

Title: The Exact Renormalization Group - Lecture 3: The derivative Expansion

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Abstract: One of the main strengths of the ERG is that it admits nonperturbative approximation schemes which preserve renormalizability. I will introduce a particularly powerful scheme, the derivative expansion.

# The Exact Renormalization Group: Introduction & Applications

## Lecture 3: The Derivative Expansion

Oliver J. Rosten

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April / May 2008

## Useful Literature

- T. R. Morris, "Elements of the continuous renormalization group," *Prog. Theor. Phys.* **131** (1998) 395, [hep-th/9802039](#).
- C. Bagnuls and C. Bervillier, "Exact renormalization group equations: An introductory review," *Phys. Rept.* **348** (2001) 91, [hep-th/0002034](#).
- J. Berges, N. Tetradis, and C. Wetterich, "Non-perturbative renormalization flow in quantum field theory and statistical physics," *Phys. Rept.* **363** (2002) 223, [hep-ph/0005122](#).

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- J. Berges, N. Tetradis, and C. Wetterich, “Non-perturbative renormalization flow in quantum field theory and statistical physics,” *Phys. Rept.* **363** (2002) 223, [hep-ph/0005122](#).

## Continued...

- A. Hasenfratz and P. Hasenfratz, "Renormalization Group Study Of Scalar Field Theories," Nucl. Phys. B **270** (1986) 687 [Helv. Phys. Acta **59** (1986) 833].
- R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, "Scheme Independence And The Exact Renormalization Group," *Phys. Lett. B* **347** (1995) 80, hep-th/9411122.
- T. R. Morris, "Derivative expansion of the exact renormalization group," *Phys. Lett. B* **329** (1994) 241, hep-ph/9403340.
- T. R. Morris, "On truncations of the exact renormalization group," *Phys. Lett. B* **334** (1994) 355, hep-th/9405190.
- C. Wetterich, "Exact evolution equation for the effective potential," *Phys. Lett. B* **301** (1993) 90.

# Useful Concepts from Earlier Lectures

*(Faint, illegible text, likely bleed-through from the reverse side of the slide)*

## Useful Concepts from Earlier Lectures

- To conveniently see fixed point behaviour etc., we need to **rescale** to dimensionless units

- Renormalized Trajectories (RT)s exhibit self-similarity

$$S_\Lambda[\varphi] = S[\varphi](g_1(\Lambda), \dots, g_d(\Lambda), \gamma(\Lambda))$$

- There are two common forms for the ERG equation involving

$$\Gamma_\Lambda[\varphi] = \Gamma[\varphi] + \int d^d x \left[ \frac{1}{2} \varphi(x) \left( \frac{1}{\Lambda^2} \square + \frac{1}{\Lambda} \square \right) \varphi(x) + \frac{1}{\Lambda} \varphi(x) \square \varphi(x) \right]$$

$$= \Gamma[\varphi] + \int d^d x \left[ \frac{1}{2} \varphi(x) \left( \frac{1}{\Lambda^2} \square + \frac{1}{\Lambda} \square \right) \varphi(x) + \frac{1}{\Lambda} \varphi(x) \square \varphi(x) \right]$$

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  - The Wilsonian effective action (Polchinski-like)

$$-\Lambda \partial_\Lambda S_\Lambda = \frac{1}{2} \frac{\delta S_\Lambda}{\delta \varphi} \cdot \Delta_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \Delta_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

- The Effective Average Action

$$\frac{\partial \Gamma_\Lambda^{\text{int}}[\varphi_c]}{\partial \Lambda} = \frac{1}{2} \text{tr} \left[ \frac{\partial \Delta_{\text{int}}^{-1}}{\partial \Lambda} \cdot \left( \Delta_{\text{int}}^{-1} + \frac{\delta^2 \Gamma_\Lambda^{\text{int}}}{\delta \varphi_c \delta \varphi_c} \right)^{-1} \right]$$

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Linearizing about a fixed point, operators are classified as

- Scaling operators—these are physical
  - Relevant
  - Marginal
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- 1 Preamble
- 2 Motivation
- 3 The Local Potential Approximation
  - Setting up the Approximation Scheme
  - Fixed Points and their Properties
- 4 Beyond Leading Order
  - Set-up
  - Reparametrization Invariance
- 5 Conclusion



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# The Starting Point

Consider a system of  $N$  particles with positions  $\mathbf{r}_i$  and masses  $m_i$ . The total potential energy is  $V(\{\mathbf{r}_i\})$ . The Hamiltonian is  $H = \sum_i \frac{p_i^2}{2m_i} + V(\{\mathbf{r}_i\})$ . The partition function is  $Z = \int \prod_i d\mathbf{r}_i d\mathbf{p}_i e^{-\beta H}$ .

$$\ln Z = -\beta \langle H \rangle + \ln \int \prod_i d\mathbf{r}_i d\mathbf{p}_i e^{-\beta H}$$

The local potential approximation is  $V(\{\mathbf{r}_i\}) \approx \sum_i v(\mathbf{r}_i)$ .

$$Z \approx \int \prod_i d\mathbf{r}_i d\mathbf{p}_i e^{-\beta \sum_i \left( \frac{p_i^2}{2m_i} + v(\mathbf{r}_i) \right)}$$

The partition function factorizes:

# The Starting Point

- A flow equation with the anomalous dimension explicitly and **conveniently** included is a cousin of the Polchinski equation:

$$\left( -\Lambda \partial_\Lambda + \frac{\gamma}{2} \varphi \cdot \frac{\delta}{\delta \varphi} \right) S = \frac{1}{2} \frac{\delta S}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

- The ERG kernels, which incorporate UV regularization
- $\Sigma = S - 2\hat{S}$ ,  $\hat{S}$  is the seed action
- The simplest choice for the seed action is

$$\hat{S} = \frac{1}{2} \varphi \cdot C_{UV}^{-1}(\rho) p^2 \cdot \varphi$$

- We will also choose

$$\Delta(\rho, \Lambda) = \frac{C_{UV}}{\rho^2}$$

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# Notation

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \dots$   
 $\mathcal{H}_0 = \frac{p^2}{2m} + V_0(r)$   
 $\mathcal{H}_1 = V_1(r)$   
 $\mathcal{H}_2 = V_2(r)$

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots$$

Energy eigenvalues  $E = E_0 + E_1 + E_2 + \dots$

Wavefunction  $\psi = \psi_0 + \psi_1 + \psi_2 + \dots$

# Notation

- The vertices of the Wilsonian effective action are defined according to

$$S = \sum_{n=2}^{\infty} \left( \prod_{i=1}^n \int \frac{d^D p_i}{(2\pi)^D} \right) S^{(n)}(p_1, \dots, p_n) \varphi(p_1) \cdots \varphi(p_n) (2\pi)^D \delta^{(D)}(p_1 + \dots + p_n)$$

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- $\int_p \equiv \int \frac{d^D p_i}{(2\pi)^D}$
- $\delta(p) \equiv (2\pi)^D \delta^{(D)}(p)$

# Rescaling to Dimensionless Variables

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## Rescaling the Field

- The field has already been rescaled by  $\sqrt{Z}$
- Now scale out the engineering dimension:

$$\bullet \left( -\Lambda \partial_\Lambda + \frac{\gamma}{2} \varphi \cdot \frac{\delta}{\delta \varphi} \right) S \rightarrow \left( -\Lambda \partial_\Lambda - \frac{D+2-\gamma}{2} \varphi \cdot \frac{\delta}{\delta \varphi} \right) S$$

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- $\varphi(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip \cdot x} \varphi(p) \Rightarrow [\varphi(p)] = -(D + 2)/2$

- $\varphi \rightarrow \varphi \Lambda^{-(D+2)/2}$

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## Rescaling Momenta

- $p \rightarrow p\Lambda$
- This has three effects on the vertices

$$\int_{p_1, \dots, p_n} S^{(n)}(p_1, \dots, p_n) \varphi(p_1) \cdots \varphi(p_n) \hat{\delta}(p_1, \dots, p_n)$$

- $-\Lambda \partial_\Lambda S \rightarrow$   

$$\left( -\Lambda \partial_\Lambda + D_\varphi \cdot \frac{\delta}{\delta \varphi} - D + \int \frac{d^D p}{(2\pi)^D} \varphi(p) p^\mu \frac{\partial}{\partial p^\mu} \frac{\delta}{\delta \varphi(p)} \right) S$$

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# Rescaling to Dimensionless Variables

## Putting Everything Together

- Write  $-\Lambda \partial_\Lambda = \partial_t$ , with  $t = \ln \mu/\Lambda$
- Note that

$$\begin{aligned}\dot{\Delta} &= -\Lambda \partial_\Lambda C_{UV}(p^2/\Lambda^2)/p^2 = 2C'_{UV}(p^2/\Lambda^2)/\Lambda^2 \\ &\rightarrow 2C'_{UV}(p^2)/\Lambda^2\end{aligned}$$

- $(\partial_t + d_\varphi \Delta_\varphi + \Delta_\partial - D)S = \frac{\delta S}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{\delta}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi}$



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# Rescaling to Dimensionless Variables

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# Motivation

Why do we need this?

... ..

... ..

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# Motivation

- The ERG equation (ERGE)
  - Is an integro-differential equation
  - Contains variational derivatives
  - Is non-linear
  - Applied to problems where there is no small parameter
- Exact solutions to the ERGE are known only in special cases
- For problems with no small parameter, a reliable truncation scheme is required
- For finding non-trivial fixed points, truncating the infinite tower of equations arising from expanding in powers of the field is known to be unreliable  
T. R. Morris, Phys. Lett. B 334 (1994) 355.
- An alternative is to expand in derivatives of the field:

$$S_\Lambda[\varphi] \sim \int d^D x \left[ V_\Lambda(\varphi) + \frac{1}{2} (\partial_\mu \varphi)^2 K_\Lambda(\varphi) + \mathcal{O}(\partial^4) \right]$$

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## Justification for the Derivative Expansion

- It hopefully captures the essentials of the long distance physics
- (But fixed points are scale independent?!)
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  - an infinite number of interactions are included
- It works in practise!
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# Outline of this Lecture

- 1 Preamble
- 2 Motivation
- 3 The Local Potential Approximation**
  - Setting up the Approximation Scheme
  - Fixed Points and their Properties
- 4 Beyond Leading Order
  - Set-up
  - Reparametrization Invariance
- 5 Conclusion



# The Approximation Scheme

1.  $\Psi(\mathbf{r}) = \sum_{\mathbf{k}} \tilde{\Psi}_{\mathbf{k}}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}$  (Fourier transform)

$$\tilde{\Psi}_{\mathbf{k}}(\mathbf{r}) = \int d\mathbf{r}' \Psi(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

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- At lowest order, the derivative expansion reduces to The Local Potential Approximation (LPA):

$$S_\Lambda[\varphi] \sim \int d^D x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + V_\Lambda(\varphi) \right]$$

- The local potential depends on the field but not its derivatives
- The kinetic term does not renormalize, so the anomalous dimension is zero
- Writing

$$S = \sum_{n=2}^{\infty} \int_{p_1, \dots, p_n} S^{(n)}(p_1, \dots, p_n) \varphi(p_1) \cdots \varphi(p_n) \delta(p_1 + \dots + p_n)$$

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- It was then rediscovered several times!
  
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- There are several different versions of the LPA, corresponding to using different flow equations as a starting point

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# The Hasenfratz Projection Method

1. Motivation

2. The Local Potential Approximation

3. The Hasenfratz Projection Method

4. Numerical Results

# The Hasenfratz Projection Method

- Introduce a projector,  $P(x)$ :

$$P(x)G[\varphi] = e^{x\partial/\partial\varphi(0)} G[\varphi] \Big|_{\varphi=0}$$

- $P(x)G_1[\varphi] \cdots G_m[\varphi] = P(x)G_1[\varphi] \cdots P(x)G_m[\varphi]$

- $P(x)S[\varphi] = \sum_{n=2}^{\infty} S^{(n)}(0, \dots, 0) x^n \delta(0)$

How do we compute  $P(x)G[\varphi]$  in practice? We can compute  $G[\varphi]$  in the  $\varphi=0$  limit and then project it back to  $\varphi \neq 0$  using the projector  $P(x)$ .

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- The  $\hat{\delta}(0)$  can be regularized by working in a finite box (Hasenfratz & Hasenfratz)

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# Projection of the Flow Equation

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- $\int_p \frac{\delta S}{\delta \varphi(-p)} C'_{UV}(p) \frac{\delta S}{\delta \varphi(p)} \rightarrow C'_{UV}(0) V'^2 \equiv -K_0 V'^2$

## Quantum Term

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$c'$

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$$\frac{\delta S}{\delta \varphi} \quad C' \quad \frac{\delta S}{\delta \varphi}$$

$$S^{(n)}(p, -p) \rightarrow \frac{1}{2} p^2 + S^{(n)}(0)$$

$$S^{(n)}(p_1, \dots, p_n) = S^{(n)}(0, \dots, 0)$$



# Projection of the Flow Equation

## Classical Term

- $P(x) \frac{\delta S}{\delta \varphi} = \sum_{n=2}^{\infty} S^{(n)}(p, 0, \dots, 0) n x^{n-1} \hat{\delta}(p) = V'(x, t) \hat{\delta}(p)$
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- $\dot{V} = I_0 V''' - K_0 V'^2 + \frac{2-D}{2} x V' + DV$ 
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  - Projection of  $d_\varphi \Delta_\varphi$  (rem:  $\gamma = 0$ )
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## Set-up

- Set  $\dot{V} = 0$
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- $V_{\star}'(0) = 0$
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- The system is a fixed point manifold that cannot be set to zero
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- Choose a value of  $V_+(0)$
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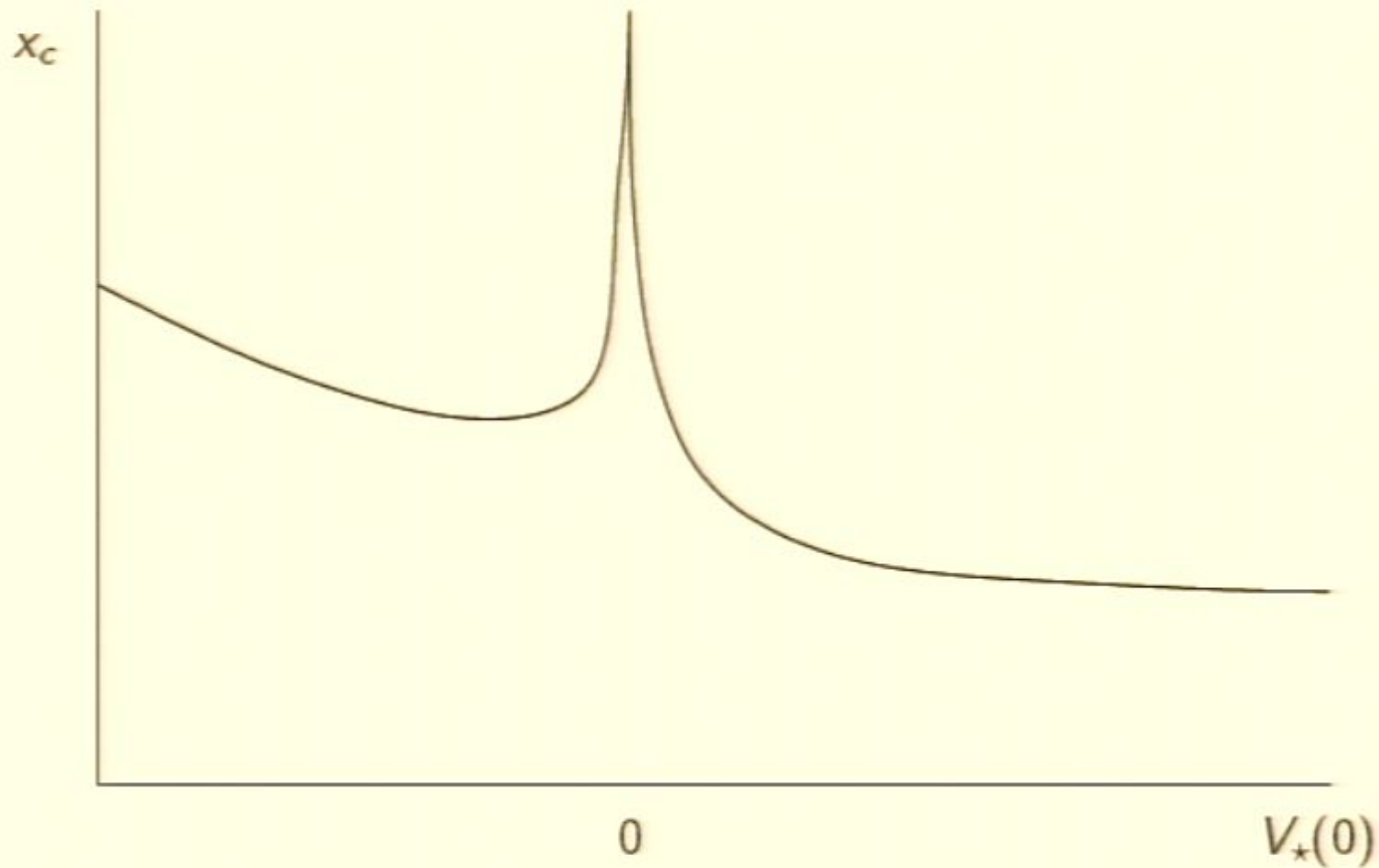
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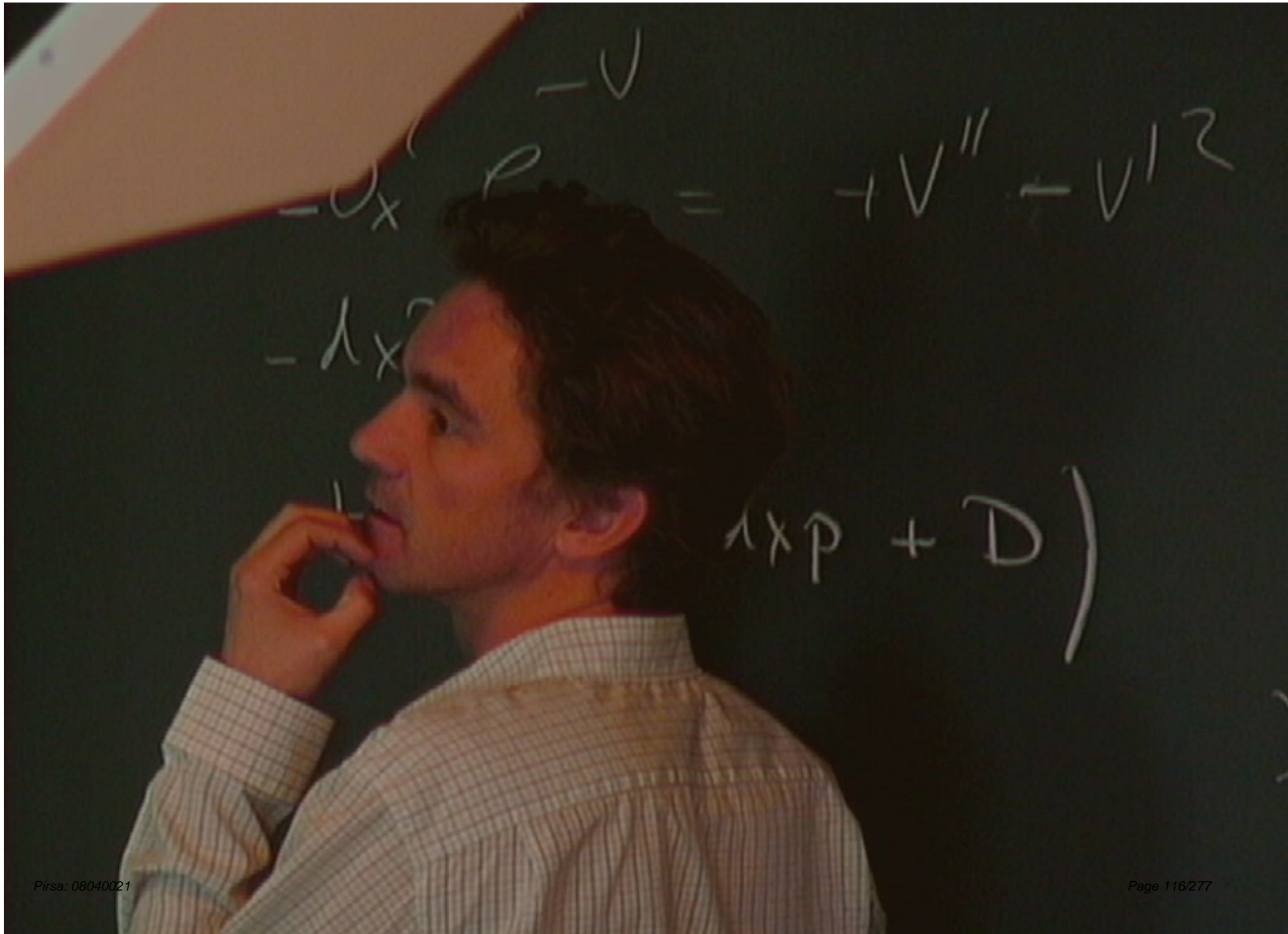
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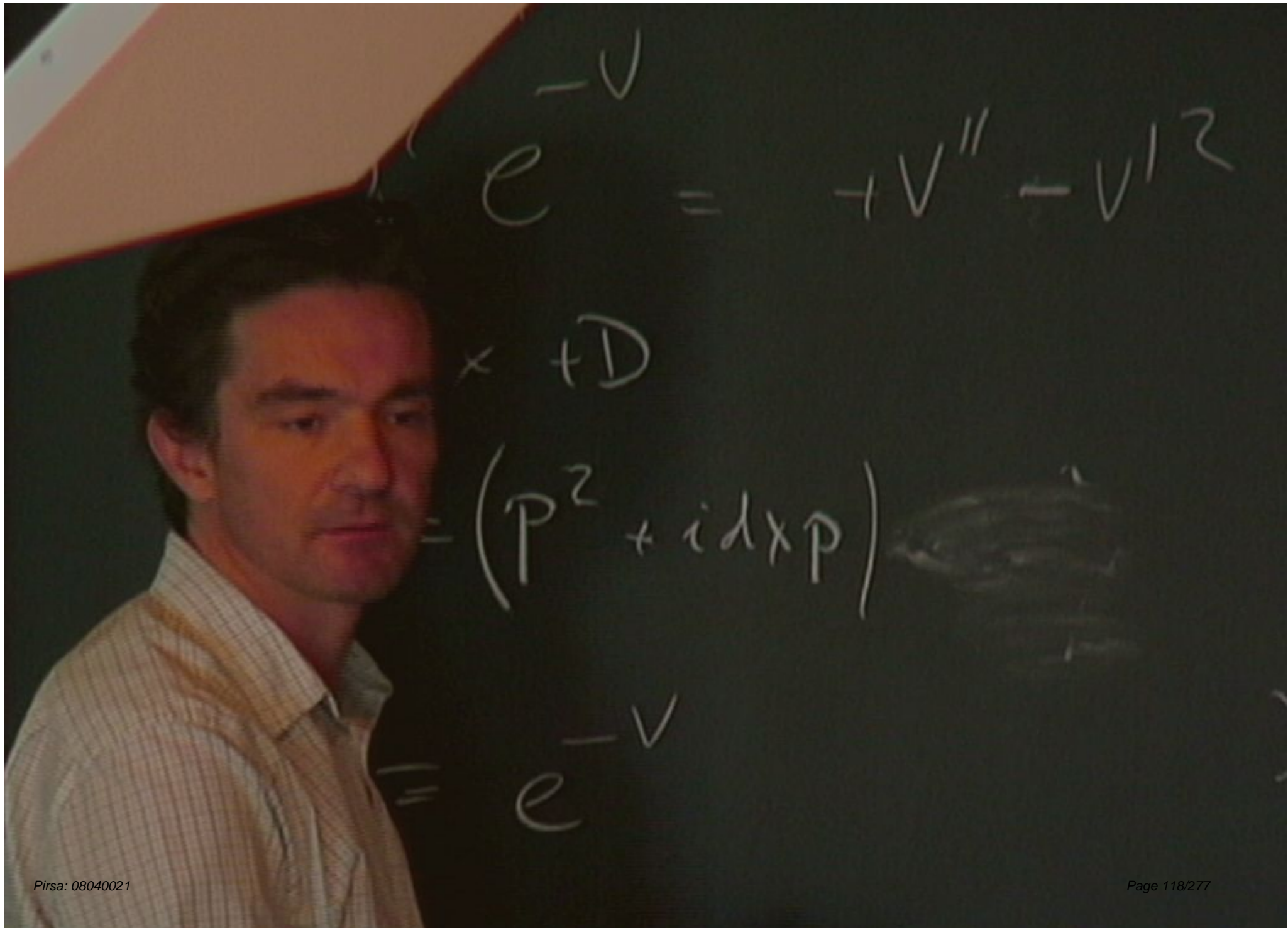


$$\partial_x (e^{-V}) = +V'' - V' \cdot 2$$

$$i \hbar \partial_x + D$$

$$H = \left( P^2 + i \hbar \partial_x P + D \right)$$

$$\equiv e^{-V}$$



$$e^{-V} = +V'' - V'^2$$

$$x + D$$

$$(P^2 + i\alpha x p)$$

$$= e^{-V}$$

$$-\partial_x (e^{-V}) = +V'' - V'{}^2$$

$$-\lambda x \partial_x + D$$

$$H = (P^2 + i\lambda x P)$$

$$P = e^{-V}$$

) 24

$$i\partial_t \psi = P$$
$$-\partial_x^2 e^{-V} = +V'' - V'^2$$

$$-\lambda x \partial_x + D$$

$$H = (P^2 + i\lambda x P)$$

$$\rho = e^{-V}$$

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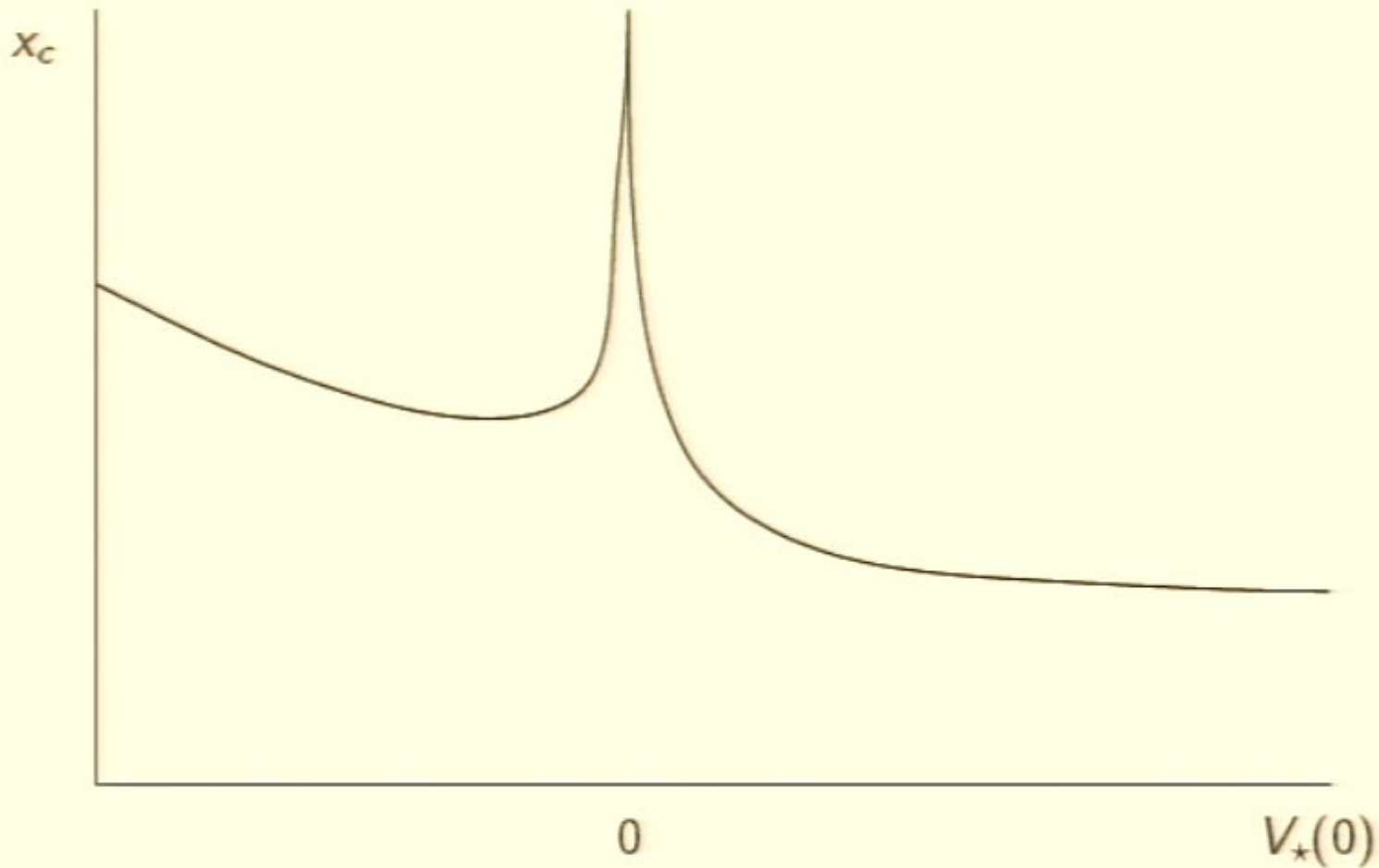
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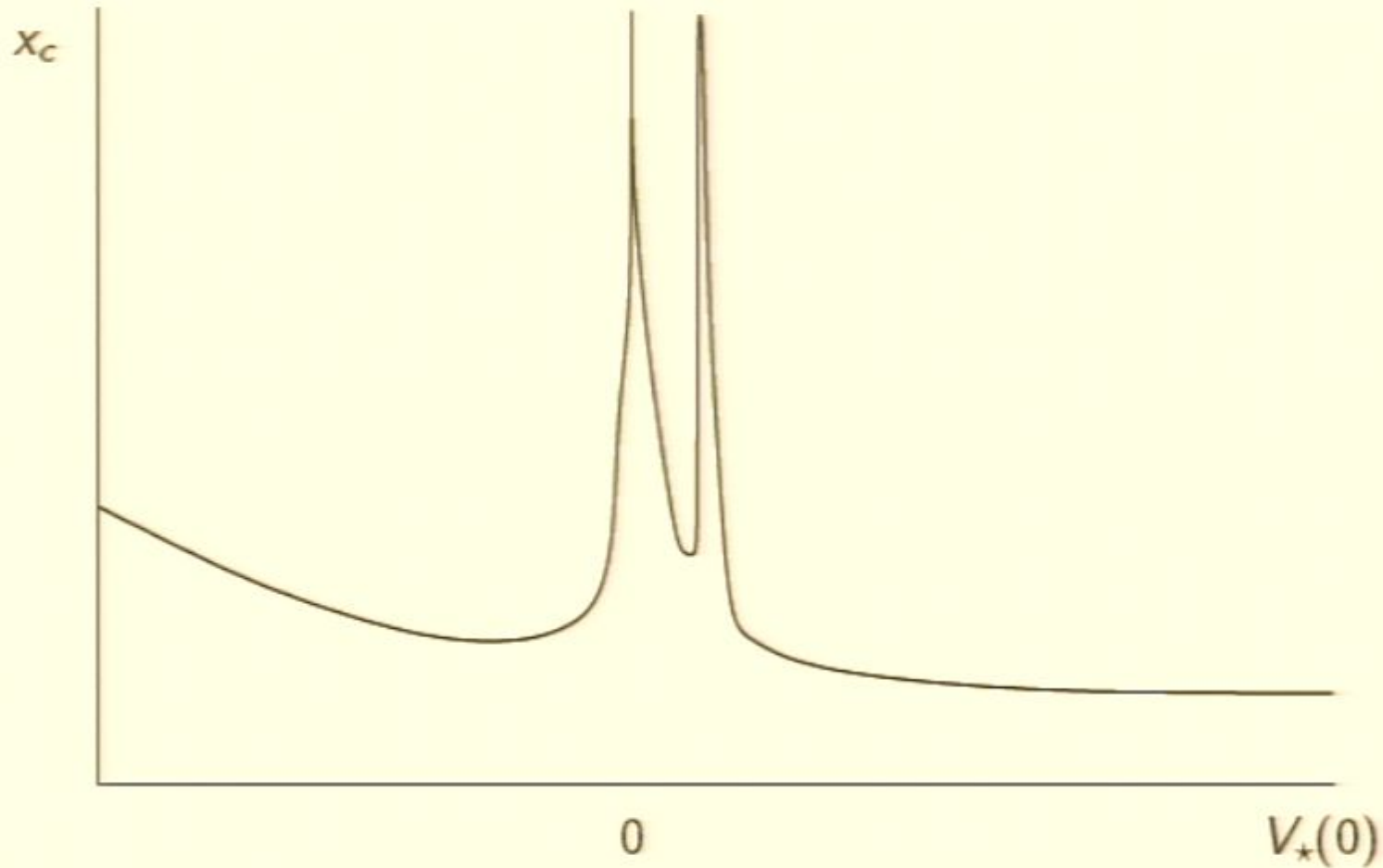
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# Typical Numerical Results

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# Alternative Boundary Conditions

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- For globally non-singular solutions, the large  $x$  behaviour is

$$V(x) \sim x^2$$

- Impose this as a pair of conditions for  $x \rightarrow \pm\infty$
- Two conditions for a second order differential equation  
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# What Does the LPA do for Us?

1. **What is the LPA?** The LPA is a systematic expansion of the effective action in powers of  $\hbar$ . It is the leading order approximation to the full theory.

2. **Why is it important?** The LPA provides a simple and powerful tool for studying the non-perturbative dynamics of quantum field theories. It is particularly useful for studying the formation of bound states and the emergence of new phases of matter.

3. **How is it used?** The LPA is used to calculate the effective action of a theory, which is then used to study the dynamics of the theory. It is often used in conjunction with other techniques, such as the renormalization group and the epsilon expansion.

4. **What are its limitations?** The LPA is only a leading order approximation, and it may not be accurate enough for some problems. It also does not capture all of the non-perturbative effects of a theory.

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3. **How is it used?** The LPA is used to calculate the effective action by expanding the path integral in powers of  $\hbar$ . This leads to a series of equations that can be solved order by order.

4. **What are the limitations?** The LPA is only valid for small values of  $\hbar$ . It is not applicable to strongly coupled systems or to systems with a large number of degrees of freedom.

5. **What are the applications?** The LPA has been used to study a wide range of phenomena, including the renormalization group, the effective potential, and the non-perturbative aspects of quantum field theory.

## What Does the LPA do for Us?

- '[It allows for] an exhaustive search for continuum limits in the entire infinite dimensional space of all possible potentials  $V(x)$ '  
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# The Scaling Operators



# The Scaling Operators

## Linearizing About a Fixed Point

- LPA:  $\dot{V} = V'' - V'^2 - dxV' + DV$
- Away from fixed point:  $V(x, t) = V_*(x) + v(x, t)$
- Treating  $v$  as small, linearize:

$$\dot{v} = v'' - 2V'_*v' - dxv' + Dv$$

- Separate variables:  $v(x, t) = \alpha e^{\lambda t} u(x)$
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- Note: at any finite distance away from the FP, the linear approximation will break down for large  $x$

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- $u'' - 2V'_+ u' - dxu' + (D - \lambda)u = 0$
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- But what about the other?
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- $u'' - 2V'_*u' - dxu' + (D - \lambda)u = 0$
- Note: at any **finite** distance away from the FP, the linear approximation will break down for large  $x$

# The Scaling Operators

## Another Paradox!

- $u'' - 2V'_+ u' - dxu' + (D - \lambda)u = 0$
- We have a two parameter continuum of solutions
- One corresponds to the normalization - OK
- But what about the other?
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- The resolution is subtle!

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- **The resolution is subtle!**

# Example: The Gaussian Fixed Point

Consider the  $\phi^4$  theory in  $d$  dimensions

$$\mathcal{L} = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right]$$

At  $d = 4$ , the theory is renormalizable

At  $d < 4$ , the theory is non-renormalizable

At  $d > 4$ , the theory is super-renormalizable

At  $d = 2$ , the theory is

exactly marginal (Gaussian fixed point)

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- After linearization and separation of variables

$$V(x, t) = V_*(x) + \alpha e^{\lambda t} u(x)$$

- $u'' - 2V_*'u' - dxu' + (D - \lambda)u = 0$
- At the Gaussian fixed point, take  $V_* = 0$
- $u'' - dxu' + (D - \lambda)u = 0$
- Shifting  $x \rightarrow x/\sqrt{d} \dots$
- ... yields Hermite's Equation

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# Polynomial Form of the Potential

- Hermite's equation  $u'' - 2xu' + 2nu = 0$  has polynomial solutions
- For  $n$  integer,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

- Comparing with  $u'' - 2xu' + 2\frac{2(D-\lambda)}{D-2}u = 0$  implies quantization of  $\lambda$
- For an even potential,  $n = 2k$

$$\lambda_k = D - k(D - 2)$$

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# So Where's The Problem?

What is the problem? What is the problem? What is the problem?

$$\frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi$$

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- We found polynomial solutions for which the eigenvalues are quantized
- But there are also nonpolynomial solutions!
  - $\psi_{\text{nonpolynomial}}$  is a bound state
  - $\psi_{\text{nonpolynomial}}$  is not normalizable
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Morris, Prog. Theor. Phys. **131** (1998) 395

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- Solutions for  $u(x)$  which are not bounded by polynomials are irrelevant to continuum physics

- Health warning:

Halpern & Huang "Fixed point structure of scalar fields,"  
Phys. Rev. Lett. 74 (1995) 3526 hep-th/9406199

incorrectly concluded that there are non-trivial RTs leaving  
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# Large Field Behaviour

- We want to determine which perturbations can be identified with renormalized couplings
- We need to analyse the evolution of trajectories a finite distance from the fixed point
- We are interested in the large field behaviour, so the linearization is not valid
- If we use the ERG equation for the effective average action, the large field behaviour comes out very simply

$$\bullet \quad \dot{V} + dxV' - DV = -\frac{1}{\sqrt{2 + V''}} \stackrel{x \rightarrow \pm\infty}{\sim} 0$$

$$\bullet \quad V(x, t) \sim e^{Dt} V(xe^{-dt}, 0)$$

• Writing  $V(x, t) = V_*(x) + v(x, t)$  (true even if  $v$  is big)

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- We need to analyse the evolution of trajectories a **finite distance from the fixed point**
- We are interested in the large field behaviour, so the linearization is not valid
- If we use the ERG equation for the effective average action, the large field behaviour comes out very simply
- $\dot{V} + dxV' - DV = -\frac{1}{\sqrt{2+V''}} \xrightarrow{x \rightarrow \pm\infty} 0$
- $V(x, t) \sim e^{Dt} V(xe^{-dt}, 0)$
- Writing  $V(x, t) = V_*(x) + v(x, t)$  (true even if  $v$  is big)
- $v(x, t, \alpha) \sim \alpha e^{Dt} u(e^{-dt} x)$ , with  $v(x, 0, \alpha) = \alpha u(x)$

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$$V = V_*(\lambda) + V(\lambda, t)$$

$$J_1(\varphi) = J(g_1(\lambda), \dots, \lambda(\lambda))$$

~~$\lambda_0$~~

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$$v(x, t, \alpha) \sim \alpha e^{Dt} u(e^{-dt} x)$$

## Polynomial Perturbations

- Try  $u(y) \sim y^p$
- $v(x, t, \alpha) \sim \alpha e^{t(D-pd)} x^p$
- This is self-similar evolution:  $v(x, t, \alpha) = v(x, g(t))$
- Choosing  $p = (D - \lambda)/d$ , we can identify the coupling consistently with the linearized theory:

## Nonpolynomial Perturbations

Self-similar evolution of  $v(x, t, \alpha)$  in the large field limit

The coupling  $\alpha$  is identified with the linearized theory

# Large Field Behaviour

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Self-similar evolution:  $v(x, t, \alpha) = v(x, g(t))$

The self-similar flow  $g(t)$  is determined by the linearized theory

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- The trajectory does not sink back into the fixed point for  $t \rightarrow -\infty$



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# Outline of this Lecture

- 1 Preamble
- 2 Motivation
- 3 The Local Potential Approximation
  - Setting up the Approximation Scheme
  - Fixed Points and their Properties
- 4 Beyond Leading Order
  - Set-up
  - Reparametrization Invariance
- 5 Conclusion

# Basic Structure

To see a concrete example, we use

$$V(x) = \frac{1}{2} \mu \omega^2 x^2 + \frac{1}{4} \lambda x^4$$

The two minima are at  $x = \pm \sqrt{-2\lambda/\mu\omega^2}$  and  $\hbar$

The energy spectrum is approximately  $E_n \approx \hbar\omega(n + \frac{1}{2}) + \frac{\hbar^2\omega^2}{8\mu\lambda}$

The energy difference between the two minima is  $\Delta E \approx \hbar\omega$

Using the WKB approximation, we can estimate the tunneling probability

# Basic Structure

- At the first order beyond LPA, we take

$$S_\Lambda[\varphi] \sim \int d^D x \left[ V_\Lambda(\varphi) + \frac{1}{2} (\partial_\mu \varphi)^2 K_\Lambda(\varphi) + O(\partial^4) \right]$$

- The flow equation yields a pair of second order coupled equations for  $V$  and  $K$
- The kinetic term now renormalized, so we must include the anomalous dimension
- As before, the form of the equations depends on the choice of ERG
- Unlike the LPA, it is not possible to completely remove cutoff dependence.

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# Boundary Conditions

The ground state wave function  $\psi_0(x)$  is real

$$\psi_0(x) = \sqrt{\rho(x)} e^{i\theta(x)}$$

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Boundary conditions are

1)  $\psi_0(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$   
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# Boundary Conditions

- The coupled equations have the generic form

$$\begin{aligned}\dot{V}(x, t) + d_\varphi x V' - DV &= \dots \\ \dot{K}(x, t) + \gamma K &= \dots\end{aligned}$$

- $d_\varphi = (D - 2 + \gamma)/2$
- For the Legendre transformed version, with the cutoff preferred by Morris, the right-hand sides are sub-leading for large field
- At a fixed point, for large  $x$ ,

$$V \sim x^{D/d_\varphi}, \quad K \sim x^{-\gamma/d_\varphi}$$

- Demanding this behaviour for  $x \rightarrow \pm\infty$  provides four conditions for the pair of second order equations
- But how does  $\gamma_*$  get determined?

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# Reparametrization Invariance

- Physics does not depend on the normalization of  $\varphi$
- In the partition function, we can send  $\varphi \rightarrow \Omega\varphi$  whilst leaving  $J \cdot \varphi$  alone
- More generally, physics does not change under the shift

$$\varphi \rightarrow \varphi + \text{local function of } \varphi$$

- Analogously, the ERGE has a reparametrization invariance
- Fixed point solutions are not unique
- Each fixed point belongs to a line of equivalent fixed points
- Each member of the family has the same spectrum of scaling operators and eigenvalues
- The unphysical redundant operators arise because of this symmetry

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# Determination of the Anomalous Dimension

From the renormalization group equation for the coupling constant  $g$  and the anomalous dimension  $\gamma$  of the operator  $\mathcal{O}$  we obtain

$$\frac{d}{d\ln\mu} g = \beta(g) = -\epsilon g + \beta_1 g^2 + \beta_2 g^3 + \dots$$

For a fixed point  $g^*$  we have  $\beta(g^*) = 0$

# Determination of the Anomalous Dimension

- Four boundary conditions + Reparametrization Invariance quantizes  $\gamma_*$

- Equivalently, we can choose an extra boundary condition:

$\lim_{\mu \rightarrow 0} \gamma_* = \gamma_{*0}$  (fixed)

or  $\gamma_{*0} = 0$

- F. J. Wegner, J. Phys C7 1974, 2098:

The Dimensional-regularization method allows the determination of the anomalous dimension  $\gamma_*$  of the coupling constant  $g$  in the  $\epsilon$ -expansion.

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  - e.g.  $K_*(\rho) = 1$
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The dimensionless renormalized coupling  $\tilde{g}$  is a function of the dimensionless renormalized mass  $\tilde{m}$  and the dimensionless renormalized length  $\tilde{l}$ :

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The Dimensional Reduction Hypothesis (DRH) states that the critical exponents of a system in  $d$  dimensions are equal to those of the corresponding system in  $d^*$  dimensions.

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# Example of Reparametrization Invariance

Example of reparametrization invariance

Consider the Lagrangian  $\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2} m \dot{\phi}^2 - V(\phi)$  with  $V(\phi) = \frac{1}{2} m \omega^2 \phi^2$ . The action is  $S = \int dt \mathcal{L}(\phi, \dot{\phi})$ .

The Lagrangian is invariant under the reparametrization  $t \rightarrow \tau$  with  $\phi(t) \rightarrow \tilde{\phi}(\tau)$ .

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## Example of Reparametrization Invariance

- Take the rescaled ERGE

$$(\partial_t + d_\varphi \Delta_\varphi + \Delta_\partial - D) S = \frac{\delta S}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{\delta}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

- Take the simplest seed action  $\hat{S} = 1/2 \varphi \cdot \Delta^{-1} \cdot \varphi$
- Look for critical Gaussian fixed point solutions

- $z' = z^2 C'_{UV}$

- $S = \frac{1}{2} \varphi \cdot \frac{ap^2}{C_{UV}(a - C_{UV})} \cdot \varphi, \quad a > 1$

- There is a line of equivalent fixed points, parametrized by  $a$

## Example of Reparametrization Invariance

- Take the rescaled ERGE

$$(\partial_t + d_\varphi \Delta_\varphi + \Delta_\partial - D) S = \frac{\delta S}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{\delta}{\delta \varphi} \cdot C'_{UV} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

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# Reparametrization Invariance versus Convergence



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- Consider the general Gaussian solution

$$S = \frac{1}{2} \varphi \cdot \frac{ap^2}{C_{UV}(a - C_{UV})} \cdot \varphi$$

- A derivative expansion will break reparametrization invariance for most cutoff functions
- There are two exceptions

1. The cutoff function is a constant and the derivative expansion is exact

2. The cutoff function is linear in the momentum squared and the derivative expansion is exact

- Unfortunately, in both cases, the momentum scale / derivative expansion is not convergent

T. R. Morris and J. E. Tigue, "Convergence of derivative expansions in scalar field theory," *Int. J. Mod. Phys. A* **16** (2001) 2095, hep-th/0102027.

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2. The cutoff function is a constant

The first exception is the one that is followed in the literature

The second one is not followed

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- The derivative expansion is intrinsically nonperturbative
- It permits searches over an infinite dimensional space for fixed point solutions
- The identification of non-trivial fixed points is robust
- The computation of critical exponents has decent to excellent accuracy

T. R. Morris, "The Renormalization group and two-dimensional multicritical effective scalar field theory," *Phys. Lett. B* **345** (1995) 139 hep-th/9410141.

## The Cons

• The renormalization group is a powerful tool for understanding the behavior of physical systems at different scales.



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 2. The cutoff function is a function of the dimensionless operator  $\frac{p^2}{a}$   
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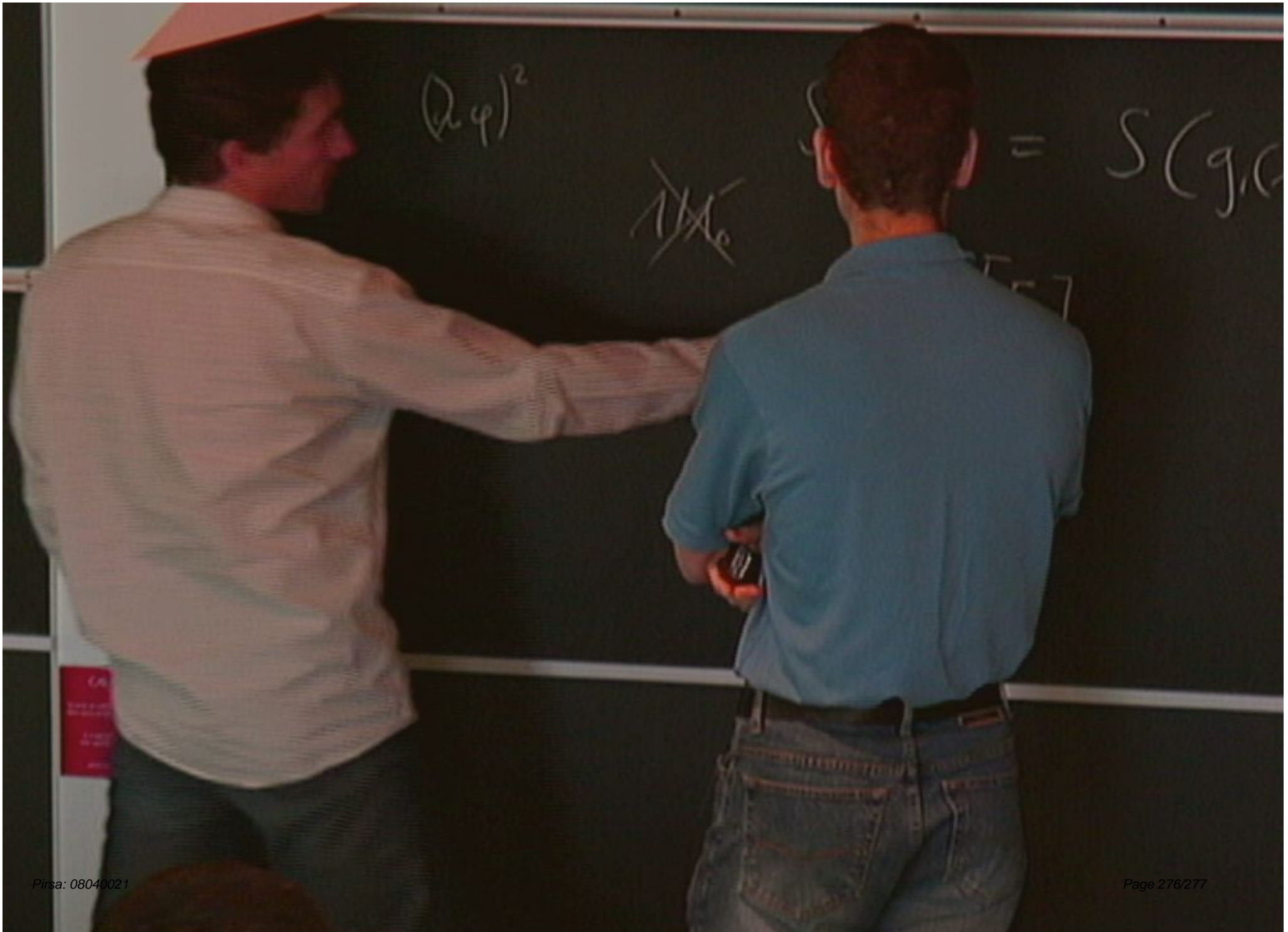
2. The cutoff function is a function of  $\partial^2$  and  $\partial^4$  only (e.g.  $\Lambda^2 - \partial^2 + \partial^4$ )

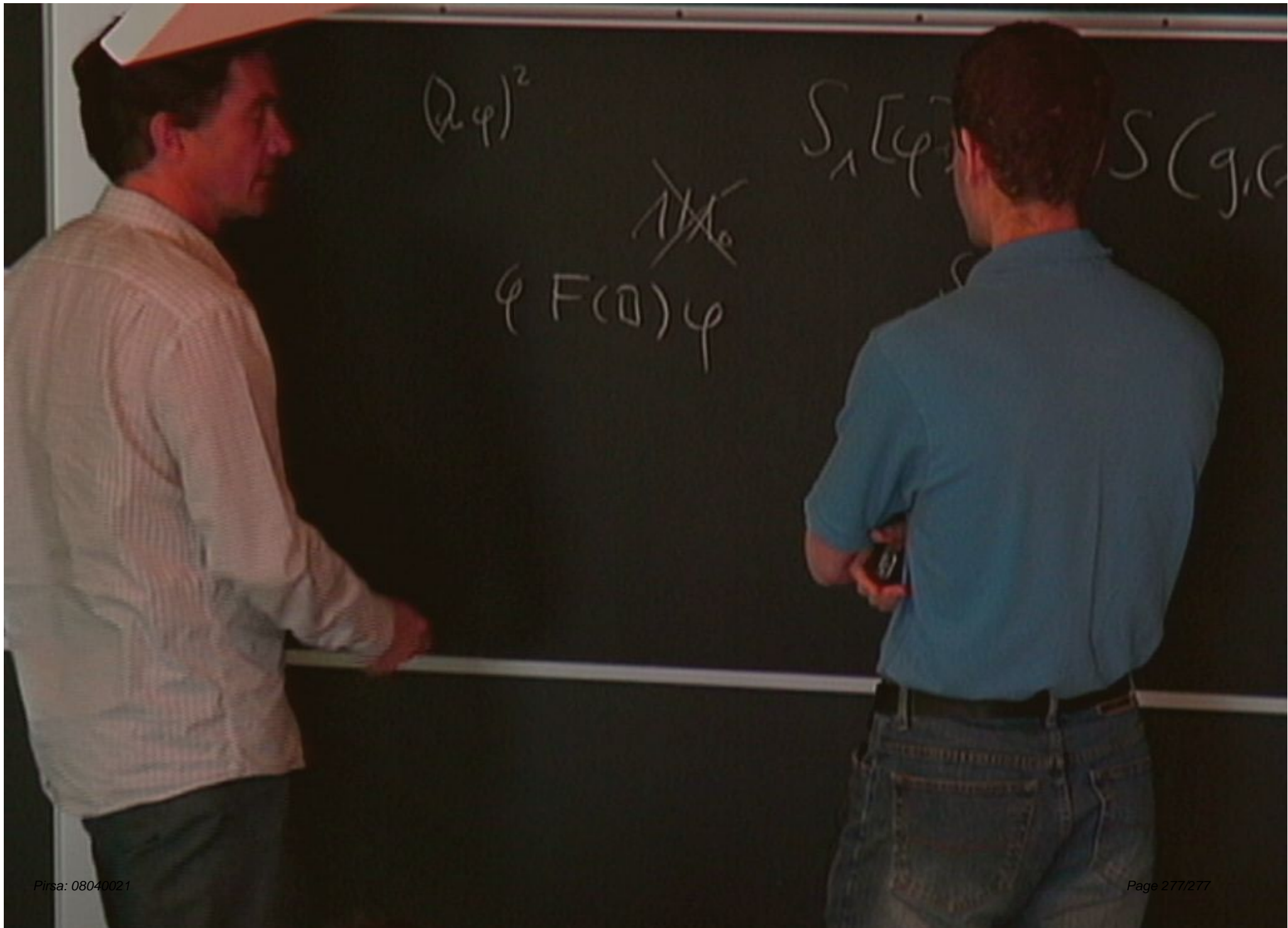
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~~11/10~~

$$S_1[\varphi] = S(g_1(x), \dots, \alpha(x))$$
$$S[\text{Eins}]$$





$$(\rho\varphi)^2$$

$$\cancel{1/\lambda_0}$$

$$\varphi F(\square)\varphi$$

$$S_1[\varphi]$$

$$S(g, \square)$$