Title: Wald Entropy and the Einstein Equation of State

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Abstract:

Wald Entropy + the Einstein Equation of State

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PLAN

(1) The Einstein eq^{n} as an eq^{n} of state

(2) The general case?

(3) Wald entropy

1 The Einstein eq$^{\underline{n}}$ of state

[Jacobson gr-qc/9504004]

Black holes

Einstein eq$^{\underline{n}}$

↓

1st law BH mechanics [global statement]

↓

Identify $S \propto A$, $T \propto \kappa$

How does classical GR know about Hawking radiation?

Jacobson: Reverse the logic!

Declare $S \propto A$, $T \propto \kappa$ for local causal horizon

↓

Local thermodynamic equilibrium:

- $\delta Q = T \partial S$
- 1st law $[\nabla_a T^{ab} = 0]$

↓

Einstein eq$^{\underline{n}}$ as eq$^{\underline{n}}$ of state

$$S(E,V) \Rightarrow TdS = T\frac{\partial S}{\partial E}\Big|_V dE + T\frac{\partial S}{\partial V}\Big|_E dV$$

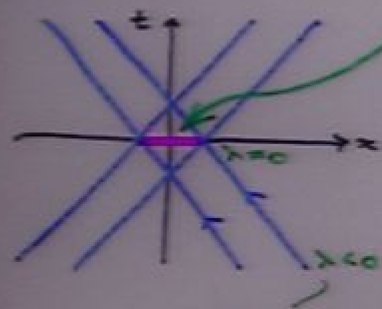
$$1^{st} \text{ law} \Rightarrow TdS = dE + p\,dV$$

$$\frac{1}{T} = \frac{\partial S}{\partial E}\Big|_V \quad \text{and} \quad p = T\frac{\partial S}{\partial V}\Big|_E \quad [\text{eq}^{\underline{n}} \text{ of state}]$$

The argument in a nutshell:

Need to specify
(a) meaning of local equilibrium
(b) T
(c) δQ
(d) δS

(a) Local eq$^{\underline{n}}$



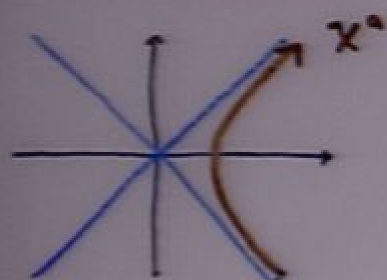
Spacelike 2-surface element P.

Choose so past null normal congruence to one side has

$$\begin{cases} \theta = 0 \\ \sigma_{ab} = 0 \end{cases} \quad \text{at } P.$$

$\Rightarrow$ Defines "system" (i.e. part of spacetime beyond past horizon) instantaneously in eq$^{\underline{n}}$.

(b) Temperature



Approx. Killing field generating boosts orthogonal to P + vanishing at P.

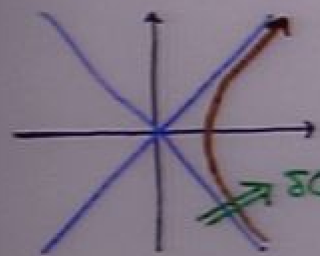
Unruh: $T = \dfrac{\hbar \kappa}{2\pi}$

accel$^{\underline{n}}$ of orbit on which $\chi^2 = -1$

$$\kappa \to \infty \quad \begin{cases} T \to \infty \\ \delta Q \to \infty \\ \dfrac{\delta Q}{T} \text{ finite} \end{cases}$$

(c) Heat flux

$$\delta Q = \int_H T^{ab} \chi_a \, d\Sigma_b$$



Boost-energy current $j^a = T^{ab}\chi_b$

$$\nabla_a j^a = T^{ab} \nabla_a \chi_b = 0$$

$$d\Sigma_b = k_b \, d\lambda \, dA$$

horizon surface element; affine tangent vector; cross-sectional area of congruence

on horizon $\chi_a = -\lambda \kappa \, k_a$

$$\Longrightarrow \delta Q = -\kappa \int_H T_{ab} k^a k^b \lambda \, d\lambda \, dA$$

$\left[\frac{\delta Q}{T} \text{ indept. of } \kappa \right]$ ✓

(d) Entropy change

[along horizon segment]

$$\delta S = \alpha \, \delta A = \alpha \int_H \theta \, d\lambda \underbrace{\sqrt{h}\, d^2\Sigma}_{dA}$$

Expansion

$$\theta = \frac{1}{\sqrt{h}} \frac{d\sqrt{h}}{d\lambda}$$

Raychaudhuri

$$\frac{d\theta}{d\lambda} = \underbrace{-\tfrac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab}}_{O(\lambda^2)} - R_{ab} k^a k^b$$

$$\theta = -R_{ab} k^a k^b \lambda + O(\lambda^2)$$

$$\Longrightarrow \delta S = -\alpha \int_H R_{ab} k^a k^b \lambda \, d\lambda \, dA$$

Now impose Clausius: $\boxed{\delta S = \frac{\delta Q}{T}}$

$\Rightarrow$ Matter flux associated with focussing of horizon generators.

$$\Rightarrow \quad \alpha R_{ab} k^a k^b = \frac{2\pi}{\hbar} T_{ab} k^a k^b \qquad \forall \text{ null } k^a$$

$$R_{ab} + \Phi g_{ab} = \frac{2\pi}{\hbar\alpha} T_{ab}$$

Impose $\nabla_a T^{ab} = 0$ [1st law: energy conservation]

$$0 = \tfrac{1}{2}\nabla_b R + \nabla_b \Phi \quad \text{so} \quad \Phi = -\tfrac{1}{2}R + \Lambda$$

cosmological constant!

$$\Longrightarrow \quad G_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\alpha} T_{ab}$$

The Einstein eqn of state.

N.B. $\alpha = \frac{1}{4\hbar G_N} = \frac{1}{4 l_p^2}$

$\boxed{Q:}$ Was this a fluke?

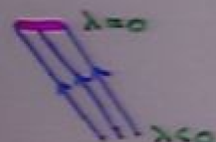
Does it work for any gravitational theory?

$\boxed{2}$ Beyond Einstein gravity:

Non-eq$^{\underline{m}}$ thermodynamics

[Eling, Guedens, Jacobson
gr-qc/0602001]

1. Taylor expand:



$$\delta S = S(0) - S(\lambda) = -\lambda \frac{dS}{d\lambda}\Big|_0 - \frac{1}{2}\lambda^2 \frac{d^2S}{d\lambda^2}\Big|_0 \cdots$$

$$\delta Q = Q(0) - Q(\lambda) = -\lambda \underbrace{\frac{dQ}{d\lambda}\Big|_0}_{0} - \frac{1}{2}\lambda^2 \underbrace{\frac{d^2Q}{d\lambda^2}\Big|_0}_{-\frac{2\pi\kappa}{\hbar}\int T_{ab}k^a k^b \, dA} \cdots$$

Before
$$\delta Q = -\frac{2\pi\kappa}{\hbar}\int T_{ab}k^a k^b \lambda \, d\lambda \, dA$$

2. For general $S = \int s \underbrace{\sqrt{h}\, d^2\Sigma}_{dA}$

entropy density

$$\frac{dS}{d\lambda} = \int \left(\frac{ds}{d\lambda} + s\theta\right)\sqrt{h}\, d^2\Sigma \quad \Rightarrow \quad \theta = -\frac{1}{s}\frac{ds}{d\lambda} \text{ at } \lambda = 0 \, !$$

Nonzero expansion at P in general

$$\boxed{\text{Exponential relaxation to equilibrium:}}$$

Killing time v $\quad (\lambda = -e^{-v})$

$$\tilde{\theta} = \theta \frac{d\lambda}{dv} = \theta e^{-v}$$

$\lambda \to 0$
$v \to -\infty$

Einstein
$\theta \sim \lambda$; $\underline{\tilde{\theta} \sim e^{-2v}}$

$$\delta S = \frac{\delta Q}{T}$$

General case
$\theta \sim$ const.; $\underline{\tilde{\theta} \sim e^{-v}}$

$$\delta S = \frac{\delta Q}{T} + \delta S_{\text{internal}}$$

At next order

$$\left.\frac{d^2S}{d\lambda^2}\right|_0 = \int\left(\frac{d^2s}{d\lambda^2} + \frac{ds}{d\lambda}\Theta + s\frac{d\Theta}{d\lambda}\right)\sqrt{h}\,d^2\Sigma$$

$$= \int\left[(\nabla_a\nabla_b s - sR_{ab})k^a k^b \underbrace{- \tfrac{3}{2}s\Theta^2}\right]\sqrt{h}\,d^2\Sigma$$

For special case $\mathcal{L}(R)$ (hence $s(R)$) Jacobson & co. show Bianchi identity only closes without this term:

$\delta S = \frac{\delta Q}{T} + \delta S_i$ Internal entropy production due to bulk viscosity

$\eta = \frac{3}{2} s T$

[(expansion)^2 ✓]

$$\boxed{\text{The general problem is still open!}}$$

* Need to know connection between $\mathcal{L}$ and s

* Deeper conceptual picture

Wald Entropy

$s = \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}$

How to adapt from BH setup to local acceln horizon?

$\boxed{3}$ Wald Entropy for BHs

Ingredients: * Stationary BH with bifurcate Killing horizon.
* General (diff. inv.) Lagrangian

Wald $\Longrightarrow$ BH entropy is Noether charge associated with horizon Killing field.

[gr-qc/9307038]

Expressible as local geometrical quantity.

$\boxed{3 \text{ steps}}$

① Diff. inv. ξ^a $\Rightarrow$ conserved current $j = dQ$, surface charge

$$\int_\Sigma j = \int_{\partial\Sigma} Q$$

[= Noether charge when ξ^a Killing vector]

② $\delta j \Rightarrow$ symplectic current $\Rightarrow \delta H$

Evaluate for ξ^a = horizon Killing field $\Longrightarrow$ 1^{st} law $\int_{\Sigma_0} = \int_\infty$

$$\kappa \delta S = \delta \int_{\Sigma_0} Q[\xi]$$

③ Eliminate dependence on ξ^a

$Q[\xi] \to \kappa \tilde{Q}$

* $\nabla_a \nabla_b \xi_c = -R^d{}_{abc}\xi_d$
* $\xi^a = 0$ on Σ_0 (bifurcation surface)
* $\nabla_a \xi_b = \kappa\, \epsilon_{ab}$ on Σ_0

N.B. Binormal $\epsilon_{ab} = \xi_{[a} N_{b]}$ $\quad \xi_a N^a = -1$

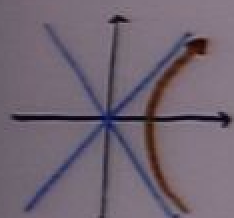
auxiliary null vector

$$\boxed{\text{RESULT:} \quad S = 2\pi \int_{\Sigma_0} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \sqrt{h}\, d^2\Omega \qquad [\text{K,M,J!}]}$$

later: arbitrary cross-section

[more generally, corr for R_{abcd}]

Application to local Rindler horizon?



* Also have bifurcate Killing horizon:

  Identify entropy density as Noether charge density w.r.t. boost Killing field.

* Taylor expansion about bifurcation surface:

  $\epsilon_{ab} = \nabla_a \chi_b$ at $\lambda = 0$

  Killing id's $\Longrightarrow$ can prove $\left.\frac{d\epsilon_{ab}}{d\lambda}\right|_0 = \left.\frac{d^2\epsilon_{ab}}{d\lambda^2}\right|_0 = 0$

So far $\delta S = \frac{\delta Q}{T} + \delta S_i \Longrightarrow s R_{ab} - \nabla_a \nabla_b s = \frac{2\pi}{\hbar} T_{ab} + \Phi g_{ab}$

Fix Φ via Bianchi id. $\quad 0 = \frac{1}{2} s\, \partial_a R + \partial_a (\Box s - \Phi)$

?

Goal:

Input $S = S_{Wald}$

$T \propto \kappa$

* $\delta S = \dfrac{\delta Q}{T} + \delta S_i$
* $\nabla_a T^{ab} = 0$

Correct gravitational field eq$^{\underline{n}}$s as eq$^{\underline{n}}$s of state.

[For ANY theory]

Conclusion?

Watch this space!

[+ feed me coffee ...]