Title: The Exact Renormalization Group - Lecture 3: The derivative Expansion

Date: Apr 30, 2008 10:30 AM

URL: http://pirsa.org/08040021

Abstract: One of the main strengths of the ERG is that it admits nonperturbative approximation schemes which preserve renormalizability. I will introduce a particularly powerful scheme, the derivative expansion.

The Exact Renormalization Group: Introduction & Applications

Lecture 3: The Derivative Expansion

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Dublin Institute for Advanced Studies

April / May 2008

Useful Literature

- T. R. Morris, "Elements of the continuous renormalization group," Prog. Theor. Phys. **131** (1998) 395, hep-th/9802039.
- C. Bagnuls and C. Bervillier, "Exact renormalization group equations: An introductory review," Phys. Rept. **348** (2001) 91, hep-th/0002034.
- J. Berges, N. Tetradis, and C. Wetterich, "Non-perturbative renormalization flow in quantum field theory and statistical physics," Phys. Rept. **363** (2002) 223, hep-ph/0005122.

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Continued. . .

- A. Hasenfratz and P. Hasenfratz, "Renormalization Group Study Of Scalar Field Theories," Nucl. Phys. B **270** (1986) 687 [Helv. Phys. Acta **59** (1986) 833].
- R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, "Scheme Independence And The Exact Renormalization Group," *Phys. Lett.* **B 347** (1995) 80, hep-th/9411122.
- T. R. Morris, "Derivative expansion of the exact renormalization group," Phys. Lett. B **329** (1994) 241, hep-ph/9403340.
- T. R. Morris, "On truncations of the exact renormalization group," Phys. Lett. B **334** (1994) 355, hep-th/9405190.
- C. Wetterich, "Exact evolution equation for the effective potential," Phys. Lett. B **301** (1993) 90.

Useful Concepts from Earlier Lectures

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- To conveniently see fixed point behaviour etc., we need to rescale to dimensionless units
- Renormalized Trajectories (RT)s exhibit self-similarity

$$S_\Lambda[\varphi] = S[\varphi](g_1(\Lambda), \ldots, g_n(\Lambda), \gamma(\Lambda))$$

- There are two common forms for the ERG equation involving

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  - The Wilsonian effective action (Polchinski-like)

$$-\Lambda\partial_\Lambda S_\Lambda = \frac{1}{2}\frac{\delta S_\Lambda}{\delta\varphi}\cdot\Delta_{UV}\cdot\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\Delta_{UV}\cdot\frac{\delta\Sigma}{\delta\varphi}$$

  - The Effective Average Action

$$\frac{\partial\Gamma^{\rm int}_\Lambda[\varphi_c]}{\partial\Lambda} = \frac{1}{2}\mathrm{tr}\left[\frac{\partial\Delta^{-1}_{\rm IR}}{\partial\Lambda}\cdot\left(\Delta^{-1}_{\rm IR} + \frac{\delta^2\Gamma^{\rm int}_\Lambda}{\delta\varphi_c\delta\varphi_c}\right)^{-1}\right]$$

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Linearizing about a fixed point, operators are classified as

- Scaling operators—these are physical
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1. Preamble
2. Motivation
3. The Local Potential Approximation
   - Setting up the Approximation Scheme
   - Fixed Points and their Properties
4. Beyond Leading Order
   - Set-up
   - Reparametrization Invariance
5. Conclusions

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The Starting Point

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- A flow equation with the anomalous dimension explicitly and conveniently included is a cousin of the Polchinski equation:

$$\left(-\Lambda\partial_\Lambda + \frac{\gamma}{2}\varphi \cdot \frac{\delta}{\delta\varphi}\right) S = \frac{1}{2}\frac{\delta S}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi}$$

  - The ERG kernels, which incorporate UV regularization
  - $\Sigma = S - 2\hat{S}$, $\hat{S}$ is the seed action

- The simplest choice for the seed action is

$$\hat{S} = \frac{1}{2}\varphi \cdot C_{UV}^{-1}(p)p^2 \cdot \varphi$$

- We will also choose

$$\Delta(p,\Lambda) = \frac{C_{UV}}{p^2}$$

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$$S = \sum_{n=2}^{\infty} \left(\prod_{i=1}^{n} \int \frac{d^D p_i}{(2\pi)^D} \right) S^{(n)}(p_1, \ldots, p_n) \varphi(p_1) \cdots \varphi(p_n) (2\pi)^D \delta^{(D)}(p_1 + \ldots + p_n)$$

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  - $\int_p \equiv \int \frac{d^D p_i}{(2\pi)^D}$
  - $\hat{\delta}(p) \equiv (2\pi)^D \delta^{(D)}(p)$

Rescaling to Dimensionless Variables

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Rescaling the Field

- The field has already been rescaled by $\sqrt{Z}$
- Now scale out the engineering dimension:

- $\left(-\Lambda\partial_\Lambda + \frac{\eta}{2}\varphi \cdot \frac{\delta}{\delta\varphi}\right) S \to \left(-\Lambda\partial_\Lambda - \frac{D+2-\eta}{2}\varphi \cdot \frac{\delta}{\delta\varphi}\right) S$

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- This has three effects on the vertices

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    - Compensates $\int_{p_1} \to \int_{p_1}\Lambda^D$
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Putting Everything Together

- Write $-\Lambda\partial_\Lambda = \partial_t$, with $t = \ln \mu/\Lambda$
- Note that

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  - Field counting operator
  - Momentum scale counting operator

$$\int \frac{d^Dp}{(2\pi)^D}\varphi(p)p^\mu \frac{\partial'}{\partial p^\mu}\frac{\delta}{\delta\varphi(p)} = \int \frac{d^Dp}{(2\pi)^D}\varphi(p)p^\mu \frac{\partial}{\partial p^\mu}\frac{\delta}{\delta\varphi(p)} + D$$

Rescaling to Dimensionless Variables

Putting Everything Together

- Write $-\Lambda\partial_\Lambda = \partial_t$, with $t = \ln \mu/\Lambda$
- Note that

$$\begin{aligned}\dot{\Delta} &= -\Lambda\partial_\Lambda C_{\mathrm{UV}}(p^2/\Lambda^2)/p^2 = 2C'_{UV}(p^2/\Lambda^2)/\Lambda^2 \\ &\to 2C'_{\mathrm{UV}}(p^2)/\Lambda^2\end{aligned}$$

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- The ERG equation (ERGE)
  - Is an integro-differential equation
  - Contains variational derivatives
  - Is non-linear
  - Applied to problems where there is no small parameter
- Exact solutions to the ERGE are known only in special cases
- For problems with no small parameter, a reliable truncation scheme is required
- For finding non-trivial fixed points, truncating the infinite tower of equations arising from expanding in powers of the field is known to be unreliable
  T. R. Morris, Phys. Lett. B **334** (1994) 355.
- An alternative is to expand in derivatives of the field:

$$S_\Lambda[\varphi] \sim \int d^D x \left[V_\Lambda(\varphi) + \frac{1}{2}(\partial_\mu \varphi)^2 K_\Lambda(\varphi) + O\left(\partial^4\right) \right]$$

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Justification for the Derivative Expansion

- It hopefully captures the essentials of the long distance physics
- (But fixed points are scale independent?!)
- Even at lowest order,

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Outline of this Lecture

1. Preamble
2. Motivation
3. The Local Potential Approximation
   - Setting up the Approximation Scheme
   - Fixed Points and their Properties
4. Beyond Leading Order
   - Set-up
   - Reparametrization Invariance
5. Conclusion

The Approximation Scheme

The Approximation Scheme

- At lowest order, the derivative expansion reduces to The Local Potential Approximation (LPA):

$$S_\Lambda[\varphi] \sim \int d^D x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + V_\Lambda(\varphi) \right]$$

- The local potential depends on the field but not its derivatives
- The kinetic term does not renormalize, so the anomalous dimension is zero
- Writing

$$S = \sum_{n=2}^{\infty} \int_{p_1,\ldots,p_n} S^{(n)}(p_1,\ldots,p_n)\varphi(p_1)\cdots\varphi(p_n)\hat{\delta}(p_1+\cdots+p_n)$$

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the approximation consists of taking

- $S^{(2)}(p,-p) = \frac{1}{2}p^2 + S^{(2)}(0)$
- $S^{(n>2)}(p_1, \ldots, p_n) = S^{(n>2)}(0, \ldots, 0)$

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History

- The LPA was first written down in J. F. Nicoll et al., Phys. Rev. Lett. 33 (1974) 540.
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$$P(x)G[\varphi] = e^{x\partial/\partial\varphi(0)}G[\varphi]\Big|_{\varphi=0}$$

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Classical Term

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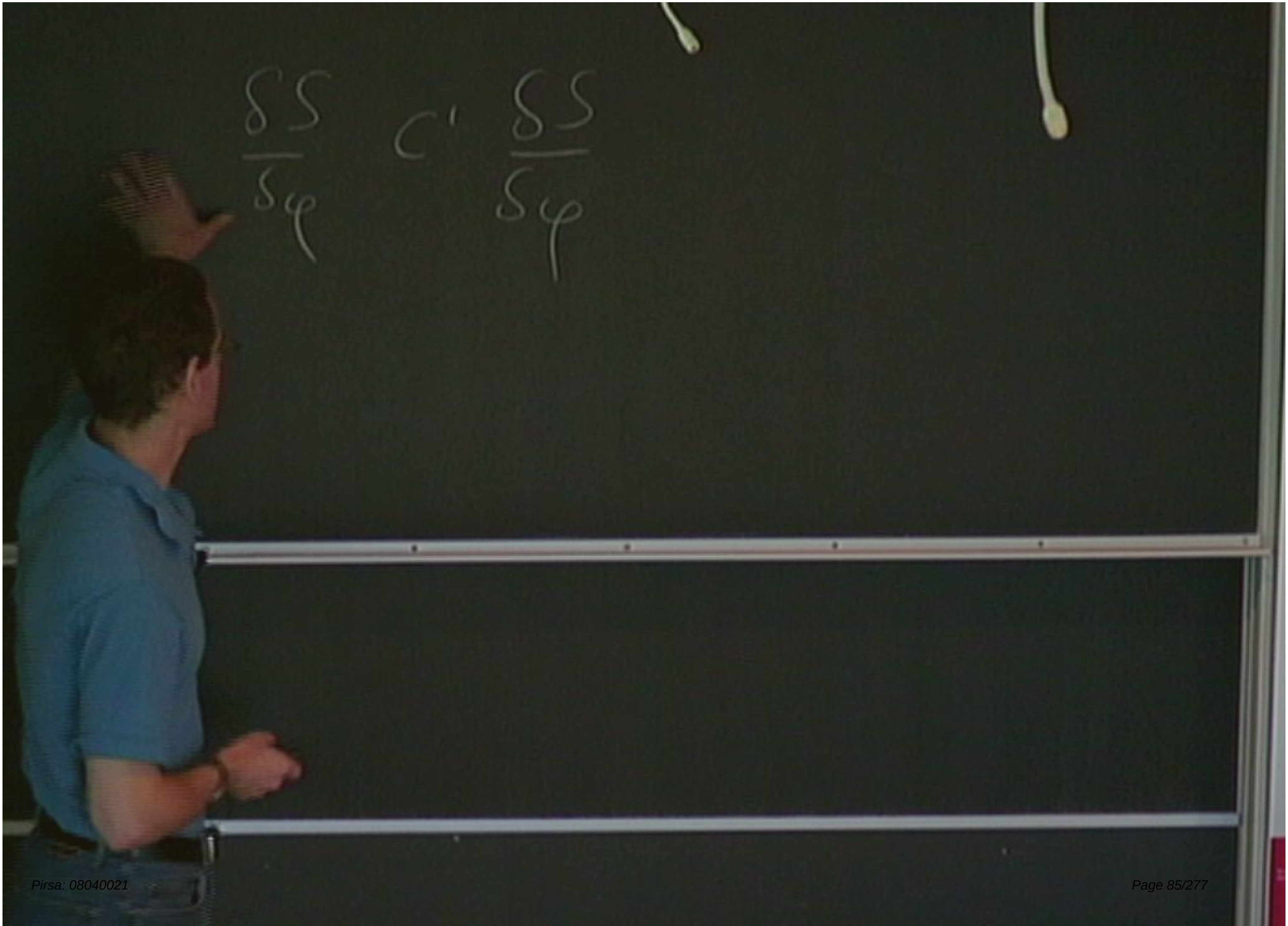
- $\int_p \frac{\delta S}{\delta\varphi(-p)} C'_{\mathrm{UV}}(p) \frac{\delta S}{\delta\varphi(p)} \to C'_{\mathrm{UV}}(0) V'^2 \equiv -K_0 V'^2$

Quantum Term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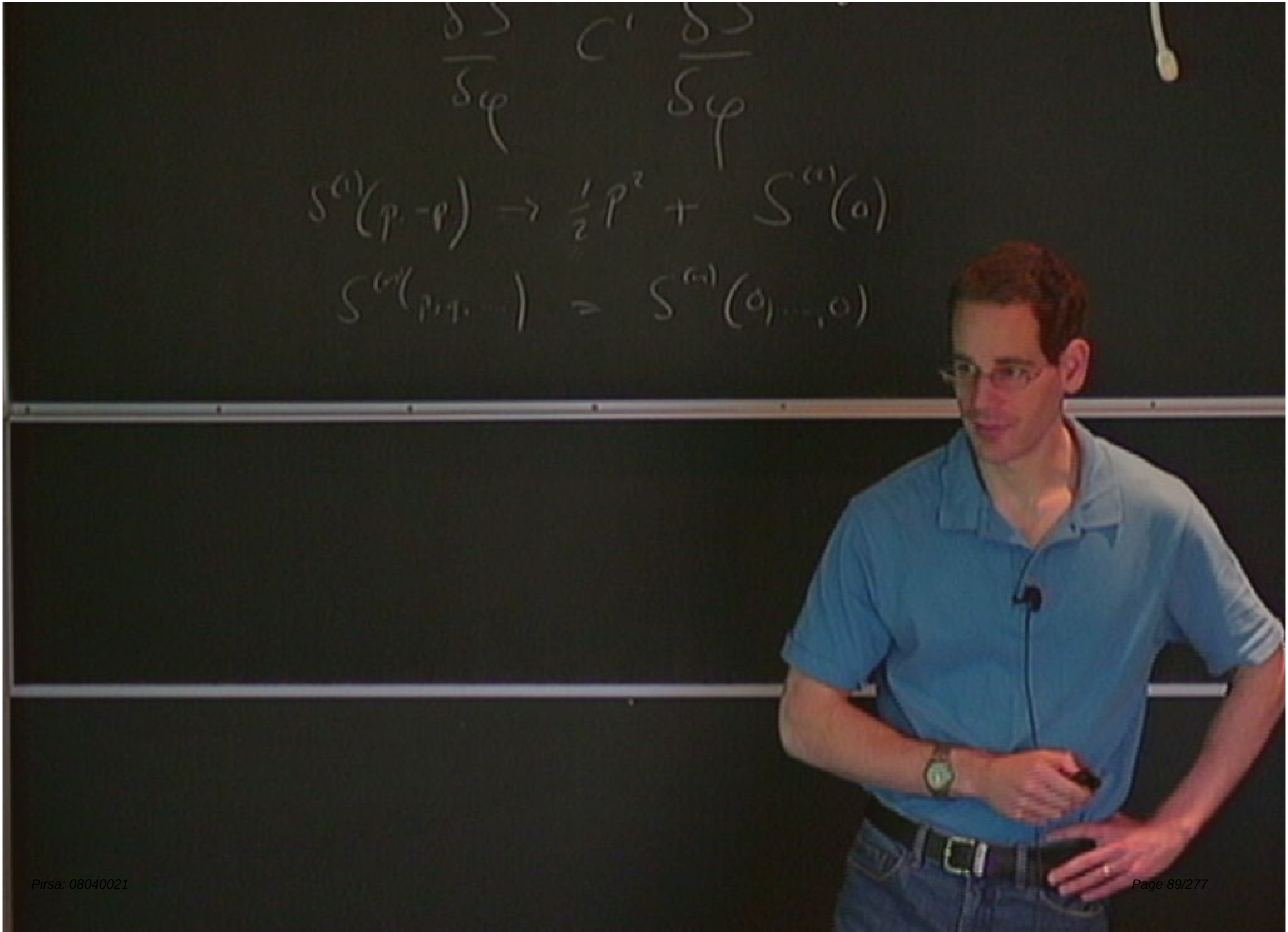
δS/δφ   C' δS/δφ
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- $\dot{V} = V'' - V'^2 - dxV' + DV$
  - $d \equiv (D-2)/2$
  - Non-linear p.d.e
  - Manifestly independent of the cutoff function

Finding Fixed Points

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- Set $\dot{V} = 0$
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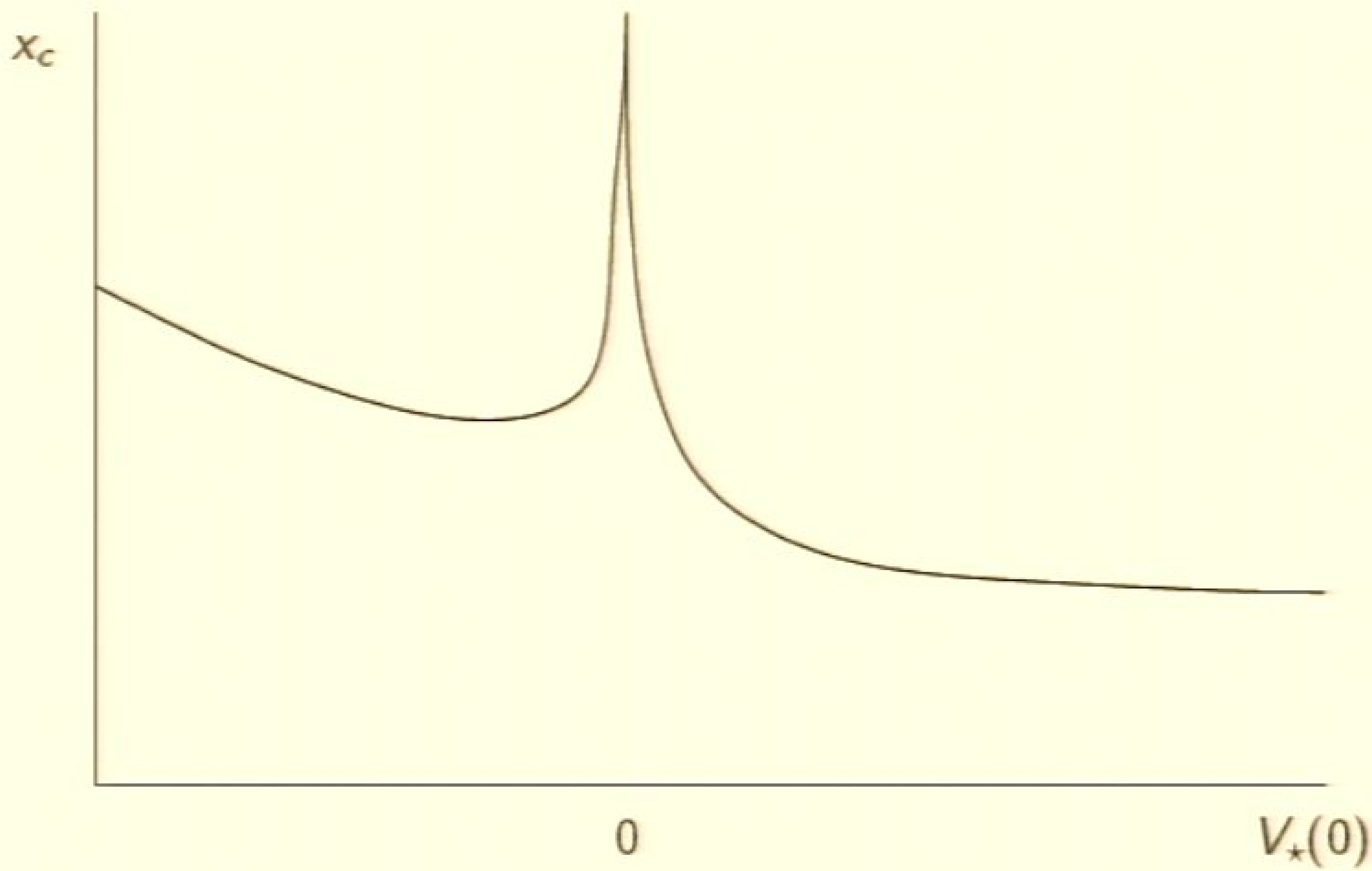

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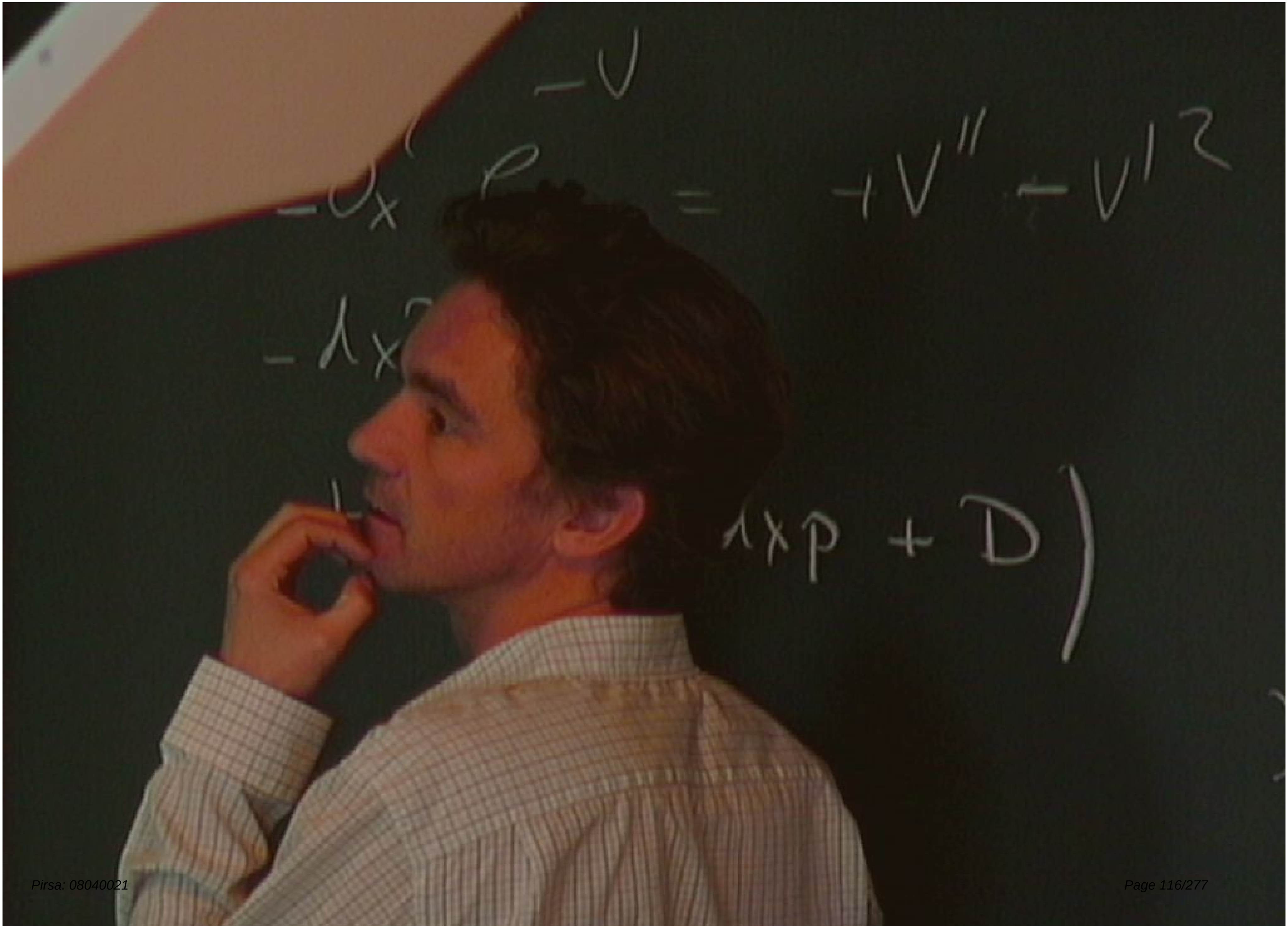
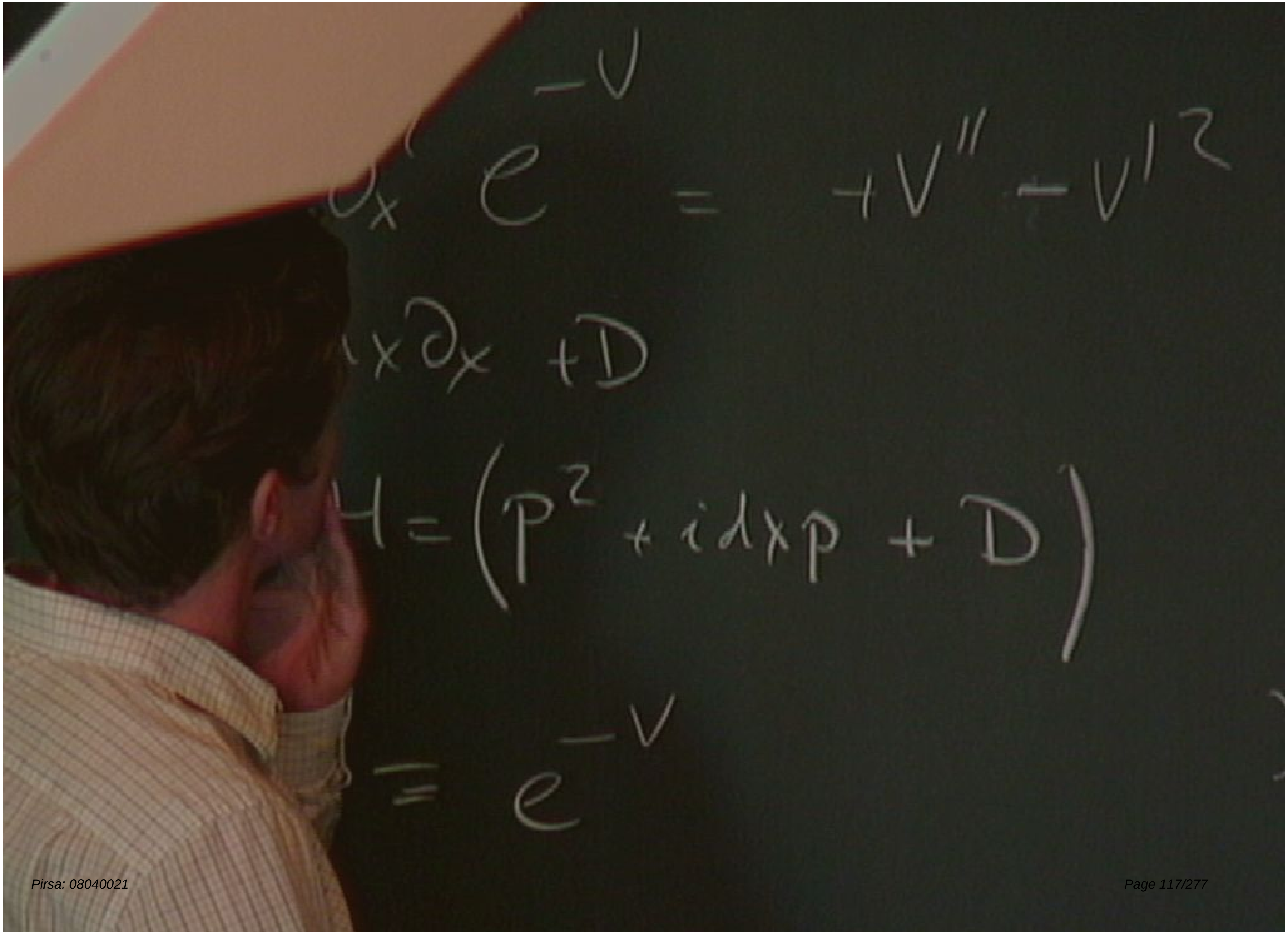
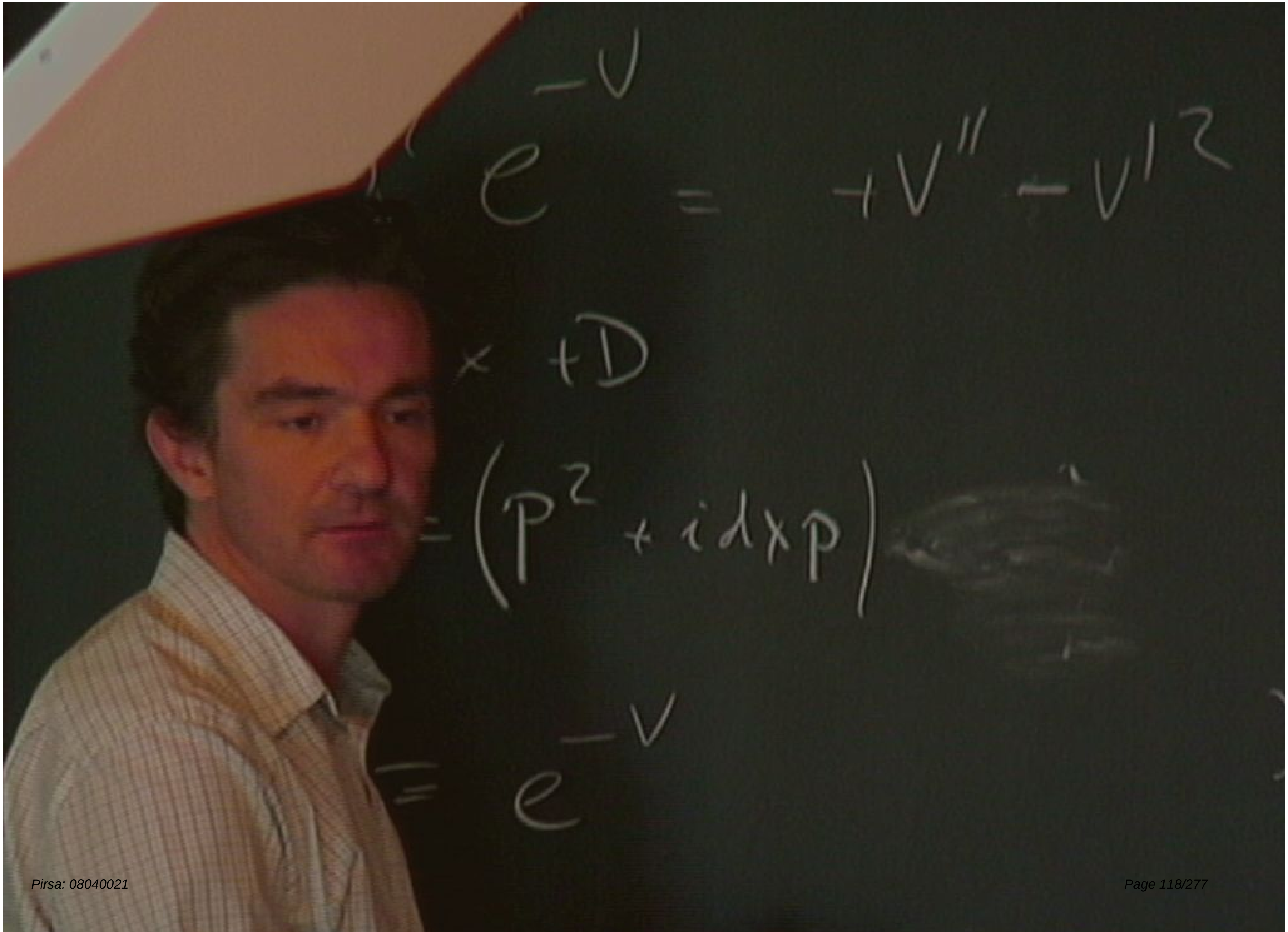

−V
= −V'' + V'2
λxp + D)


∂x e−V = +V'' − V'2
x∂x +D
H = (p2 + iλxp + D)
= e−V


e^-V = +V'' - V'^2
+D
= (P^2 + idxp)
= e^-V

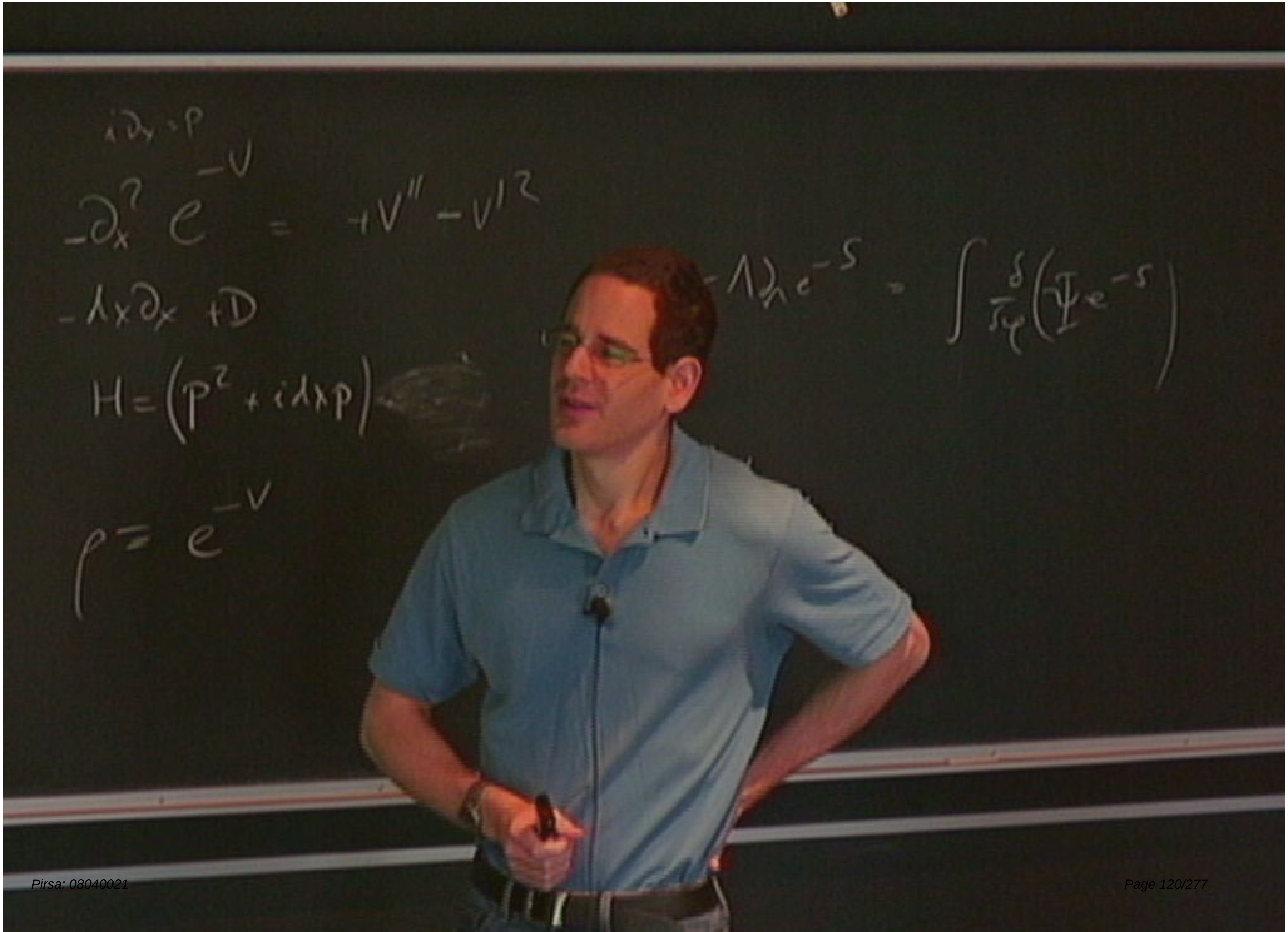
$$-\partial_x (e^{-V} = +V'' - V'^2$$

$$-\lambda x \partial_x + D$$

$$H = \left(p^2 + i\lambda x p\right)$$

$$\rho = e^{-V}$$

$$\partial_t \psi$$



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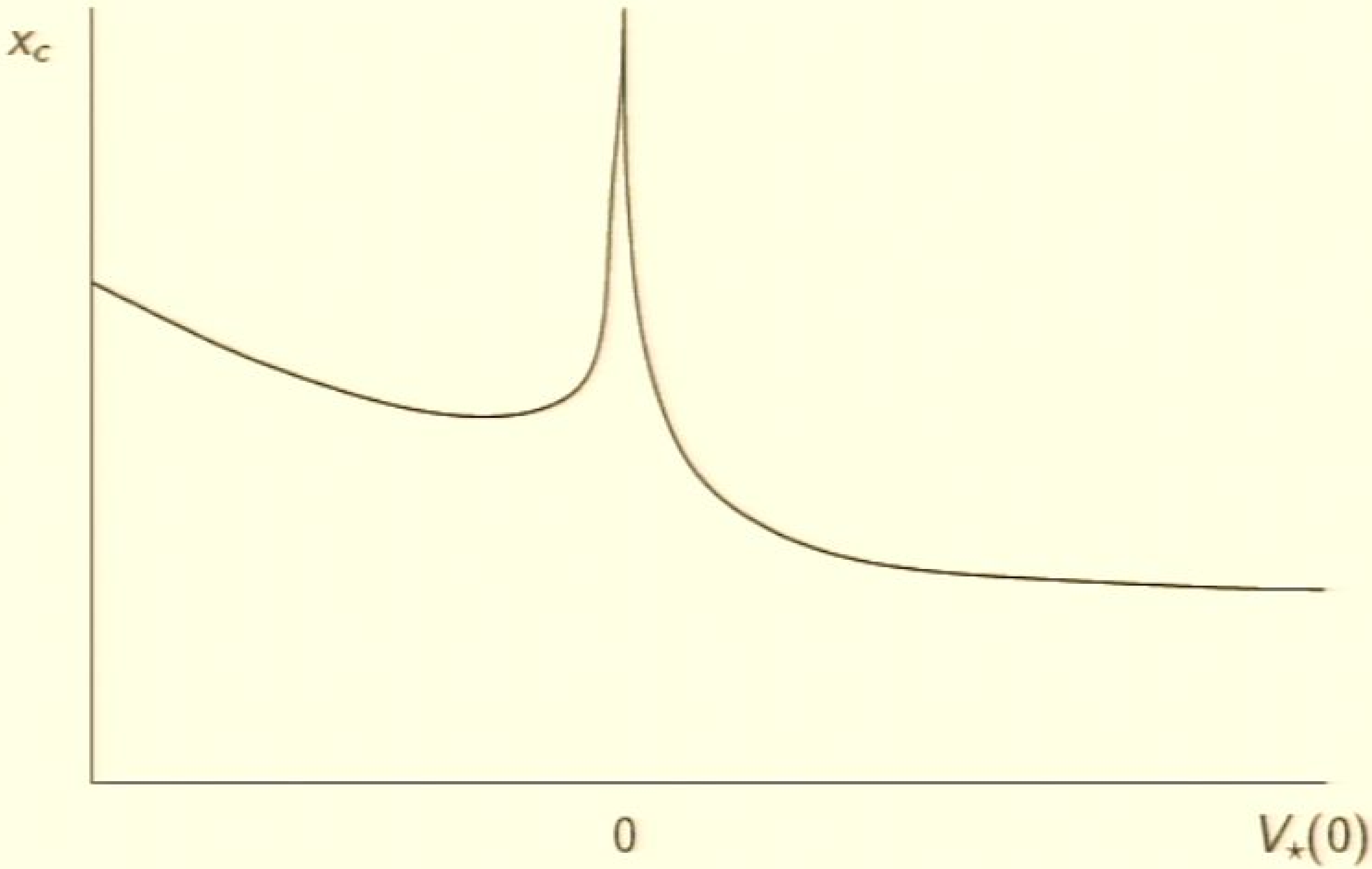

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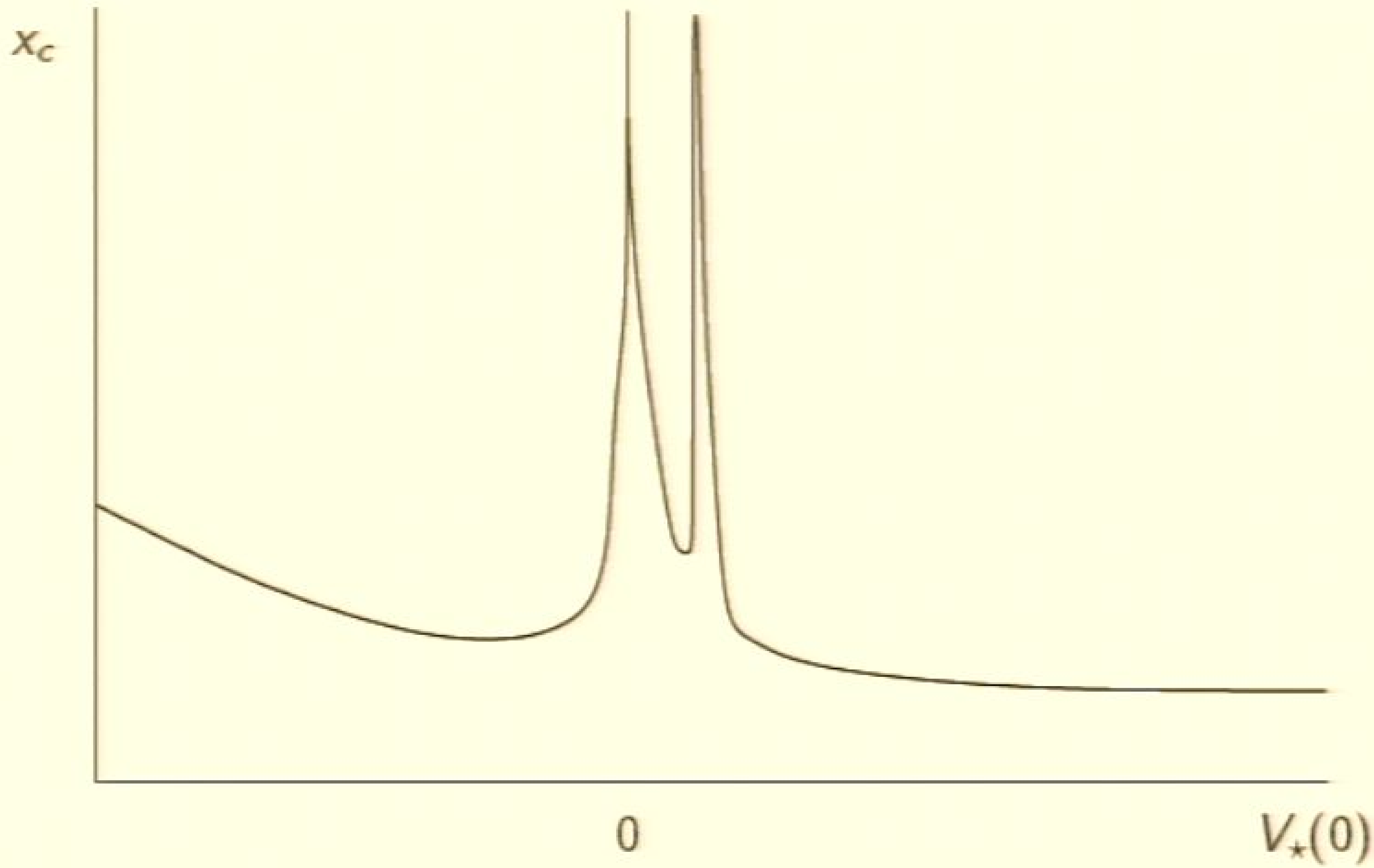

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- Recall the fixed point equation:

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- For globally non-singular solutions, the large x behaviour is

$$V(x) \sim x^2$$

- Impose this as a pair of conditions for $x \to \pm\infty$
- Two conditions for a second order differential equation
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The Scaling Operators

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Linearizing About a Fixed Point

- LPA: $\dot{V} = V'' - V'^2 - dxV' + DV$
- Away from fixed point: $V(x,t) = V_*(x) + v(x,t)$
- Treating v as small, linearize:

$$\dot{v} = v'' - 2V_*'v' - dxv' + Dv$$

- Separate variables: $v(x,t) = \alpha\, e^{\lambda t} u(x)$
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- Note: at any finite distance away from the FP, the linear approximation will break down for large x

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- After linearization and separation of variables

$$V(x,t) = V_\star(x) + \alpha e^{\lambda t} u(x)$$

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- $u'' - dxu' + (D - \lambda)u = 0$
- Shifting $x \to x/\sqrt{d}$ ...
- ...yields Hermite's Equation

$$u'' - 2xu' + 2\frac{2(D-\lambda)}{D-2}u = 0$$

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Polynomial Form of the Potential

- Hermite's equation $u'' - 2xu' + 2nu = 0$ has polynomial solutions
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$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

- Comparing with $u'' - 2xu' + 2\frac{2(D-\lambda)}{D-2} u = 0$ implies quantization of λ.
- For an even potential, $n = 2k$

$$\lambda_k = D - k(D-2)$$

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$D = 4$

- $\lambda_k = 4 - 2k$
- Relevant direction: $k = 1$ (mass term)
- Marginal direction: $k = 2$ (φ^4 term)
- Everything else irrelevant

Relevance / Irrelevance

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- $\lambda_k = D - k(D-2)$
- Directions with
  - $\lambda > 0$ are relevant
  - $\lambda < 0$ are irrelevant
  - $\lambda = 0$ are marginal

$D = 4$

- $\lambda_k = 4 - 2k$
- Relevant direction: $k = 1$ (mass term)
- Marginal direction: $k = 2$ (φ^4 term)
- Everything else irrelevant

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Morris, Prog. Theor. Phys. **131** (1998) 395

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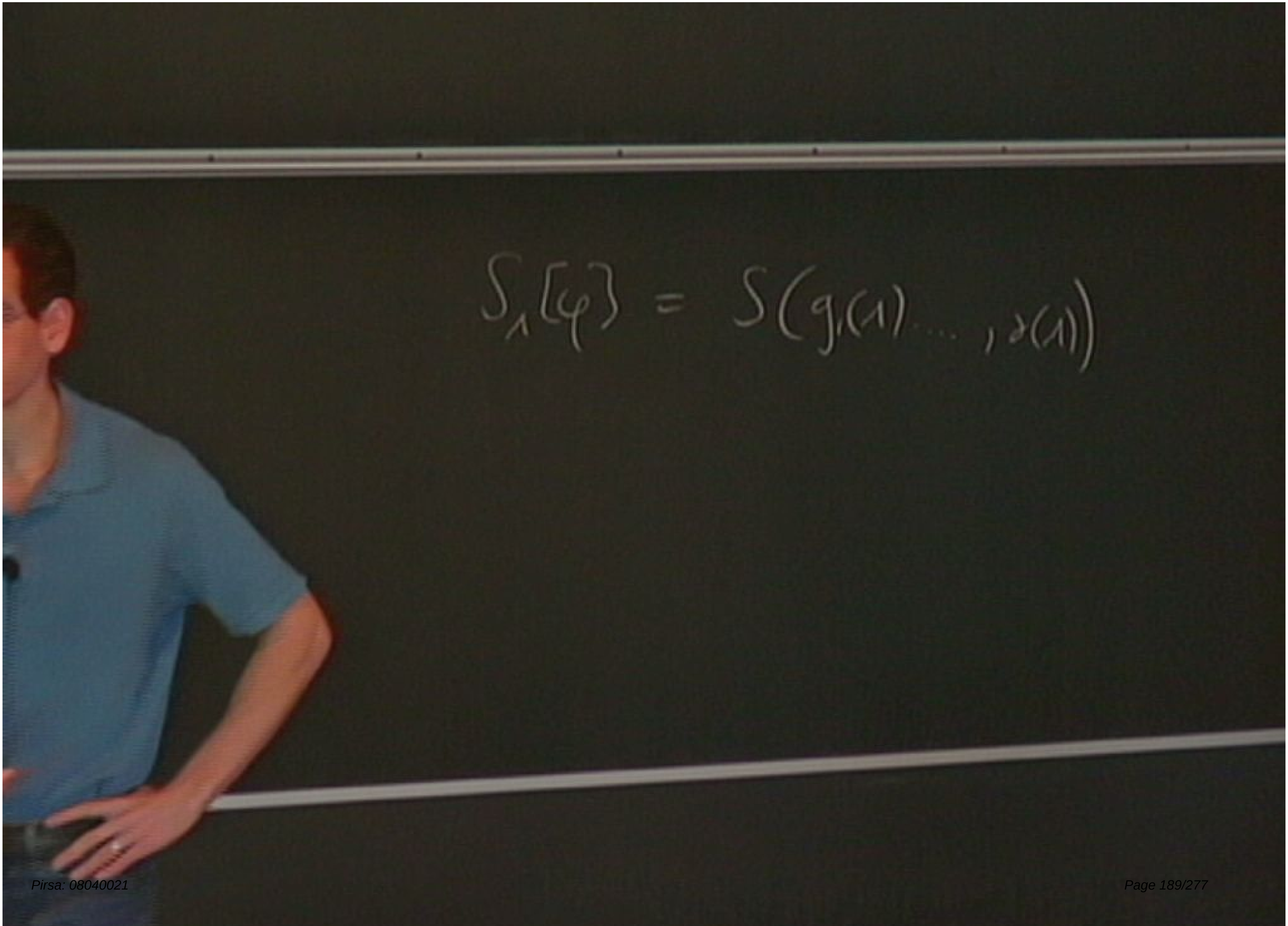
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Large Field Behaviour

- We want to determine which perturbations can be identified with renormalized couplings
- We need to analyse the evolution of trajectories a finite distance from the fixed point
- We are interested in the large field behaviour, so the linearization is not valid
- If we use the ERG equation for the effective average action, the large field behaviour comes out very simply
- $\dot{V} + dxV' - DV = -\frac{1}{\sqrt{2+V''}} \overset{x\to\pm\infty}{\sim} 0$
- $V(x,t) \sim e^{Dt} V(xe^{-dt}, 0)$
- Writing $V(x,t) = V_*(x) + v(x,t)$ (true even if v is big)
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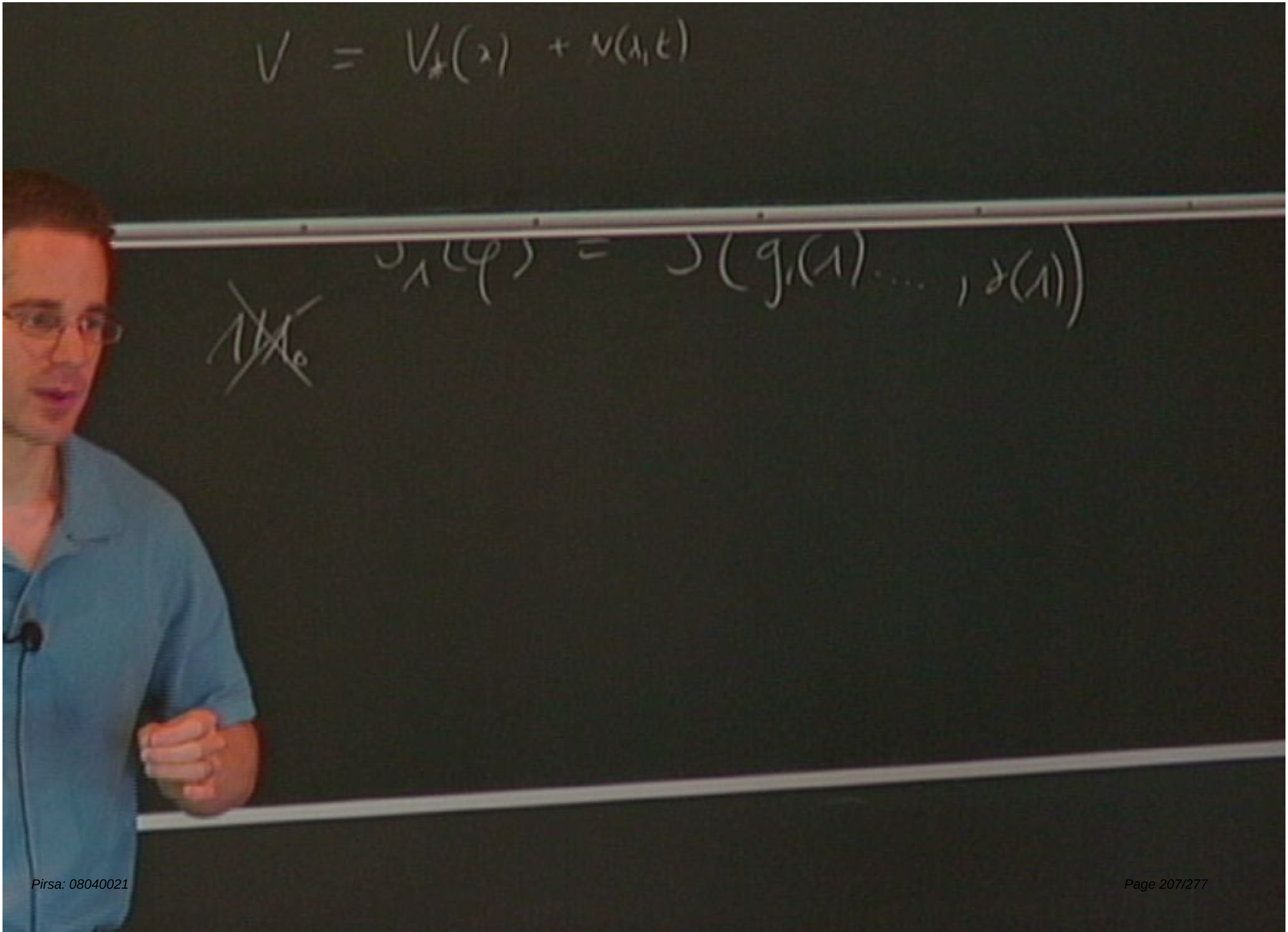
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Outline of this Lecture

1. Preamble
2. Motivation
3. The Local Potential Approximation
   - Setting up the Approximation Scheme
   - Fixed Points and their Properties
4. Beyond Leading Order
   - Set-up
   - Reparametrization Invariance
5. Conclusion

Basic Structure

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- At the first order beyond LPA, we take

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- The coupled equations have the generic form

$$\begin{aligned} \dot{V}(x,t) + d_\varphi x V' - DV &= \dots \\ \dot{K}(x,t) + \gamma K &= \dots \end{aligned}$$

- $d_\varphi = (D - 2 + \gamma)/2$
- For the Legendre transformed version, with the cutoff preferred by Morris, the right-hand sides are sub-leading for large field
- At a fixed point, for large x,

$$V \sim x^{D/d_\varphi}, \qquad K \sim x^{-\gamma_*/d_\varphi}$$

- Demanding this behaviour for $x \to \pm\infty$ provides four conditions for the pair of second order equations
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- Physics does not depend on the normalization of φ
- In the partition function, we can send $\varphi \to \Omega\varphi$ whilst leaving $J \cdot \varphi$ alone
- More generally, physics does not change under the shift

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- Analogously, the ERGE has a reparametrization invariance
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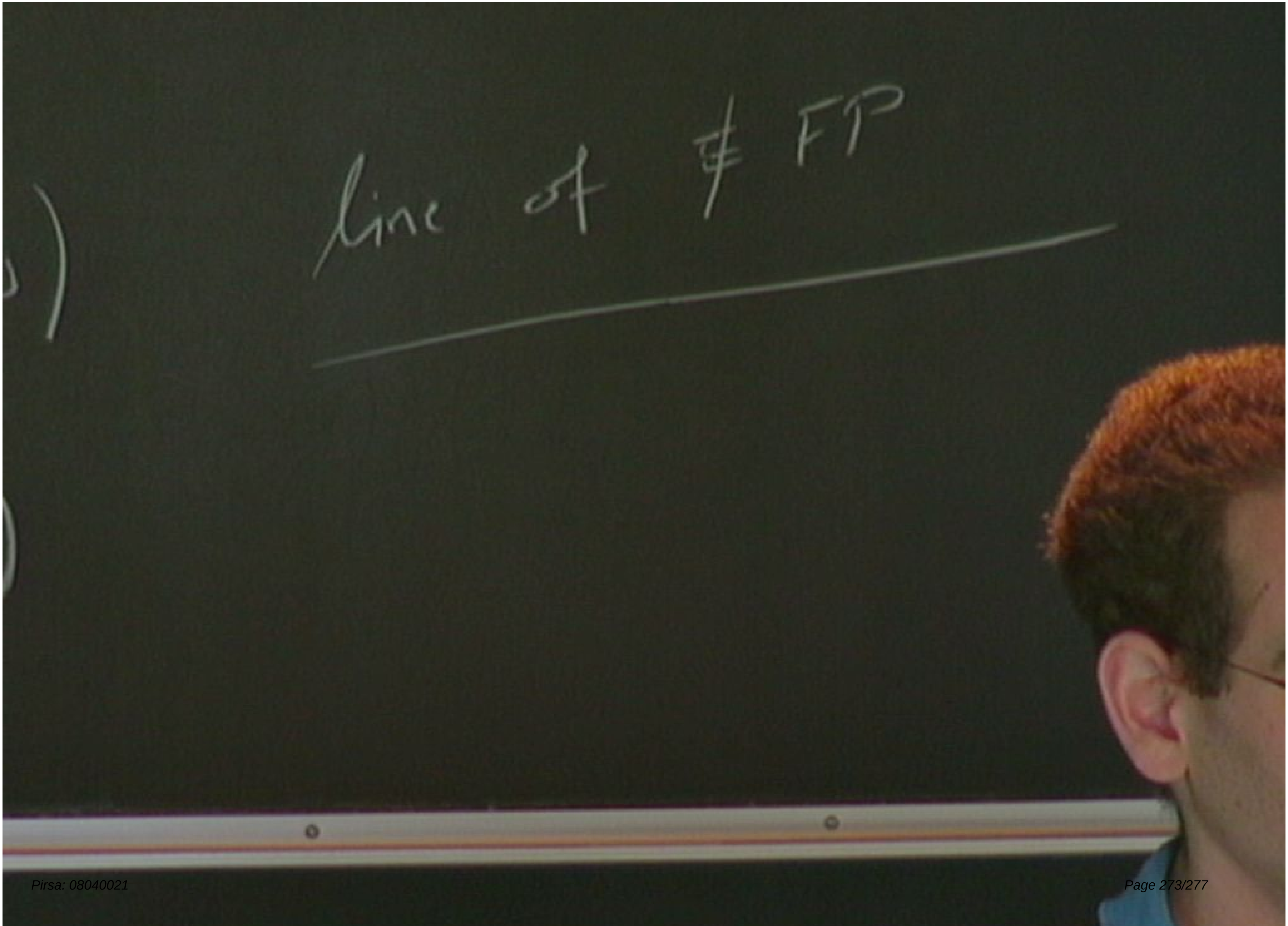
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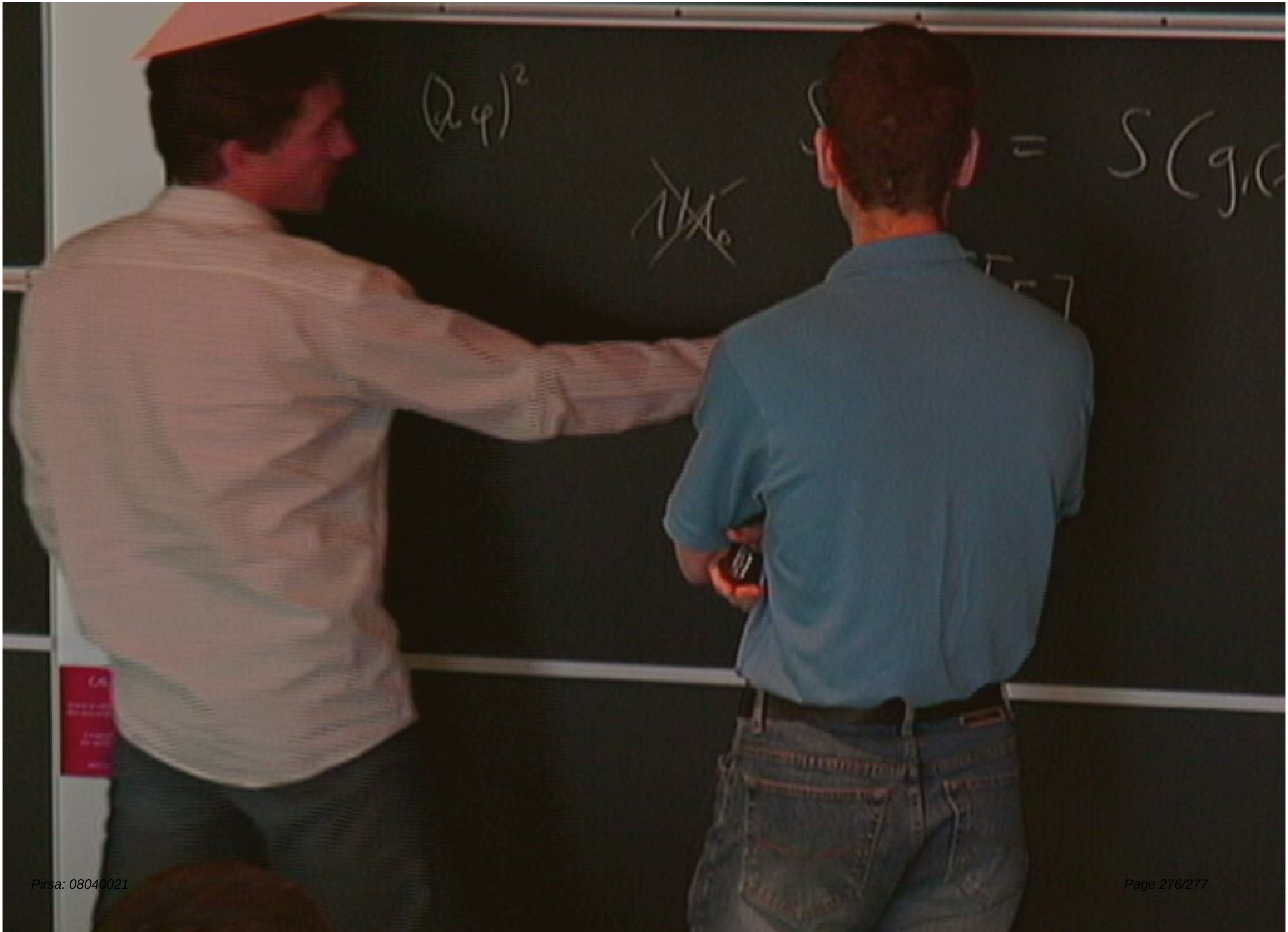

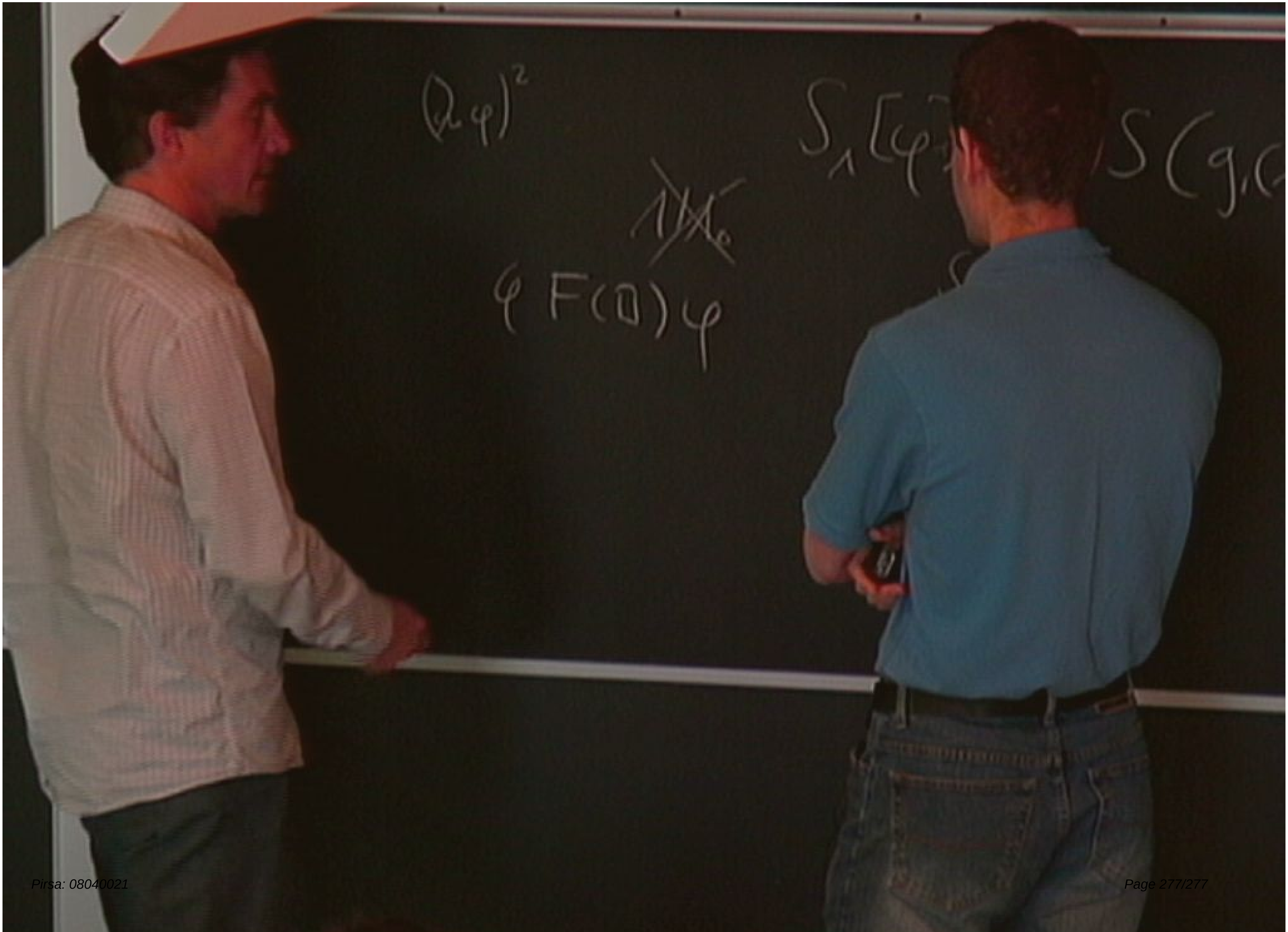
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