

Title: Advanced General Relativity - Lecture 12B

Date: Apr 09, 2008 04:00 PM

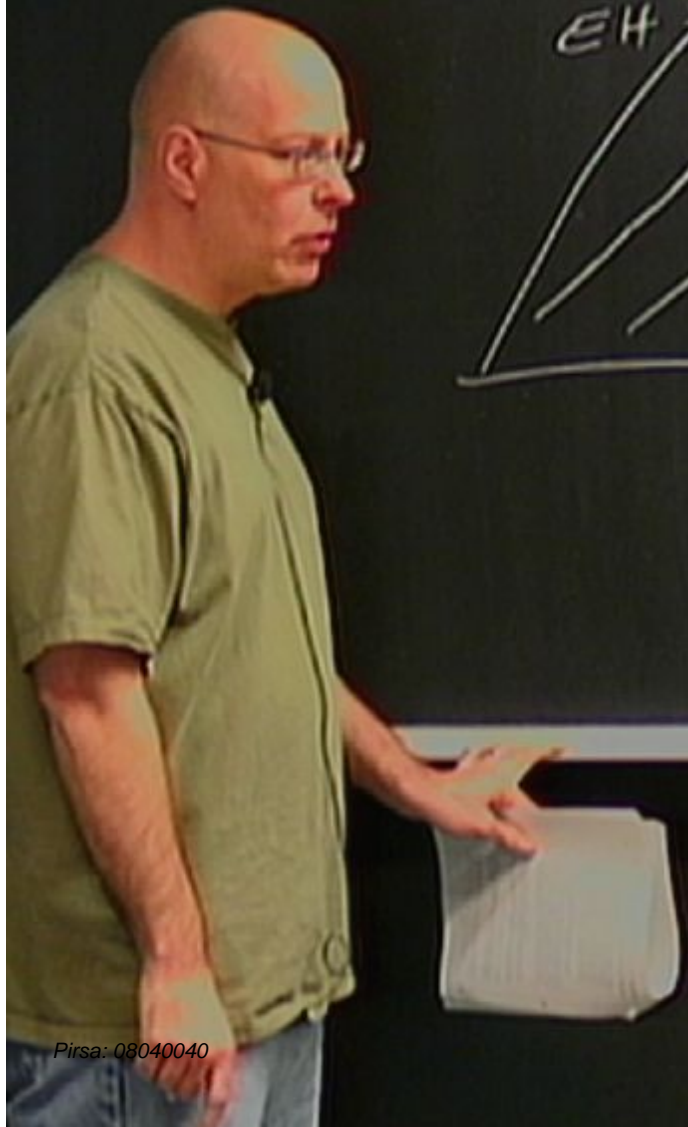
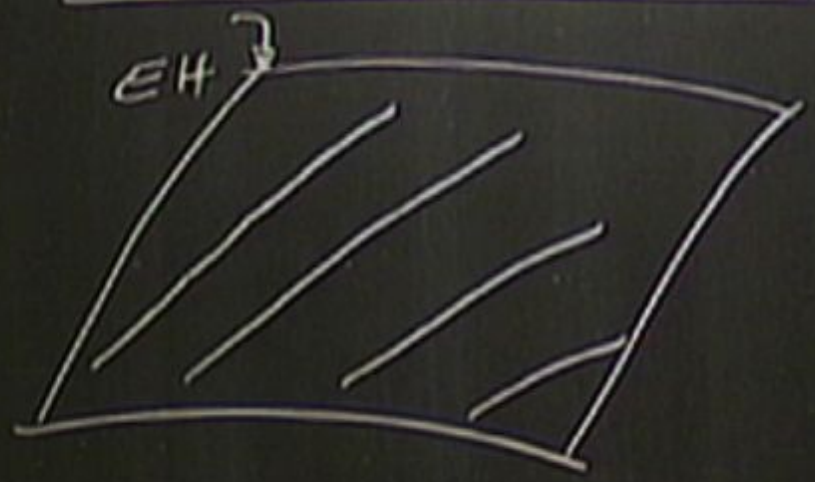
URL: <http://pirsa.org/08040040>

Abstract: Advanced General Relativity

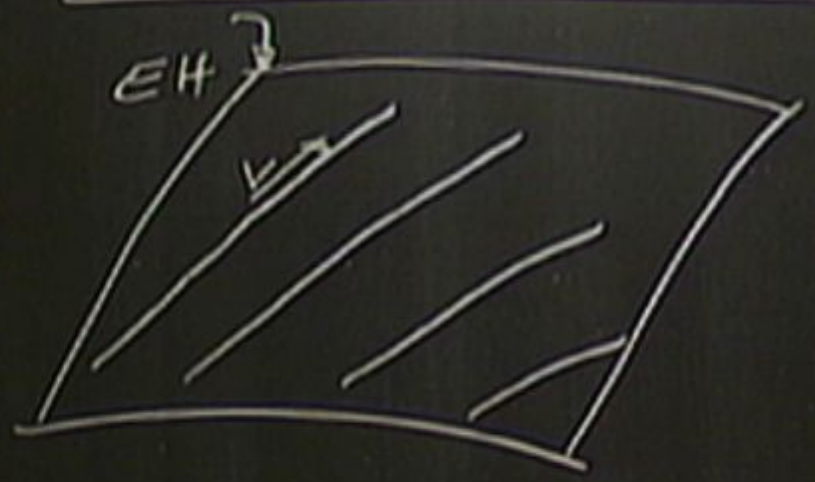
# LAWS OF BH MECHANICS



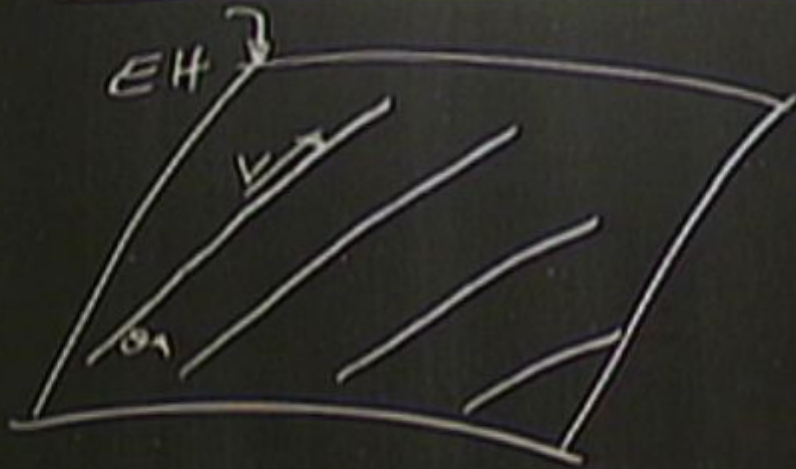
# LAWS OF BH MECHANICS



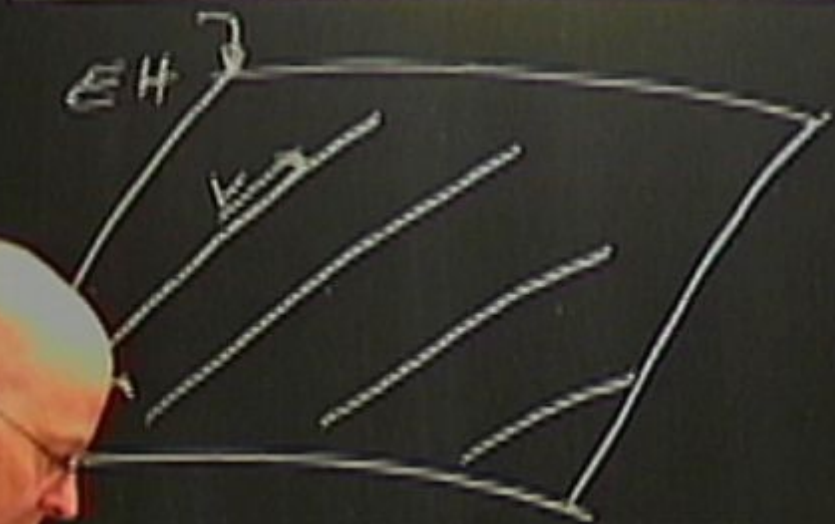
# LAWS OF BH MECHANICS



# LAWS OF BH MECHANICS



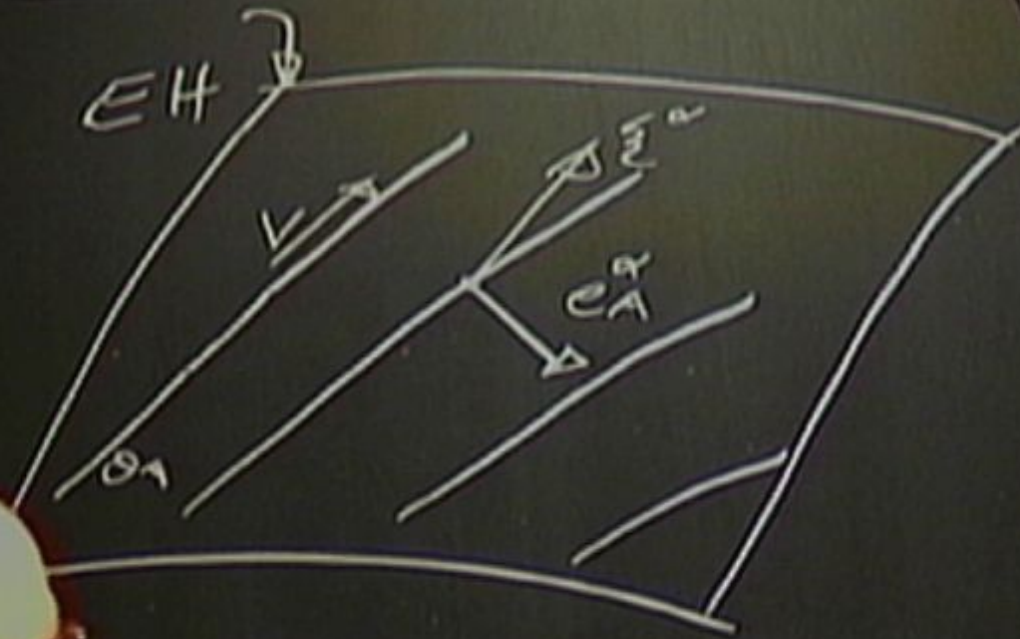
# LAWS OF BH MECHANICS



coordinate system on H  
( $V, \theta^A$ )

$$ds^2 = \sigma_{AB} d\theta^A d\theta^B$$

# LAWS OF BH MECHANICS



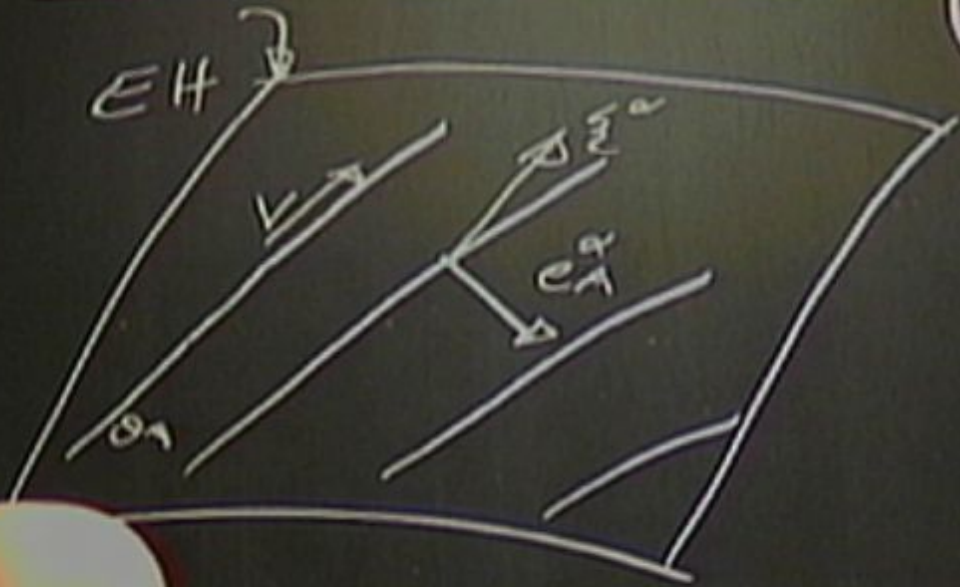
coordinate system on H  
(V,  $\theta^A$ )

$$ds^2 = g_{AB} d\theta^A d\theta^B$$

$$e^{\alpha}_A = \left( \frac{\partial X^{\alpha}}{\partial \theta^A} \right)$$

$$e^{\alpha}_V = \left( \frac{\partial X^{\alpha}}{\partial V} \right)$$

# LAWS OF BH MECHANICS



coordinate system on H  
 $(V, \theta^A)$

$$ds^2 = g_{AB} d\theta^A d\theta^B$$

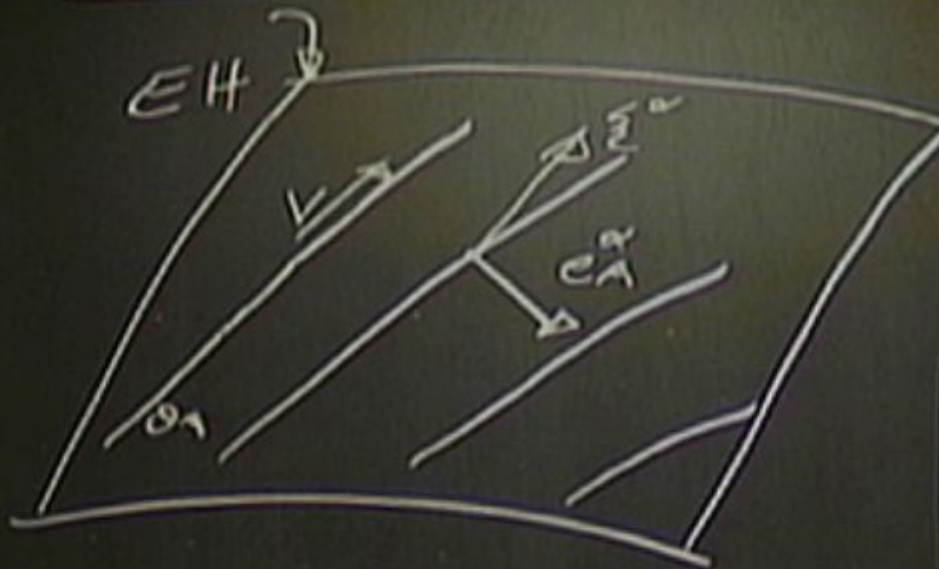
$$\xi^\alpha = \left( \frac{\partial X^\alpha}{\partial V} \right)_{\theta^A}$$

$$e^\alpha_A = \left( \frac{\partial X^\alpha}{\partial \theta^A} \right)_V$$

$$\left. \begin{array}{l} N^\alpha \\ N_\alpha N^\alpha = 0 \\ N_\alpha \xi^\alpha = -1 \end{array} \right\}$$



# LAWS OF BH MECHANICS



coordinate system on  $H$   
 $(V, \theta^A)$

$$ds^2 = \sigma_{AB} d\theta^A d\theta^B$$

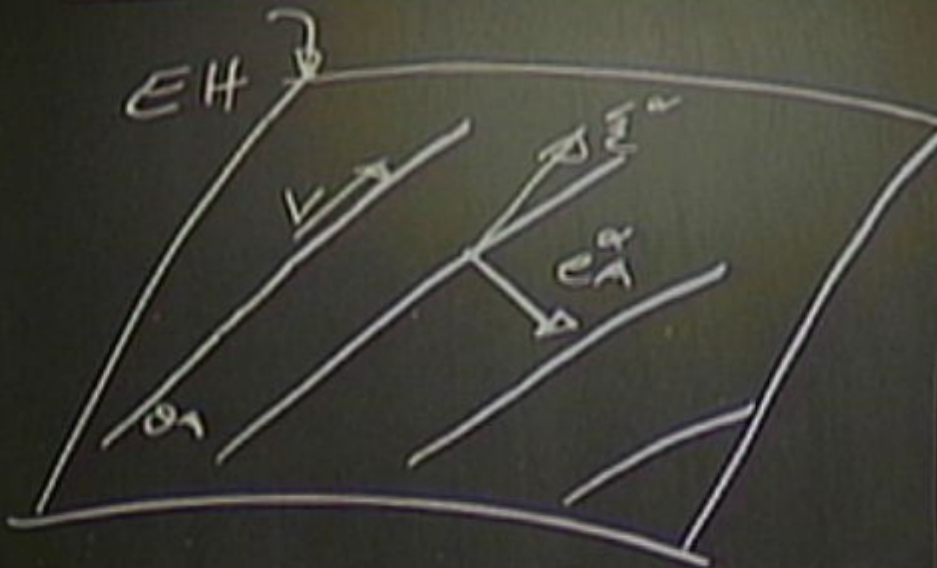
$$\xi^\alpha = \left( \frac{\partial X^\alpha}{\partial V} \right)_{\theta^A}$$

$$e_A^\alpha = \left( \frac{\partial X^\alpha}{\partial \theta^A} \right)_V$$

$$\left. \begin{array}{l} N^\alpha \\ N_\alpha N^\alpha = 0 \\ N_\alpha \xi^\alpha = -1 \end{array} \right\}$$

$$\int_{\partial V} \xi^\alpha p_\alpha = - \int \left( N^\alpha p_\alpha - N^\alpha \xi^\beta p_\beta + \sigma^{AB} e_A^\alpha e_B^\beta p_\alpha \right)$$

# LAWS OF BH MECHANICS



coordinate system on H  
(V,  $\theta^A$ )

$$dS^2 = \sigma_{AB} d\theta^A d\theta^B$$


$$\xi^{\alpha} = \left( \frac{\partial X^{\alpha}}{\partial \theta^A} \right)_{\theta^A}$$

$$e^{\alpha}_A = \left( \frac{\partial X^{\alpha}}{\partial \theta^A} \right)_V$$

$$\left. \begin{array}{l} N^{\alpha} \\ N_{\alpha} N^{\alpha} = 0 \\ N_{\alpha} \xi^{\alpha} = -1 \end{array} \right\}$$

$$\xi^{\alpha} = -\xi^{\alpha} N^{\beta} - N^{\alpha} \xi^{\beta} + \sigma^{AB} e^{\alpha}_A e^{\beta}_B$$


$$d\xi^{\alpha} = -\xi^{\alpha} dS dV$$



$$\mathcal{L}^{\alpha\beta} = -\xi^{\alpha} N^{\beta} - N^{\alpha} \xi^{\beta} + \sigma^{AB} e_A^{\alpha} e_B^{\beta}$$

$$\delta \mathcal{L}_{\alpha} = -\xi_{\alpha} \delta S \delta V$$

$$\delta S = \sqrt{\sigma} \delta^2 \theta$$



$$\mathcal{L}^{\alpha\beta} = - \left\{ N^{\beta} - N^{\alpha} \xi^{\beta} + \sigma^{AB} e_A^{\alpha} e_B^{\beta} \right.$$

$$\begin{aligned} d\mathcal{L}_{\alpha} &= - \left\{ \xi_{\alpha} dS + dV \right. \\ dS &= \sqrt{\sigma} d^2\theta \end{aligned}$$

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\text{op}}\sigma_{\text{op}} - 8\pi T_{\text{op}} \xi \xi'$$

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\text{app}}\sigma_{\text{app}} - 8\pi T_{\text{app}} \xi \xi'$$

$$\theta = \frac{1}{S_C} \frac{d}{dV} S_C$$

$e_A e_B^T$

$$e_A = \left( \frac{\partial X}{\partial \theta^A} \right)_{\downarrow} N_{\alpha} N_{\alpha}^T$$

$$\sum_{\alpha} \alpha_{; \beta} + \sum_{\alpha} \alpha_{; \beta} = 0$$
$$\sum_{\alpha} \alpha_{; \beta} \sum_{\alpha} \alpha^{\beta} = \tau \sum_{\alpha} \alpha^{\beta}$$

Zeroth law

$K$  is a constant over  $E \#$



## Zeroth law

$\kappa$  is a constant over  $E, H$

$$\left(\frac{\partial \kappa}{\partial V}\right)_{\partial A} = 0$$

$$\left(\frac{\partial \kappa}{\partial \partial A}\right)_V = 0$$

Zeroth law

$K$  is a constant over  $E_H$

$$\left(\frac{\partial K}{\partial V}\right)_{\partial^A} = 0$$
$$\left(\frac{\partial K}{\partial \partial^A}\right)_V = 0$$

Decomposition of  $\Sigma_{exp}$  on  $E_H$ :

Zeroth law

$K$  is a constant over  $EH$

$$\left(\frac{\partial K}{\partial V}\right)_{\partial A} = 0$$

$$\left(\frac{\partial K}{\partial \partial A}\right)_V = 0$$

Decomposition of  $\xi_{\alpha\beta}$  on  $EH$ :

$$\xi_{\alpha\beta} = a \xi_{\alpha\beta} + b \xi_{\alpha} N_{\beta} + c^{\lambda} \xi_{\lambda\alpha}$$

## Zeroth law

$\mathcal{K}$  is a constant over  $EH$

$$\left(\frac{\partial \mathcal{K}}{\partial V}\right)_{\partial^{\alpha}} = 0$$

$$\left(\frac{\partial \mathcal{K}}{\partial \partial^{\alpha}}\right)_{V} = 0$$

Decomposition of  $\xi_{\alpha\beta}$  on  $EH$ :

$$\begin{aligned}\xi_{\alpha\beta} = & a \xi_{\alpha} \xi_{\beta} + b \xi_{\alpha} N_{\beta} + c^A \xi_{\alpha} e_{A\beta} \\ & + d N_{\alpha} \xi_{\beta} + e N_{\alpha} N_{\beta} + f^A N_{\alpha} e_{A\beta} \\ & + g^A e_{A\alpha} \xi_{\beta} + h^A e_{A\alpha} N_{\beta} + \int^{\Lambda B} e_{\Lambda\alpha} e_{B\beta}\end{aligned}$$

## Zeroth law

$K$  is a constant over  $E \cup H$

$$\left(\frac{\partial K}{\partial V}\right)_{\partial \Omega} = 0$$

$$\left(\frac{\partial K}{\partial \Omega}\right)_V = 0$$

Decomposition of  $\xi_{\alpha\beta}$  on  $E \cup H$ :

$$\begin{aligned}\xi_{\alpha\beta} = & a \xi_{\alpha} \xi_{\beta} + b \xi_{\alpha} N_{\beta} + c^A \xi_{\alpha} e_{A\beta} \\ & + d N_{\alpha} \xi_{\beta} + e N_{\alpha} N_{\beta} + f^A N_{\alpha} e_{A\beta} \\ & + g^A e_{A\alpha} \xi_{\beta} + h^A e_{A\alpha} N_{\beta} + \int^{AB} e_{A\alpha} e_{B\beta}\end{aligned}$$

Decomposition of  $\Sigma_{\alpha\beta}$  on  $EH$ :

$$\begin{aligned} \Sigma_{\alpha\beta} = & a \Sigma_{\alpha} \Sigma_{\beta} + b \Sigma_{\alpha} N_{\beta} + c^A \Sigma_{\alpha} e_{A\beta} \\ & + d N_{\alpha} \Sigma_{\beta} + e N_{\alpha} N_{\beta} + f^A N_{\alpha} e_{A\beta} \\ & + g^A e_{A\alpha} \Sigma_{\beta} + h^A e_{A\alpha} N_{\beta} + \delta^{AB} e_{A\alpha} e_{B\beta} \end{aligned}$$

$$\left(\frac{\partial}{\partial A}\right)_V = 0$$

$$1 - \frac{\sum^\alpha}{\sum^\alpha \sum^{\alpha;\beta}}$$

+  $\Delta$

$$v = 0$$

$\xi^\alpha$  is null

$$0 = \xi^\alpha \xi_\alpha \xi_\beta \xi^\beta$$

$\xi^\alpha$



$$\left( \frac{\partial K}{\partial \alpha^A} \right)_V = 0$$

1-  $\xi^\alpha$  is null

$$0 = \xi^\alpha \xi_{\alpha;\rho} \xi^\rho$$

$$0 = \xi^\alpha \xi_{\alpha;\rho} e^\rho_\mu$$

$$+ \xi^A e_{A\alpha}$$

1-  $\sum x$  is null

$$0 = \sum^{\alpha} \sum_{\alpha; \beta} \sum^{\beta}$$

$$0 = \sum^{-x} \sum_{\alpha; \beta} e^{\beta_A}$$

Decomposition of  $\xi_{\alpha\beta}$  on  $EH$ :

$$\xi_{\alpha\beta} = a \xi_{\alpha} \xi_{\beta} + b \xi_{\alpha} N_{\beta} + c^A \xi_{\alpha} e_{\beta}^A + d N_{\alpha} \xi_{\beta} + e \cancel{N_{\alpha} N_{\beta}} + f^A N_{\alpha} e_{\beta}^A + g^A e_{\alpha}^A \xi_{\beta} + h^A e_{\alpha}^A N_{\beta} + \cancel{j^{AB} e_{\alpha}^A e_{\beta}^B}$$

$\xi^{\tau}$  is null

$$\begin{aligned} &= \xi^{\alpha} \xi_{\alpha\beta} \xi^{\beta} = e \\ &= \xi^{\tau} \xi_{\tau\beta} e^{\beta}_n \end{aligned}$$



over  $EH$

Decomposition of  $\xi_{\alpha\beta}$  on  $EH$ :

$$\begin{aligned} \xi_{\alpha\beta} = & a \xi_{\alpha} \xi_{\beta} + b \xi_{\alpha} N_{\beta} + c^A \xi_{\alpha} e_{A\beta} \\ & + d N_{\alpha} \xi_{\beta} + e \cancel{\Delta_{\alpha} N_{\beta}} + f^A N_{\alpha} e_{A\beta} \\ & + g^A e_{A\alpha} \xi_{\beta} + h^A e_{A\alpha} N_{\beta} + \cancel{j^{AB} e_{A\alpha} e_{\beta}^B} \end{aligned}$$

1-  $\xi^{\alpha}$  is null

$$0 = \xi^{\alpha} \xi_{\alpha\beta} \xi^{\beta} = e$$

$$0 = \xi^{\alpha} \xi_{\alpha\beta} e_{\beta}^{\gamma} = -f^A \sigma_{AB}$$

$$\underline{2 - \sum^{\alpha} \rho \xi^{\beta} = k \xi^{\alpha}}$$

CAUTION

$$2 - \sum_{\alpha} \zeta^{\alpha} \zeta^{\beta} = 12 \zeta^{\alpha}$$

---

$$\sum_{\alpha} \zeta^{\alpha} \zeta^{\beta} = -6 \zeta^{\alpha}$$

$$\underline{a - \sum_{\alpha} \xi^{\alpha} \xi^{\beta} = k \xi^{\beta}}$$

$$\sum_{\alpha} \xi^{\alpha} \xi^{\beta} = -b \xi^{\alpha} - h^A c_{A\alpha} \Rightarrow b = -k$$

Decomposition of  $\Sigma_{\alpha\beta}$  on  $EH$ :

over  $EH$

$$\begin{aligned} \Sigma_{\alpha\beta} = & a \Sigma_{\alpha} \xi^{\beta} - k \Sigma_{\alpha} N^{\beta} + c^A \xi_{\alpha} e_{\beta A} \\ & + d N_{\alpha} \xi^{\beta} + e \cancel{\Delta_{\alpha} N^{\beta}} + f^A \cancel{N_{\alpha} e_{\beta A}} \\ & + g^A e_{\alpha A} \xi^{\beta} + h^A \cancel{e_{\alpha A} N^{\beta}} \end{aligned}$$

1-  $\xi^{\alpha}$  is null

$$0 = \xi^{\alpha} \Sigma_{\alpha\beta} \xi^{\beta} = e$$

$$0 = \xi^{\alpha} \xi_{\alpha} e^{\beta}_{\beta} = -f^A \sigma_{AB}$$



$$+ \cancel{\sum^{\alpha} \epsilon_{A\alpha} \eta^{\mu} p} + \cancel{h^{\mu} \epsilon_{A\alpha} N^{\mu} p} + \epsilon_{A\alpha} \epsilon^{\mu} p$$

1 -  $\underline{\xi^{\alpha} \epsilon_{\alpha} \eta^{\mu}}$

$$0 = \xi^{\alpha} \epsilon_{\alpha} \eta^{\mu} \xi^{\beta} p = e$$

$$0 = \xi^{\alpha} \epsilon_{\alpha} \eta^{\mu} \epsilon^{\beta} p = -\int^A \sigma_{AB}$$

$$r - h^A \epsilon_{A\alpha} \Rightarrow b = -k \quad h^A = 0$$



$$2- \underline{\sum_{\alpha \beta} \xi^{\alpha} \xi^{\beta} = k \xi^{\alpha}}$$

$$\sum_{\alpha \beta} \xi^{\alpha} \xi^{\beta} = -b \xi^{\alpha} - h^A e_{A\alpha} \Rightarrow b = -k \quad h^A = 0$$

3-

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\text{eff}}\sigma_{\text{eff}} - 8\pi T_{\text{eff}} \xi^{\text{eff}}$$

$$\theta = \frac{1}{dS} \frac{d\theta}{dV} dS$$

$$2 - \underline{\xi^{\alpha}{}_{;\rho} \xi^{\rho} = k \xi^{\alpha}}$$

$$\xi^{\alpha}{}_{;\rho} \xi^{\rho} = -b \xi^{\alpha} - h^A \epsilon_{A\alpha} \Rightarrow b = -k \quad h^A = 0$$

3 - expansion, shear, rotation vanish,  
 $\rightarrow j^{AB} = 0$

4 -  $\xi_\alpha$  is Killing vector

$$\xi(\alpha; p) = 0$$

4-  $\xi_\alpha$  is Killing vector

$$\xi(\alpha; p) = 0$$

4-  $\xi_\alpha$  is Killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\xi_{\alpha;\beta} = -\kappa (\xi_\alpha N_\beta - N_\alpha \xi_\beta)$$

4-  $\xi_\alpha$  is Killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\xi_{\alpha;\beta} = -\kappa (\xi_\alpha N_\beta - N_\alpha \xi_\beta) + C^A (\xi_\alpha R_{A\beta} - R_{A\alpha} \xi_\beta)$$





$$dZ_\alpha = - \sum_\alpha dS dV$$

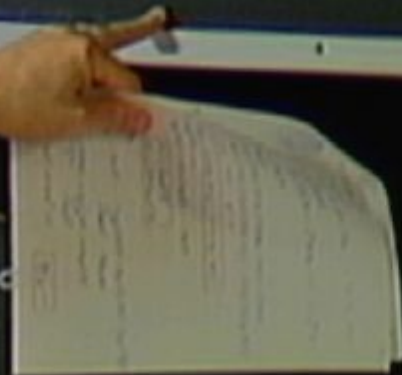
$$dS = \sqrt{\sigma} d^2\theta$$

$$\sum_\alpha \gamma_\alpha + \sum_\alpha \gamma_\alpha = 0$$

$$\sum_\alpha \gamma_\alpha \gamma_\alpha = \pi \sum_\alpha \gamma_\alpha$$

$$\frac{d\theta}{dV} = k\theta - \sigma^{-1/2} \theta^2$$

$$\theta = \frac{1}{\sqrt{\sigma}}$$

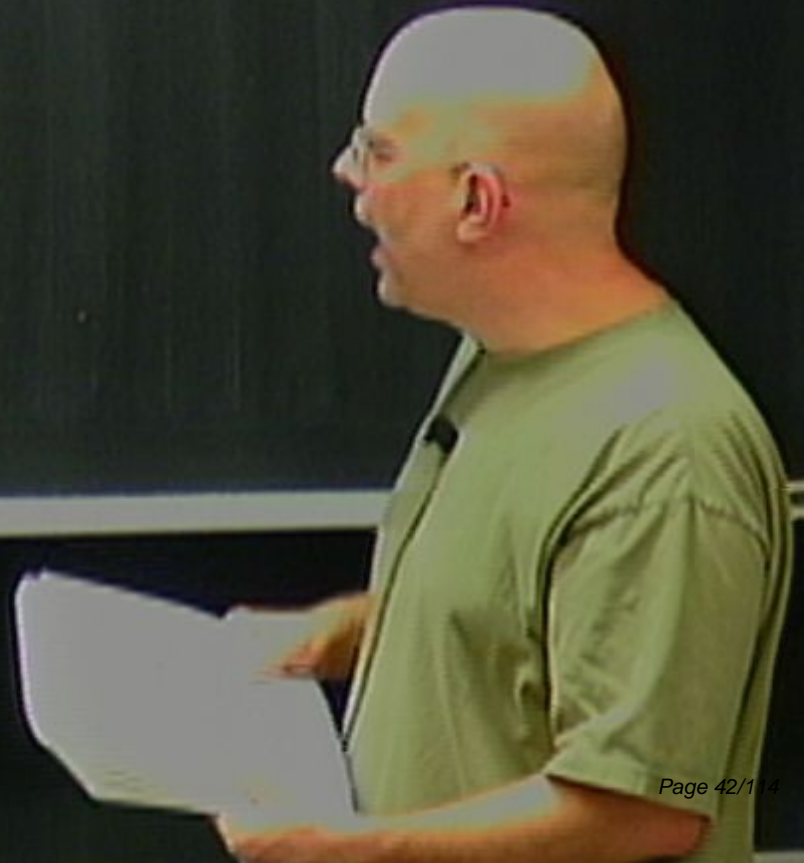


4-  $\xi_\alpha$  is Killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\xi_{\alpha;\beta} = -\kappa (\xi_\alpha N_\beta - N_\alpha \xi_\beta) + C^A (\xi_\alpha R_{A\beta} - R_{A\alpha} \xi_\beta)$$

$$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$$



4-  $\xi_\alpha$  is killing vector

$$\xi_{(\alpha;\beta)} = 0$$

$$\xi_{\alpha;\beta} = -\kappa (\xi_\alpha N_\beta - N_\alpha \xi_\beta) + C^A (\xi_\alpha R_{A\beta} - R_{A\alpha} \xi_\beta)$$

$$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$$

4-  $\xi_\alpha$  is killing vector

$$\xi_{(\alpha;\beta)} = 0$$

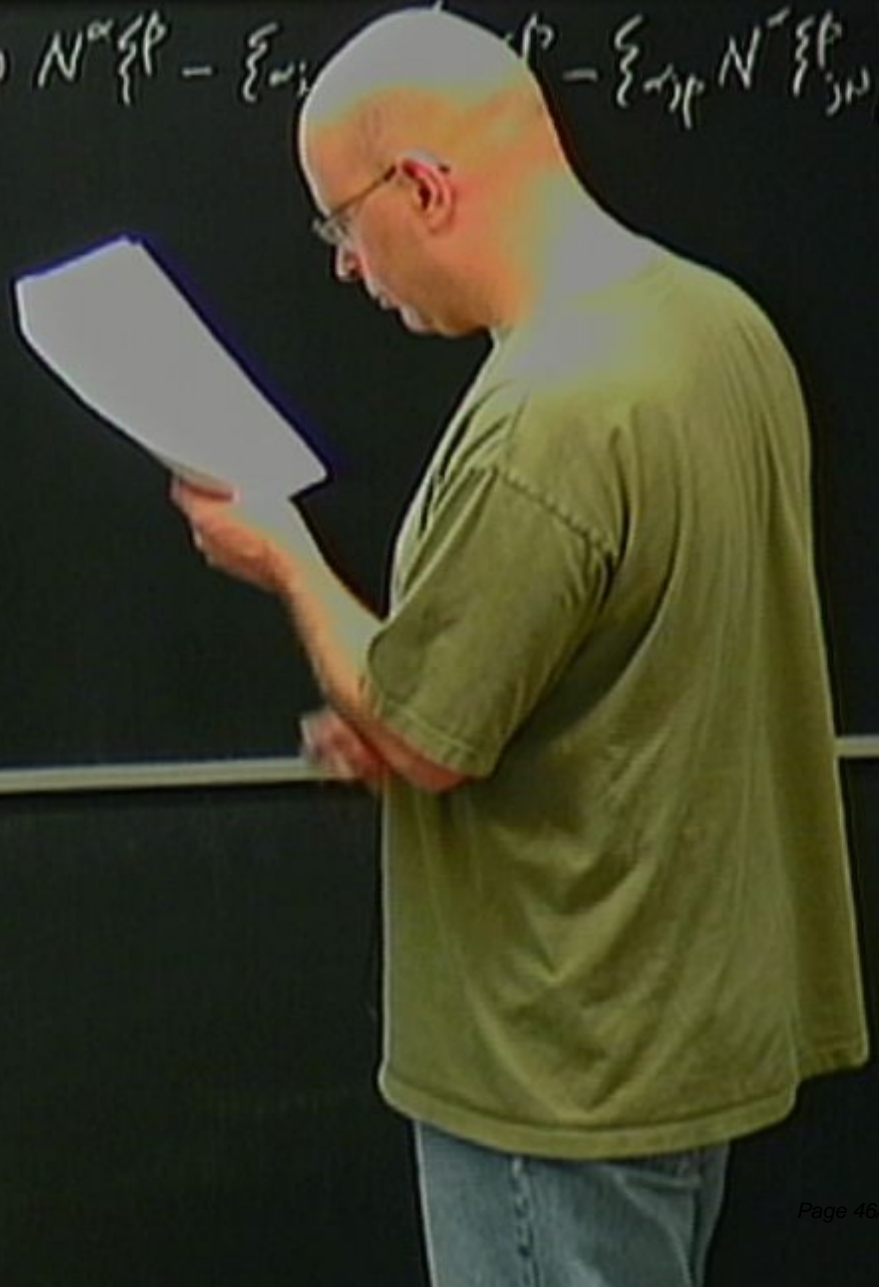
$$\xi_{\alpha;\beta} = -\kappa (\xi_\alpha N_\beta - N_\alpha \xi_\beta) + C^A (\xi_\alpha R_{A\beta} - R_{A\alpha} \xi_\beta)$$

$$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$$

$$\left( \frac{\partial k}{\partial V} \right)_{\partial A} = k_{,\mu} \xi^{\mu}$$

$$\left(\frac{\partial k}{\partial v}\right)_{0\alpha} = k_{,\mu} \xi^\mu = \left( -\xi_{\alpha;\mu} N^\mu \xi^\alpha - \xi_{\alpha;\mu} N^\mu \xi^\alpha - \xi_{\alpha;\mu} N^\mu \xi^\alpha \right) \xi^\mu$$

$$= R_{\alpha\beta\mu\nu} \xi^\nu$$



$$\begin{aligned}
 \left( \frac{\partial k}{\partial V} \right)_{0^{\wedge}} &= k_{\mu\nu} \xi^{\mu} = \left( -\xi_{\alpha\beta\mu} N^{\alpha} \xi^{\beta} - \xi_{\alpha\beta\mu} N^{\alpha}_{;\nu} \xi^{\beta} - \xi_{\alpha\beta\mu} N^{\alpha} \xi^{\beta}_{;\nu} \right) \xi^{\mu} \\
 &= \left( R_{\alpha\beta\mu\nu} \xi^{\nu} N^{\alpha} \xi^{\beta} \xi^{\mu} - \xi_{\alpha\beta\mu} \xi^{\beta} N^{\alpha}_{;\nu} \xi^{\mu} - \xi_{\alpha\beta\mu} N^{\alpha} \xi^{\beta}_{;\nu} \xi^{\mu} \right)
 \end{aligned}$$

$$\left(\frac{\partial k}{\partial V}\right)_{0^{\wedge}} = k_{,\mu} \xi^{\mu} = \left( -\xi^{\alpha;\beta\mu} N^{\alpha} \xi^{\beta} - \xi^{\alpha;\beta} N^{\alpha}_{;\mu} \xi^{\beta} - \xi^{\alpha;\beta} N^{\alpha} \xi^{\beta}_{;\mu} \right) \xi^{\mu}$$

$$= \left( R_{\alpha\beta\mu\nu} \xi^{\nu} N^{\alpha} \xi^{\beta} \xi^{\mu} - \xi^{\alpha;\beta} \xi^{\beta} N^{\alpha}_{;\mu} \xi^{\mu} - \xi^{\alpha;\beta} N^{\alpha} \xi^{\beta}_{;\mu} \xi^{\mu} \right)$$



$$\begin{aligned}
\left(\frac{\partial k}{\partial V}\right)_{0^{\wedge}} &= k_{;\mu} \xi^{\mu} = \left( -\xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta} - \xi_{\alpha;\beta\mu} N^{\alpha}{}_{;\nu} \xi^{\beta} - \xi_{\alpha;\beta\mu} N^{\alpha}{}_{;\nu} \xi^{\beta}{}_{;\rho} \xi^{\rho} \right) \xi^{\mu} \\
&= \left( \cancel{R_{\alpha\beta\mu\nu} \xi^{\nu}} N^{\alpha} \xi^{\beta} \xi^{\mu} - \underbrace{\xi_{\alpha;\beta\mu} \xi^{\beta} N^{\alpha}{}_{;\nu} \xi^{\nu}}_{k \xi_{\alpha}} - \xi_{\alpha;\beta\mu} N^{\alpha} \underbrace{\xi^{\beta}{}_{;\nu} \xi^{\nu}}_{k \xi^{\beta}} \right) \xi^{\mu} \\
&= -k \underbrace{\xi_{\alpha} N^{\alpha}{}_{;\nu} \xi^{\nu}}_{-\xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta}} - k \underbrace{\xi_{\alpha;\beta\mu} \xi^{\beta} N^{\alpha}}_{k \xi_{\alpha}} \\
&= -k \xi_{\alpha} N^{\alpha}
\end{aligned}$$



$$\left(\frac{\partial k}{\partial V}\right)_{0^{\wedge}} = k_{;\mu} \xi^{\mu} = \left( -\xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta} - \xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta} - \xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta} \right) \xi^{\mu}$$

$$= \left( \cancel{R_{\alpha\beta\mu\nu} \xi^{\nu}} N^{\alpha} \xi^{\beta} - \underbrace{\xi_{\alpha;\beta\mu} \xi^{\beta} N^{\alpha} \xi^{\mu}}_{k \xi_{\alpha}} - \xi_{\alpha;\beta\mu} N^{\alpha} \underbrace{\xi^{\beta} \xi^{\mu}}_{k \xi^{\beta}} \right)$$

$$= -k \underbrace{\xi_{\alpha} N^{\alpha} \xi^{\mu}}_{-\xi_{\alpha;\beta\mu} N^{\alpha} \xi^{\beta}} - k \underbrace{\xi_{\alpha;\beta\mu} \xi^{\beta} N^{\alpha}}_{k \xi_{\alpha}}$$

$$= -k \xi_{\alpha} N^{\alpha}$$

$$= k^{\wedge} \xi_{\alpha} N^{\alpha} - k^2 \xi_{\alpha} N^{\alpha} = 0$$

$$\left(\frac{\partial t}{\partial \omega^A}\right)_V = k_{in} e^A =$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{in} e^x_A = \left( R_{\mu\nu} \xi^\mu N^\nu \xi^\rho e^{\rho}_A - k \right)$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_V = k_{\mu\nu} e^{\nu}_A = (R_{\mu\rho\nu\sigma} \xi^\sigma N^\alpha \xi^\rho e^{\mu}_A - k \xi_\alpha N^\alpha_{;\mu} e^{\mu}_A)$$

$$\left(\frac{\partial k}{\partial \omega^A}\right)_V = k_{,A} e^A = \left( R_{\mu\nu\rho\sigma} \xi^\mu N^\nu \xi^\rho e^A - k \xi_\mu N^\nu \xi^\mu e^A - \xi_{\mu\rho} N^\nu \xi^\rho \xi^\mu e^A \right)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}^A}\right)_V = \kappa_{, \mu} e^{\mu}_A = \left( -R_{\mu\nu\rho\sigma} \xi^{\nu} N^{\mu} \xi^{\rho} e^{\mu}_A - \kappa \xi_{\mu} N^{\mu} \xi^{\nu} e^{\nu}_A - \xi_{\mu} \xi^{\nu} N^{\mu} \xi^{\rho} e^{\nu}_A \right)$$

$$\begin{aligned}
 \left( \frac{\partial \mathcal{L}}{\partial \omega^A} \right)_V &= \kappa_{\mu\nu} e^{\nu A} = \left( R_{\mu\rho\sigma\tau} \xi^{\rho} N^{\sigma} \xi^{\tau} e^{\mu A} - \kappa \xi_{\alpha} N^{\mu} \xi^{\nu} e^{\mu A} - \xi_{\alpha\beta} N^{\sigma} \xi^{\rho} \xi^{\nu} e^{\mu A} \right) \\
 &= -R_{\mu\rho\sigma\tau} e^{\mu A} \xi^{\rho} N^{\sigma} \xi^{\tau} + \kappa N_{\alpha} \xi^{\mu} \xi^{\nu} e^{\mu A} - N_{\alpha} \xi^{\mu} \xi^{\rho} \xi^{\nu} e^{\mu A}
 \end{aligned}$$



$$\begin{aligned}
 \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}^A} \right)_V &= k_{1\mu} \dot{\alpha}^A = \left( R_{\mu\alpha\beta\gamma} \xi^\mu N^\alpha \xi^\beta \dot{\alpha}^A - k \xi_\alpha N^\alpha \dot{\alpha}^A - \xi_{\alpha\beta} N^\alpha \xi^\beta \dot{\alpha}^A \right) \\
 &= -R_{\mu\alpha\beta\gamma} \dot{\alpha}^A \xi^\mu N^\alpha \xi^\beta + k N_\alpha \underbrace{\xi^\alpha}_{\dot{\alpha}^A} \dot{\alpha}^A - N_\alpha \xi^\alpha \underbrace{\xi^\beta}_{\dot{\alpha}^A} \dot{\alpha}^A
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\partial \mathcal{L}}{\partial \omega^{\alpha}}\right)_{\nu} &= \kappa_{\mu\nu} \vec{e}^{\alpha} = \left( R_{\mu\alpha\nu\beta} \xi^{\beta} N^{\alpha} \xi^{\rho} e^{\mu}_{\rho} - \kappa \xi_{\alpha} N^{\alpha} \xi^{\mu} e^{\nu}_{\mu} - \xi_{\alpha} \xi^{\rho} N^{\alpha} \xi^{\beta} \xi^{\mu} e^{\nu}_{\rho} \right) \\
 &= -R_{\mu\alpha\nu\beta} e^{\mu}_{\rho} \xi^{\rho} N^{\alpha} \xi^{\beta} + \kappa N_{\alpha} \underbrace{\xi^{\alpha} \xi^{\mu} e^{\nu}_{\mu}}_{\substack{\text{CD} \\ \sigma^{\alpha\beta}}} - N_{\alpha} \xi^{\alpha} \xi^{\rho} \underbrace{\xi^{\beta} \xi^{\mu} e^{\nu}_{\rho}}_{\substack{\text{CD} \\ \sigma^{\alpha\beta}}}
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\partial \mathcal{L}}{\partial \omega^{\alpha}}\right)_{\nu} &= \kappa_{\mu\nu} e^{\mu\alpha} = \left( -R_{\mu\nu\rho\sigma} \xi^{\rho} N^{\sigma} \xi^{\beta} e^{\mu\alpha} - \kappa \xi_{\alpha} N^{\sigma} \xi^{\beta} e^{\mu\alpha} - \xi_{\alpha\beta} N^{\sigma} \xi^{\beta} \xi^{\rho} e^{\mu\alpha} \right) \\
 &= -R_{\mu\nu\rho\sigma} e^{\mu\alpha} \xi^{\rho} N^{\sigma} \xi^{\beta} + \kappa N_{\alpha} \xi^{\beta} \xi^{\rho} e^{\mu\alpha} - N_{\alpha} \xi^{\beta} \xi^{\rho} \xi^{\sigma} e^{\mu\alpha} \\
 &\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{\epsilon^{\sigma\alpha\beta} \xi^{\rho}}
 \end{aligned}$$

$$\begin{aligned}
 \left( \frac{\partial k}{\partial \omega^{\alpha}} \right)_{\nu} &= k_{\mu\nu} e^{\mu\alpha} = \left( -R_{\mu\nu\rho\sigma} \xi^{\rho} N^{\sigma} \xi^{\beta} e^{\mu\alpha} - k \xi_{\alpha} N^{\sigma} \xi^{\beta} e^{\mu\alpha} - \xi_{\alpha\beta} N^{\sigma} \xi^{\beta} \xi^{\mu} e^{\mu\alpha} \right) \\
 &= -R_{\mu\nu\rho\sigma} e^{\mu\alpha} \xi^{\rho} N^{\sigma} \xi^{\beta} + k N_{\alpha} \underbrace{\xi^{\sigma} \xi^{\beta} e^{\mu\alpha}}_{\xi^{\sigma} \xi^{\beta} e^{\mu\alpha}} - N_{\alpha} \xi^{\sigma} \xi^{\beta} \underbrace{\xi^{\mu} e^{\mu\alpha}}_{\xi^{\mu} e^{\mu\alpha}} \\
 &\qquad\qquad\qquad c^{\sigma\alpha\beta} \xi^{\rho} = c^{\alpha\beta\rho}
 \end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial \mathcal{L}}{\partial \omega^{\alpha}}\right)_{\nu} &= \kappa_{\mu\nu} e^{\mu}_{\alpha} = \left( R_{\mu\nu\rho\sigma} \xi^{\sigma} N^{\alpha} \xi^{\rho} e^{\mu}_{\alpha} - \kappa \xi_{\alpha} N^{\alpha} \xi^{\mu} e^{\mu}_{\alpha} - \xi_{\alpha} \xi^{\rho} N^{\alpha} \xi^{\rho} e^{\mu}_{\alpha} \right) \\
&= -R_{\mu\nu\rho\sigma} e^{\mu}_{\alpha} \xi^{\sigma} N^{\alpha} \xi^{\rho} + \kappa N_{\alpha} \underbrace{\xi^{\sigma} \xi^{\mu} e^{\mu}_{\alpha}}_{\xi^{\sigma} \xi^{\mu} e^{\mu}_{\alpha}} - N_{\alpha} \xi^{\rho} \xi^{\rho} \underbrace{\xi^{\rho} e^{\mu}_{\alpha}}_{\xi^{\rho} e^{\mu}_{\alpha}} \\
&= -R_{\mu\nu\rho\sigma} e^{\mu}_{\alpha} \xi^{\sigma} N^{\alpha} \xi^{\rho} - \cancel{\frac{1}{2} \kappa} - \cancel{c_{\alpha} \xi^{\sigma}} \quad c^{\rho} \sigma_{\alpha\beta} \xi^{\rho} = c_{\alpha} \xi^{\rho}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial k}{\partial \omega^A}\right)_V &= k_{,\mu} e^{\mu A} = (R_{\mu\nu\rho\sigma} \xi^\nu N^\alpha \xi^\rho e^{\mu A} - k \xi_{,\alpha} N^\alpha e^{\mu A} - \xi_{\alpha;\beta} N^\alpha \xi^\beta e^{\mu A}) \\
&= -R_{\mu\nu\rho\sigma} e^{\mu A} \xi^\nu N^\alpha \xi^\rho + k N_\alpha \underbrace{\xi^\alpha_{;\mu} e^{\mu A}}_{C^A \xi^\alpha} - N_\alpha \xi^\alpha_{;\beta} \underbrace{\xi^\beta e^{\mu A}}_{C^B \sigma_{\alpha\beta} \xi^\beta - C^A \xi^\beta} \\
&= -R_{\mu\nu\rho\sigma} e^{\mu A} \xi^\nu N^\alpha \xi^\rho - \cancel{C^A k} - \cancel{C^A k} \xi^\alpha N_\alpha
\end{aligned}$$

$$\boxed{\left(\frac{\partial k}{\partial \omega^A}\right)_V = -R_{\mu\nu\rho\sigma} e^{\mu A} \xi^\nu N^\alpha \xi^\rho}$$

$$\begin{aligned}
 \left(\frac{\partial \mathcal{L}}{\partial \omega^A}\right)_V &= \kappa_{\mu\nu} e^{\nu A} = \left( R_{\mu\alpha\beta\gamma} \xi^\mu N^\alpha \xi^\beta e^{\nu A} - \kappa \xi_\alpha N^\alpha{}_{;\mu} e^{\nu A} - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^{\nu A} \right) \\
 &= -R_{\mu\alpha\beta\gamma} e^{\nu A} \xi^\mu N^\alpha \xi^\beta + \kappa N_\alpha \underbrace{\xi^\alpha{}_{;\mu} e^{\nu A}}_{C^A \xi^\alpha} - N_\alpha \xi^\alpha{}_{;\mu} \underbrace{\xi^\beta{}_{;\nu} e^{\nu A}}_{C^B \sigma_{\alpha\beta} \xi^\beta} = C^A \xi^\alpha - C^B \sigma_{\alpha\beta} \xi^\beta = C^A \xi^\alpha \\
 &= -R_{\mu\alpha\beta\gamma} e^{\nu A} \xi^\mu N^\alpha \xi^\beta - \cancel{C^A \xi^\alpha} - \cancel{C^B \sigma_{\alpha\beta} \xi^\beta} = C^A \xi^\alpha
 \end{aligned}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \omega^A}\right)_V = -R_{\mu\alpha\beta\gamma} e^{\nu A} \xi^\mu N^\alpha \xi^\beta$$

$$= -R_{\mu\nu\alpha\beta} e^{\mu}_A \xi^{\nu} N^{\tau} \xi^{\beta} + \kappa N_{\alpha} \xi^{\tau} e^{\mu}_A$$

$$= -R_{\mu\nu\alpha\beta} e^{\mu}_A \xi^{\nu} N^{\tau} \xi^{\beta} - \cancel{\gamma_A \kappa} - \cancel{C_A \kappa}$$

$$\left( \frac{\partial \kappa}{\partial \theta^A} \right)_{\nu} = -R_{\mu\nu\alpha\beta} e^{\mu}_A \xi^{\nu} N^{\tau} \xi^{\beta}$$

$$\mathcal{E}^{\nu\alpha} = -\xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\tau} + \sigma^{\beta\gamma} e^{\nu}_{\beta} e^{\alpha}_{\gamma}$$



$$= - R_{\mu\nu\rho\sigma} e^{\mu}{}_{\alpha} \xi^{\nu} N^{\sigma} \xi^{\rho} - \cancel{g_{\alpha\beta}} - \cancel{g_{\alpha\beta}} \xi^{\nu} N_{\nu}$$

$$\left( \frac{\partial K}{\partial \dot{\alpha}^A} \right)_{\nu} = - R_{\mu\nu\rho\sigma} e^{\mu}{}_{\alpha} \xi^{\nu} N^{\sigma} \xi^{\rho}$$

$$\xi^{\nu\alpha} = - \xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{\beta\gamma} e^{\nu}{}_{\beta} e^{\alpha}{}_{\gamma}$$

$$\xi^{\nu} N^{\alpha} = - \xi^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{\beta\gamma} e^{\nu}{}_{\beta} e^{\alpha}{}_{\gamma}$$



$$\left( \frac{\partial K}{\partial \omega^A} \right)_V = - R_{\mu\nu\rho} e^{\nu\alpha} \underbrace{\xi^\nu N^\tau}_{\xi^\nu N^\tau} \xi^\rho$$

$\gamma^A \xi^\nu N^\alpha$

$$\exists^{\nu\alpha} = - \xi^\nu N^\alpha - N^\nu \xi^\alpha + \sigma^{BC} e^{\nu B} e^{\alpha C}$$

$$\xi^\nu N^\tau = - \exists^{\nu\tau} - N^\nu \xi^\tau + \sigma^{BC} e^{\nu B} e^{\tau C}$$

$$\frac{\partial K}{\partial \omega^A} = - R_{\mu\nu\rho} e^{\nu\alpha} \xi^\rho - R_{\mu\nu\rho} e^{\nu B} e^{\alpha C} \xi^\rho$$

$$\frac{dk}{dt} = -R_{mp} e^{\alpha} \xi^{\beta} - R_{mp} e^{\alpha} e^{\nu} e^{\epsilon} \xi^{\beta} \sigma^{\beta \epsilon}$$

$$= - R_{\mu\nu\rho\sigma} e^{\mu\alpha} \xi^{\nu} N^{\sigma\beta} - \frac{1}{2} \epsilon^{\alpha\beta\gamma} - \frac{1}{2} \epsilon^{\alpha\beta\gamma} N^{\alpha}$$

$$\left( \frac{\partial K}{\partial \dot{\alpha}^A} \right)_V = - R_{\mu\nu\rho\sigma} e^{\mu\alpha} \xi^{\nu} N^{\sigma\beta}$$

$$\xi^{\nu\alpha} = \xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma_{BC} e^{\nu B} e^{\alpha C}$$

$$\xi^{\nu} N^{\alpha} = -\xi^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma_{BC} e^{\nu B} e^{\alpha C}$$

$$\frac{\partial K}{\partial \dot{\alpha}^A} = - R_{\mu\nu\rho\sigma} e^{\mu\alpha} \xi^{\nu} - R_{\mu\nu\rho\sigma} e^{\mu\alpha} e^{\nu B} e^{\sigma C} \xi^{\beta} \sigma_{BC}$$

$$\frac{\partial k}{\partial x} = - R_{\mu\nu} e^{\nu}{}_{A} \xi^{\mu} - R_{\mu\nu\rho\sigma} e^{\nu}{}_{B} e^{\rho}{}_{C} \xi^{\mu} \xi^{\sigma} \beta^{BC}$$

Recall:  $\xi^{\mu} \xi_{\mu} = 0$



$$\frac{dx}{dt} = -R_{\text{up}} e^{\gamma} \xi^{\uparrow} - R_{\text{down}} e^{\gamma} e^{\beta} e^{\alpha} \xi^{\downarrow} \sigma^{\text{BC}}$$

$$R_{\text{null}} : \xi_{\text{up}} e^{\gamma} e^{\beta} - 0$$

$$\frac{dk}{d\lambda} = -R_{\tau\rho} e^{\tau A} \xi^{\rho} - R_{\mu\nu\rho\sigma} e^{\mu A} e^{\nu B} e^{\rho C} \xi^{\sigma} \sigma^{BC}$$

Recall:  $\xi_{\alpha;\beta} e^{\alpha A} e^{\beta B} = 0$

$$\Rightarrow (\xi_{\alpha;\beta} e^{\alpha A} e^{\beta B})_{;\delta} e^{\delta C} = 0$$

$$0 = -R_{\alpha\beta} e^{\alpha} e^{\beta} \xi^{\gamma} - R_{\mu\nu\rho\sigma} e^{\mu} e^{\nu} e^{\rho} e^{\sigma} \xi^{\alpha} \xi^{\beta} \sigma^{BC}$$

Recall:  $\xi_{\alpha;\beta} e^{\alpha} e^{\beta} = 0$

$$\Rightarrow (\xi_{\alpha;\beta} e^{\alpha} e^{\beta})_{;\gamma} e^{\gamma} = 0$$

$$\hookrightarrow R_{\alpha\beta\gamma\delta} e^{\alpha} e^{\beta} e^{\gamma} e^{\delta} \xi^{\delta} \equiv 0$$



$$\frac{\partial k}{\partial x^\alpha} = -R_{\alpha\beta} e^{\beta A} \xi^{\beta} - R_{\mu\nu\rho\sigma} e^{\mu A} e^{\nu B} e^{\rho C} \xi^{\beta} \sigma^{BC}$$

$$R_{\text{kill}} : \xi_{\alpha;\beta} e^{\beta A} e^{\beta B} = 0$$

$$\Rightarrow (\xi_{\alpha;\beta} e^{\beta A} e^{\beta B})_{;\delta} e^{\delta C} = 0$$

$$\hookrightarrow R_{\alpha\beta\gamma\delta} e^{\beta A} e^{\gamma B} e^{\delta C} \xi^{\delta} \equiv 0$$

$$\dot{N} = -\gamma N + N^2 \xi + \sigma_{DC} e_{DC}$$

$$\frac{dk}{dt} = -R_{sp} e^{\gamma t} \xi - R_{sp} e^{\gamma t} \sigma_{DC}$$

Real:  $\xi = \text{exp}(\gamma t)$   
 $\Rightarrow (\text{exp}(\gamma t))_{t=0}$   
 $\hookrightarrow R_{sp}$

$$\frac{\partial k}{\partial \omega^2} = -8\pi T_{\text{exp}}$$



$$\frac{\partial k}{\partial \omega^{\alpha}} = -8\pi T_{\alpha\beta} \epsilon^{\alpha} \xi^{\beta}$$

$$= -k \xi^{\alpha} N$$

$$\frac{\partial k}{\partial \omega^\alpha} = -8\pi T_{\alpha\beta} e^{\gamma A} \xi^\beta = 8\pi j_\alpha e^{\gamma A}$$

$$j_\alpha = -T_{\alpha\beta} \xi^\beta$$

$$= -\mu \xi_\alpha N$$

$$= \mu \xi_\alpha N$$

$$\frac{\partial k}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha}_A \xi^{\beta} = 8\pi j_{\alpha} e^{\alpha}_A$$

$$j_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0$$

$$\frac{\partial K}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha A} \xi^{\beta} = 8\pi j_{\alpha} e^{\alpha A}$$

$$j_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow j_{\alpha} \xi^{\alpha} = 0.$$

$$\frac{\partial k}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha A} \xi^{\beta} = 8\pi j_{\alpha} e^{\alpha A}$$

$$j_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow j_{\alpha} \xi^{\alpha} = 0.$$

$$\Rightarrow j^{\alpha} = a \xi^{\alpha} + b^A e^{\alpha A}$$



$$\frac{\partial K}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\beta A} \xi^\alpha = 8\pi j_\alpha e^{\alpha A}$$

$$j_\alpha = -T_{\alpha\beta} \xi^\beta$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = 0 \Rightarrow j_\alpha \xi^\alpha = 0.$$

$$\Rightarrow j^\alpha = a \xi^\alpha + b e^{\alpha A}$$

Dominant energy condition:  $j^\alpha$  must be timelike or null.

$$\frac{\partial K}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha}_A \xi^{\beta} = 8\pi \zeta_{\alpha} e^{\alpha}_A$$

$$\zeta_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow \zeta_{\alpha} \xi^{\alpha} = 0.$$

$$\Rightarrow \zeta^{\alpha} = a \xi^{\alpha} + b^A e^{\alpha}_A$$

Dominant energy condition:  $\zeta^{\alpha}$  must be timelike or null.  
 $\Rightarrow b^A = 0$

$$= \frac{1}{2} (K_{\alpha\beta\gamma\delta} N^{\alpha} \xi^{\beta} \xi^{\gamma} - \underbrace{\{a_{\alpha\beta}\}}_{\text{trace}} N_{\gamma\delta} \xi^{\alpha} \xi^{\beta} - \underbrace{\{a_{\alpha\beta}\}}_{\text{trace}} N_{\gamma\delta} \xi^{\alpha} \xi^{\beta})$$

$$\frac{\partial k}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha}_A \xi^{\beta} = 8\pi \zeta_{\alpha} e^{\alpha}_A$$

$$\zeta_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow \zeta_{\alpha} \xi^{\alpha} = 0.$$

$$\Rightarrow \zeta^{\alpha} = a \xi^{\alpha} + b^A e^{\alpha}_A$$

Dominant energy condition:  $\zeta^{\alpha}$  must be timelike or null.  
 $\Rightarrow b^A = 0$

$$\frac{\partial k}{\partial \omega^A} = -8\pi T_{\alpha\beta} e^{\alpha}_A \xi^{\beta} = 8\pi \zeta_{\alpha} e^{\alpha}_A$$

$$\zeta_{\alpha} = -T_{\alpha\beta} \xi^{\beta}$$

$$T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = 0 \Rightarrow \zeta_{\alpha} \xi^{\alpha} = 0.$$

$$\Rightarrow \zeta^{\alpha} = a \xi^{\alpha} + b^A e^{\alpha}_A \quad \text{if } \zeta^{\alpha} = b^A b^B \sigma_{AB} \neq 0$$

Dominant energy condition:  $\zeta^{\alpha}$  must be timelike or null.  
 $\Rightarrow b^A = 0$

$$= \cancel{K_{\alpha\beta\mu\nu} \xi^{\alpha} \xi^{\beta} \xi^{\mu} \xi^{\nu}} - \underbrace{\zeta_{\alpha\beta} \xi^{\alpha} \xi^{\beta}}_{k \xi_{\alpha}} N_{,\mu} \xi^{\mu} - \zeta_{\alpha\beta} N_{,\mu} \underbrace{\xi^{\alpha} \xi^{\beta}}_{k \xi^{\beta}}$$

$$= -k \underbrace{\xi_{\alpha} N_{,\mu} \xi^{\mu} \xi^{\alpha}} - k \underbrace{\xi_{\alpha\beta} \xi^{\alpha} \xi^{\beta}} N^{\alpha}$$

$$\frac{\partial K}{\partial \omega^A} = -8\pi T_{0p} e^{\lambda} \xi^p = 8\pi \zeta_{\alpha} e^{\lambda}$$

$$\zeta_{\alpha} = -T_{0p} \xi^p$$

$$T_{0p} \xi^p = 0 \Rightarrow \zeta_{\alpha} \xi^{\alpha} = 0.$$

$$\Rightarrow \zeta^{\alpha} = a \xi^{\alpha} + b^A e_A^{\alpha}$$

Dominant energy condition:  $\zeta^{\alpha}$  must be timelike or null  $\Rightarrow b^A = 0$

$$\zeta^{\alpha} = a \xi^{\alpha}$$

$b^A b^B \sigma_{AB} \neq 0$

$$= -\kappa \xi_{\alpha} N^{\alpha} = \kappa \xi_{\alpha} N^{\alpha}$$

$$|\alpha| \neq 0 \Rightarrow \dots$$

$$\Rightarrow j^\alpha = a \xi^\alpha + b^A e^A_\alpha$$

Dominant energy condition:  $j^\alpha$  must be timelike or null.  
 $\Rightarrow b^A = 0$

$$j^\alpha = a \xi^\alpha \Rightarrow \left( \frac{\partial \mathcal{L}}{\partial \theta^A} \right)_V = 0$$

$$\left( \frac{\partial \mathcal{L}}{\partial V} \right)_{\theta^A} = k_{11} \xi^\mu = \left( -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta - \xi_{\alpha;\beta\mu} N^\alpha_{;\nu} \xi^\beta - \xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \right)$$

$$= \left( -R_{\alpha\beta\mu\nu} \xi^\nu N^\alpha \xi^\beta \xi^\mu - \underbrace{\xi_{\alpha;\beta\mu} \xi^\beta N^\alpha_{;\nu} \xi^\mu}_{k \xi^\alpha} - \xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu \right)$$

$$= -k \underbrace{\xi_\alpha N^\alpha_{;\beta\mu} \xi^\beta \xi^\mu}_{\alpha < \mu} - k \underbrace{\xi_{\alpha;\beta\mu} \xi^\beta \xi^\mu}_{k \xi^\alpha} N^\alpha$$

# First law

During a quasi-static process :

## First law

During a quasi-static process :

$$\delta M = \frac{k}{8\pi} \delta A + \Omega + \delta J$$



## First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega \delta J + \delta U$$

## First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega \delta J$$

infinitesimal Top

## First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega_H \delta J$$

infinitesimal  $T_{\mu\nu}$

$$\delta M = - \int_H T_{\mu\nu} \xi^\mu d\Sigma^\nu$$

## First law

During a quasi-static process:

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta J$$

infinitesimal  $T_{\alpha\beta}$

$$\delta M = - \int_H T_{\alpha}^{\alpha} + P \delta \Sigma_{\alpha}$$

$$\delta J = \int_H T_{\alpha}^{\alpha} Q^{\beta} \delta \Sigma_{\alpha}$$

## First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega_H \delta J$$

infinitesimal  $T_{\alpha\beta}$

$$\delta M = - \int_H T_{\alpha}^{\alpha} \eta^{\beta} d\Sigma_{\alpha}$$

$$\delta J = \int_H T_{\alpha}^{\alpha} \eta^{\beta} d\Sigma_{\alpha}$$

$$\delta M - \Omega_H \delta J = - \int_H T_{\alpha}^{\alpha} (\eta^{\beta} + \Omega_H \eta^{\beta}) d\Sigma_{\alpha}$$

Individual Top

$$\delta M = - \int_H T_{\alpha}^{\alpha} + p \int \delta \Sigma_{\alpha}$$

$$\delta S = \int_H T_{\alpha}^{\alpha} q^{\beta} \int \delta \Sigma_{\alpha}$$

$$\delta M - \Omega_H \delta S = - \int_H T_{\alpha}^{\alpha} \underbrace{(p + \Omega_H q^{\beta})}_{\xi^{\beta}} \underbrace{\int \delta \Sigma_{\alpha}}_{-\xi_{\alpha} \int \delta V}$$

$$\delta M - \Omega_H \delta S = \int_H T_{\alpha\beta} \xi^{\alpha} \xi^{\beta} \int \delta V$$

$$\delta M - \Omega_H \delta S = \int_H T_{\alpha\beta} \xi^\alpha \eta^\beta dS dV$$

$$T_{\alpha\beta} \xi^\alpha \eta^\beta = \frac{1}{8\pi} \left( \frac{\delta\theta}{\delta V} - \theta \right)$$

$$= \frac{\kappa}{8\pi} \left( \int_H \theta dS dV - \frac{1}{8\pi} \int_H \frac{\delta\theta}{\delta V} dS dV \right)$$

$$\delta M - \Omega_H \delta T = \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \delta S \delta V$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{8\pi} \left( k \theta - \frac{\delta \theta}{\delta V} \right)$$

$$= \frac{k}{8\pi} \int_H \theta \delta S \delta V - \frac{1}{8\pi} \int_H \frac{\delta \theta}{\delta V} \delta S \delta V$$



$$\delta M - \Omega_H \delta S = \int_H T_{\alpha\beta} \xi^\alpha \zeta^\beta \delta S \delta V$$

$$T_{\alpha\beta} \xi^\alpha \zeta^\beta = \frac{1}{8\pi} \left( k\theta - \frac{\delta\theta}{\delta V} \right)$$

$$= \frac{k}{8\pi} \int_H \theta \delta S \delta V - \frac{1}{8\pi} \int_H \frac{\delta\theta}{\delta V} \delta S \delta V$$

$\int_{S^2} \theta \delta S \Big|_{V=-\infty}^{V=\infty}$

$$\delta M - \Omega_H \delta S = \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \delta S \delta V$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{8\pi} \left( k\theta - \frac{\delta\theta}{\delta V} \right)$$

$$= \frac{k}{8\pi} \int_H \underbrace{\theta \delta S \delta V}_{\frac{\delta}{\delta V} \delta S} - \frac{1}{8\pi} \int_H \underbrace{\frac{\delta\theta}{\delta V} \delta S \delta V}_{\int_{S^2} \theta \delta S \Big|_{V=V_0}^{V=V_1}}$$

$$\delta M - \Omega_H \delta S = \int_H T_{\alpha\beta} \xi^\alpha \zeta^\beta \partial S \partial V$$

$$T_{\alpha\beta} \xi^\alpha \zeta^\beta = \frac{1}{8\pi} \left( k\theta - \frac{\partial\theta}{\partial V} \right)$$

$$= \frac{k}{8\pi} \int_H \underbrace{\theta \partial S \partial V}_{\frac{\partial\theta}{\partial V} \partial S} - \frac{1}{8\pi} \int_H \underbrace{\frac{\partial\theta}{\partial V} \partial S \partial V}_{\int_{V=-\infty}^{V=\infty} \partial S}$$

$$\delta M - \Omega_H \delta J = \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \sqrt{-g} d^3x$$

$$T_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{8\pi} \left( k \partial_\alpha \partial_\beta \phi - \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \right)$$

$$= \frac{k}{8\pi} \int_H \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x - \frac{1}{8\pi} \int_H \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x$$

$\int_{\partial V} \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x$   
 $\left. \begin{array}{l} \alpha = \nu \\ \nu = -\infty \end{array} \right\} \int_{\partial V} \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x$   
 $\left. \begin{array}{l} \alpha = \nu \\ \nu = \infty \end{array} \right\} \int_{\partial V} \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x$

$$= \frac{k}{8\pi} \int_H \partial_\alpha \phi \partial_\beta \phi \sqrt{-g} d^3x$$

Second law

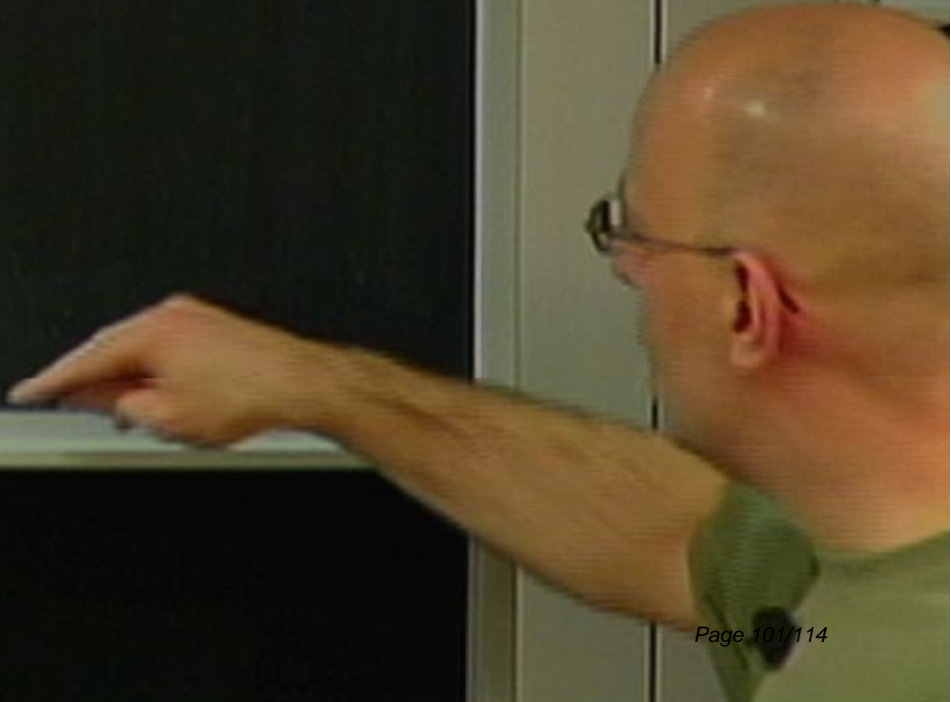
$$\delta A > 0$$

$$\sum \alpha$$

$$\alpha$$

$$\left. \begin{matrix} \sum \alpha \\ \sum \beta \end{matrix} \right\}$$

$$- \sum \alpha \delta S \delta V$$



Second law

$$\boxed{\delta A > 0}$$

$\sum x$

$x$

$\sum x$

$\sum x$

8

Second law

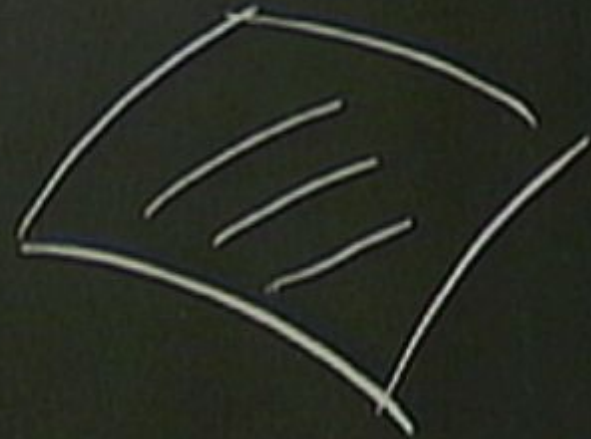
SA 7/0

$$dS = \sqrt{\sigma} d^2\theta$$

$$\sum_{ip} \xi^i \xi^p = t \xi^T$$

$$S = k\theta - \frac{1}{2}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} - 8\pi T_{\mu\nu} \xi^\mu \xi^\nu$$

$$\theta = \frac{1}{\sqrt{\sigma}} \frac{d}{dV} dS$$



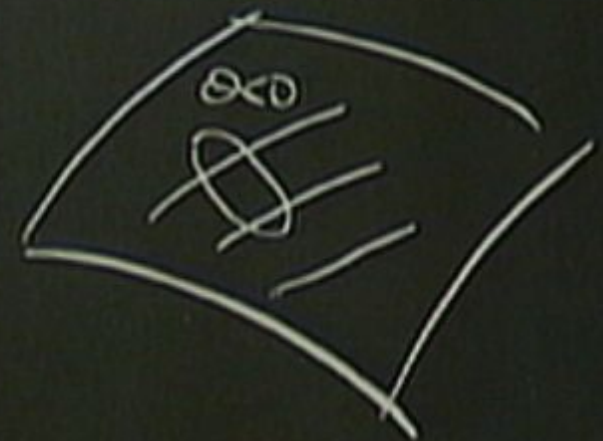


$$dS = \sqrt{\sigma} d^2\theta$$

$$\sum_{ip} \xi^i \xi^p = t \xi^T$$

$$k\theta - \frac{1}{2}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} - 8\pi T_{\mu\nu} \xi^\mu \xi^\nu$$

$$\frac{d}{dV} dS$$

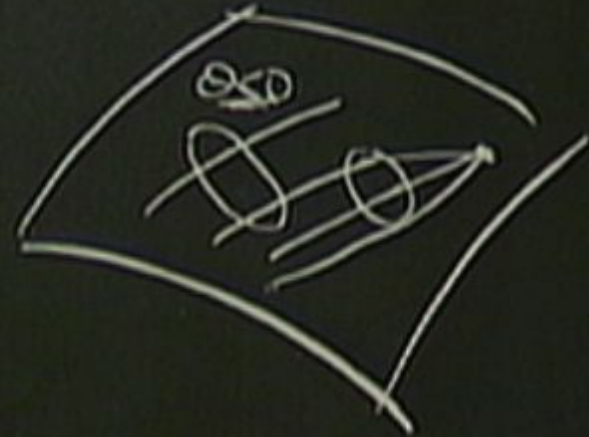


$$dS = \sqrt{\sigma} d^2\theta$$

$$\sum_{ip} \xi^i \xi^p = t \xi^T$$

$$\frac{d\theta}{dt} \theta - \frac{1}{2} \theta^2 - \sigma^{ap} \sigma_{ap} - 8\pi T_{ap} \xi^a \xi^p$$

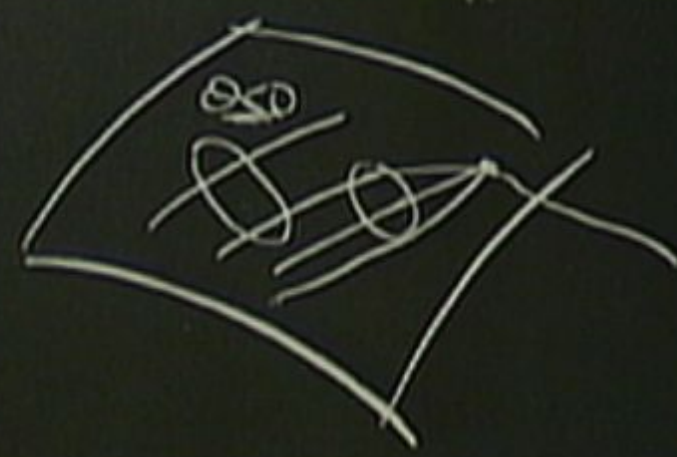
$$S \frac{d}{dt} dS$$



$$dS = \sqrt{\sigma} d^2\theta$$

$$\sum_{ip} \xi^i \xi^p = t \xi^T$$

$$\frac{\partial \theta}{\partial V} \left( \dots - \frac{1}{2} \theta^2 - \sigma^{ap} \sigma_{ap} - 8\pi T_{ap} \xi^a \xi^b \right)$$

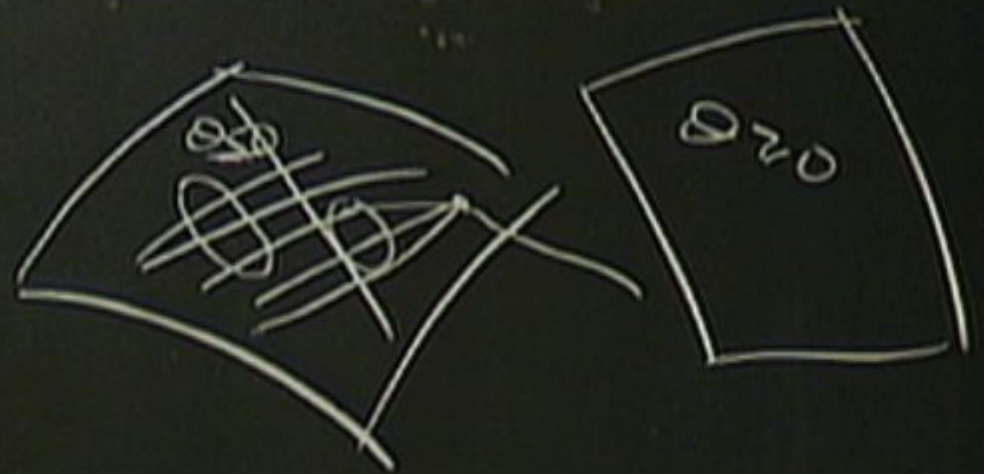


$$\partial_\mu = \nabla_\mu \partial_\nu$$

$$\sum_{j \neq i} \xi_j^{\mu} \xi_i^{\nu} = t \xi^{\mu \nu}$$

$$\frac{\delta \mathcal{L}}{\delta \mathcal{V}} = -\frac{1}{2} \Theta^2 - \sigma^{\mu \nu} \sigma_{\mu \nu} - 8\pi T_{\mu \nu} \xi^{\mu} \xi^{\nu}$$

$$\Theta = \frac{\delta \mathcal{L}}{\delta \mathcal{V}}$$

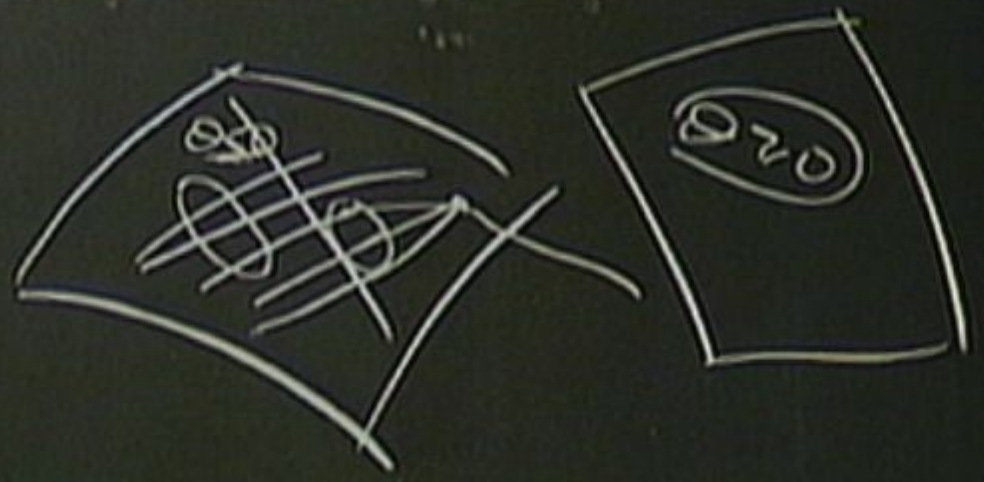


$$\partial \theta = \sqrt{\sigma} \partial \theta$$

$$\sum_{ip} \xi_i \xi_p = t \xi^T$$

$$\frac{\partial \theta}{\partial V} = k\theta - \frac{1}{2}\theta^2 - \sigma^{op}\sigma_{op} - 8\pi T_{op} \xi^T \xi^i$$

$$\theta = \frac{1}{\partial S} \frac{\partial}{\partial V} \partial S$$

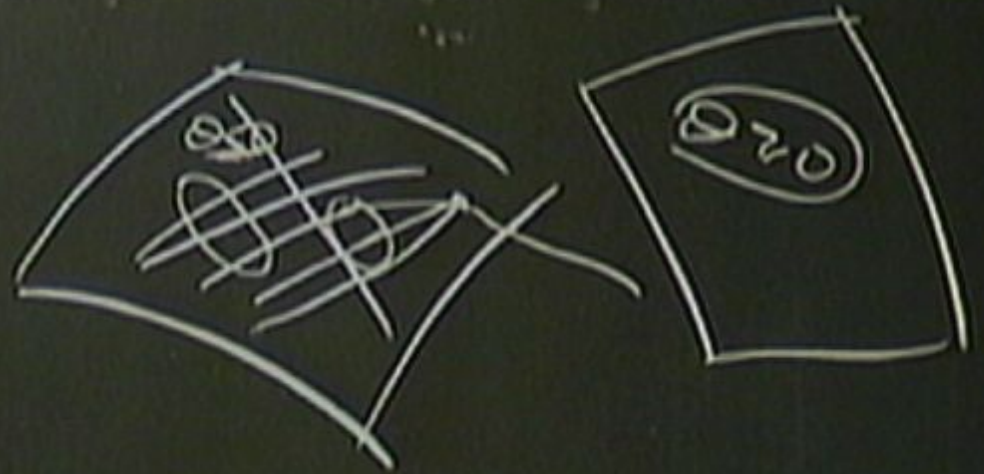


$$\partial \bar{\partial} = \nabla \sigma \partial \bar{\partial}$$

$$\sum_{j,p} \xi^{j,p} \xi^{p,j} = t \xi^T$$

$$\frac{\partial \theta}{\partial V} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\mu\nu} \sigma_{\mu\nu} - 8\pi T_{\mu\nu} \xi^{\mu} \xi^{\nu}$$

$$\theta = \frac{1}{\partial S} \frac{\partial}{\partial V} \partial S$$



First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega_H \delta S$$

infinitesimal  $T_{\alpha\beta}$

$$\delta M = - \int_H T_{\alpha}^{\alpha} + P \delta \Sigma_{\alpha}$$

$$\delta S = \int_H T_{\alpha}^{\alpha} Q^{\beta} \delta \Sigma_{\alpha}$$

$$\delta M - \Omega_H \delta S = - \int_H T_{\alpha}^{\alpha} \underbrace{(P + \Omega_H Q^{\beta})}_{\delta P} \delta \Sigma_{\alpha} \quad \underbrace{\delta \Sigma_{\alpha}}_{\delta S \delta V}$$

Sand law

$$\delta A \approx 0$$

$$= \frac{k}{8\pi} \int_H \delta S$$

# First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega_H \delta S$$

infinitesimal  $T_{\alpha\beta}$

$$\delta M = - \int_H T_{\alpha\beta} n^{\alpha} \xi^{\beta} d\Sigma_{\alpha}$$

$$\delta S = \int_H T_{\alpha\beta} \chi^{\alpha} \xi^{\beta} d\Sigma_{\alpha}$$

$$\delta M - \Omega_H \delta S = - \int_H T_{\alpha\beta} \underbrace{(\xi^{\alpha} + \Omega_H \chi^{\alpha})}_{\xi^{\alpha}} \underbrace{\xi^{\beta}}_{-\xi^{\beta}} d\Sigma_{\alpha}$$

Second law

$$\delta A \geq 0$$

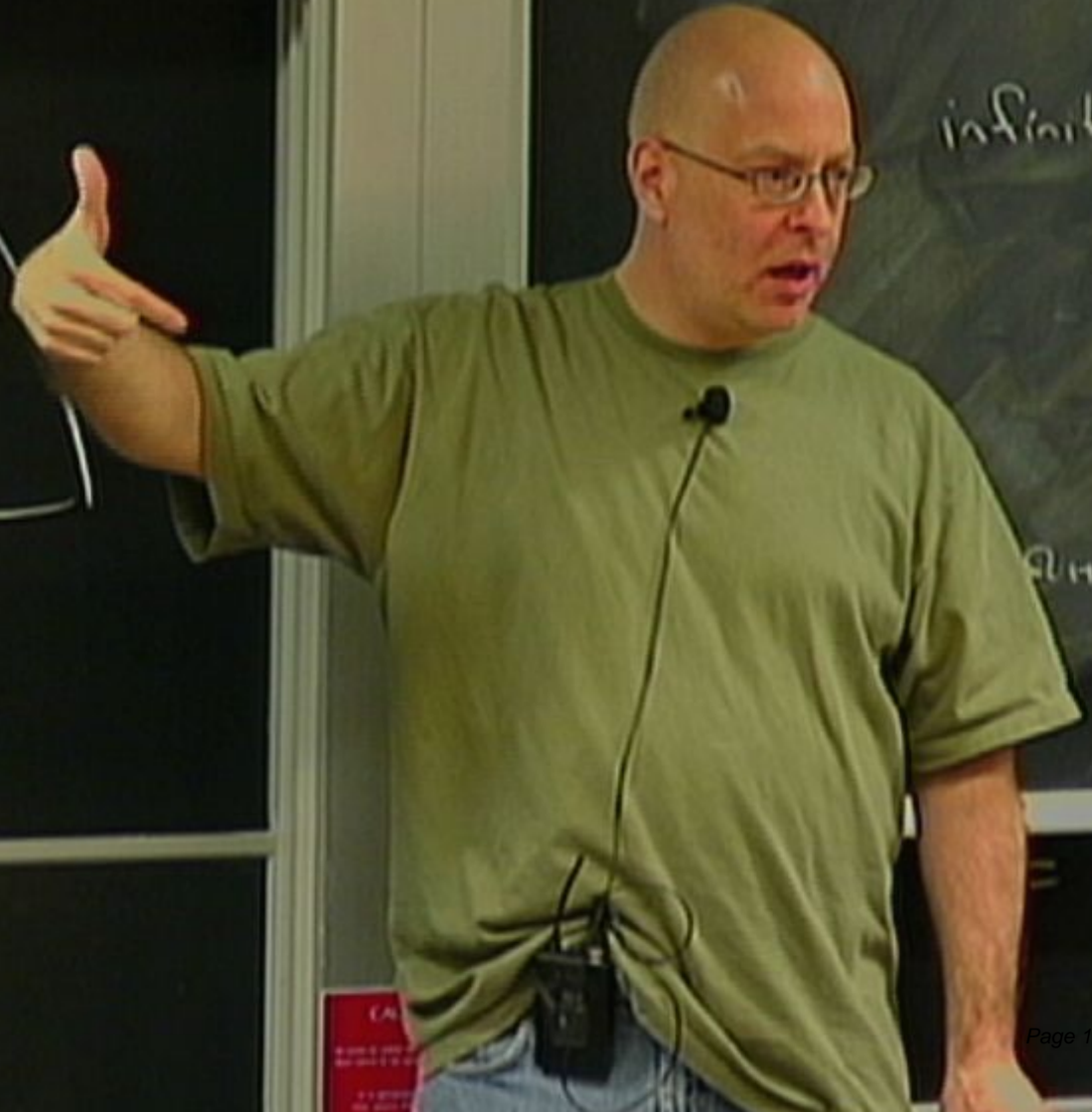
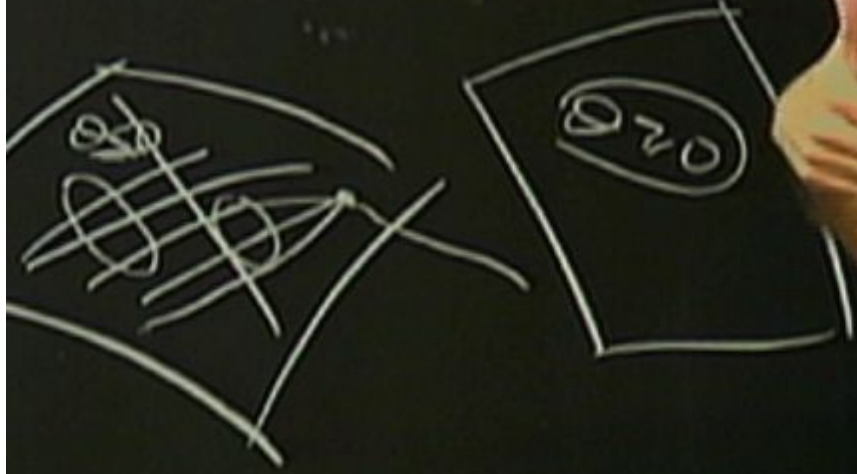


$$\sum_{\alpha} \omega^{\alpha} / \nu \quad \sum_{\alpha} N_{\alpha} \xi = -1$$

$$\sum_{\alpha} \gamma_{\alpha} + \sum_{\beta} \rho_{\beta} = 0$$

$$\sum_{\alpha} \gamma_{\alpha} \xi^{\alpha} = t c \xi^{\alpha}$$

$$-8\pi T_{\alpha\beta} \xi^{\alpha} \xi^{\beta}$$



First law

During a th

infinitesimal

$\delta M$

$\delta T$

$\delta S =$

$\frac{12}{8\pi}$

# First law

During a quasi-static process:

$$\delta M = \frac{k}{8\pi} \delta A + \Omega_H \delta S$$

infinitesimal Top

$$\delta M = - \int_H T_{\alpha}^{\alpha} + P^{\alpha} d\Sigma_{\alpha}$$

$$\delta S = \int_H T_{\alpha}^{\alpha} Q^{\alpha} d\Sigma_{\alpha}$$

$$\delta M - \Omega_H \delta S = - \int_H T_{\alpha}^{\alpha} \underbrace{(P^{\alpha} + \Omega_H Q^{\alpha})}_{\xi^{\alpha}} d\Sigma_{\alpha} = - \int_H \xi^{\alpha} dS dV$$

Second law

$$\delta A \geq 0$$

$$= \frac{k}{8\pi} \int_H \xi^{\alpha} dS$$