Title: Topos theory in the foundations of physics

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Abstract: At a very basic level, physics is about what we can say about propositions like \'A has a value in S\' (or \'A is in S\' for short), where A is some physical quantity like energy, position, momentum etc. of a physical system, and S is some subset of the real line. In classical physics, given a state of the system, every proposition of the form \'A is in S\' is either true or false, and thus classical physics is realist in the sense that there is a \'way things are\'. In contrast to that, quantum theory only delivers a probability of \'A is in S\' being true. The usual instrumentalist interpretation of the formalism leading to these probabilities involves an external observer, measurements etc.In a future theory of quantum gravity/cosmology, we will have to treat the whole universe as a quantum system, which renders instrumentalism meaningless, since there is no external observer. Moreover, space-time presumably does not have a smooth continuum structure at small scales, and possibly physical quantities will take their values in some other mathematical structure than the real numbers, which are the \'mathematical continuum\'. In my talk, I will show how the use of topos theory, which is a branch of category theory, may help to formulate physical theories in a way that (a) is neo-realist in the sense that all propositions \'A is in S\' do have truth values and (b) does not depend fundamentally on the continuum in the form of the real numbers. After introducing topoi and their internal logic, I will identify suitable topoi for classical and quantum physics and show which structures within these topoi are of physical significance. This is still very far from a theory of quantum gravity, but it can already shed some light on ordinary quantum theory, since we avoid the usual instrumentalism. Moreover, the formalism is general enough to allow for major generalisations. I will conclude with some more general remarks on related developments.

Topos Theory in the Foundations of Physics

Colloquium, Perimeter Institute
5. March 2008

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"A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it."

(Unknown)

Motivation

The biggest open problem in fundamental physics is the formulation of a predictive, experimentally testable theory of **quantum gravity** and **quantum cosmology**. This has proven to be very hard.

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Apart from technical questions, there are a number of deep conceptual problems. Two of them are:

- The mathematical formalism of quantum theory is usually interpreted in an **instrumentalist** manner.
- All physical structures used are based on the idea of a **continuum**. Mathematically, the real and complex numbers are at the heart of all constructions (Hilbert spaces, operators, path integrals, strings,...).

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- A more **realist** formulation of QT is needed.

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- This means that when we try to formulate QG using mathematical structures based on the real and complex numbers, we possibly make a category error.
- But: all major approaches to QG use more-or-less standard quantum theory.
- Ideally, we would like a framework for the formulation of physical theories that does not fundamentally depend on the real numbers.

Topos theory as a new mathematical framework

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In this talk, I want to show that topos theory allows to formulate physical theories in a way that

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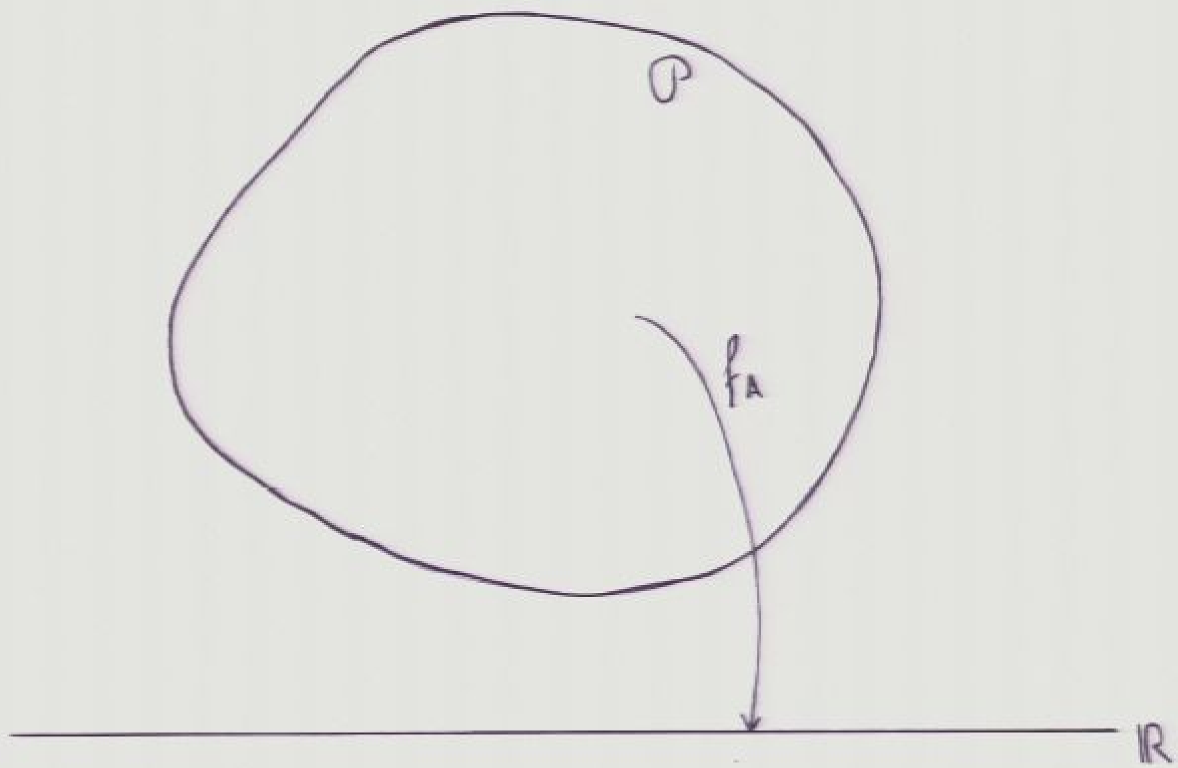
- does not (fundamentally) depend on the real numbers.

Of course, a theory of quantum gravity is still a long way off. I will mostly concentrate on ordinary (algebraic) quantum theory.

State spaces and Boolean logic

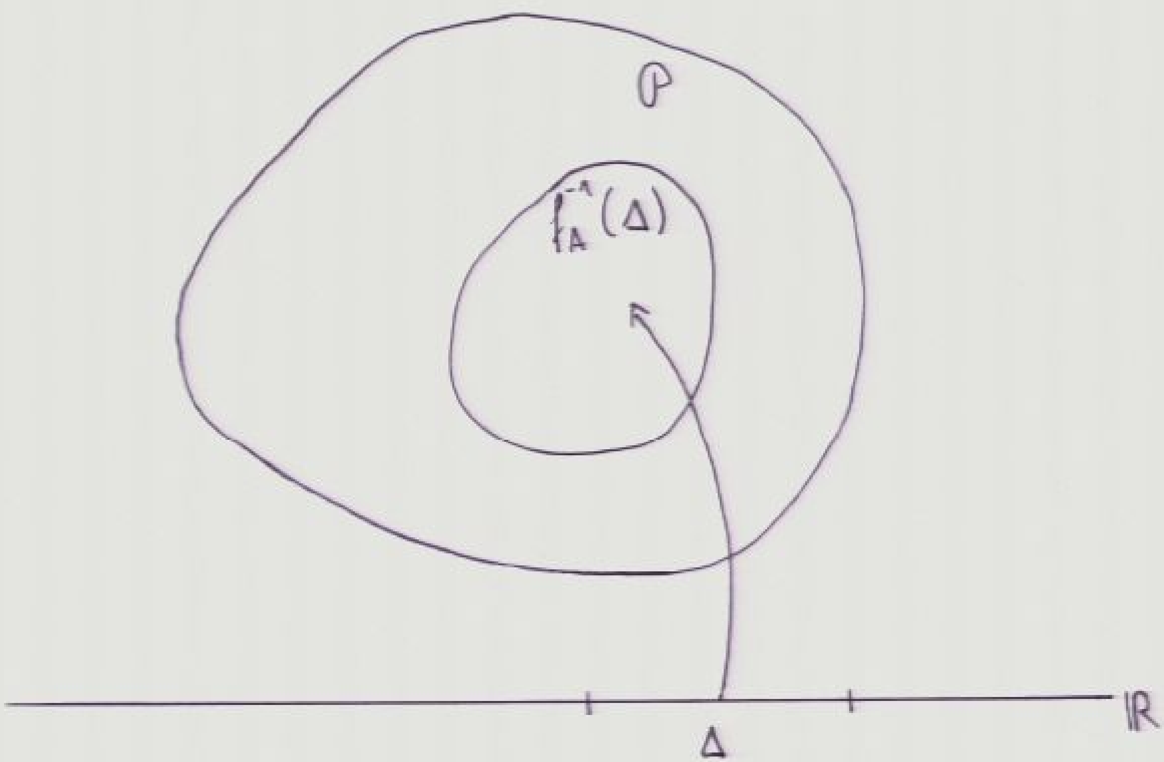
In classical physics, physical quantities/observables A are described by real-valued functions f_A on the state space $\mathcal{P}$, that is, mappings



Points of $\mathcal{P}$ are states. In a given state, all physical quantities have values.

State spaces and Boolean logic

One can also consider inverse images of (Borel) subsets $\Delta \subseteq \mathbb{R}$:



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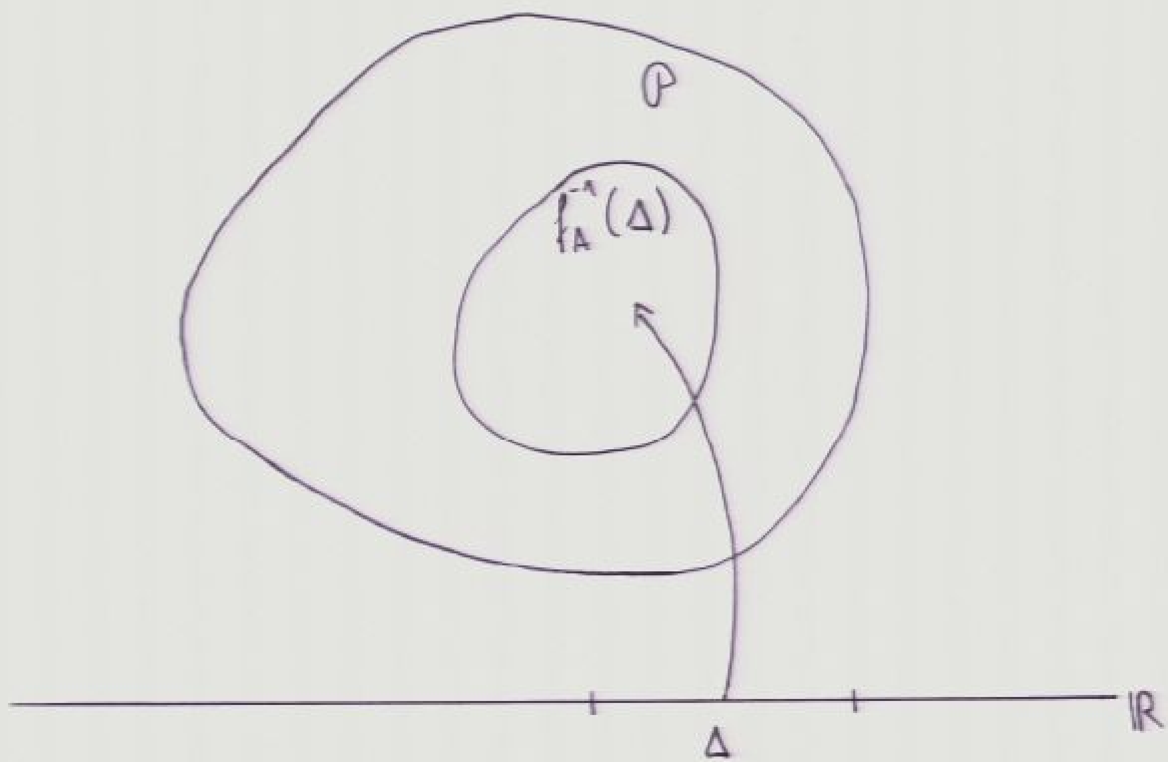

Such a subset of the state space $\mathcal{P}$ corresponds to a **proposition** "$A \in \Delta$", that is, "the physical quantity A has a value lying in the set Δ".

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- $f_A^{-1}(\Delta_1) \cap f_B^{-1}(\Delta_2)$ corresponds to the proposition "$A \in \Delta_1$ **and** $B \in \Delta_2$"
- $f_A^{-1}(\Delta_1) \cup f_B^{-1}(\Delta_2)$ corresponds to the proposition "$A \in \Delta_1$ **or** $B \in \Delta_2$"
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Classical physics is based on a **Boolean logical structure**, namely the Boolean algebra of subsets of state space.

Realism, Boolean logic and sets

Classical physics is a **realist** theory. In a given state $s \in \mathcal{P}$, all physical quantities have values, and all propositions have truth-values. Logical formulas involving propositions can be manipulated according to the rules of a **deductive system**. These are the rules of classical, Boolean logic, which is closely tied to the use of sets:

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- Stone '36: Every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of a suitable space.

The Kochen-Specker theorem

Problem: Is there a realist formulation of quantum theory similar to classical physics?

More concretely, is there a 'state space' for a quantum system such that physical quantities are real-valued functions on this space? We require that the self-adjoint operators in the von Neumann algebra $\mathcal{R}$ of physical quantities correspond to functions on the (hypothetical) state space.

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Kochen, Specker 1967: If the von Neumann algebra $\mathcal{R}$ of physical quantities of a quantum system consists of all bounded operators on Hilbert space, $\mathcal{R} = \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, then there exists no state space model of QT (under very natural conditions). It is impossible to assign real values to all physical quantities at once.

D 2005: This also holds for all von Neumann algebras $\mathcal{R}$ without summands of type I_1 and I_2, i.e., for all quantum systems with symmetries and/or superselection rules.

The Kochen-Specker theorem

The spectral theorem shows that to each proposition "$A \in \Delta$", there exists a projection operator $\widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$. (Projection operators are self-adjoint and idempotent.)

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This means that we cannot use Boolean logic to describe quantum systems in a realist manner.

Ordinary Quantum Logic

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- At first sight, this is similar to a classical propositional calculus with the Hilbert space $\mathcal{H}$ taking the role of the quantum state space analogue.
- Severe interpretational problem: if $\dim \mathcal{H} > 1$, then $L(\mathcal{H})$ is **non-distributive**. Example: the "quantum breakfast"

$$E \wedge (B \vee S) \neq (E \wedge B) \vee (E \wedge S).$$

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- There are many further developments in **quantum logic**, but these are somewhat detached from physics.
- In particular, a viable deductive system is lacking.

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Given a physical system S, look for suitable objects $\Sigma, \mathcal{R}$ in a category (bear with me!) such that

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We want the subobjects of Σ to have a logical structure, while allowing for something more general than Boolean logic. We also want a deductive system. This suggests the use of **topoi**.

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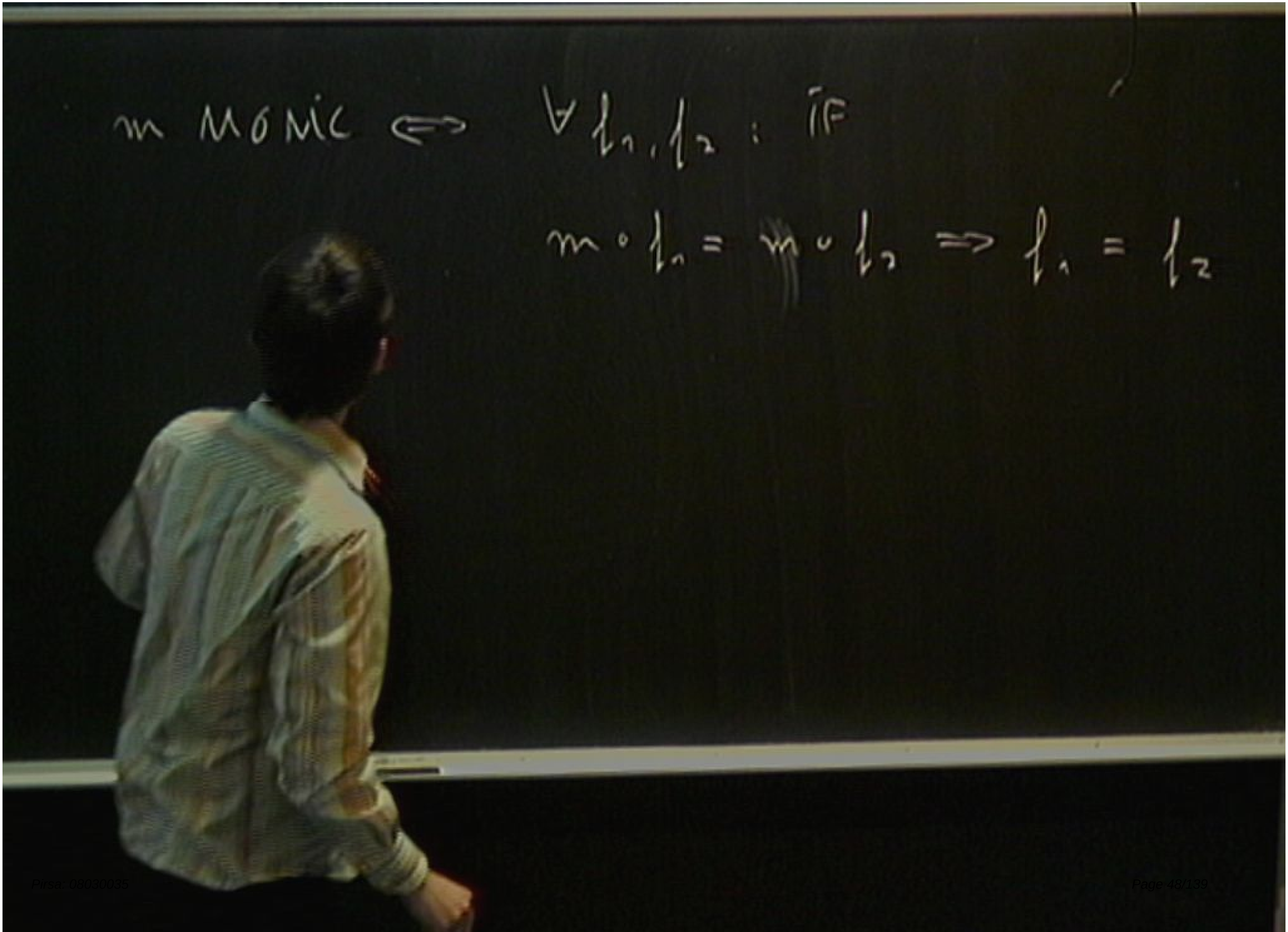
Examples of categories

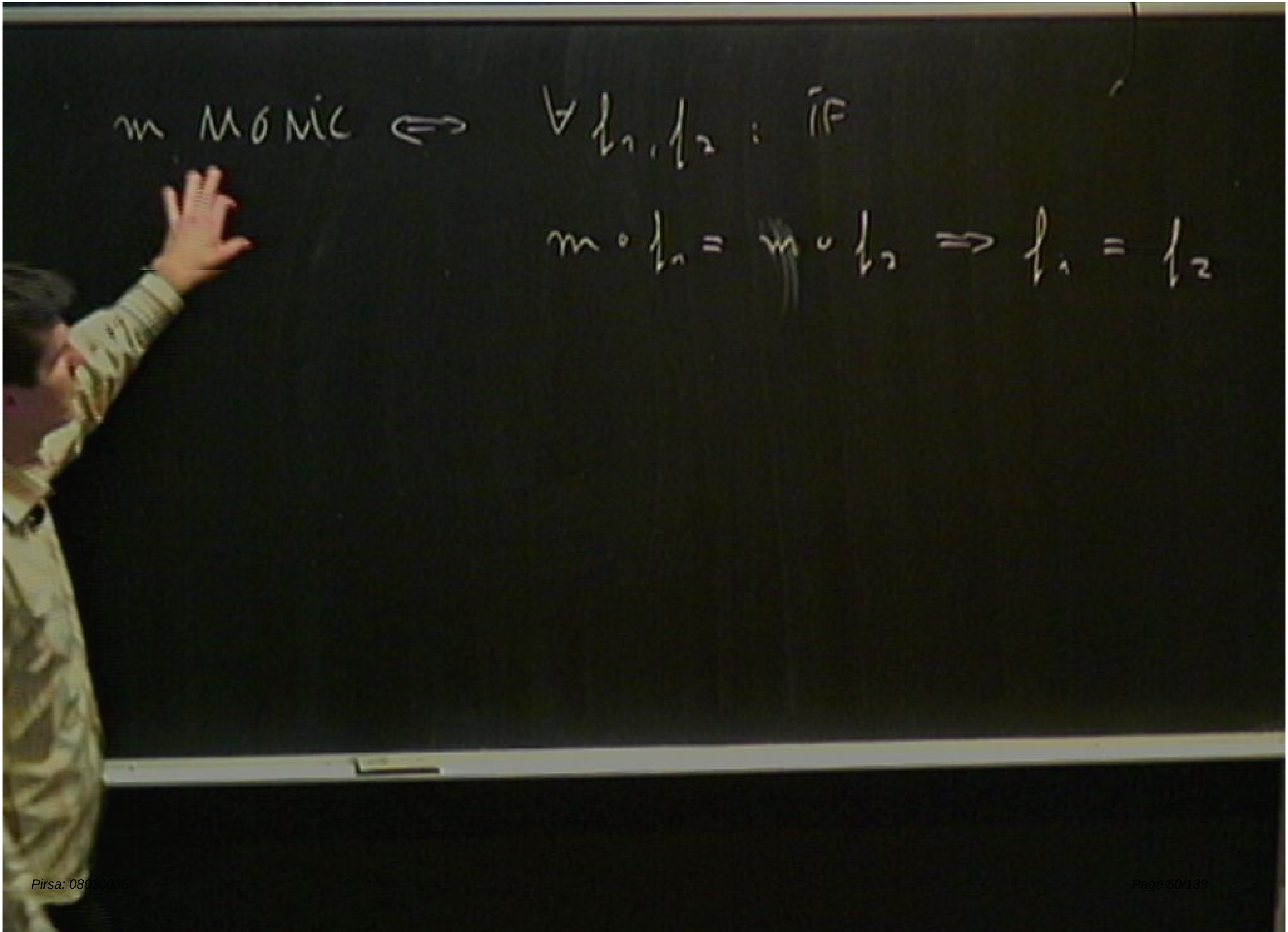
Many mathematical structures are categories:

- a set is a category: objects are elements, arrows are (only) identities


m MONIC ⟺ ∀ f1, f2 : IF
m ∘ f1 = m ∘ f2 ⟹ f1 = f2

m MONIC $\Longleftrightarrow$ $\forall f_1, f_2$: IF

$$m \circ f_1 = m \circ f_2 \Rightarrow f_1 = f_2$$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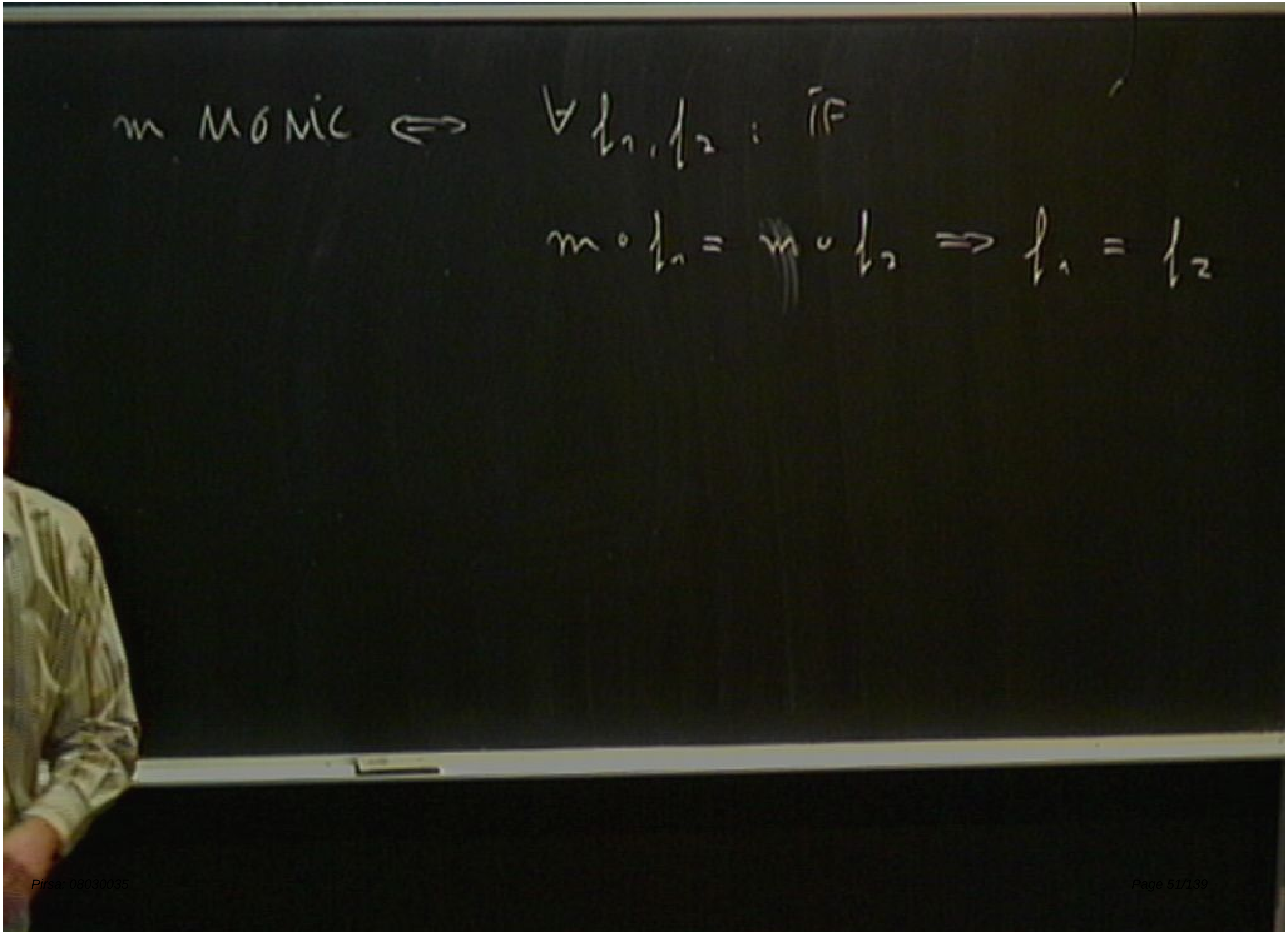
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- a partially ordered set is a category: objects are elements of the poset, an arrow $s : A \to B$ exists if and only if $A \leq B$.
- (small) sets and functions form a category Sets. This category usually serves as our mathematical universe.

Definition of a topos

A **topos** is a category that is similar to Sets. Specifically, in Sets, we can form new sets from given ones in several ways: let S, T be two sets, then there is the cartesian product $S \times T$, the disjoint union $S \amalg T$ and the exponential S^T, the set of all functions from T to S.

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Moreover, each topos has a so-called **subobject classifier**. This is a special object Ω in the topos that generalises the set $\{0, 1\}$ of truth-values in the category Sets. (Of course, Sets is a topos, too.)

The subobject classifier

The textbook definition is:

Def.: In a category $\mathcal{C}$ with finite limits, a **subobject classifier** is a monic, true : $1 \to \Omega$, such that to every monic $S \to X$ in $\mathcal{C}$ there is a unique morphism ϕ which, with the given monic, forms a pullback diagram:

$$
\begin{array}{ccc}
S & \longrightarrow & 1 \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\
X & \xrightarrow[\phi]{} & \Omega
\end{array}
$$

For the case of sets and functions, this boils down to something very familiar.

Topos logic

The subobject classifier endows a topos with an **internal logic**, which is of *intuitionistic* type. This means that the law of excluded middle need not hold: if E is a proposition, then

$$E \vee \neg E \leq 1,$$

instead of $E \vee \neg E = 1$ as in Boolean logic. Here, 1 represents the truth-value 'totally true'.

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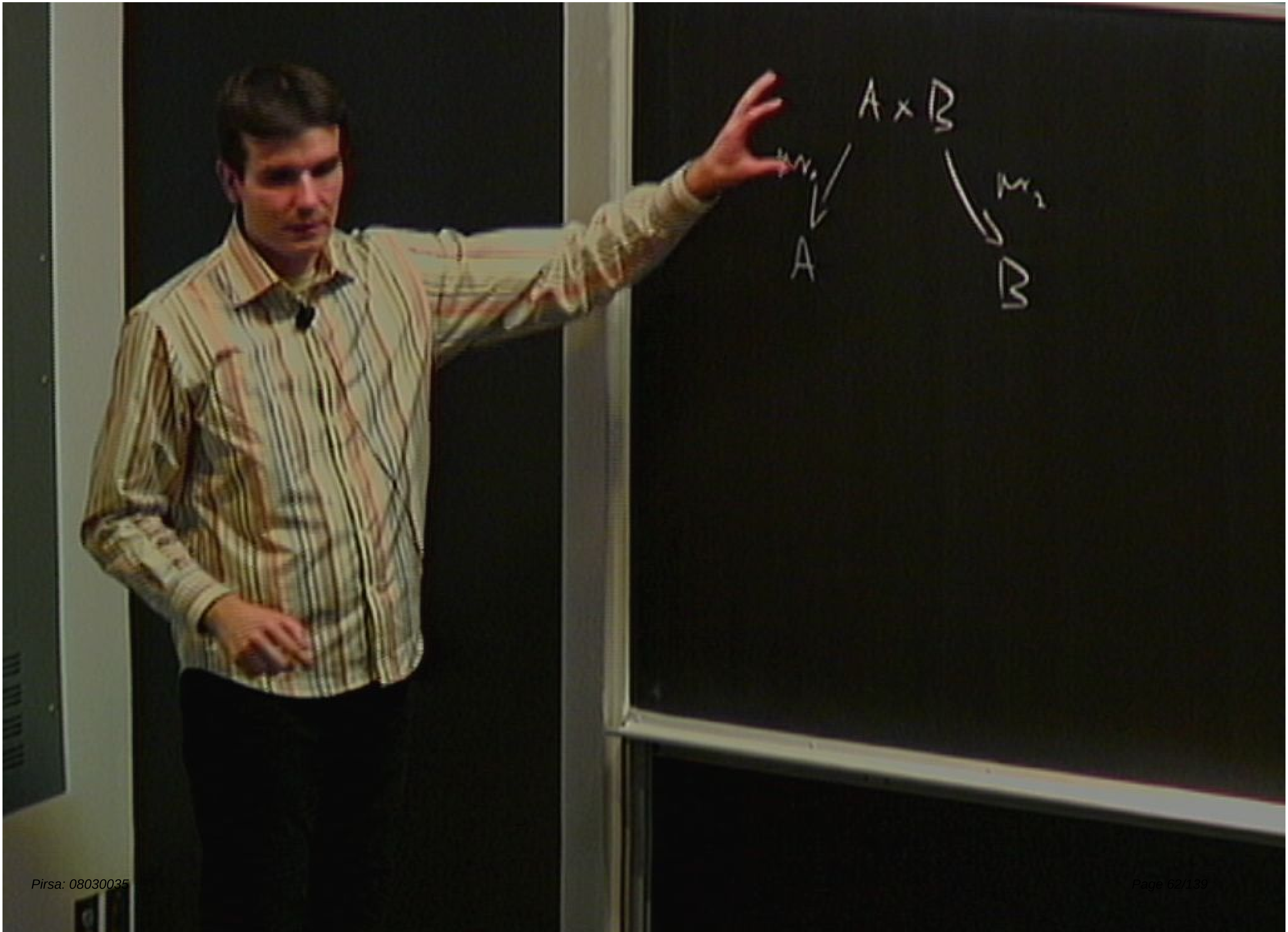
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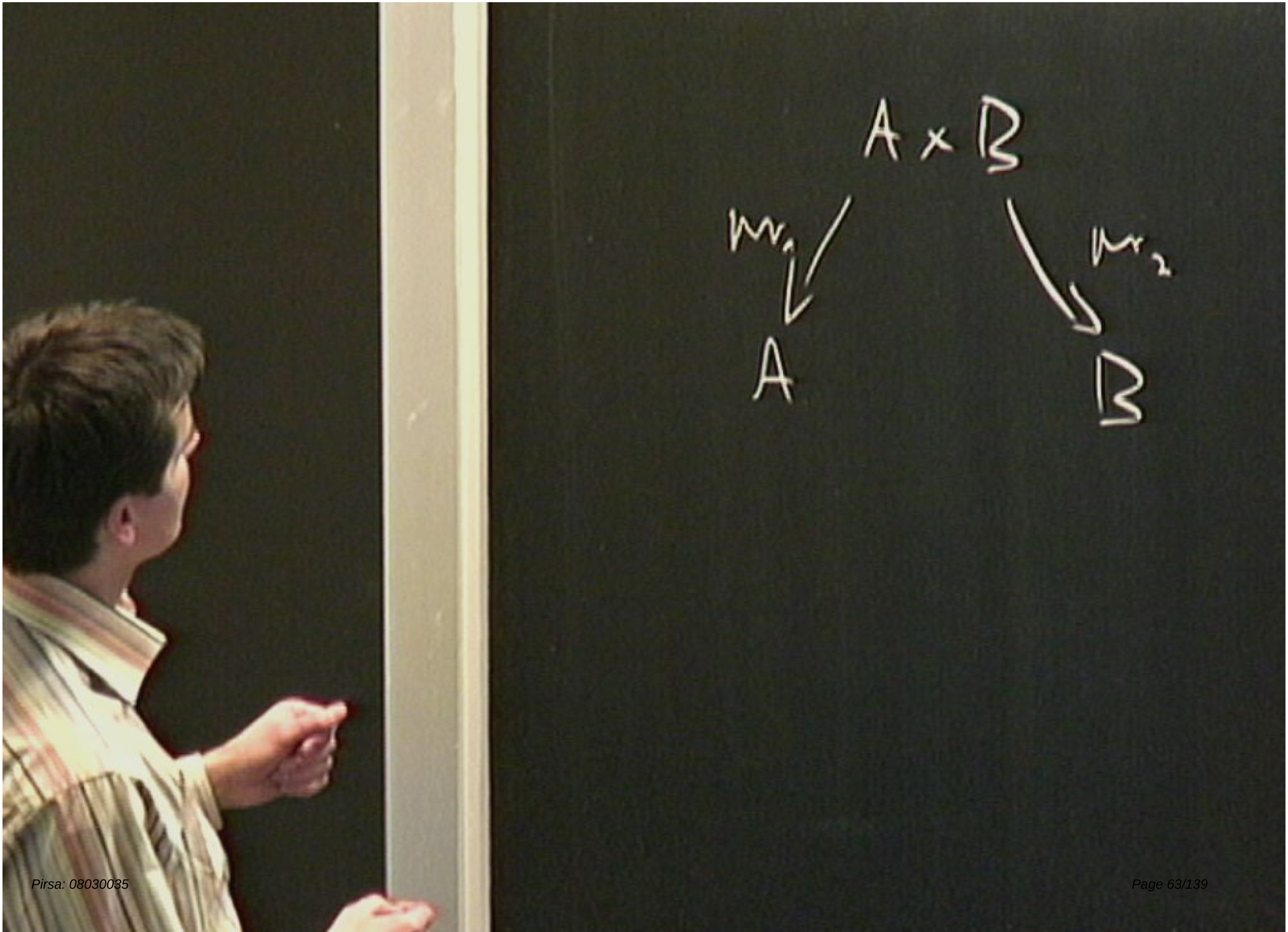
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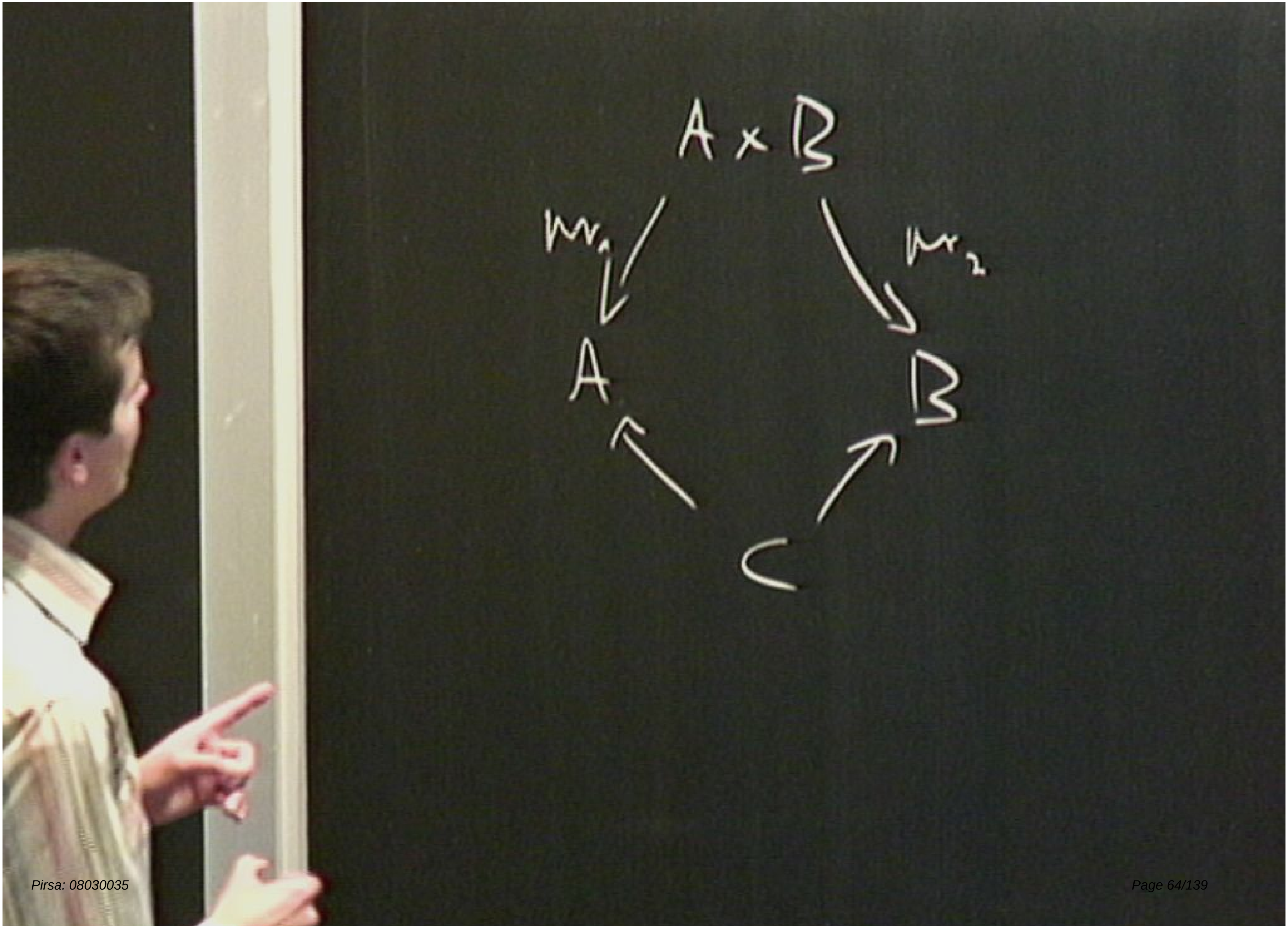
A × B
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pr2
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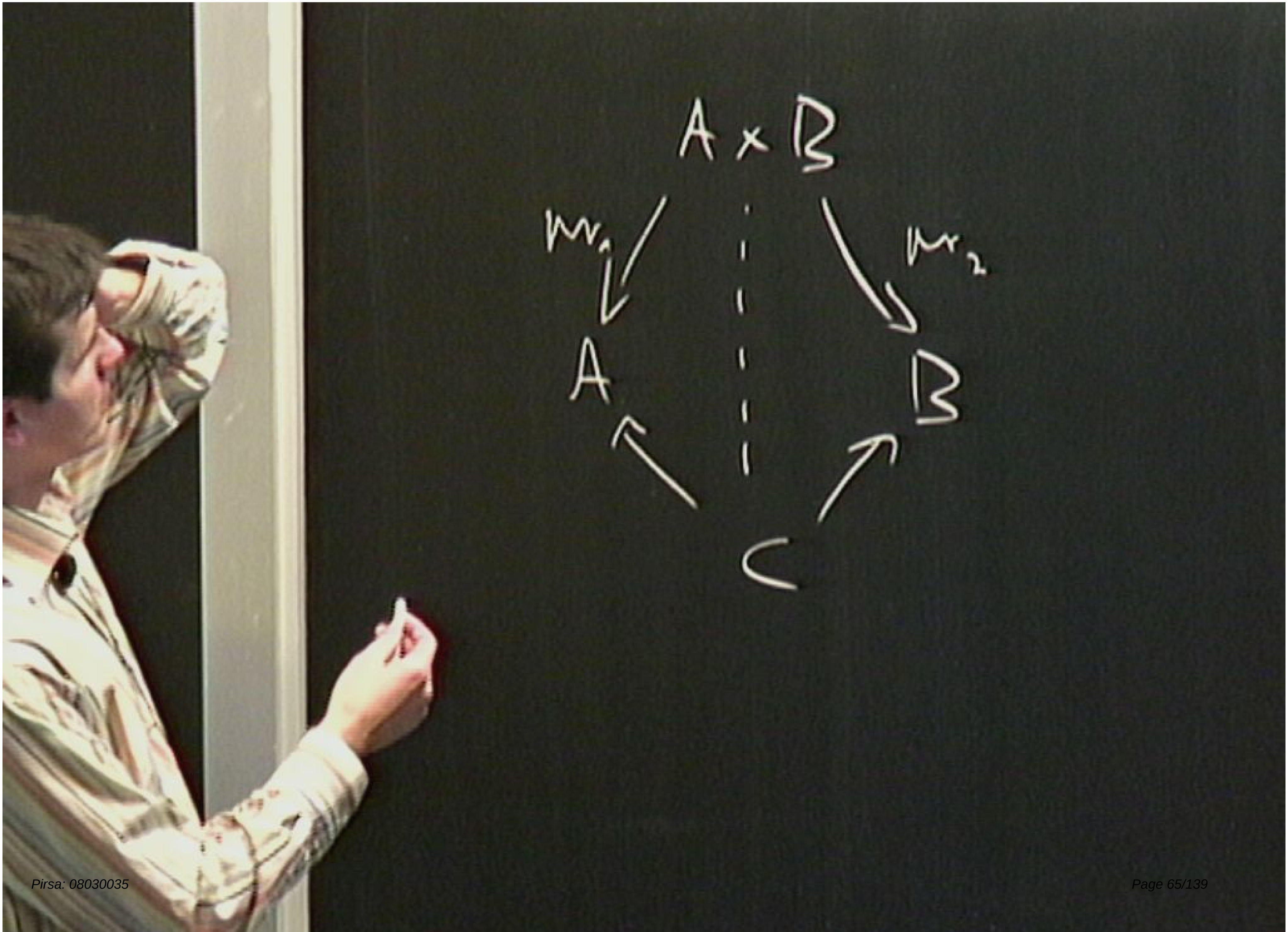
C


A × B
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B
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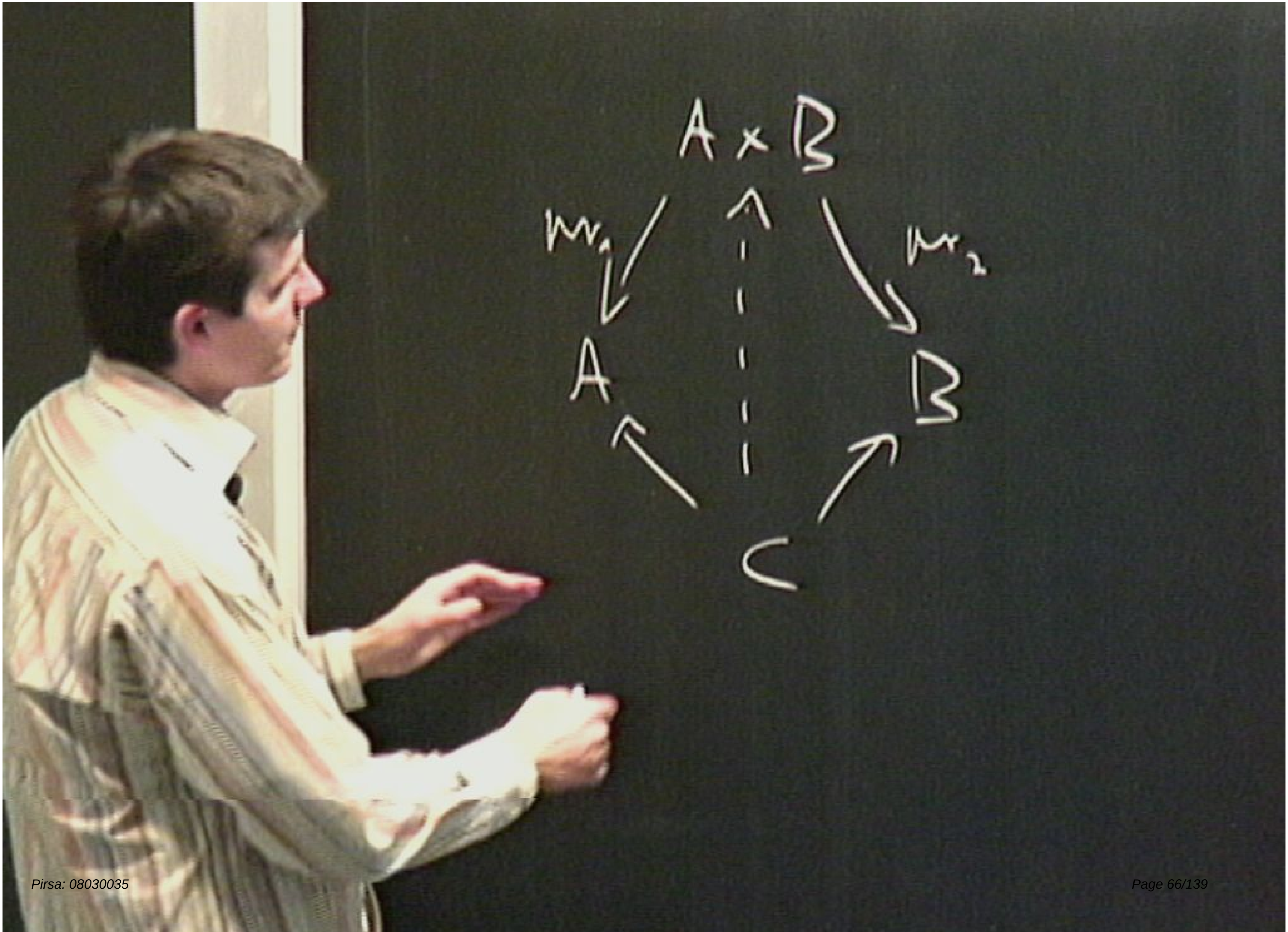

A × B
pr₁
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A
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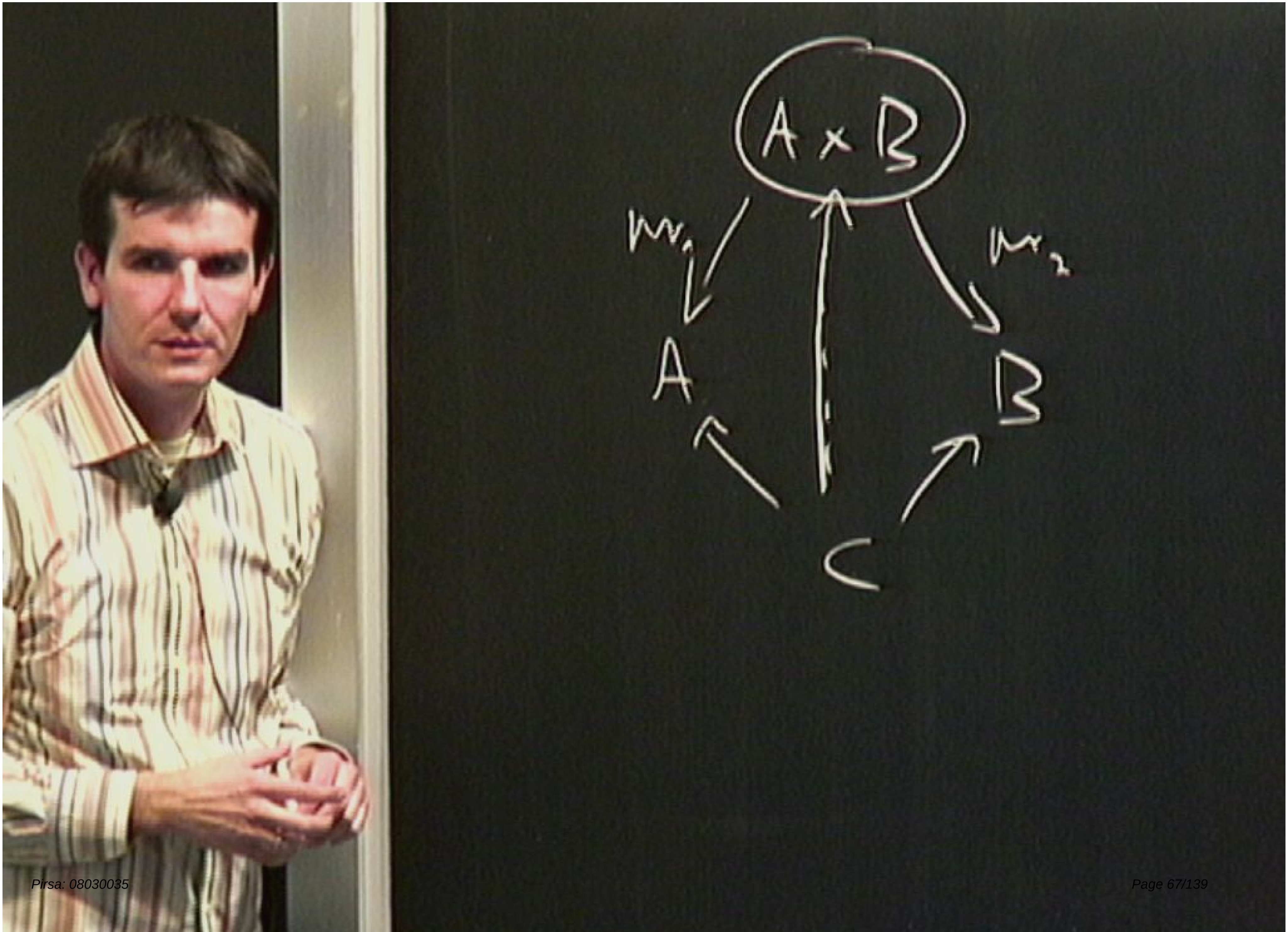
A × B
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Topos logic

The subobject classifier endows a topos with an **internal logic**, which is of *intuitionistic* type. This means that the law of excluded middle need not hold: if E is a proposition, then

$$E \vee \neg E \leq 1,$$

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This type of logic seems very reasonable with respect to propositions about quantum systems. Moreover, we have distributivity:

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The subsets of a set form a Boolean algebra, and the subobjects of an object in a topos form a **Heyting algebra**.

Contexts or Weltanschauungen

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- Commutative subalgebras of $\mathcal{R}$ are called **contexts**. They are like 'classical snapshots' of the quantum system.
- Some kind of *contextual* model of QT is needed (but with good control of the relations between contexts).

The context category

All constructions we will use work for an arbitrary von Neumann algebra $\mathcal{R}$. For simplicity, we concentrate on $\mathcal{R} = \mathcal{B}(\mathcal{H})$. The Hilbert space $\mathcal{H}$ can be infinite-dimensional.

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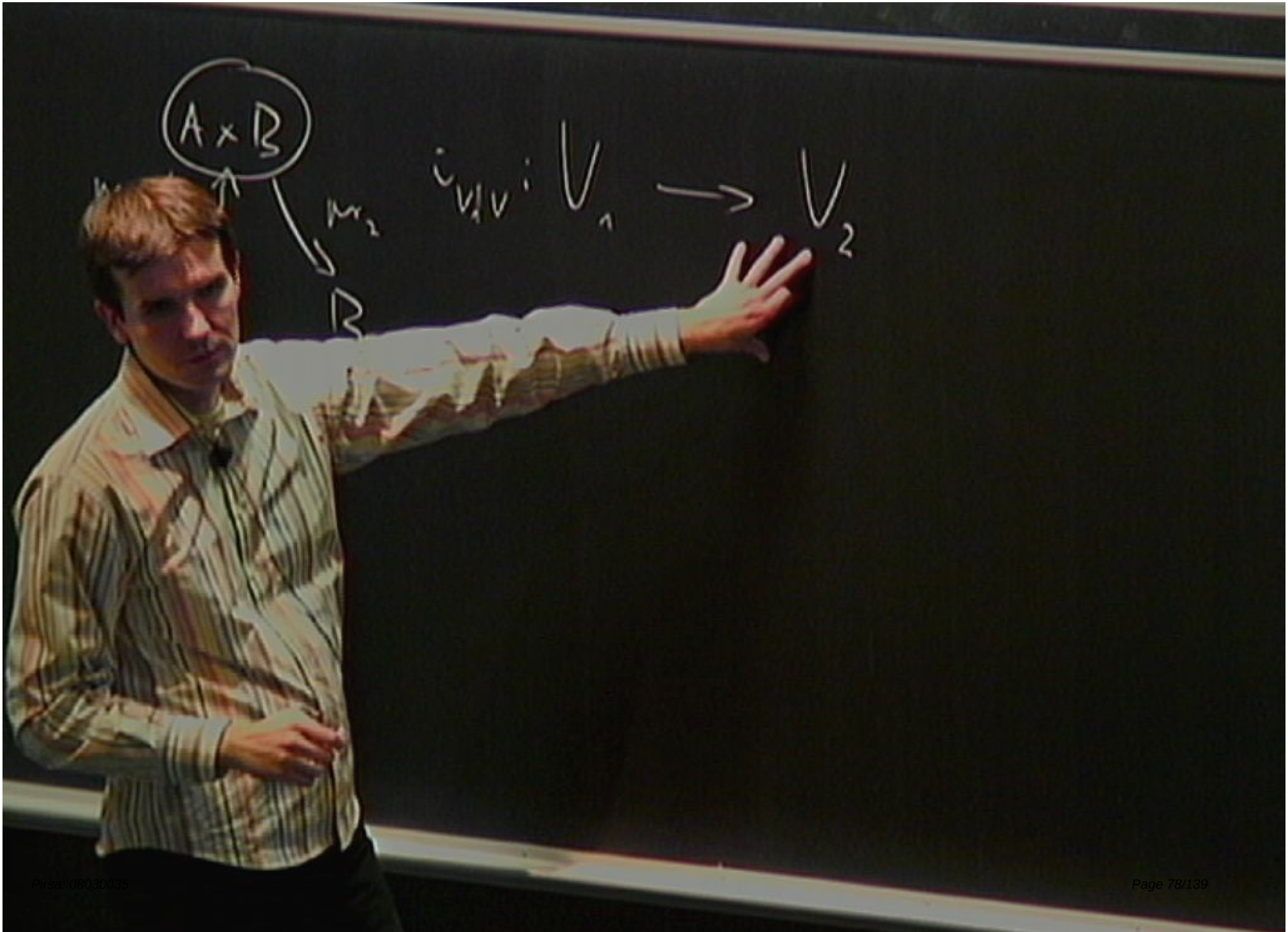
We consider the category $\mathcal{V}(\mathcal{H})$ of non-trivial commutative von Neumann subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ of physical quantities. This is a partially ordered set under inclusion and is called the **context category**.


A × B
B
V1 → V2

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This category 'knows about' the relations between the commutative subalgebras of $\mathcal{B}(\mathcal{H})$: if $V_1, V_2 \in \mathcal{V}(\mathcal{H})$ and $V := V_1 \cap V_2$, then there are arrows $i_{VV_1} : V \to V_1$ and $i_{VV_2} : V \to V_2$.

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Going from a commutative algebra V to a smaller algebra $V' \subset V$ is a process of **coarse-graining**: V' contains less self-adjoint operators than V, so we can describe less physics in it.

Gel'fand spectrum and Gel'fand transformation

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$$\begin{aligned} V &\longrightarrow C(\underline{\Sigma}_V) \\ \widehat{A} &\longmapsto \overline{A}, \end{aligned}$$

where, for all $\lambda \in \underline{\Sigma}_V$, we have $\overline{A}(\lambda) := \lambda(\widehat{A})$. If $\widehat{A}$ is self-adjoint, then $\overline{A}$ is real-valued.

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The Gel'fand spectrum $\underline{\Sigma}_V$ has all the properties of a **local state space** at V.

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We now form a global object from all the local state spaces: to each $V \in \mathcal{V}(\mathcal{H})$, we assign its Gel'fand spectrum $\underline{\Sigma}_V$.

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$\underline{\Sigma}$ is a contravariant functor from the context category $\mathcal{V}(\mathcal{H})$ to the category Sets, i.e., a **presheaf over** $\mathcal{V}(\mathcal{H})$. We regard the **spectral presheaf** $\underline{\Sigma}$ as a quantum analogue of state space.

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The presheaves over the context category $\mathcal{V}(\mathcal{H})$ form a topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$. This is the topos associated to the quantum physical system, and $\underline{\Sigma}$ is the state object within this topos.

Reformulation of the Kochen-Specker theorem

Does $\underline{\Sigma}$ have elements, so is it like a set? An element x of a *set* S can be seen as a function $x : 1 \to S$, where $1 = \{*\}$ denotes a set with one element.

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Isham, Butterfield '98: a global element of $\underline{\Sigma}$ would allow to assign values to all physical quantities at once, which is impossible due to the Kochen-Specker theorem. So we have a reformulation of the KS theorem:

Thm.: The spectral presheaf $\underline{\Sigma}$ has no global elements.

Representation of physical quantities

We want to represent physical quantities as arrows in our quantum topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$. These arrows will go from $\underline{\Sigma}$, the state object, to some quantity-value object $\underline{\mathcal{R}}$, yet to be specified.

f_1, f_2

$$m \circ f_1 = m \circ f_2$$

$$\underline{\Sigma} \longrightarrow \underline{\mathcal{R}}$$

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For each self-adjoint operator $\widehat{A}$ and each context V, we have to define a function

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and for each $\widehat{A}$, these functions must fit together to form a natural transformation.

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Daseinisation of self-adjoint operators

Let $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$. From the spectral family $\widehat{E}^A = (\widehat{E}^A_\lambda)_{\lambda\in\mathbb{R}}$, we obtain a new spectral family in $\mathcal{P}(V)$ by defining

$$\forall\lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta(\widehat{A})_V} := \bigvee\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \leq \widehat{E}^A_\lambda\}.$$

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Similarly, we can define

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The quantity-value object $\underline{\mathbb{R}^{\leftrightarrow}}$

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Clearly, there is a similar construction giving a presheaf $\underline{\mathbb{R}^{\preceq}}$ of order-preserving functions, and for each $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$, a natural transformation $\breve{\delta}^{i}(\widehat{A}) : \underline{\Sigma} \to \underline{\mathbb{R}^{\preceq}}$, constructed using inner daseinisation of $\widehat{A}$.

We can combine both presheaves into one presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$. We consider this presheaf as the quantity value-object for QT.

Physical quantities, i.e., self-adjoint operators $\widehat{A}$, are represented by arrows $\breve{\delta}(\widehat{A}) : \underline{\Sigma} \to \underline{\mathbb{R}^{\leftrightarrow}}$. This is structurally similar to classical physics (but quite different from ordinary quantum theory).

Subobjects from inverse images

A proposition of the form "$A \in \Delta$" refers to the real numbers, since $\Delta \subset \mathbb{R}$. The real numbers lie *outside* the topos $\mathrm{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$ (resp. the formal language, see later).

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Now that we have defined $\underline{\mathbb{R}^{\leftrightarrow}}$, we can construct subobjects of $\underline{\Sigma}$ by taking inverse images: let $\underline{\Theta}$ be a subobject of $\underline{\mathbb{R}^{\leftrightarrow}}$, then $\breve{\delta}(\widehat{A})^{-1}(\underline{\Theta})$ is a subobject of $\underline{\Sigma}$.

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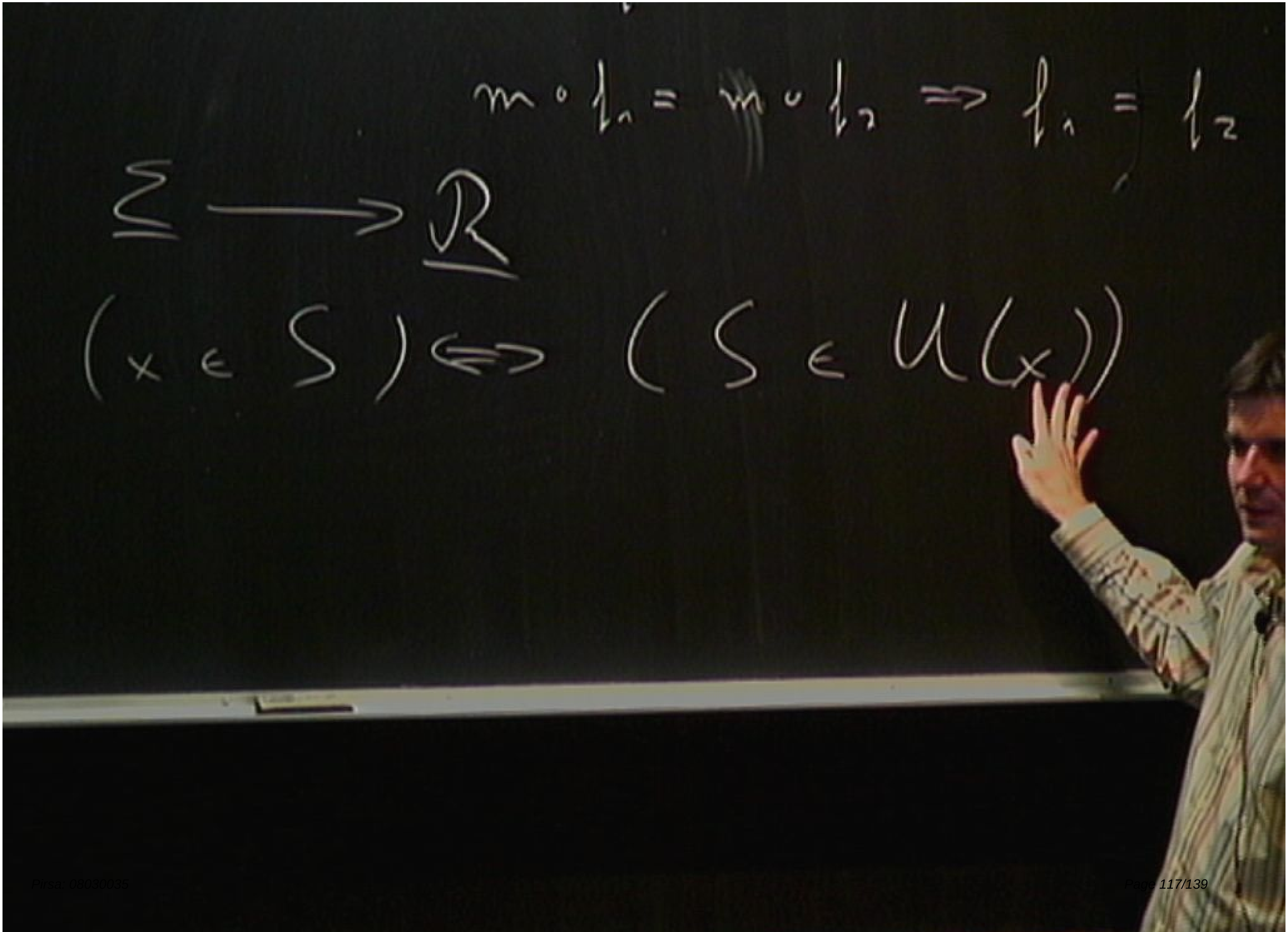
In this way, we get a topos-internal construction of propositions that do not refer to the real numbers. The 'meaning' of such propositions must be discussed from 'within the topos'.

Pure states and truth objects

In classical theory, a pure state is nothing but a point of state space.

Since the spectral presheaf $\underline{\Sigma}$ has no points, we must use another description for (pure) states, namely by certain elements of $P(P\underline{\Sigma})$. (In classical theory, both descriptions agree.)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Let ψ be a unit vector in Hilbert space. For each $V \in \mathcal{V}(\mathcal{H})$, we define

$$\begin{aligned} \mathbb{T}^{\psi}(V) &:= \{S \subseteq \underline{\Sigma}_V \mid \langle\psi|\widehat{P}_S|\psi\rangle = 1\} \\ &= \{S \subseteq \underline{\Sigma}_V \mid \widehat{P}_S \geq \delta(\widehat{P}_{\psi})_V\}. \end{aligned}$$

The subobject classifier in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$

The subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of **sieves**.

A sieve in a poset like $\mathcal{V}(\mathcal{H})$ is particularly simple: let $V \in \mathcal{V}(\mathcal{H})$. A sieve σ on V is a collection of subalgebras $V' \subseteq V$ such that, whenever $V' \in \sigma$ and $V'' \subset V'$, then $V'' \in \sigma$ (so σ is a downward closed set).

The **maximal sieve** on V is $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

A truth-value is a global element of the presheaf $\underline{\Omega}$.

The global element consisting entirely of maximal sieves is interpreted as 'totally true', the global element consisting of empty sieves as 'totally false'. There are many other global elements of $\underline{\Omega}$, interpreted as truth-values between 'totally false' and 'totally true'.

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A truth-value is a global element of the presheaf $\underline{\Omega}$.

The global element consisting entirely of maximal sieves is interpreted as 'totally true', the global element consisting of empty sieves as 'totally false'. There are many other global elements of $\underline{\Omega}$, interpreted as truth-values between 'totally false' and 'totally true'.

Truth values from truth objects

We saw that subobjects of $\underline{\Sigma}$ represent propositions about the physical system under consideration, and that states are represented by truth objects.

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- overcome the direct dependence on the continuum: $\underline{\mathcal{R}}$ is *not* the real-number object in the topos
- achieve a structural similarity between classical and quantum physics, previously not given.

Related work and outlook

Recently, Klaasman, Spitters and Heunen realised that the spectral presheaf can be understood as the Gel'fand spectrum of an topos-internal $C*$-algebra. This is based on major work by Mulvey and Banaschewski.

Open problems and goals

There are many interesting open questions in the topos programme. Some of the things we are working on are:

- description of time evolution
- topos formulation of uncertainty relations
- composite systems and entanglement
- internal vs. external formulations
- space-time concepts
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References

A. Döring, "Kochen-Specker theorem for von Neumann algebras", Int. J. Theor. Phys. **44**, 139-160 (2005)

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Formal languages

There is a very elegant way of describing what we are doing: viz to construct a theory of a physical system S is equivalent to finding a representation in a topos of a certain formal language, $\mathcal{L}(S)$, that is attached to S.

- The language $\mathcal{L}(S)$ will depend on the physical system S, but not on the theory type (classical, quantum, ...).
- The representation will depend on the theory type.
- We allow for a logic that is not Boolean, but still is a deductive system. We choose *intuitionistic* axioms for the language.

For quantum theory, we choose a representation in the topos $\mathrm{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$. *For classical theory, one uses* Sets. *Most importantly, the whole topos scheme allows for major generalisations; in future theories other topoi will play a role.*

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