Title: Probing local non--Gaussianities in CMB within a Bayesian framework

Date: Mar 09, 2008 09:00 AM

URL: http://pirsa.org/08030056

Abstract:

Probing local non–Gaussianities in CMB within a Bayesian framework

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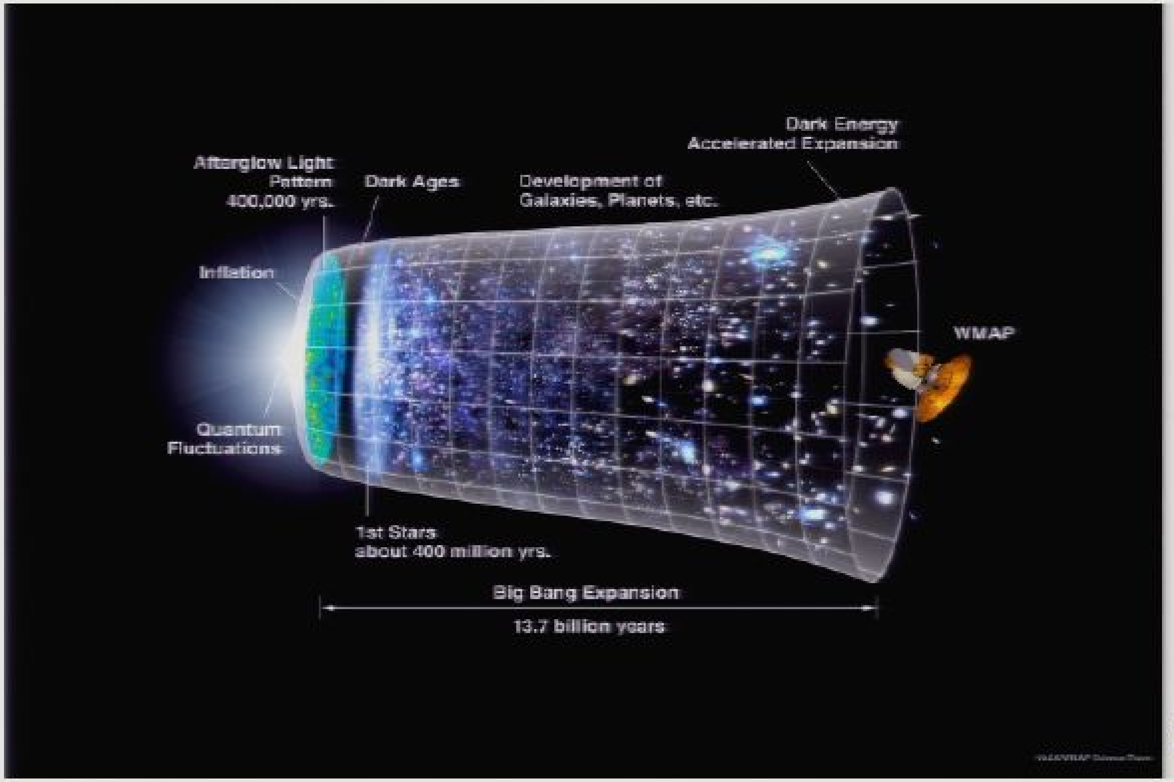
Outline

- Introduction
- Traditional (frequentist) approach
- New Bayesian approach
- Test on simulated data
- Summary

Motivation



NASA
WMAP team

$$\Rightarrow \Omega_\Lambda,\ \Omega_m,\ \sigma_8,\ \tau,\ n_s,\ H_0,\ \ldots$$

Notation

$$a_{\ell m} = \frac{2b_\ell}{\pi} \int k^2 dk\, r^2 dr \left[\, \Phi_{\ell m}(r)\, g_\ell^{adi}(k) + S_{\ell m}(r)\, g_\ell^{iso}(k) \,\right] j_\ell(kr) + n_{\ell m}$$

$\Phi_{\ell m}(r)$	: Adiabatic perturbation
$S_{\ell m}(r)$	: Isocurvature perturbation
$g_\ell(k)$	: Radiation transfer functions
$n_{\ell m}$	: Noise

Local Non–Gaussianity

$$\Phi_{NL}(r) = \Phi_L(r) + f_{NL}\left[\Phi_L^2(r) - \langle \Phi_L^2(r) \rangle\right] + g_{NL}\Phi_L^3(r) + \dots$$

$\Phi_{NL}(r)$: Non–Gaussian potential

$\Phi_L(r)$: Gaussian contribution

f_{NL} : Amplitude of non–Gaussianity

Local Non–Gaussianity

$$\Phi_{NL}(r) = \Phi_L(r) + f_{NL}\left[\Phi_L^2(r) - \langle \Phi_L^2(r) \rangle\right] + g_{NL}\Phi_L^3(r) + \ldots$$

$\Phi_{NL}(r)$: Non–Gaussian potential

$\Phi_L(r)$: Gaussian contribution

f_{NL} : Amplitude of non–Gaussianity, constant

Predicted by multi–field inflation models

(e.g. Moroi & Takahashi 2001; Enqvist & Sloth 2002; Lyth et al. 2003)

or string theory

(e.g. Khoury et al. 2001; Steinhardt & Turok 2002)

Traditional methods

Calculate the bispectrum

$$\langle \Phi(\mathbf{k_1})\Phi(\mathbf{k_2})\Phi(\mathbf{k_3})\rangle = (2\pi)^3\,\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,F(k_1,k_2,k_3)$$

where $F(k_1,k_2,k_3) = F(k_1,k_2,k_3,\mathbf{f_{NL}})$

Introduction

Frequentist approach
- Traditional methods
- Some caveats

Bayesian approach

Test on simulated data

Summary

Traditional methods

Calculate the bispectrum

$$\langle \Phi(\mathbf{k_1})\Phi(\mathbf{k_2})\Phi(\mathbf{k_3})\rangle = (2\pi)^3\,\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\,F(k_1,k_2,k_3)$$

where $F(k_1,k_2,k_3) = F(k_1,k_2,k_3,\mathbf{f_{NL}})$

or the KSW–estimator *(Komatsu et al. 2005; Yadav et al. 2007)*

$$f_{NL} \propto \hat{S}_{prim} = \frac{1}{f_{sky}} \int r^2 dr \int d^2\hat{n}\, A(\hat{n},r)B(\hat{n},r)B(\hat{n},r)$$

$$\begin{aligned}
A(\hat{n},r) &= \textstyle\sum_{lm} C_\ell^{-1} a_{\ell m} Y_{\ell m}(\hat{n})\alpha_\ell(r) \\
B(\hat{n},r) &= \textstyle\sum_{lm} C_\ell^{-1} a_{\ell m} Y_{\ell m}(\hat{n})\beta_\ell(r) \\
\alpha_\ell(r) &= \tfrac{2b_\ell}{\pi} \int k^2 dk\, g_\ell(k)\, j_\ell(kr) \\
\beta_\ell(r) &= \tfrac{2b_\ell}{\pi} \int k^2 dk\, g_\ell(k)\, j_\ell(kr)\, \mathcal{P}(k)
\end{aligned}$$

Some caveats

Properties of the estimator:

- Variance larger than predicted by Fisher matrix
- Uses information from 3–point–function only

Ambiguity in defining the estimator:

- Uncertainties in systematic effects of instrument
- How to account for errors in foreground substraction
- Uncertainties in cosmological parameters

Some caveats

Properties of the estimator:

- Variance larger than predicted by Fisher matrix
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Ambiguity in defining the estimator:

- Uncertainties in systematic effects of instrument
- How to account for errors in foreground substraction
- Uncertainties in cosmological parameters

Can we improve the analysis?

Bayesian approach (warm up)

The data model

$$
\begin{aligned}
s_{\ell m} &= \frac{2}{\pi}\int k^2 dk\, r^2 dr\, \Phi_{\ell m}(r)\, g_\ell(k)\, j_\ell(kr) \\
&\approx \sum_{r_i} M(r_i)\Phi_{\ell m}(r_i) \\
&\equiv M\Phi_{\ell m}
\end{aligned}
$$

Bayesian approach (warm up)

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\begin{aligned}
s_{\ell m} &= \frac{2}{\pi} \int k^2 dk\, r^2 dr\, \Phi_{\ell m}(r)\, g_\ell(k)\, j_\ell(kr) \\
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&\equiv M \Phi_{\ell m}
\end{aligned}
$$

$$
\begin{aligned}
d_i &= B_i s + f_i + n_i \\
d &= B M \Phi + n
\end{aligned}
$$

B : Beam convolution matrix
M : Transfer matrix

Bayesian approach I

Aim: Construct the posterior density $P(f_{NL}|d)$

Bayesian approach I

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The joint probability

$$P(d, \Phi_L, f_{NL}, \Omega_\Lambda, \Omega_m, \tau, \ldots)$$

Bayesian approach I

Aim: Construct the posterior density $P(f_{NL}|d)$

The joint probability

$$P(d, \Phi_L, f_{NL}, \theta) = P(d|\Phi_L, f_{NL}, \theta)\, P(\Phi_L|\theta)\, P(\theta)\, P(f_{NL})$$

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$$P(d, \Phi_L, f_{NL}, \theta) = P(d|\Phi_L, f_{NL}, \theta)\, P(\Phi_L|\theta)\, P(\theta)\, P(f_{NL})$$

$$P(\Phi_L|\theta) \quad = \quad \frac{1}{\sqrt{|2\pi P_\Phi|}}\, e^{-1/2\, \Phi_L^T P_\Phi^{-1} \Phi_L}$$

Bayesian approach I

Aim: Construct the posterior density $P(f_{NL}|d)$

The joint probability

$$P(d, \Phi_L, f_{NL}, \theta) = P(d|\Phi_L, f_{NL}, \theta)\, P(\Phi_L|\theta)\, P(\theta)\, P(f_{NL})$$

$$\begin{aligned} P(\Phi_L|\theta) &= \frac{1}{\sqrt{|2\pi P_\Phi|}} \mathrm{e}^{-1/2\, \Phi_L^T P_\Phi^{-1} \Phi_L} \\ P(d|\Phi_L, f_{NL}, \theta) &= \frac{1}{\sqrt{|2\pi N|}} \mathrm{e}^{-1/2 \left[d - BM\Phi_L - f_{NL} BM(\Phi_L^2 - \langle\Phi_L^2\rangle)\right]^T N^{-1}} \\ & \qquad \times \left[d - BM\Phi_L - f_{NL} BM(\Phi_L^2 - \langle\Phi_L^2\rangle)\right] \end{aligned}$$

Bayesian approach II

Aim: Construct the posterior density $P(f_{NL}|d)$

Exact expression

$$P(d, \Phi_L, f_{NL}, \theta) \propto \mathrm{e}^{-1/2\,[\,(d - BM\Phi_L - f_{NL}BM(\Phi_L^2 - \langle\Phi_L^2\rangle))^T N^{-1} \times (d - BM\Phi_L - f_{NL}BM(\Phi_L^2 - \langle\Phi_L^2\rangle)) + \Phi_L^T P_\Phi^{-1}\Phi_L\,]}$$

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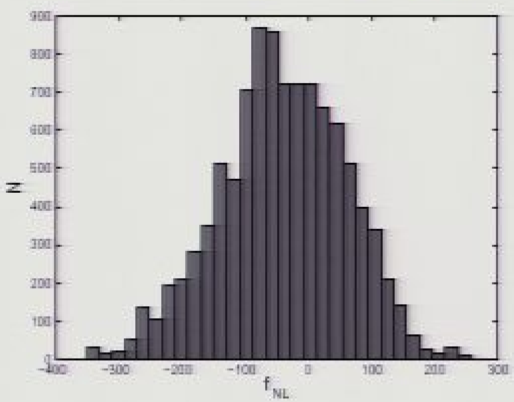
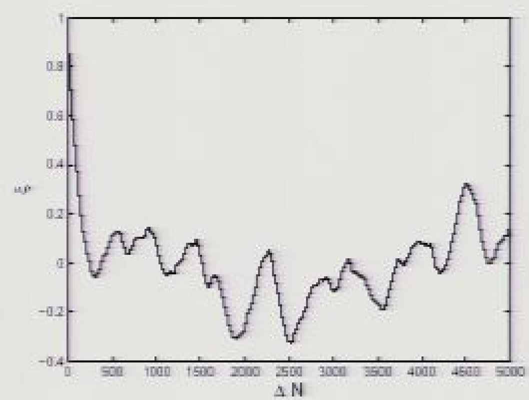
Hamiltonian sampling?





Bayesian approach II

Aim: Construct the posterior density $P(f_{NL}|d)$

Exact expression

$$P(d, \Phi_L, f_{NL}, \theta) \propto \mathrm{e}^{-1/2\left[(d - BM\Phi_L - f_{NL}BM(\Phi_L^2 - \langle\Phi_L^2\rangle))^T N^{-1} \times (d - BM\Phi_L - f_{NL}BM(\Phi_L^2 - \langle\Phi_L^2\rangle)) + \Phi_L^T P_\Phi^{-1} \Phi_L\right]}$$

$$P(f_{NL}|d, \theta) = \int d\Phi_{NL}\, P(f_{NL}|\Phi_{NL}, \theta)\, P(\Phi_{NL}|d, \theta)$$

Bayesian approach II

Aim: Construct the posterior density $P(f_{NL}|d)$

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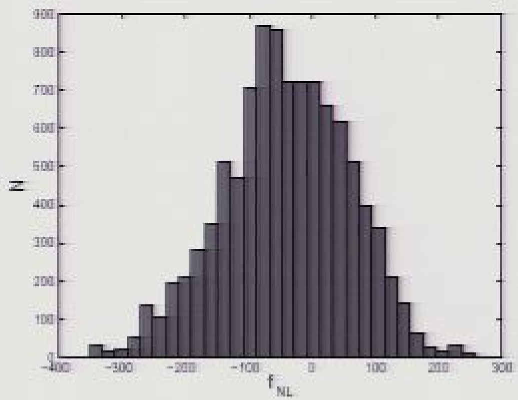
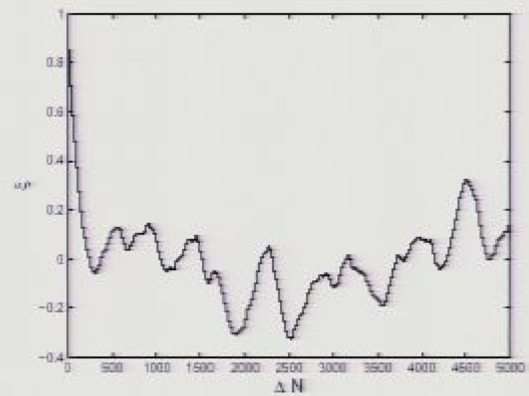
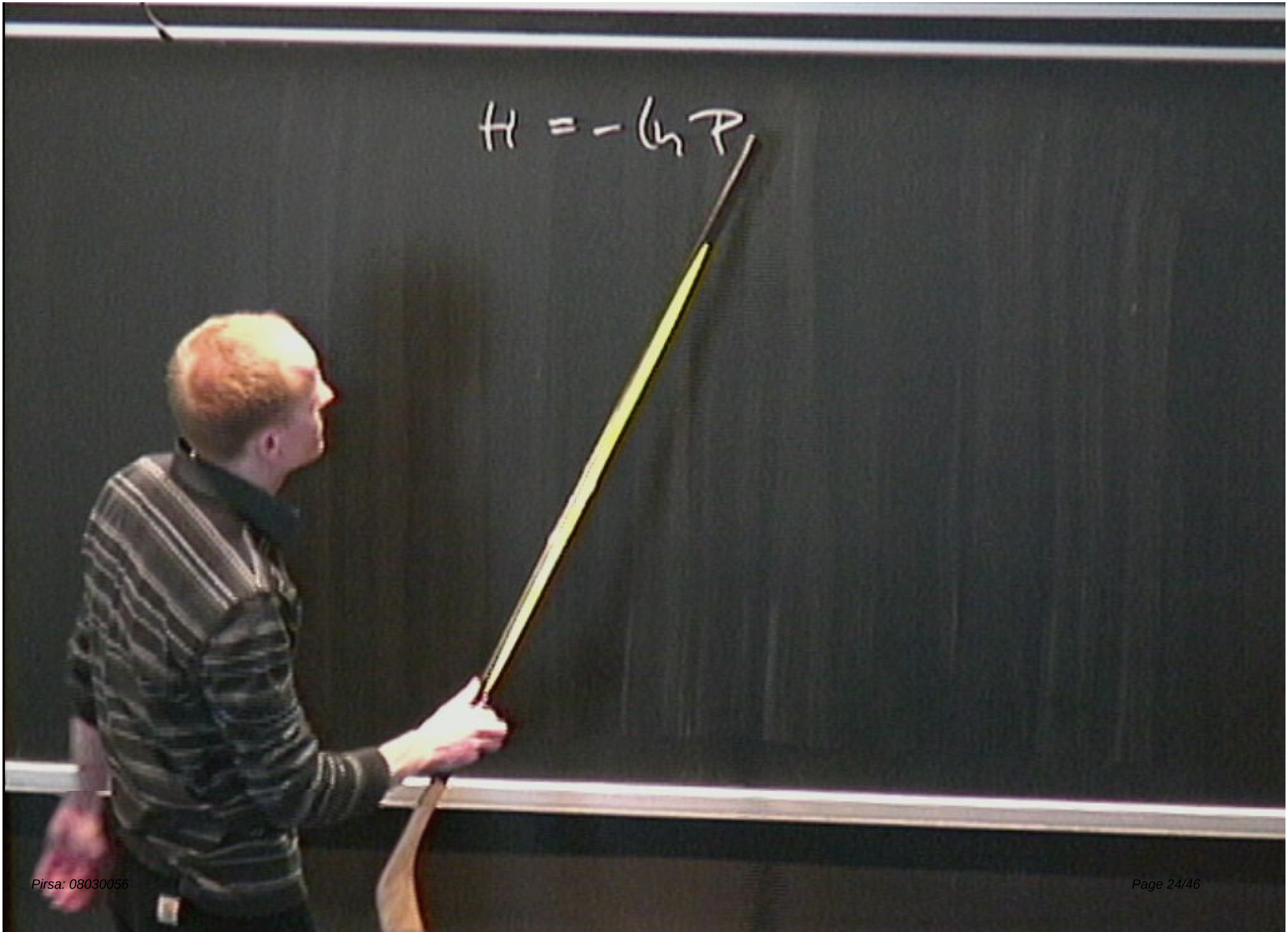
$$P(d,\Phi_L,f_{NL},\theta) \propto \mathrm{e}^{-1/2[\,(d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))^T N^{-1}} \\ \times(d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))+\Phi_L^T P_\Phi^{-1}\Phi_L\,]$$

Hamiltonian sampling?






$H = -\ln P$

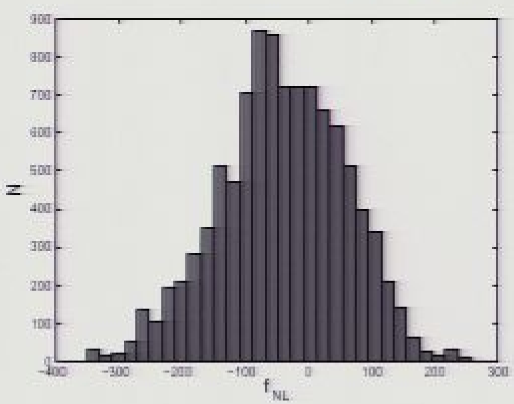
Bayesian approach II

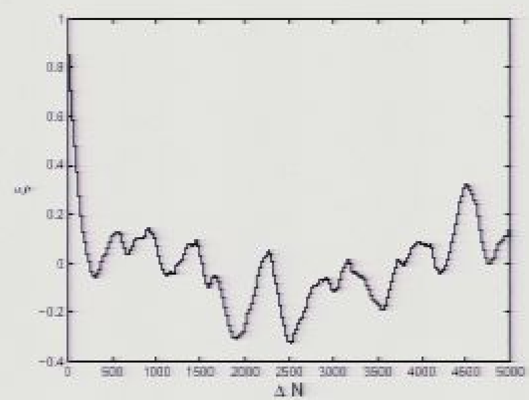
Aim: Construct the posterior density $P(f_{NL}|d)$

Exact expression

$$P(d, \Phi_L, f_{NL}, \theta) \propto \mathrm{e}^{-1/2\left[(d - BM\Phi_L - f_{NL} BM(\Phi_L^2 - \langle\Phi_L^2\rangle))^T N^{-1} \times (d - BM\Phi_L - f_{NL} BM(\Phi_L^2 - \langle\Phi_L^2\rangle)) + \Phi_L^T P_\Phi^{-1} \Phi_L\right]}$$

Hamiltonian sampling?





Bayesian approach II

Aim: Construct the posterior density $P(f_{NL}|d)$

Exact expression

$$P(d,\Phi_L,f_{NL},\theta) \propto e^{-1/2[(d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))^T N^{-1} \times (d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))+\Phi_L^T P_\Phi^{-1}\Phi_L]}$$

$$P(f_{NL}|d,\theta) = \int d\Phi_{NL}\, P(f_{NL}|\Phi_{NL},\theta)\, P(\Phi_{NL}|d,\theta)$$

Bayesian approach II

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$$P(d,\Phi_L,f_{NL},\theta) \propto \mathrm{e}^{-1/2[\,(d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))^T N^{-1} \times (d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle)) + \Phi_L^T P_\Phi^{-1}\Phi_L\,]}$$

$$P(d,\Phi_{NL},\theta) \propto \mathrm{e}^{-1/2[\,(d-BM\Phi_{NL})^T N^{-1}(d-BM\Phi_{NL}) + \Phi_L^T P_\Phi^{-1}\Phi_L\,]}$$

Bayesian approach II

Aim: Construct the posterior density $P(f_{NL}|d)$

Exact expression

$$P(d,\Phi_L,f_{NL},\theta) \propto \mathrm{e}^{-1/2[\,(d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))^T N^{-1} \times (d-BM\Phi_L-f_{NL}BM(\Phi_L^2-\langle\Phi_L^2\rangle))+\Phi_L^T P_\Phi^{-1}\Phi_L\,]}$$

$$P(d,\Phi_{NL},\theta) \propto \mathrm{e}^{-1/2[\,(d-BM\Phi_{NL})^T N^{-1}(d-BM\Phi_{NL})+\Phi_{NL}^T P_\Phi^{-1}\Phi_{NL}\,]}$$

$\Rightarrow$ Conservative estimate of f_{NL}

Bayesian approach III

Aim: Construct the posterior density $P(f_{NL}|d)$

Gaussian conditional probability $P(\Phi_{NL}|d,\theta)$

$$\langle \Phi_{NL} \rangle = \left[M^T B^T N^{-1} BM + P_\Phi^{-1} \right]^{-1} M^T B^T N^{-1} d$$

$$\langle (\Phi_{NL} - \langle \Phi_{NL} \rangle)^2 \rangle = \left[M^T B^T N^{-1} BM + P_\Phi^{-1} \right]^{-1}$$

Bayesian approach IV

Aim: Construct the posterior density $P(f_{NL}|d)$

Conditional probability $P(f_{NL}|\Phi_{NL},\theta)$

$$\begin{aligned} P(f_{NL}|\Phi_{NL},\theta) &= \int d\Phi_L \, P(f_{NL}|\Phi_L,\Phi_{NL}) \, P(\Phi_L|\theta) \\ &= \int d\Phi_L \, \delta(\Phi_{NL} - \Phi_L - f_{NL}(\Phi_L^2 - \langle\Phi_L^2\rangle)) \, P(\Phi_L|\theta) \end{aligned}$$

Bayesian approach IV

Aim: Construct the posterior density $P(f_{NL}|d)$

Conditional probability $P(f_{NL}|\Phi_{NL},\theta)$

$$\begin{aligned} P(f_{NL}|\Phi_{NL},\theta) &= \int d\Phi_L \, P(f_{NL}|\Phi_L,\Phi_{NL}) \, P(\Phi_L|\theta) \\ &= \int d\Phi_L \, \delta(\Phi_{NL} - \Phi_L - f_{NL}(\Phi_L^2 - \langle\Phi_L^2\rangle) \, P(\Phi_L|\theta) \end{aligned}$$

$$P(f_{NL}|\Phi_{NL},\theta) \propto \left| \prod_i \frac{1}{1+2f_{NL}(\tilde{\Phi}_L)_i} \right| \mathrm{e}^{-1/2\,\tilde{\Phi}_L^T P_\Phi^{-1} \tilde{\Phi}_L}$$

where $\tilde{\Phi}_L = \frac{1}{2f_{NL}}\left[-1 + \sqrt{1+4f_{NL}(\Phi_{NL}+f_{NL}\langle\Phi_L^2\rangle)}\right]$

Sampling from the PDFs

Aim: Construct the posterior density $P(f_{NL}|d)$

Sample from conditional probabilities numerically

Sampling from the PDFs

Aim: Construct the posterior density $P(f_{NL}|d)$

Sample from conditional probabilities numerically

$$\begin{array}{ccc} \Phi^i_{NL} & \hookleftarrow & P(\Phi_{NL}|d,\theta) \\ f^i_{NL} & \hookleftarrow & P(f_{NL}|\Phi^i_{NL},\theta) \end{array}$$

$\int d\theta$

$\Rightarrow$ The $\{f^i_{NL}\}$ are samples from $P(f_{NL}|d,\theta)$

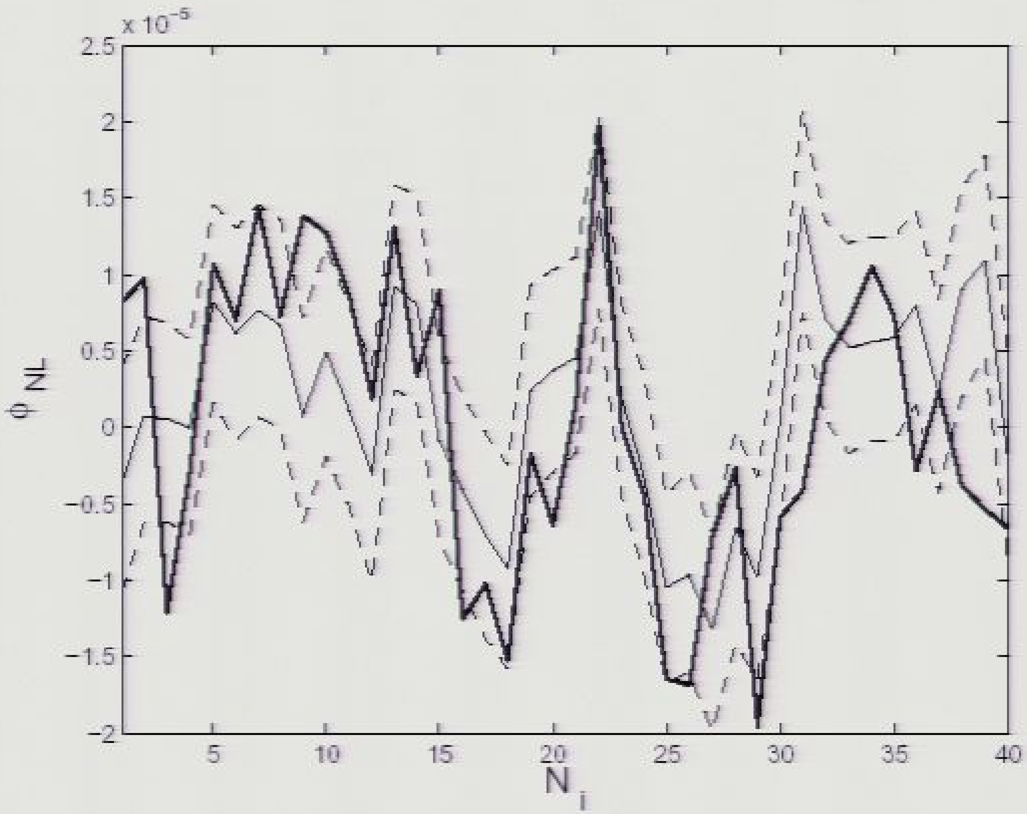
Advantages

- Data and foregrounds can be modeled jointly
- Techniques to account for instrumental effects and foregrounds already exists in a Bayesian framework
- Further extensions possible: e.g. marginalization over cosmology

Model setup

- Data vector with $N_{pix} = 10^6$
- Sparse signal covariance matrix
- Reconstruction on one shell
- Neglecting beam convolution
- Gaussian white noise

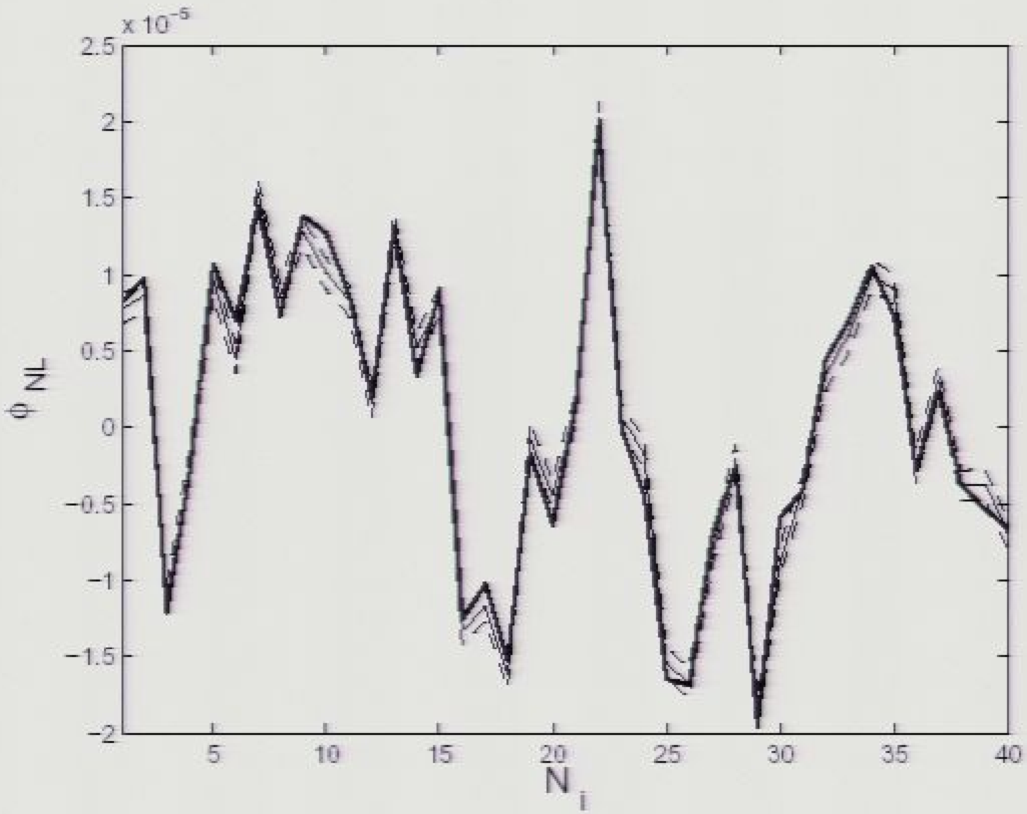
Results I



$S/N = 1$

Reconstruction of Φ_{NL} from 1000 samples, $f_{NL} = 200$

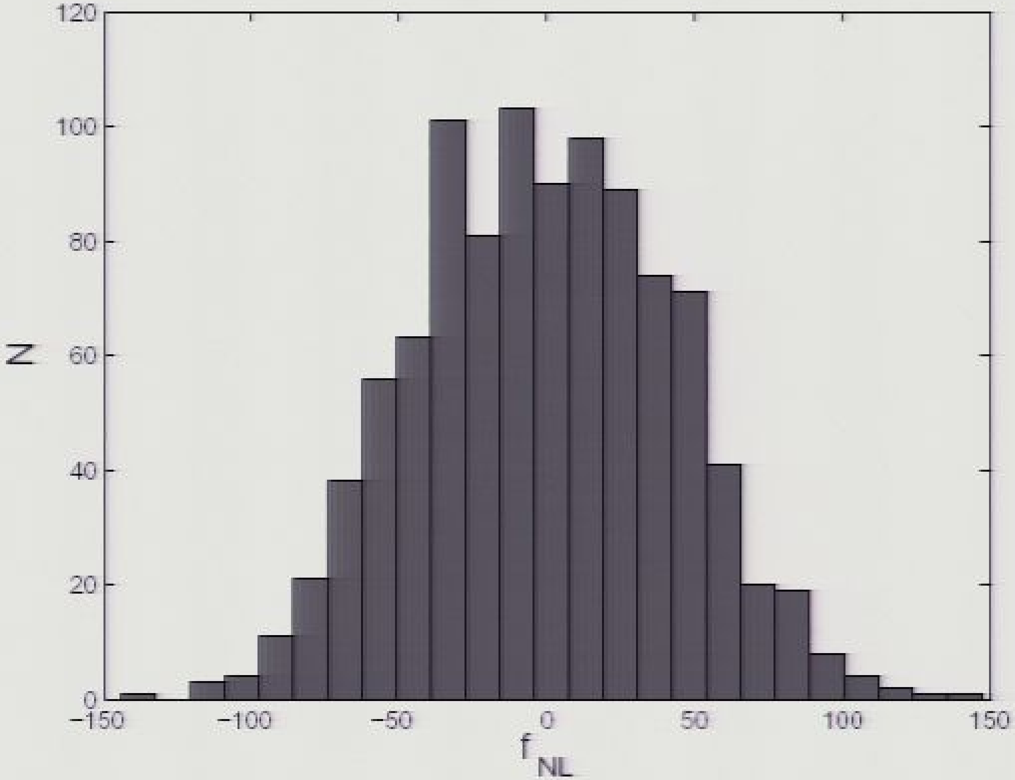
Results I



$S/N = 10$

Reconstruction of Φ_{NL} from 1000 samples, $f_{NL} = 200$

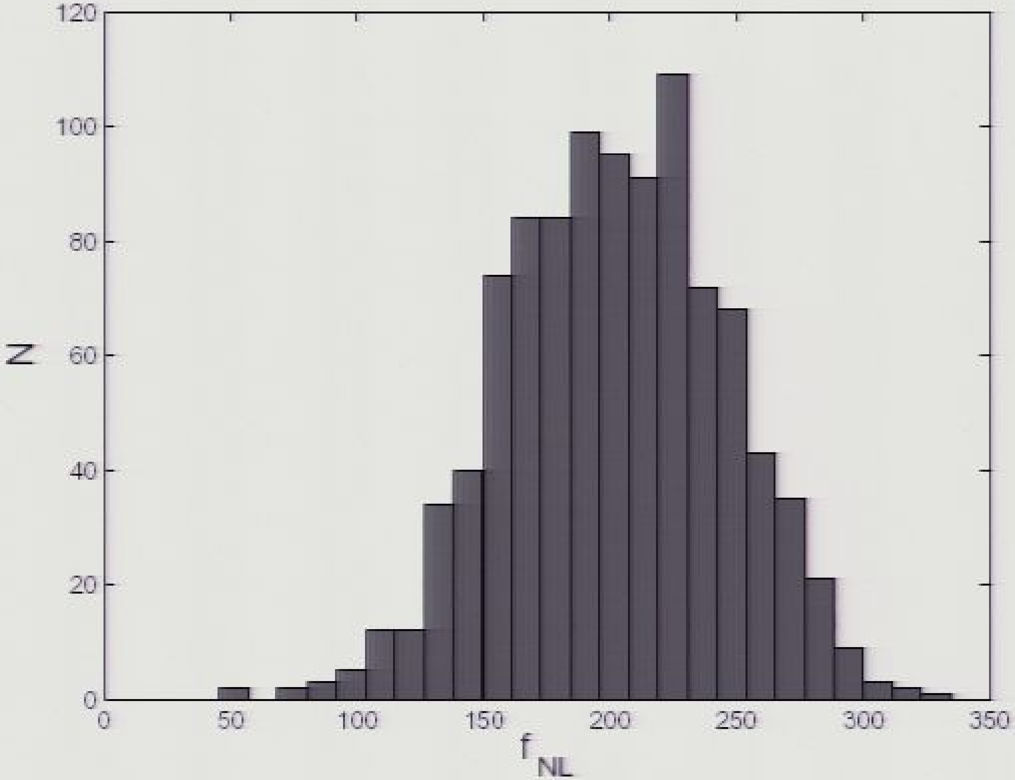
Results II



$f_{NL} = 0$

Reconstruction of f_{NL} from 1000 samples

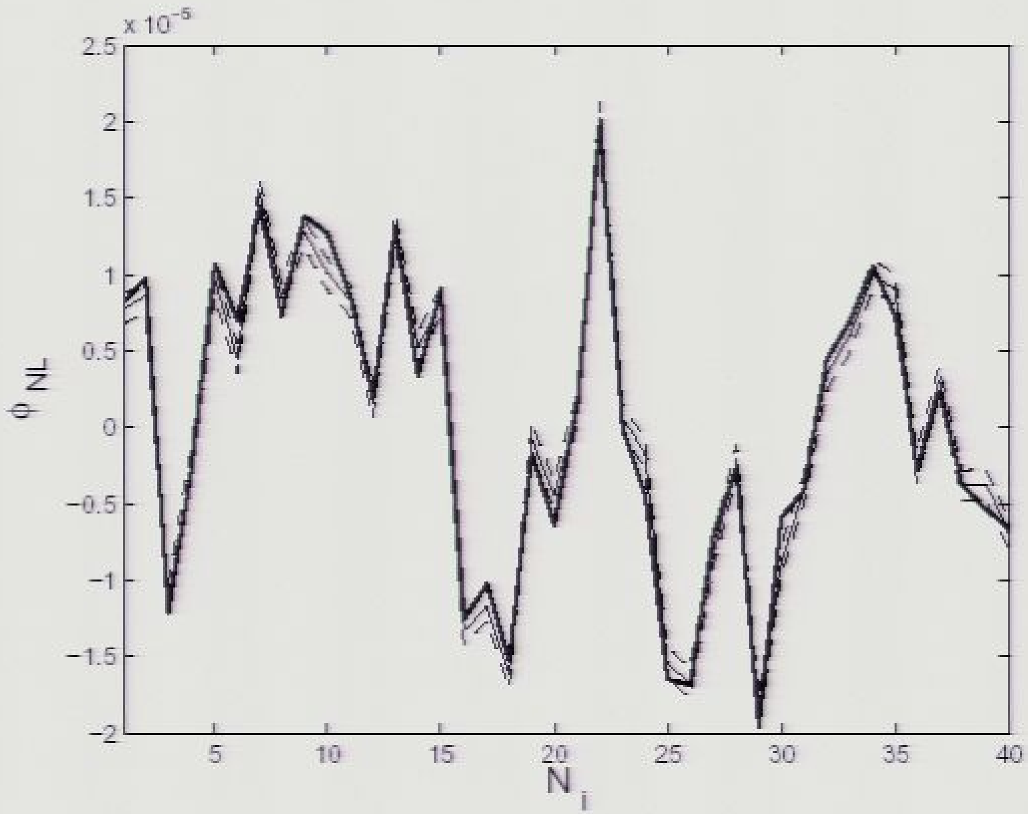
Results II



$f_{NL} = 200$

Reconstruction of f_{NL} from 1000 samples

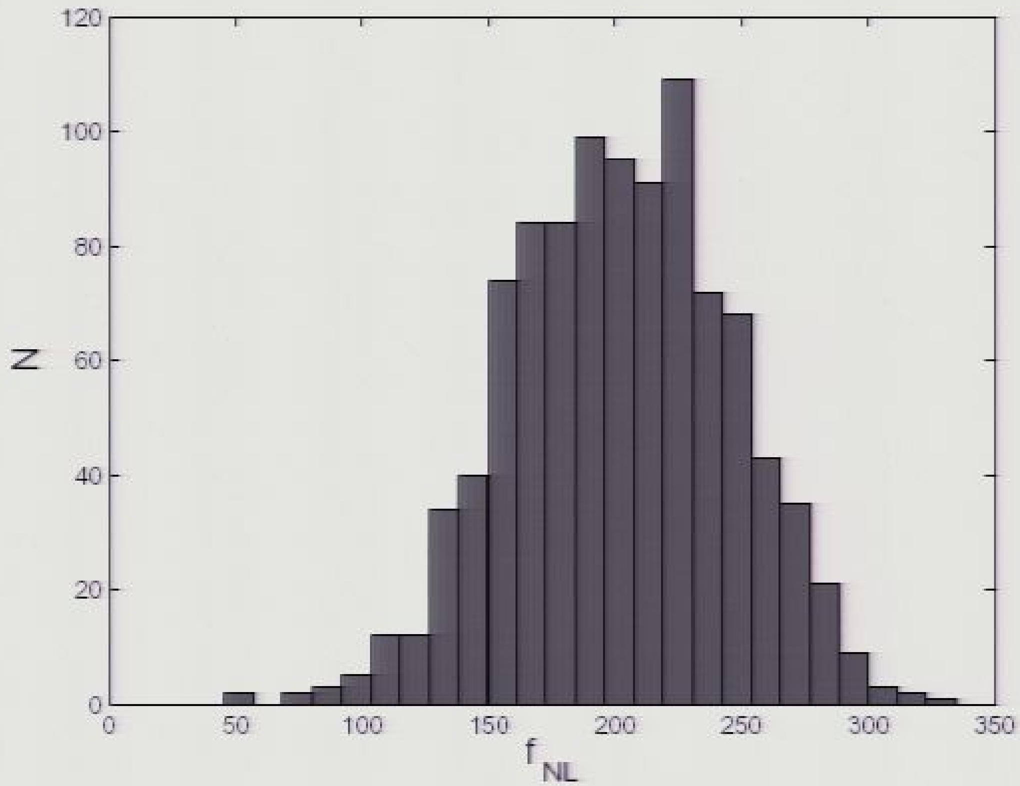
Results I



$S/N = 10$

Reconstruction of Φ_{NL} from 1000 samples, $f_{NL} = 200$

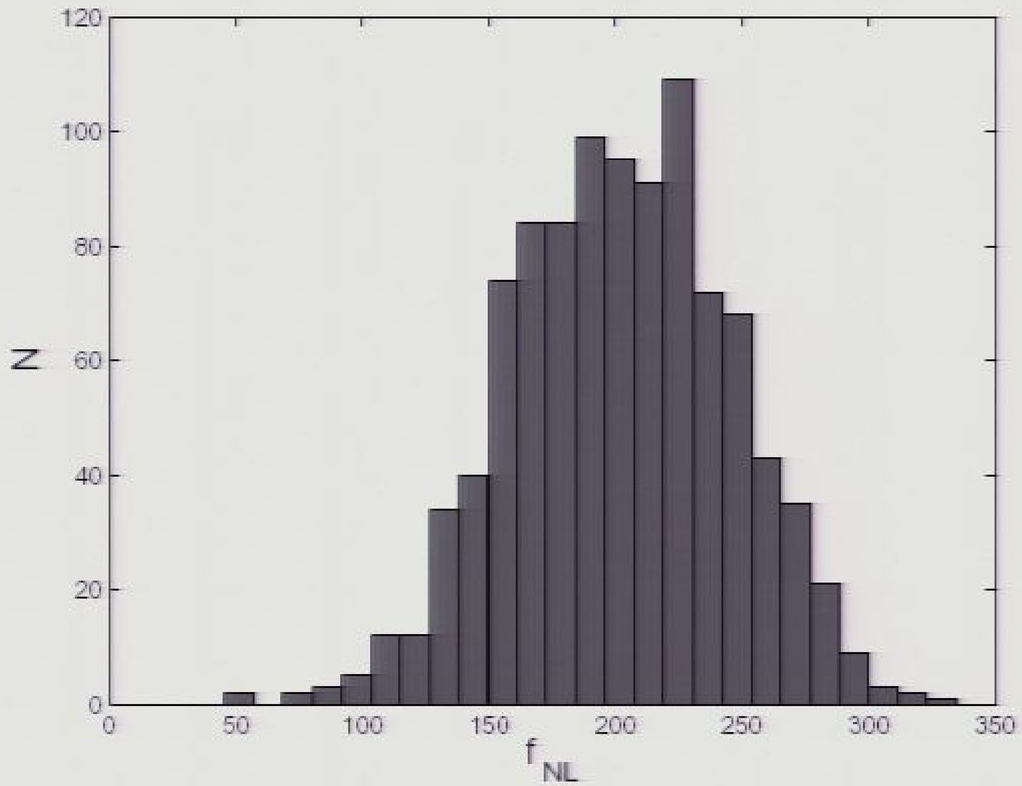
Results II



$f_{NL} = 200$

Reconstruction of f_{NL} from 1000 samples

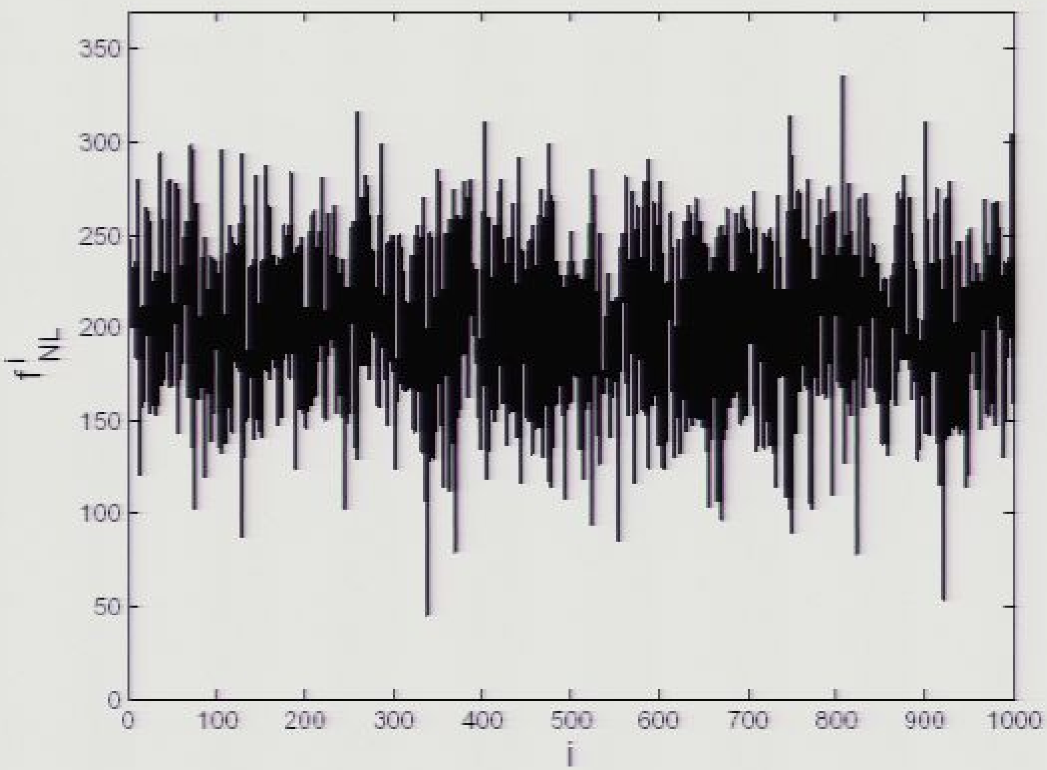
Results II



$f_{NL} = 200$

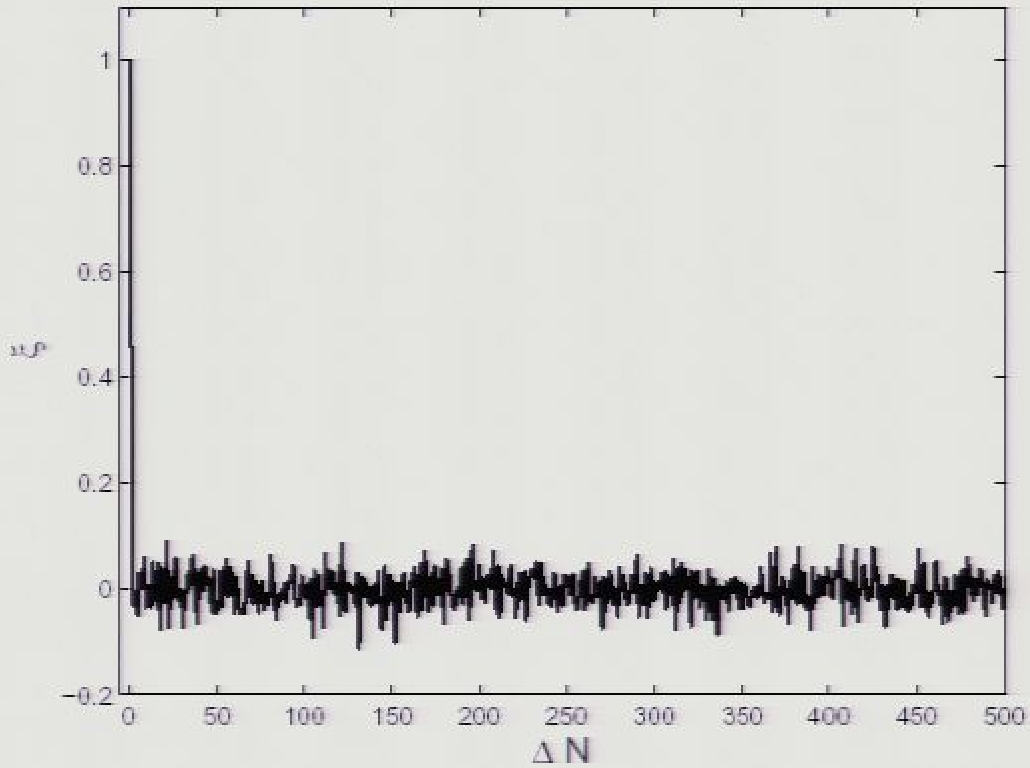
Reconstruction of f_{NL} from 1000 samples

Properties of the sampler



f_{NL} chain of the 1000 samples

Properties of the sampler



Autocorrelation of the f_{NL} chain

Summary

- The frequentist approach suffers from some drawbacks
- Bayesian methods have been applied to powerspectum analysis succesfully
- They are applicable to the detection of f_{NL}
- The Bayesian approach allows proper treatment of uncertainties

No Signal
VGA-1