Title: Advanced General Relativity - Lecture 12B

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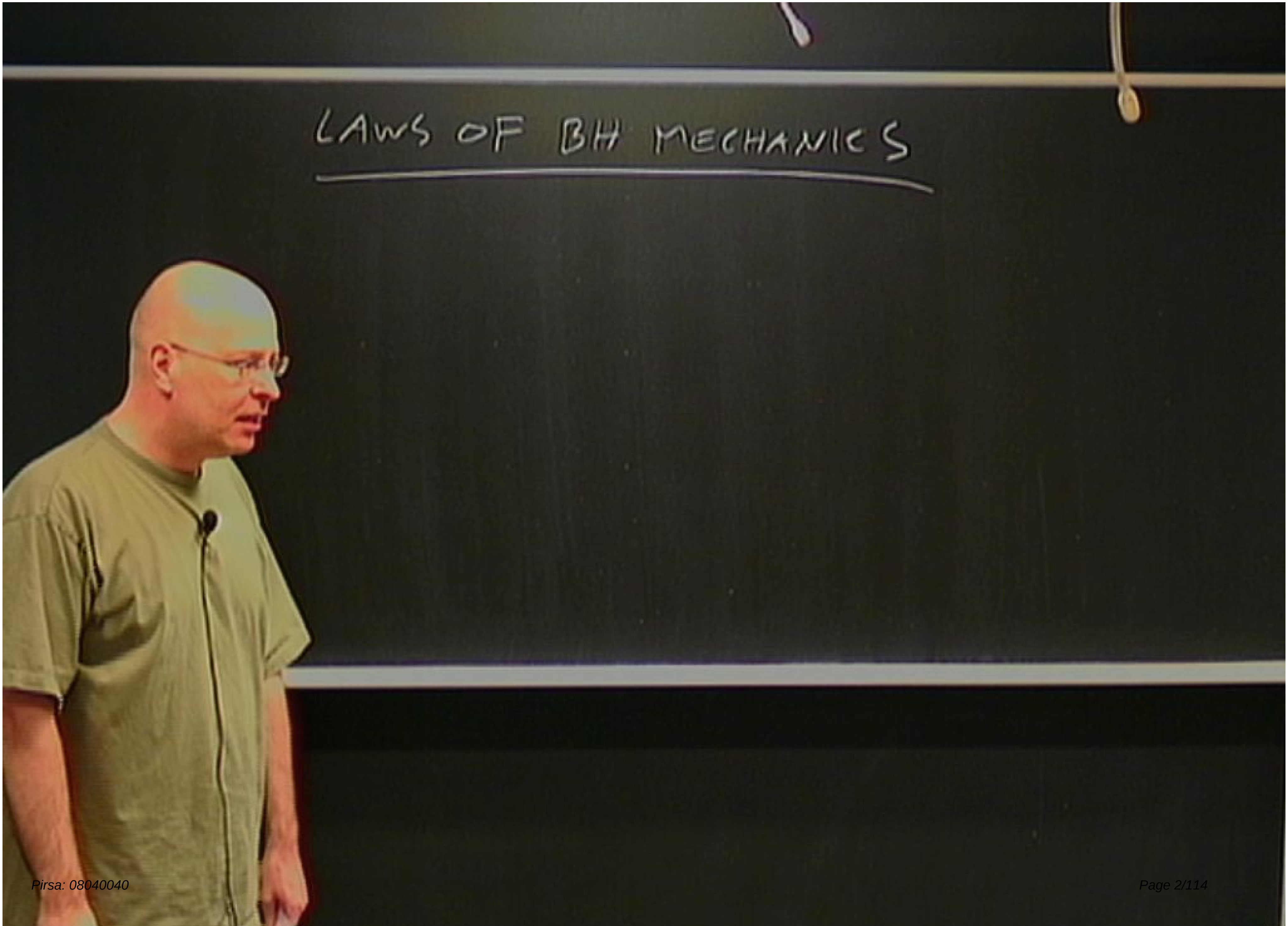
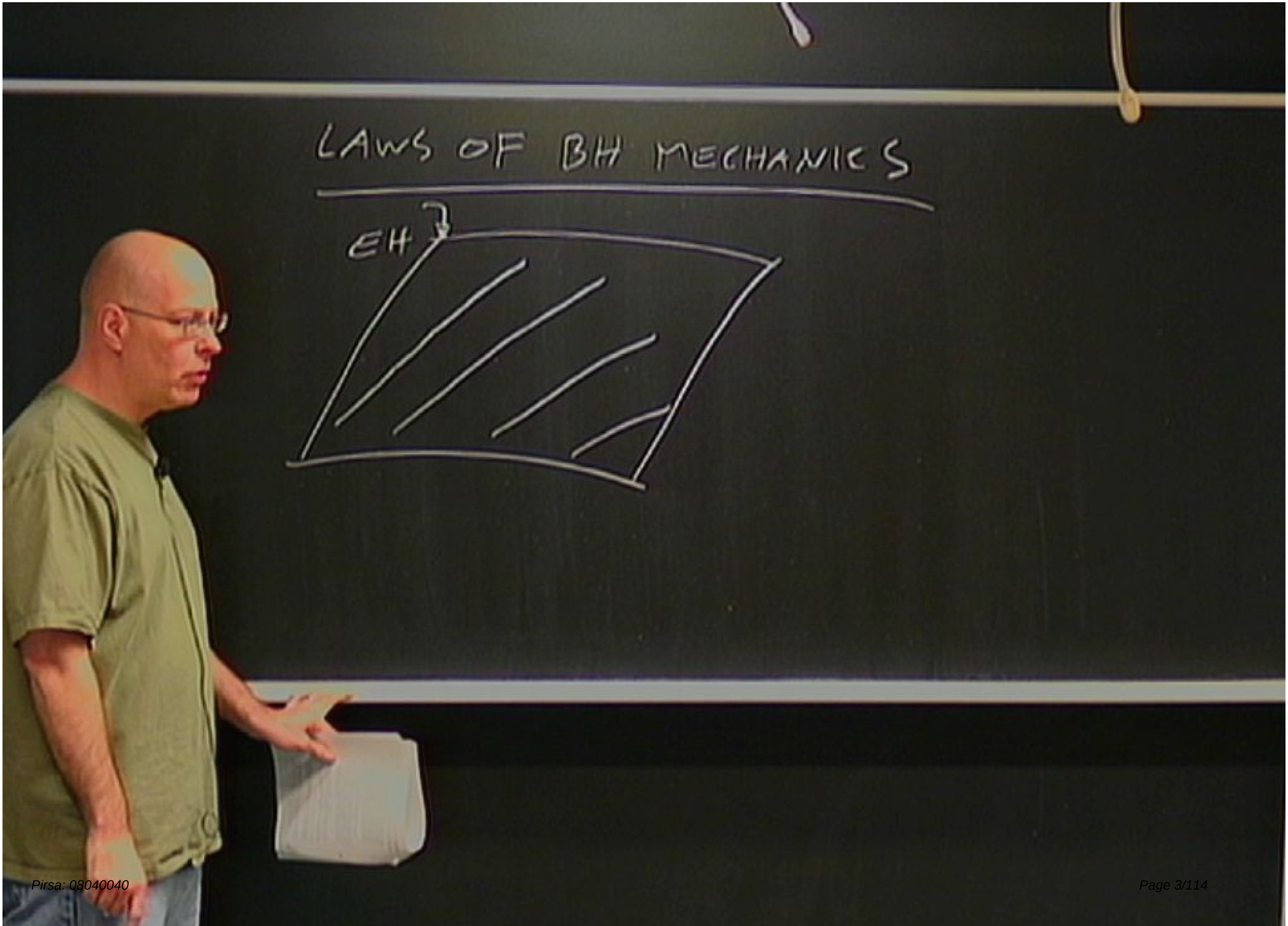
Abstract: Advanced General Relativity


LAWS OF BH MECHANICS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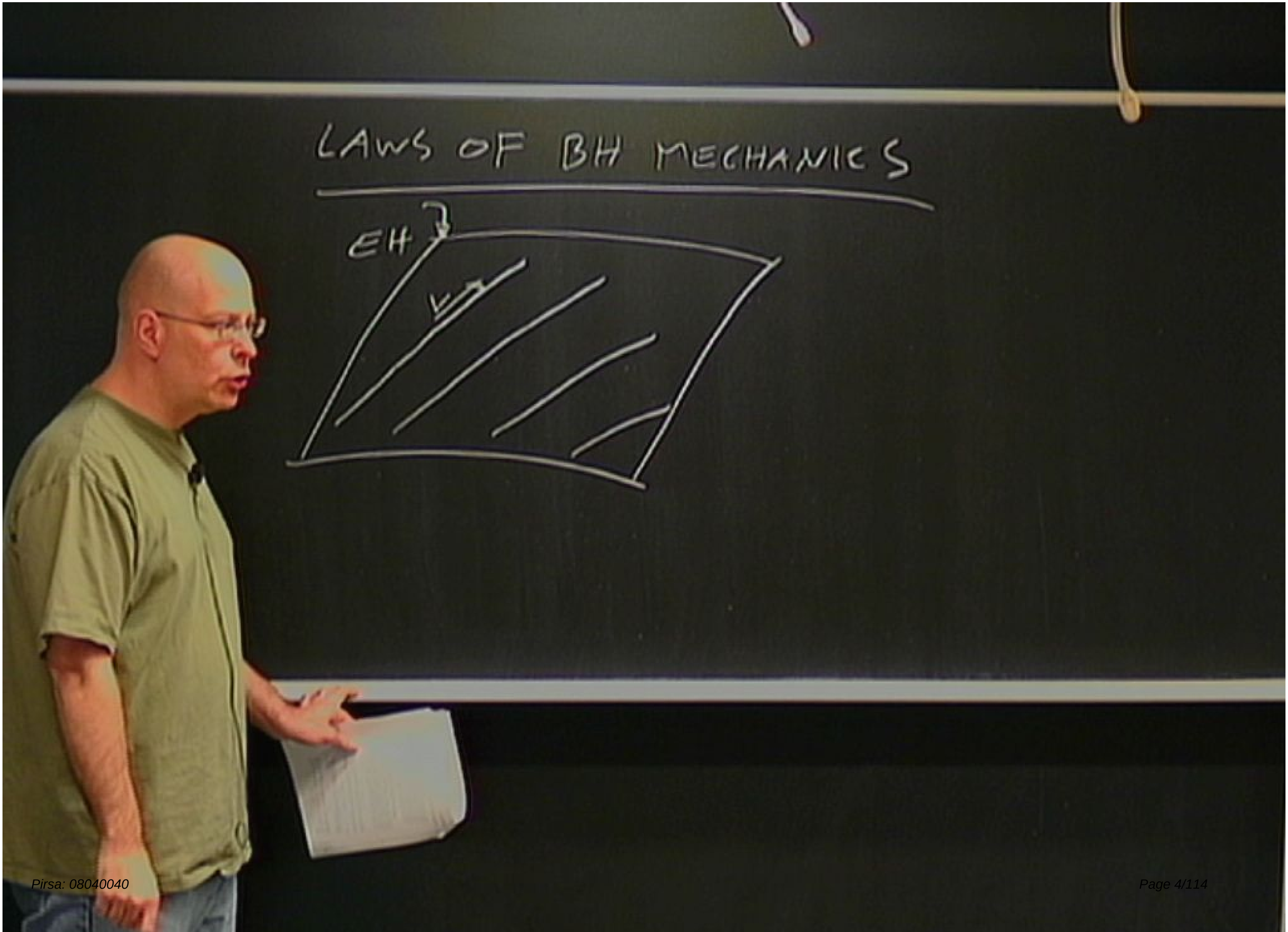
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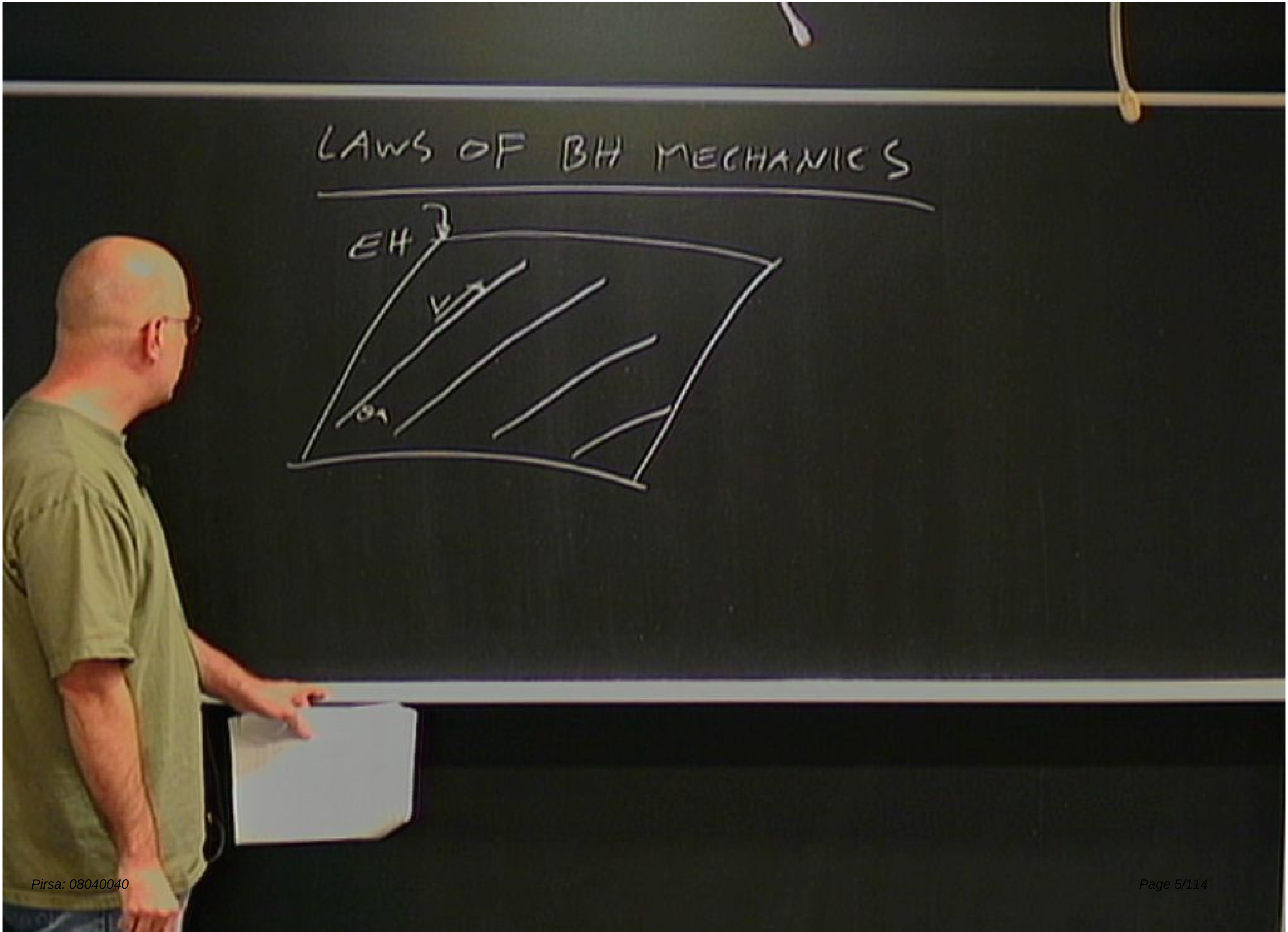
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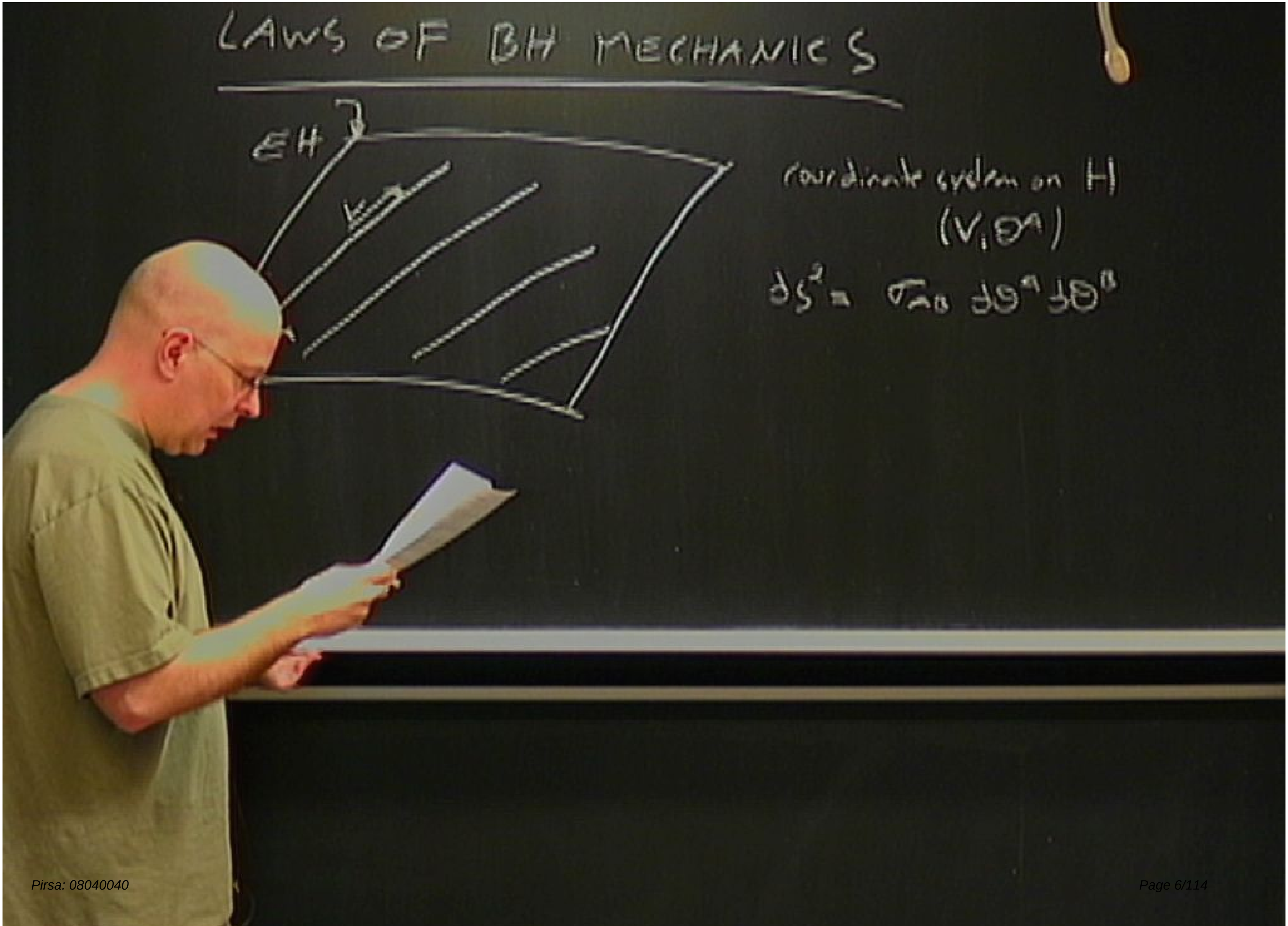
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coordinate system on H
(V, θ^A)
ds² = σ_AB dθ^A dθ^B

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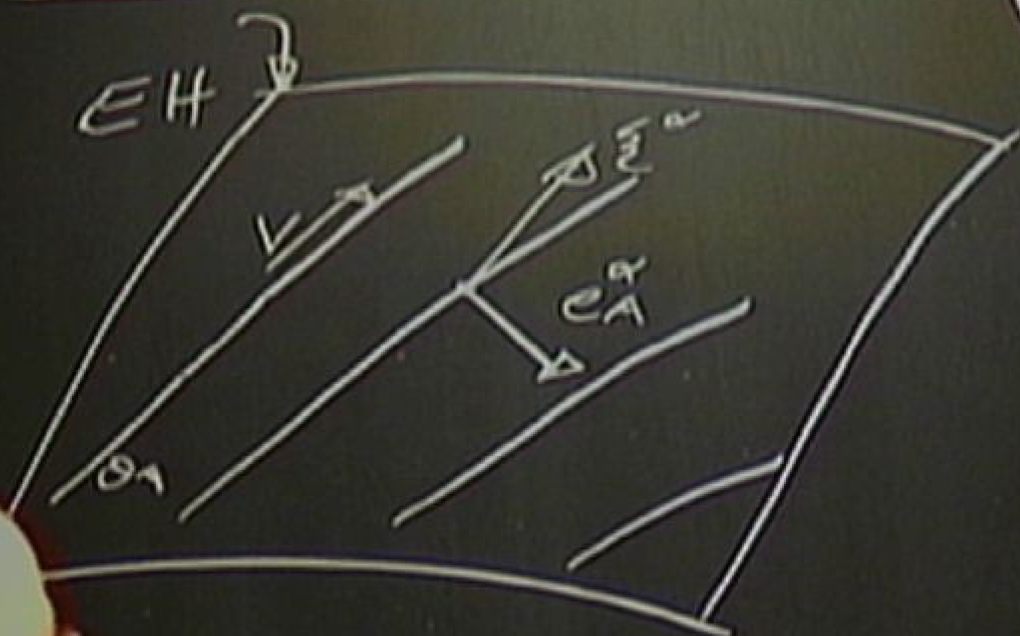


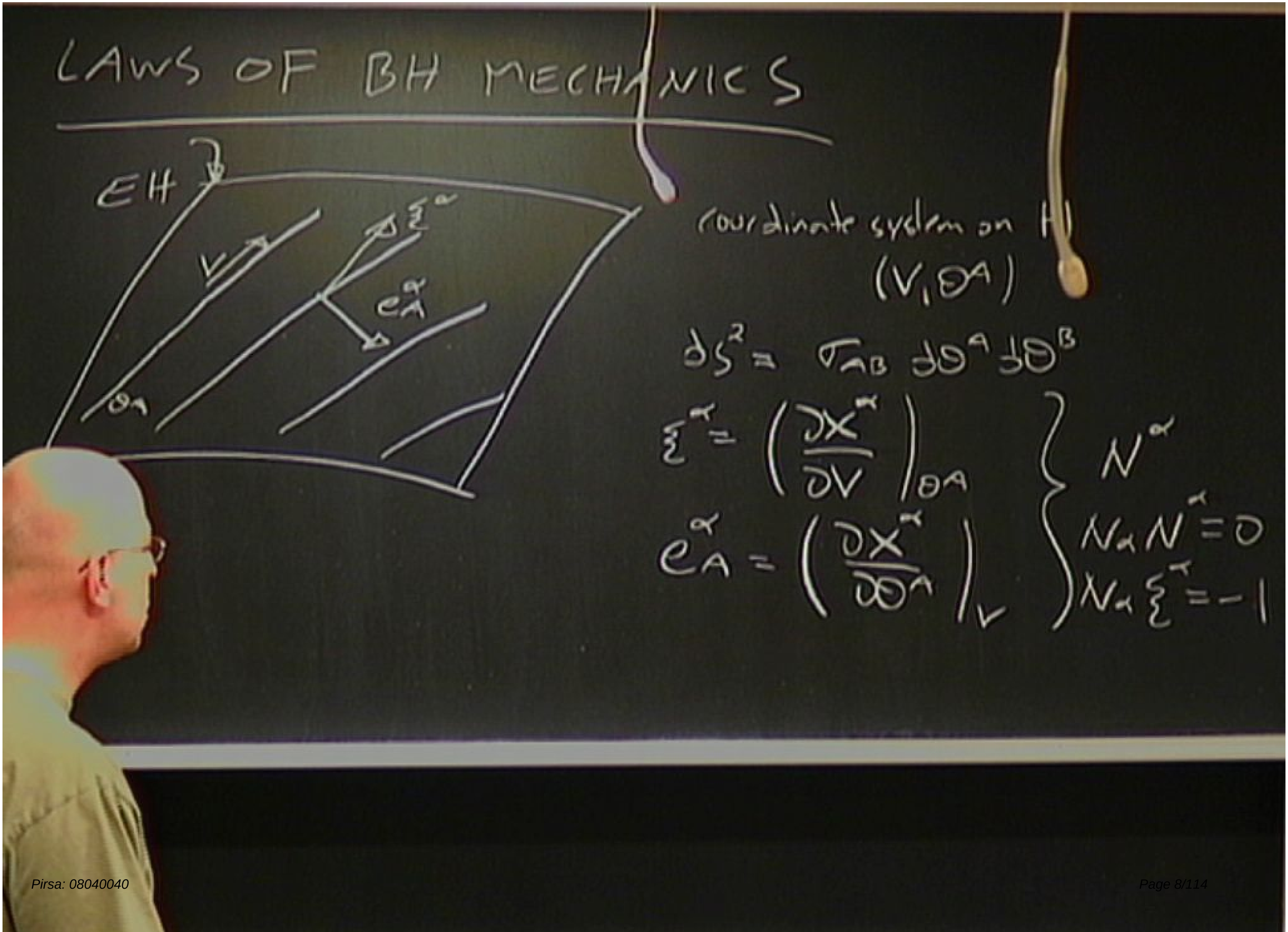
coordinate system on H

$$(v, \theta^A)$$

$$ds^2 = \sigma_{AB}\, d\theta^A\, d\theta^B$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v} \right)_{\theta^A}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A} \right)_{v}$$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v
ξ^α
e^α_A
θ^A
coordinate system on H
(v, θ^A)
$ds^2 = \sigma_{AB}\, d\theta^A d\theta^B$
$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$
$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$
N^α
$N_\alpha N^\alpha = 0$
$N_\alpha \xi^\alpha = -1$

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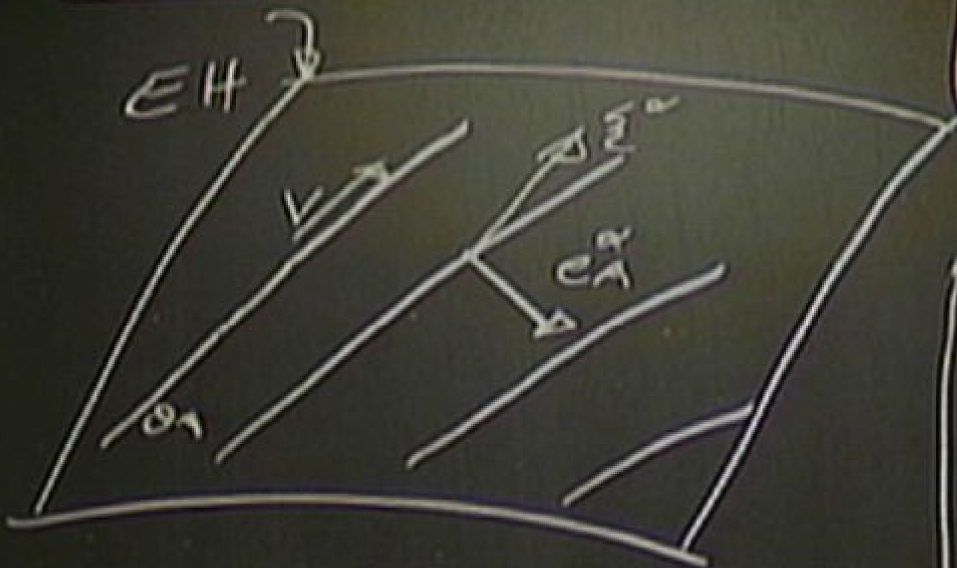

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e_A^\alpha e_B^\beta$$

coordinate system on H

(v, θ^A)

$$ds^2 = \sigma_{AB}\, d\theta^A d\theta^B$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$$

$$e_A^\alpha = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

N^α

$N_\alpha N^\alpha = 0$

$N_\alpha \xi^\alpha = -1$

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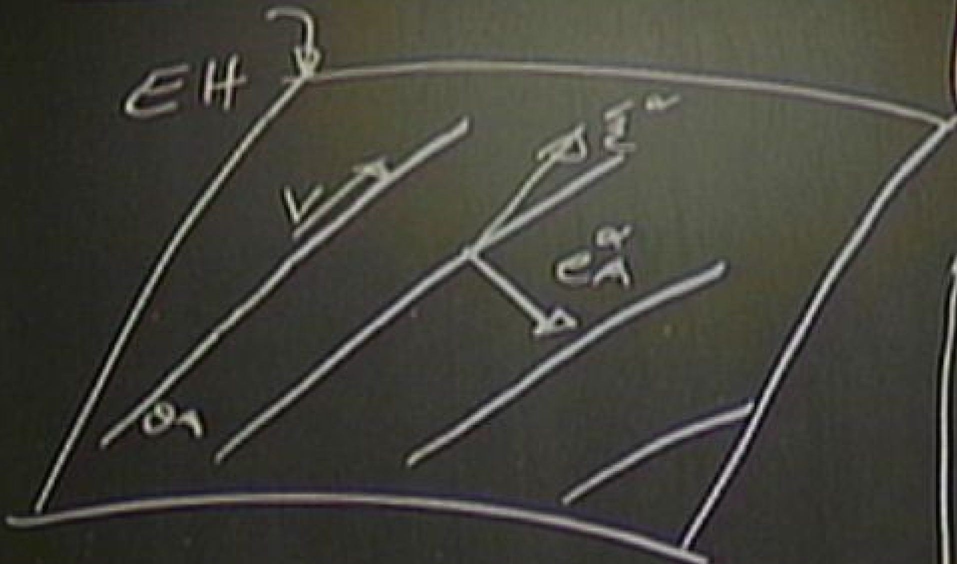


coordinate system on H

(v, θ^A)

$$ds^2 = \sigma_{AB}\, d\theta^A d\theta^B$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$$

$$e_A^\alpha = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

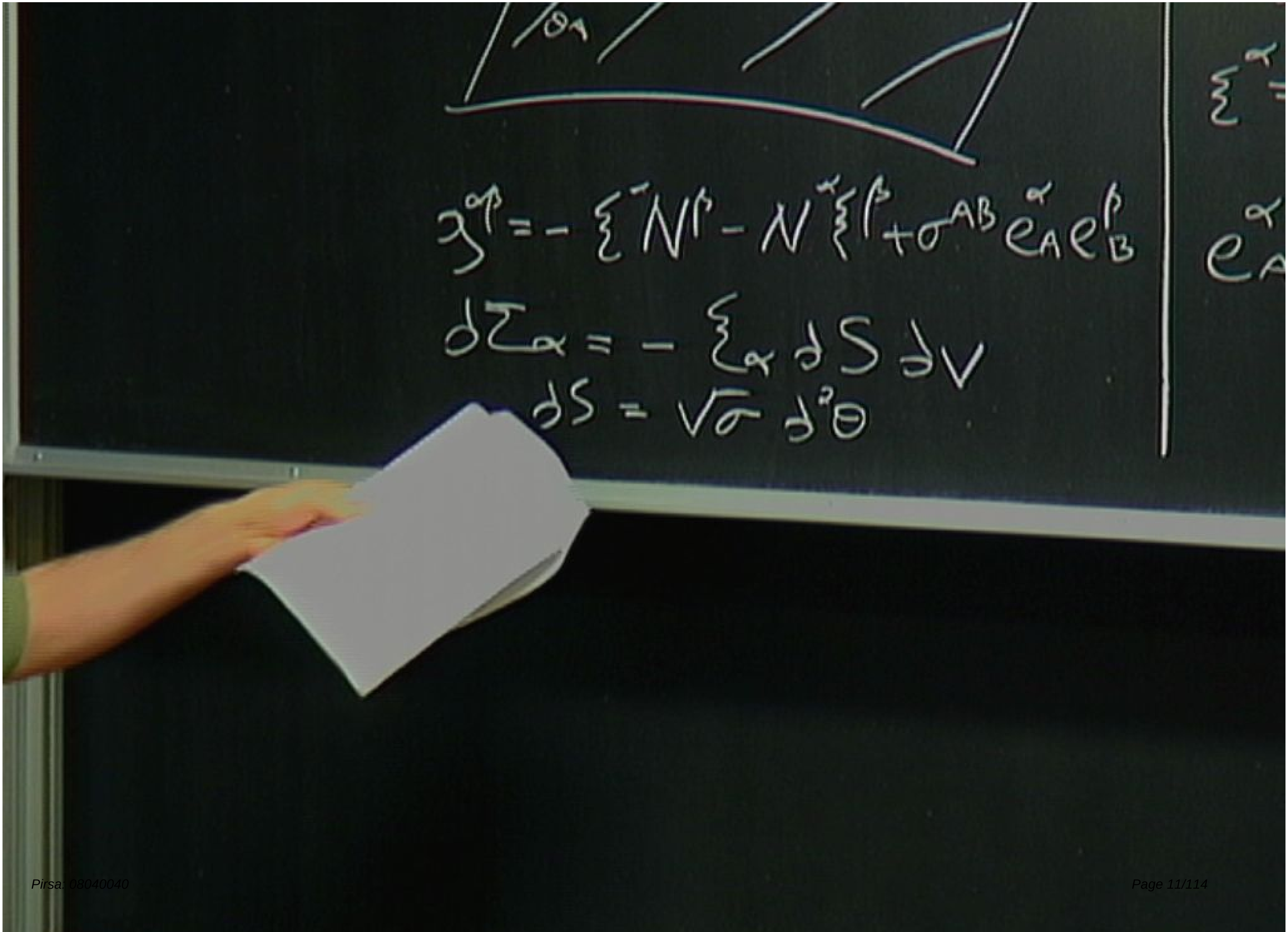
$$N^\alpha$$

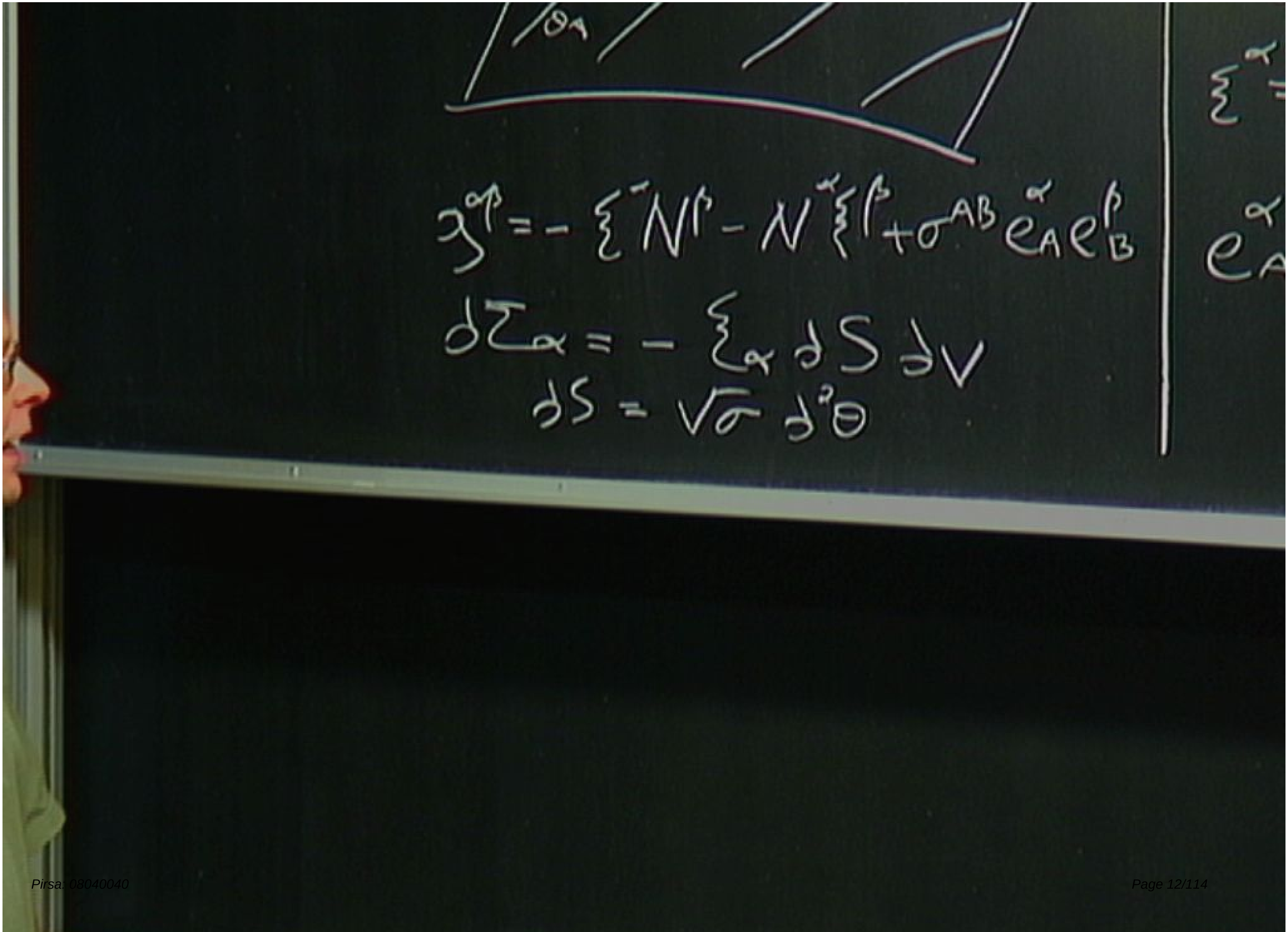
$$N_\alpha N^\alpha = 0$$

$$N_\alpha \xi^\alpha = -1$$

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e_A^\alpha e_B^\beta$$

$$d\Sigma_\alpha = -\xi_\alpha\, dS\, dv$$


θ_A
$\gamma^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$
$d\Sigma_\alpha = -\xi_\alpha \, dS \, dV$
$dS = \sqrt{\sigma} \, d^2\theta$


θ_A
$\gamma^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$
$d\Sigma_\alpha = -\xi_\alpha \, dS \, dV$
$dS = \sqrt{\sigma} \, d^2\theta$
$\xi^\alpha =$
e^α_A

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$$

$$\frac{d\theta}{d\lambda} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$$

$$\theta = \frac{1}{\delta S}\frac{d}{d\lambda}\delta S$$

$e_A e'_B$

$$e_A = \left(\frac{\partial x}{\partial \theta^A}\right)_V$$

$N_\alpha N$

$N_\alpha \xi$

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

$$\xi^\alpha{}_{;\beta}\,\xi^\beta = \kappa\,\xi^\alpha$$

Zeroth law

$$\boxed{\kappa \text{ is a constant over EH}}$$

Zeroth law

$$\boxed{\kappa \text{ is a constant over EH}}$$

$$\left(\frac{\partial \kappa}{\partial V}\right)_{\theta^A} = 0$$

$$\left(\frac{\partial \kappa}{\partial \theta^A}\right)_{V} = 0$$

Zeroth law

$$\boxed{\kappa \text{ is a constant over EH}}$$

$$\boxed{\begin{aligned} \left(\frac{\partial \kappa}{\partial V}\right)_{\theta^A} &= 0 \\ \left(\frac{\partial \kappa}{\partial \theta^A}\right)_{V} &= 0 \end{aligned}}$$

Decomposition of $\Sigma_{\alpha\beta}$ on EH:

Zeroth law

$\boxed{\kappa \text{ is a constant over EH}}$

$$\boxed{\begin{aligned} \left(\frac{\partial \kappa}{\partial V}\right)_{\theta^A} &= 0 \\ \left(\frac{\partial \kappa}{\partial \theta^A}\right)_{V} &= 0 \end{aligned}}$$

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e$$

Zeroth law

$\boxed{\kappa \text{ is a constant over EH}}$

$$\left(\frac{\partial \kappa}{\partial V}\right)_{\theta^A} = 0$$

$$\left(\frac{\partial \kappa}{\partial \theta^A}\right)_{V} = 0$$

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$
$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e$$

Zeroth law

$$\boxed{\kappa \text{ is a constant over EH}}$$

$$\boxed{\left(\frac{\partial\kappa}{\partial v}\right)_{\theta^A} = 0 \qquad \left(\frac{\partial\kappa}{\partial\theta^A}\right)_{v} = 0}$$

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha\xi_\beta + b\,\xi_\alpha N_\beta + c^A\xi_\alpha e_{A\beta} + d\,N_\alpha\xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta} + g^A e_{A\alpha}\xi_\beta + h^A e_{A\alpha}N_\beta + j^{AB} e_{A\alpha} e$$

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$
$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + \ell^{AB} e_{A\alpha} e_{B\beta}$$

$$\left.\frac{\quad}{\partial A}\right)_V = 0$$

$+\, g^A$

1 - $\underline{\xi^\alpha \text{ is null}}$

$\xi^\alpha \xi_{\alpha;\beta}$

$$\Big)_{\nu} = 0$$

$$+ g^{A}$$

1 – $\underline{\xi^{\alpha} \text{ is null}}$

$$0 = \xi^{\alpha} \xi_{\alpha;\beta} \xi^{\beta}$$

$$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_V = 0}$$

$+\, \eta^A e_{A\alpha}$

1 – $\underline{\xi^\alpha \text{ is null}}$

$$0 = \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta$$

$$0 = \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_A$$

1 – ξ^α is null

$$0 = \xi^\alpha \xi_{\alpha;\beta} \xi^\beta$$

$$0 = \xi^\alpha \xi_{\alpha;\beta} e^\beta{}_A$$

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha\xi_\beta + b\,\xi_\alpha N_\beta + c^A\xi_\alpha e_{A\beta}$$

$$+ d\,N_\alpha\xi_\beta + \cancel{e\,N_\alpha N_\beta} + f^A N_\alpha e_{A\beta}$$

$$+ g^A e_{A\alpha}\xi_\beta + h^A e_{A\alpha}N_\beta + s^{AB} e_{A\alpha}e_{B\beta}$$

ξ^α is null

$$= \xi^\alpha\xi_{\alpha;\beta}\xi^\beta = e$$

$$= \xi^\alpha\xi_{\alpha;\beta}e^\beta_A$$

over EH

Decomposition of $\xi_{\alpha;\beta}$ on EH:

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$

$$+ d\,N_\alpha \xi_\beta + \cancel{e\,N_\alpha N_\beta} + f^A N_\alpha e_{A\beta}$$

$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + i^{AB} e_{A\alpha} e_{B\beta}$$

1- $\underline{\xi^\alpha \text{ is null}}$

$$0 = \xi^\alpha \xi_{\alpha;\beta} \xi^\beta = e$$

$$0 = \xi^\alpha \xi_{\alpha;\beta} e^\beta_B = -f^A \sigma_{AB}$$

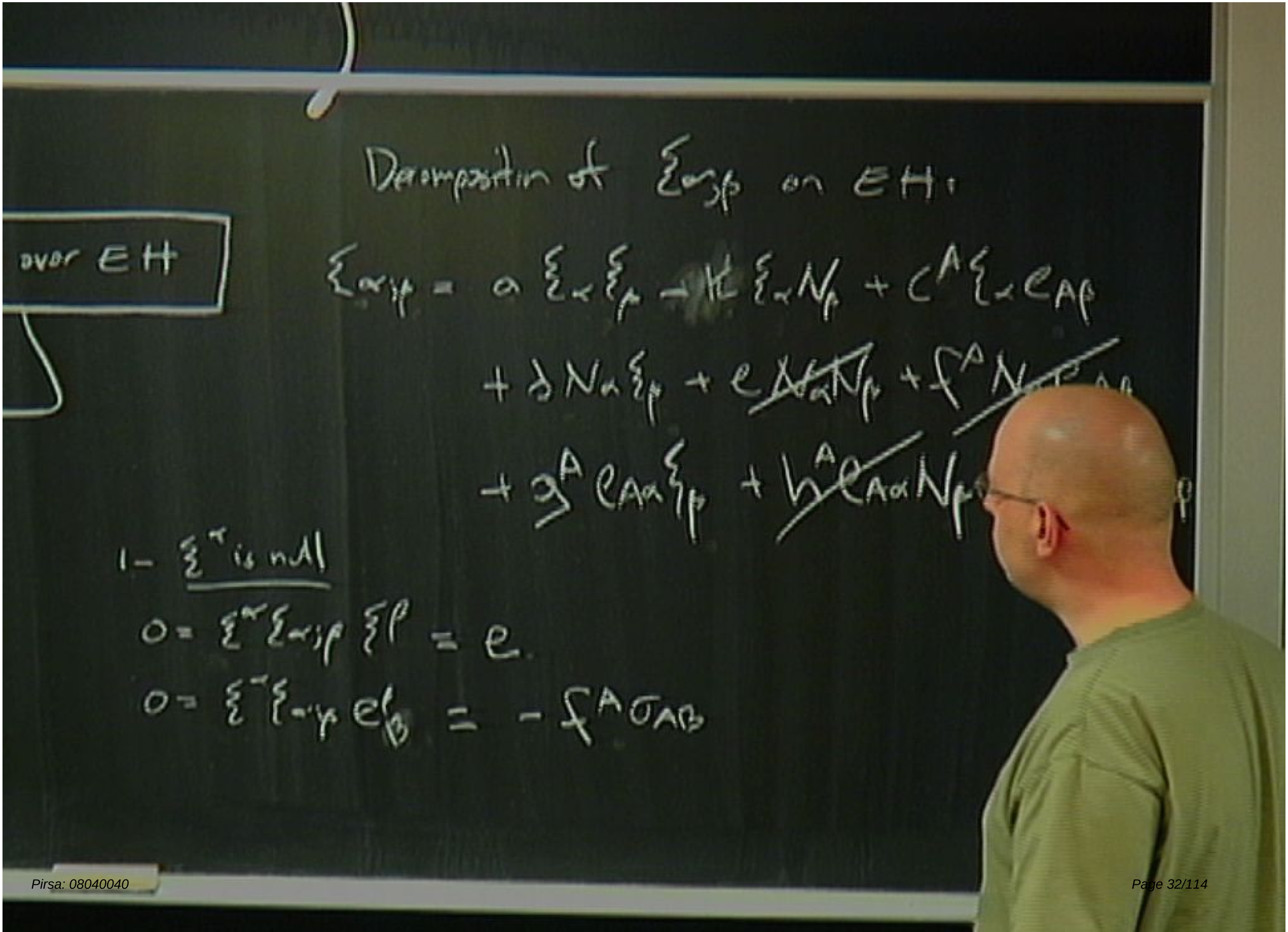
$$2- \ \xi^{\alpha}{}_{;\beta}\, \xi^{\beta} = k\, \xi^{\alpha}$$

$$a - \quad \xi^{\alpha}{}_{;\beta}\,\xi^{\beta} = \kappa\,\xi^{\alpha}$$

$$\xi_{\alpha;\beta}\,\xi^{\beta} = -\,b\,\xi_{\alpha}$$

a- $\underline{\xi^{\alpha}{}_{;\beta}\,\xi^{\beta} = k\,\xi^{\alpha}}$

$$\xi_{\alpha;\beta}\,\xi^{\beta} = -\,b\,\xi_{\alpha} - h^{A}\,e_{A\alpha} \Rightarrow b = -k$$


over EH
Decomposition of $\Sigma_{\alpha\beta}$ on EH:
$\Sigma_{\alpha\beta} = a\,\xi_\alpha\xi_\beta + b\,\xi_\alpha N_\beta + c^A\xi_\alpha e_{A\beta}$
$+ d\,N_\alpha\xi_\beta + e\,\xi_\alpha N_\beta + f^A N_\alpha e_{A\beta}$
$+ g^A e_{A\alpha}\xi_\beta + h^A e_{A\alpha}N_\beta$
1- ξ^α is null
$0 = \xi^\alpha\Sigma_{\alpha\beta}\xi^\beta = e$
$0 = \xi^\alpha\Sigma_{\alpha\beta}e^\beta_B = -f^A\sigma_{AB}$

$$+ g^{\alpha} e_{A\alpha} \xi_\beta + h\, e_{A\alpha} N_\beta + f\, e_{A\alpha} e_\beta$$

1 – ξ^α is null

$$0 = \xi^\alpha \xi_{\alpha;\beta} \xi^\beta = e$$

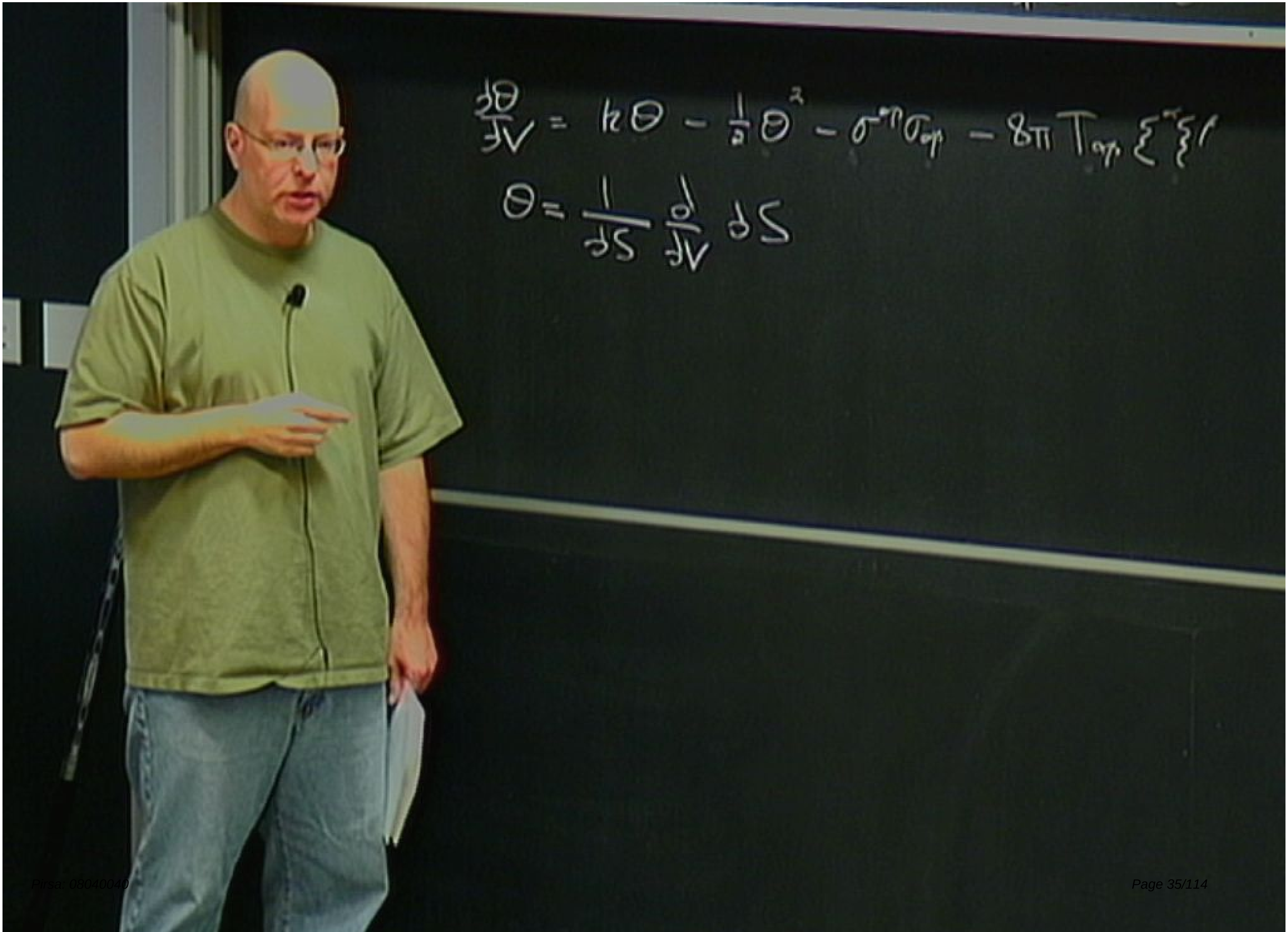
$$0 = \xi^\alpha \xi_{\alpha;\gamma} e^\gamma_B = - f^A \sigma_{AB}$$

$$- h^A e_{A\alpha} \Rightarrow b = -k \qquad h^A = 0$$

2- $\underline{\xi^\alpha{}_{;\beta}\,\xi^\beta = k\,\xi^\alpha}$

$\xi_{\alpha;\beta}\,\xi^\beta = -b\,\xi_\alpha - h^A e_{A\alpha} \Rightarrow b = -k \quad h^A = 0$

3-

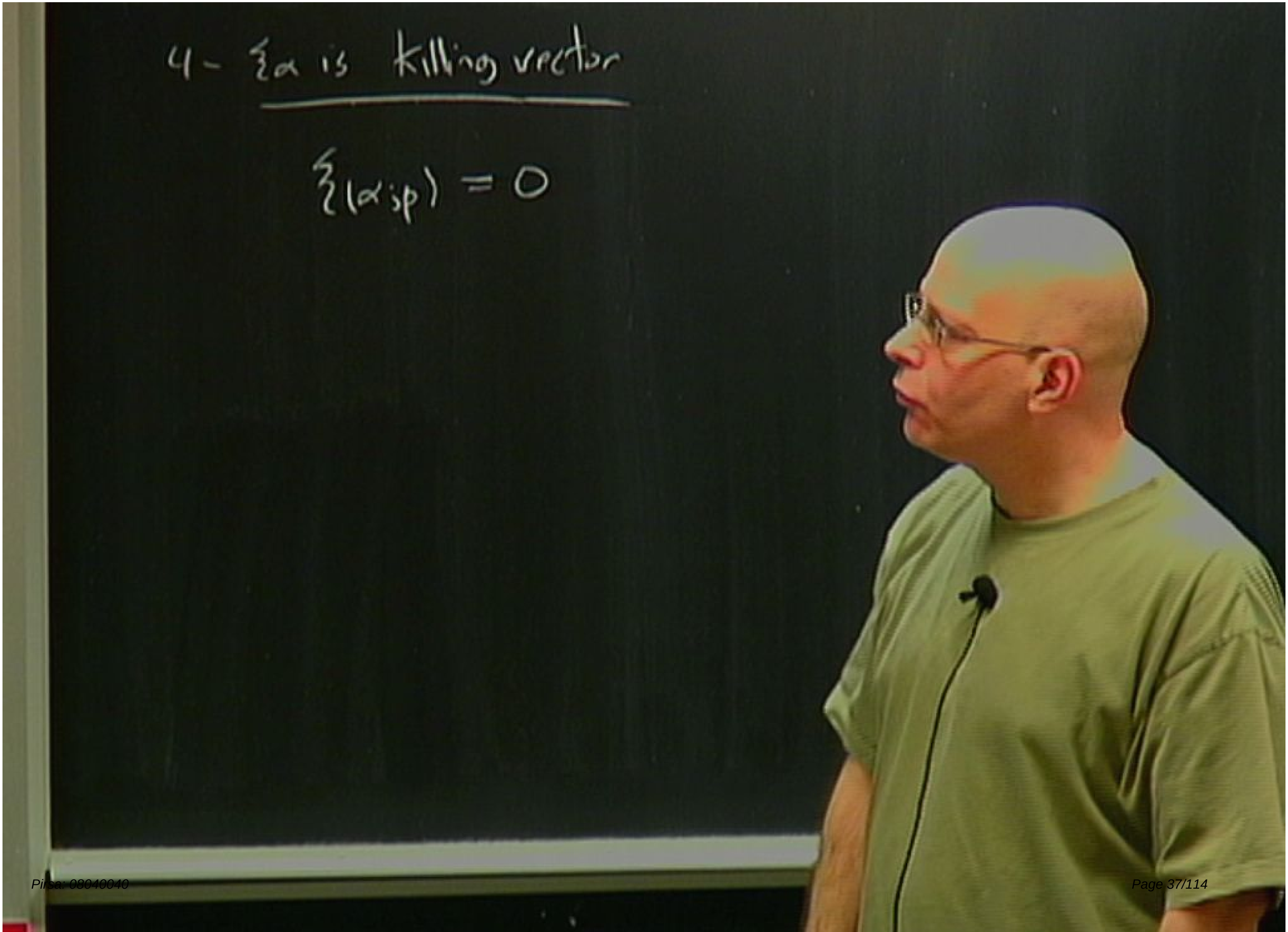

$\frac{\partial\theta}{\partial V} = k\theta - \frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} - 8\pi T_{ab}\xi^a\xi^b$
$\theta = \frac{1}{\partial S}\frac{\partial}{\partial V}\partial S$

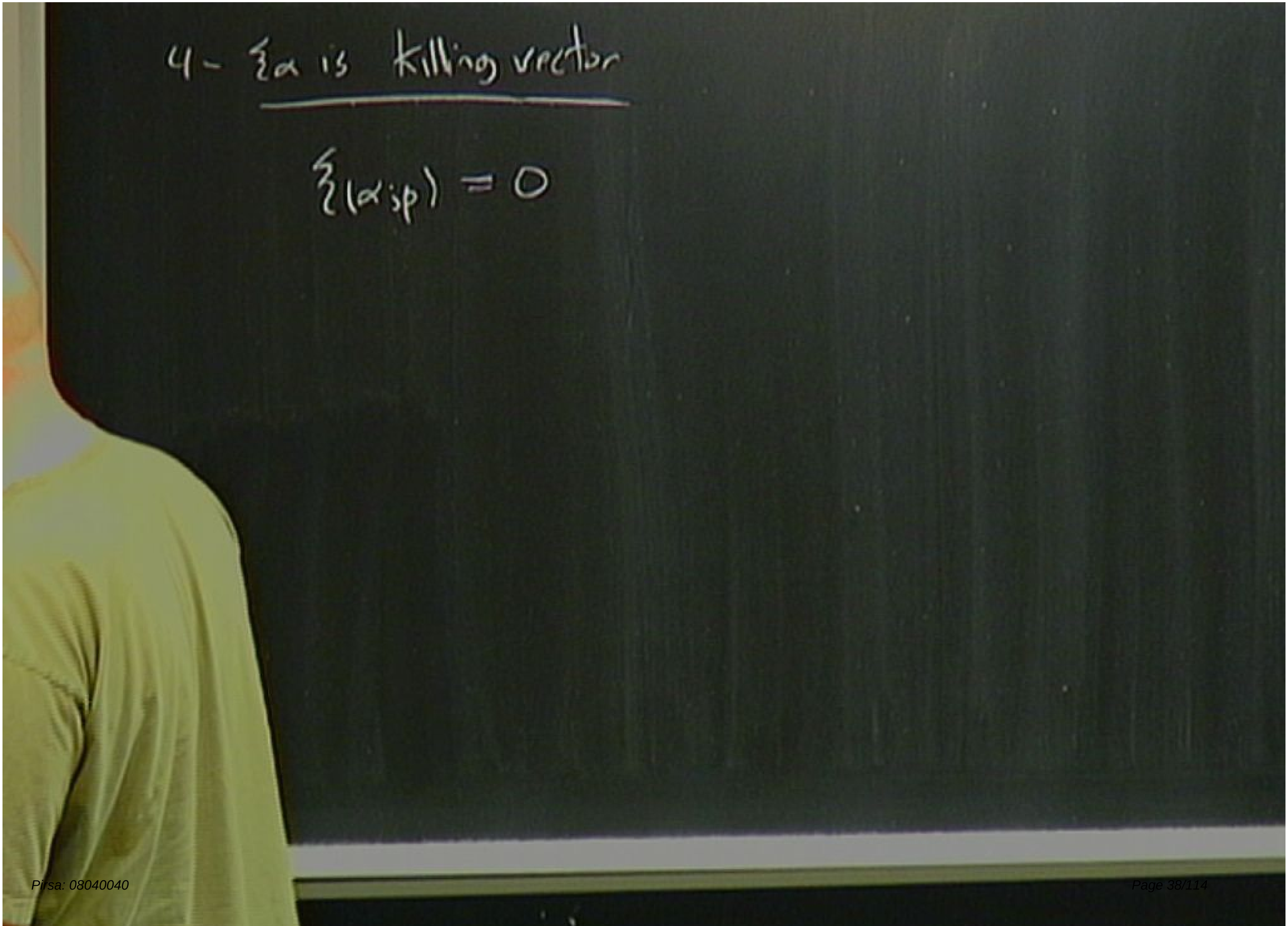
2 - $\underline{\xi^{\alpha}{}_{;\beta}\,\xi^{\beta} = k\,\xi^{\alpha}}$

$\xi_{\alpha;\beta}\,\xi^{\beta} = -\,b\,\xi_{\alpha} - h^{A}\,e_{A\alpha} \Rightarrow b = -k \quad h^{A} = 0$

3 — expansion, shear, rotation vanish.

$\rightarrow \; j^{AB} = 0$


4 - ξ_α is killing vector
$\xi_{(\alpha;\beta)} = 0$


4 - ξ_α is killing vector
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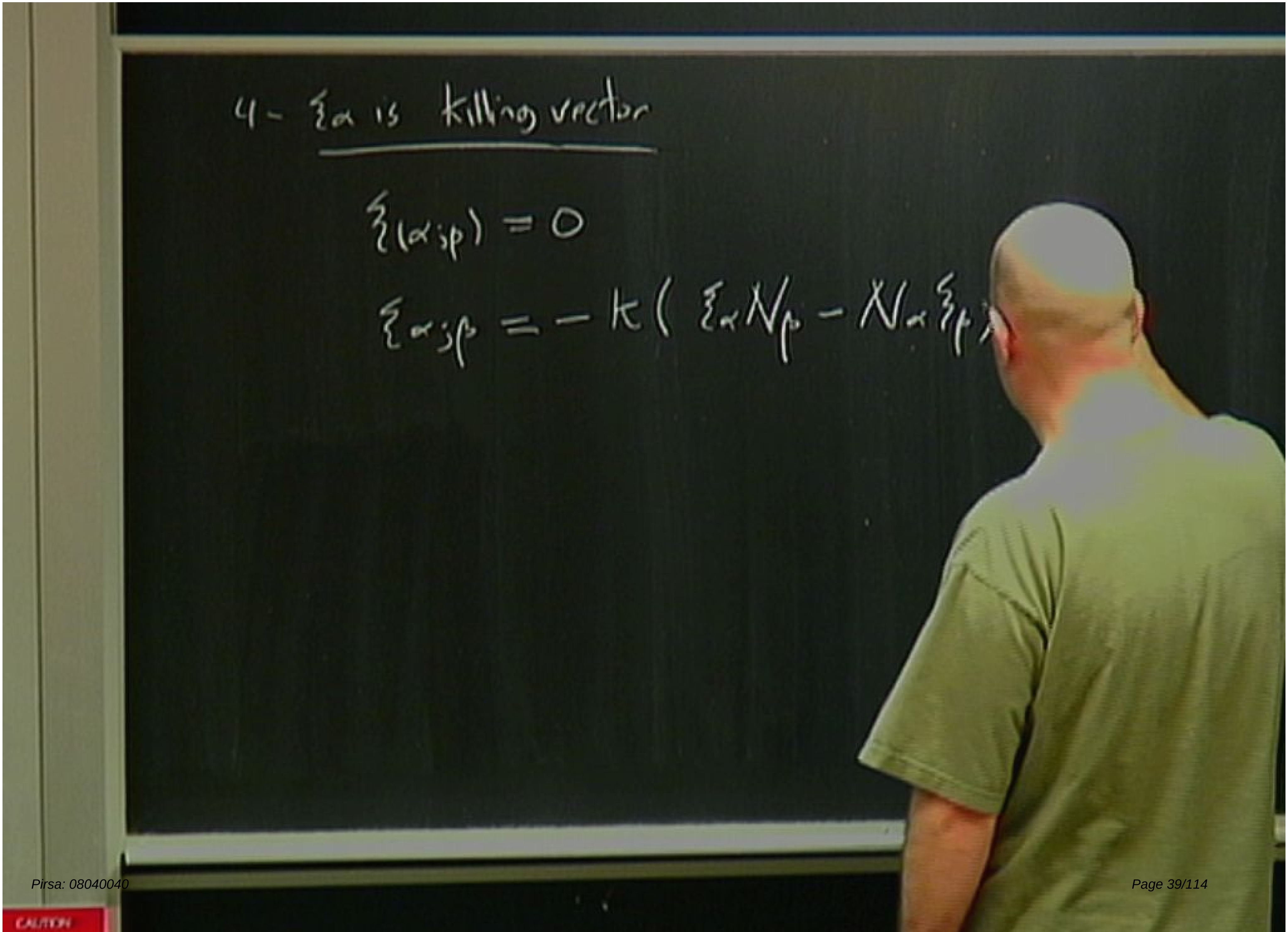
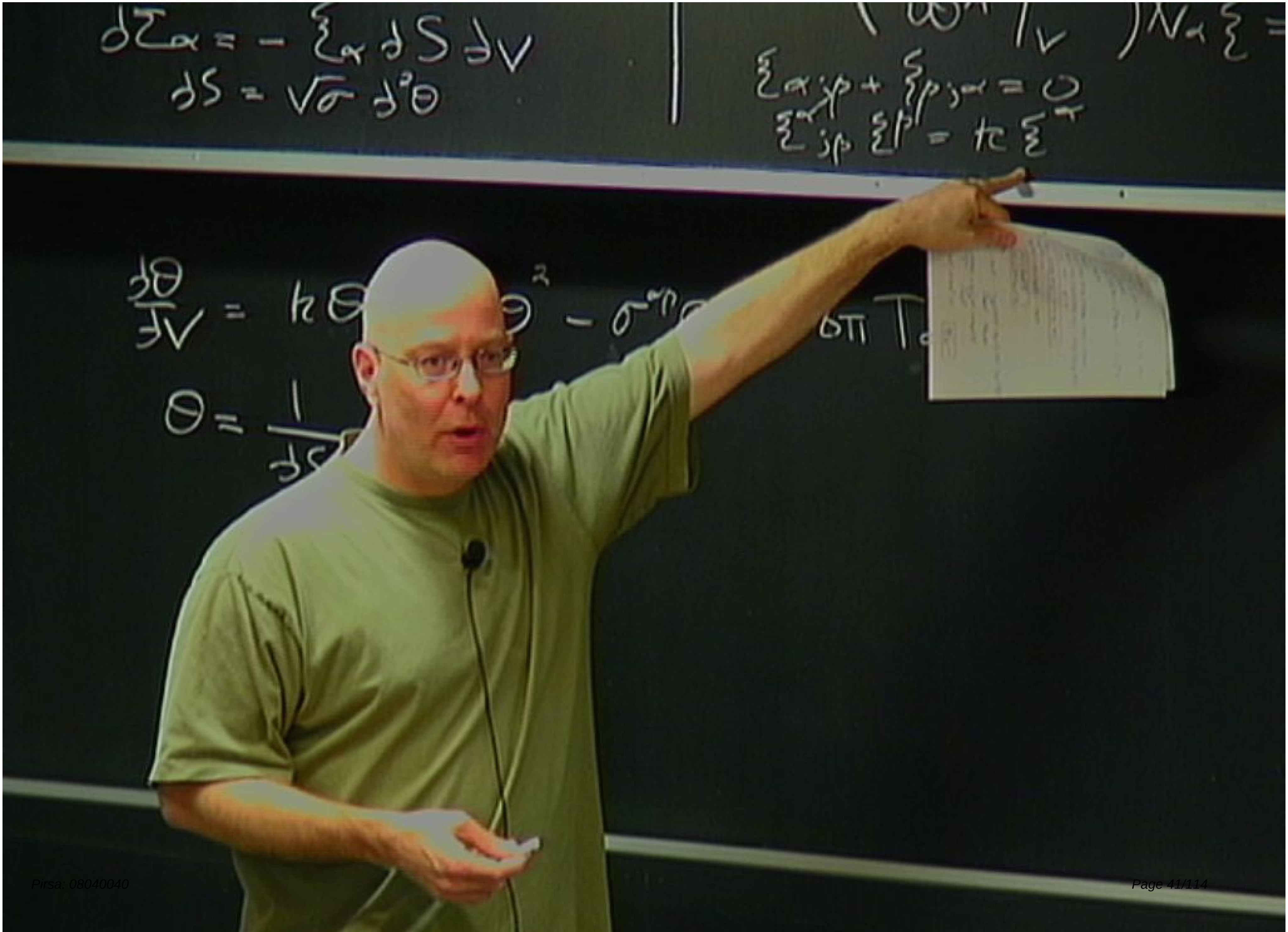
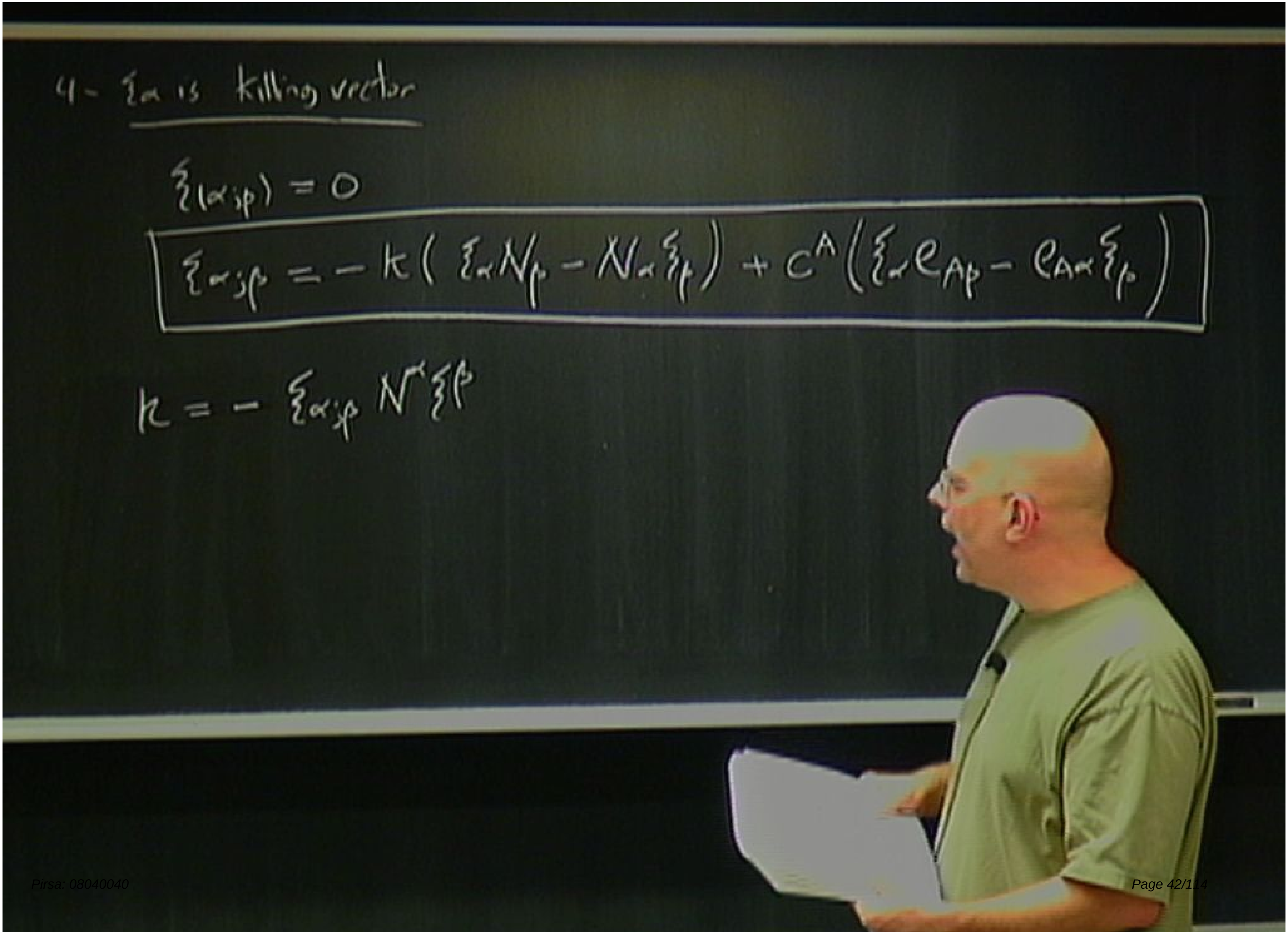

4 - ξ_α is killing vector
$\xi_{(\alpha;\beta)} = 0$
$\xi_{\alpha;\beta} = -k(\xi_\alpha N_\beta - N_\alpha \xi_\beta)$

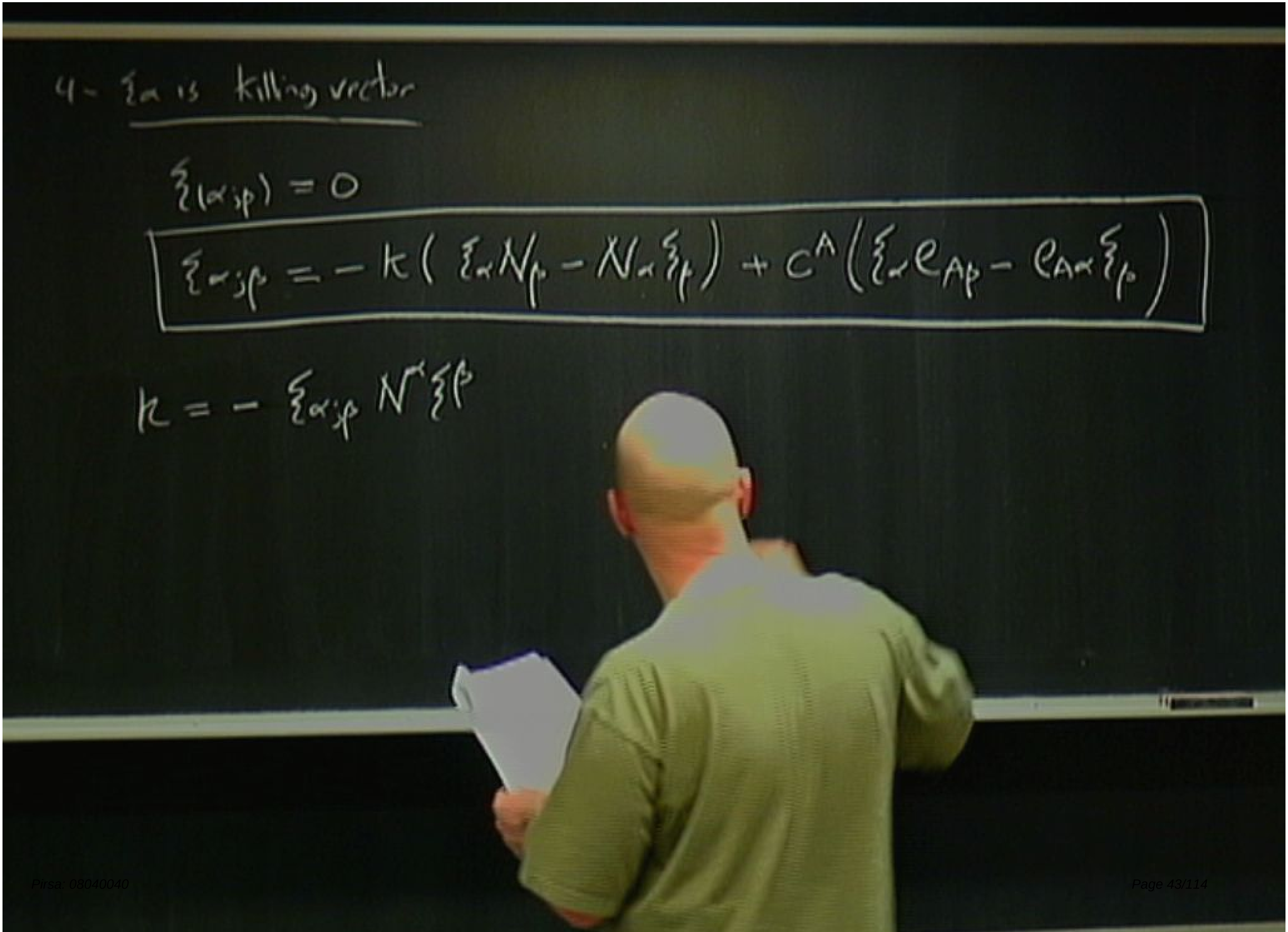
4- ξ_α is killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + c^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$


dΣα = − ξα dS dV
dS = √σ d²θ
ξα;β + ξβ;α = 0
ξα;β ξβ = κ ξα


4- ξ_α is killing vector
$\xi_{(\alpha;\beta)} = 0$
$\boxed{\xi_{\alpha;\beta} = -\kappa(\xi_\alpha N_\beta - N_\alpha \xi_\beta) + c^A(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta)}$
$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$


4 - ξ_α is killing vector
$\xi_{(\alpha;\beta)} = 0$
$\boxed{\xi_{\alpha;\beta} = -k(\xi_\alpha N_\beta - N_\alpha \xi_\beta) + c^A(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta)}$
$k = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$

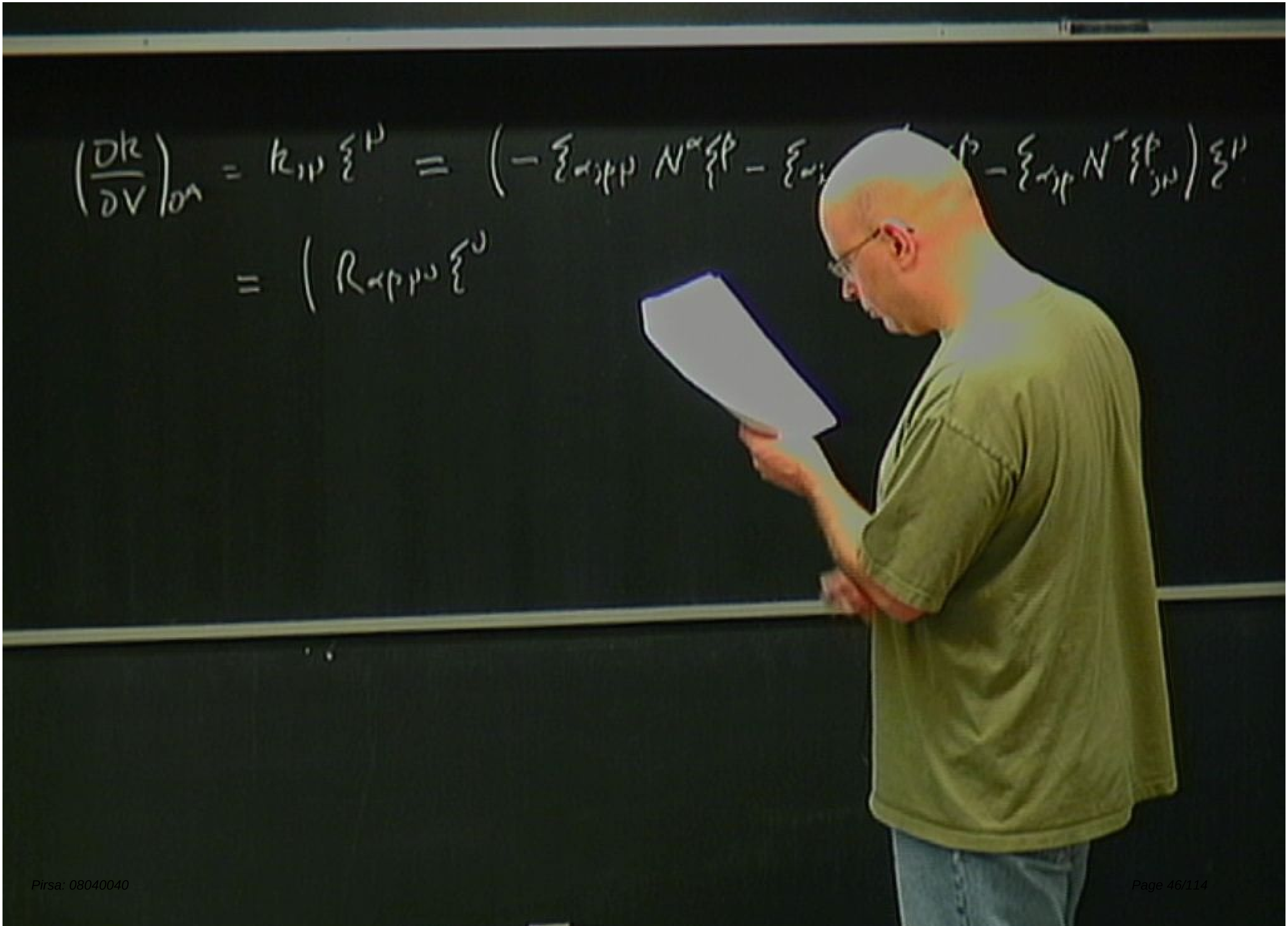
4 - ξ_α is killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + c^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$

$$\boxed{k = -\xi_{\alpha;\beta} N^\alpha \xi^\beta}$$

$$\left(\frac{\partial k}{\partial V}\right)_{\partial A} = k_{,\mu}\,\xi^{\mu}$$


$\left(\frac{Dk}{Dv}\right)_{on} = k_{;\mu}\,\xi^{\mu} = \left(-\xi_{\alpha;\beta\mu}\,N^{\alpha}\xi^{\beta} - \xi_{\alpha;}\right.$
$\xi^{\beta} - \xi_{\alpha;\beta}\,N^{\alpha}\,\xi^{\beta}_{;\mu}\Big)\,\xi^{\mu}$
$= \Big(\,R_{\alpha\beta\mu\nu}\,\xi^{\nu}$

$$\left(\frac{\partial k}{\partial v}\right)_{on} = k_{;\mu}\xi^\mu = \left(-\xi_{\alpha;\rho\mu} N^\alpha \xi^\rho - \xi_{\alpha;\rho} N^\alpha_{;\mu}\xi^\rho - \xi_{\alpha;\rho} N^\alpha \xi^\rho_{;\mu}\right)\xi^\mu$$

$$= \left(R_{\alpha\rho\mu\nu}\xi^\nu N^\alpha \xi^\rho\xi^\mu - \xi_{\alpha;\rho}\xi^\rho N^\alpha_{;\mu}\xi^\mu - \xi_{\alpha;\rho} N^\alpha \xi^\rho_{;\mu}\xi^\mu \right)$$

$$\left(\frac{Dk}{Dv}\right)_{on} = k_{;\mu}\xi^{\mu} = \left(-\xi_{\alpha;\beta\mu}N^{\alpha}\xi^{\beta} - \xi_{\alpha;\beta}N^{\alpha}_{;\mu}\xi^{\beta} - \xi_{\alpha;\beta}N^{\alpha}\xi^{\beta}_{;\mu}\right)\xi^{\mu}$$

$$= \left(R_{\alpha\beta\mu\nu}\xi^{\nu}N^{\alpha}\xi^{\beta}\xi^{\mu} - \xi_{\alpha;\beta}\xi^{\beta}N^{\alpha}_{;\mu}\xi^{\mu} - \xi_{\alpha;\beta}N^{\alpha}\xi^{\beta}_{;\mu}\xi^{\mu}\right)$$

$$\left(\frac{\partial \kappa}{\partial V}\right)_{\partial n} = \kappa_{;\mu}\,\xi^{\mu} = \left(-\xi_{\alpha;\beta\mu} N^{\alpha}\xi^{\beta} - \xi_{\alpha;\beta} N^{\alpha}{}_{;\mu}\,\xi^{\beta} - \xi_{\alpha;\beta} N^{\alpha}\,\xi^{\beta}{}_{;\mu}\right)\xi^{\mu}$$

$$= \left(\cancel{R_{\alpha\beta\mu\nu}\,\xi^{\nu}}\, N^{\alpha}\xi^{\beta}\xi^{\mu} - \underbrace{\xi_{\alpha;\beta}\,\xi^{\beta}}_{\kappa\,\xi_{\alpha}} N^{\alpha}{}_{;\mu}\,\xi^{\mu} - \xi_{\alpha;\beta} N^{\alpha}\, \underbrace{\xi^{\beta}{}_{;\mu}\,\xi^{\mu}}_{\kappa\,\xi^{\beta}} \right)$$

$$= -\kappa\, \underbrace{\xi_{\alpha} N^{\alpha}{}_{;\mu}\,\xi^{\mu}}_{\substack{-\xi_{\alpha;\mu} N^{\alpha}\xi^{\mu} \\ = -\kappa\,\xi_{\alpha} N^{\alpha}}} - \kappa\, \underbrace{\xi_{\alpha;\beta}\,\xi^{\beta}}_{\kappa\,\xi_{\alpha}} N^{\alpha} =$$

$$\left(\frac{Dk}{DV}\right)_{on} = k_{;\mu}\xi^{\mu} = \left(-\xi_{\alpha;\beta\mu} N^{\alpha}\xi^{\beta} - \xi_{\alpha;\beta} N^{\alpha}_{;\mu}\xi^{\beta} - \xi_{\alpha;\beta} N^{\alpha}\xi^{\beta}_{;\mu}\right)\xi^{\mu}$$

$$= \left(\cancel{R_{\alpha\beta\mu\nu}\xi^{\nu}N^{\alpha}\xi^{\beta}\xi^{\mu}} - \underbrace{\xi_{\alpha;\mu}\xi^{\mu}}_{k\xi_{\alpha}} N^{\alpha}_{;\nu}\xi^{\nu} - \xi_{\alpha;\beta}N^{\alpha}\underbrace{\xi^{\beta}_{;\mu}\xi^{\mu}}_{k\xi^{\beta}}\right)$$

$$= -k\underbrace{\xi_{\alpha}N^{\alpha}_{;\mu}\xi^{\mu}}_{\substack{-\xi_{\alpha;\mu}N^{\alpha}\xi^{\mu} \\ = -k\xi_{\alpha}N^{\alpha}}} - k\underbrace{\xi_{\alpha;\beta}\xi^{\beta}}_{k\xi_{\alpha}} N^{\alpha}$$

$$= k^{2}\xi_{\alpha}N^{\alpha} - k^{2}\xi_{\alpha}N^{\alpha} = 0$$

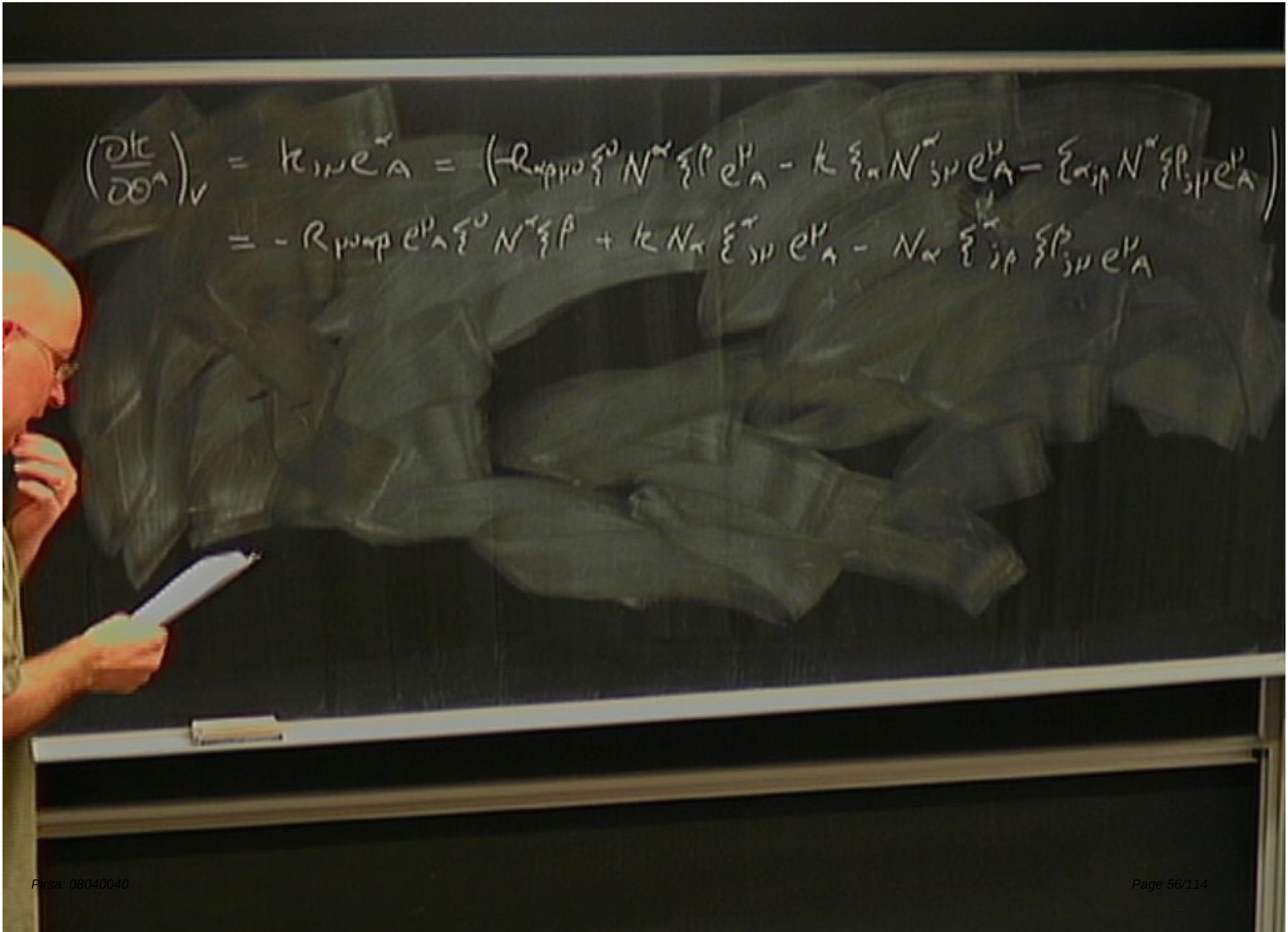
$$\left(\frac{\partial K}{\partial \theta^A}\right)_V = K_{,N} e^{\alpha}_{A} =$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{;\alpha} e^\alpha_A = \left(R_{\alpha\beta\mu\nu} \xi^\nu N^\alpha \xi^\beta e^\mu_A - k \right.$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{;\mu} e^\mu_A = \left(R_{\alpha\beta\mu\sigma} \xi^\sigma N^\alpha \xi^\beta e^\mu_A - k \xi_\alpha N^\alpha_{;\mu} e^\mu_A \right.$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{;\mu} e^\mu_A = \left(R_{\alpha\rho\mu\sigma}\xi^\sigma N^\alpha \xi^\rho e^\mu_A - k\,\xi_\alpha N^\alpha{}_{;\mu} e^\mu_A - \xi_{\alpha;\rho} N^\alpha \xi^\rho{}_{;\mu} e^\mu_A \right)$$

$$\left(\frac{Dk}{D\theta^A}\right)_V = k_{;\mu} e^{\mu}_{A} = \left(-R_{\alpha\beta\mu\nu}\xi^{\nu} N^{\alpha} \xi^{\beta} e^{\mu}_{A} - k\,\xi_{\alpha} N^{\alpha}_{\ ;\mu} e^{\mu}_{A} - \xi_{\alpha;\beta} N^{\alpha} \xi^{\beta}_{\ ;\mu} e^{\mu}_{A} \right)$$


$\left(\frac{\partial k}{\partial \theta^A}\right)_v = k_{;\nu} e^\nu{}_A = \left(-R_{\alpha\beta\mu\nu} \xi^\nu N^\alpha \xi^\beta e^\mu{}_A - k\, \xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu{}_A\right)$
$= -R_{\mu\nu\alpha\beta} e^\mu{}_A \xi^\nu N^\alpha \xi^\beta + k N_\alpha \xi^\alpha{}_{;\mu} e^\mu{}_A - N_\alpha \xi^\alpha{}_{;\beta} \xi^\beta{}_{;\mu} e^\mu{}_A$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_v = k_{;\nu} e^\nu_A = \left(-R_{\alpha\beta\mu\nu} \xi^\nu N^\alpha \xi^\beta e^\mu_A - k\, \xi_\alpha N^\alpha_{;\mu} e^\mu_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} e^\mu_A \right)$$

$$= -R_{\mu\nu\alpha\beta}\, e^\mu_A \xi^\nu N^\alpha \xi^\beta + k N_\alpha \underbrace{\xi^\alpha_{;\mu} e^\mu_A} - N_\alpha \xi^\alpha_{;\beta} \underbrace{\xi^\beta_{;\mu} e^\mu_A}$$

$$\left(\frac{\partial k}{\partial \sigma^A}\right)_v = k_{;\nu} e^\nu_A = \left(-R_{\alpha\beta\mu\nu}\xi^\nu N^\alpha \xi^\beta e^\mu_A - k\,\xi_\alpha N^\alpha{}_{;\mu} e^\mu_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu_A \right)$$

$$= -R_{\mu\nu\alpha\beta} e^\mu_A \xi^\nu N^\alpha \xi^\beta + k N_\alpha \underbrace{\xi^\alpha{}_{;\mu} e^\mu_A} - N_\alpha \xi^\alpha{}_{;\beta} \underbrace{\xi^\beta{}_{;\mu} e^\mu_A}_{c^B \sigma_{AB}}$$

$$\left(\frac{\partial k}{\partial \sigma^A}\right)_V = k_{;\mu} e^\mu_A = \left(-R_{\alpha\rho\mu\nu}\xi^\nu N^\alpha \xi^\rho e^\mu_A - k\,\xi_\alpha N^\alpha{}_{;\mu} e^\mu_A - \xi_{\alpha;\rho} N^\alpha \xi^\rho{}_{;\mu} e^\mu_A\right)$$

$$= -R_{\mu\nu\alpha\rho} e^\mu_A \xi^\nu N^\alpha \xi^\rho + k N_\alpha \underbrace{\xi^\alpha{}_{;\mu} e^\mu_A} - N_\alpha \xi^\alpha{}_{;\rho} \underbrace{\xi^\rho{}_{;\mu} e^\mu_A}_{c^B \sigma_{AB} \xi^\rho}$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_v = k_{;\mu} e^\mu{}_A = \left(-R_{\alpha\rho\mu\sigma}\xi^\sigma N^\alpha \xi^\rho e^\mu{}_A - k\,\xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\rho} N^\alpha \xi^\rho{}_{;\mu} e^\mu{}_A\right)$$

$$= -R_{\mu\sigma\alpha\rho}\, e^\mu{}_A \xi^\sigma N^\alpha \xi^\rho + k N_\alpha \underbrace{\xi^\alpha{}_{;\mu} e^\mu{}_A} - N_\alpha \xi^\alpha{}_{;\rho} \underbrace{\xi^\rho{}_{;\mu} e^\mu{}_A}_{c^B \sigma_{AB} \xi^\rho = c_A \xi^\rho}$$

$$\left(\frac{\partial k}{\partial \sigma^A}\right)_v = k_{;\mu} e^\mu{}_A = \left(-R_{\alpha\beta\mu\nu} \xi^\nu N^\alpha \xi^\beta e^\mu{}_A - k\, \xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu{}_A \right)$$

$$= - R_{\mu\nu\alpha\beta} e^\mu{}_A \xi^\nu N^\alpha \xi^\beta + k N_\alpha \underbrace{\xi^\alpha{}_{;\mu} e^\mu{}_A}_{c_A \xi^\alpha} - N_\alpha \xi^\alpha{}_{;\beta} \underbrace{\xi^\beta{}_{;\mu} e^\mu{}_A}_{c^B \sigma_{AB} \xi^\beta = c_A \xi^\beta}$$

$$= - R_{\mu\nu\alpha\beta} e^\mu{}_A \xi^\nu N^\alpha \xi^\beta - \cancel{c_A k} - c_A \cancel{k \xi^\alpha N_\alpha}$$

$$\left(\frac{\partial k}{\partial \sigma^A}\right)_V = k_{;\mu} e^\mu_A = \left(-R_{\alpha\beta\mu\sigma} \xi^\sigma N^\alpha \xi^\beta e^\mu_A - k\, \xi_\alpha N^\alpha_{;\mu} e^\mu_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} e^\mu_A\right.$$

$$= -R_{\mu\sigma\alpha\beta} e^\mu_A \xi^\sigma N^\alpha \xi^\beta + k N_\alpha \underbrace{\xi^\alpha_{;\mu} e^\mu_A}_{c_A \xi^\alpha} - N_\alpha \xi^\alpha_{;\beta} \underbrace{\xi^\beta_{;\mu} e^\mu_A}_{c^B \sigma_{AB} \xi^\beta = c_A \xi^\beta}$$

$$= -R_{\mu\sigma\alpha\beta} e^\mu_A \xi^\sigma N^\alpha \xi^\beta - \cancel{c_A k} - c_A \cancel{k\, \xi^\alpha} N_\alpha$$

$$\boxed{\left(\frac{\partial k}{\partial \sigma^A}\right)_V = -R_{\mu\sigma\alpha\beta} e^\mu_A \xi^\sigma N^\alpha \xi^\beta}$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{;\nu} e^{\nu}_{A} = \left(-R_{\alpha\beta\mu\nu}\xi^{\nu} N^{\alpha} \xi^{\beta} e^{\mu}_{A} - k\,\xi_{\alpha} N^{\alpha}_{\ ;\mu} e^{\mu}_{A} - \xi_{\alpha;\beta} N^{\alpha} \xi^{\beta}_{\ ;\mu} e^{\mu}_{A}\right)$$

$$= -R_{\mu\nu\alpha\beta} e^{\mu}_{A} \xi^{\nu} N^{\alpha} \xi^{\beta} + k N_{\alpha} \underbrace{\xi^{\alpha}_{\ ;\mu} e^{\mu}_{A}}_{c_A \xi^{\alpha}} - N_{\alpha} \xi^{\alpha}_{\ ;\beta} \underbrace{\xi^{\beta}_{\ ;\mu} e^{\mu}_{A}}_{c^B \sigma_{AB} \xi^{\beta} = c_A \xi^{\beta}}$$

$$= -R_{\mu\nu\alpha\beta} e^{\mu}_{A} \xi^{\nu} N^{\alpha} \xi^{\beta} - \cancel{c_A k} - c_A \cancel{k\, \xi^{\alpha} N_{\alpha}}$$

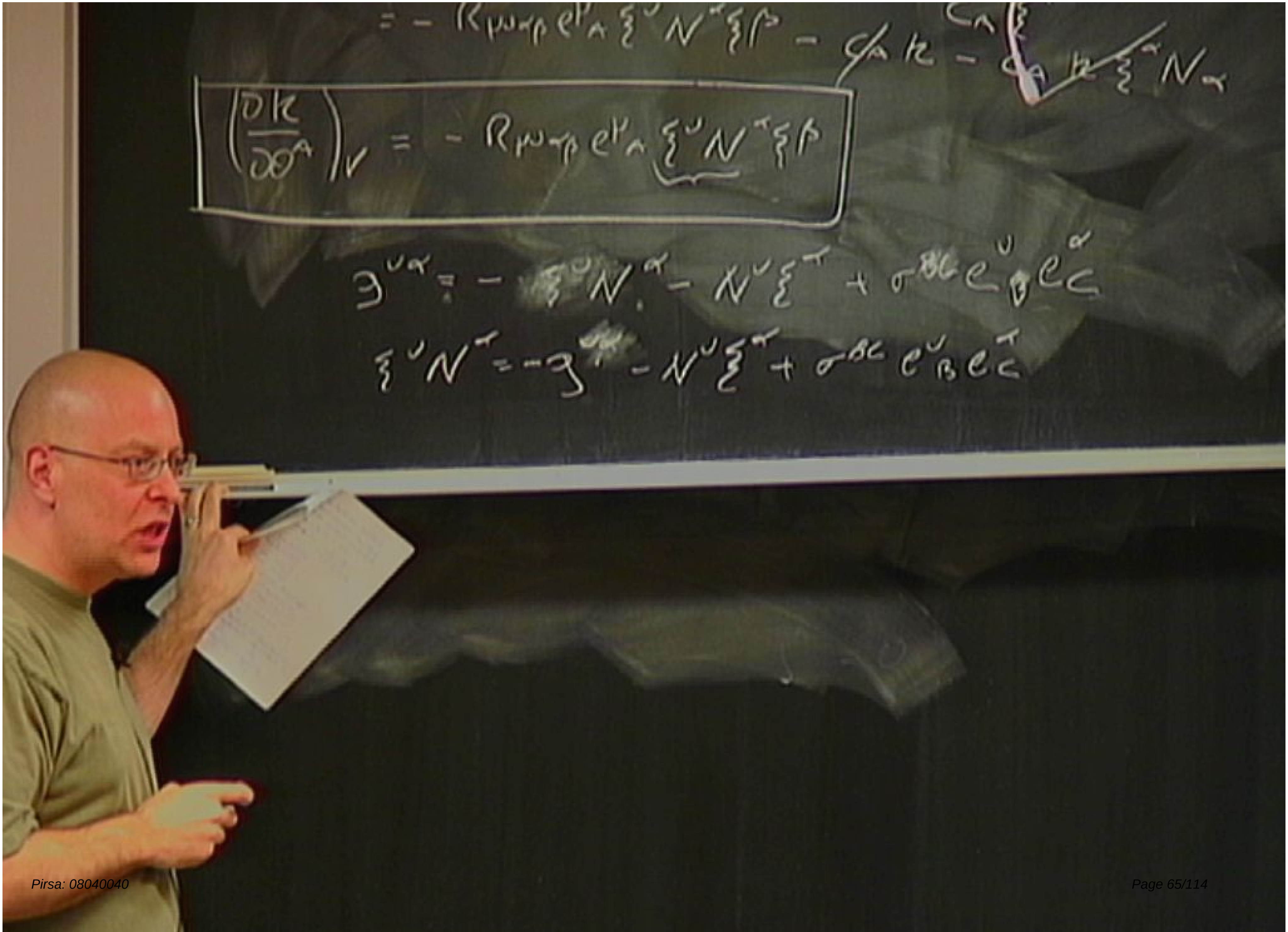
$$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_V = -R_{\mu\nu\alpha\beta} e^{\mu}_{A} \xi^{\nu} N^{\alpha} \xi^{\beta}}$$

$$= -R_{\mu\nu\alpha\beta}\, e^{\mu}{}_{A}\, \xi^{\nu} N^{\alpha} \xi^{\beta} + k N_{\alpha}\, \underbrace{\xi^{\alpha}{}_{;\mu}\, e^{\mu}{}_{A}}_{c_A \xi^{\alpha}} -$$

$$= -R_{\mu\nu\alpha\beta}\, e^{\mu}{}_{A}\, \xi^{\nu} N^{\alpha} \xi^{\beta} - \cancel{c_A k} - c_A \cancel{k}$$

$$\boxed{\left(\frac{\partial k}{\partial \theta^{A}}\right)_{v} = -R_{\mu\nu\alpha\beta}\, e^{\mu}{}_{A}\, \xi^{\nu} N^{\alpha} \xi^{\beta}}$$

$$g^{\nu\alpha} = -\xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B}\, e^{\alpha}{}_{C}$$

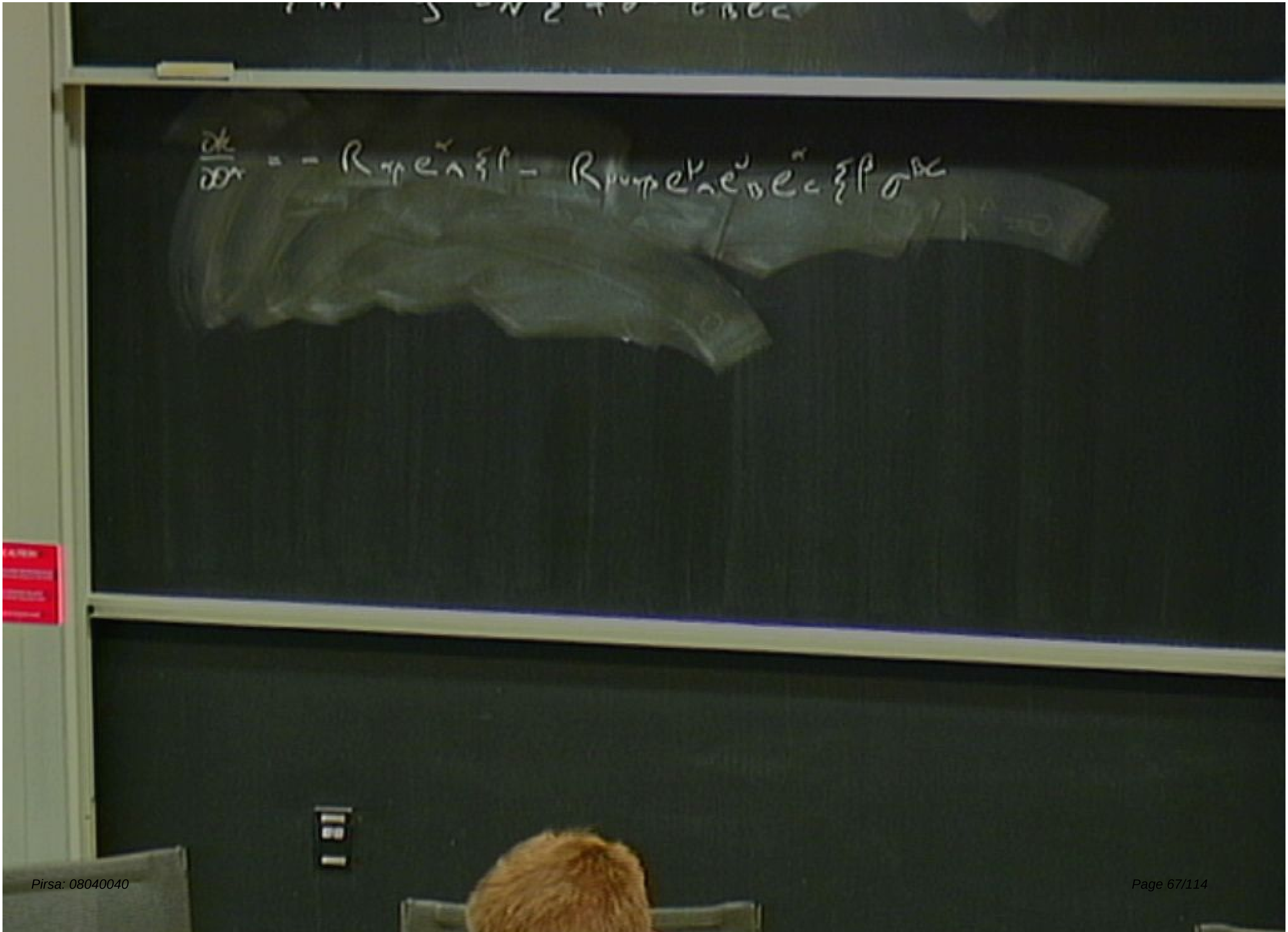

$= - R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} \xi^{\nu} N^{\alpha} \xi^{\beta} - \ldots$
$\boxed{\left(\frac{\partial k}{\partial \theta^{A}}\right)_{V} = - R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} \underbrace{\xi^{\nu} N^{\alpha}} \xi^{\beta}}$
$g^{\nu\alpha} = - \xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$
$\xi^{\nu} N^{\alpha} = - g^{\nu\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$

$$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_V = -R_{\mu\nu\alpha\beta}\, e^\mu{}_A\, \underbrace{\xi^\nu N^\alpha}\, \xi^\beta}$$

$$g^{\nu\alpha} = -\xi^\nu N^\alpha - N^\nu \xi^\alpha + \sigma^{BC} e^\nu{}_B e^\alpha{}_C$$

$$\xi^\nu N^\alpha = -g^{\nu\alpha} - N^\nu \xi^\alpha + \sigma^{BC} e^\nu{}_B e^\alpha{}_C$$

$$\frac{\partial k}{\partial \theta^A} = -R_{\alpha\beta} e^\alpha{}_A \xi^\beta - R_{\mu\nu\alpha\beta} e^\mu{}_A e^\nu{}_B e^\alpha{}_C \xi^\beta$$


$\frac{\partial k}{\partial x^\mu} = - R_{\mu\rho} e^\alpha{}_A \xi^\rho - R_{\nu\mu\rho} e^\nu{}_A e^\mu{}_B e^\alpha{}_C \xi^\rho \sigma^{BC}$

$$= - R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} \xi^{\nu} N^{\alpha} \xi^{\beta} - \cancel{c_A k} - c_A \cancel{k \xi^{\alpha} N_{\alpha}}$$

$$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_V = - R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} \underbrace{\xi^{\nu} N^{\alpha}} \xi^{\beta}}$$

$$g^{\nu\alpha} = - \xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$$

$$\xi^{\nu} N^{\alpha} = - g^{\nu\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$$

$$\frac{\partial k}{\partial \theta^A} = - R_{\alpha\beta} e^{\alpha}{}_{A} \xi^{\beta} - R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} e^{\nu}{}_{B} e^{\alpha}{}_{C} \xi^{\beta} \sigma^{BC}$$

$$\boxed{\frac{\partial K}{\partial \sigma^A} = - R_{\alpha\beta} e^\alpha_A \xi^\beta - R_{\mu\nu\alpha\beta} e^\mu_A e^\nu_B e^\alpha_C \xi^\beta \sigma^{BC}}$$

Recall: $\xi_{\alpha;\beta} e^\alpha_A e^\beta_B = 0$

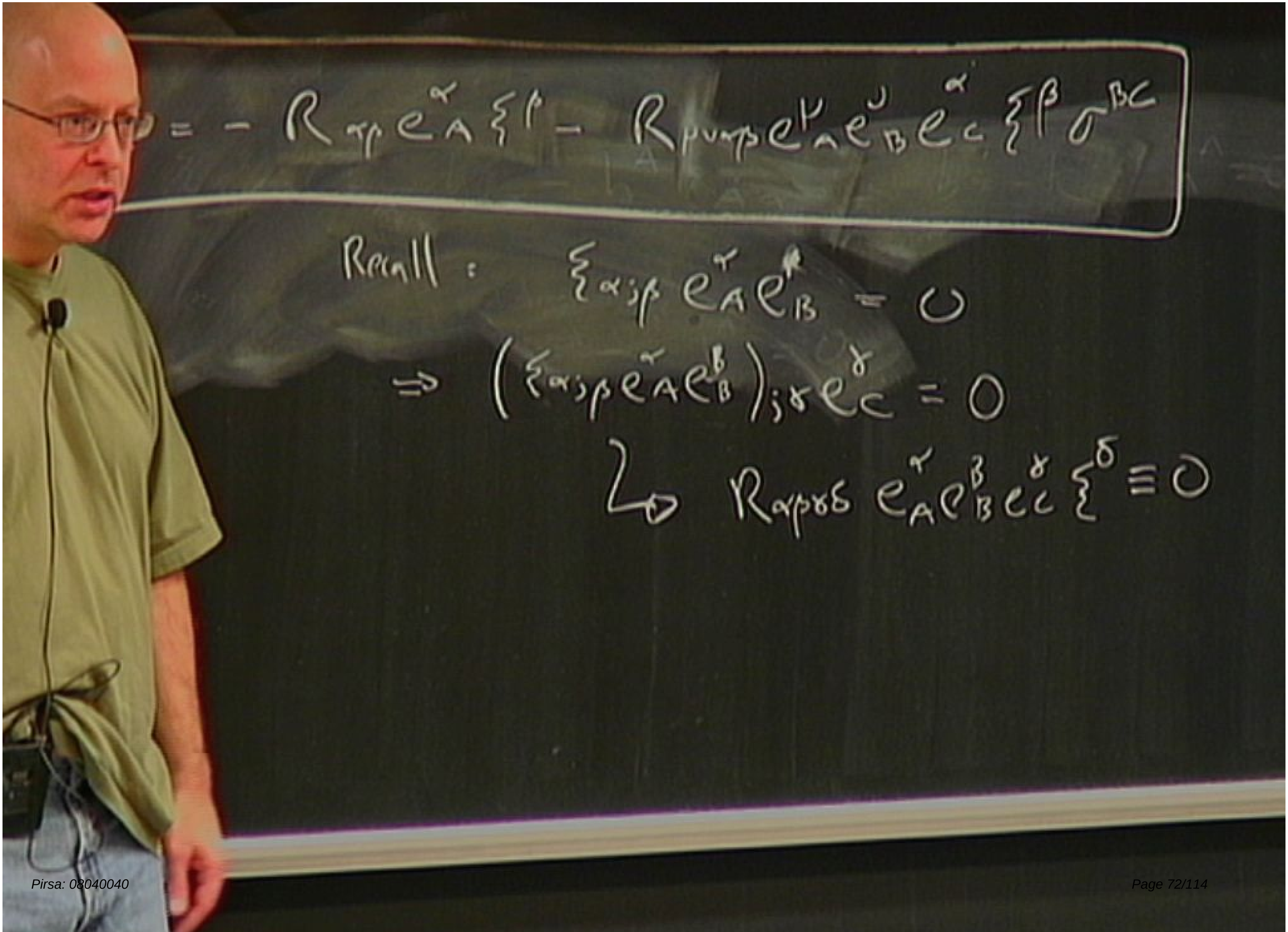
$$\boxed{\frac{\partial H}{\partial \sigma^A} = - R_{\alpha\beta} e^{\alpha}_{A} \xi^{\beta} - R_{\mu\nu\alpha\beta} e^{\mu}_{A} e^{\nu}_{B} e^{\alpha}_{C} \xi^{\beta} \sigma^{BC}}$$

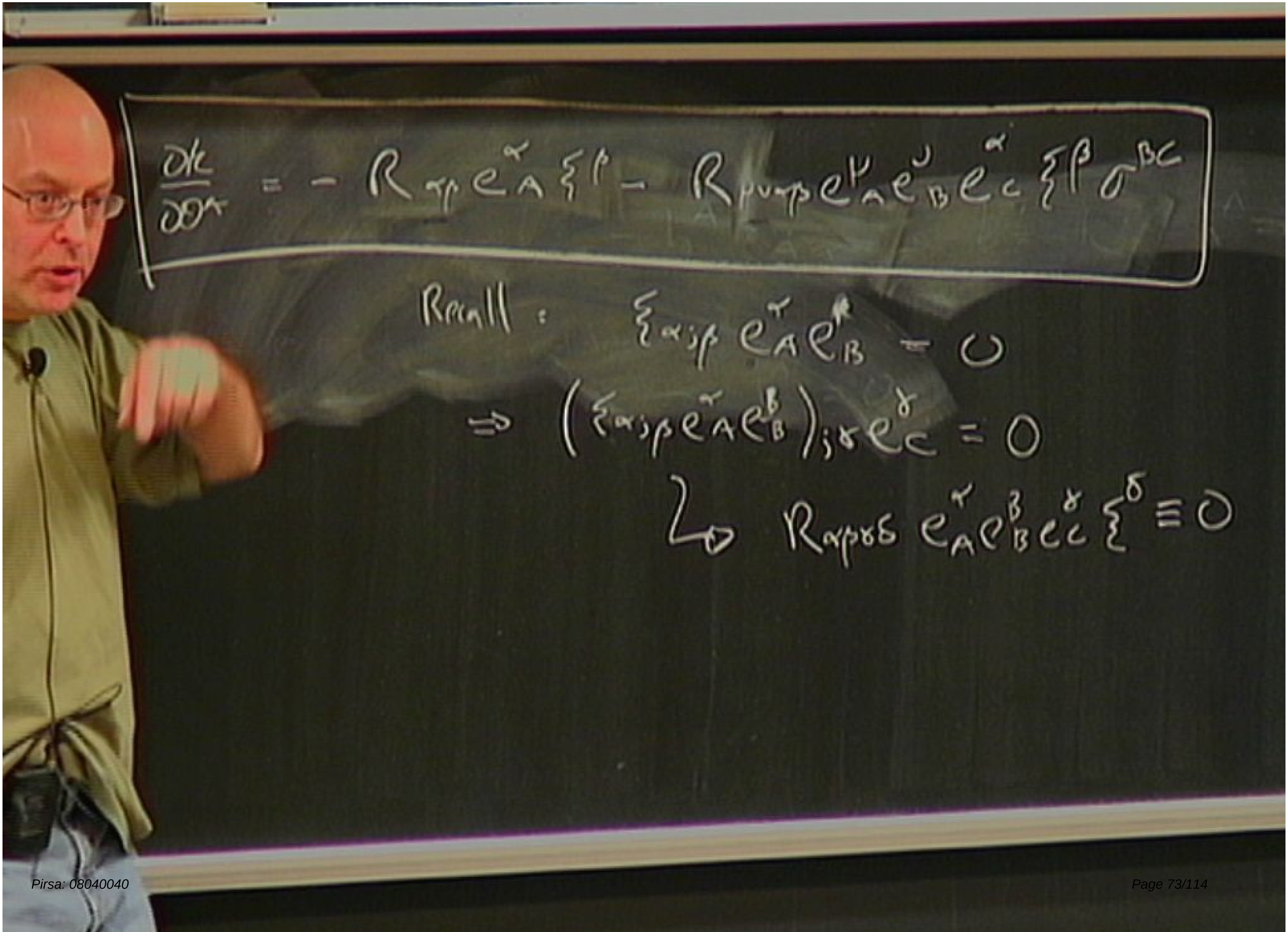
Recall: $\xi_{\alpha;\beta} e^{\alpha}_{A} e^{\beta}_{B} = 0$

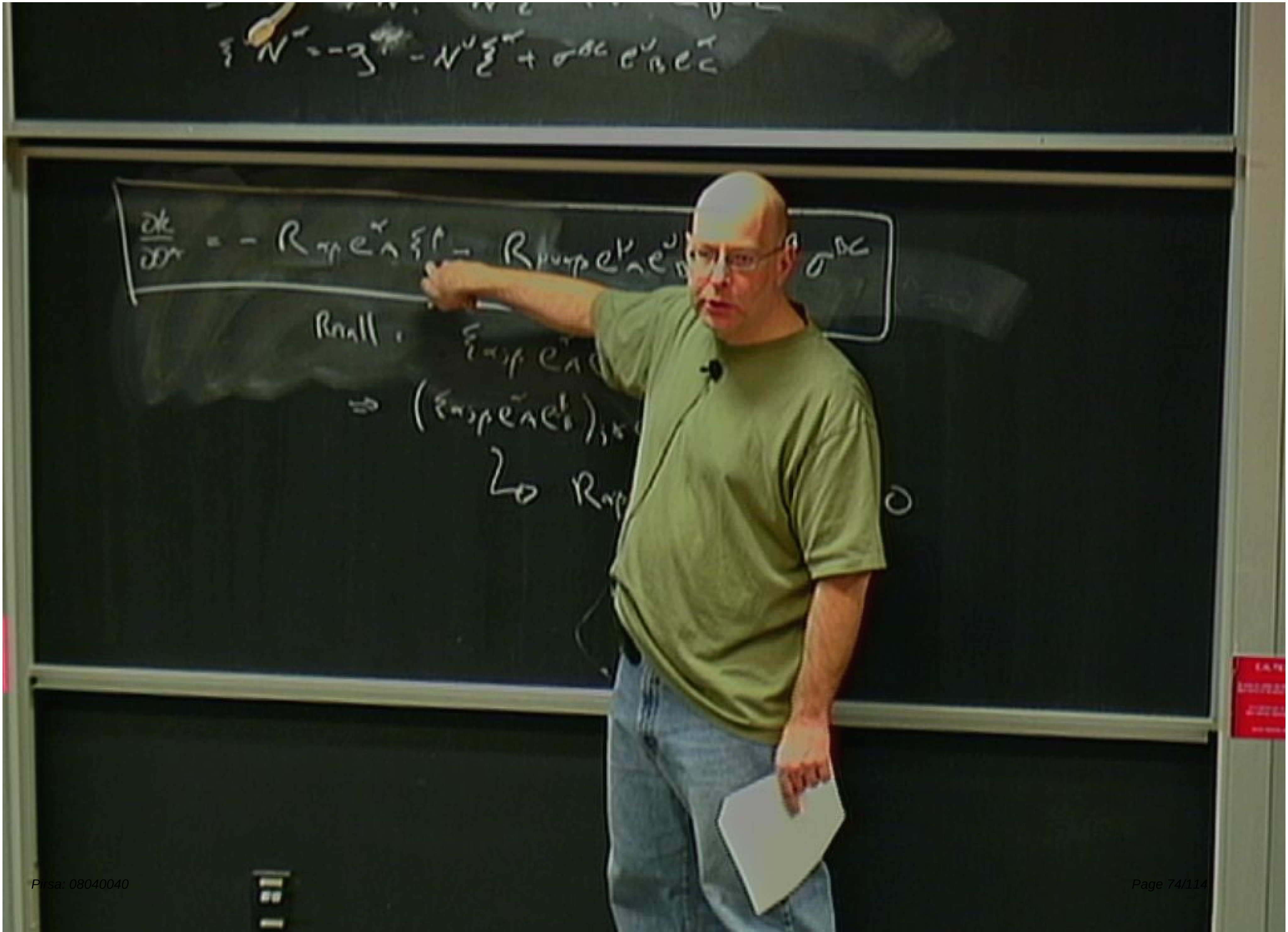
$$\boxed{\frac{\partial k}{\partial \sigma^A} = -R_{\alpha\beta}\, e^\alpha_A\, \xi^\beta - R_{\mu\nu\alpha\beta}\, e^\mu_A\, e^\nu_B\, e^\alpha_C\, \xi^\beta\, \sigma^{BC}}$$

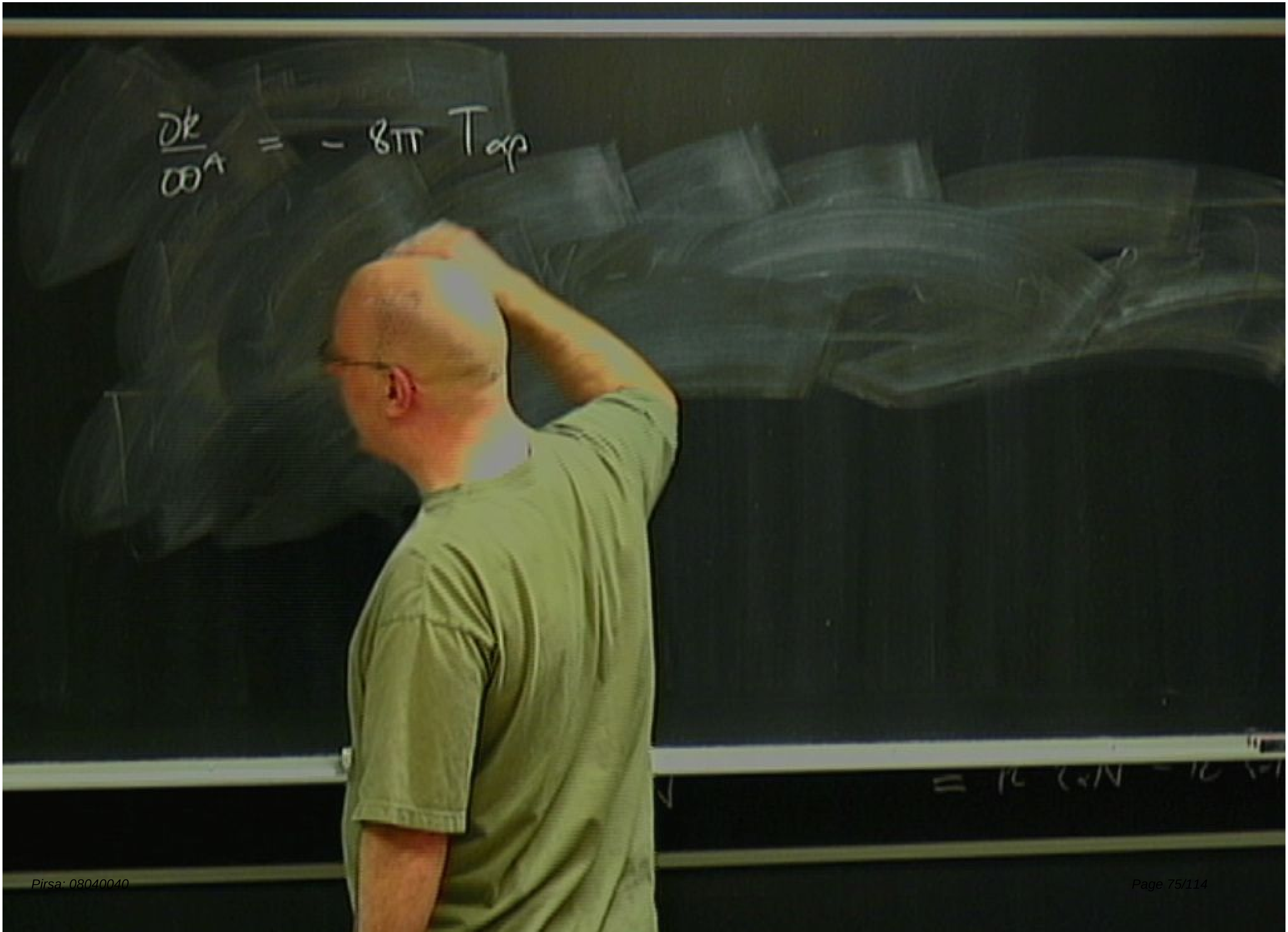
Recall : $\xi_{\alpha;\beta}\, e^\alpha_A e^\beta_B = 0$

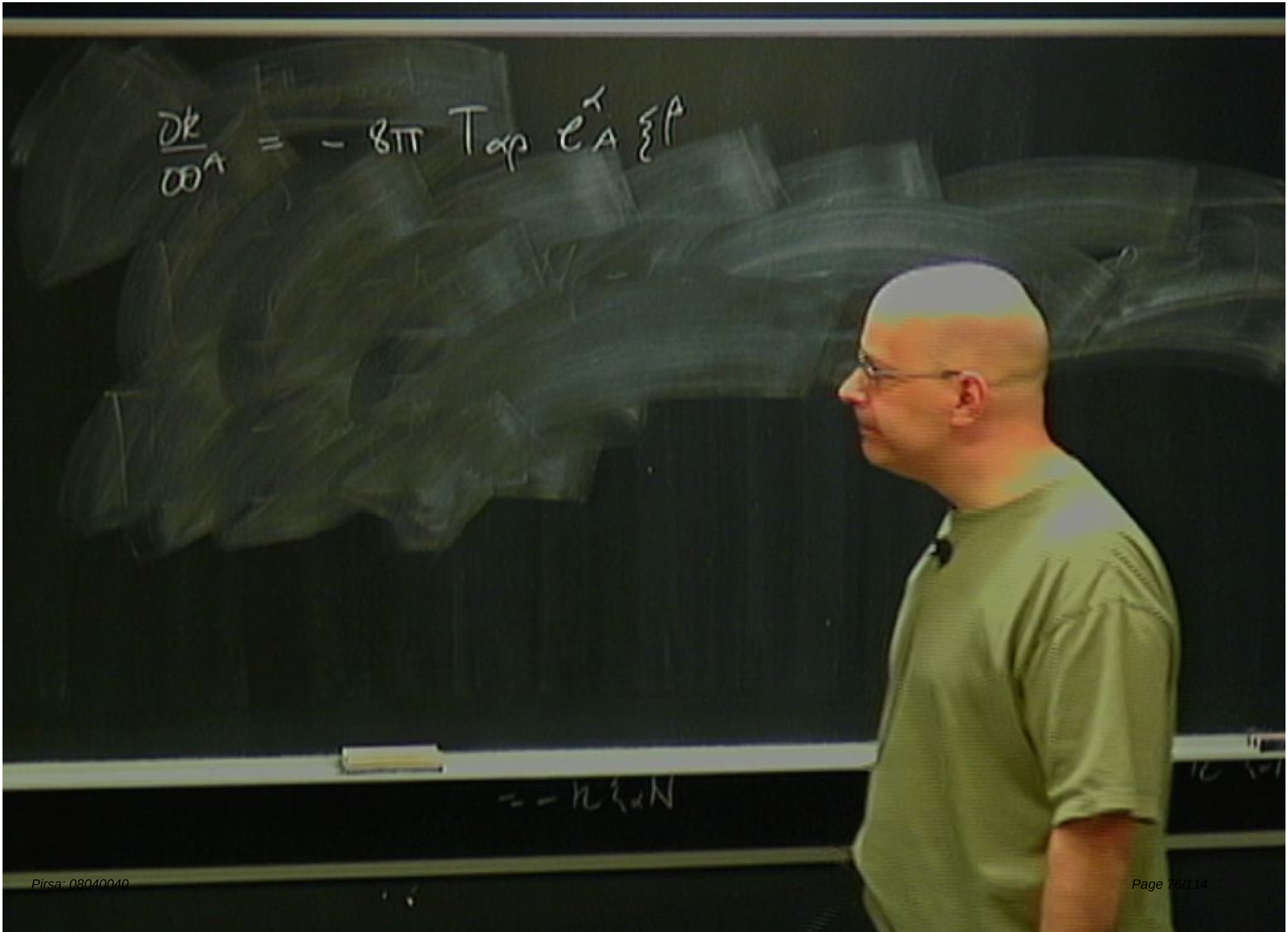
$$\Rightarrow \left(\xi_{\alpha;\beta}\, e^\alpha_A e^\beta_B\right)_{;\gamma} e^\gamma_C = 0$$

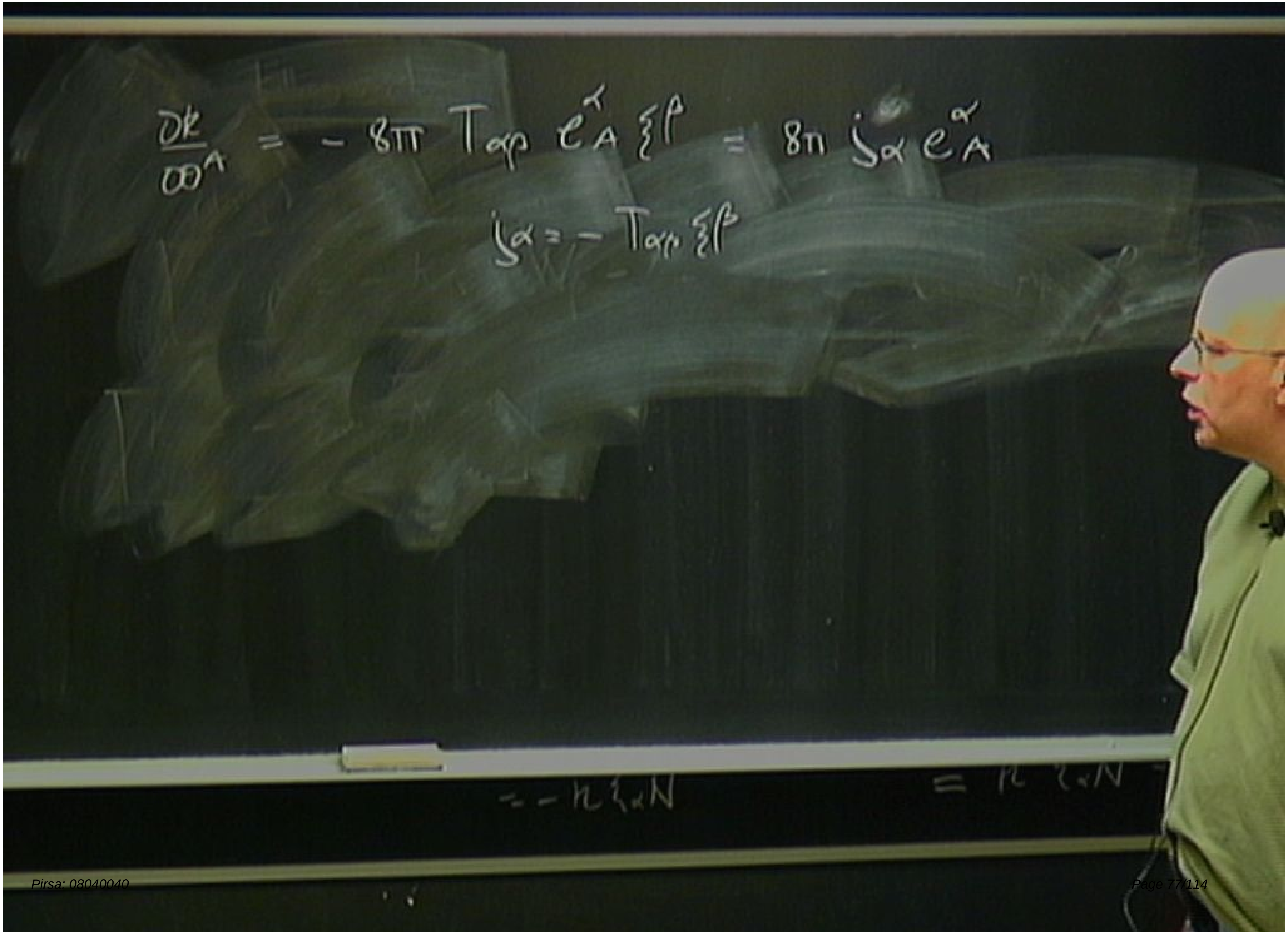

$= - R_{\alpha\beta} e^{\alpha}_{A} \xi^{\beta} - R_{\mu\nu\alpha\beta} e^{\mu}_{A} e^{\nu}_{B} e^{\alpha}_{C} \xi^{\beta} \sigma^{BC}$
Recall : $\xi_{\alpha;\beta} e^{\alpha}_{A} e^{\beta}_{B} = 0$
$\Rightarrow \left(\xi_{\alpha;\beta} e^{\alpha}_{A} e^{\beta}_{B} \right)_{;\gamma} e^{\gamma}_{C} = 0$
$R_{\alpha\beta\gamma\delta} e^{\alpha}_{A} e^{\beta}_{B} e^{\gamma}_{C} \xi^{\delta} \equiv 0$

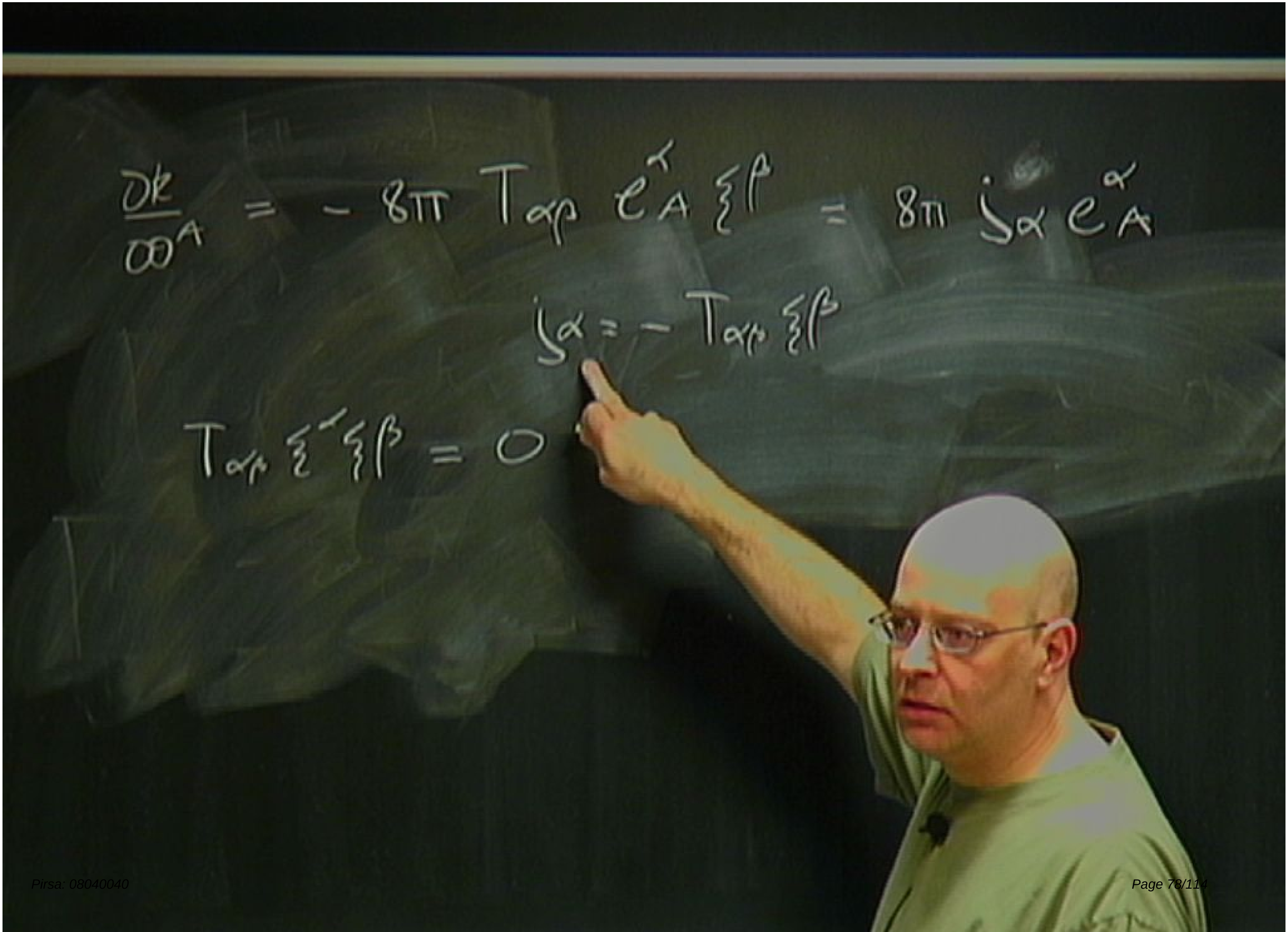

$\frac{\partial K}{\partial \theta^A} = -R_{\alpha\beta} e^\alpha_A \xi^\beta - R_{\mu\nu\alpha\beta} e^\mu_A e^\nu_B e^\alpha_C \xi^\beta \sigma^{BC}$
Recall: $\xi_{\alpha;\beta} e^\alpha_A e^\beta_B = 0$
$\Rightarrow (\xi_{\alpha;\beta} e^\alpha_A e^\beta_B)_{;\gamma} e^\gamma_C = 0$
$\hookrightarrow R_{\alpha\beta\gamma\delta} e^\alpha_A e^\beta_B e^\gamma_C \xi^\delta \equiv 0$




$\frac{\partial k}{\partial \theta^A} = -8\pi T_{\alpha\beta}$


$\frac{\partial k}{\partial \theta^A} = -8\pi T_{\alpha\beta} e^\alpha_A \xi^\beta$
$= -h \xi_\alpha N$


$\frac{\partial \kappa}{\partial \sigma^A} = -8\pi T_{\alpha\beta} e^\alpha_A \xi^\beta = 8\pi j_\alpha e^\alpha_A$
$j_\alpha = -T_{\alpha\beta}\xi^\beta$
$= -\kappa \xi_\alpha N$
$= \kappa \xi_\alpha N$


$\frac{\partial \kappa}{\partial \theta^A} = -8\pi T_{\alpha\beta} e^\alpha_A \xi^\beta = 8\pi j_\alpha e^\alpha_A$
$j_\alpha = -T_{\alpha\beta}\xi^\beta$
$T_{\alpha\beta}\xi^\alpha\xi^\beta = 0$

$$\frac{\partial k}{\partial \Theta^A} = -8\pi\, T_{\alpha\beta}\, e^\alpha_A\, \xi^\beta = 8\pi\, j_\alpha\, e^\alpha_A$$

$$j_\alpha = -T_{\alpha\beta}\,\xi^\beta$$

$$T_{\alpha\beta}\,\xi^\alpha\xi^\beta = 0 \Rightarrow j_\alpha\,\xi^\alpha = 0.$$

$$\frac{\partial K}{\partial \theta^A} = -8\pi \, T_{\alpha\beta} \, e^{\alpha}_{A} \, \xi^{\beta} = 8\pi \, j_{\alpha} \, e^{\alpha}_{A}$$

$$j_{\alpha} = -T_{\alpha\beta} \, \xi^{\beta}$$

$$T_{\alpha\beta} \, \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow j_{\alpha} \, \xi^{\alpha} = 0.$$

$$\Rightarrow j^{\alpha} = a \, \xi^{\alpha} + b^{A} e^{\alpha}_{A}$$

$$\frac{\partial k}{\partial \theta^A} = -8\pi T_{\alpha\beta} e^{\alpha}_{A} \xi^{\beta} = 8\pi j_\alpha e^{\alpha}_{A}$$

$$j_\alpha = -T_{\alpha\beta}\xi^\beta$$

$$T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \Rightarrow j_\alpha \xi^\alpha = 0.$$

$$\Rightarrow j^\alpha = a\xi^\alpha + b^A e^{\alpha}_{A}$$

Dominant energy condition: j^α must be timelike or null.

$$\frac{\partial k}{\partial \omega^A} = -8\pi\, T_{\alpha\beta}\, e_A^\alpha\, \xi^\beta = 8\pi\, j_\alpha\, e_A^\alpha$$

$$j_\alpha = -T_{\alpha\beta}\,\xi^\beta$$

$$T_{\alpha\beta}\,\xi^\alpha\xi^\beta = 0 \Rightarrow j_\alpha\,\xi^\alpha = 0.$$

$$\Rightarrow j^\alpha = a\,\xi^\alpha + b^A\, e_A^\alpha$$

Dominant energy condition: j^α must be timelike or null.

$$\Rightarrow b^A =$$

$$\frac{\partial \kappa}{\partial \theta^A} = -8\pi\, T_{\alpha\beta}\, e^{\alpha}_{A}\, \xi^{\beta} = 8\pi\, j_\alpha e^{\alpha}_A$$

$$j_\alpha = -T_{\alpha\beta}\,\xi^\beta$$

$$T_{\alpha\beta}\,\xi^\alpha \xi^\beta = 0 \Rightarrow j_\alpha \xi^\alpha = 0.$$

$$\Rightarrow j^\alpha = a\,\xi^\alpha + b^A e^{\alpha}_{A}$$

Dominant energy condition: j^α must be timelike or null.

$$\Rightarrow b^A = 0$$

$$= \left[-\mathcal{L}_{\alpha\beta\gamma\delta}\, \xi \, N \, \xi \, \xi - \xi_{\alpha;\beta}\, \xi^\beta N_{;\gamma}\, \xi^\gamma - \xi_\alpha \ldots \right.$$

$$\frac{\partial k}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^\alpha_A \xi^\beta = 8\pi j_\alpha e^\alpha_A$$

$$j_\alpha = -T_{\alpha\beta}\xi^\beta$$

$$T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \Rightarrow j_\alpha \xi^\alpha = 0.$$

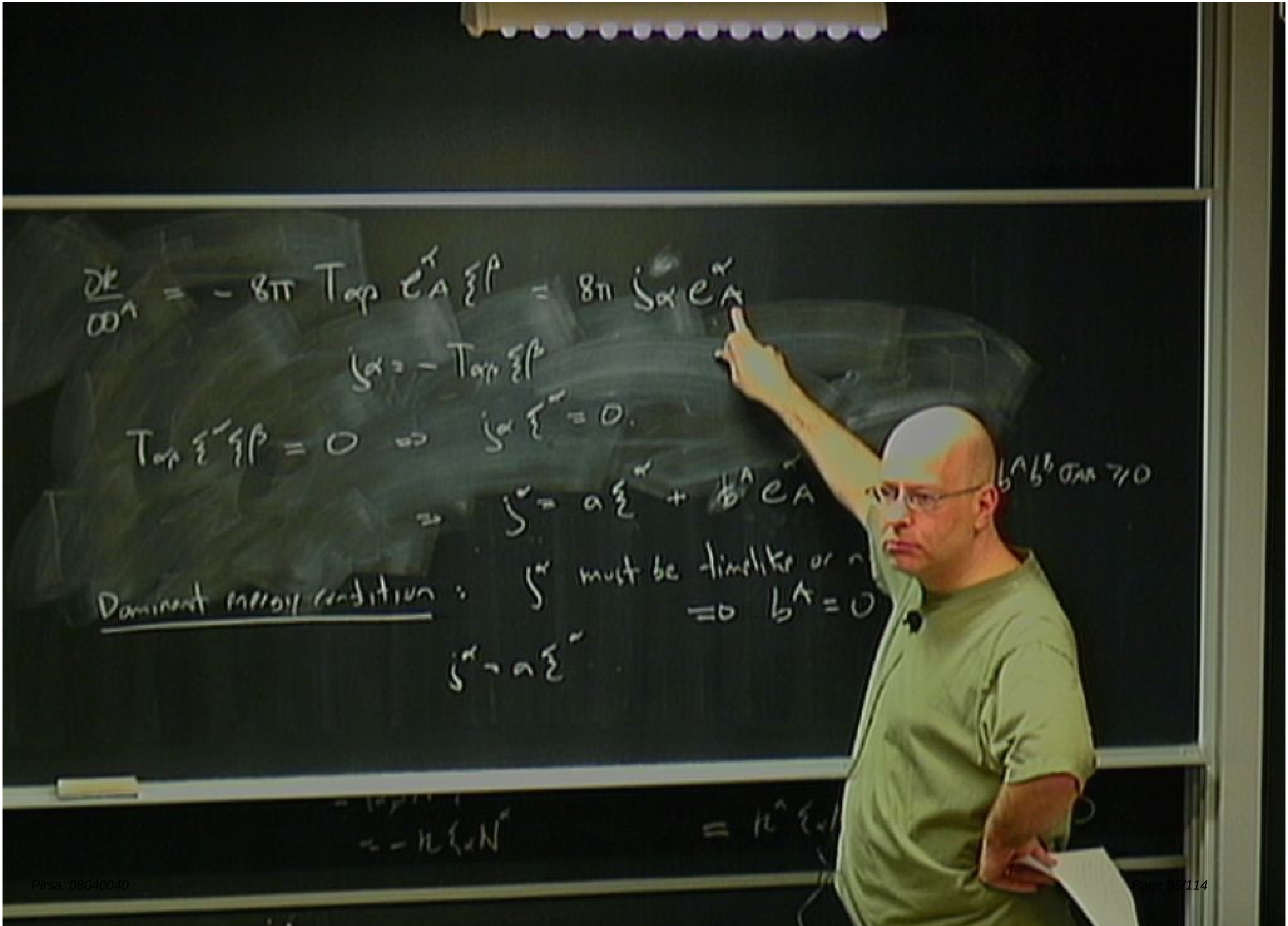
$$\Rightarrow j^\alpha = a\xi^\alpha + b^A e^\alpha_A \qquad j_\alpha j^\alpha = b^A b^B \sigma_{AB} \geq 0$$

Dominant energy condition: j^α must be timelike or null.

$$\Rightarrow b^A = 0$$

$$= \left(\cancel{-R_{\alpha\beta\gamma\mu}\xi^\gamma} N^\alpha \xi^\beta \xi^\mu - \underbrace{\xi_{\alpha;\beta}\xi^\beta}_{k\xi_\alpha} N^\alpha_{;\mu}\xi^\mu - \xi_{\alpha;\beta} N^\alpha \underbrace{\xi^\beta_{;\mu}\xi^\mu}_{k\xi^\beta} \right)$$

$$= -k\,\xi_\alpha N^\alpha_{;\mu}\xi^\mu - k\,\xi_{\alpha;\beta}\xi^\beta N^\alpha$$


$\frac{\partial \kappa}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^\alpha_A \xi^\beta = 8\pi j_\alpha e^\alpha_A$
$j_\alpha = -T_{\alpha\beta}\xi^\beta$
$T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \Rightarrow j_\alpha \xi^\alpha = 0.$
$\Rightarrow j^\alpha = a\xi^\alpha + b^A e^\alpha_A$
$b^A b^B \sigma_{AB} \geq 0$
Dominant energy condition: j^α must be timelike or n
$\Rightarrow b^A = 0$
$j^\alpha = a\xi^\alpha$
$= -\kappa \xi_\alpha N^\alpha$

$$\Rightarrow \quad \xi^\alpha = a\xi^\alpha + b^A e_A^\alpha$$

<u>Dominant energy condition</u> : j^α must be timelike or null.

$$\Rightarrow \quad b^A = 0$$

$$j^\alpha = a\xi^\alpha$$

$$\Rightarrow \boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_V = 0}$$

$$\left(\frac{\partial k}{\partial V}\right)_{\theta^A} = k_{;\nu}\xi^\nu = \left(-\xi_{\alpha;\rho\nu} N^\alpha \xi^\rho - \xi_{\alpha;\rho} N^\alpha_{\ ;\nu}\xi^\rho - \xi_{\alpha;\ldots}\right.$$

$$= \left(-\cancel{R_{\alpha\rho\rho\nu}\xi^\nu} N^\alpha \xi^\rho \xi^\nu - \underbrace{\xi_{\alpha;\rho}\xi^\rho}_{k\xi_\alpha} N^\alpha_{\ ;\nu}\xi^\nu - \xi_{\alpha;\rho} N^\alpha \ldots\right.$$

$$= -k\,\underbrace{\xi_\alpha N^\alpha_{\ ;\nu}\xi^\nu}_{\ldots} - k\,\underbrace{\xi_{\alpha;\rho}\xi^\rho}_{k\xi_\alpha} N^\alpha$$

First law

During a quasi-static process :

First law

During a quasi-static process : $\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J$

First law

During a quasi-static process:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$$

First law

During a quasi-static process:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$$

infinitesimal Top

First law

During a quasi-static process :

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$$

infinitesimal Tap

$$\delta M = -\int_H T^{\alpha}{}_{\beta}\, t^{\beta}\, d\Sigma_{\alpha}$$

First law

During a quasi-static process:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$$

infinitesimal $T_{\alpha\beta}$

$$\delta M = -\int_H T^{\alpha}_{\ \beta}\, t^{\beta}\, d\Sigma_{\alpha}$$

$$\delta J = \int_H T^{\alpha}_{\ \beta}\, \phi^{\beta}\, d\Sigma_{\alpha}$$

First law

During a quasi-static process :

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$$

infinitesimal $T_{\alpha\beta}$

$$\delta M = -\int_H T^\alpha_{\ \beta}\, t^\beta\, d\Sigma_\alpha$$

$$\delta J = \int_H T^\alpha_{\ \beta}\, \phi^\beta\, d\Sigma_\alpha$$

$$\delta M - \Omega_H\,\delta J = -\int_H T^\alpha_{\ \beta}\left(t^\beta + \Omega_H\,\phi^\beta\right) d\Sigma_\alpha$$

infinitesimal $T_{\alpha\beta}$

$$\delta M = -\int_H T^\alpha_{\ \beta}\, t^\beta\, d\Sigma_\alpha$$

$$\delta J = \int_H T^\alpha_{\ \beta}\, \phi^\beta\, d\Sigma_\alpha$$

$$\delta M - \Omega_H \delta J = -\int_H T^\alpha_{\ \beta} \big(\underbrace{t^\beta + \Omega_H \phi^\beta}_{\xi^\beta} \big) \underbrace{d\Sigma_\alpha}_{-\xi_\alpha\, dS\, dV}$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta}\, \xi^\alpha \xi^\beta\, dS\, dV$$

$$\delta M - \Omega_H \, \delta J = \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \, dS \, dV$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{8\pi} \left(\kappa \theta - \frac{d\theta}{dV} \right)$$

$$= \frac{\kappa}{8\pi} \int_H \theta \, dS \, dV - \frac{1}{8\pi} \int_H \frac{d\theta}{dV} \, dS \, dV$$

$$\delta M - \Omega_H \, \delta J = \int_H T_{\alpha\beta}\, \xi^\alpha \xi^\beta \, dS \, dV$$

$$T_{\alpha\beta}\, \xi^\alpha \xi^\beta = \frac{1}{8\pi}\left(k\theta - \frac{d\theta}{dV} \right)$$

$$= \frac{\kappa}{8\pi} \int_H \theta \, dS \, dV - \frac{1}{8\pi} \int_H \frac{d\theta}{dV} \, dS \, dV$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta}\, \xi^\alpha \xi^\beta \, dS\, dV$$

$$T_{\alpha\beta}\, \xi^\alpha \xi^\beta = \frac{1}{8\pi}\left(\kappa\theta - \frac{\partial \theta}{\partial V} \right)$$

$$= \frac{\kappa}{8\pi} \int_H \theta \, dS\, dV - \frac{1}{8\pi} \underbrace{\int_H \frac{\partial \theta}{\partial V}\, dS\, dV}_{\int \theta\, dS \Big|_{V=-\infty}^{V=\infty}}$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta}\xi^\alpha \xi^\beta \, dS \, dV$$

$$T_{\alpha\beta}\xi^\alpha\xi^\beta = \frac{1}{8\pi}\left(\kappa\theta - \frac{d\theta}{dV}\right)$$

$$= \frac{\kappa}{8\pi}\int_H \underbrace{\theta \, dS \, dV}_{\frac{d}{dV} dS} - \frac{1}{8\pi}\cancel{\underbrace{\int_H \frac{d\theta}{dV} dS \, dV}_{\int \theta \, dS \Big|_{V=-\infty}^{V=\infty}}}$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \, dS \, dV$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{8\pi} \left(\kappa\theta - \frac{d\theta}{dV} \right)$$

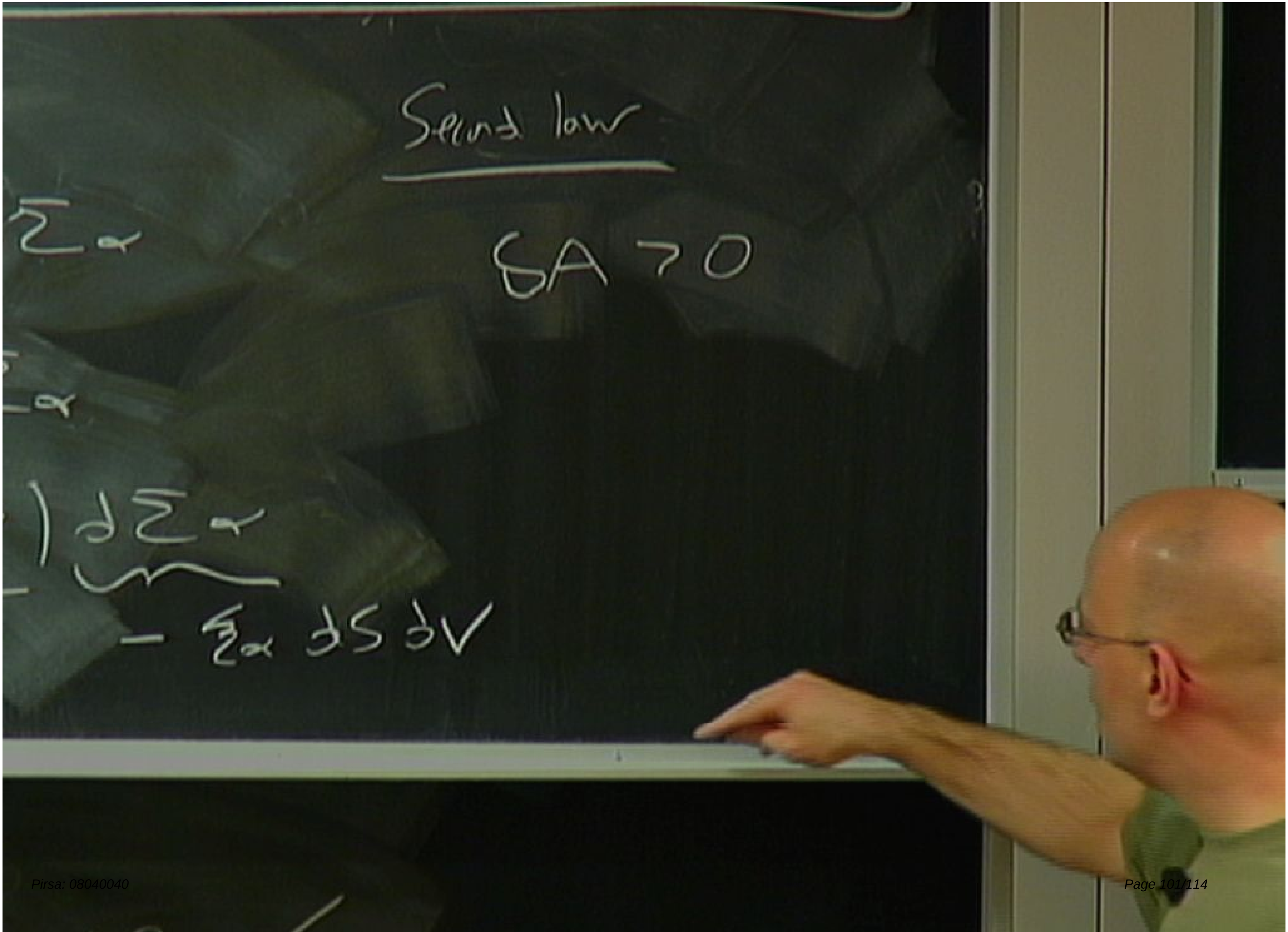
$$= \frac{\kappa}{8\pi} \int_H \underbrace{\theta \, dS}_{\frac{d}{dV} dS} \, dV - \frac{1}{8\pi} \underbrace{\cancel{\int_H \frac{d\theta}{dV} \, dS \, dV}}_{\int \theta \, dS \Big|_{V=-\infty}^{V=\infty}}$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta}\xi^\alpha \xi^\beta \, dS \, dV$$

$$T_{\alpha\beta}\xi^\alpha\xi^\beta = \frac{1}{8\pi}\left(\kappa\theta - \frac{d\theta}{dV} \right)$$

$$= \frac{\kappa}{8\pi}\int_H \underbrace{\theta \, dS \, dV}_{\frac{d}{dV} dS} - \frac{1}{8\pi}\cancel{\int_H \frac{d\theta}{dV} \, dS \, dV} \qquad \int \theta \, dS \Big|_{V=-\infty}^{V=\infty}$$

$$= \frac{\kappa}{8\pi}\int_H dS \Big|_{V=-\infty}^{V=\infty}$$


Second law
δA > 0
) ∂Σα
− ξα ∂S ∂V

Second law

$$\boxed{\delta A \geq 0}$$

Σ_α

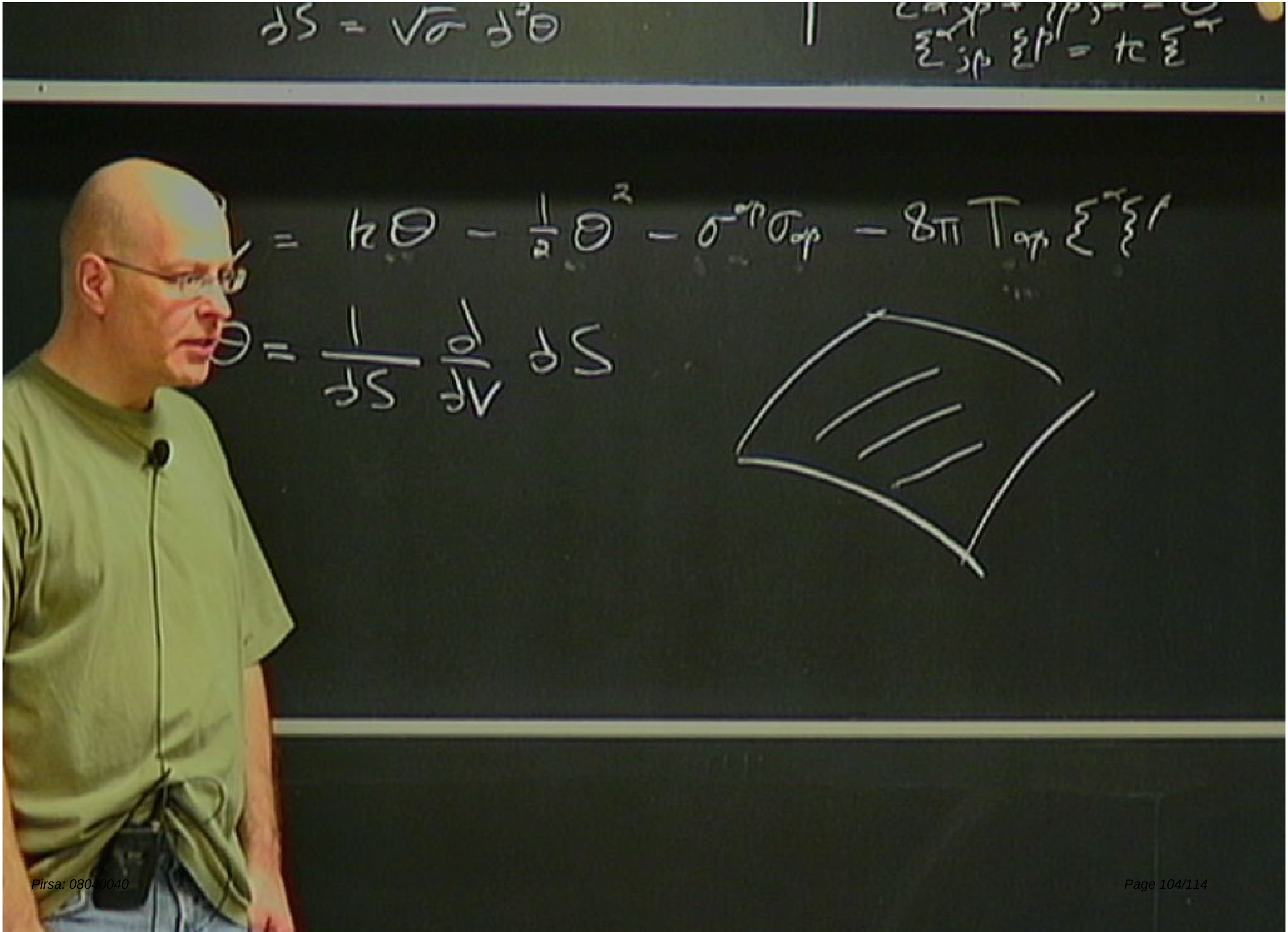
Σ_α

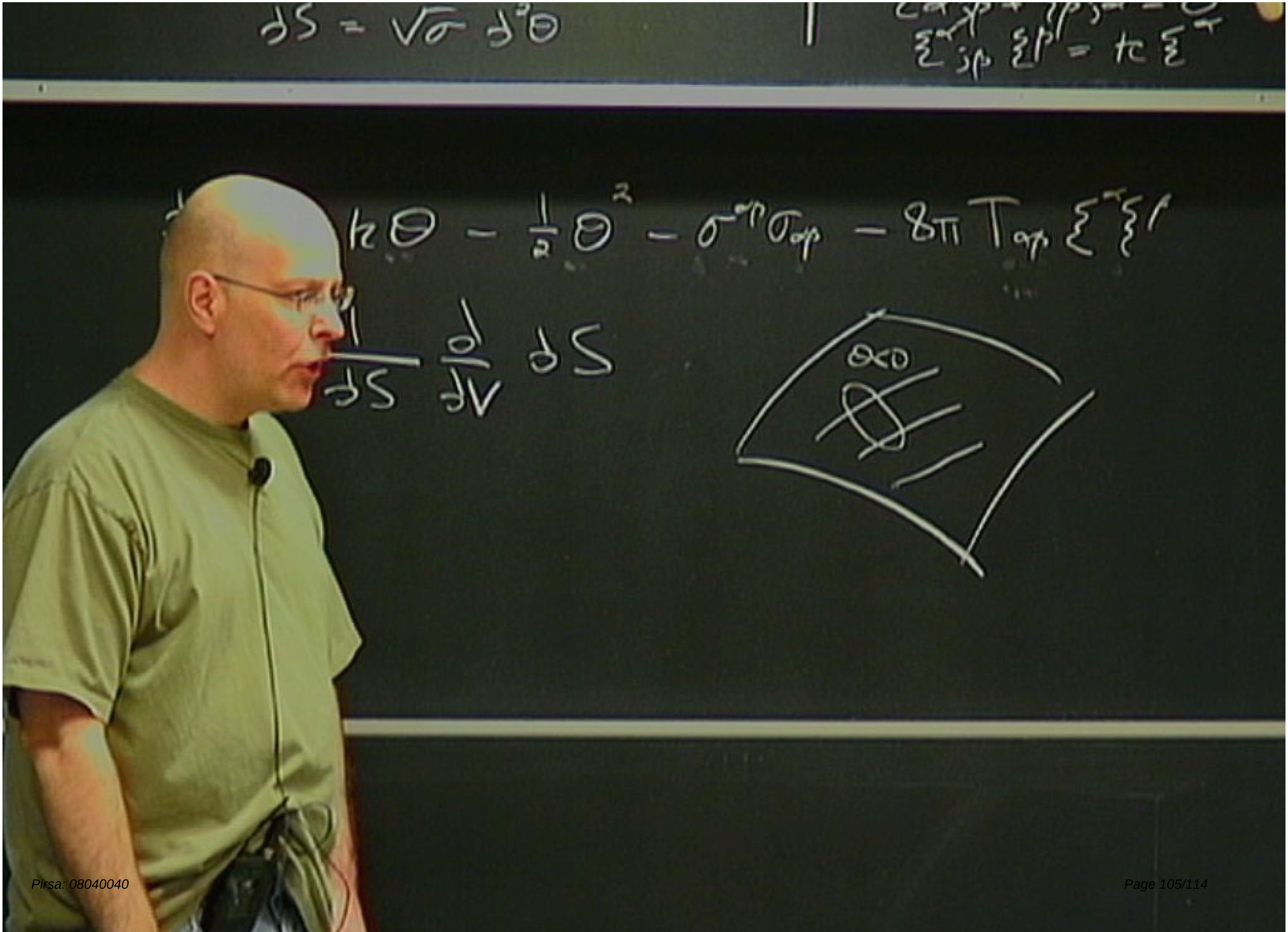
$) \, \partial \Sigma_\alpha$

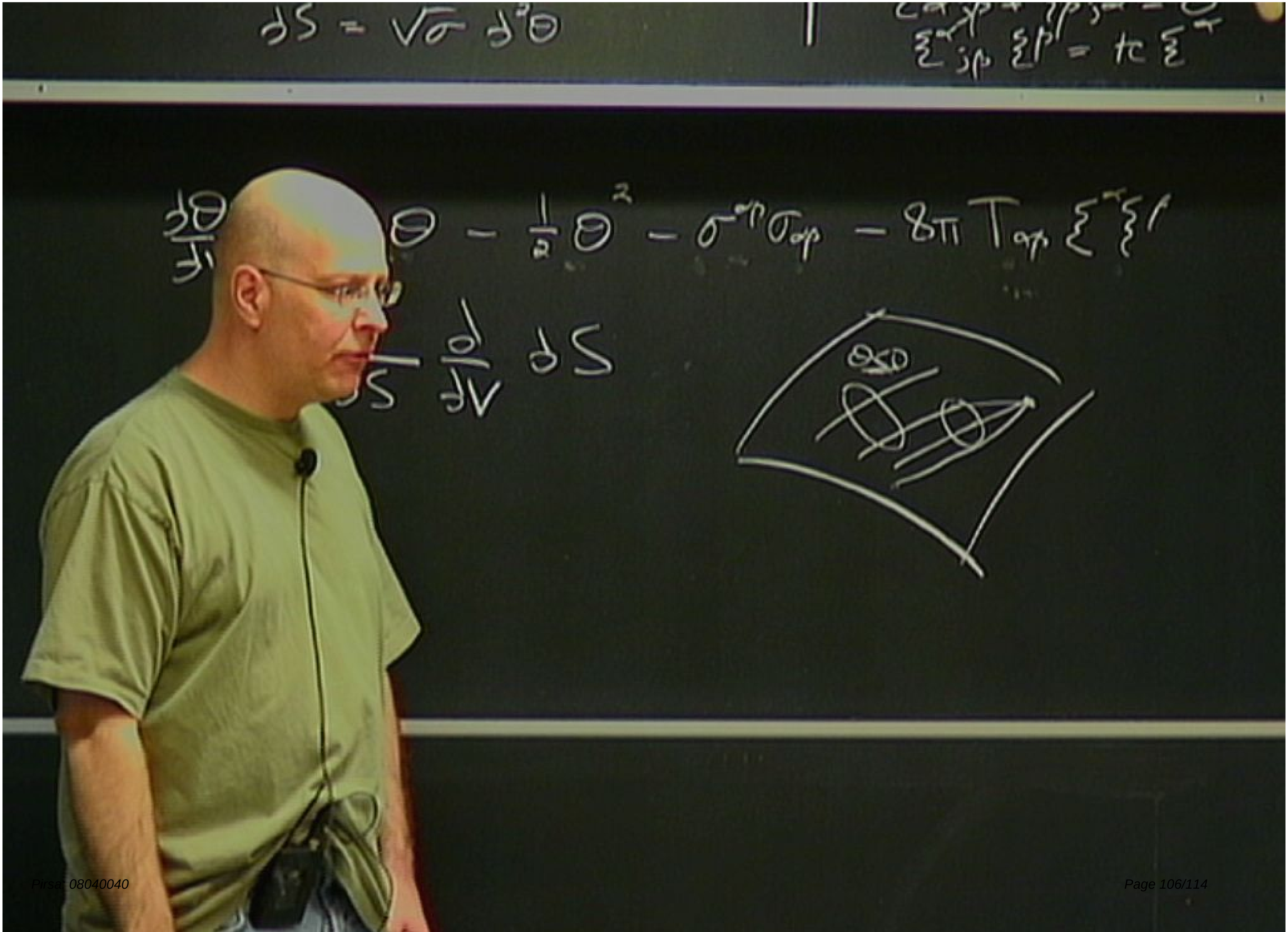
$- \, \xi_\alpha \, \partial S \, \partial V$

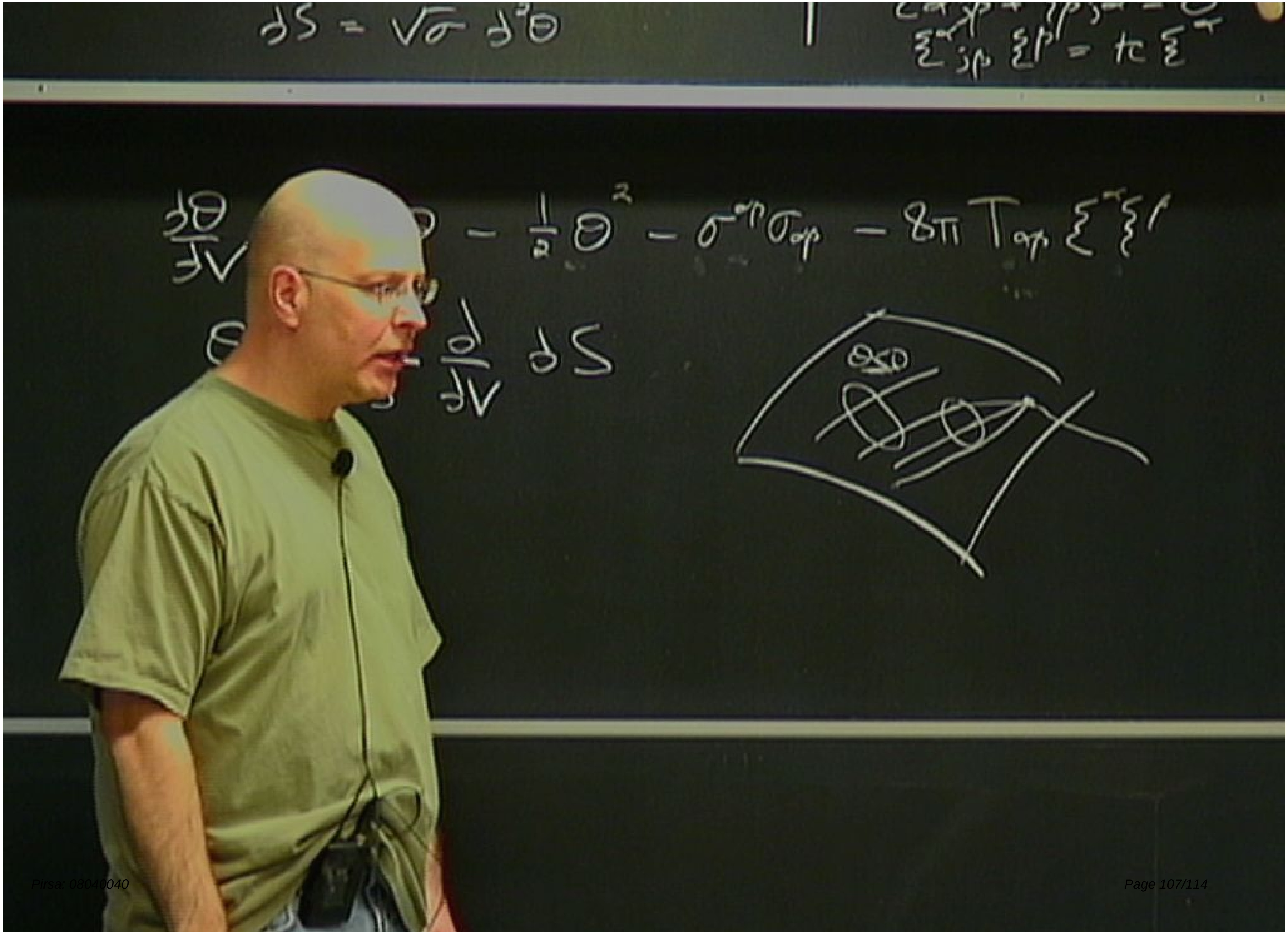
Second law

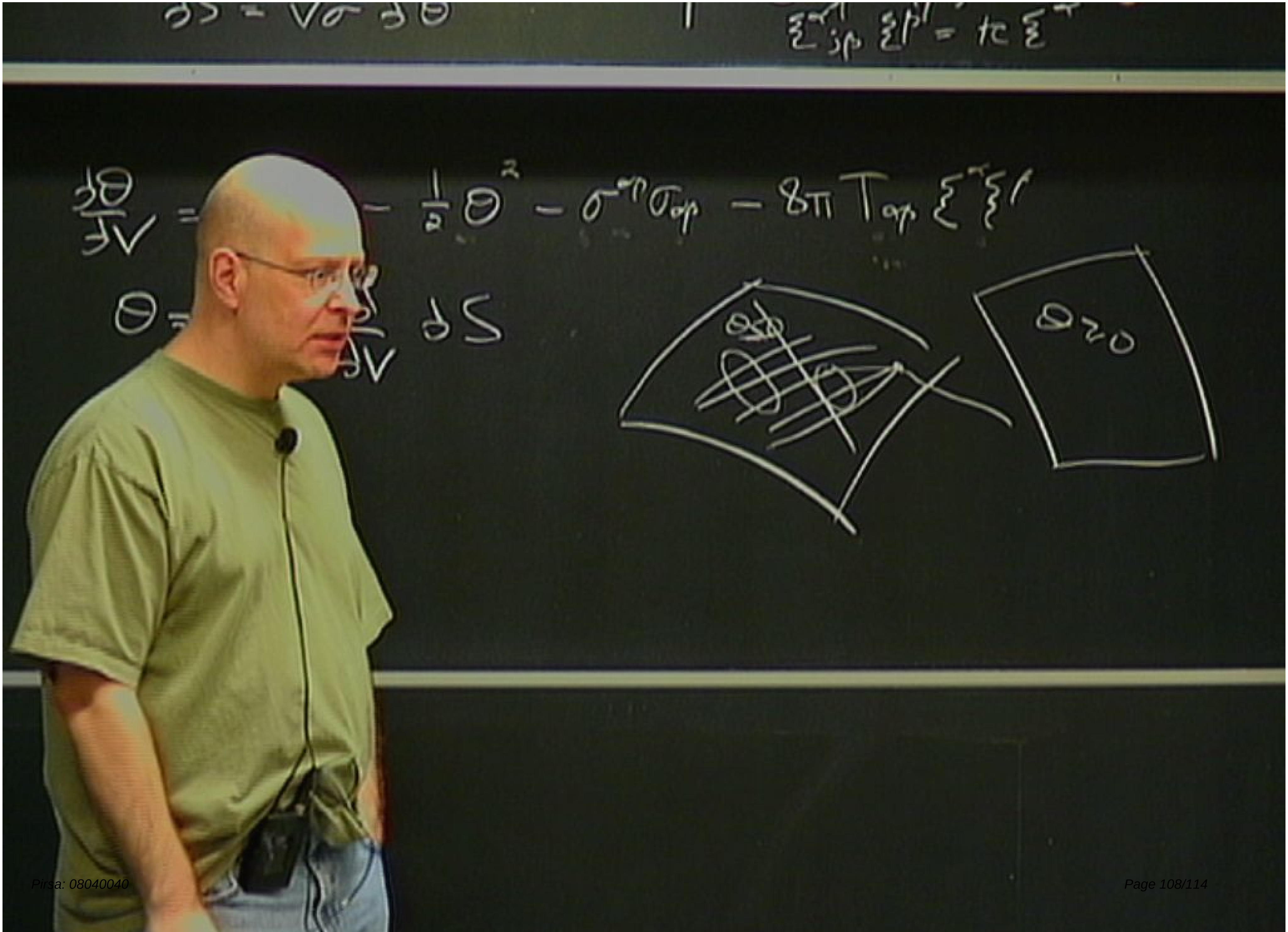
$$\boxed{\delta A \geq 0}$$


$dS = \sqrt{\sigma}\, d^2\theta$
$\xi^{\alpha}{}_{;\beta}\, \xi^{\beta} = \kappa\, \xi^{\alpha}$
$= \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^{\alpha}\xi^{\beta}$
$\theta = \frac{1}{dS}\frac{d}{d\lambda} dS$


$dS = \sqrt{\sigma}\, d^2\theta$
$\xi^{\alpha}{}_{;\beta}\, \xi^{\beta} = \kappa\, \xi^{\alpha}$
$\kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi\, T_{\alpha\beta}\, \xi^{\alpha}\xi^{\beta}$


dS = √σ d²θ
− 1/2 θ² − σ^{αβ}σ_{αβ} − 8π T_{αβ} ξ^α ξ^β
dS


dS = √σ d²θ
ξ^α;β ξ^β = κ ξ^α
− ½ θ² − σ^αβ σ_αβ − 8π T_αβ ξ^α ξ^β



$$dS = \sqrt{\sigma}\, d\theta \qquad \xi^{\alpha}{}_{;\beta}\, \xi^{\beta} = \kappa\, \xi^{\alpha}$$

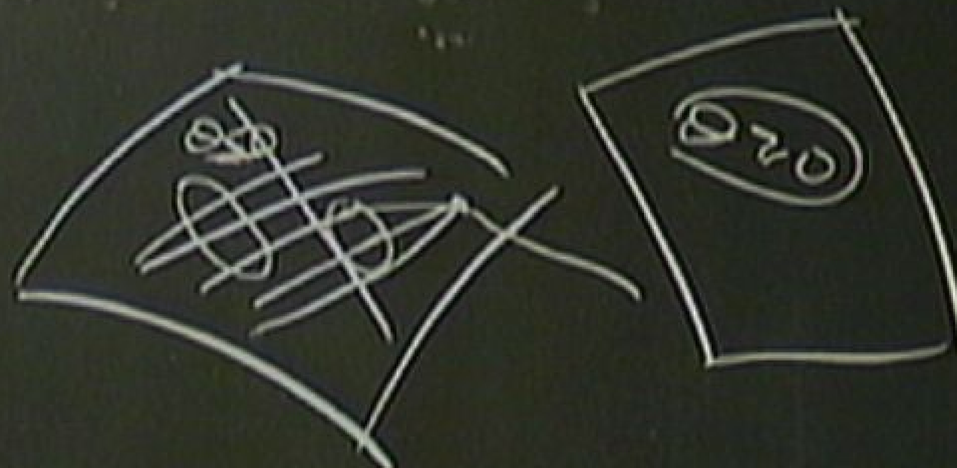
$$\frac{d\theta}{dV} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\, \xi^{\alpha}\xi^{\beta}$$

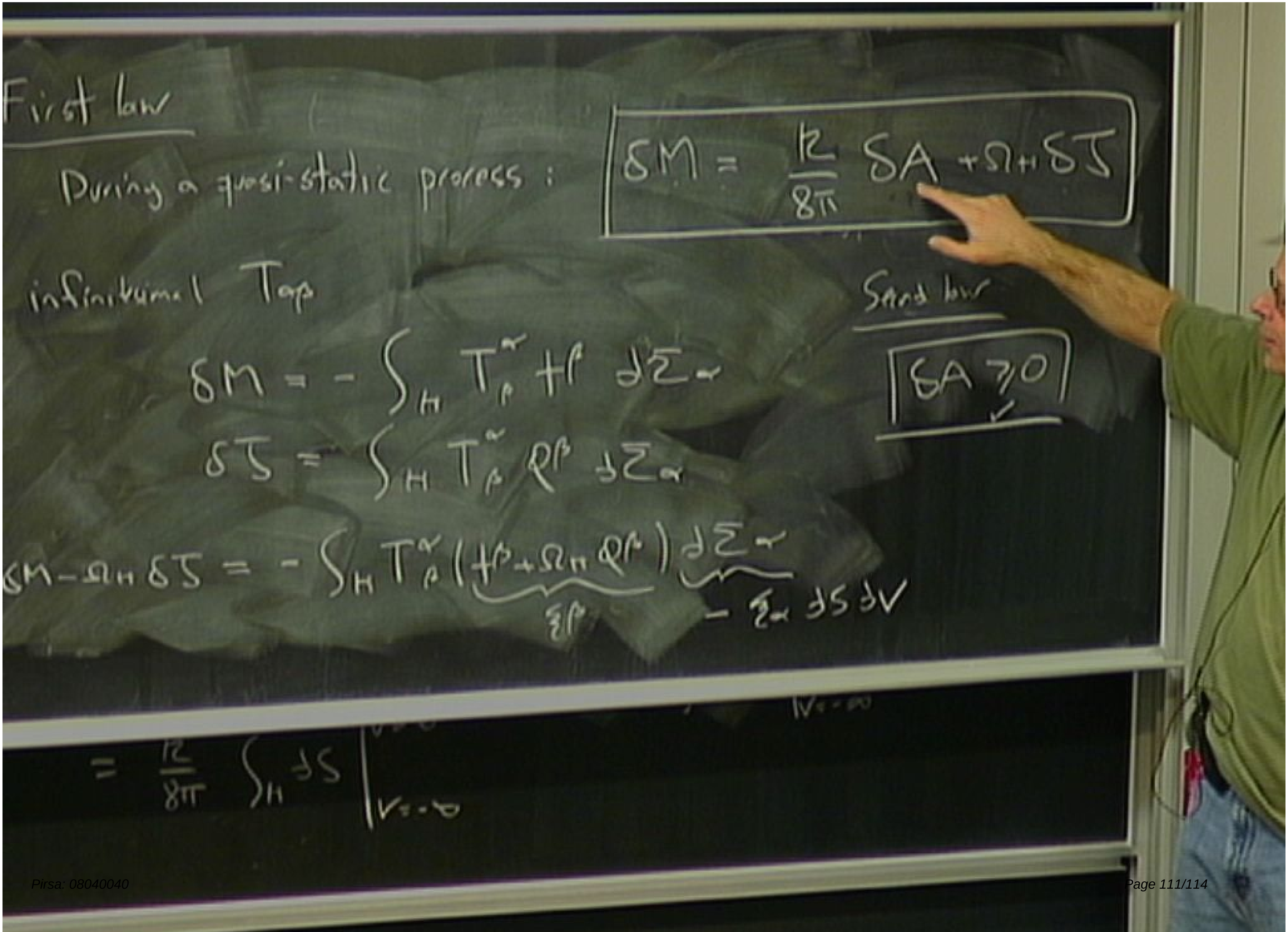
$$\theta = \frac{1}{dS}\frac{d}{dV}\, dS$$

$dS = \sqrt{\sigma}\, d^2\theta$ $\qquad \xi^{\alpha}{}_{;\beta}\, \xi^{\beta} = \kappa\, \xi^{\alpha}$

$$\frac{d\theta}{dV} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}$$

$$\theta = \frac{1}{dS}\frac{d}{dV}\, dS$$




First law
During a quasi-static process :
$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_H\,\delta J}$
infinitesimal T_{ab}
$\delta M = -\int_H T^{\alpha}{}_{\beta}\, t^{\beta}\, d\Sigma_{\alpha}$
$\delta J = \int_H T^{\alpha}{}_{\beta}\, \varphi^{\beta}\, d\Sigma_{\alpha}$
$\delta M - \Omega_H \delta J = -\int_H T^{\alpha}{}_{\beta} \underbrace{(t^{\beta} + \Omega_H \varphi^{\beta})}_{\xi^{\beta}} \underbrace{d\Sigma_{\alpha}}_{-\xi_{\alpha}\, dS\, dV}$
Second law
$\boxed{\delta A \geq 0}$ ✓
$= \frac{\kappa}{8\pi} \int_H dS \Big|_{V=-\infty}$

First law

During a quasi-static process:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J}$$

infinitesimal Tap

$$\delta M = -\int_H T^\alpha{}_\beta\, t^\beta\, d\Sigma_\alpha$$

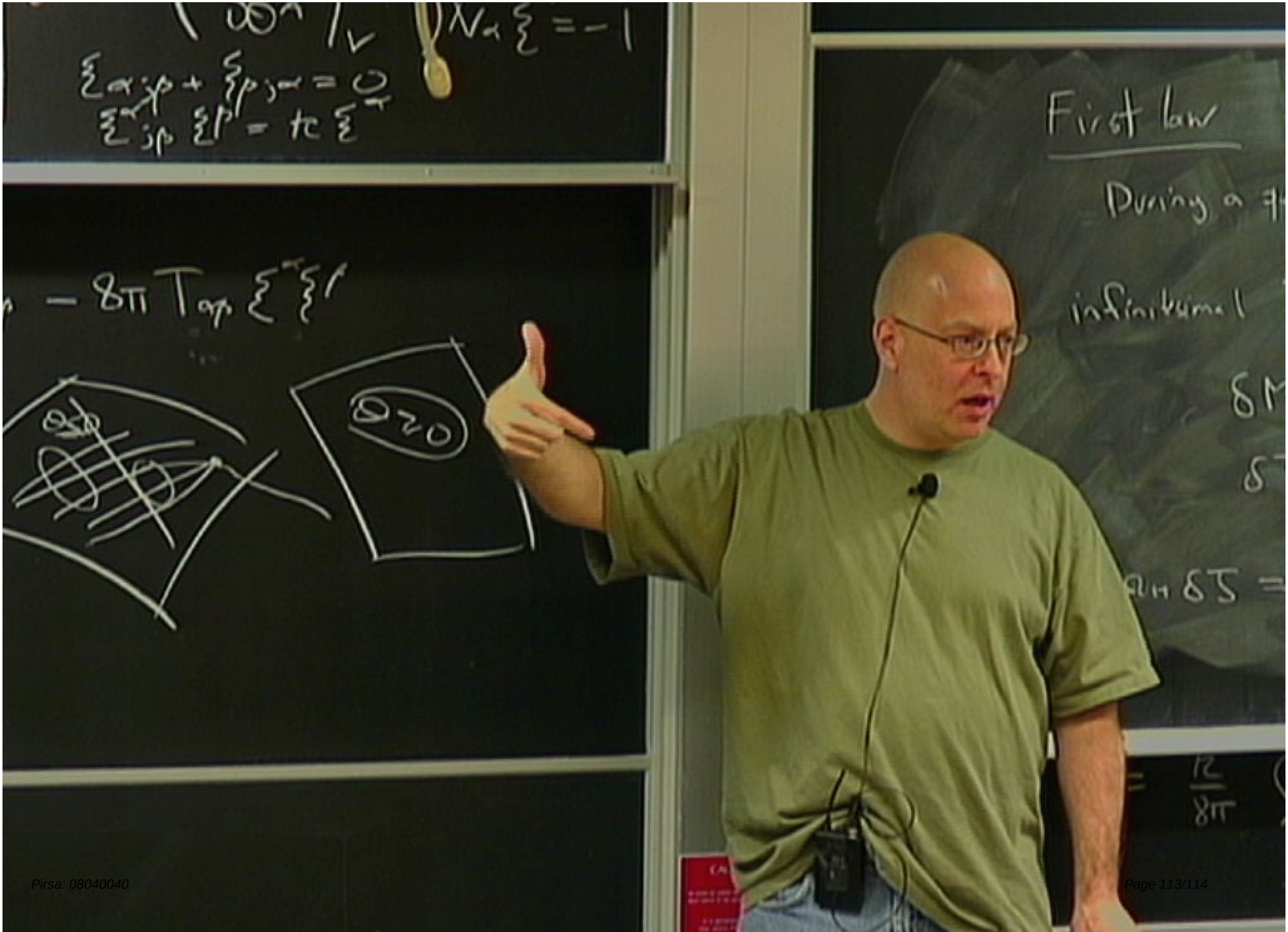
$$\delta J = \int_H T^\alpha{}_\beta\, \varphi^\beta\, d\Sigma_\alpha$$

$$\delta M - \Omega_H \delta J = -\int_H T^\alpha{}_\beta \underbrace{(t^\beta + \Omega_H \varphi^\beta)}_{\xi^\beta} \underbrace{d\Sigma_\alpha}_{-\xi_\alpha\, dS\, dV}$$

Second law

$$\boxed{\delta A \geq 0}\ \checkmark$$

$$= \frac{\kappa}{8\pi}\int_H dS\,\Big|_{v=-\infty}$$


First law
During a
infinitesimal

First law

During a quasi-static process:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J}$$

infinitesimal Tap

$$\delta M = -\int_H T^\alpha_\beta\, t^\beta\, d\Sigma_\alpha$$

$$\delta J = \int_H T^\alpha_\beta\, \phi^\beta\, d\Sigma_\alpha$$

$$\delta M - \Omega_H \delta J = -\int_H T^\alpha_\beta \underbrace{(t^\beta + \Omega_H \phi^\beta)}_{\xi^\beta} \underbrace{d\Sigma_\alpha}_{-\xi_\alpha\, dS\, dV}$$

Second law

$$\boxed{\delta A \geq 0}$$

✓

$$= \frac{\kappa}{8\pi}\int_H dS \Big|_{v=-\infty}^{v=\infty}$$