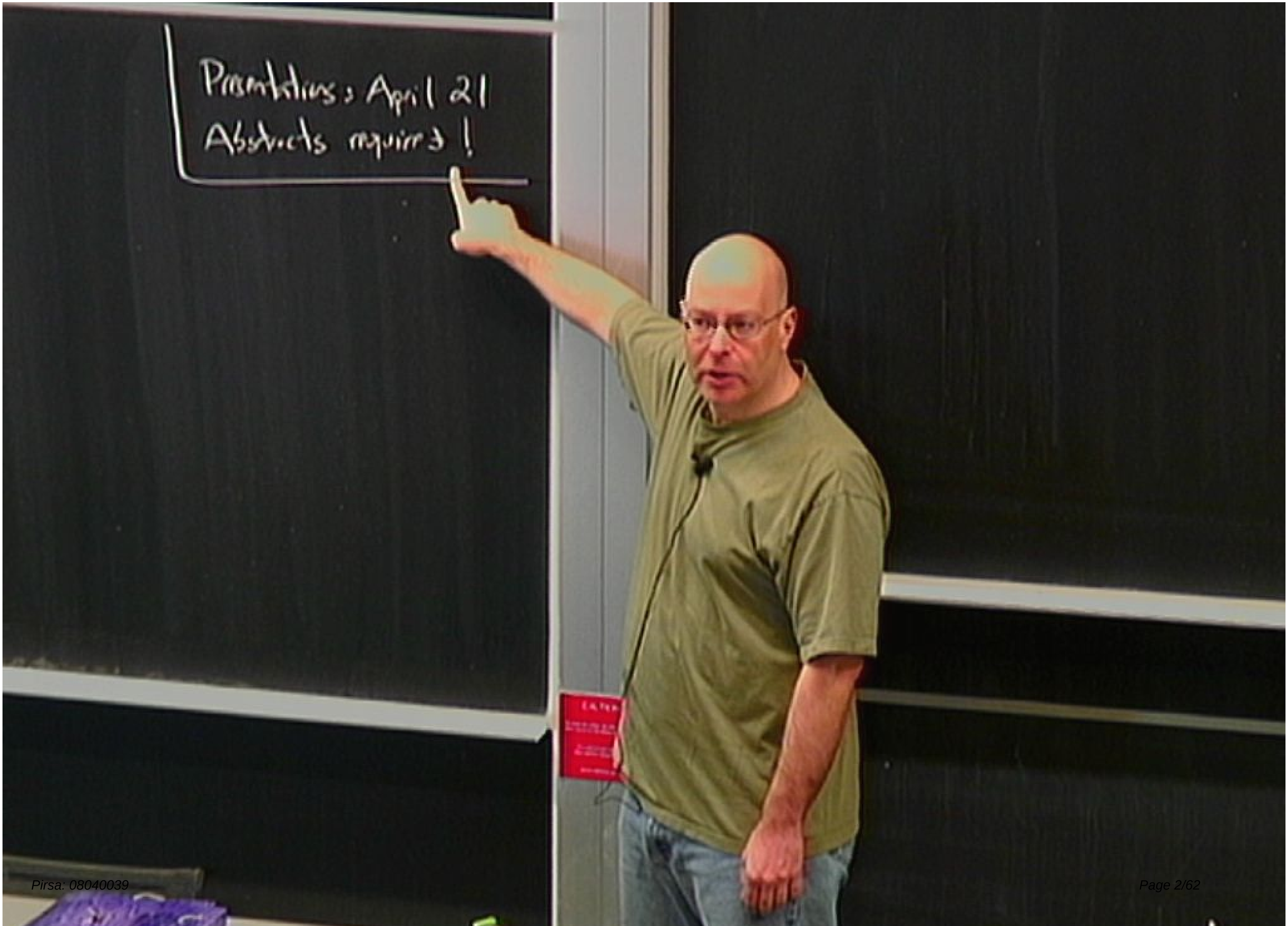
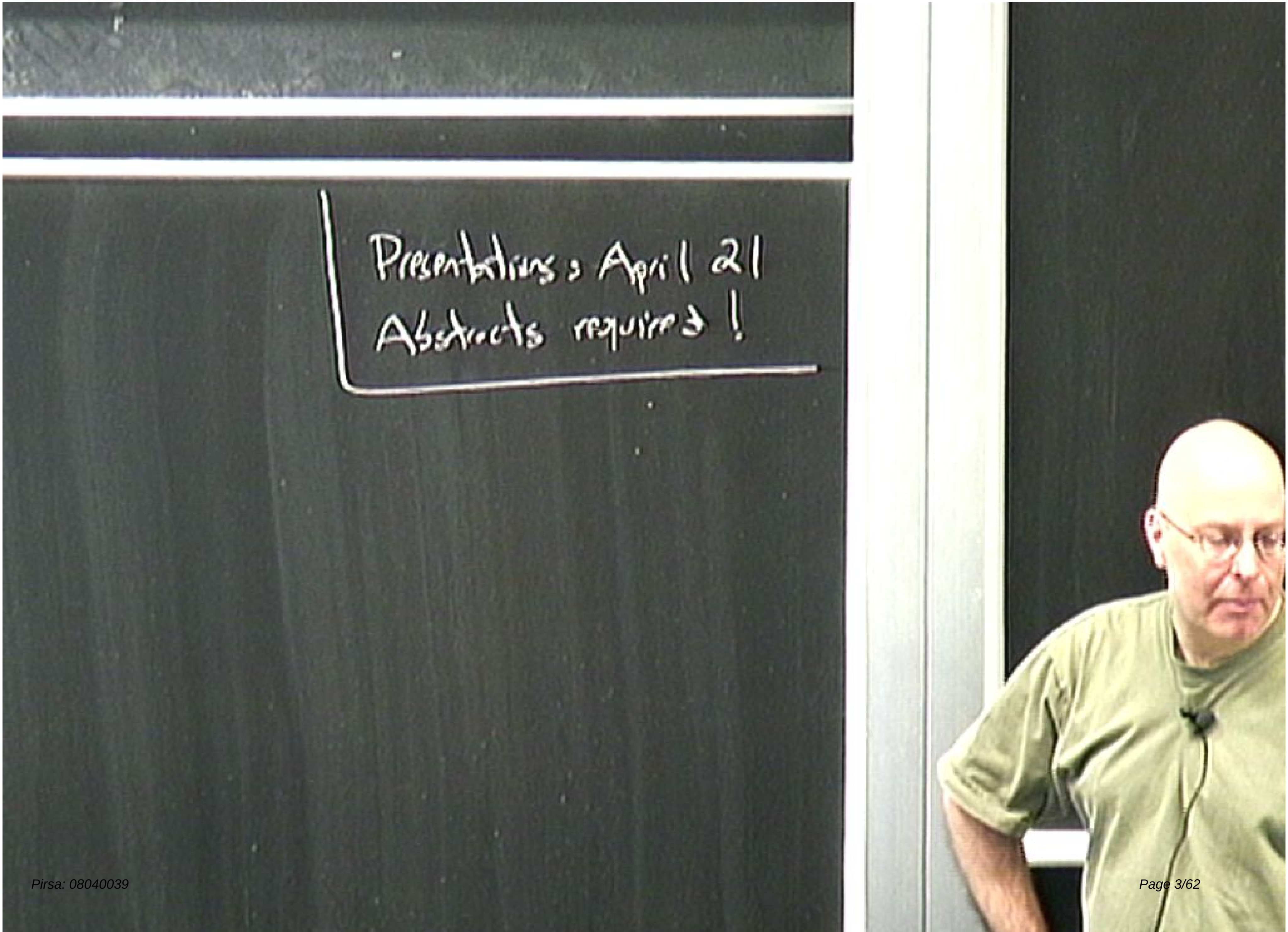
Title: Advanced General Relativity - Lecture 12A

Date: Apr 09, 2008 10:30 AM

URL: http://pirsa.org/08040039

Abstract: Advanced General Relativity


Presentations: April 21
Abstracts required !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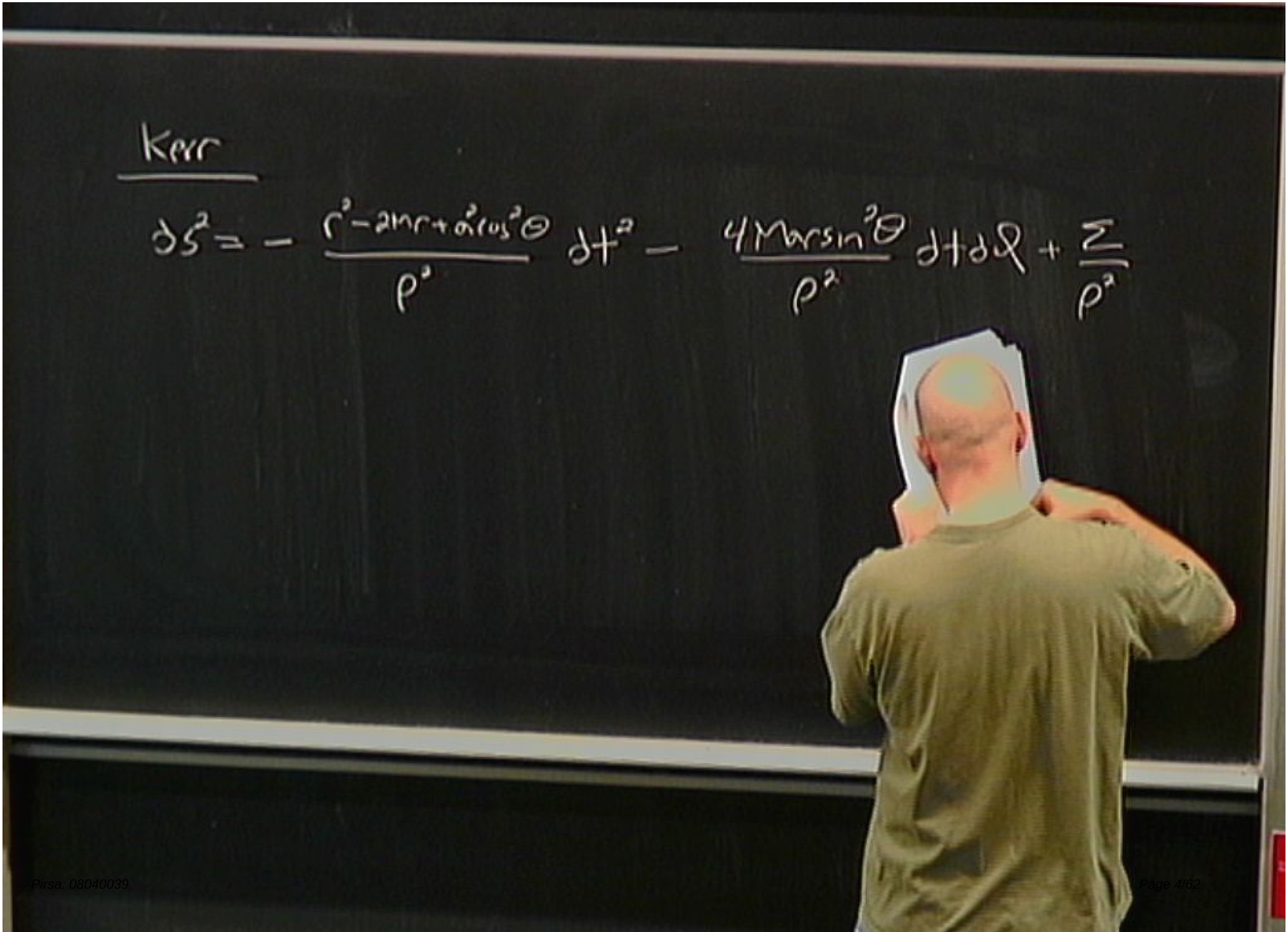
Kerr
$ds^2 = -\frac{r^2 - 2Mr + a^2\cos^2\theta}{\rho^2}\,dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\varphi + \frac{\Sigma}{\rho^2}$

Kerr

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$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

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$$+ \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2$$

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$$\text{Killing}\begin{cases} t^\mu = (1,0,0,0) \\ \varphi^\mu = (0,0,0,1) \end{cases}$$

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curvature singularity at $\rho^2 = 0$

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curvature singularity at $\rho^2 = 0$

coordinate singularity at $\Delta = 0$

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coordinate singularity at $\Delta = 0$

$\hookrightarrow r_\pm = M \pm \sqrt{M^2 - a^2}$

Kerr

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$M^2 \geq a^2$

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$J = aM$

Static observers: $r = \text{const}$, $\theta = \text{const}$, $\varphi = \text{const}$

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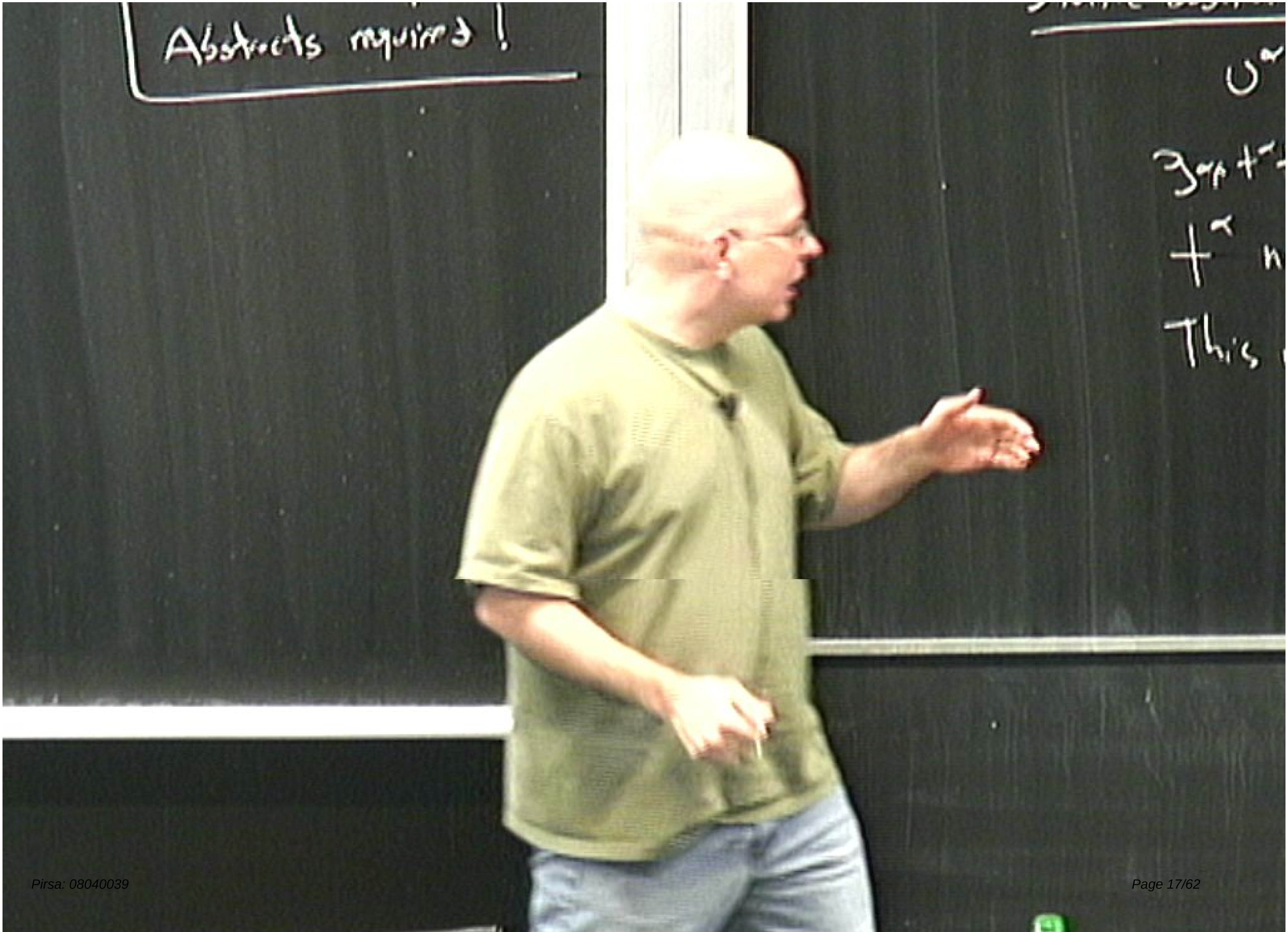
Abstracts required !
This

Static observers: $r = \text{const}$, $\theta = \text{const}$, $\varphi = \text{const}$

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t^α not hypersurface orthogonal.

This is not EH — Static limit

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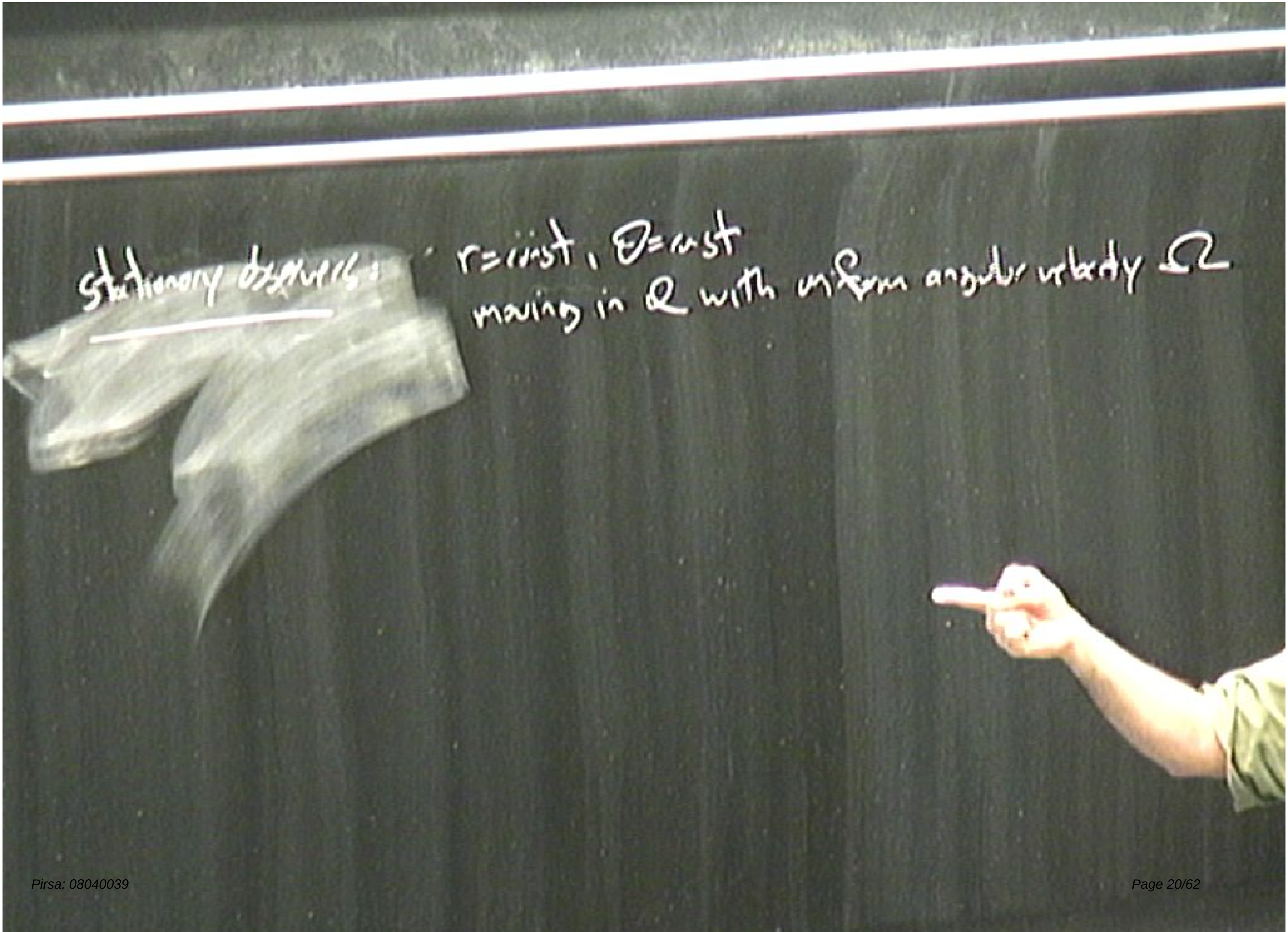
t^α not hypersurface orthogonal.

This is not EH — Static limit (no static observers can exist within static limit)


stationary observers: r = const, θ = const
moving in Q with uniform angular velocity Ω

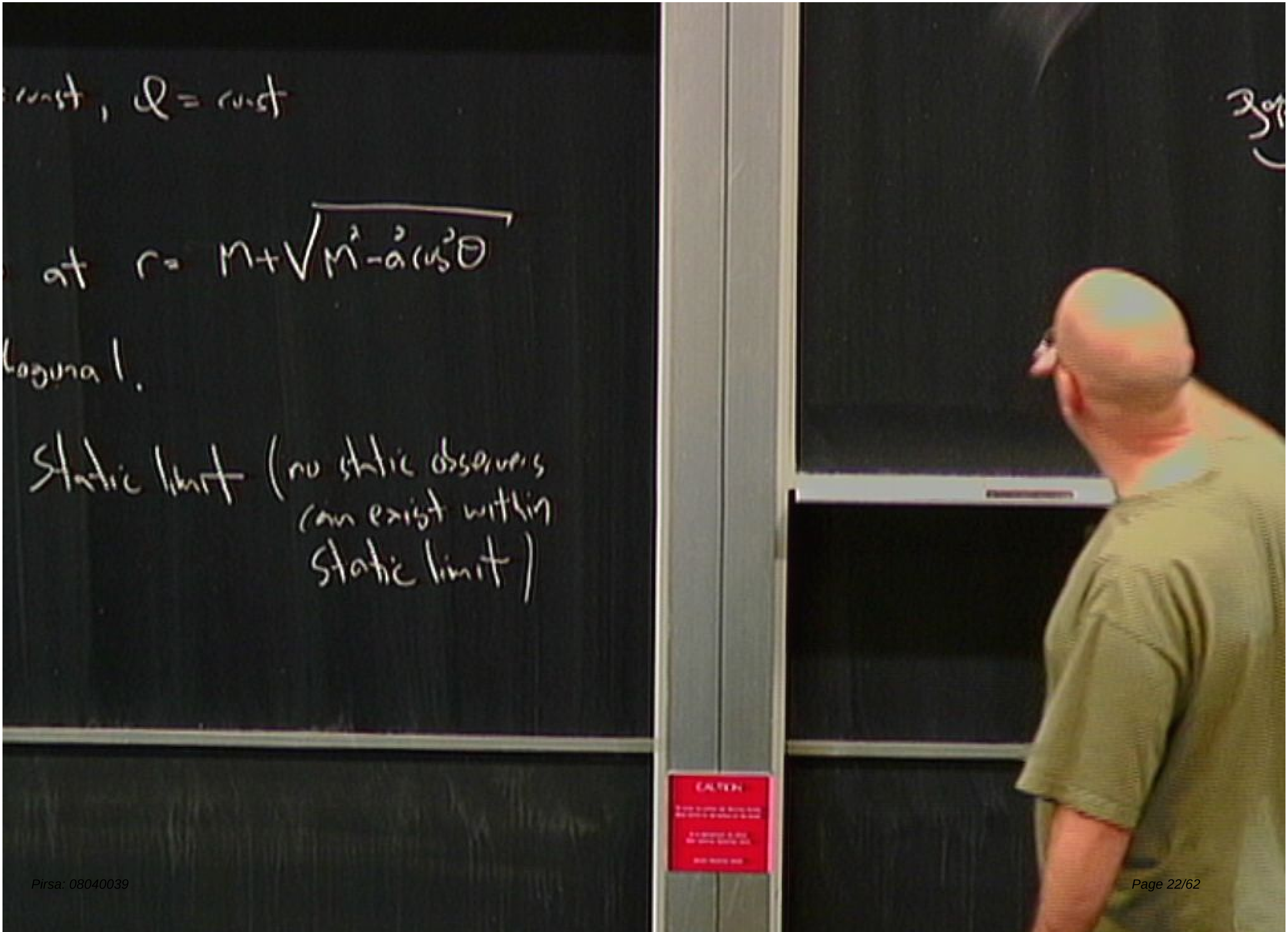
stationary observers: $r = \text{const}$, $\theta = \text{const}$

moving in ϕ with uniform angular velocity Ω

$$u^\alpha \propto \underbrace{t^\alpha + \Omega \phi^\alpha}_{\text{timelike}}$$

$$\underbrace{g_{\alpha\beta}(t^\alpha + \Omega \phi^\alpha)(t^\beta + \Omega \phi^\beta)}_{\text{quadratic in } \Omega} < 0$$

$$\longrightarrow \Omega_- < \Omega < \Omega_+$$


const , Q = const
at r = M+√(M²−a²cos²θ)
agonal.
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static limit)
CAUTION

stationary observers: $r = \text{const}$, $\theta = \text{const}$
moving in φ with uniform angular velocity Ω

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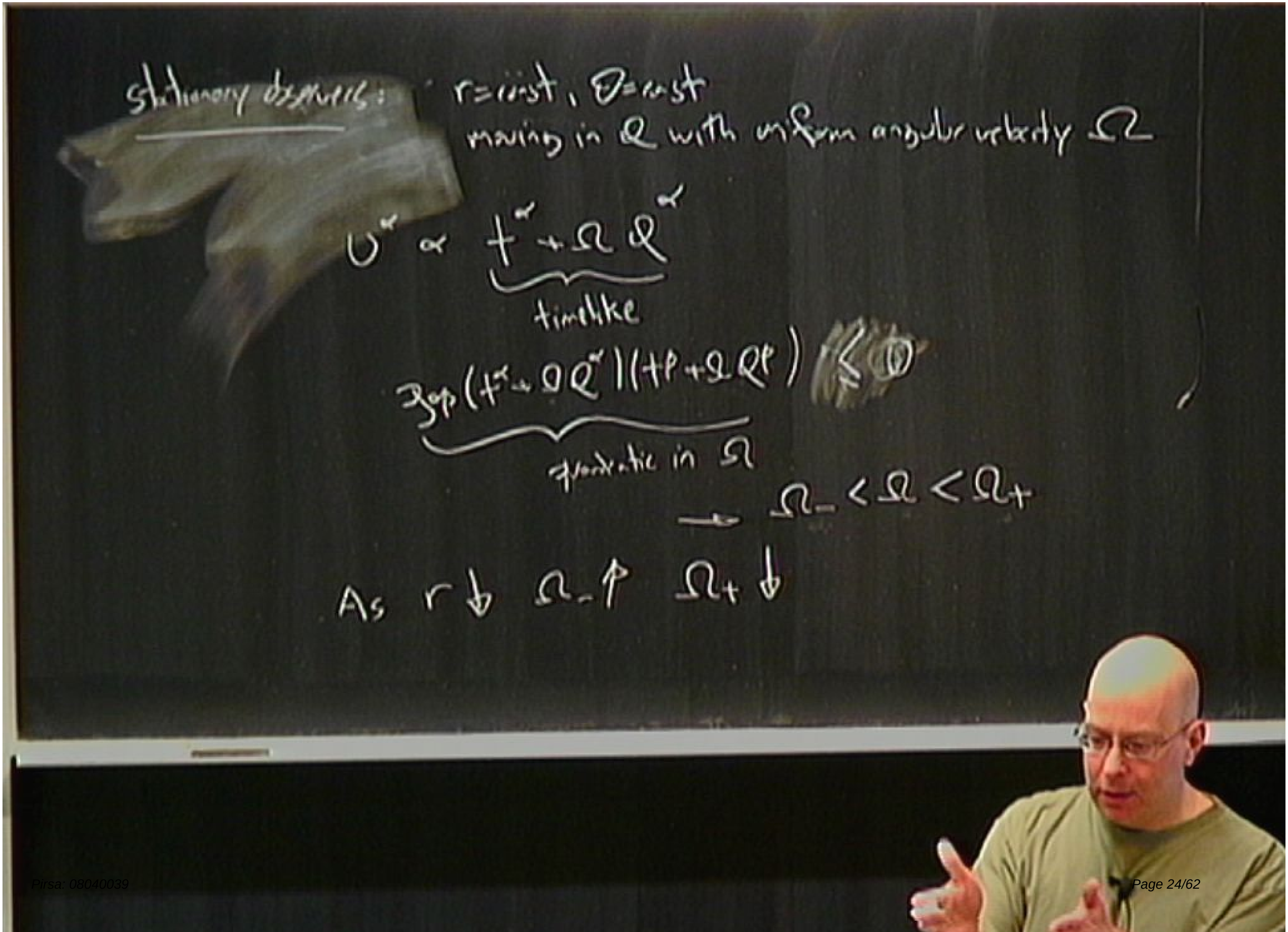
As $r \downarrow$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As $r \downarrow$ $\Omega_- \uparrow$ $\Omega_+ \downarrow$

stationary observers: $r = \text{const}$, $\theta = \text{const}$

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$$\to \Omega_- < \Omega < \Omega_+$$

As $r \downarrow$ $\Omega_- \uparrow$ $\Omega_+ \downarrow$

when $r \to r_+ = M + \sqrt{M^2 - a^2}$

$$\Omega_- = \Omega_+ = \Omega_H$$

stationary observers: $r = \text{const}$, $\theta = \text{const}$

moving in φ with uniform angular velocity Ω

$$u^\alpha \propto \underbrace{t^\alpha + \Omega \varphi^\alpha}_{\text{timelike}}$$

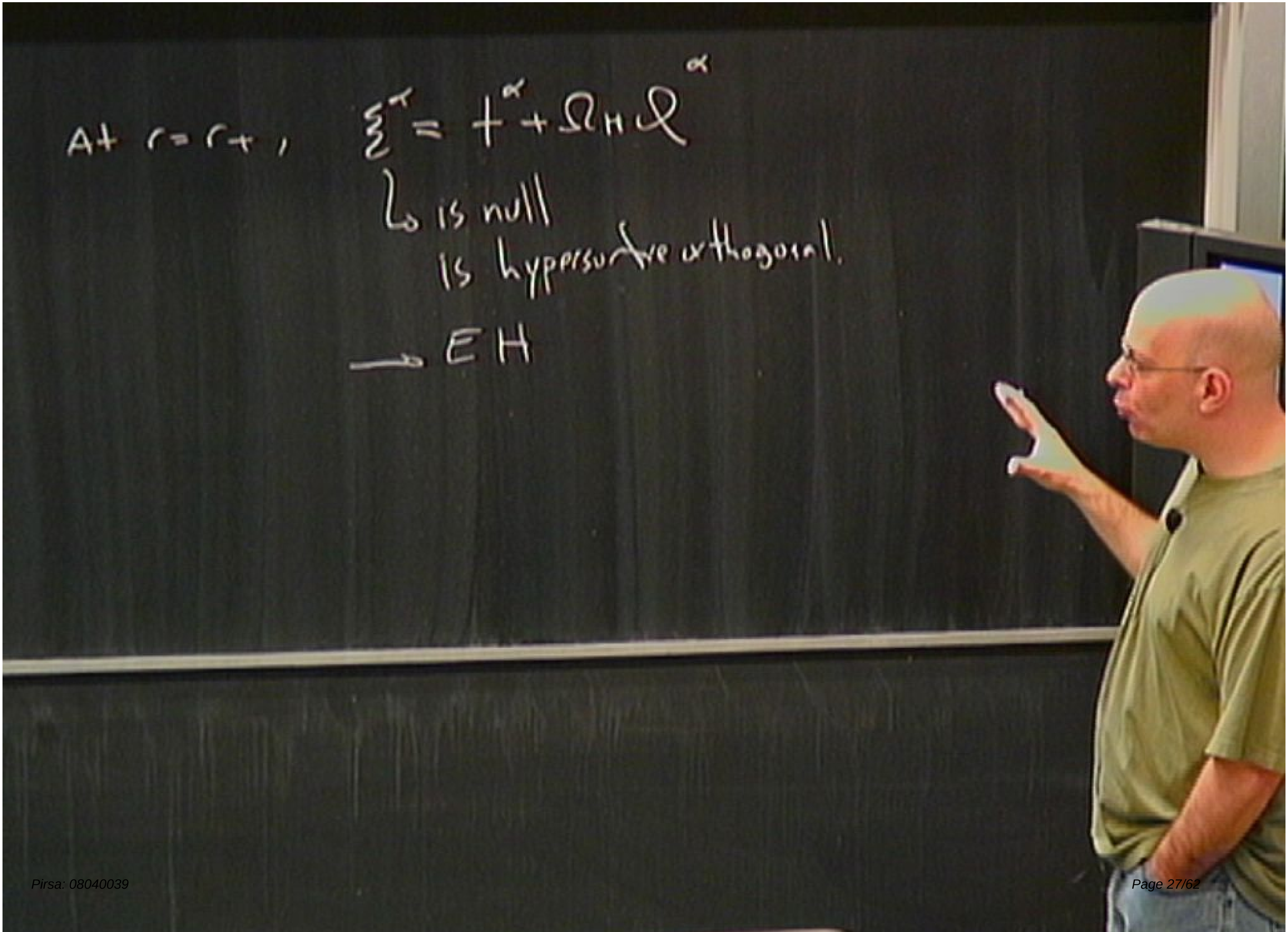
$$\underbrace{g_{\alpha\beta}(t^\alpha + \Omega\varphi^\alpha)(t^\beta + \Omega\varphi^\beta)}_{\text{quadratic in } \Omega} \leq 0$$

$$\rightarrow \Omega_- < \Omega < \Omega_+$$

As $r \downarrow$ $\Omega_- \uparrow$ $\Omega_+ \downarrow$

when $r \to r_+ = M + \sqrt{M^2 - a^2}$

$$\Omega_- = \Omega_+ = \Omega_H = \frac{a}{r_+^2 + a^2}$$


At $r = r_+$,
$\xi^\alpha = t^\alpha + \Omega_H \phi^\alpha$
is null
is hypersurface orthogonal.
→ EH

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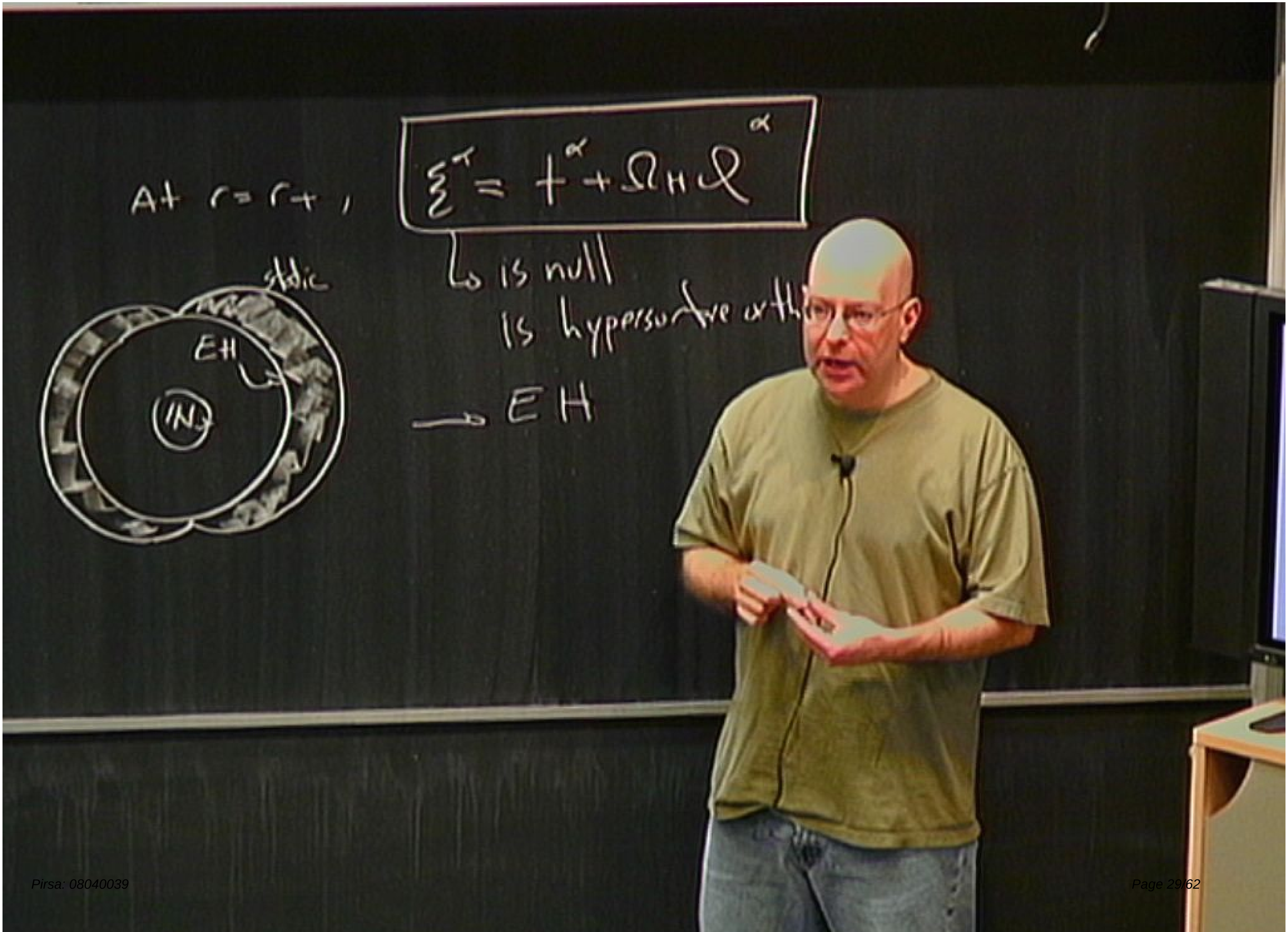
is hypersurface orthogonal.

$\longrightarrow$ EH


At r=r+,
ξ^α = t^α + Ω_H φ^α
is null
is hypersurface orth
→ EH
static
EH
IN

EH = causal boundary = null hypersurface
generated by null
geodesics without
endpoints

congruence of null generators cannot have
future caustics

Presentations: April 21
Abstracts required!

EH = causal boundary = null hypersurface generated by null geodesics without endpoints

congruence of null generators cannot have future caustics

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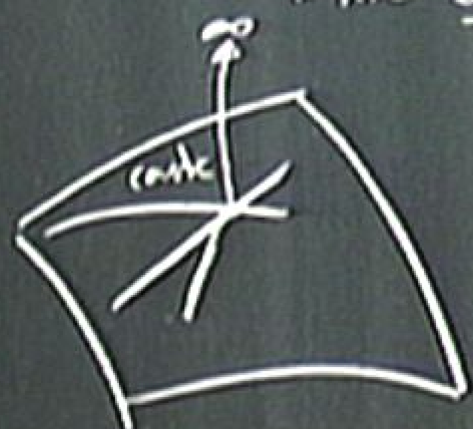
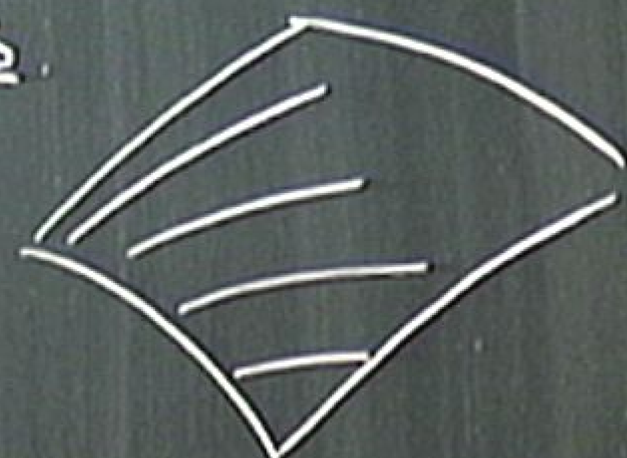
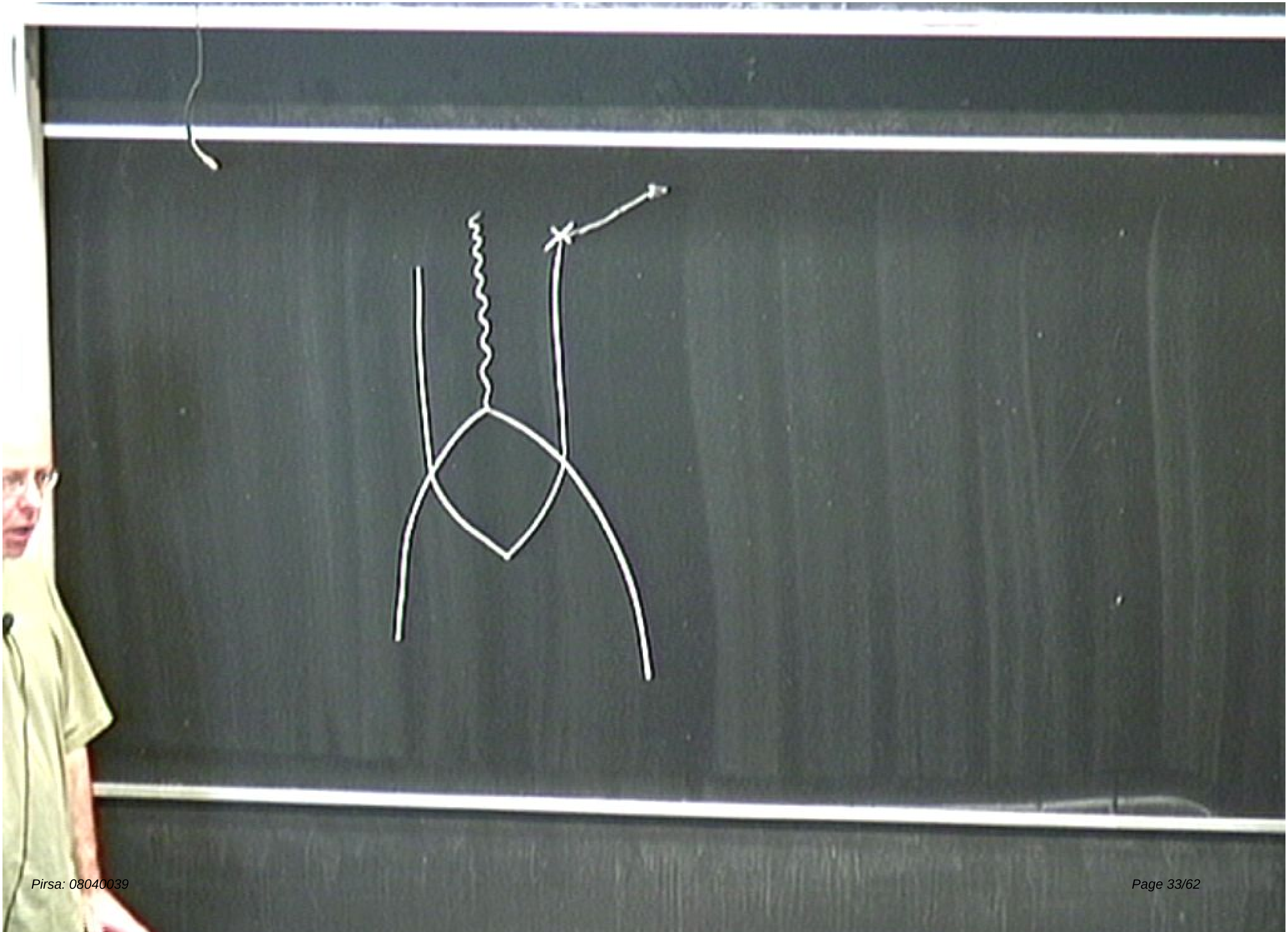


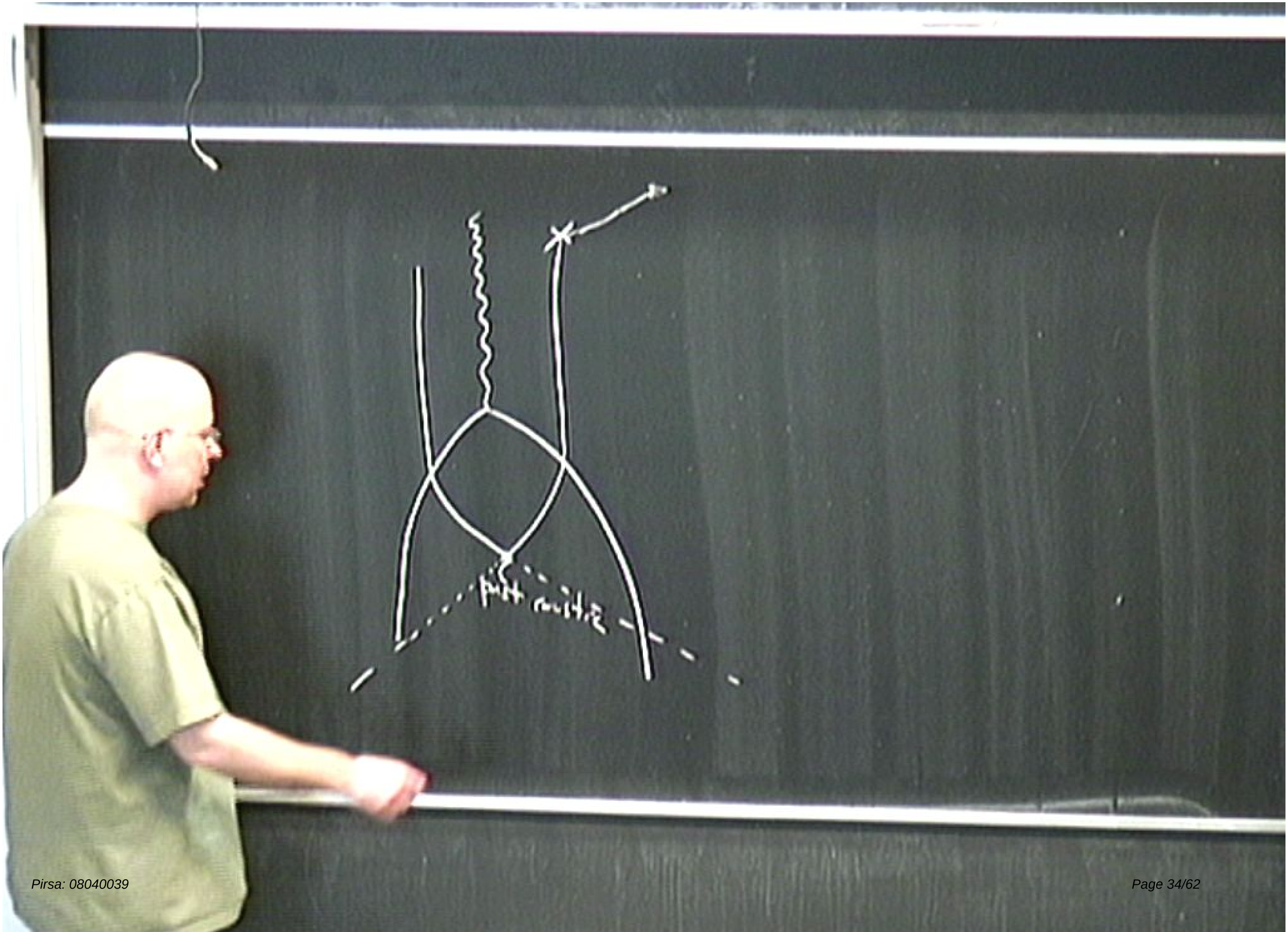


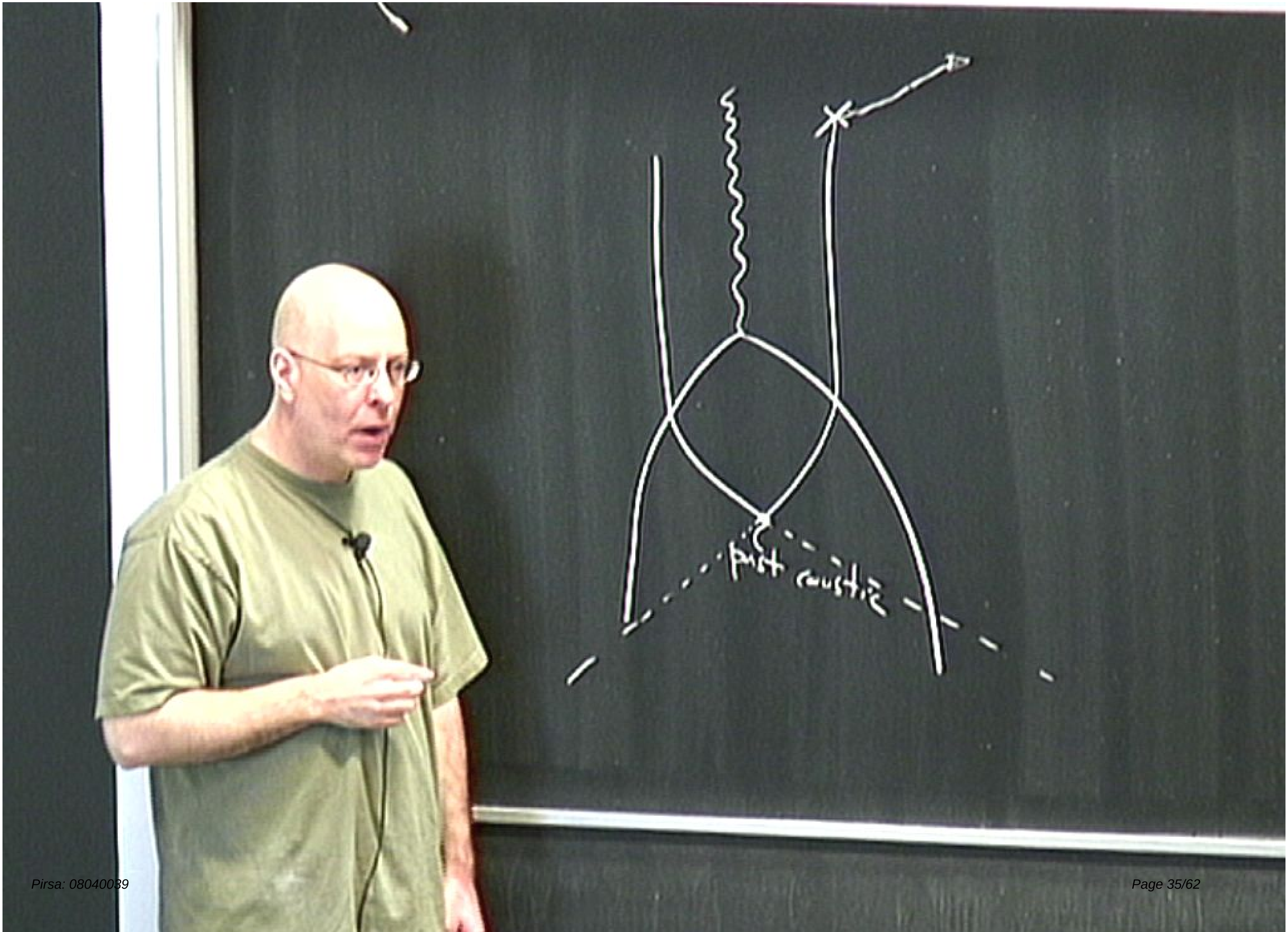
EH = causal boundary = null hypersurface generated by null geodesics without endpoints

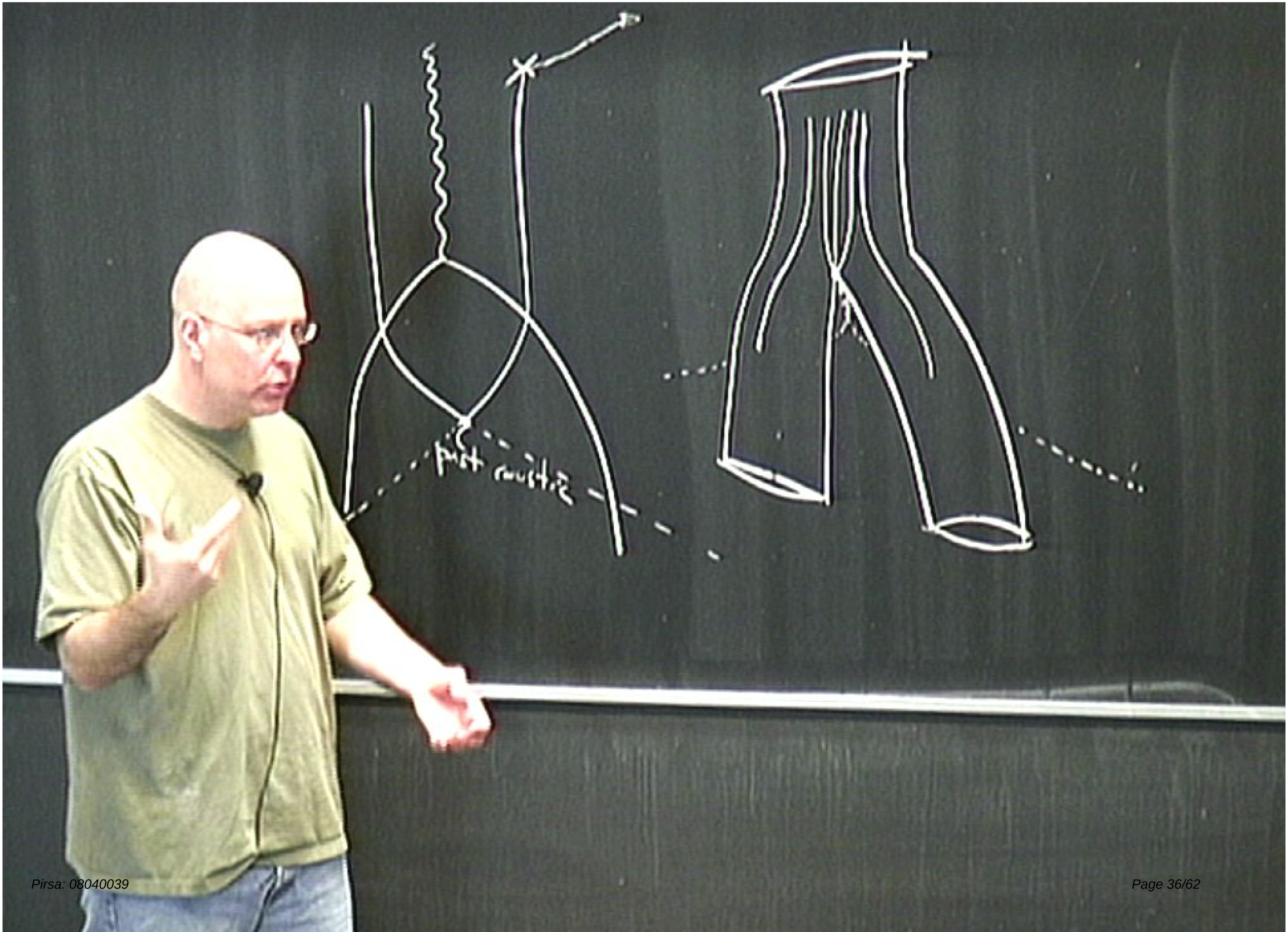
Presentations: April 21
Abstracts required!

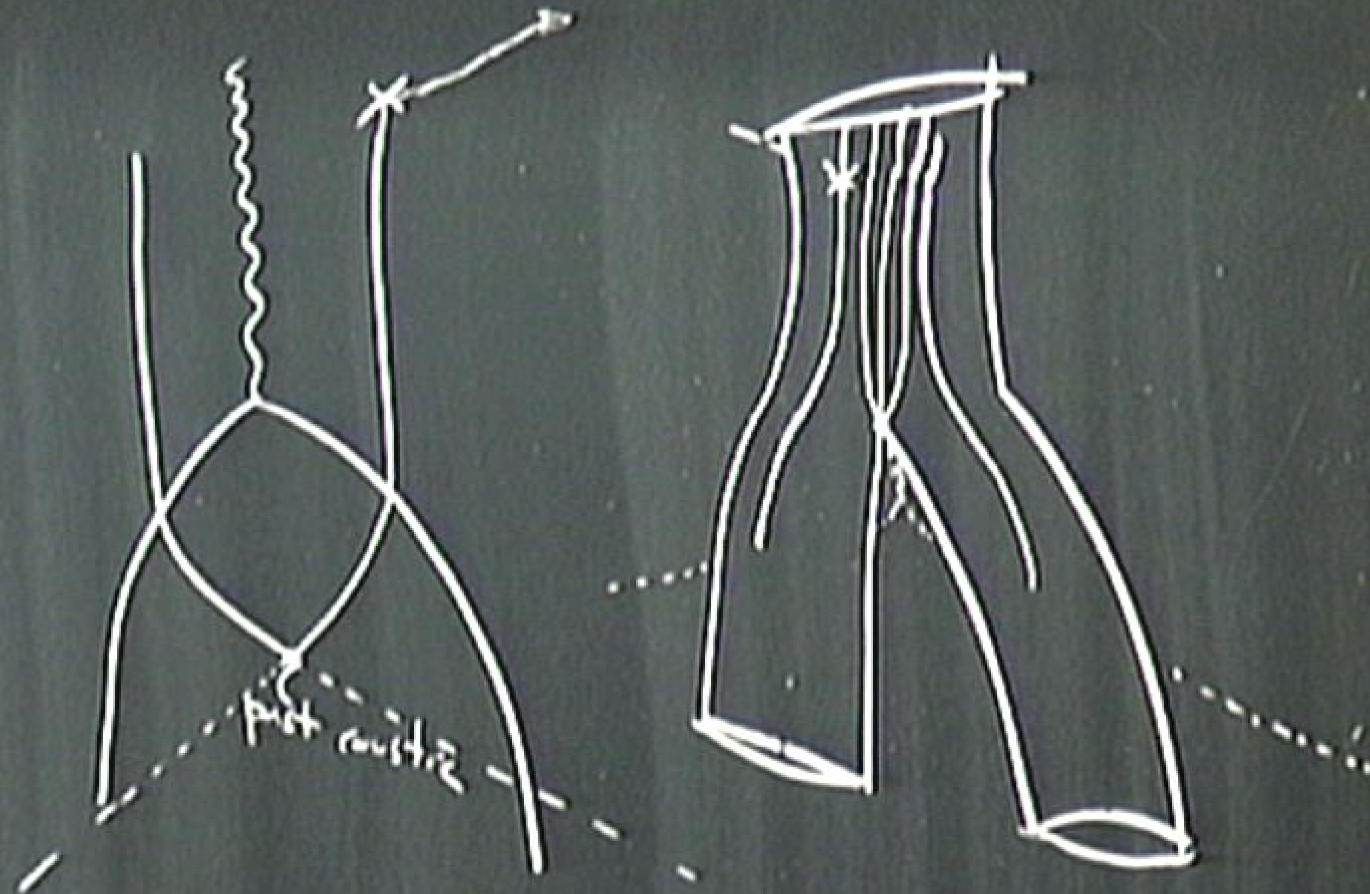
congruence of null generators cannot have <u>future caustics</u>






past caustic





Stationary BH

Hawking: static & nonrotating or $\underbrace{\text{axially symmetric}}_{\text{rotating}}$

rotating $\longrightarrow$ t^α, Q^α

Stationary BH

Hawking : static or $\underbrace{\text{axially symmetric}}_{\text{rotating}}$
↳ nonrotating

rotating $\longrightarrow$ t^α , φ^α

$$\xi^\alpha = t^\alpha + \Omega_H \varphi^\alpha$$

Stationary BH

Hawking : static or $\underbrace{\text{axially symmetric}}$ rotating

↳ nonrotating

rotating $\longrightarrow$ t^α , ϕ^α

$\xi^\alpha = t^\alpha + \Omega_H \phi^\alpha$ is null on EH

Stationary BH

Hawking: static or $\underbrace{\text{axially symmetric}}_{\text{rotating}}$

nonrotating

rotating $\rightarrow$ t^α, φ^α

$\xi^\alpha = t^\alpha + \Omega_H \varphi^\alpha$ is null on EH

EH is generated by null geodesics to which ξ^α is tangent

forced to rotate.

$$\Phi^2 = -g_{\alpha\beta}\,\xi^\alpha\xi^\beta$$

$$\text{EH}: \quad \Phi^2 = 0$$

$\Phi = -g_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}$

EH : $\Phi = 0$

normal to EH $\Leftarrow \partial_\alpha \Phi = -\left(g_{\mu\nu}\,\xi^{\mu}\xi^{\nu}\right)_{,\alpha}$

$\Phi = -g_{\sigma\rho}\,\xi^\sigma \xi^\rho$

EH : $\Phi = 0$

normal to EH $\Leftarrow$ $\partial_\alpha \Phi = -\left(g_{\mu\nu}\,\xi^\mu \xi^\nu\right)_{;\alpha}$

$= -2\,\xi^\mu\,\xi_{\mu;\alpha}$

$\Phi = - g_{\alpha\beta} \xi^\alpha \xi^\beta$

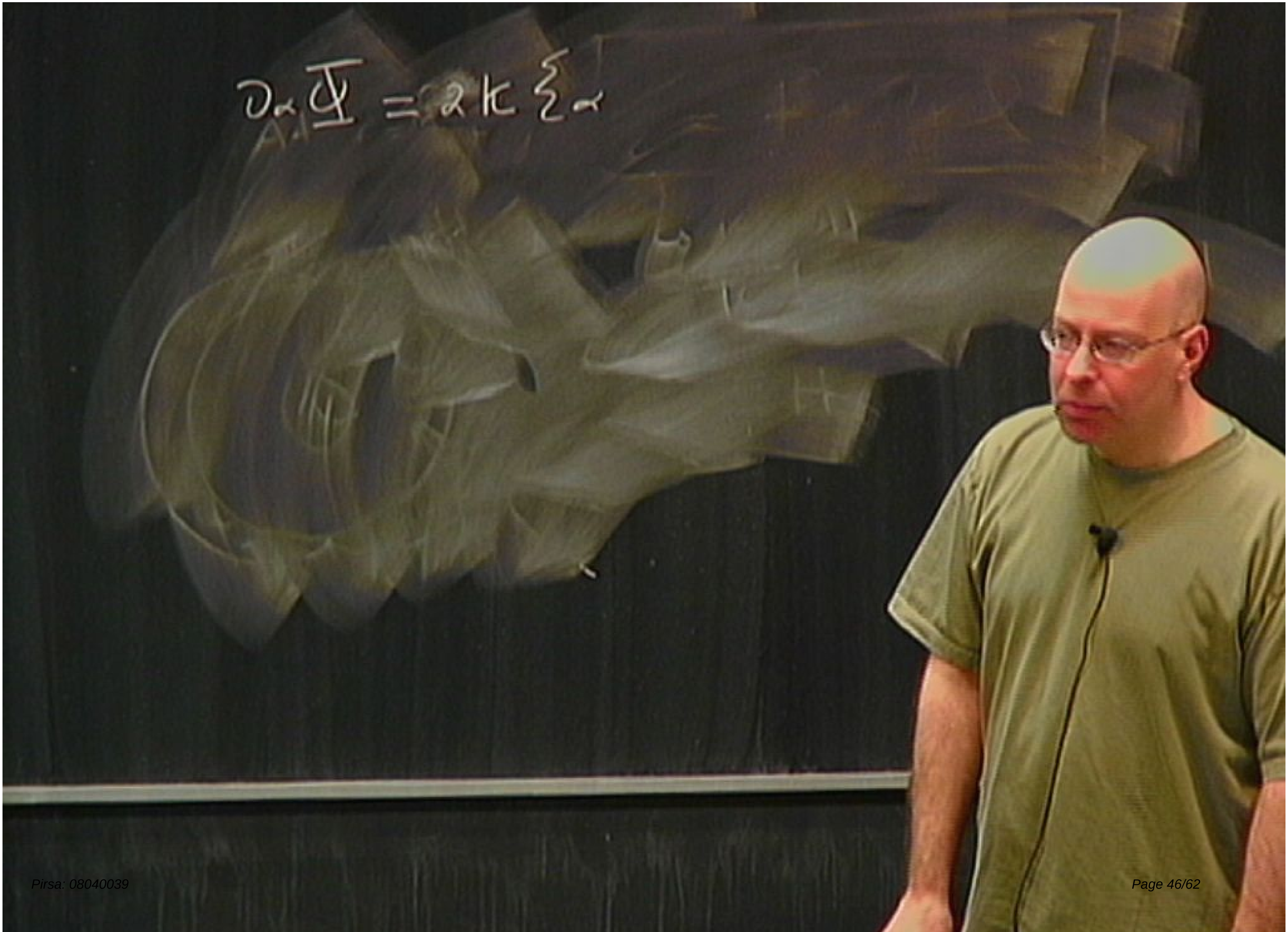
EH : $\Phi = 0$

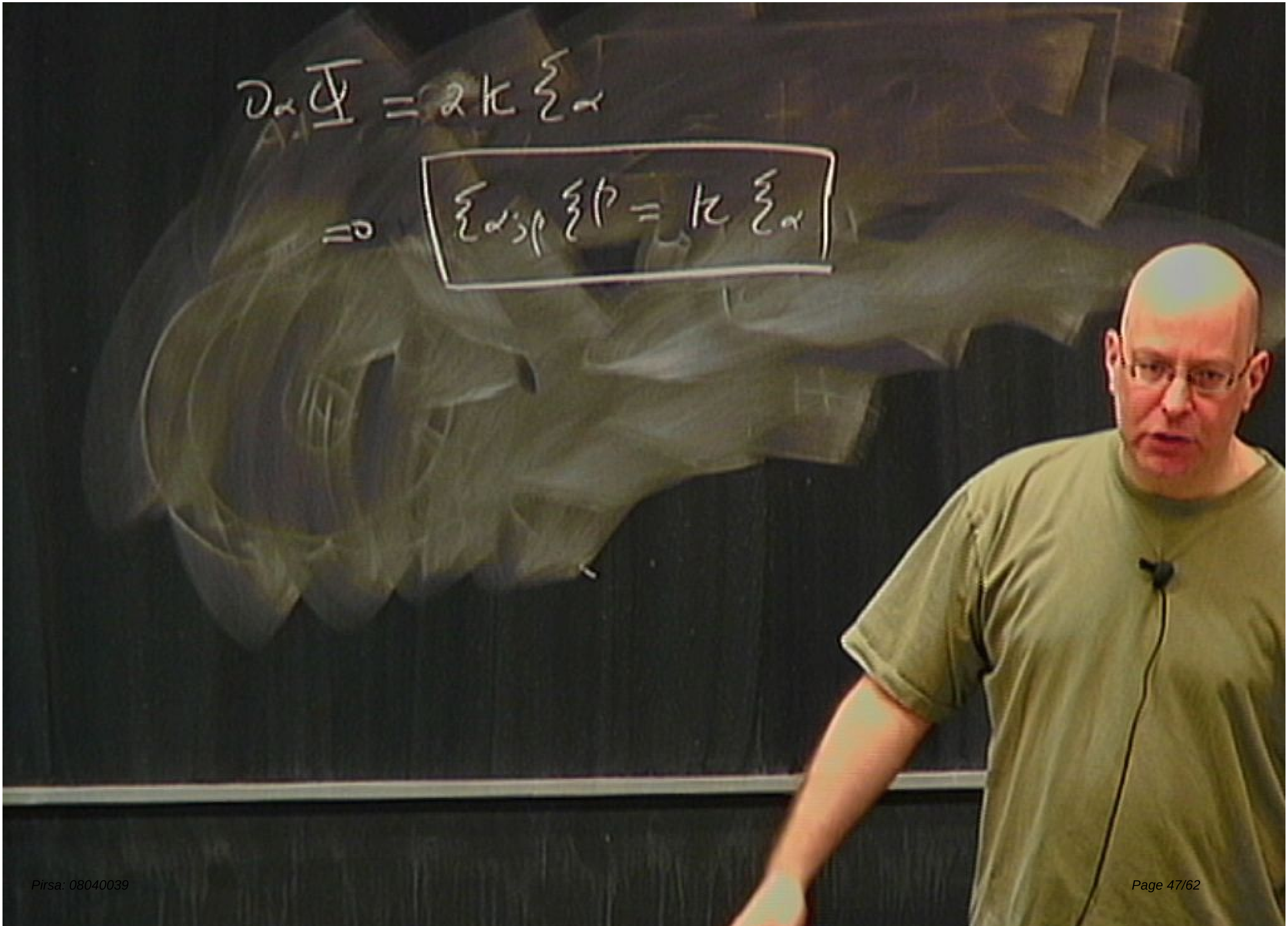
normal to EH $\leftarrow \partial_\alpha \Phi = - (g_{\mu\nu} \xi^\mu \xi^\nu)_{;\alpha}$

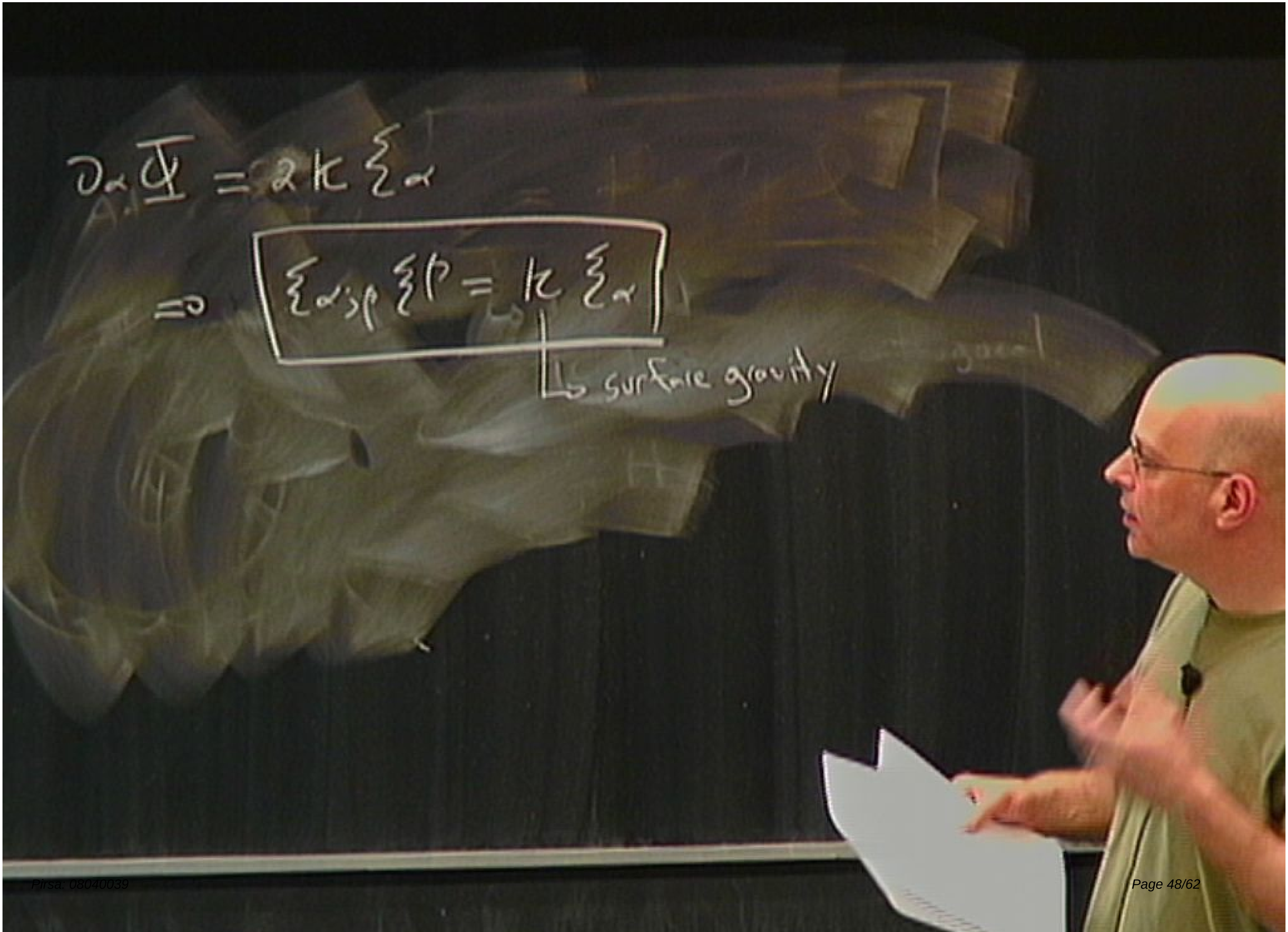
$= -2 \xi^\mu \xi_{\mu;\alpha}$

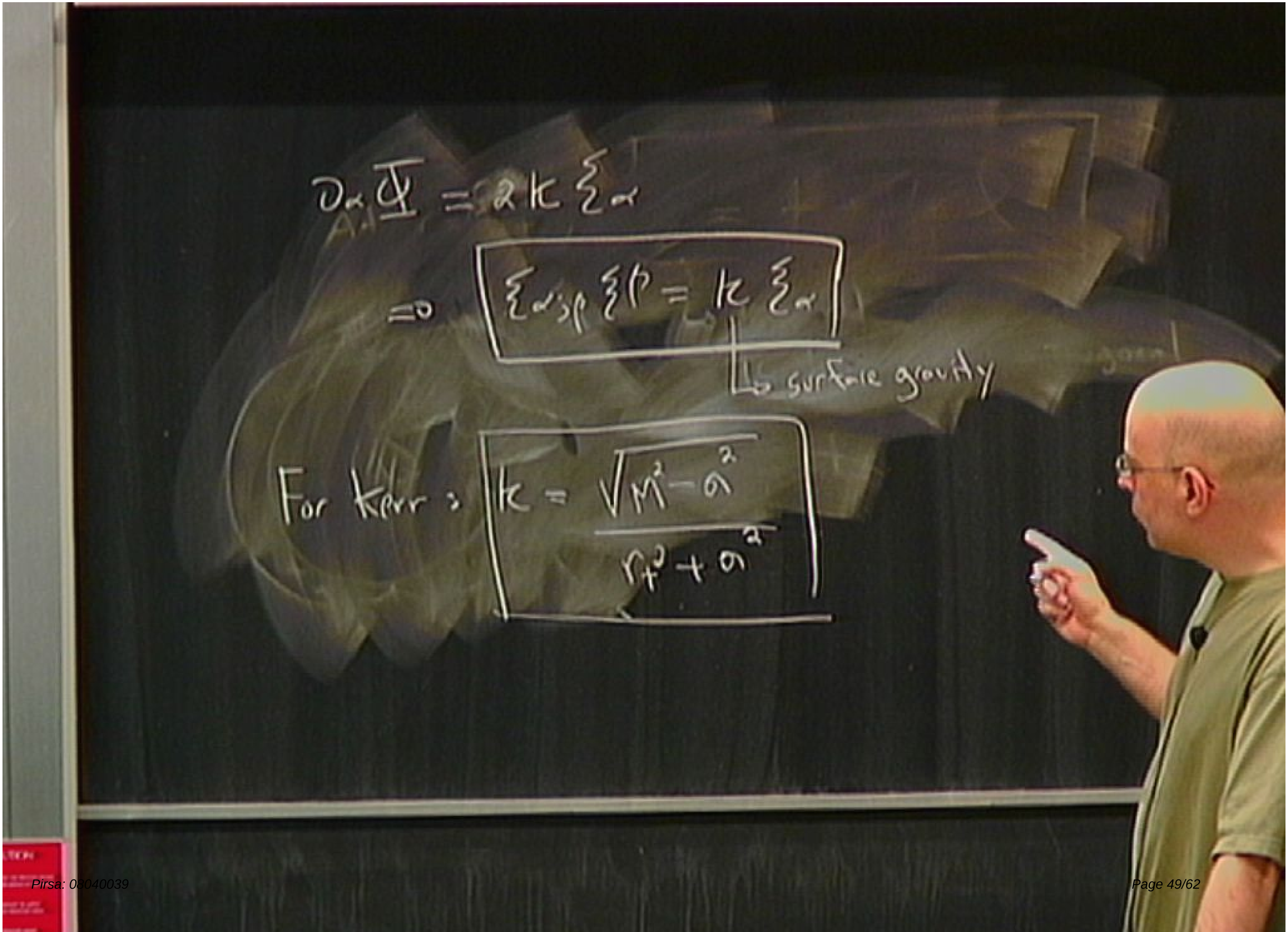
$= 2 \xi^\mu \xi_{\alpha;\mu}$

$= 2 \xi_{\alpha;\mu} \xi^\mu$


$\partial_\alpha \Phi = 2\kappa \xi_\alpha$


$\partial_\alpha \Phi = 2\kappa \xi_\alpha$
$\Rightarrow \boxed{\xi_{\alpha;\beta} \xi^\beta = \kappa \xi_\alpha}$


$\partial_\alpha \Phi = 2\kappa \xi_\alpha$
$\Rightarrow \boxed{\xi_{\alpha;\beta}\xi^\beta = \kappa \xi_\alpha}$
surface gravity


$\partial_\alpha \Phi = 2\kappa \xi_\alpha$
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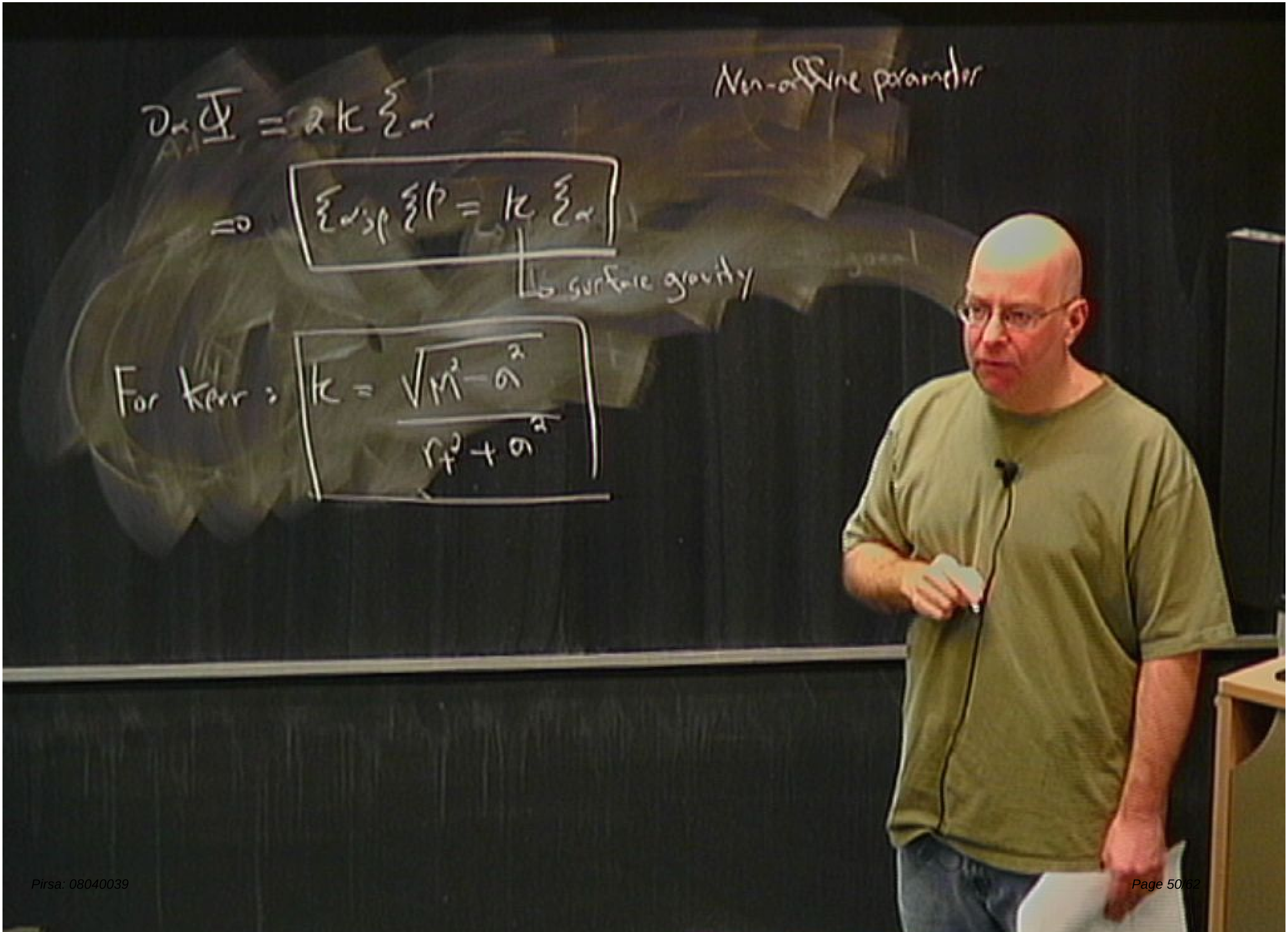
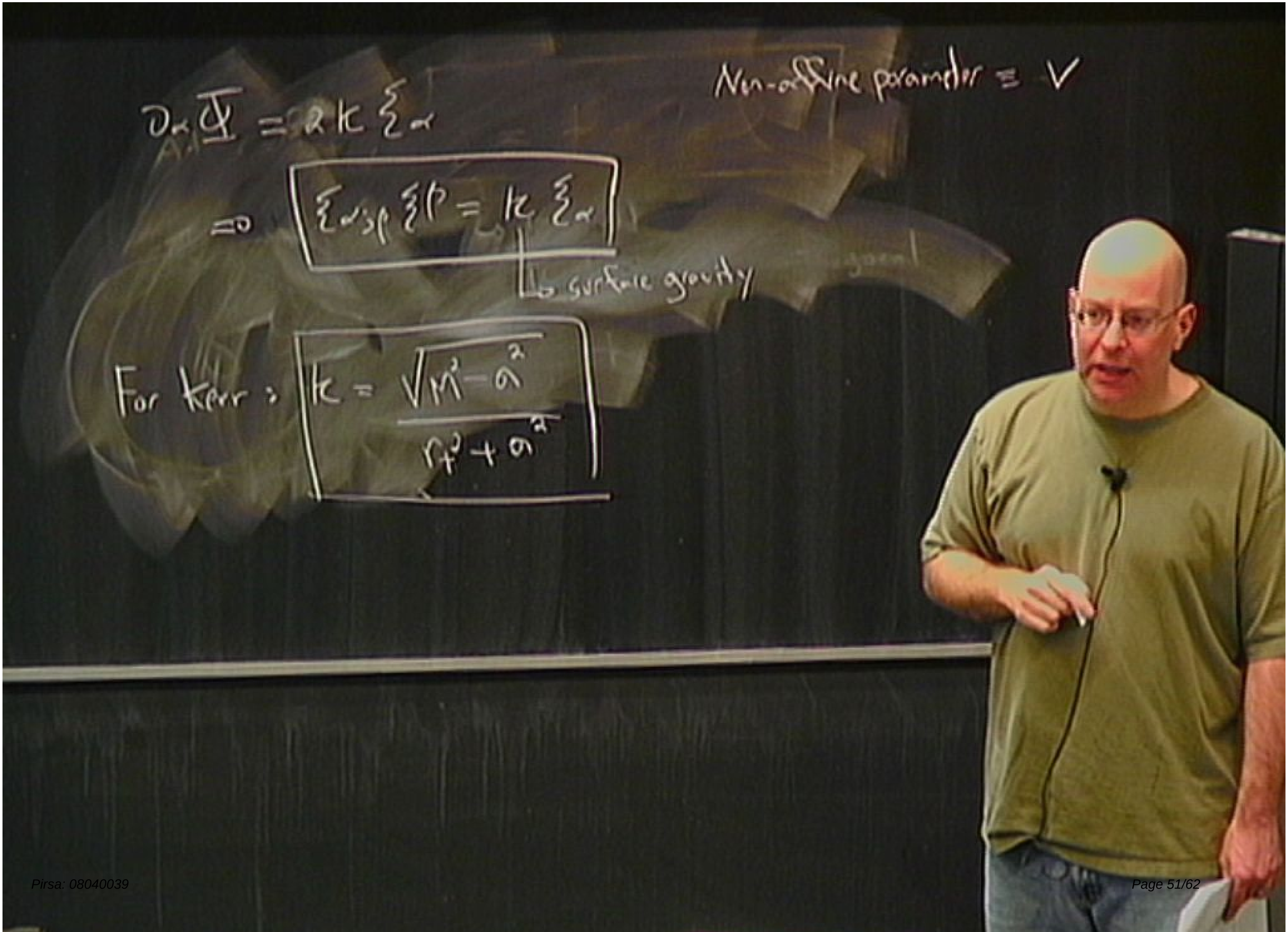
surface gravity
For Kerr: $\boxed{\kappa = \frac{\sqrt{M^2-a^2}}{r_+^2+a^2}}$



Non-affine parameter
$\nabla_\alpha \Phi = 2\kappa \xi_\alpha$
$\Rightarrow \xi_{\alpha;\beta}\xi^\beta = \kappa \xi_\alpha$
surface gravity
For Kerr: $\kappa = \frac{\sqrt{M^2-a^2}}{r_+^2+a^2}$


Non-affine parameter $\equiv V$
$\partial_\alpha \Phi = 2\kappa \xi_\alpha$
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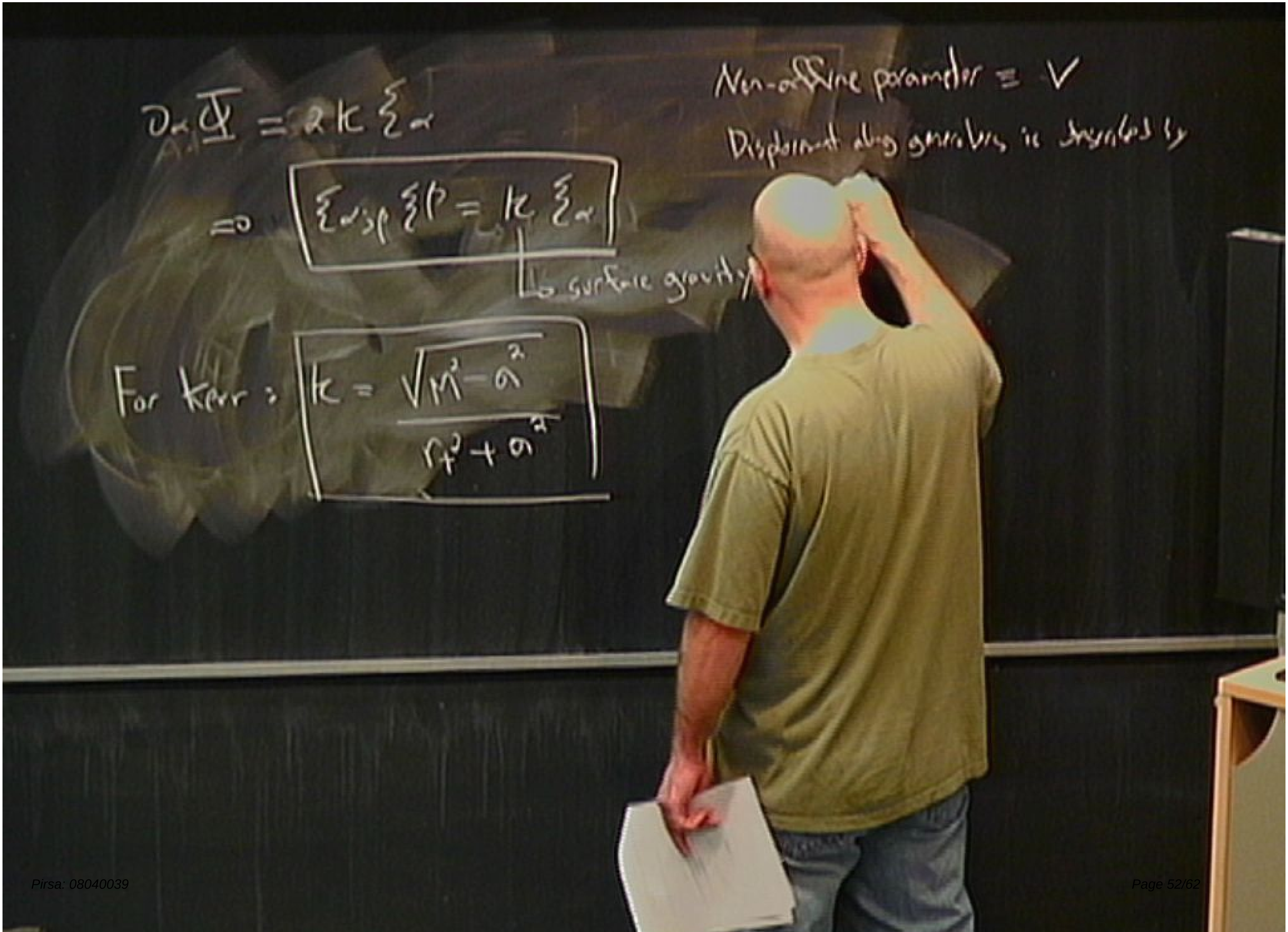
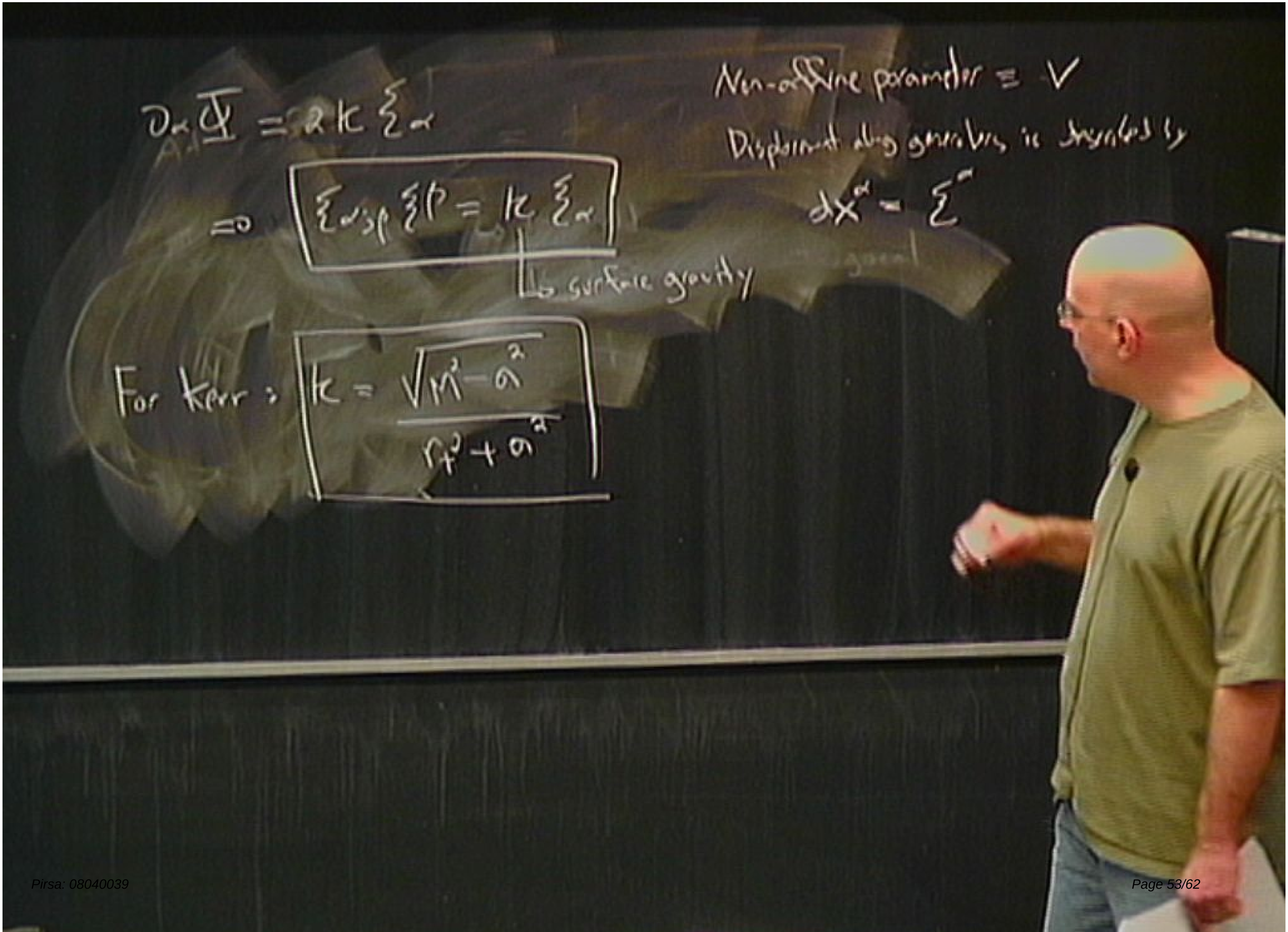

Non-affine parameter $\equiv$ V
Displacement along generators is
$D_\alpha \underline{\Phi} = 2\kappa \xi_\alpha$
$\Rightarrow \boxed{\xi_{\alpha;\beta}\xi^\beta = \kappa \xi_\alpha}$
surface gravity
For Kerr : $\boxed{\kappa = \frac{\sqrt{M^2-a^2}}{r_+^2+a^2}}$


Non-affine parameter $\equiv V$
Displacement along generators is generated by
$dx^\alpha = \xi^\alpha$
$\partial_\alpha \Phi = 2\kappa \xi_\alpha$
$\Rightarrow \xi_{\alpha;\beta}\xi^\beta = \kappa \xi_\alpha$
surface gravity
For Kerr: $\kappa = \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2}$

$$\partial_\alpha \Phi = 2\kappa\, \xi_\alpha$$

$$\Rightarrow \boxed{\xi_{\alpha;\beta}\,\xi^\beta = \kappa\, \xi_\alpha}$$

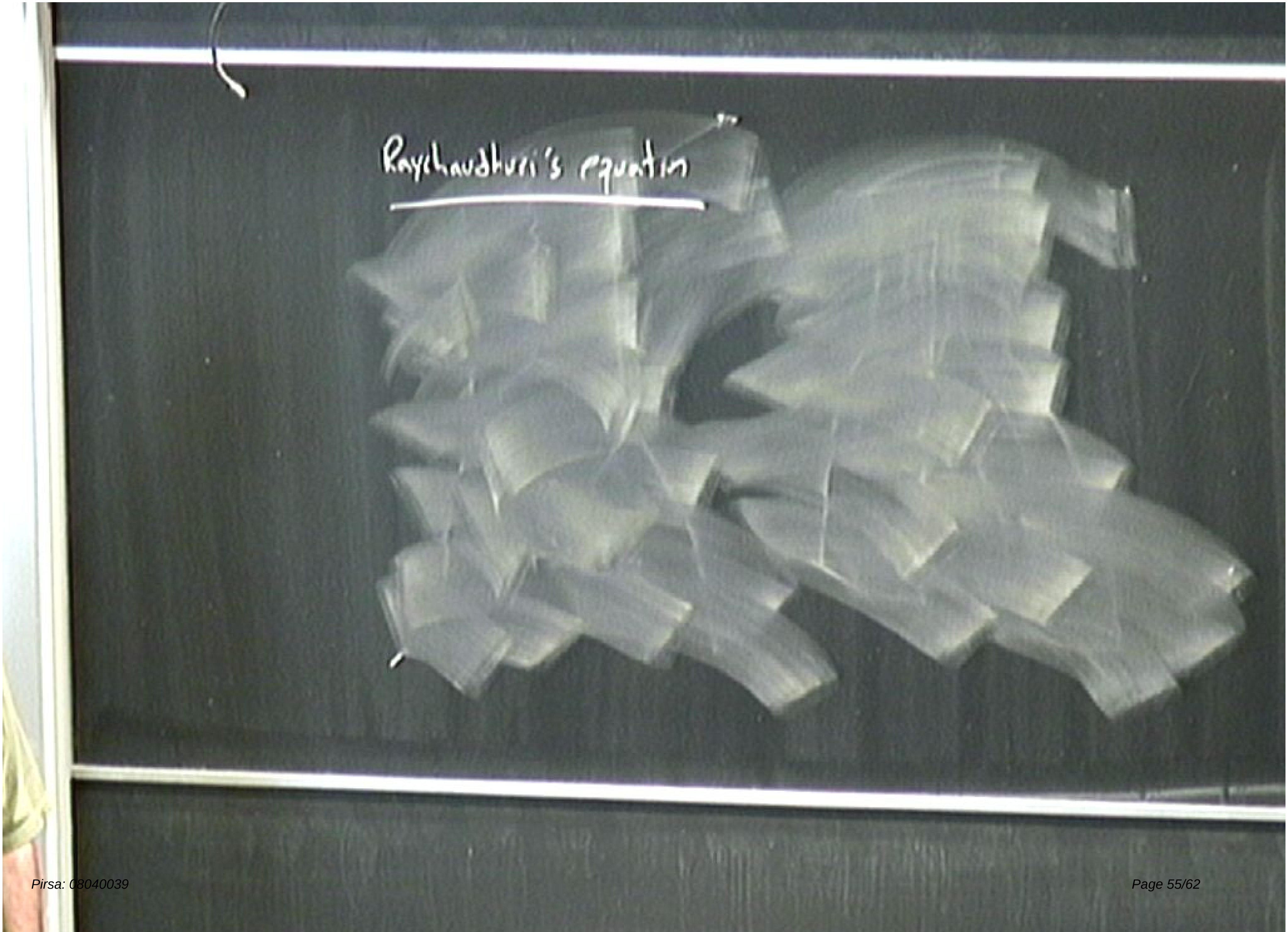
κ → surface gravity

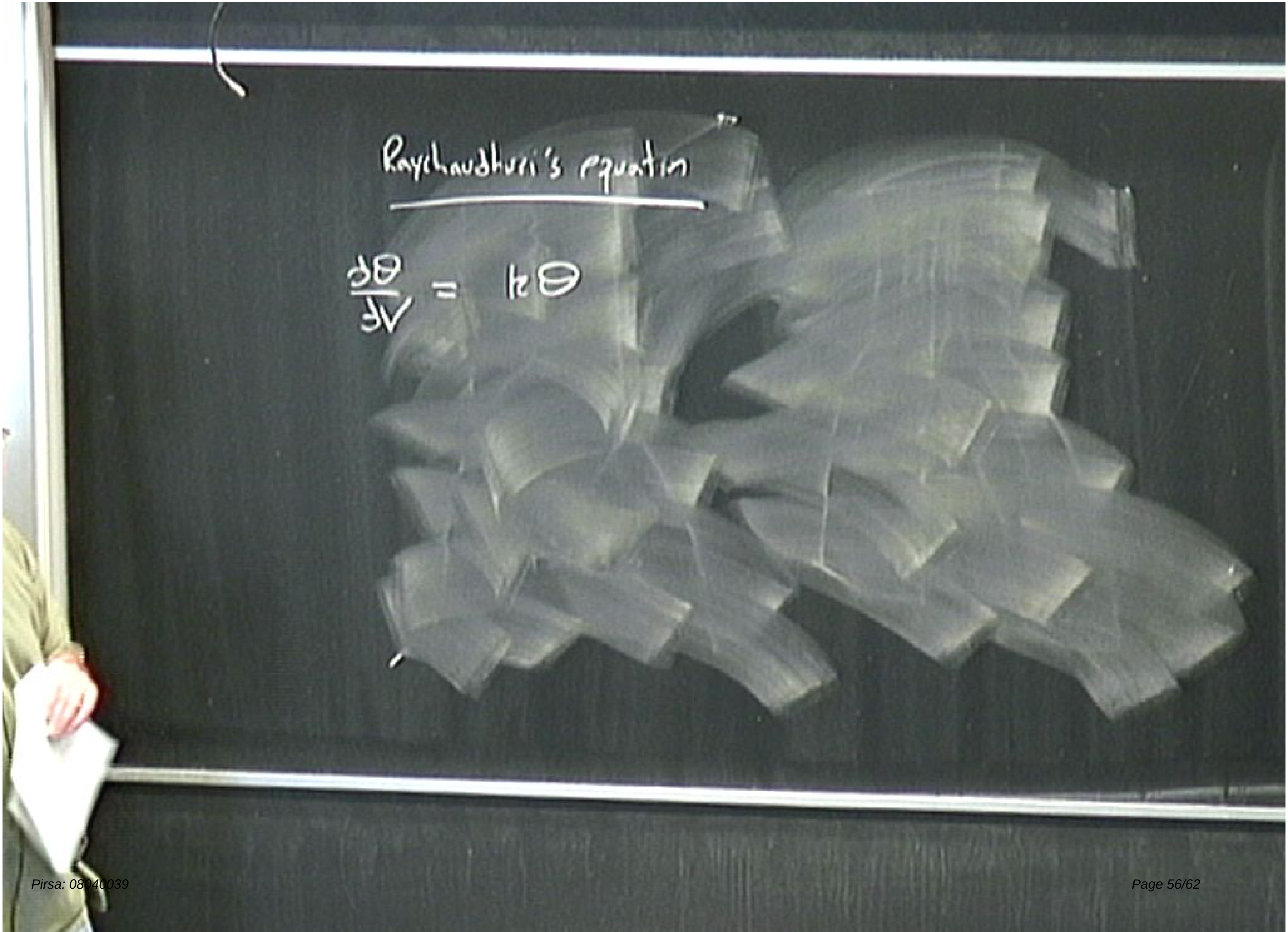
For Kerr: $\boxed{\kappa = \dfrac{\sqrt{M^2 - a^2}}{r_+^2 + a^2}}$

Non-affine parameter $\equiv V$

Displacement along generators is described by

$$dx^\alpha = \xi^\alpha\, dV$$


Raychaudhuri's equation


Raychaudhuri's equation
dθ/dV = kθ

Raychaudhuri's equation

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 8\pi T_{\mu\nu}\xi^\mu\xi^\nu$$

Raychaudhuri's equation

$$\boxed{\frac{d\theta}{dV} = \cancel{\kappa\theta} - \cancel{\tfrac{1}{2}\theta^2} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}$$

Stationary horizon: $\theta = 0 = \dfrac{d\theta}{dV}$

$$\underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\geq 0} + 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta = 0$$

Raychaudhuri's equation

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Raychaudhuri's equation

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Stationary horizon: $\theta = 0 = \frac{d\theta}{dV}$

$$\underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\geq 0} + 8\pi \underbrace{T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0 \text{ (null energy condition)}} = 0$$

$\sigma_{\alpha\beta} = 0$

Raychaudhuri's equation

$$\boxed{\frac{d\theta}{dV} = \cancel{k\theta} - \cancel{\tfrac{1}{2}\theta^2} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}$$

Stationary horizon: $\theta = 0 = \frac{d\theta}{dV}$

$$\underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\geq 0} + 8\pi \underbrace{T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0 \text{ (null energy condition)}} = 0$$

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