Title: Topos theory in the formulation of theories of physics

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Abstract: Chris Isham in pre-recorded video, with Andreas Doring fielding questions and clarifications. Like watching commentators Scott Hamilton and Katarina Witt analyze Kristi Yamaguchi\'s performance at the World Figure Skating Championships for CBS News, join us for something different in quantum foundations. Chris Isham parries the intricacies of topos theory; Andreas Doring shows us how to see the moves in slow motion. Bring your own popcorn and plenty of questions.

TOPOS THEORY IN THE FORMULATION OF THEORIES OF PHYSICS

January 2008

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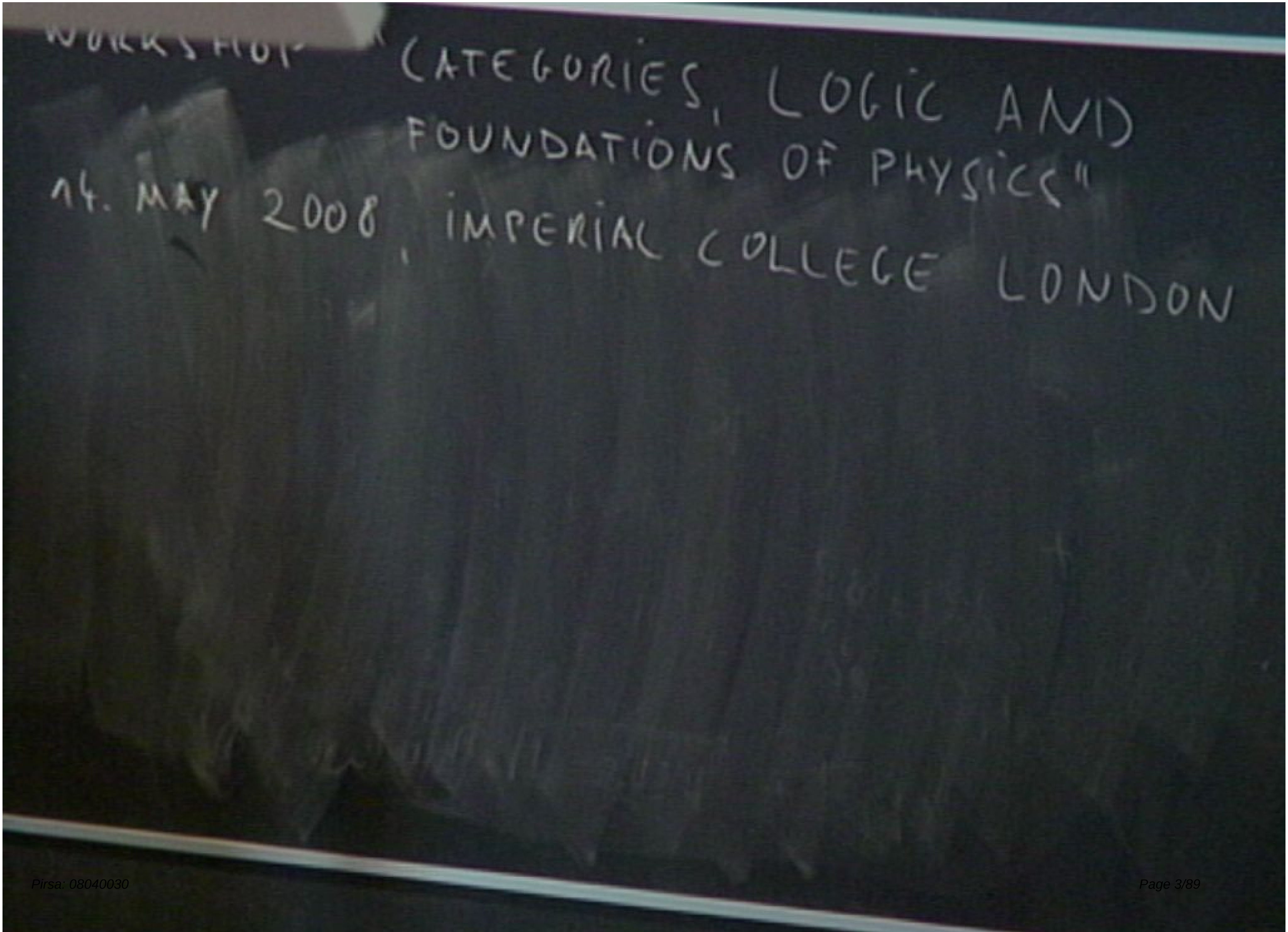
WORKSHOP "CATEGORIES, LOGIC AND
FOUNDATIONS OF PHYSICS"
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I. The Challenge of Quantum Gravity

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   - Space-time is represented by a *differentiable manifold*
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   - But: 'reality of space/time points' is *very* dubious.
     - $\mathrm{Diff}(M)$ invariance/covariance.
     - Use of $\mathbb{R}$: a highly abstract concept.

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The Unholy Trinity

The triangle of

real-world data $\leftrightarrow$ mathematics $\leftrightarrow$ conceptual framework

underpins all theoretical physics.

But: in QG, the first is largely missing. Consequently:

- Would we recognise the 'correct' theory if we saw it?
- What makes any particular idea a 'good' one?
  - Appointments and promotions
  - Getting research grants

In practice:

- Work by analogy with other theories (e.g., QFT)
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How can this be applied to space and time themselves?

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But: It is often asserted that classical space and time will 'emerge' from the correct QG formalism in some limit.

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2. Interpretational: *instrumentalism* versus *realism*.

   We want to talk about 'the way things are' in regard to space and time.

II. Problem of Real Numbers in QG-related QT

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(ii) as the values of *probabilities*;

(iii) as a fundamental ingredient in mathematical models of *space and time* via the ideas of differential geometry.

The use of $\mathbb{R}$ (and $\mathbb{C}$) in standard quantum theory is a reflection of (i) and (ii); and, indirectly, of (iii) too.

2. Why are Physical Quantities Assumed Real-Valued?

Traditionally, quantities are measured with rulers and pointers.

- Thus there is a direct link between the 'quantity-value space' and the assumed structure of *physical* space.
- Thus we have a potential 'category error' at L_P: if physical space is not based on $\mathbb{R}$, we should not assume *a priori* that physical quantities are real-valued.

If the quantity-value space is *not* $\mathbb{R}$, then what is the status of the Hilbert-space formalism?

3. Why Are Probabilities Assumed Real Numbers?

Relative-frequency interpretation: $\frac{N_i}{N}$ tends to $r \in [0, 1]$ as $N \to \infty$.

- This statement is instrumentalist. It does not work if there is no classical spatio-temporal background in which measurements could be made.

- In 'realist' interpretations, probability is often interpreted as *propensity* (*latency*, *potentiality*).

  – But why should a propensity be a real number in $[0, 1]$?

  – Minimal requirement is, presumably, an ordered set, but this need not even be *totally* ordered.

The Big Problem

Standard QT is grounded in Newtonian space and time; or SR.

How can the formalism be modified, or generalised, so as

(i) to be'realist'; and

(ii) not to be dependent *a priori* on real and complex numbers?

- For example, if we have a given 'non-standard' background $\mathcal{C}$, what is the quantum formalism that is *adapted* to $\mathcal{C}$?

- Very difficult: usual Hilbert-space formalism is very rigid

What *are* the basic principles of 'quantum theory'; or beyond?

III. Theories of a Physical System

"From the range of the basic questions of metaphysics we shall here ask this one question: "What is a thing?" The question is quite old. What remains ever new about it is merely that it must be asked again and again."

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"A thing is always something that has such and such properties, always something that is constituted in such and such a way. This something is the bearer of the properties; the something, as it were, that underlies the qualities." **Heidegger**

In constructing a theory of some branch of physics, the key ingredients are the *mathematical models* for

1. Space, time, or space-time;
2. Physical quantities;
3. 'States' of a system—"The way things are" (or *history* equivalent)

Some key questions:

- What is meant by a 'value' of a 'physical quantity'?
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The Realism of Classical Physics:

1. A physical quantity A is represented by $\tilde{A} : \mathcal{S} \to \mathbb{R}$.

2. A state $s \in \mathcal{S}$ specifies 'how things are': i.e., the value of any physical quantity A in that state is $\tilde{A}(s) \in \mathbb{R}$.

This is how ('naive') realism enters into classical physics.

3. The subset $\tilde{A}^{-1}(\Delta) \subseteq \mathcal{S}$ represents the proposition "$A \in \Delta$": asserts that the system has a certain *property*.

Thus, the *mathematical* structure of set theory, implies that, of necessity, the propositions in classical physics form a *Boolean logic*.

The collection of such propositions forms a *deductive system*: i.e., there is a sequent calculus for constructing proofs.

Failure of Naive Realism in Quantum Physics

> **Kochen-Specker theorem:** it is impossible to assign consistent true-false values to all the propositions in quantum theory.

Equivalently: it is not possible to assign consistent values to all the physical quantities in a quantum theory.

Conclusion:

- From Heidegger's perspective, there is 'no way things are'.
- Instead an *instrumentalist* interpretation is used.

Representing Physical Quantities

Let A be a physical quantity in a system S, with associated propositions "$A \in \Delta$", where $\Delta \subseteq \mathbb{R}$.

Classical theory of S:

- State space is a symplectic manifold, $\mathcal{S}$
- $A \rightsquigarrow \tilde{A} : \mathcal{S} \to \mathbb{R}$
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- $A \rightsquigarrow \breve{A} : \Sigma \to \mathcal{R}$.
- For Δ a sub-object of $\mathcal{R}$, "$A \in \Delta$" $\rightsquigarrow$ a sub-object of Σ?

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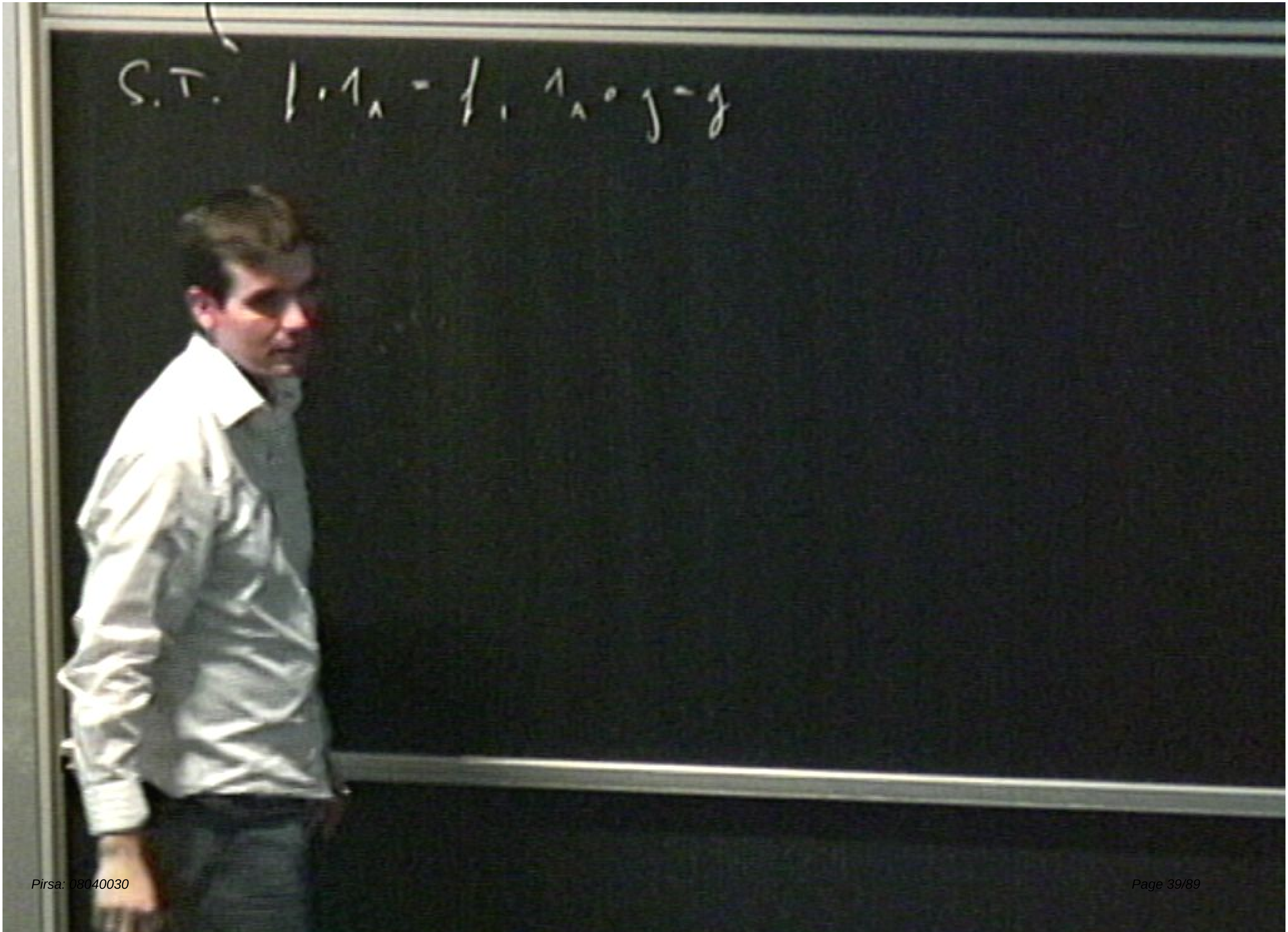
A CATEGORY $\mathcal{C}$ CONSISTS OF OBJECTS $A, B, \ldots$ AND ARROWS $f: A \longrightarrow B$, S.T. IF $f: A \to B$, $g: B \longrightarrow C$, THEN $g \circ f: A \longrightarrow C$,


S.T. $f \circ 1_A = f$, $1_A \circ g = g$

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EXS. (a)

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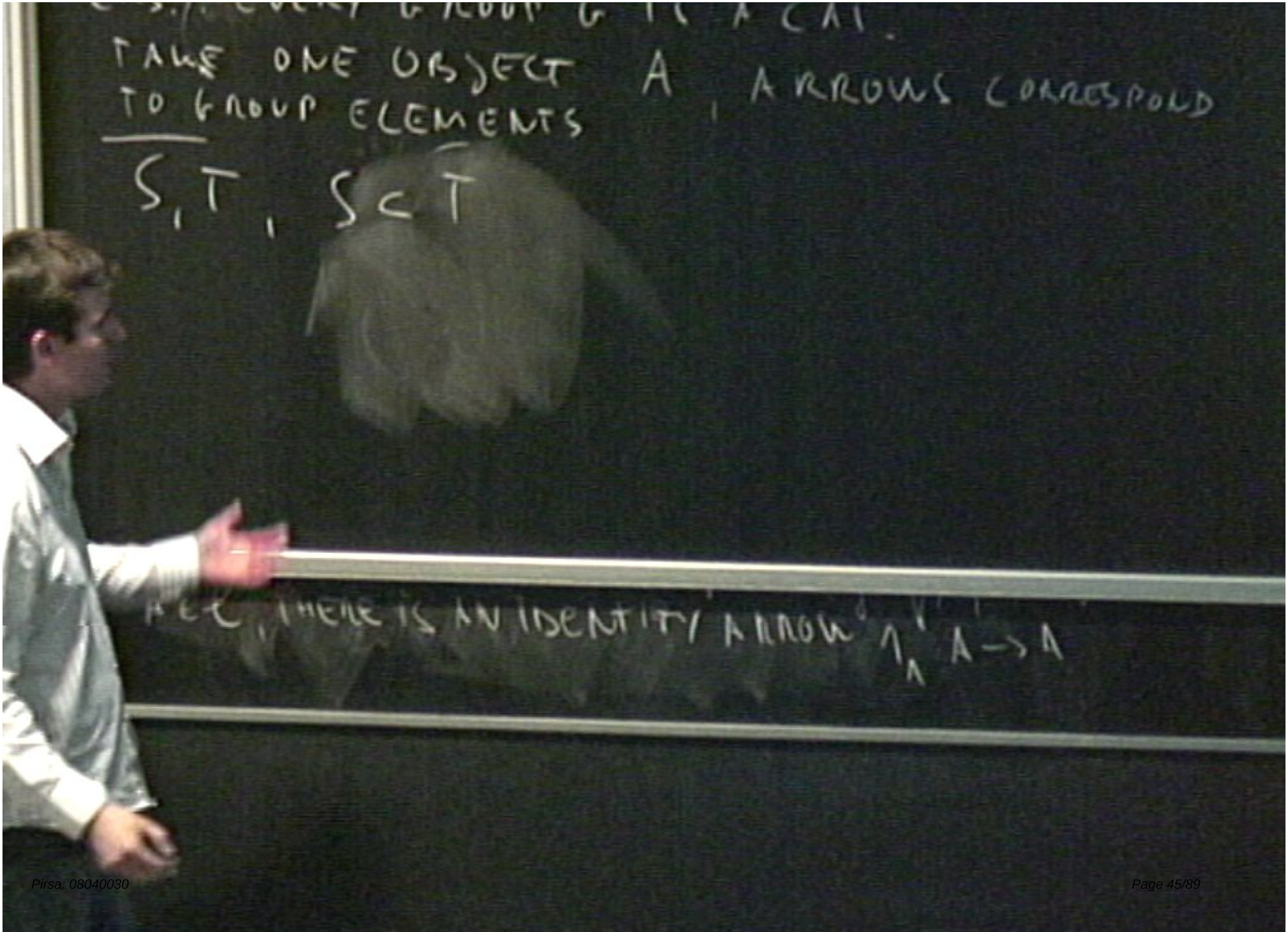
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S, T, S⊂T
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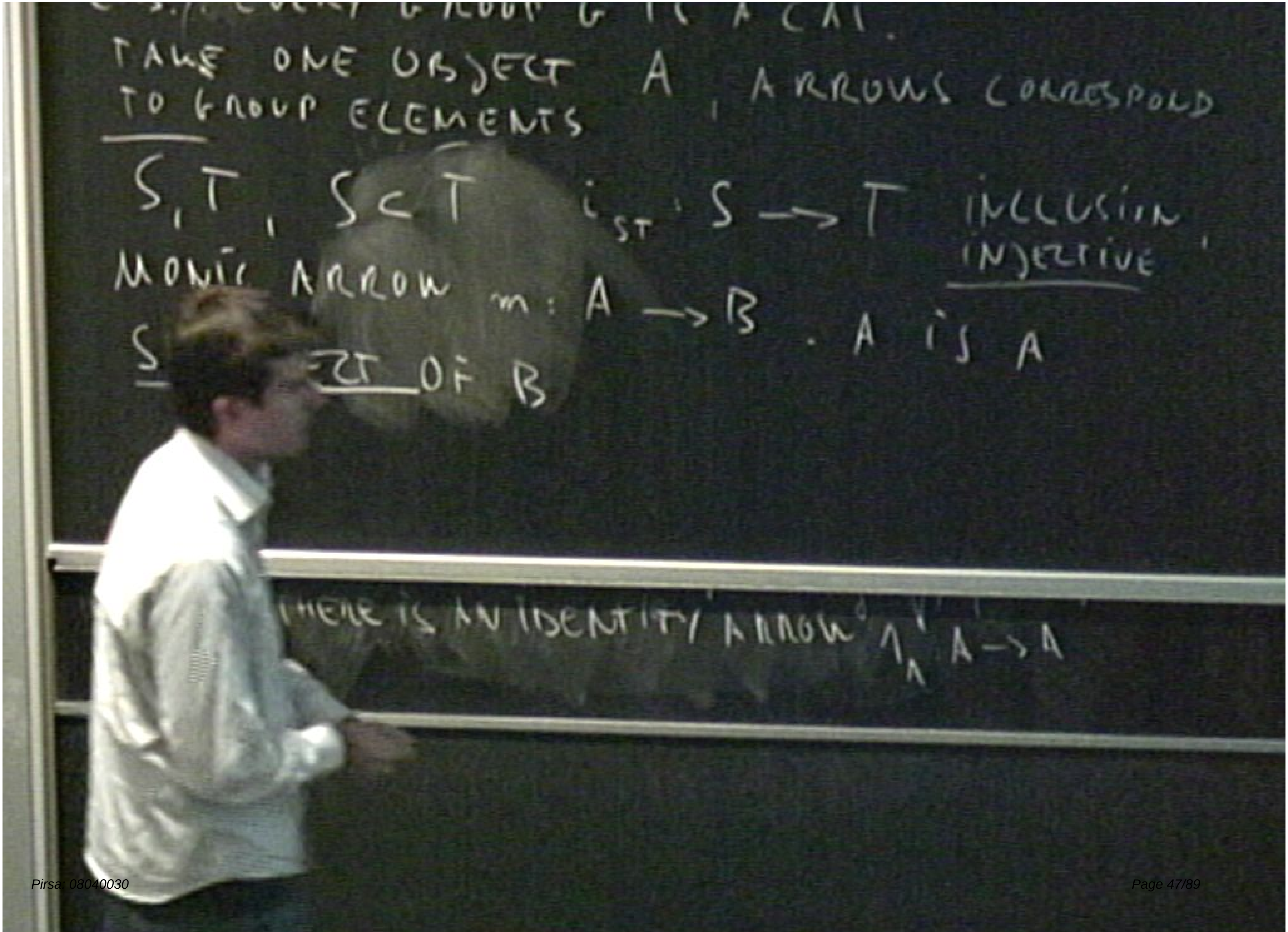
S, T, S ⊂ T i_{ST}: S → T INCLUSION, INJECTIVE

A ∈ C, THERE IS AN IDENTITY ARROW 1_A: A → A


TAKE ONE OBJECT A, ARROWS CORRESPOND
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MONIC ARROW m: A → B. A IS A
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S, T, $S \subset T$ $i_{ST} : S \to T$ INCLUSION, INJECTIVE,

MONIC ARROW $m : A \to B$. A IS A SUBOBJECT OF B

$A \in \mathcal{C}$, THERE IS AN IDENTITY ARROW $1_A : A \to A$

IV. Introducing Topos Theory

Does such 'categorification' work?

1. Not in general: usually, sub-objects of an object do not have a logical structure. However, they *do* in a *topos*!

2. A topos is a category that 'behaves much like **Sets**'. In particular there are:
   - $0, 1$; pull-backs & push-outs (hence, products & co-products)
   - Exponentiation:

$$\mathrm{Hom}(C, A^B) \simeq \mathrm{Hom}(C \times B, A)$$

   - A 'sub-object classifier', Ω: to any sub-object A of B, $\exists \chi_A : B \to \Omega$ such that $A = \chi_A^{-1}(1)$.

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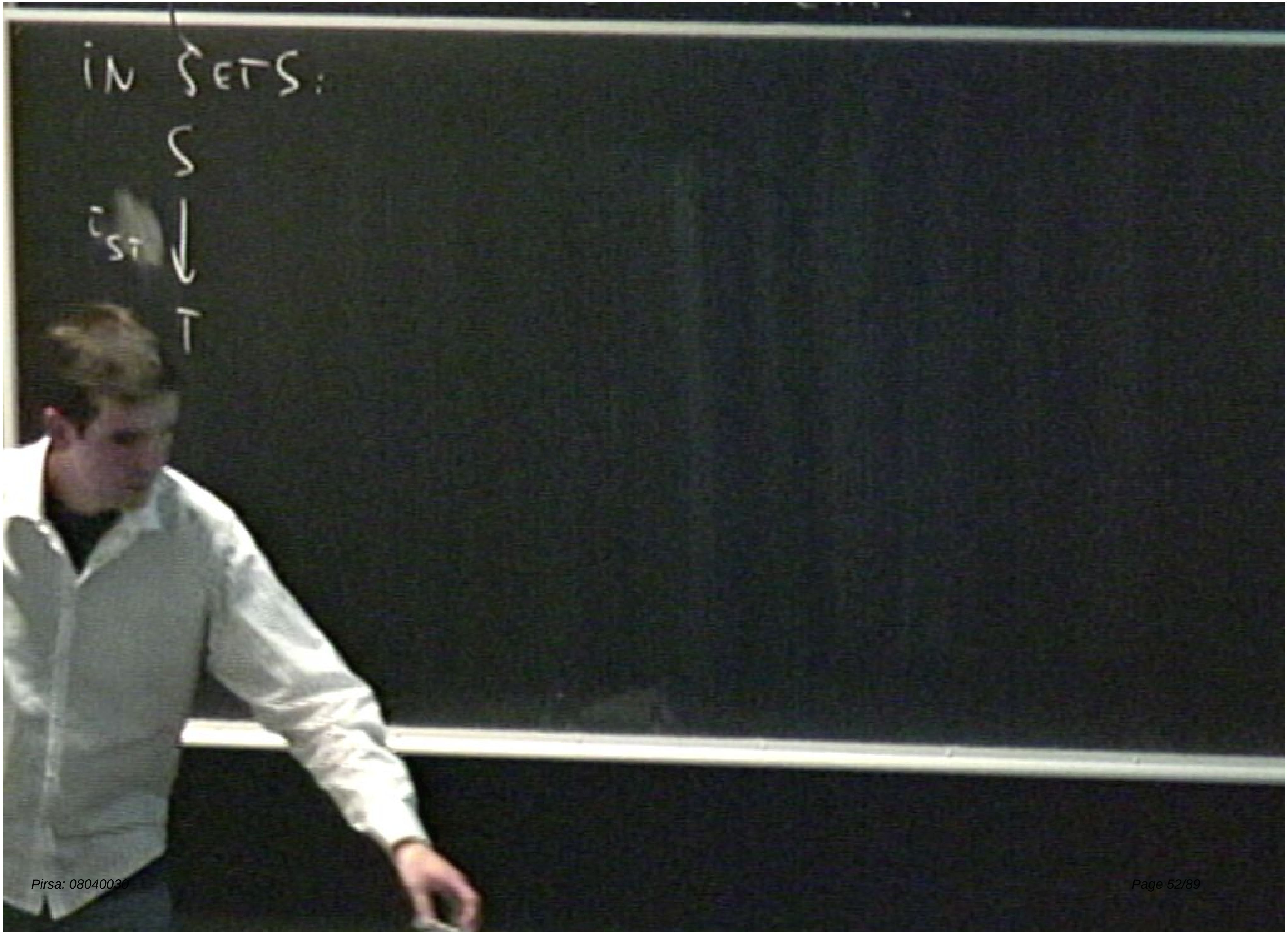
IN SETS:
S
i_{ST} ↓
T

in SETS:

$$\begin{array}{l} S \\ \downarrow i_{ST} \\ T \longrightarrow \{0,1\} \end{array}$$

IN SETS:

$$S \xrightarrow{i_{ST}} T \xrightarrow{\chi_S} \{0,1\}$$

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$$\{x\} = 1$$

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$$\begin{array}{ccc} S & \longrightarrow & \{x\} = 1 \\ \downarrow & & \\ T & \xrightarrow[\chi_S]{} & \{0,1\} \end{array}$$

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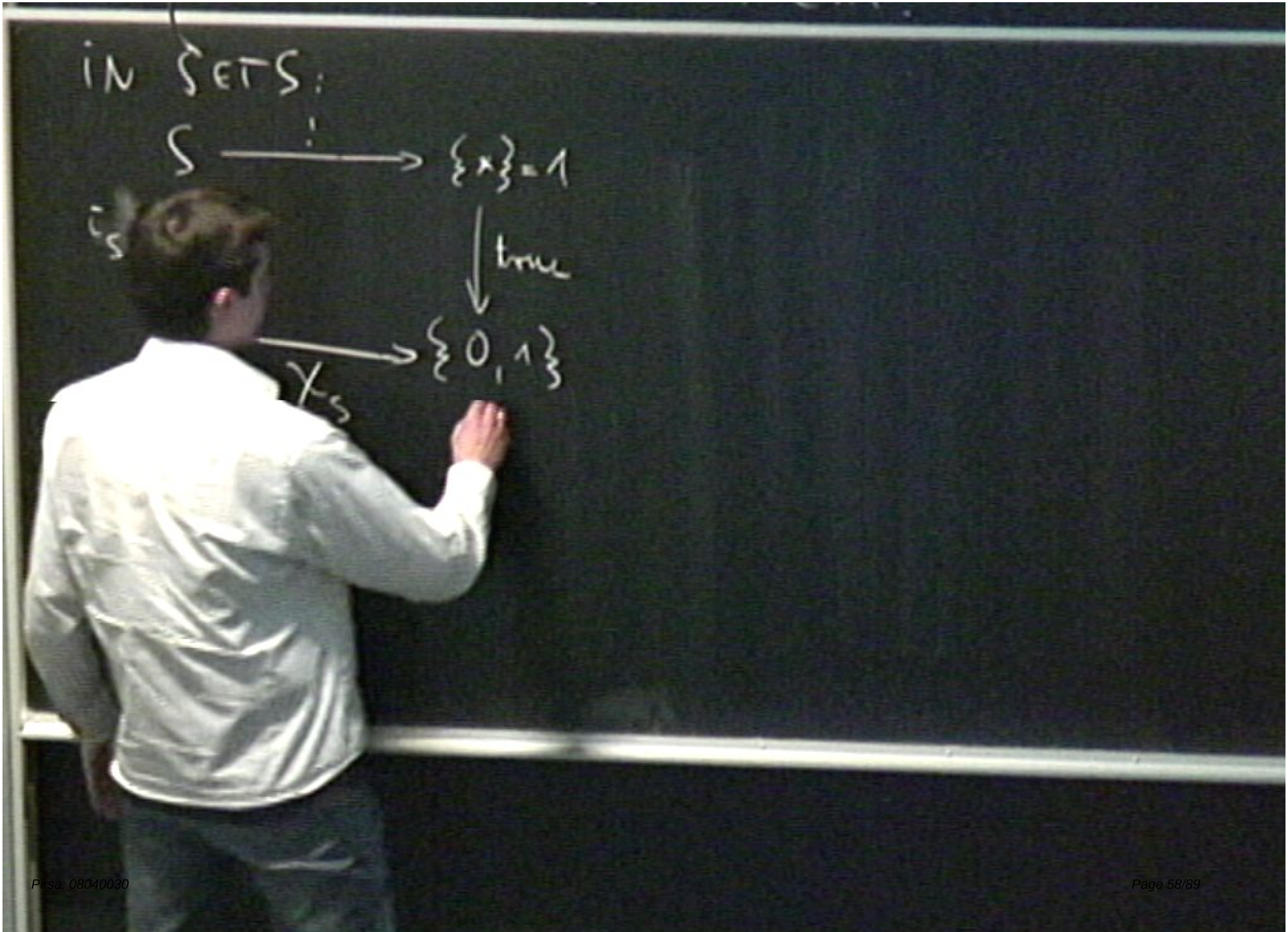
in SETS:
S
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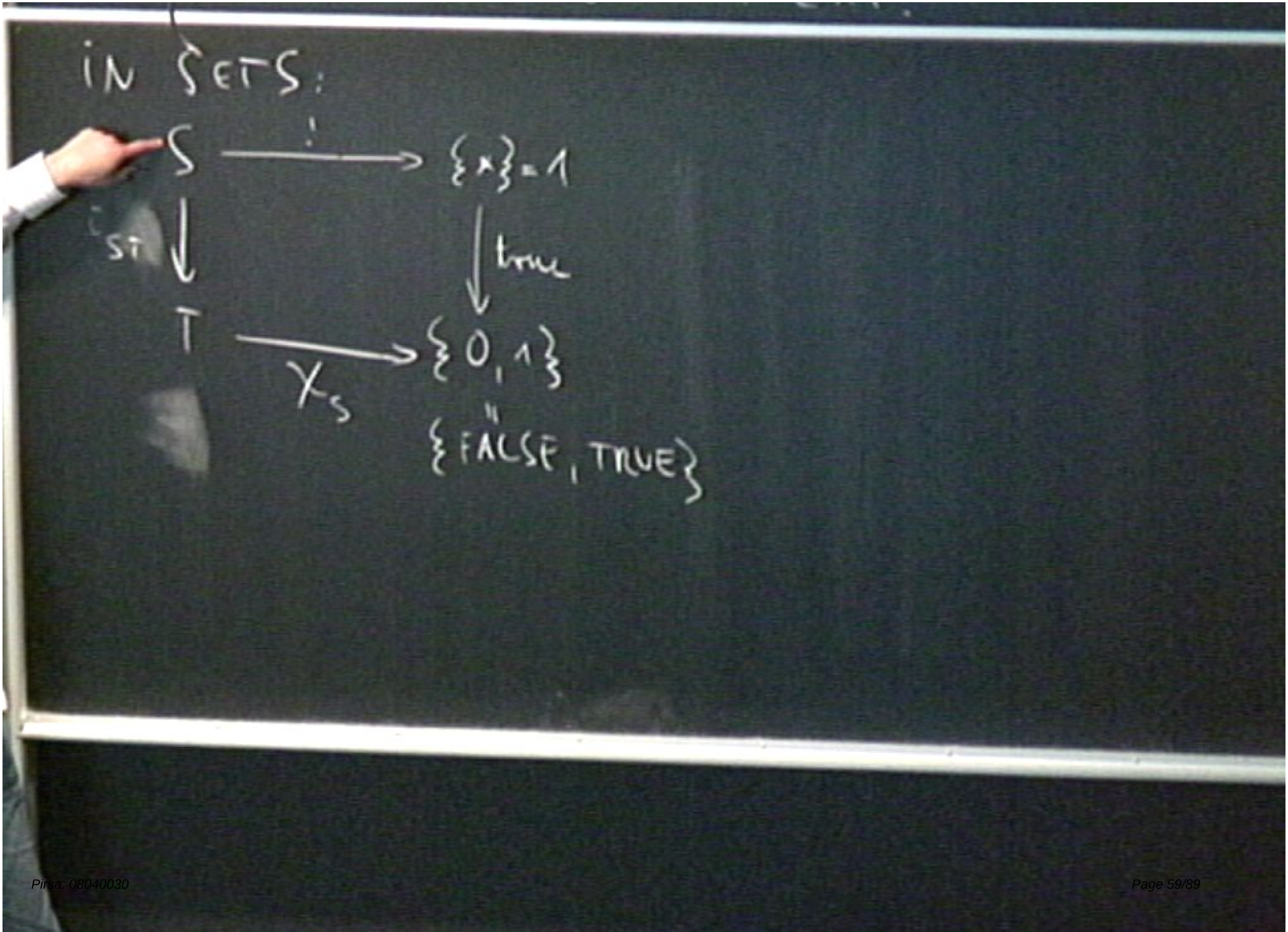
{x}=1
true
{0,1}
i_S
χ_S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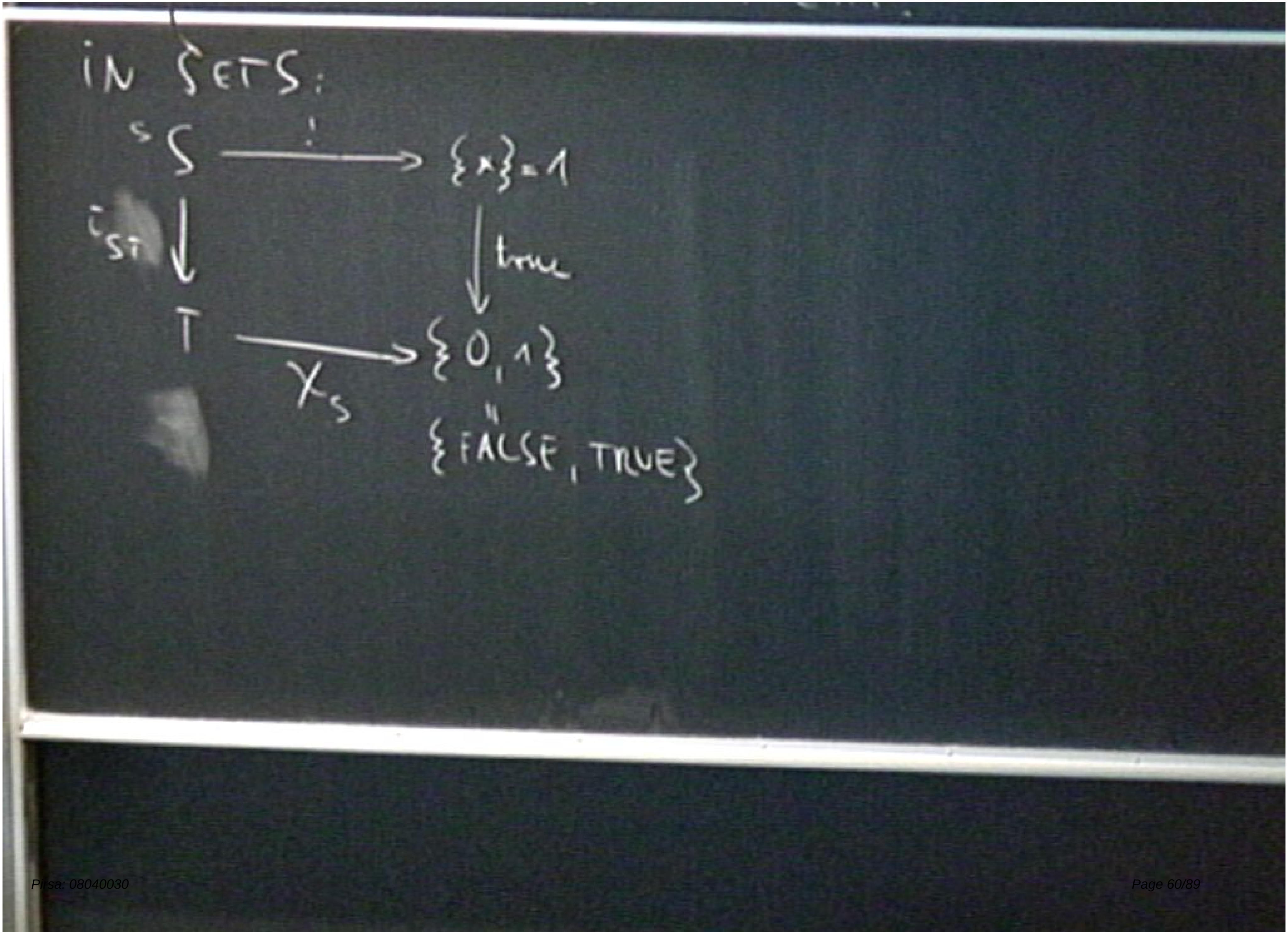
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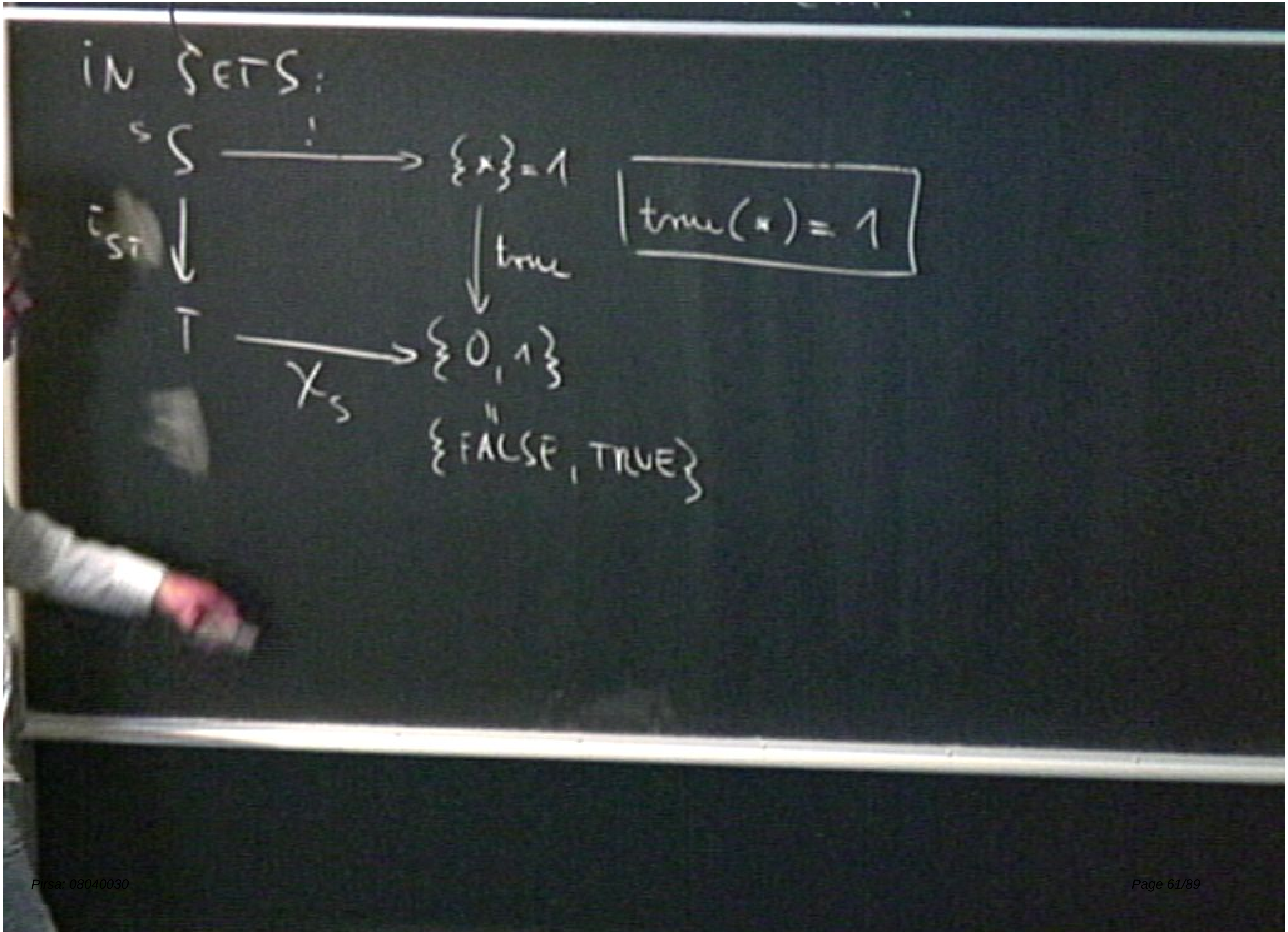
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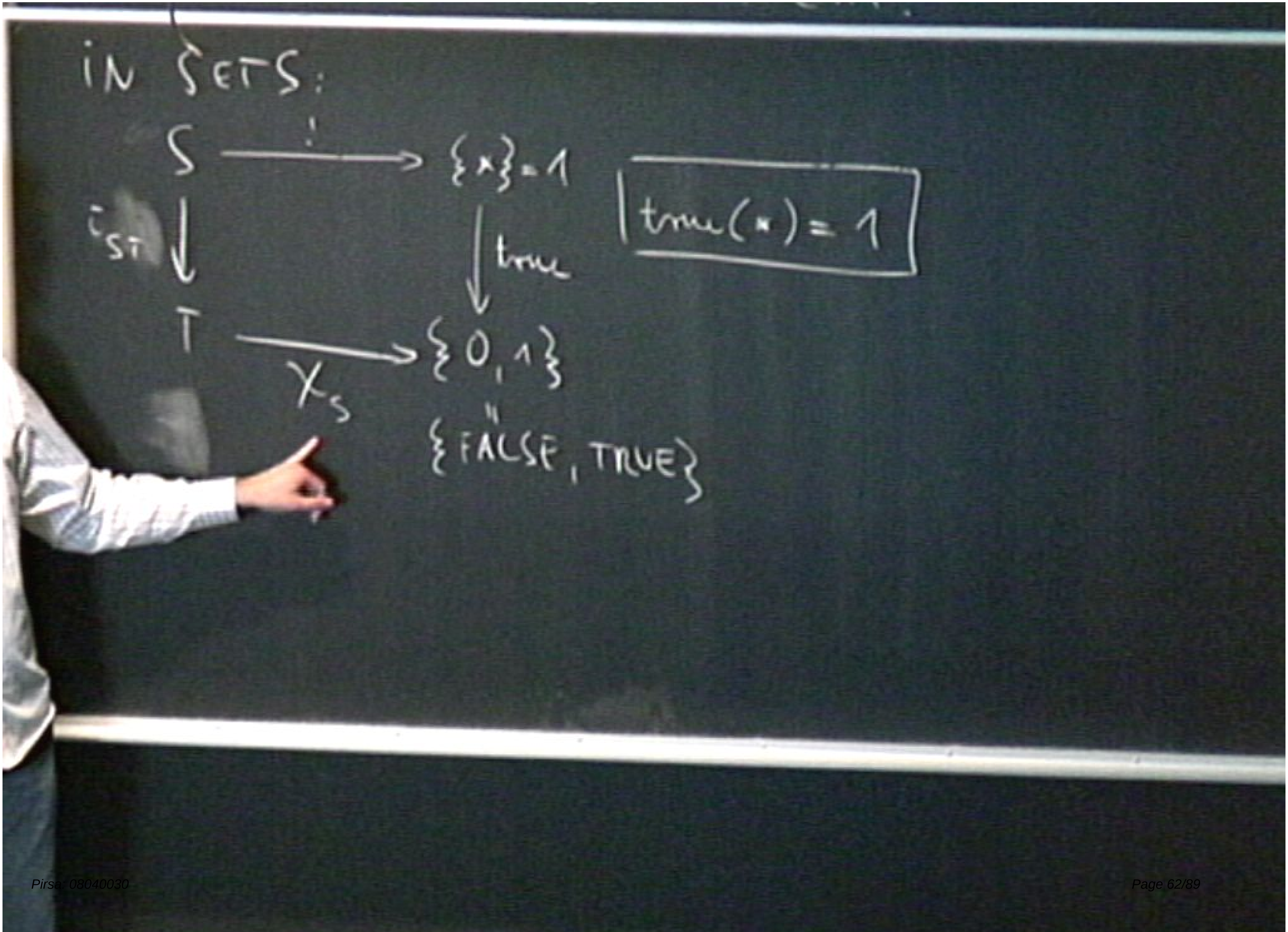
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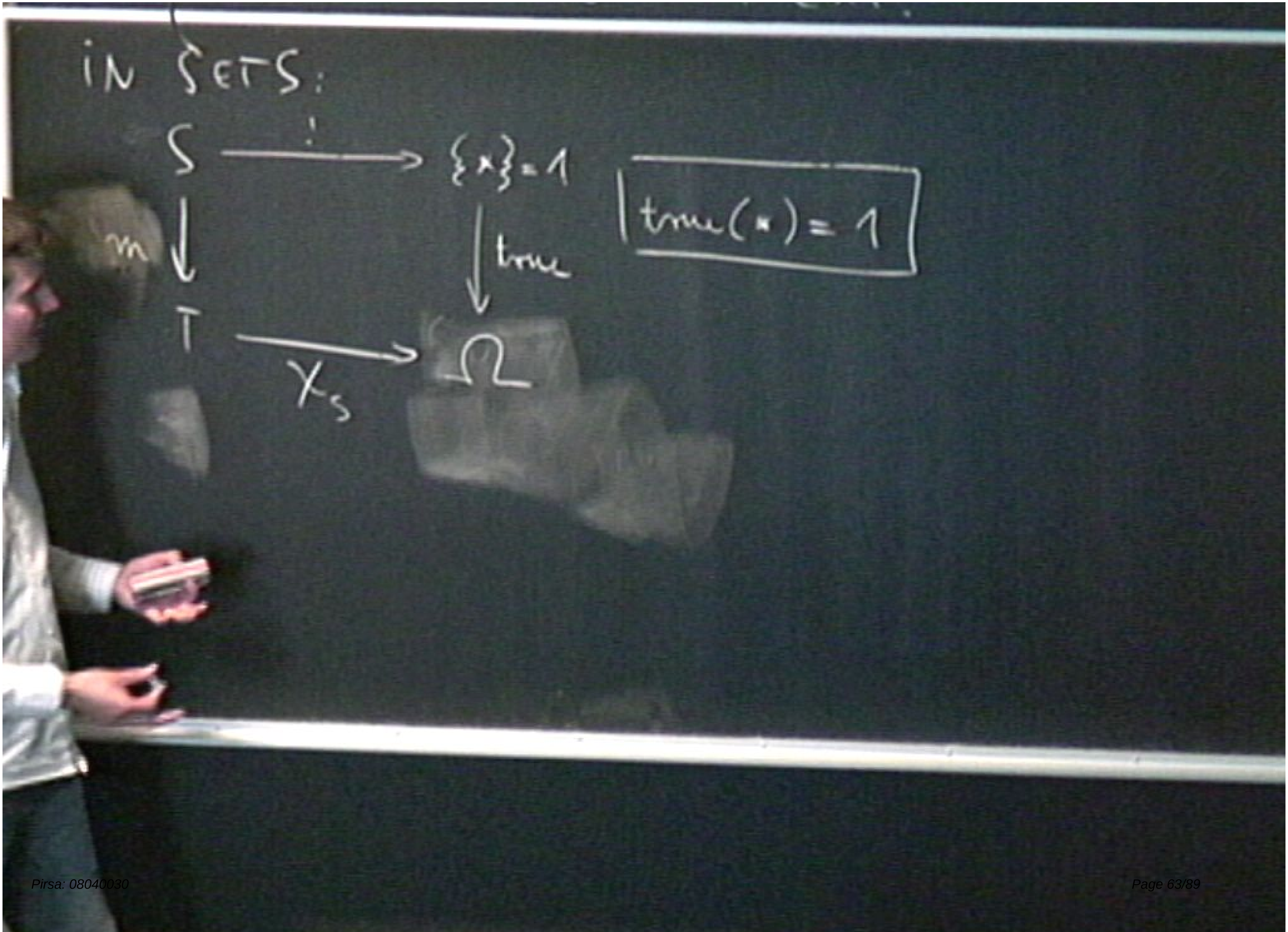
in SETS:
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χS
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in SETS:

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ T & \xrightarrow[\chi_S]{} & \Omega \end{array}$$

$$S = \chi_S^{-1}(1)$$

The Logical Structure of Sub-objects

In a topos:

1. The collection, $\mathrm{Sub}(A)$, of sub-objects of an object A forms a *Heyting algebra*.
2. The same applies to $\Gamma\Omega := \mathrm{Hom}(1, \Omega)$, 'global elements'

A Heyting algebra is a distributive lattice, $\mathfrak{H}$, with 0 and 1, and such that to each $\alpha, \beta \in \mathfrak{H}$ there exists $\alpha \Rightarrow \beta \in \mathfrak{H}$ such that

$$\gamma \preceq (\alpha \Rightarrow \beta) \text{ iff } \gamma \wedge \alpha \preceq \beta.$$

- Negation is defined as $\neg\alpha := (\alpha \Rightarrow 0)$.
- *Excluded middle* may not hold: there may exist $\alpha \in \mathfrak{H}$ such that $\alpha \vee \neg\alpha \prec 1$.

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The Mathematics of ‘Neo-Realism’

- **In set theory:** let $K \subseteq X$ and $x \in X$. Consider the proposition “$x \in K$”. The truth value is

$$\nu(x \in K) = \begin{cases} 1 & \text{if } x \text{ belongs to } K; \\ 0 & \text{otherwise.} \end{cases}$$

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$$\nu(x \in K) := \chi_K \circ \ulcorner x \urcorner$$

where $\chi_K \circ \ulcorner x \urcorner : 1 \to \Omega$. Thus the 'generalised truth value' of "$x \in K$" belongs to the Heyting algebra $\Gamma\Omega$.

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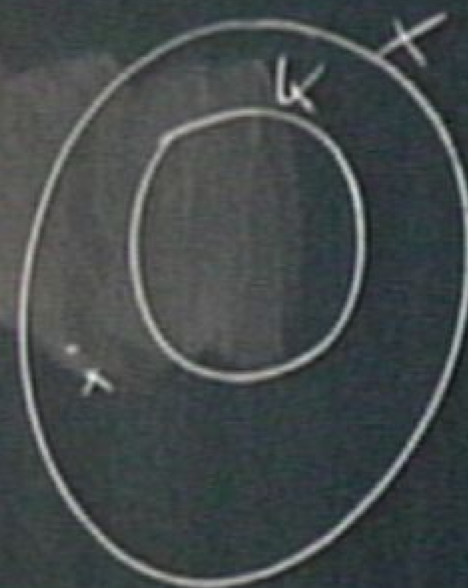
IN SETS:

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ T & \xrightarrow[\chi_S]{} & \Omega \end{array}$$



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Our Main Contention

For a given theory-type, each system S to which the theory is applicable can be formulated and interpreted neo-realistically within the framework of a particular topos $\tau_\phi(S)$.

Different theory-types have *different* topoi representations ϕ.

Conceptually, this structure is 'neo-realist' in the sense:

1. A physical quantity, A, is represented by an arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ where $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are two special objects in the topos $\tau_\phi(S)$.
2. Propositions about S are represented by sub-objects of $\Sigma_{\phi,S}$. These form a Heyting algebra.
3. The topos analogue of a state is a 'truth object' or 'pseudo-state'.

   Propositions are assigned truth values in $\Gamma\Omega_{\tau_\phi(S)}$.

- Thus a theory expressed in this way *looks* like classical physics except that classical physics always employs the topos **Sets**, whereas other theories—including quantum theory—use a different topos.

  If desired a space/time object can be included too, possibly using SDG.


EXS.: EVERY GROUP G IS A CAT
IN SETS:
S
1
true
Ω
Σ STATE OBJECT
KS ⟺ Σ HAS NO POINTS

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- A topos can be used as a *foundation* for mathematics itself, just as set theory is used in the foundations of 'normal' (or 'classical') mathematics.

- In fact, any topos has an 'internal language' that is similar to the formal language on which set theory is based.

This is used to *interpret* the theory in a 'neo-realist' way.

The Idea of a Truth Object

In set theory, basic *mathematical* propositions are of the form

- "$x \in K$"; or, equivalently,
- "$\{x\} \subseteq K$"

In classical physics, a truth value is assigned to propositions by specifying a micro-state, $s \in \mathcal{S}$. Then, truth value of "$A \in \Delta$" is

$$\nu(A \in \Delta; s) = \begin{cases} 1 & \text{if } \tilde{A}(s) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

- But in a topos, the state object $\Sigma_{\phi,S}$ may have *no* global elements.

  For example, this *is* the case' in quantum theory.

- So, what is the analogue of a state in a general topos?

In Classical Physics

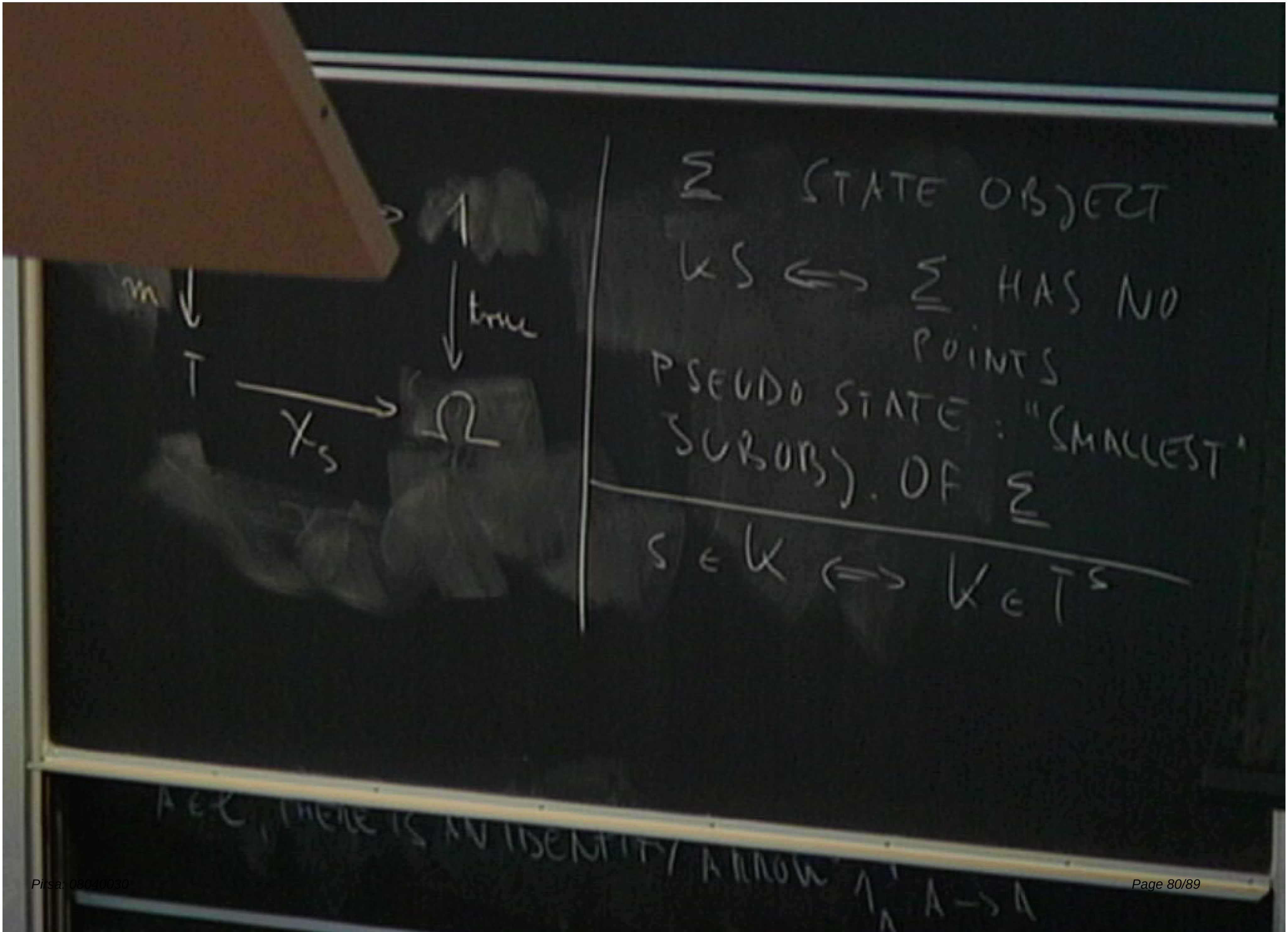
For each $s \in \mathcal{S}$, the *truth object*, T^s is the collection of sub-sets:

$$T^s := \{K \subseteq \mathcal{S} \mid s \in K\}$$

i.e., $T^s \subseteq P\mathcal{S}$, or, equivalently, $T^s \in PP\mathcal{S}$. Then

$$\begin{aligned}
\nu(A \in \Delta; s) &= \begin{cases} 1 & \text{if } s \in \tilde{A}^{-1}(\Delta); \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \tilde{A}^{-1}(\Delta) \in T^s; \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \{s\} \subseteq \tilde{A}^{-1}(\Delta); \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

which exploits the fact that $\{s\} = \bigcap_{K \in T^s} K$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In a General Topos

Version 1: A truth object is $\ulcorner T \urcorner : 1_\tau \to PP\Sigma_{\phi,S}$, a sub-object of $P\Sigma_{\phi,S}$. Then, if $K \in \mathrm{Sub}(\Sigma_{\phi,S})$, the mathematical proposition "$K \in T$" has truth value $\nu(K \in T) \in \Gamma\Omega_{\phi,S}$.

Version 2: A truth object is $\ulcorner \mathfrak{w} \urcorner : 1_\tau \to P\Sigma_{\phi,S}$, a sub-object of $\Sigma_{\phi,S}$. Then if $K \in \mathrm{Sub}(\Sigma_{\phi,S})$, the mathematical proposition "$\mathfrak{w} \subseteq K$" has truth value $\nu(\mathfrak{w} \subseteq K) \in \Gamma\Omega_{\phi,S}$.

To relate them we define

$$\mathfrak{w}^T := \bigwedge_{K \in T} K$$

which is the topos analogue of $\{s\} = \bigcap_{K \in T^s} K$.

We can think of $\mathfrak{w}^T$ as being a 'pseudo-state' of the system.

V. Formal Languages

There is a very elegant way of describing what we are doing. Namely, to construct a theory of a system S is equivalent to finding a *representation*, ϕ, in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

V. Formal Languages

There is a very elegant way of describing what we are doing. Namely, to construct a theory of a system S is equivalent to finding a *representation*, ϕ, in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

- The language, $\mathcal{L}(S)$ depends on the physical system, S, but not on the theory type (classical, quantum,...).

  However, the representation *does* depend on theory type.

The Language $\mathcal{L}(S)$

The language $\mathcal{L}(S)$ of a system S is *typed*. It includes:

- A symbol Σ: the linguistic precursor of the state object.
- A symbol $\mathcal{R}$: the linguistic precursor of the quantity-value object.
- A set, $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ of 'function symbols' $A : \Sigma \to \mathcal{R}$: the linguistic precursors of physical quantities.
- A symbol Ω: the linguistic precursor of the sub-object classifier.
- A 'set builder' $\{\tilde{x} \mid \omega\}$. This is a term of type PT, where $\tilde{x}$ is a variable of type T, and ω is a term of type Ω.
- May also include symbols for space/time

Representing the Language $\mathcal{L}(S)$

Next step: find a representation of $\mathcal{L}(S)$ in a suitable topos.

A classical theory of S: The representation σ is:

- The topos $\tau_\sigma(S)$ is **Sets**.
- Σ is represented by a symplectic manifold $\Sigma_{\sigma,S}$ (was $\mathcal{S}$).
- $\mathcal{R}$ is represented by the real numbers $\mathbb{R}$; i.e., $\mathcal{R}_{\sigma,S} := \mathbb{R}$.
- The function symbols $A : \Sigma \to \mathcal{R}$ become functions $A_{\sigma,S} : \Sigma_{\sigma,S} \to \mathbb{R}$ (was $\tilde{A}$)
- Ω is represented by the set $\{0, 1\}$ of truth values.

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The Topos of Quantum Theory

- We focus on the intrinsic *contextuality* implied by the Kocken-Specher theorem.

- In standard theory, we can potentially assign 'actual values' only to members of a commuting set of operators.

  This is used in *modal* interpretations: for example, Bohm.

We think of such a set as a *context*, in which to view the system.
A context is a 'classical snapshot', or 'world-view'.

- We want to consider all contexts at once! A theory of 'many world-views'.

  This motivates considering the topos of presheaves over the category of abelian subalgebras of $\mathcal{B}(\mathcal{H})$: a poset under the operation of sub-algebra inclusion.

The state object that represents symbol Σ is the '*spectral presheaf*' $\underline{\Sigma}$. For each abelian subalgebra V, $\underline{\Sigma}(V)$ is the spectrum of V.

The K-S theorem is equivalent to the statement that $\underline{\Sigma}$ has no global elements.

- $\underline{\Sigma}$ replaces the (non-existent) state space
- A proposition represented by a projector $\hat{P}$ in QT is mapped to a sub-object $\delta(\hat{P})$ of $\underline{\Sigma}$: '*daseinisation*'.
- The quantity-value symbol $\mathcal{R}$ is represented by a presheaf $\underline{\mathcal{R}}$. This is *not* the real-number object in the topos.
- Physical quantities: arrows $\breve{A} : \underline{\Sigma} \to \underline{\mathcal{R}}$. Constructed from the Gel'fand transforms of the spectra in $\underline{\Sigma}$.
- Each state $|\psi\rangle$ gives a truth object $\underline{T}^{|\psi\rangle}$, and associated pseudo-state $\underline{\mathfrak{w}}^{|\psi\rangle} := \delta(\,|\psi\rangle\langle\psi|\,)$.