Title: Exact results for supersymmetric Wilson loops in four-dimensional N=2 and N=4 super Yang-Mills theory

Date: Apr 29, 2008 11:00 AM

URL: http://pirsa.org/08040015

Abstract: TBA

Exact results for supersymmetric Wilson loops in four-dimensional N=2 and N=4 super Yang-Mills theory

Vasily Pestun

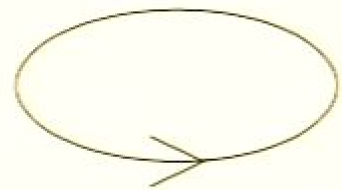
29 April, 2008

Erickson-Semenoff-Zarembo/Drukker-Gross conjecture

Supersymmetric circular Wilson loop in the four-dimensional $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ or S^4

Erickson-Semenoff-Zarembo/Drukker-Gross conjecture

Supersymmetric circular Wilson loop in the four-dimensional $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ or S^4



- J. K. Erickson, G. W. Semenoff and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory", hep-th/0003055
- N. Drukker and D. J. Gross, "An exact prediction of N = 4 SUSYM theory for string theory", hep-th/0010274

The conjecture is

$$\left\langle \mathrm{tr}_R \, \mathrm{P}\exp \oint_C A_\mu dx^\mu + i\Phi ds \right\rangle_{SYM} = \left\langle \mathrm{tr}_R \, e^{2\pi i r a} \right\rangle_{MM}$$

The conjecture is

$$\boxed{\langle \mathrm{tr}_R \, \mathrm{Pexp} \oint_C A_\mu dx^\mu + i\Phi ds \rangle_{SYM} = \langle \mathrm{tr}_R \, e^{2\pi i r a} \rangle_{MM}}$$

where $\langle \rangle_{SYM}$ is $\mathcal{N} = 4$ SYM in $d = 4$

$$Z_{SYM} = \int [DA \, D\Phi \, D\Psi] e^{-\frac{1}{2g_{YM}^2}(\frac{1}{2}F^2 + (D\Phi)^2 + \dots)}$$

The conjecture is

$$\boxed{\langle \mathrm{tr}_R \, \mathrm{Pexp} \oint_C A_\mu dx^\mu + i\Phi ds \rangle_{SYM} = \langle \mathrm{tr}_R \, e^{2\pi i r a} \rangle_{MM}}$$

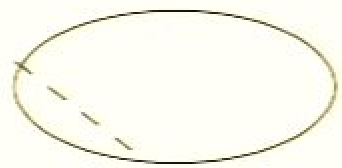
where $\langle\rangle_{SYM}$ is $\mathcal{N} = 4$ SYM in $d = 4$

$$Z_{SYM} = \int [DA \, D\Phi \, D\Psi] e^{-\frac{1}{2g_{YM}^2}(\frac{1}{2}F^2 + (D\Phi)^2 + \dots)}$$

and $\langle\rangle_{MM}$ is Gaussian Matrix Model in $d = 0$

$$Z_{MM} = \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)}$$

Perturbative argument

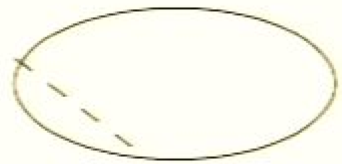


In Feynman gauge

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x-x')^2}$$

$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x-x')^2}$$

Perturbative argument



In Feynman gauge

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x - x')^2}$$

$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x - x')^2}$$

Hence, for $x_1 = r \cos\phi, x_2 = r \sin\phi$ we get

$$\langle A_\mu(\phi)\dot{x}^\mu \, A_\nu(\phi')\dot{x}^\nu + i\Phi(\phi) i\Phi(\phi') \rangle = \frac{1}{4\pi^2 r^2} \frac{cos(\phi - \phi') - 1}{4 sin^2 \frac{\phi - \phi'}{2}} = -\frac{1}{8\pi^2 r}$$

The conjecture is

$$\boxed{\langle \mathrm{tr}_R \, \mathrm{Pexp} \oint_C A_\mu dx^\mu + i\Phi ds \rangle_{SYM} = \langle \mathrm{tr}_R \, e^{2\pi i r a} \rangle_{MM}}$$

where $\langle\rangle_{SYM}$ is $\mathcal{N} = 4$ SYM in $d = 4$

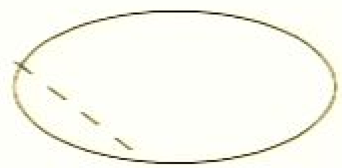
$$Z_{SYM} = \int [DA \, D\Phi \, D\Psi] e^{-\frac{1}{2g_{YM}^2}(\frac{1}{2}F^2 + (D\Phi)^2 + \dots)}$$

and $\langle\rangle_{MM}$ is Gaussian Matrix Model in $d = 0$

$$Z_{MM} = \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)}$$

.

Perturbative argument

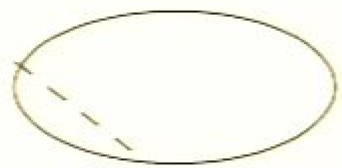


In Feynman gauge

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x - x')^2}$$

$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x - x')^2}$$

Perturbative argument



In Feynman gauge

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{4\pi^2} \frac{g_{\mu\nu}}{(x-x')^2}$$

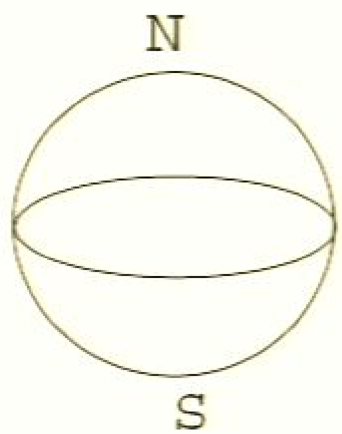
$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x-x')^2}$$

Hence, for $x_1 = r\cos\phi, x_2 = r\sin\phi$ we get

$$\langle A_\mu(\phi)\dot{x}^\mu \, A_\nu(\phi')\dot{x}^\nu + i\Phi(\phi) i\Phi(\phi') \rangle = \frac{1}{4\pi^2 r^2} \frac{cos(\phi-\phi')-1}{4sin^2\frac{\phi-\phi'}{2}} = -\frac{1}{8\pi^2 r}$$

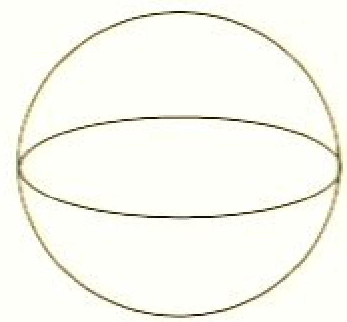
The $\mathcal{N} = 4$ SYM on S^4

Consider a four-sphere S^4 with a Wilson loop placed on the equator



The $\mathcal{N} = 4$ SYM on S^4

Consider a four-sphere S^4 with a Wilson loop placed on the equator



In $d = 10$ $\mathcal{N} = 1$ SYM notations $A_M = \{A_\mu, \Phi_A\}$. The action is

$$S = \frac{1}{2g_{YM}^2} \int \sqrt{g} d^4x \left(\frac{1}{2} F_{MN}^2 - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A \right)$$

Exact results for supersymmetric Wilson loops in four-dimensional N=2 and N=4 super Yang-Mills theory

Vasily Pestun

29 April, 2008

The conjecture is

$$\boxed{\langle \mathrm{tr}_R \, \mathrm{Pexp} \oint_C A_\mu dx^\mu + i\Phi ds \rangle_{SYM} = \langle \mathrm{tr}_R \, e^{2\pi i r a} \rangle_{MM}}$$

where $\langle\rangle_{SYM}$ is $\mathcal{N} = 4$ SYM in $d = 4$

$$Z_{SYM} = \int [DA\, D\Phi\, D\Psi] e^{-\frac{1}{2g_{YM}^2}(\frac{1}{2}F^2 + (D\Phi)^2 + \dots)}$$

and $\langle\rangle_{MM}$ is Gaussian Matrix Model in $d = 0$

$$Z_{MM} = \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)}$$

.

The partition function of the theory is well defined in IR: there are no zero modes for the physical, non-twisted theory on a four-sphere.

The partition function of the theory is well defined in IR: there are no zero modes for the physical, non-twisted theory on a four-sphere. Thanks to the $\mathcal{N} = 4$ supersymmetry the partition function is also finite in UV.

Let us try to compute it!

$\mathcal{N} = 4$ superconformal symmetry

The action is invariant under the fermionic supersymmetry

$$\delta A_M = \varepsilon \Gamma_M \psi$$
$$\delta \psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon$$

$\mathcal{N} = 4$ superconformal symmetry

The action is invariant under the fermionic supersymmetry

$$\delta A_M = \varepsilon \Gamma_M \psi$$
$$\delta \psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon$$

where ε is a conformal Killing spinor on S^4

$$\nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon}$$
$$\nabla_\mu \tilde{\varepsilon} = -\frac{1}{4r^2} \Gamma_\mu \varepsilon$$

The idea of localization

In some situations the integral is exactly equal to its semiclassical approximation.

The idea of localization

In some situations the integral is exactly equal to its semiclassical approximation. For example, Duistermaat-Heckman formula says

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

The idea of localization

In some situations the integral is exactly equal to its semiclassical approximation. For example, Duistermaat-Heckman formula says

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

Here (M, ω) is a symplectic manifold, $H : M \to \mathfrak{g}^*$ is a moment map for a Hamiltonian action of some torus G on M, which means $i_\phi \omega = dH(\phi)$ for $\phi \in \mathfrak{g}$. There exist non-abelian generalizations of this formula.

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

$\alpha \in \Omega(M) \otimes S(\mathfrak{g}^*)$ is a differential form on M valued in a functions on $\mathfrak{g}$

$F \xrightarrow{i} M$ is the fixed point set of G acting on M

$e(N_F)$ is the Euler class of the normal bundle of F in M.

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

$\alpha \in \Omega(M) \otimes S(\mathfrak{g}^*)$ is a differential form on M valued in a functions on $\mathfrak{g}$
$F \overset{i}{\hookrightarrow} M$ is the fixed point set of G acting on M
$e(N_F)$ is the Euler class of the normal bundle of F in M. The differential (supersymmetry operator) is

$$Q = d - \phi^a i_a$$

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

$\alpha \in \Omega(M) \otimes S(\mathfrak{g}^*)$ is a differential form on M valued in a functions on $\mathfrak{g}$

$F \stackrel{i}{\hookrightarrow} M$ is the fixed point set of G acting on M

$e(N_F)$ is the Euler class of the normal bundle of F in M. The differential (supersymmetry operator) is

$$Q = d - \phi^a i_a$$

The Q squares to a symmetry transformation

$$Q^2 = -\phi^a \mathcal{L}_{v^a},$$

where $\mathcal{L}_v$ is Lie derivative.

Atiyah-Bott-Berline-Vergne localization formula

More generally, if $Q\alpha = 0$ on a G-manifold M then

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)}$$

$\alpha \in \Omega(M) \otimes S(\mathfrak{g}^*)$ is a differential form on M valued in a functions on $\mathfrak{g}$
$F \xhookrightarrow{i} M$ is the fixed point set of G acting on M
$e(N_F)$ is the Euler class of the normal bundle of F in M. The differential (supersymmetry operator) is

$$Q = d - \phi^a i_a$$

The Q squares to a symmetry transformation

$$Q^2 = -\phi^a \mathcal{L}_{V^a},$$

where $\mathcal{L}_v$ is Lie derivative. Hence $Q^2 = 0$ on G-invariant objects

Localization argument in physical terms [Witten]

Let $QS = 0$.

Localization argument in physical terms [Witten]

Let $QS = 0$. Take V such that $Q^2 V = 0$.

Localization argument in physical terms [Witten]

Let $QS = 0$. Take V such that $Q^2 V = 0$.

$$\frac{d}{dt}\int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, V e^{S+tQV}\} = 0$$

The addition of Q-exact term to the action does not change the result of the integral.

Localization argument in physical terms [Witten]

Let $QS = 0$. Take V such that $Q^2 V = 0$.

$$\frac{d}{dt}\int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, Ve^{S+tQV}\} = 0$$

The addition of Q-exact term to the action does not change the result of the integral. As $t \to \infty$, the one-loop approximation at the critical locus of QV becomes exact!

Localization argument in physical terms [Witten]

Let $QS = 0$. Take V such that $Q^2 V = 0$.

$$\frac{d}{dt}\int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, V e^{S+tQV}\} = 0$$

The addition of Q-exact term to the action does not change the result of the integral. As $t \to \infty$, the one-loop approximation at the critical locus of QV becomes exact! Then for a sufficiently nice V the integral is computed by evaluating S at the critical points of QV and the corresponding one-loop determinants.

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = const$ on S^4
- There are no conformal transformations in Q_ε^2, but only isometry transformations

$$Q_\varepsilon^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon \Gamma^M \varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = \mathit{const}$ on S^4

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = const \quad \text{on } S^4$
- There are no conformal transformations in Q_ε^2, but only isometry transformations

$$Q_\varepsilon^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon \Gamma^M \varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = const \quad$ on S^4
- There are no conformal transformations in Q_ε^2, but only isometry transformations

$$Q_\varepsilon^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon\Gamma^M\varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.
R is an $SU(2)_L^R \subset SO(4)^R \subset SO(6)^R$ R-symmetry transformation; it acts nontrivially on four scalars $\Phi_5, \dots \Phi_8$ and fermions of $\mathcal{N} = 2$ vector multiplet

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = const \quad \text{on } S^4$
- There are no conformal transformations in Q_ε^2, but only isometry transformations

$$Q_\varepsilon^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon \Gamma^M \varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.
R is an $SU(2)_L^R \subset SO(4)^R \subset SO(6)^R$ R-symmetry transformation; it acts nontrivially on four scalars $\Phi_5, \ldots \Phi_8$ and fermions of $\mathcal{N} = 2$ vector multiplet

Choice of Killing spinor ε

Pick up Killing spinor ε such that Q_ε on S^4 resembles normal supersymmetry (not superconformal symmetry) on $\mathbb{R}^4$

- $(\varepsilon, \varepsilon) = const \quad \text{on } S^4$
- There are no conformal transformations in Q_ε^2, but only isometry transformations

$$Q_\varepsilon^2 = \mathcal{L}_v + R$$

$\mathcal{L}_v$ is a Lie derivative along the vector field $v^M = \varepsilon \Gamma^M \varepsilon$ generating rotations of S^4 and gauge transformation $[V^M A_M, \cdot]$.
R is an $SU(2)_L^R \subset SO(4)^R \subset SO(6)^R$ R-symmetry transformation; it acts nontrivially on four scalars $\Phi_5, \dots \Phi_8$ and fermions of $\mathcal{N} = 2$ vector multiplet

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives

Symmetry: $SU(2)_L \times SU(2)_R \times SU(2)^R_L \times SU(2)^R_R \times \mathrm{SO}(2)^R$

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives
Symmetry: $SU(2)_L \times SU(2)_R \times SU(2)_L^R \times SU(2)_R^R \times \mathrm{SO}(2)^R$
Bosons: $A_1, \ldots A_4,\ \Phi_5, \ldots, \Phi_8,\ \Phi_9, \Phi_0$
Fermions: $\Psi = (\psi^L, \chi^R, \psi^R, \chi^L)^t$

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives

Symmetry: $SU(2)_L \times SU(2)_R \times SU(2)_L^R \times SU(2)_R^R \times \mathrm{SO}(2)^R$

Bosons: $A_1, \ldots A_4,\ \Phi_5, \ldots, \Phi_8,\ \Phi_9, \Phi_0$

Fermions: $\Psi = (\psi^L, \chi^R, \psi^R, \chi^L)^t$

ε	Ψ	$SU(2)_L$	$SU(2)_R$	$SU(2)_L^R$	$SU(2)_R^R$	$SO(2)^R$
$*$	ψ^L	1/2	0	1/2	0	$+$
0	χ^R	0	1/2	0	1/2	$+$
$*$	ψ^R	0	1/2	1/2	0	-
0	χ^L	1/2	0	0	1/2	-

- $(A_1 \ldots A_4, \Phi_9, \Phi_0,\ \psi^L, \psi^R)$ is $\mathcal{N} = 2$ vector multiplet

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives

Symmetry: $SU(2)_L \times SU(2)_R \times SU(2)_L^R \times SU(2)_R^R \times \mathrm{SO}(2)^R$

Bosons: $A_1, \ldots A_4, \ \Phi_5, \ldots, \Phi_8, \ \Phi_9, \Phi_0$

Fermions: $\Psi = (\psi^L, \chi^R, \psi^R, \chi^L)^t$

ε	Ψ	$SU(2)_L$	$SU(2)_R$	$SU(2)_L^R$	$SU(2)_R^R$	$SO(2)^R$
$*$	ψ^L	1/2	0	1/2	0	+
0	χ^R	0	1/2	0	1/2	+
$*$	ψ^R	0	1/2	1/2	0	-
0	χ^L	1/2	0	0	1/2	-

- $(A_1 \ldots A_4, \Phi_9, \Phi_0, \ \psi^L, \psi^R)$ is $\mathcal{N} = 2$ vector multiplet
- $(\Phi_5 \ldots \Phi_8, \ \chi^L, \chi^R)$ is $\mathcal{N} = 2$ hyper multiplet

Conformal Killing spinor

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + x^\mu \Gamma_\mu \hat{\varepsilon}_c)$$

where $\hat{\varepsilon}_s = (1, 0, \dots, 0)$ and $\hat{\varepsilon}_c = \frac{1}{2r}\Gamma_{12}\hat{\varepsilon}_s$
The North and the South poles are the fixed points of Q^2 acting S^4.
The transformation Q^2 is a combination of

- an anti-self-dual Lorentz $SU(2)_L$ rotation of 12-plane and 34-plane
- gauge transformation by $[i\Phi_0 + \Phi_9 \cos\theta]$
- an R-symmetry rotation in the $SU(2)^R_L$ group

Fields and charges under the global symmetry

Reduction of $SO(10)$ gives

Symmetry: $SU(2)_L \times SU(2)_R \times SU(2)_L^R \times SU(2)_R^R \times \mathrm{SO}(2)^R$

Bosons: $A_1, \ldots A_4, \; \Phi_5, \ldots, \Phi_8, \; \Phi_9, \Phi_0$

Fermions: $\Psi = (\psi^L, \chi^R, \psi^R, \chi^L)^t$

ε	Ψ	$SU(2)_L$	$SU(2)_R$	$SU(2)_L^R$	$SU(2)_R^R$	$SO(2)^R$
$*$	ψ^L	1/2	0	1/2	0	$+$
0	χ^R	0	1/2	0	1/2	$+$
$*$	ψ^R	0	1/2	1/2	0	-
0	χ^L	1/2	0	0	1/2	-

- $(A_1 \ldots A_4, \Phi_9, \Phi_0, \; \psi^L, \psi^R)$ is $\mathcal{N} = 2$ vector multiplet
- $(\Phi_5 \ldots \Phi_8, \; \chi^L, \chi^R)$ is $\mathcal{N} = 2$ hyper multiplet

Conformal Killing spinor

$$\varepsilon = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + x^\mu \Gamma_\mu \hat{\varepsilon}_c)$$

where $\hat{\varepsilon}_s = (1, 0, \dots, 0)$ and $\hat{\varepsilon}_c = \frac{1}{2r}\Gamma_{12}\hat{\varepsilon}_s$
The North and the South poles are the fixed points of Q^2 acting S^4.
The transformation Q^2 is a combination of

- an anti-self-dual Lorentz $SU(2)_L$ rotation of 12-plane and 34-plane
- gauge transformation by $[i\Phi_0 + \Phi_9 \cos\theta]$
- an R-symmetry rotation in the $SU(2)_L^R$ group

Conformal Killing spinor

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + x^\mu \Gamma_\mu \hat{\varepsilon}_c)$$

where $\hat{\varepsilon}_s = (1, 0, \ldots, 0)$ and $\hat{\varepsilon}_c = \frac{1}{2r}\Gamma_{12}\hat{\varepsilon}_s$

The North and the South poles are the fixed points of Q^2 acting S^4.

The transformation Q^2 is a combination of

- an anti-self-dual Lorentz $SU(2)_L$ rotation of 12-plane and 34-plane
- gauge transformation by $[i\Phi_0 + \Phi_9 \cos\theta]$
- an R-symmetry rotation in the $SU(2)_L^R$ group

The action and the Wilson loop operator are Q-invariant.

$$QS = 0, \quad QW(C) = 0$$

Off-shell closure of Q_ε for $\mathcal{N}=4$ SYM on S^4

Add 7 auxiliary scalar field K_i as in [Berkovits '93] for $d=10$ $\mathcal{N}=1$ SYM on R^{10}
The action $\int \sqrt{g} d^4x \mathcal{L}$ where

$$\mathcal{L} = \frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M \Psi + \frac{2}{r^2}\Phi_A\Phi^A + K_i K_i$$

is invariant under

$$\delta A_M = \varepsilon \Gamma_M \Psi$$
$$\delta \Psi = \frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon + \frac{1}{2}\Gamma_{\mu A}\Phi^A \nabla^\mu \varepsilon + iK_i\nu_i$$
$$\delta K_i = i\nu_i \Gamma^M D_M \Psi$$

where $\{\nu_i\}$ is a set of 7 Mayorana-Weyl fermions satisfying algebraic equations

$$\varepsilon\Gamma^M \nu_i = 0, \quad \nu_i \Gamma^M \nu_j = \delta_{ij}\varepsilon\Gamma^M\varepsilon$$

Q-exact deformation of the action

$$S \to S + tQV$$

Take $V = (\psi, \overline{Q\psi})$, so $QV|_{bos} = (Q\psi, \overline{Q\psi})$.

Q-exact deformation of the action

$$S \to S + tQV$$

Take $V = (\psi, \overline{Q\psi})$, so $QV|_{bos} = (Q\psi, \overline{Q\psi})$.
After integrating out K, at the limit $t \to \infty$, only the zero mode of Φ_0 has finite quadratic action!

Q-exact deformation of the action

$$S \to S + tQV$$

Take $V = (\psi, \overline{Q\psi})$, so $QV|_{bos} = (Q\psi, \overline{Q\psi})$.
After integrating out K, at the limit $t \to \infty$, only the zero mode of Φ_0 has finite quadratic action!
The path integral is localized to the locus $\Phi_0 = const$, all other fields vanish.

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2}\int(\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

Q-exact deformation of the action

$$S \to S + tQV$$

Take $V = (\psi, \overline{Q\psi})$, so $QV|_{bos} = (Q\psi, \overline{Q\psi})$.
After integrating out K, at the limit $t \to \infty$, only the zero mode of Φ_0 has finite quadratic action!
The path integral is localized to the locus $\Phi_0 = const$, all other fields vanish.

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2}\int\left(\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots)\right)$$

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2}\int(\dots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

Then $\int DK$ gives

$$S+tQV = \frac{1}{2g_{YM}^2}\int(\dots+\frac{2}{r^2}\Phi_0^2+\frac{1}{r^2}\frac{t}{1+t}\Phi_0^2) \stackrel{t\to\infty}{=} \frac{1}{2g_{YM}^2}\int(\dots+\frac{3}{r^2}\Phi_0^2$$

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2}\int(\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

Then $\int DK$ gives

$$S+tQV = \frac{1}{2g_{YM}^2}\int(\cdots+\frac{2}{r^2}\Phi_0^2+\frac{1}{r^2}\frac{t}{1+t}\Phi_0^2) \overset{t\to\infty}{=} \frac{1}{2g_{YM}^2}\int(\cdots+\frac{3}{r^2}\Phi_0^2)$$

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2} \int (\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

Then $\int DK$ gives

$$S + tQV = \frac{1}{2g_{YM}^2} \int (\cdots + \frac{2}{r^2}\Phi_0^2 + \frac{1}{r^2}\frac{t}{1+t}\Phi_0^2) \overset{t\to\infty}{=} \frac{1}{2g_{YM}^2} \int (\cdots + \frac{3}{r^2}\Phi_0^2)$$

To evaluate the Φ_0 constant mode $\Phi_0 = a$ on S^4, we need to multiply by the $Vol(S^4) = \frac{8}{3}\pi^2 r^4$

In this way we obtain the Matrix Model action

$$S_{MM}[\Phi_0 = a] = \frac{1}{2g_{YM}^2} \times \frac{8}{3}\pi^2 r^4 \times \frac{3}{r^2}a^2 = \frac{8\pi^2 r^2}{g_{YM}^2}\Phi_0^2$$

Coincides with Erickson-Semenoff-Zarembo/Drukker-Gross matrix model

The quadratic term

Be careful with integrating out K. The deformed action

$$S + tQV = \frac{1}{2g_{YM}^2}\int(\cdots + \frac{2}{r^2}\Phi_0^2 + K^2 + t((K + \frac{1}{r}\Phi_0)^2 + \dots))$$

Then $\int DK$ gives

$$S+tQV = \frac{1}{2g_{YM}^2}\int(\cdots+\frac{2}{r^2}\Phi_0^2+\frac{1}{r^2}\frac{t}{1+t}\Phi_0^2) \overset{t\to\infty}{=} \frac{1}{2g_{YM}^2}\int(\cdots+\frac{3}{r^2}\Phi_0^2)$$

To evaluate the Φ_0 constant mode $\Phi_0 = a$ on S^4, we need to multiply by the $Vol(S^4) = \frac{8}{3}\pi^2 r^4$
In this way we obtain the Matrix Model action

$$S_{MM}[\Phi_0 = a] = \frac{1}{2g_{YM}^2} \times \frac{8}{3}\pi^2 r^4 \times \frac{3}{r^2}a^2 = \frac{8\pi^2 r^2}{g_{YM}^2}\Phi_0^2$$

Coincides with Erickson-Semenoff-Zarembo/Drukker-Gross matrix model

The one-loop determinant

At $t \to \infty$ limit we need also to compute the determinant for the fluctuations of the fields with the action $S + tQV$ near the dominant configuration $\Phi_0 = const$. Similarly to Duistermaat-Heckman formula

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

the determinant $Z_{1-loop}(a)$ can be computed as a certain product of weights of Q^2 acting to the tangents space to all fields at the locus $\Phi_0 = a$.

The one-loop determinant

At $t \to \infty$ limit we need also to compute the determinant for the fluctuations of the fields with the action $S + tQV$ near the dominant configuration $\Phi_0 = const$. Similarly to Duistermaat-Heckman formula

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)}$$

the determinant $Z_{1-loop}(a)$ can be computed as a certain product of weights of Q^2 acting to the tangents space to all fields at the locus $\Phi_0 = a$.

This is a linear problem; it can be treated by the Atiyah-Singer theorem.

Changing notations to TFT like

The SUSY transformation Q_ϵ can be conveniently written in the form of cohomological field theory (even though the theory is non-twisted)

Changing notations to TFT like

The SUSY transformation Q_ε can be conveniently written in the form of cohomological field theory (even though the theory is non-twisted)

For $M = 1 \ldots 9$ we rewrite

$$\delta A_M = \varepsilon \Gamma_M \psi$$

as

$$\delta A_M = \Psi_M$$

where we define 9 fermionic fields $\Psi_M \equiv \varepsilon \Gamma_M \psi$.

Changing notations to TFT like

The SUSY transformation Q_ε can be conveniently written in the form of cohomological field theory (even though the theory is non-twisted)

For $M = 1 \dots 9$ we rewrite

$$\delta A_M = \varepsilon \Gamma_M \psi$$

as

$$\delta A_M = \Psi_M$$

where we define 9 fermionic fields $\Psi_M \equiv \varepsilon \Gamma_M \psi$. The remaining 7 fermionic fields are superpartners of K_i. They are defined as $\chi_i \equiv (\nu_i, \Psi)$. In other words, we decompose spinor Ψ over the basis consisting of 16 orthonormal spinors $\Gamma_M \varepsilon$ and $\nu_i(\varepsilon)$. The expansion coefficients are called Ψ_M and χ_i. Then the susy transformations can be written as

$$\delta \chi_i = K_i + s_i(A_M)$$

Changing notations to TFT like

The SUSY transformation Q_ε can be conveniently written in the form of cohomological field theory (even though the theory is non-twisted)
For $M = 1\dots 9$ we rewrite

$$\delta A_M = \varepsilon \Gamma_M \psi$$

as

$$\delta A_M = \Psi_M$$

where we define 9 fermionic fields $\Psi_M \equiv \varepsilon \Gamma_M \psi$. The remaining 7 fermionic fields are superpartners of K_i. They are defined as $\chi_i \equiv (\nu_i, \Psi)$. In other words, we decompose spinor Ψ over the basis consisting of 16 orthonormal spinors $\Gamma_M \varepsilon$ and $\nu_i(\varepsilon)$. The expansion coefficients are called Ψ_M and χ_i. Then the susy transformations can be written as

$$\delta \chi_i = K_i + s_i(A_M)$$

where $s_i(A_M) = (\nu_i, \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon)$ are the

The Q-complex

In new notations the transformations look like

$$\delta A_M = \psi_M$$
$$\delta \psi_M = \mathcal{R} \cdot A_M$$
$$\delta \chi_i = H_i$$
$$\delta H_i = \mathcal{R} \cdot \chi_i$$

where $\mathcal{R}$ stands for the Q^2 action on fields.
Then some linear algebra shows that

$$Z_{1-loop} = \frac{\det \mathcal{R}|_{H_i}}{\det \mathcal{R}|_{A_M}}$$

To compute the ratio of determinants of $\mathcal{R}$ acting on the space of fields A_M and χ_i we use the Atiyah-Singer index theorem to compute the $\mathcal{R}$-equivariant character

$$ind = \mathrm{tr}_{A_M} e^{\mathcal{R}} - \mathrm{tr}_{H_i} e^{\mathcal{R}}$$

Each term is infinite dimensional and is not well-defined on its own. But the difference is well-defined.

To compute the ratio of determinants of $\mathcal{R}$ acting on the space of fields A_M and χ_i we use the Atiyah-Singer index theorem to compute the $\mathcal{R}$-equivariant character

$$ind = \mathrm{tr}_{A_M} e^{\mathcal{R}} - \mathrm{tr}_{H_i} e^{\mathcal{R}}$$

Each term is infinite dimensional and is not well-defined on its own. But the difference is well-defined. Let the space of A_M be $\Gamma(\mathcal{E}_0)$ and the space of H_i be $\Gamma(\mathcal{E}_1)$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are certain vector bundles over S^4. The usual framework for the index envolves an elliptic operator $D : \Gamma(\mathcal{E}_0) \to \Gamma(\mathcal{E}_1)$ corresponding to the linearized equations $s_i(A_M) = 0$.

To compute the ratio of determinants of $\mathcal{R}$ acting on the space of fields A_M and χ_i we use the Atiyah-Singer index theorem to compute the $\mathcal{R}$-equivariant character

$$ind = \mathrm{tr}_{A_M} e^{\mathcal{R}} - \mathrm{tr}_{H_i} e^{\mathcal{R}}$$

Each term is infinite dimensional and is not well-defined on its own. But the difference is well-defined. Let the space of A_M be $\Gamma(\mathcal{E}_0)$ and the space of H_i be $\Gamma(\mathcal{E}_1)$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are certain vector bundles over S^4. The usual framework for the index envolves an elliptic operator $D : \Gamma(\mathcal{E}_0) \to \Gamma(\mathcal{E}_1)$ corresponding to the linearized equations $s_i(A_M) = 0$.
In the present problem the equations $s_i(A_M)$ are not elliptic, but they are transversally elliptic with the respect to $U(1)$ rotations of S^4.

To compute the ratio of determinants of $\mathcal{R}$ acting on the space of fields A_M and χ_i we use the Atiyah-Singer index theorem to compute the $\mathcal{R}$-equivariant character

$$ind = \mathrm{tr}_{A_M} e^{\mathcal{R}} - \mathrm{tr}_{H_i} e^{\mathcal{R}}$$

Each term is infinite dimensional and is not well-defined on its own. But the difference is well-defined. Let the space of A_M be $\Gamma(\mathcal{E}_0)$ and the space of H_i be $\Gamma(\mathcal{E}_1)$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are certain vector bundles over S^4. The usual framework for the index envolves an elliptic operator $D : \Gamma(\mathcal{E}_0) \to \Gamma(\mathcal{E}_1)$ corresponding to the linearized equations $s_i(A_M) = 0$.
In the present problem the equations $s_i(A_M)$ are not elliptic, but they are transversally elliptic with the respect to $U(1)$ rotations of S^4.
The index is then well defined as a distribution on the group $U(1)$.

The generating function for the index is

$$-\frac{1+q^2}{(1-q)^2}$$

for the $\mathcal{N}=2$ vector multiplet, and

$$\frac{2q}{(1-q)^2}$$

for the $\mathcal{N}=2$ hyper multiplet.

The generating function for the index is

$$-\frac{1+q^2}{(1-q)^2}$$

for the $\mathcal{N}=2$ vector multiplet, and

$$\frac{2q}{(1-q)^2}$$

for the $\mathcal{N}=2$ hyper multiplet. In the $\mathcal{N}=4$ case the contributions to the index precisely cancel, hence the determinant is equal to 1.

In the case of $\mathcal{N} = 2$ theory with a matter hypermultiplet in some representation W we have

$$Z_{1-loop}^{\mathcal{N}=2,W}(ia_E) = \frac{\prod_{\alpha \in \text{weights(Ad)}} H(i\alpha \cdot a_E/\varepsilon)}{\prod_{w \in \text{weights}(W)} H(iw \cdot a_E/\varepsilon)}.$$

Here $H(z)$ is expressed in terms of the Barnes G-function ('superfactorial'):

$$H(z) = G(1+z)G(1-z) = e^{-(1+\gamma)z^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^n e^{\frac{z^2}{n}}.$$

Point instanton corrections and conclusion

Gauge theory:

$$Z_{S^4} = \int da Z_{1-loop} |Z^{inst}_{Nekr}(r^{-1}, r^{-1}, a)|^2$$

Point instanton corrections and conclusion

Gauge theory:

$$Z_{S^4} = \int da Z_{1-loop} |Z^{inst}_{Nekr}(r^{-1}, r^{-1}, a)|^2$$

OSV:

$$Z_{BH} = |Z_{top}|^2$$

The relation?