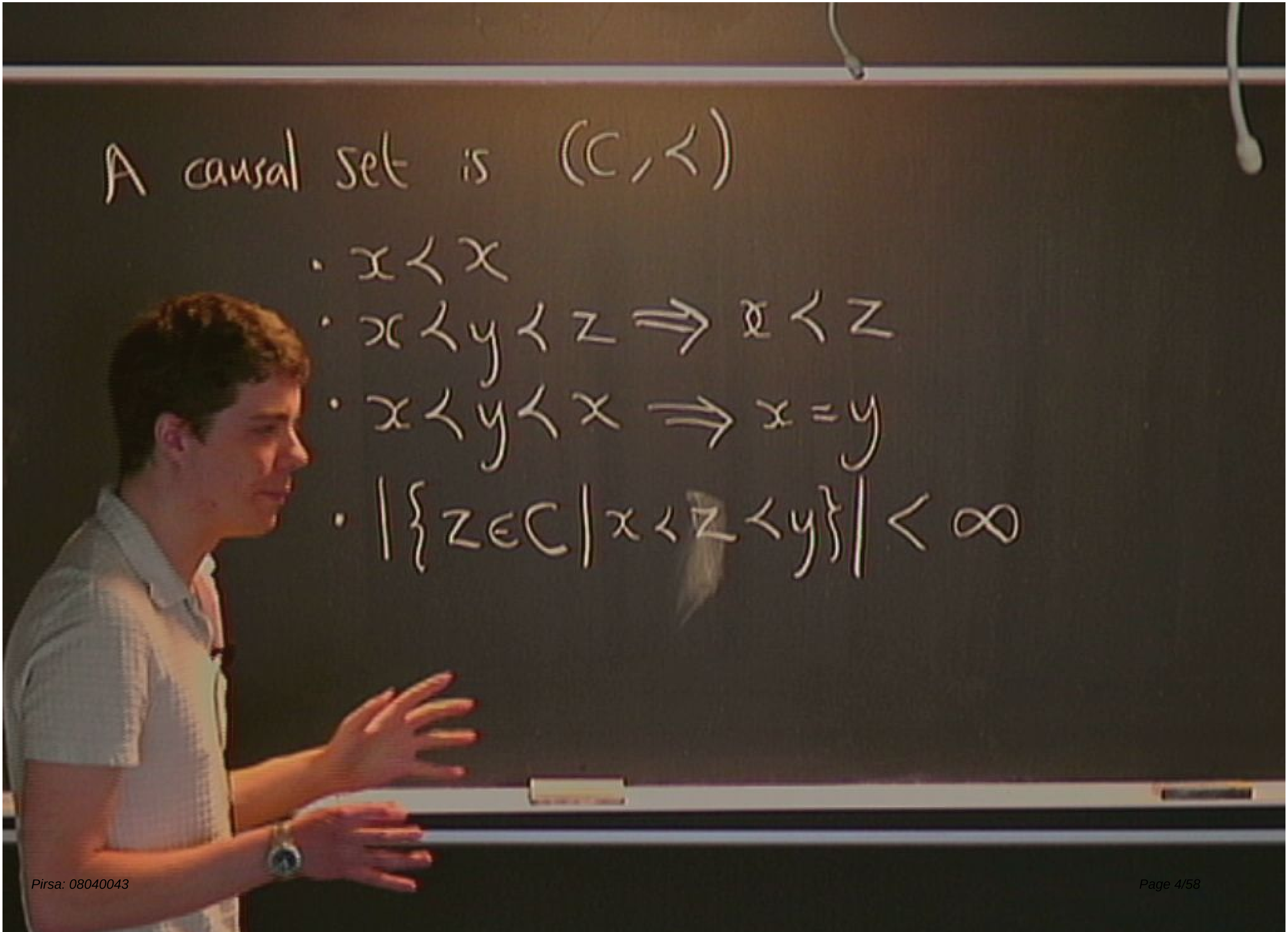
Date: Apr 14, 2008 11:00 AM

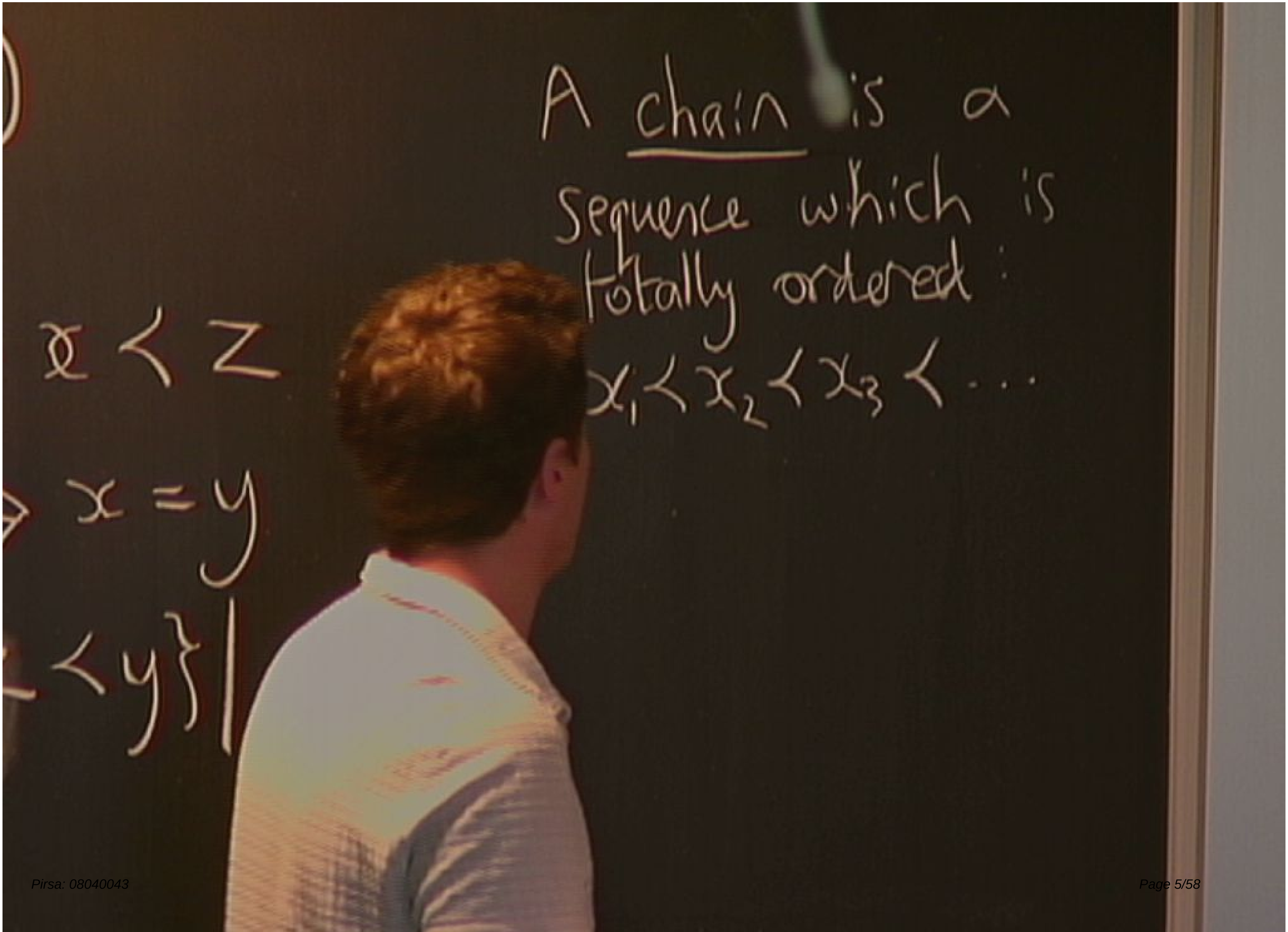
URL: http://pirsa.org/08040043

Abstract: How sure are you that spacetime is continuous? One of the more radical approaches to quantum gravity, causal set theory, models spacetime as a discrete structure: a causal set. Allowing the possibility that spacetime is discrete then how should we do physics on it? Carrying over the usual continuum descriptions in terms of differential equations seems like a difficult option. This talk begins with a brief introduction to causal sets then describes an approach to modelling the propagation of scalar particles on a causal set. We obtain the continuum causal retarded propagator by summing quantum mechanical amplitudes assigned to paths in the causal set - a kind of \'discrete path integral\' that agrees with the continuum result. The propagator so obtained should serve as a building block towards a model for quantum field theory on a causal set.


A causal set is $(C, \prec)$


A causal set is (C, <)
• x < x
• x < y < z ⇒ x < z
• x < y < x ⇒ x = y


A causal set is $(C, \prec)$
• $x \prec x$
• $x \prec y \prec z \Rightarrow x \prec z$
• $x \prec y \prec x \Rightarrow x = y$
• $|\{z \in C \mid x \prec z \prec y\}| < \infty$


A chain is a
sequence which is
totally ordered:
$x_1 < x_2 < x_3 < \dots$
$x < z$
$x = y$
$< y\}|$

A **chain** is a sequence which is totally ordered:

$$x_1 < x_2 < x_3 < \dots$$

A **link** is any relation $x < y$ with no z s.t. $x < z < y$.

$x < z$

$x = y$

$<y\}| < \infty$

A **chain** is a sequence which is totally ordered:

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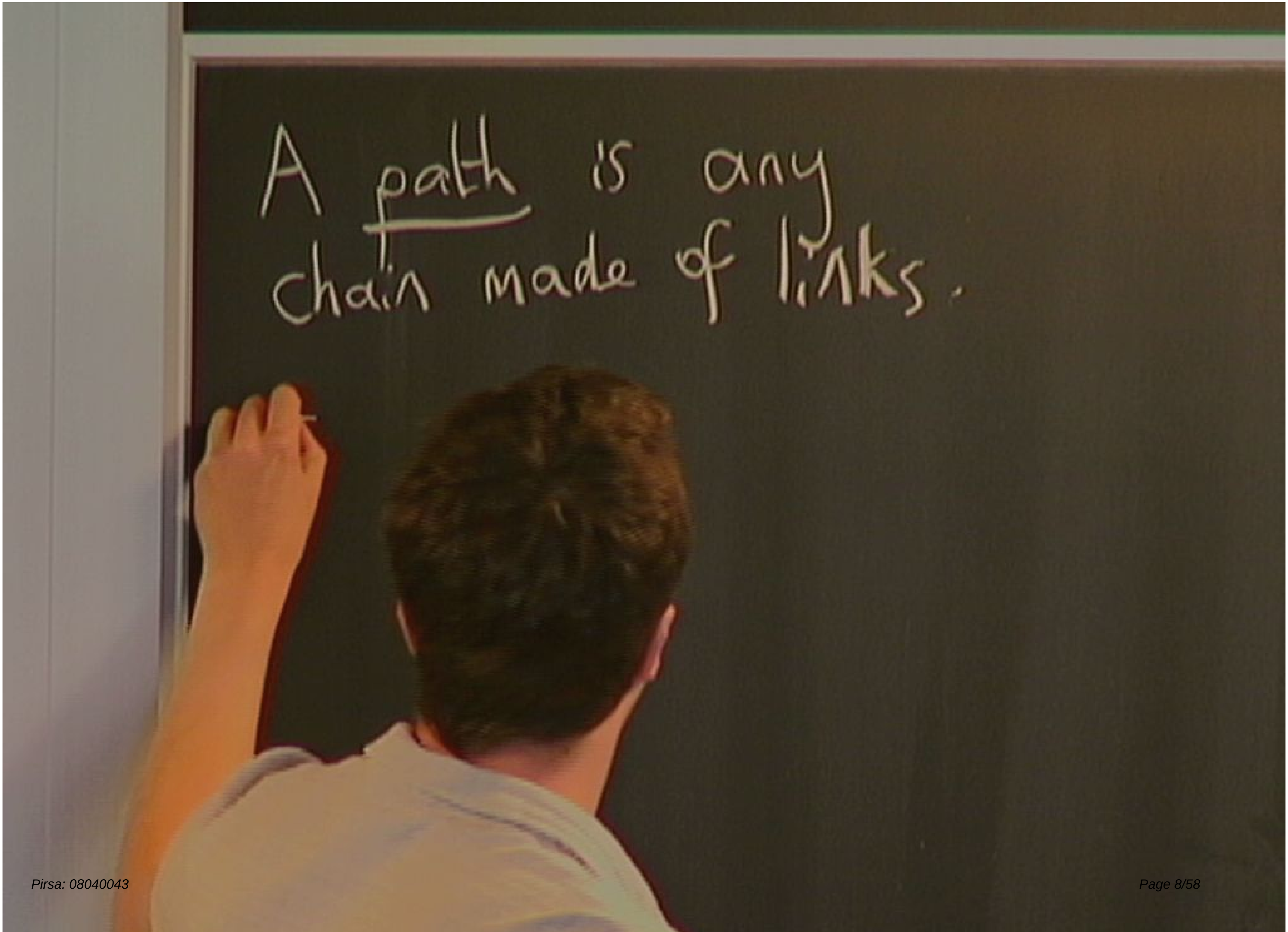
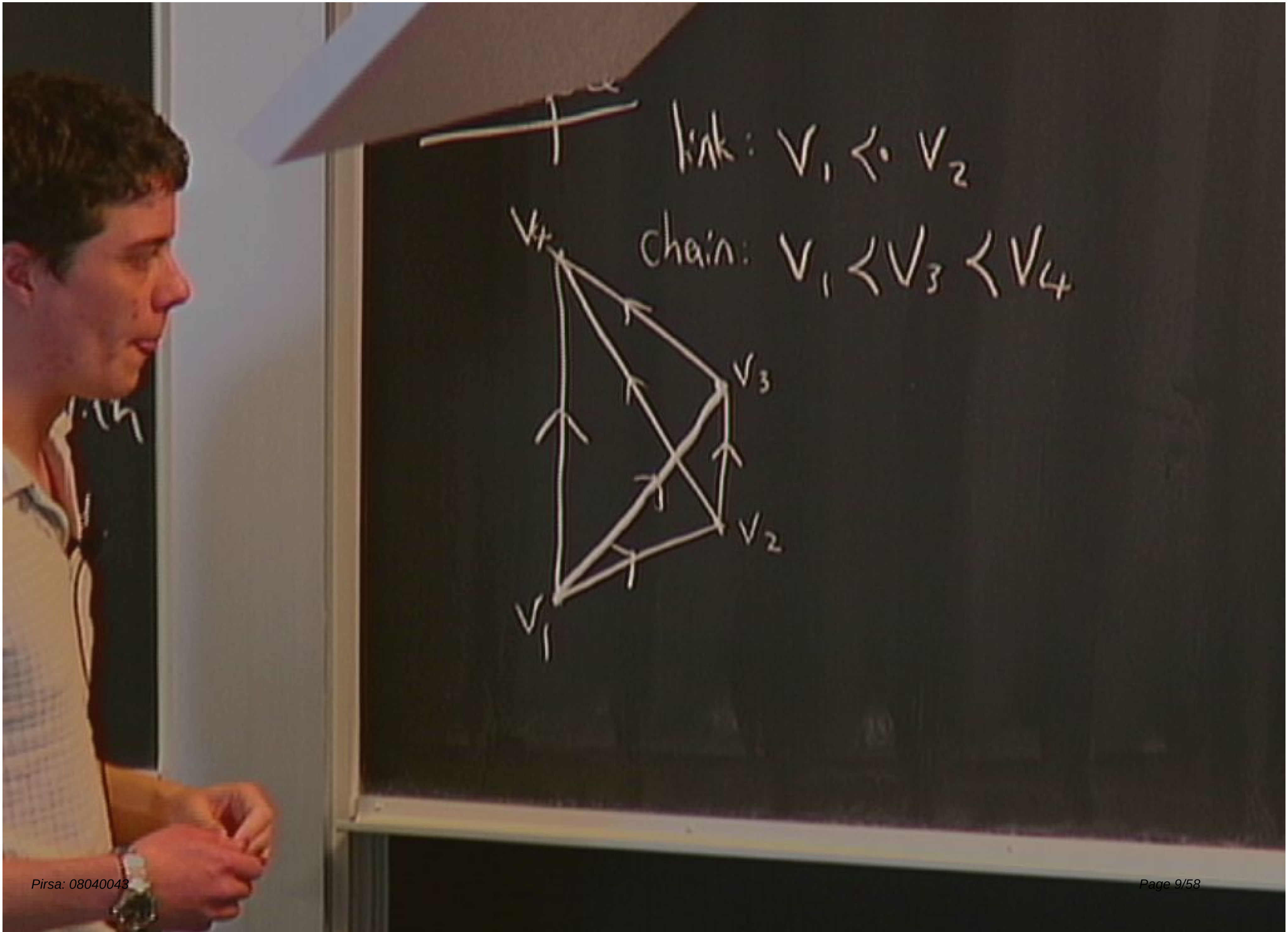
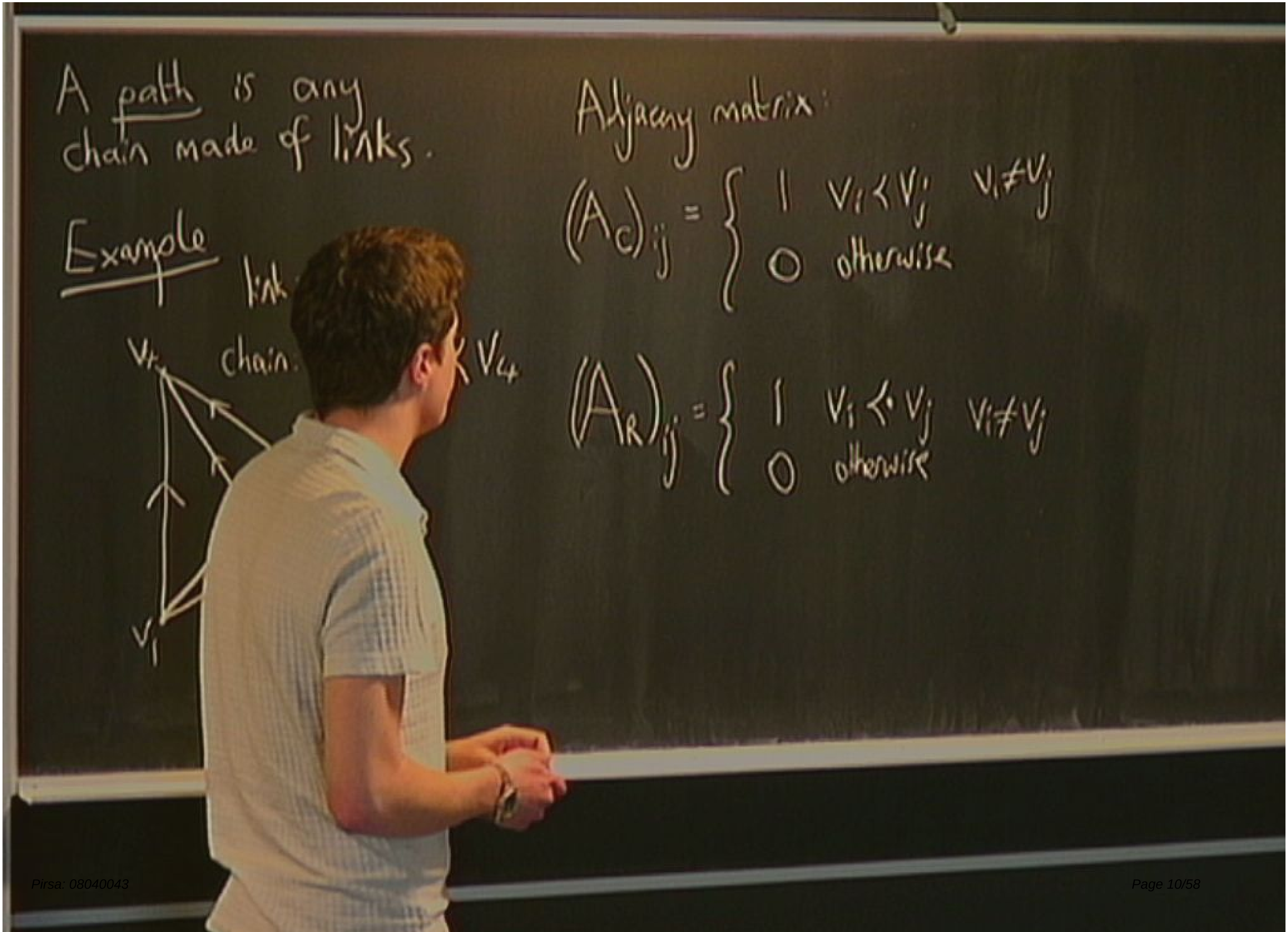
We write $x \lessdot y$


A path is any
chain made of links.


link: V1 <· V2
chain: V1 < V3 < V4
V4
V3
V2
V1

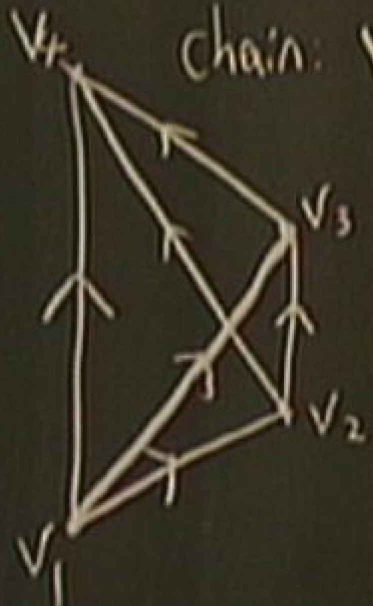

A path is any chain made of links.
Example
link
chain:
v_4
v_1
Adjacency matrix:
$(A_C)_{ij} = \begin{cases} 1 & v_i \prec v_j \quad v_i \neq v_j \\ 0 & \text{otherwise} \end{cases}$
$(A_R)_{ij} = \begin{cases} 1 & v_i \prec\!\cdot\, v_j \quad v_i \neq v_j \\ 0 & \text{otherwise} \end{cases}$

A *path* is any chain made of links.

Example

link: $v_1 \prec\!\!\cdot\; v_2$

chain: $v_1 \prec v_3 \prec v_4$



Adjacency matrix:

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$$A_C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A causal set is $(C, \preceq)$

- $x \preceq x$
- $x \preceq y \preceq z \Rightarrow x \preceq z$
- $x \preceq y \preceq x \Rightarrow x = y$
- $|\{z \in C \mid x \preceq z \preceq y\}| < \infty$

$$x \prec y \Leftrightarrow x \preceq y \;\&\; x \neq y$$

A chain is a sequence which is totally ordered: $x_1 \prec x_2 \prec x_3 \prec \cdots$

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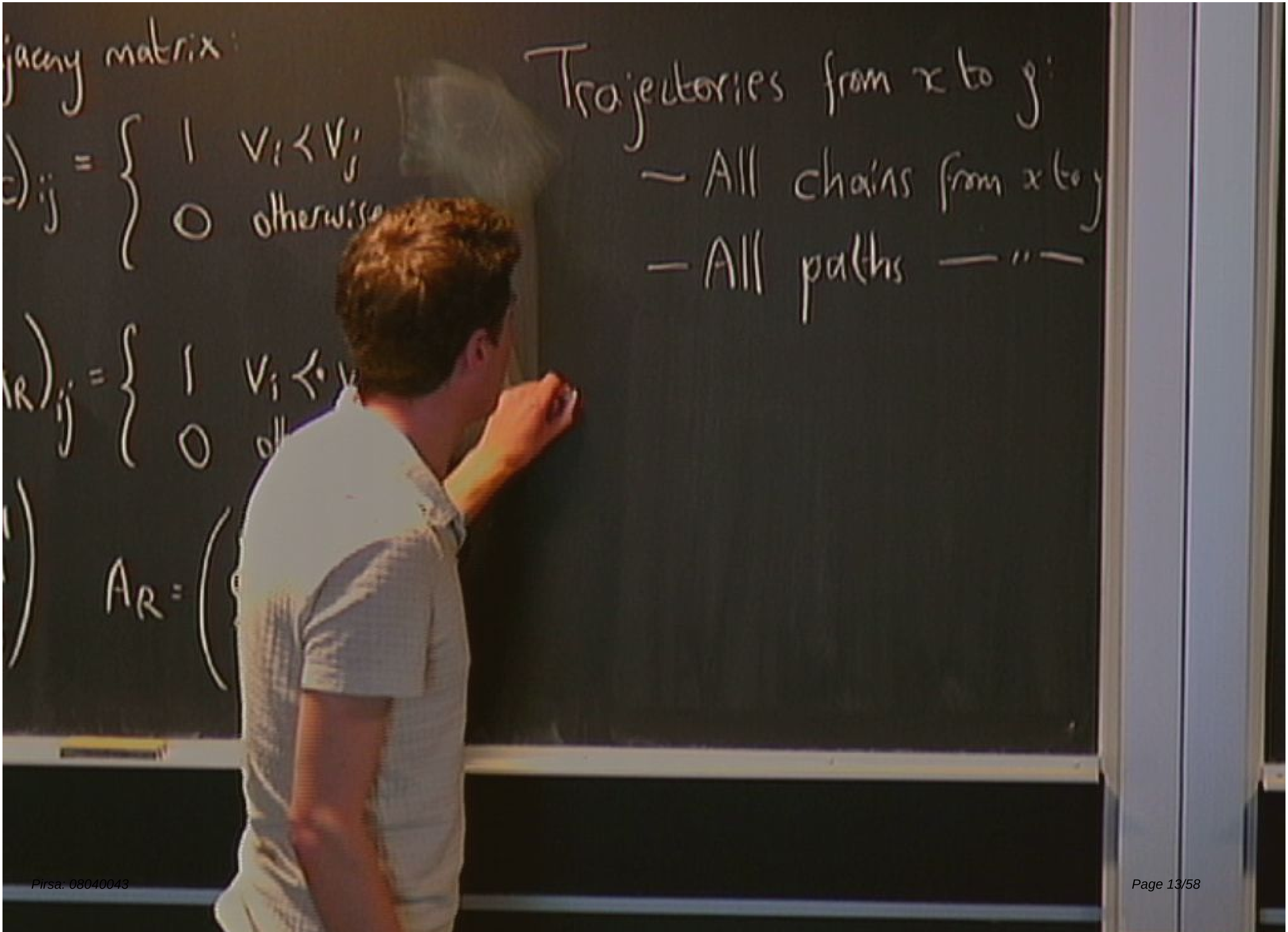
jacency matrix:
1 v_i < v_j
0 otherwise
A_R =
Trajectories from x to y:
— All chains from x to y
— All paths — " —

A path is any chain made of links.

Example

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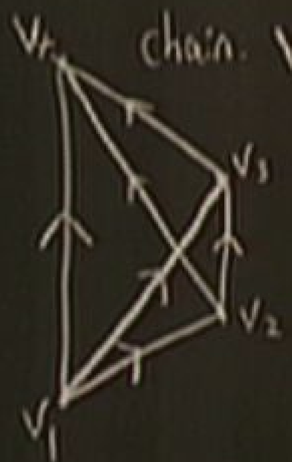

Adjacency matrix:

$$(A_C)_{ij} = \begin{cases} 1 & v_i < v_j \\ 0 & \text{otherwise} \end{cases}$$

$$(A_R)_{ij} = \begin{cases} 1 & v_i \lessdot v_j \\ 0 & \text{otherwise} \end{cases}$$

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Trajectories from x to y:

~ All chains from x to y

paths — " —

Are

$v_i < v_j$

...wise

$$\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

Trajectories from x to y:

— All chains from x to y

— All paths — " —

Amplitudes

Define: a = amp for a "hop"

b = amp for a "stop"

A path is any chain made of links.

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link: $v_1 \prec\!\!\cdot\; v_2$

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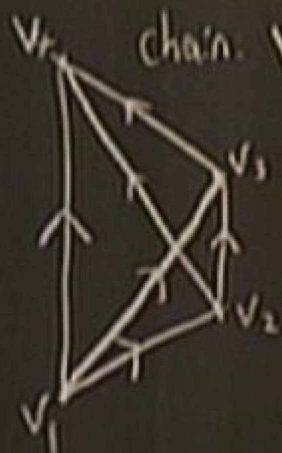

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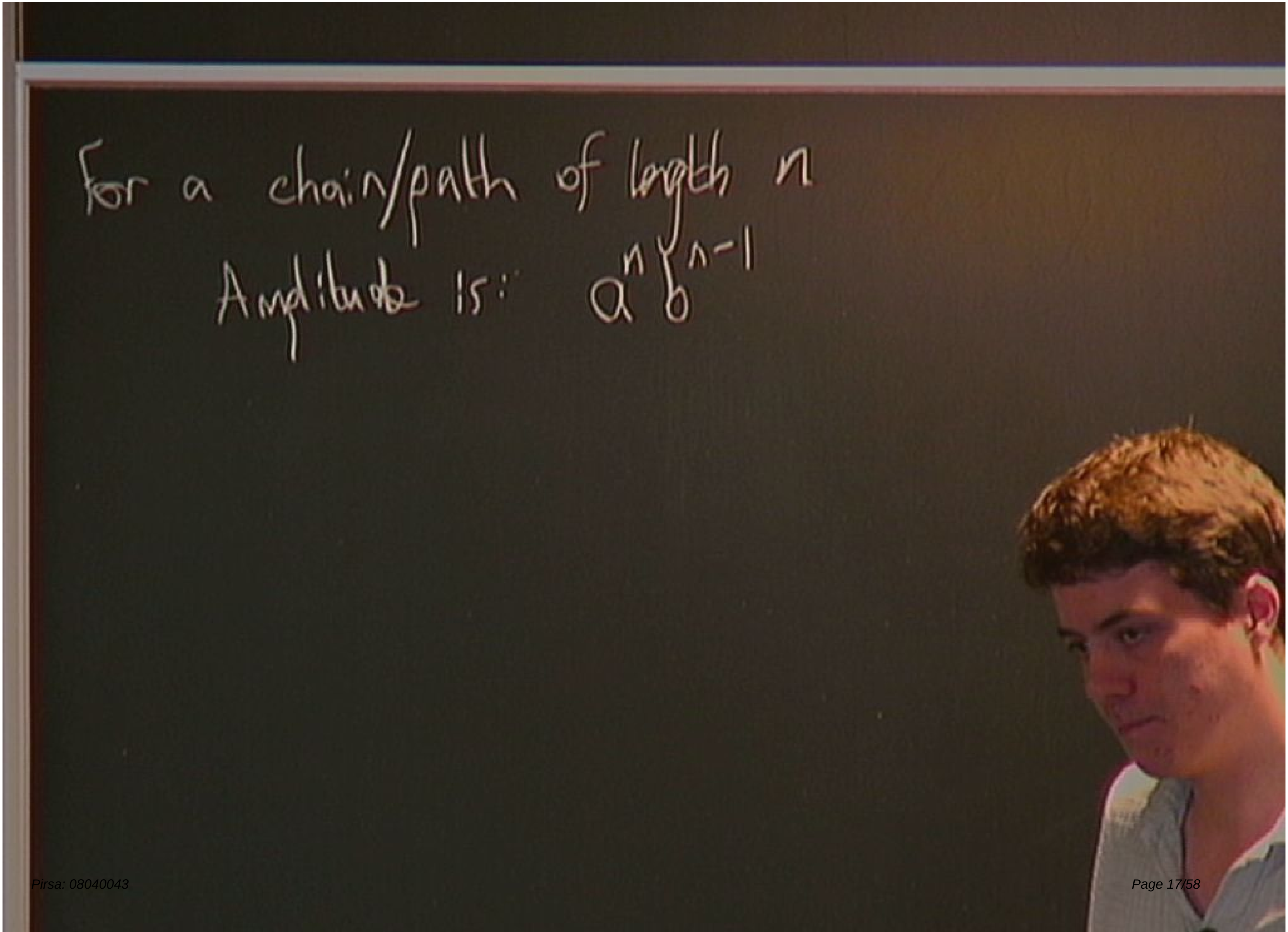
For a chain/path of length n
Amplitude is: $a^n b^{n-1}$

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To sum over chains:

$$\Phi = aA_c$$

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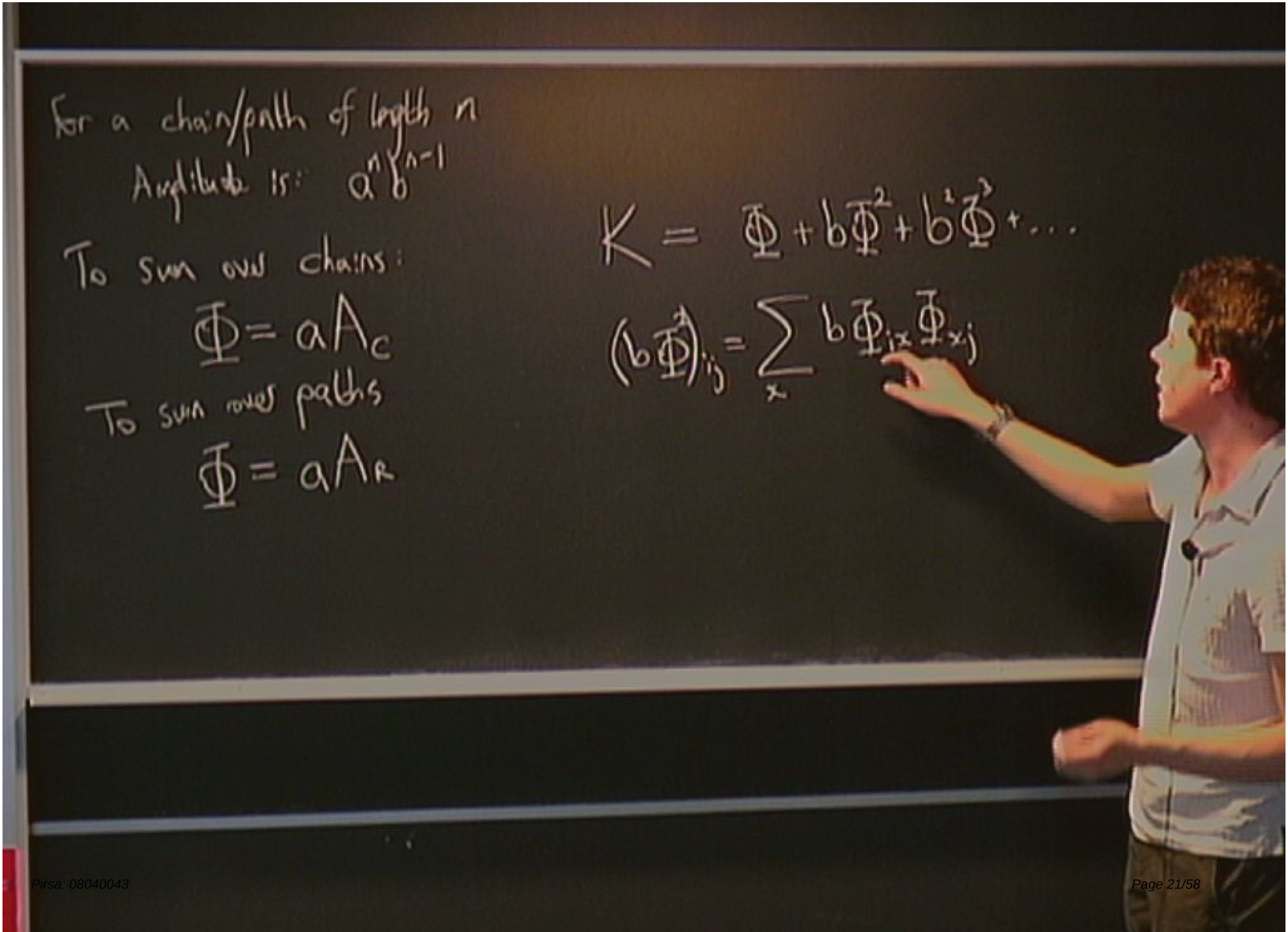
$$K = \Phi + b\Phi^2 + b^2\Phi^3 + \ldots$$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$(b\Phi^2)_{ij} = \sum_x b\Phi_{ix}\Phi_{xj}$

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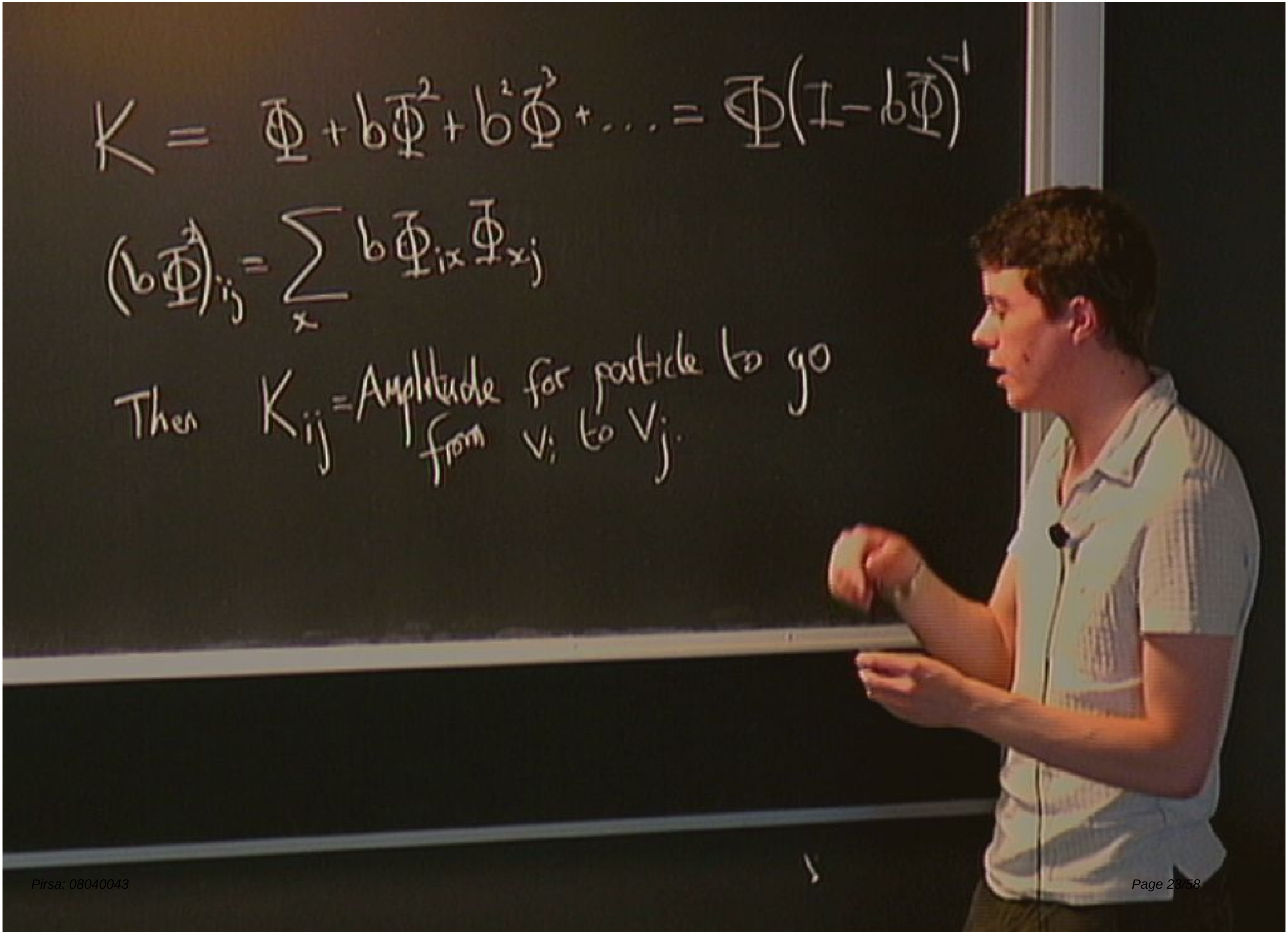
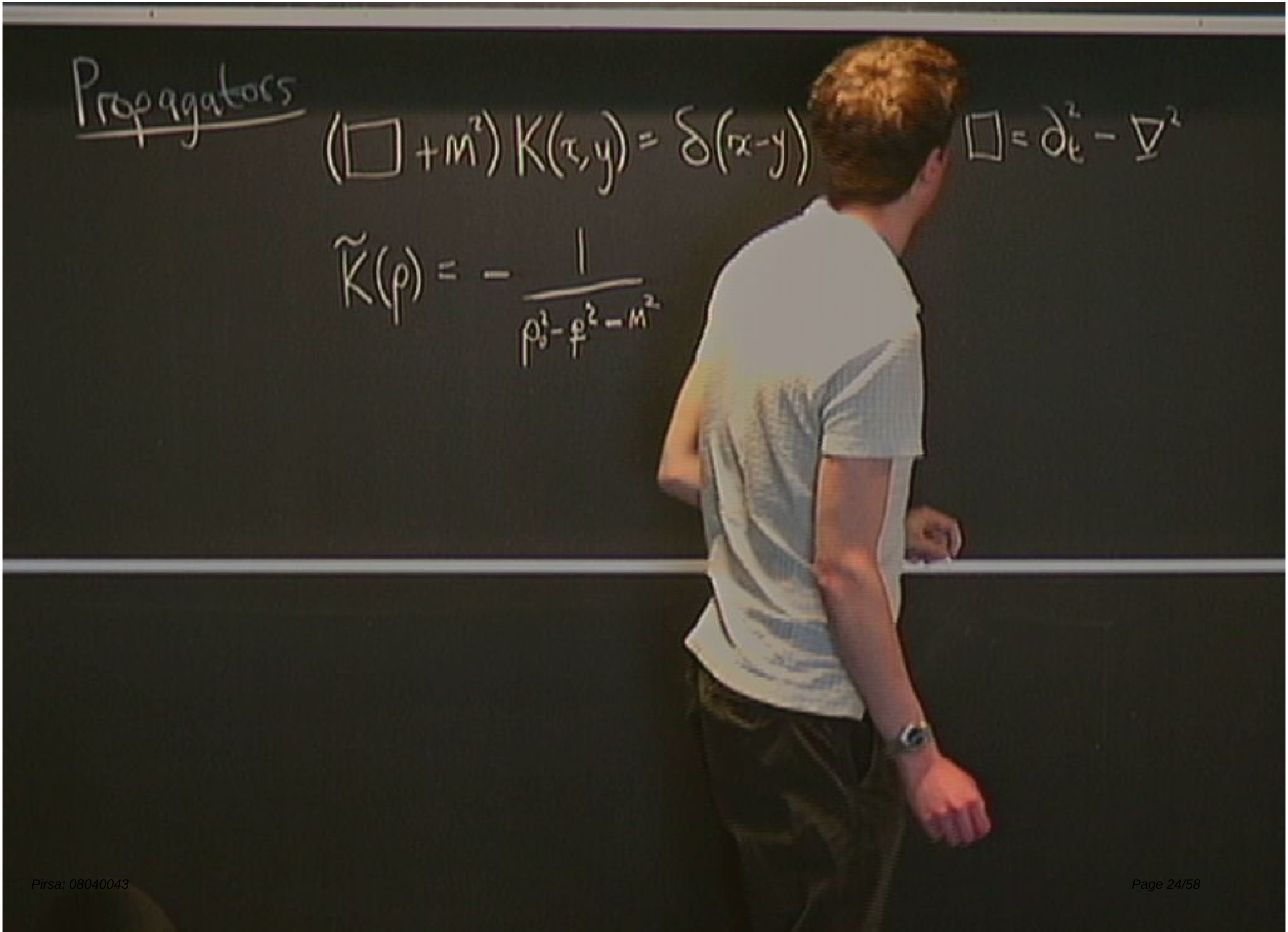
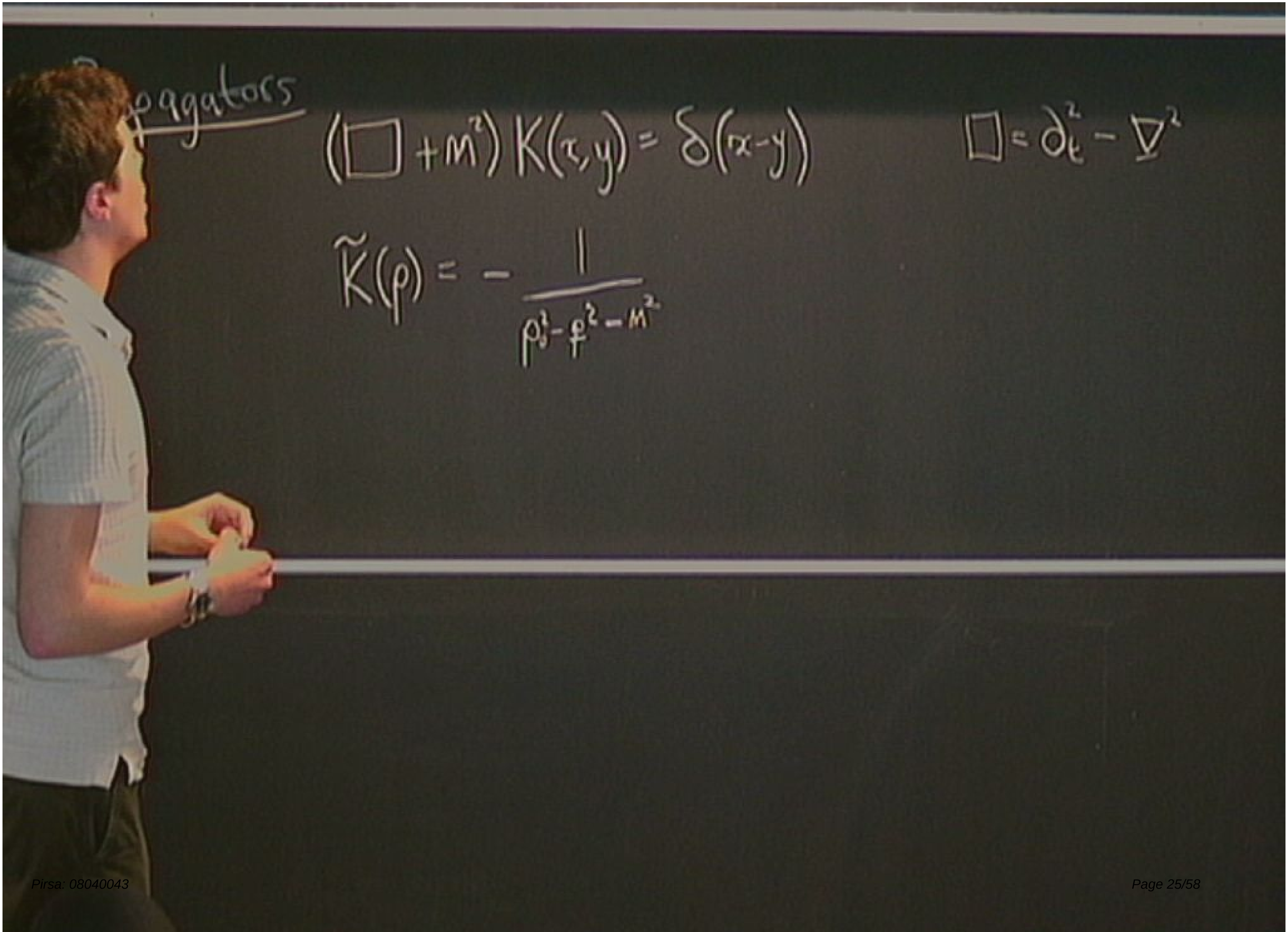


$K = \Phi + b\Phi^2 + b^2\Phi^3 + \ldots = \Phi(I - b\Phi)^{-1}$
$(b\Phi^2)_{ij} = \sum_x b\Phi_{ix}\Phi_{xj}$
Then K_{ij} = Amplitude for particle to go from v_i to v_j.


Propagators
$(\Box + m^2) K(x,y) = \delta(x-y)$
$\Box = \partial_t^2 - \nabla^2$
$\tilde{K}(p) = -\frac{1}{p_0^2 - \underline{p}^2 - m^2}$



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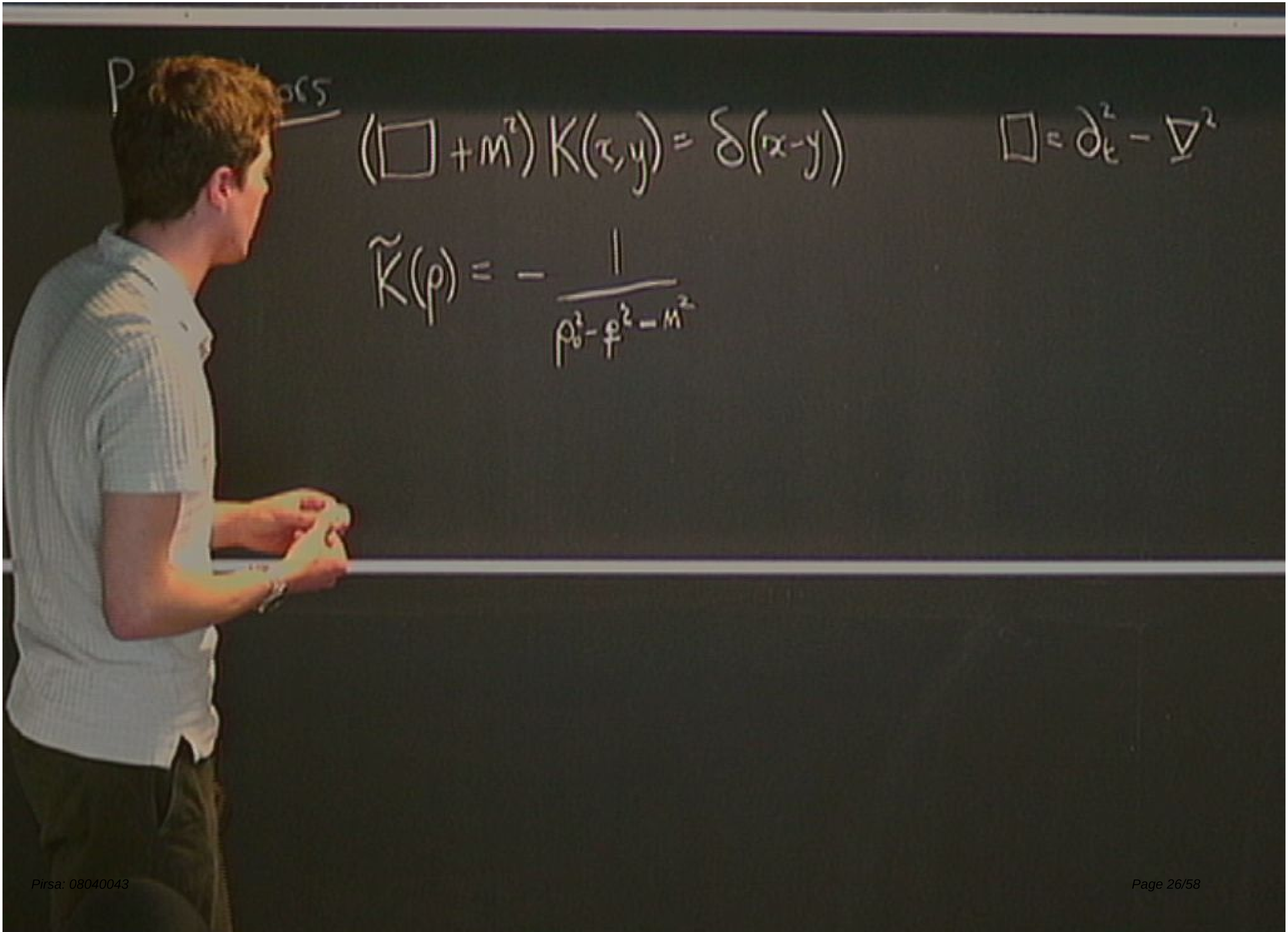
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$$K(x-y) = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)^d} \int d^d p \, \frac{e^{-i(x-y)p}}{(p_0 + i\epsilon)^2 - \mathbf{p}^2 - m^2}$$

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In 1+1 dim

$$K_m^{(2)}(x-y) = \begin{cases} \frac{1}{2} J_0(m\tau) & \text{if } x \ll y \\ 0 & \text{otherwise} \end{cases}$$

Propagators

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$$K^{(d)}_F(x-y) = \lim_{\varepsilon \to 0^+} \frac{1}{(2\pi)^d} \int d^d p \; \frac{e^{-i(x-y)p}}{(p_0 + i\varepsilon)^2 - \mathbf{p}^2 - m^2}$$

In 1+1 dim

$$K_m^{(2)}(x-y) = \begin{cases} \frac{1}{2} J_0(m\tau) & \text{if } x \lessgtr y \\ 0 & \text{otherwise} \end{cases}$$

In 3+1

$$K_m^{(4)}(x-y) = \begin{cases} \frac{\delta(\tau^2)}{2\pi} - \frac{m J_1(m\tau)}{4\pi\tau} & \text{if } x \lessgtr y \\ 0 & \text{otherwise} \end{cases}$$


Sprinking
(M, g)

Sprinkling

$$P(n, V) = \frac{(\rho V)^n}{n!} e^{-\rho V}$$

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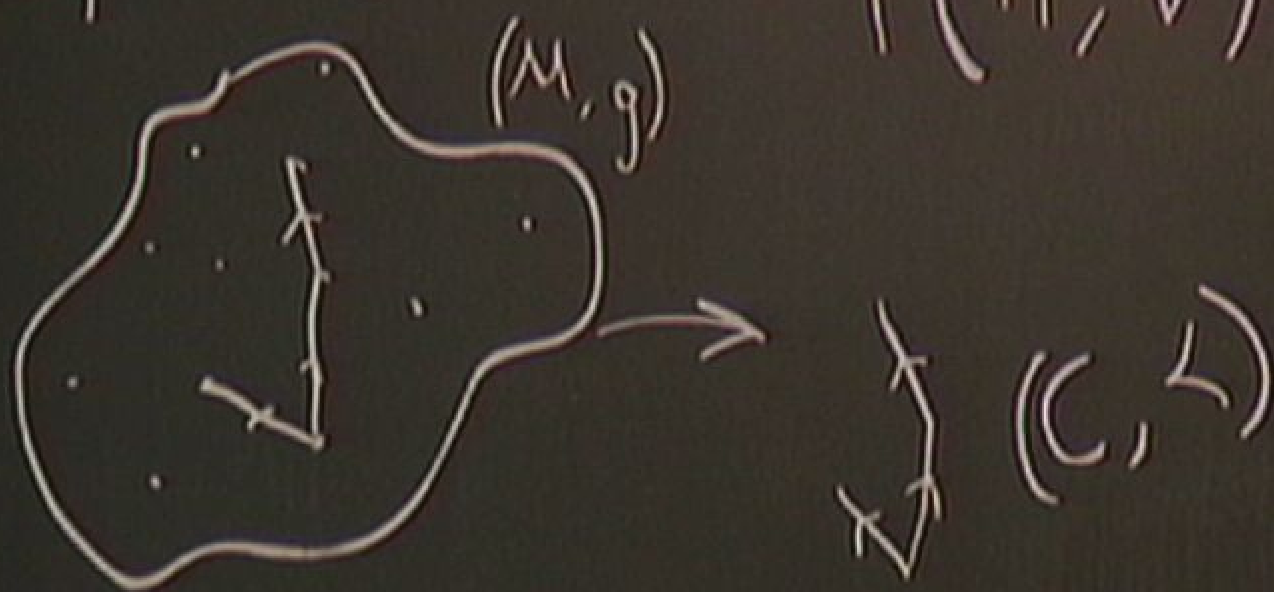
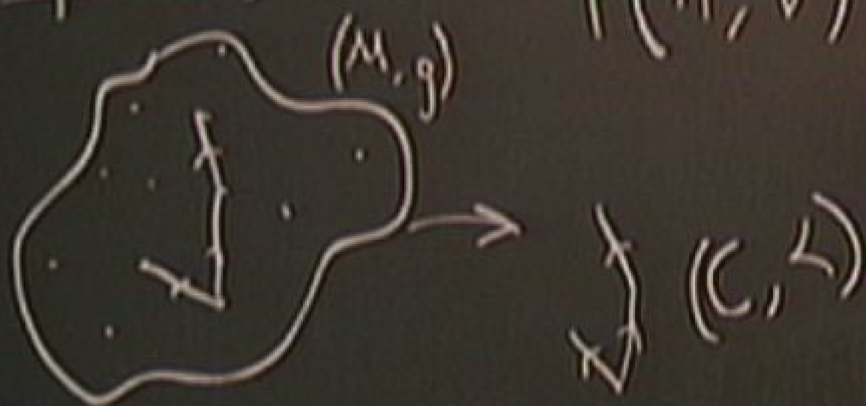


Sprinking

$$P(n,V) = \frac{(\rho V)^n}{n!} e^{-\rho V} \qquad \langle \text{Points in } V \rangle = \rho V$$



$$) = \frac{(\rho V)^n}{n!} e^{-\rho V}$$

$$\langle \text{Points in } V \rangle = \rho V$$

$$C_n = \rho^{n-1} \int \cdots \int dz_1 dz_2 \cdots \nu(x-z_1)\,\nu(z_2-z_1) \cdots \nu(z_{n-1}-y)$$

$$\nu(x-y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

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$$\nu(x-y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

$\tau)$ if $x \lessgtr y$

otherwise

$$\tilde{K}^{(2)}_m(p) = -\frac{1}{(p_0+i\epsilon)^2 - p_1^2}$$

$\dfrac{m J_1(m\tau)}{4\pi\tau}$ if $x \lessgtr y$

otherwise

$$\tilde{K}^{(4)}_m(p) = -\frac{1}{(p_0+i\epsilon)^2 - \vec{p}^{\,2}}$$

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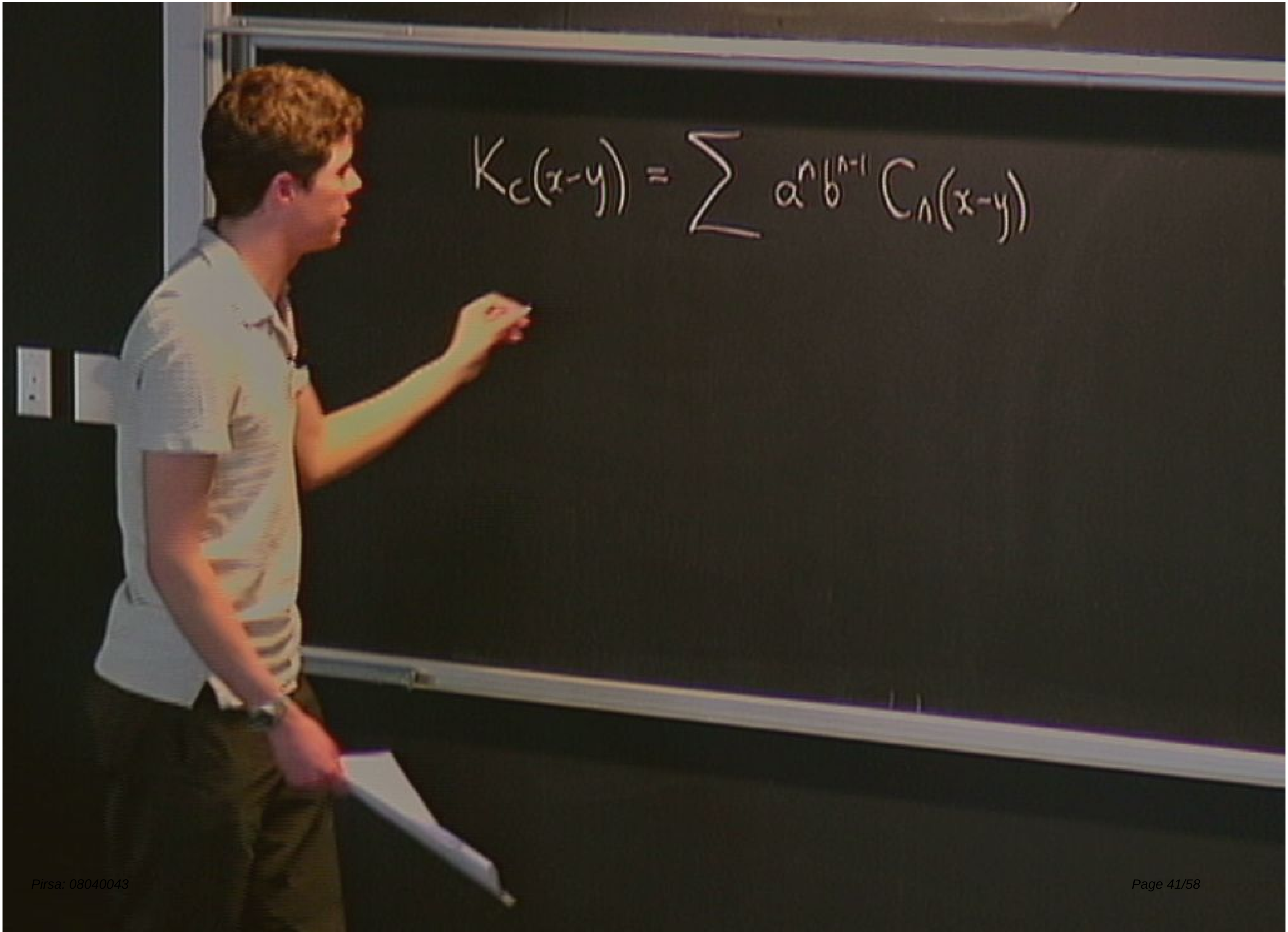
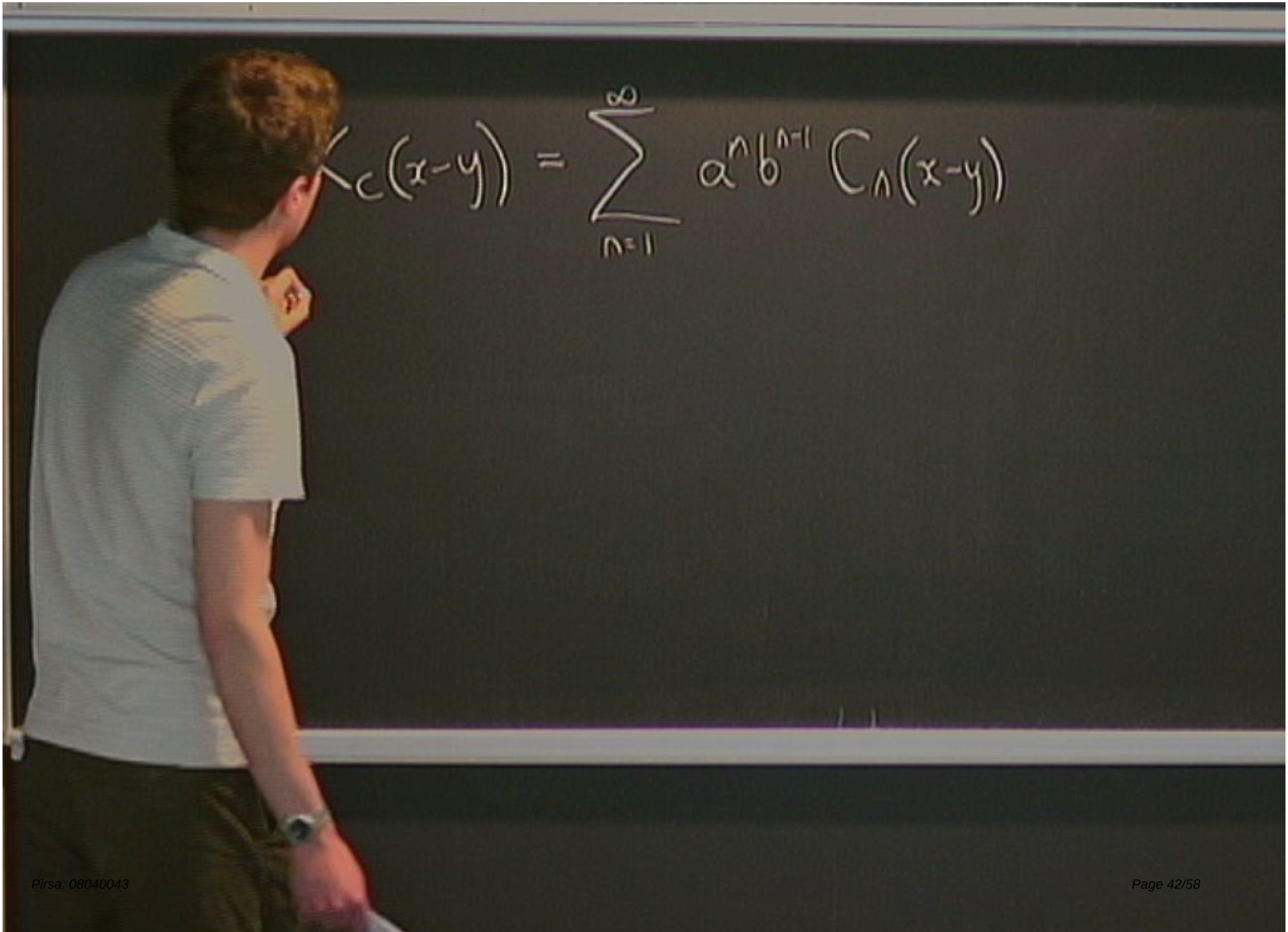

$K_c(x-y) = \sum a^n b^{n-1} C_n(x-y)$


$X_c(x-y) = \sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$

$$K_c(x-y) = \sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$$

$$K_c(x-y) = a\,v(x-y) + abe \int d^4z\, v(x-z)\, K_c(z-y)$$

$$K_c(x-y) = \sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$$

$$K_c(x-y) = a\,\mathcal{V}(x-y) + abe\int d^4z\, \mathcal{V}(x-z)\, K_c(z-y)$$

$$\tilde{K}_c(p) = \frac{a\,\tilde{\mathcal{V}}(p)}{1 - abe\,\tilde{\mathcal{V}}(p)}$$

$$K_c(x-y) = \sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$$

$$K_c(x-y) = a\,V(x-y) + abe \int d^4z\, V(x-z) K_c(z-y)$$

$$\tilde{K}_c(p) = \frac{a\,\tilde{V}(p)}{1 - abe\,\tilde{V}(p)}$$

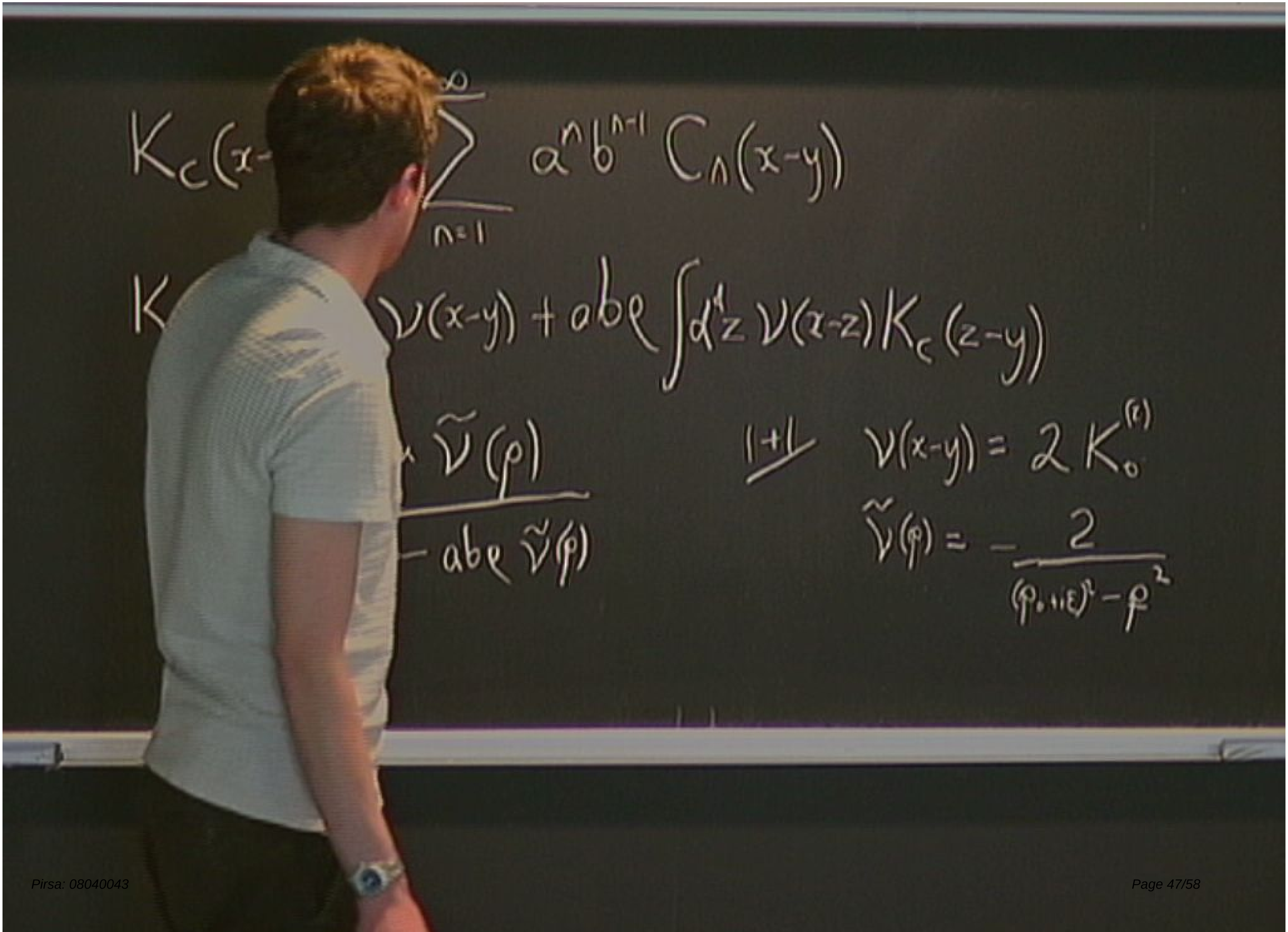
$\underline{1+1}$

$$\tilde{K}^{(2)}_m(p) = -\frac{1}{(p_0+i\epsilon)^2 - p_1^2 - m^2}$$

if $x \lesssim y$

$$\tilde{K}^{(4)}_m(p) = -\frac{1}{(p_0+i\epsilon)^2 - \vec{p}^2 - m^2}$$

otherwise



$K_c(x-$ $\sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$
K $\nu(x-y) + abe \int d^4z\, \nu(x-z) K_c(z-y)$
$\tilde{\nu}(p)$
$- abe\, \tilde{\nu}(p)$
1+1
$\nu(x-y) = 2 K_0^{(1)}$
$\tilde{\nu}(p) = - \frac{2}{(p_0+i\epsilon)^2 - p^2}$

$$K_c(x-y) = \sum_{n=1}^{\infty} a^n b^{n-1} C_n(x-y)$$

$$K_c(x-y) = a\,\mathcal{V}(x-y) + abe \int d^4z\, \mathcal{V}(x-z) K_c(z-y)$$

$$\tilde{K}_c(p) = \frac{a\,\tilde{\mathcal{V}}(p)}{1 - abe\,\tilde{\mathcal{V}}(p)}$$

$\underline{1+1}$

$$\mathcal{V}(x-y) = 2\,K_0^{(1)}$$

$$\tilde{\mathcal{V}}(p) = -\frac{2}{(p_0+i\epsilon)^2 - p^2}$$

$$K_c = \frac{\dfrac{-2a}{(p_0+i\varepsilon)^2 - p_F^2}}{1 + \dfrac{2abe}{(p_0+i\varepsilon)^2 - p_F^2}}$$

$$(z-y) = \frac{-2a}{(p_0+i\varepsilon)^2 - p_F^2 + 2abe}$$

$$) = 2K^{(z)}$$

$$\tilde{K}_c = \frac{-\dfrac{2a}{(p_0+i\varepsilon)^2-\vec{p}^2}}{1+\dfrac{2abe}{(p_0+i\varepsilon)^2-\vec{p}^2}}$$

$$(z-y) \qquad = \frac{-2a}{(p_0+i\varepsilon)^2-\vec{p}^2+2abe}$$

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$y))$

$$\tilde{K}_c = \frac{-\frac{2a}{(p_0+i\epsilon)^2 - \vec{p}^2}}{1 + \frac{2ab\epsilon}{(p_0+i\epsilon)^2 - \vec{p}^2}}$$

$$(x-z) K_c (z-y) \qquad = \left(\frac{-2a}{(p_0+i\epsilon)^2 - \vec{p}^2 + 2ab\epsilon} \right)$$

$$V(x-y) = 2 K_0^{(2)}$$

$$\tilde{V}(p) = -\frac{2}{(p_0+i\epsilon)^2 - \vec{p}^2} \qquad a:$$

$\ln\ 1+|$

$K_m^{(2)}$

$\ln\ 3+$

$K^{(4}$

CAUTION

$$x-y)$$

$$\tilde{K}_c = \frac{-\frac{2a}{(p_0+i\varepsilon)^2-p^2}}{1+\frac{2abe}{(p_0+i\varepsilon)^2-p^2}}$$

$$V(x-z)K_c(z-y) = \left(\frac{-2a}{(p_0+i\varepsilon)^2-p^2+2abe}\right)$$

$$+1$$

$$V(x-y) = 2K_0^{(z)}$$

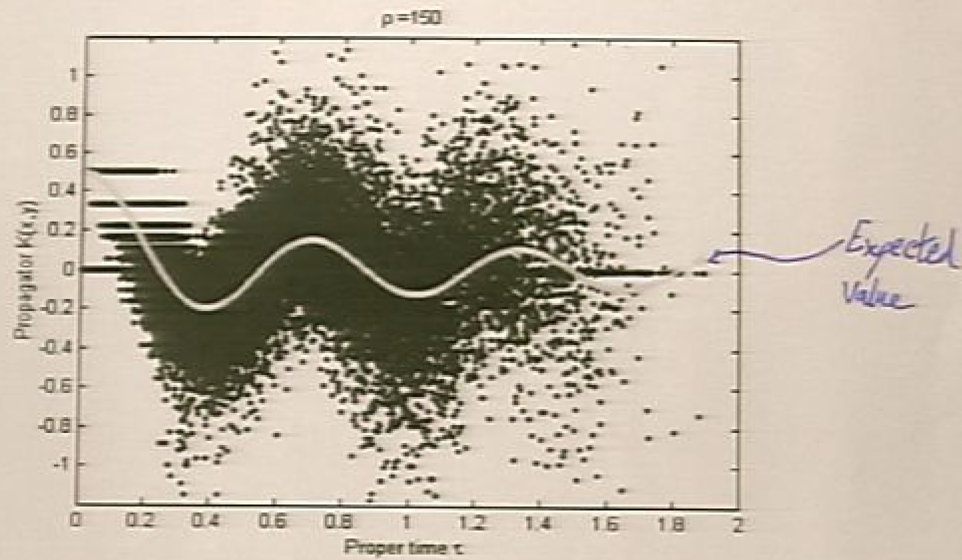
$$\tilde{V}(p) = -\frac{2}{(p_0+i\varepsilon)^2-p^2}$$

$$a = \frac{1}{2}$$

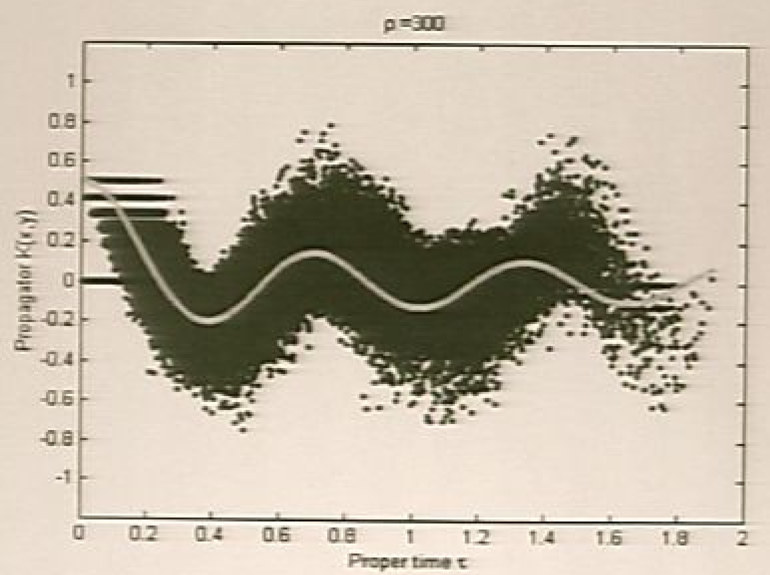
$$b = -\frac{m^2}{e}$$

3+1 Summing over paths

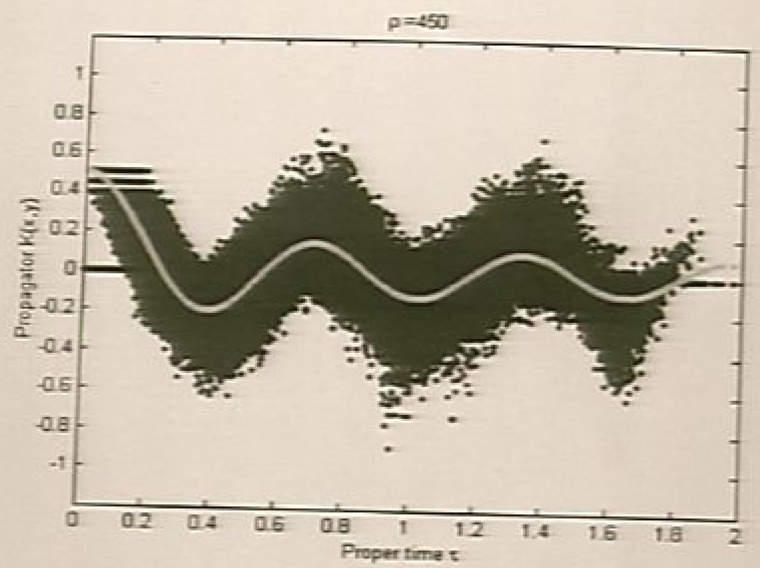
$$a = \frac{\sqrt{\epsilon}}{2\pi\sqrt{6}} \quad b = \frac{-m^2}{\epsilon}$$

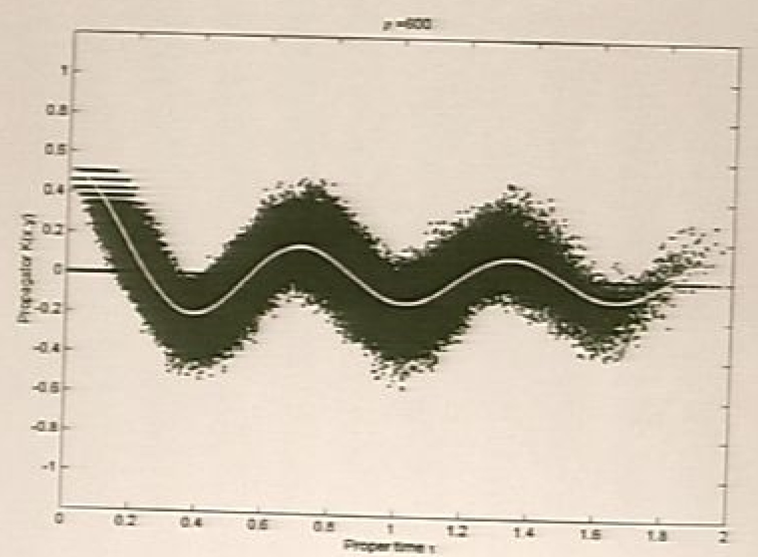


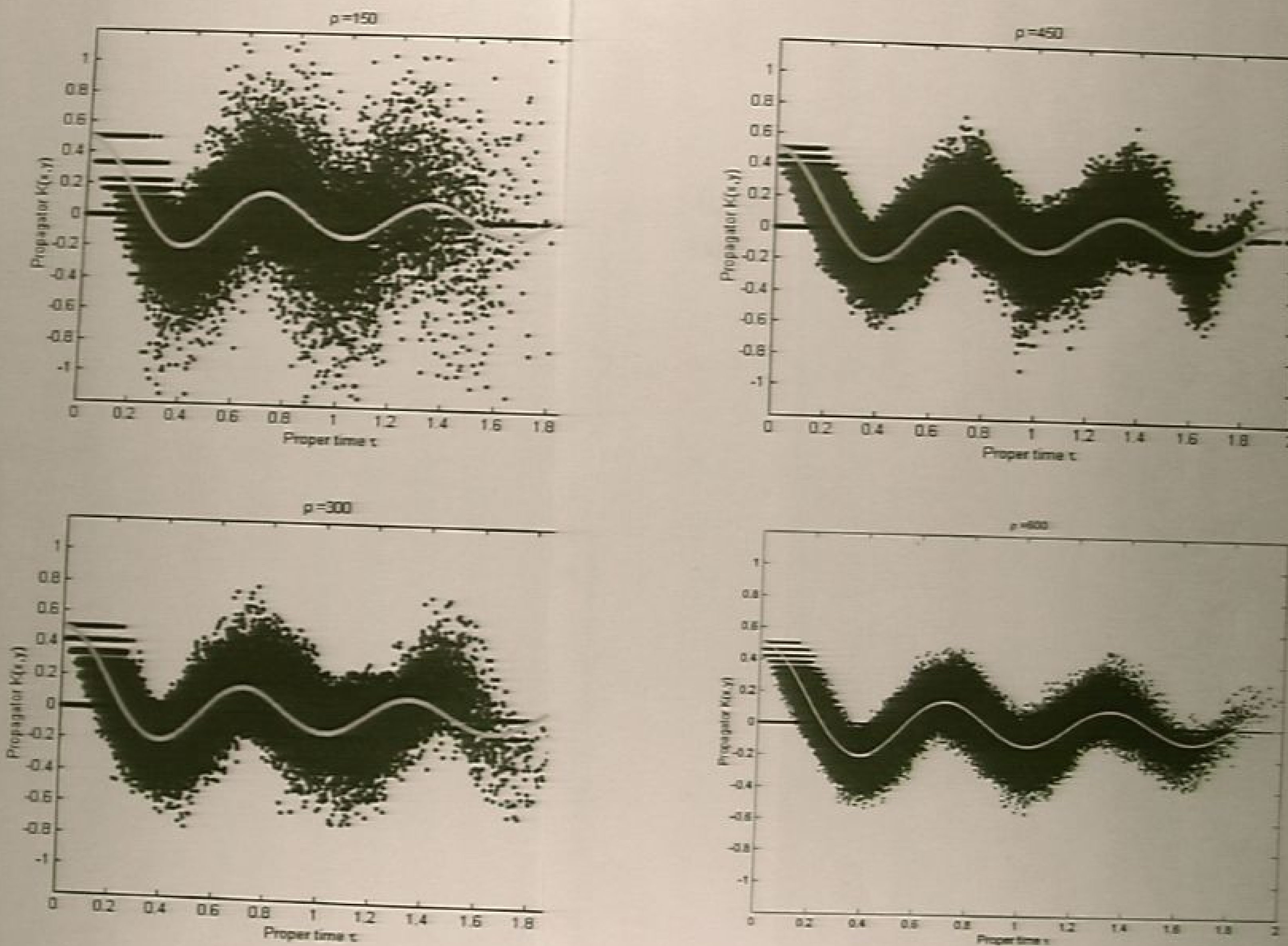
ρ =150
Propagator K(x,y)
Proper time τ
Expected
Value



ρ =300
Propagator K(x,y)
Proper time τ


ρ =450
Propagator K(x,y)
Proper time τ


Propagator K(x,y)
Proper time τ


ρ =150
Propagator K(x,y)
Proper time τ
ρ =450
Propagator K(x,y)
Proper time τ
ρ =300
Propagator K(x,y)
Proper time τ
Propagator K(x,y)
Proper time τ

ring. These are just the properties of $K_0(2, 3)$.
the other hand, according to the positron theory
ve energy states are not available to the electron
he scattering. Therefore the choice $K_+ = K_0$ is
factory. But there are other solutions of (12).
ll choose the solution defining $K_+(2, 1)$ so that
) for $t_2 > t_1$ is the sum of (3) over positive energy
nly. Now this new solution must satisfy (12) for
s in order that the representation be complete.
t therefore differ from the old solution K_0 by a
n of the homogeneous Dirac equation. It is clear
e definition that the difference $K_0 - K_+$ is the
(3) over all negative energy states, as long as
But this difference must be a solution of the
neous Dirac equation for all times and must
e be represented by the same sum over negative
tates also for $t_2 < t_1$. Since $K_0 = 0$ in this case,
s that our new kernel, *$K_+(2, 1)$, for $t_2 < t_1$ is the*
of the sum (3) over negative energy states. That is,

$$
\begin{aligned}
&= \textstyle\sum_{POS\ E_n} \phi_n(2)\bar{\phi}_n(1) \\
&\qquad \times \exp(-iE_n(t_2 - t_1)) \quad \text{for} \quad t_2 > t_1 \\
&= -\textstyle\sum_{NEG\ E_n} \phi_n(2)\bar{\phi}_n(1) \\
&\qquad \times \exp(-iE_n(t_2 - t_1)) \quad \text{for} \quad t_2 < t_1.
\end{aligned}
\tag{17}
$$

s choice of K_+ our equations such as (13) and
now give results equivalent to those of the
hole theory.
14), for example, is the correct second order
n for finding at 2 an electron originally at 1
to the positron theory may be seen as follows
Assume as a special example that $t_2 > t_1$ and
potential vanishes except in interval $t_2 - t_1$ so
d t_4 both lie between t_1 and t_2.
ppose $t_4 > t_3$ (Fig. 2(b)). Then (since $t_3 > t_1$)

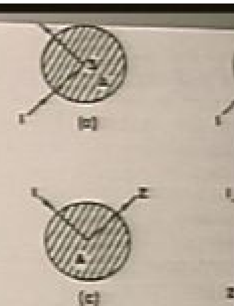
and electron of negative energy is reflected in the fact that $K_+(4, 3)$ for $t_4 < t_3$ is (minus) the sum of only negative energy components. In hole theory the real energy of these intermediate states is, of course, positive. This is true here too, since in the phases $\exp(-iE_n(t_4 - t_3))$ defining $K_+(4, 3)$ in (17), E_n is negative but so is $t_4 - t_3$. That is, the contributions vary with t_3 as $\exp(-i|E_n|(t_3 - t_4))$ as they would if the energy of the intermediate state were $|E_n|$. The fact that the entire sum is taken as negative in computing $K_+(4, 3)$ is reflected in the fact that in hole theory the amplitude has its sign reversed in accordance with the Pauli principle and the fact that the electron arriving at 2 has been exchanged with one in the sea.[6] To this, and to higher orders, all processes involving virtual pairs are correctly described in this way.

The expressions such as (14) can still be described as a passage of the electron from 1 to 3 ($K_+(3, 1)$), scattering at 3 by $A(3)$, proceeding to 4 ($K_+(4, 3)$), scattering again, $A(4)$, arriving finally at 2. The scatterings may, however, be toward both future and past times, an electron propagating backwards in time being recognized as a positron.

This therefore suggests that negative energy components created by scattering in a potential be considered as waves propagating from the scattering point toward the past, and that such waves represent the propagation of a positron annihilating the electron in the potential.[7]

[6] It has often been noted that the one-electron theory apparently gives the same matrix elements for this process as does hole theory. The problem is one of interpretation, especially in a way that will also give correct results for other processes, e.g., self-energy.

[7] The idea that positrons can be represented as electrons with proper time reversed relative to true time has been discussed by the author and others, particularly by Stückelberg. E. C. C.



(b)

(c)

Fig. 3. Several different processes ca
formula depending on the time relati
Thus $P_v|K_+^{(A)}(2, 1)|^2$ is the probabili
1 will be scattered at 2 (and no oth
(b) Electron at 1 and positron at 2
(c) A single pair at 1 and 2 is created f
at 2 is scattered to 1. ($K_+^{(A)}(2, 1)$ i
scattering in the potential to all o
constant.)

Stückelberg, Helv. Phys. Acta 15, 2
Phys. Rev. 74, 939 (1948). The fact
(proper time) increases continuously
is reflected in quantum mechanics in t
is $|E_n|(t_2 - t_1)$, always increases as th
scattering point to the next.
[8] By multiplying (12) on the righ
that $\nabla_1\delta(2, 1) = -\nabla_2\delta(2, 1)$ show t
$K_+(2, 1)(-i\nabla_1 - m) = i\delta(2, 1)$, where
1 in $K_+(2, 1)$ but is written after tha
order of the γ matrices. Multiply th
(11) (with $A = 0$, calling the variab
and integrate over a region of space-
hand side can be transformed to an
the region. The right-hand side is ψ
the region, and is zero otherwise.
surface contains a light line and hen
not concern us as these points can l
from 2 that their contribution vanis

riant expressions for the interaction of ral meson with two photons, where the lear interaction may be scalar, or the oscalar and pseudovector couplings.

the proper-time technique to be exper, apart from its value in obtaining is in a few special cases, lies in its divergent aspects of a calculation in spect to the proper-time, a parameter ference to the coordinate system or the we shall show that the customary perture of expansion in powers of the does yield gauge invariant results, that the proper-time integration is last. The technique of "invariant represents a partial realization of this thod through the use of specially ls over the conjugate quantity, the oper mass.

pendix B we shall employ the Green's electron in a weak, homogeneous, exalculate the second-order electromageby providing a simple derivation of correction to the electron magnetic

2. GENERAL THEORY

tions, commutation relations, and cure Dirac field are given by[3]

$$\partial_\mu - eA_\mu(x))\psi(x) + m\psi(x) = 0,$$
$$eA_\mu(x))\bar\psi(x)\gamma_\mu + m\bar\psi(x) = 0, \qquad (2.1)$$

$$, x_0), \bar\psi(x', x_0)\} = \gamma_0\delta(\mathbf{x}-\mathbf{x}'), \qquad (2.2)$$

$$_\mu(x) = \tfrac{1}{2}e[\bar\psi(x), \gamma_\mu\psi(x)], \qquad (2.3)$$

$$\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} = -\delta_{\mu\nu} \qquad (2.4)$$

$$\gamma_0 = -i\gamma_4, \quad \gamma_0^2 = 1. \qquad (2.5)$$

of the current operator,

$$= -e(\gamma_\mu)_{\alpha\beta}\tfrac{1}{2}[\psi_\alpha(x), \bar\psi_\beta(x)], \qquad (2.6)$$

m an explicit charge symmetrization,

$$\begin{cases} -1, & x_0 < x_0', \end{cases}$$

we have

$$(\psi_\alpha(x)\bar\psi_\beta(x'))_+\epsilon(x-x') = \begin{cases} \psi_\alpha(x)\bar\psi_\beta(x'), & x_0 > x_0' \\ -\bar\psi_\beta(x')\psi_\alpha(x), & x_0 < x_0'. \end{cases} \qquad (2.9)$$

Therefore

$$\tfrac{1}{2}[\psi_\alpha(x), \bar\psi_\beta(x)] = (\psi_\alpha(x)\bar\psi_\beta(x'))_+\epsilon(x-x')]_{x'\to x}, \qquad (2.10)$$

provided one takes the average of the forms obtained by letting x' approach x from the future, and from the past. The quantity of actual interest here is the expectation value of $j_\mu(x)$ in the vacuum of the Dirac field,

$$\langle j_\mu(x)\rangle = ie\,\mathrm{tr}\,\gamma_\mu G(x, x')]_{x'\to x}, \qquad (2.11)$$

where

$$G(x, x') = i\langle(\psi(x)\bar\psi(x'))_+\rangle\epsilon(x-x'), \qquad (2.12)$$

and tr indicates the diagonal sum with respect to the spinor indices.

The function $G(x, x')$ satisfies an inhomogeneous differential equation which is obtained by noting that

$$[\gamma(-i\partial - eA(x)) + m]G(x, x') = \langle\gamma_0\{\psi(x), \bar\psi(x')\}\rangle\delta(x_0 - x_0'), \qquad (2.13)$$

where the right side expresses the discontinuous change in form of $G(x, x')$ as x_0 is altered from $x_0' - 0$ to $x_0' + 0$. According to Eq. (2.2), therefore, we have

$$[\gamma(-i\partial - eA(x)) + m]G(x, x') = \delta(x - x'); \qquad (2.14)$$

that is, $G(x, x')$ is a Green's function for the Dirac field. We shall not discuss which particular Green's function this is, as specified by the associated boundary conditions, since no ambiguity enters if actual pair creation in the vacuum does not occur, which we shall expressly assume.

It is useful to regard $G(x, x')$ as the matrix element of an operator G, in which states are labeled by space-time coordinates as well as by the suppressed spinor indices:

$$G(x, x') = (x|G|x'). \qquad (2.15)$$

The defining differential equations for the Green's function is then considered to be a matrix element of the operator equation

is obtained from an action integral by variat $A_\mu(x)$. This is accomplished by exhibiting

$$\delta W^{(1)} = \int (dx)\delta A_\mu(x)\langle j_\mu(x)\rangle = ie\,\mathrm{Tr}\,\gamma\delta A G$$

as a total differential, subject to $\delta A_\mu(x)$ vanish infinity. In the second version of $\delta W^{(1)}$, δA_μ deno operator with the matrix elements

$$(x|\delta A_\mu|x') = \delta(x-x')\delta A_\mu(x),$$

and Tr indicates the complete diagonal sum including spinor indices and the continuous spa coordinates. Now

$$-e\gamma\delta A = \delta(\gamma\Pi + m),$$

and

$$G = \frac{1}{\gamma\Pi + m} = i\int_0^\infty ds\,\exp[-i(\gamma\Pi + m)s],$$

so that

$$ie\,\mathrm{Tr}\,\gamma\delta A G = -\delta\left[i\int_0^\infty ds\,s^{-1}\,\mathrm{Tr}\exp[-i(\gamma\Pi + m)s]\right],$$

in virtue of the fundamental property of the tra

$$\mathrm{Tr}\,AB = \mathrm{Tr}\,BA.$$

Thus, to within an additive constant,

$$W^{(1)} = i\int_0^\infty ds\,s^{-1}e^{-ims}\,\mathrm{Tr}\exp[-i\gamma\Pi s]$$
$$= \int (dx)\mathcal{L}^{(1)}(x),$$

where the lagrange function $\mathcal{L}^{(1)}(x)$ is given by

$$\mathcal{L}^{(1)}(x) = i\int_0^\infty ds\,s^{-1}e^{-ims}\,\mathrm{tr}(x|\exp[-i\gamma\Pi s]|x).$$