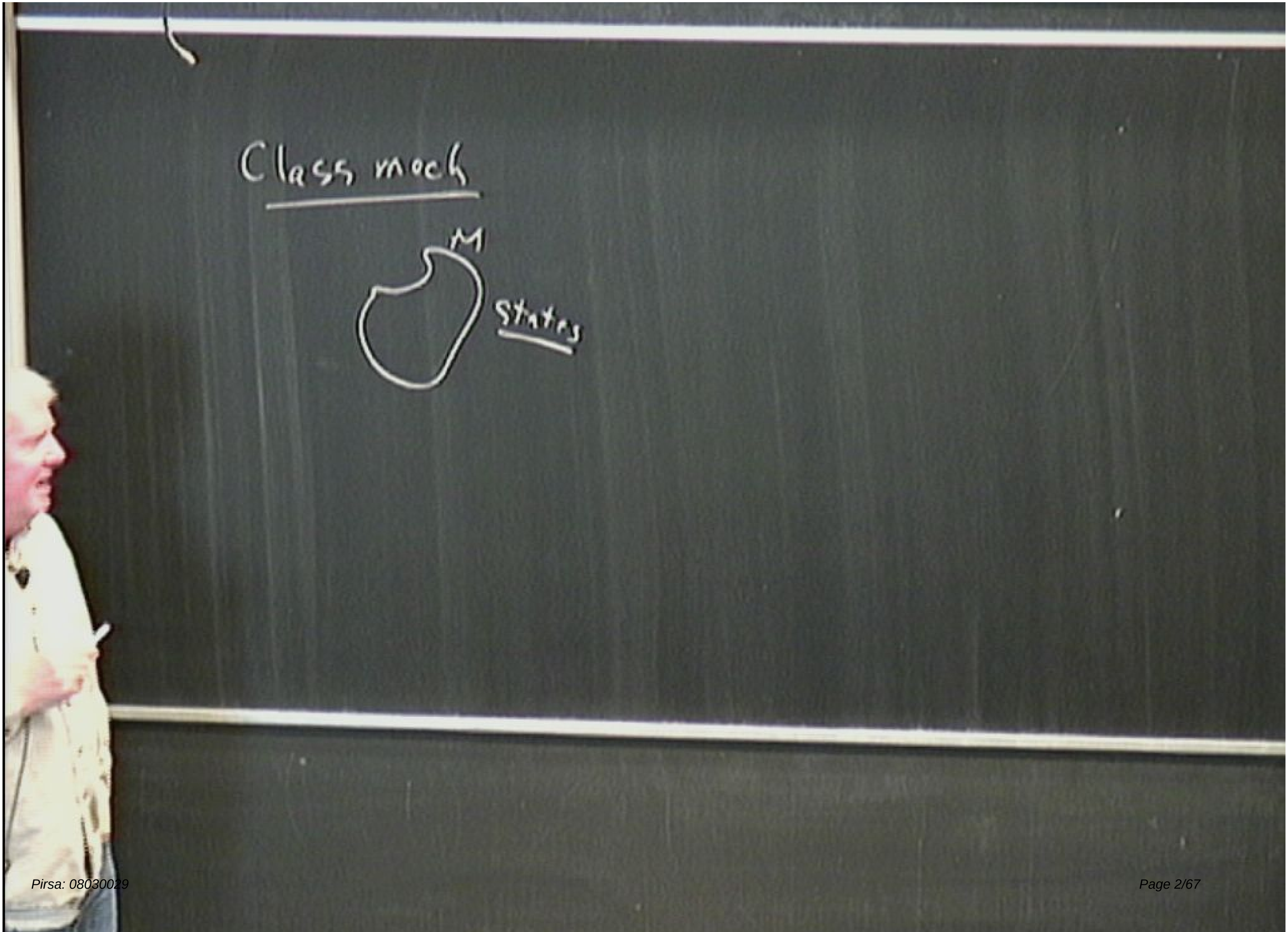
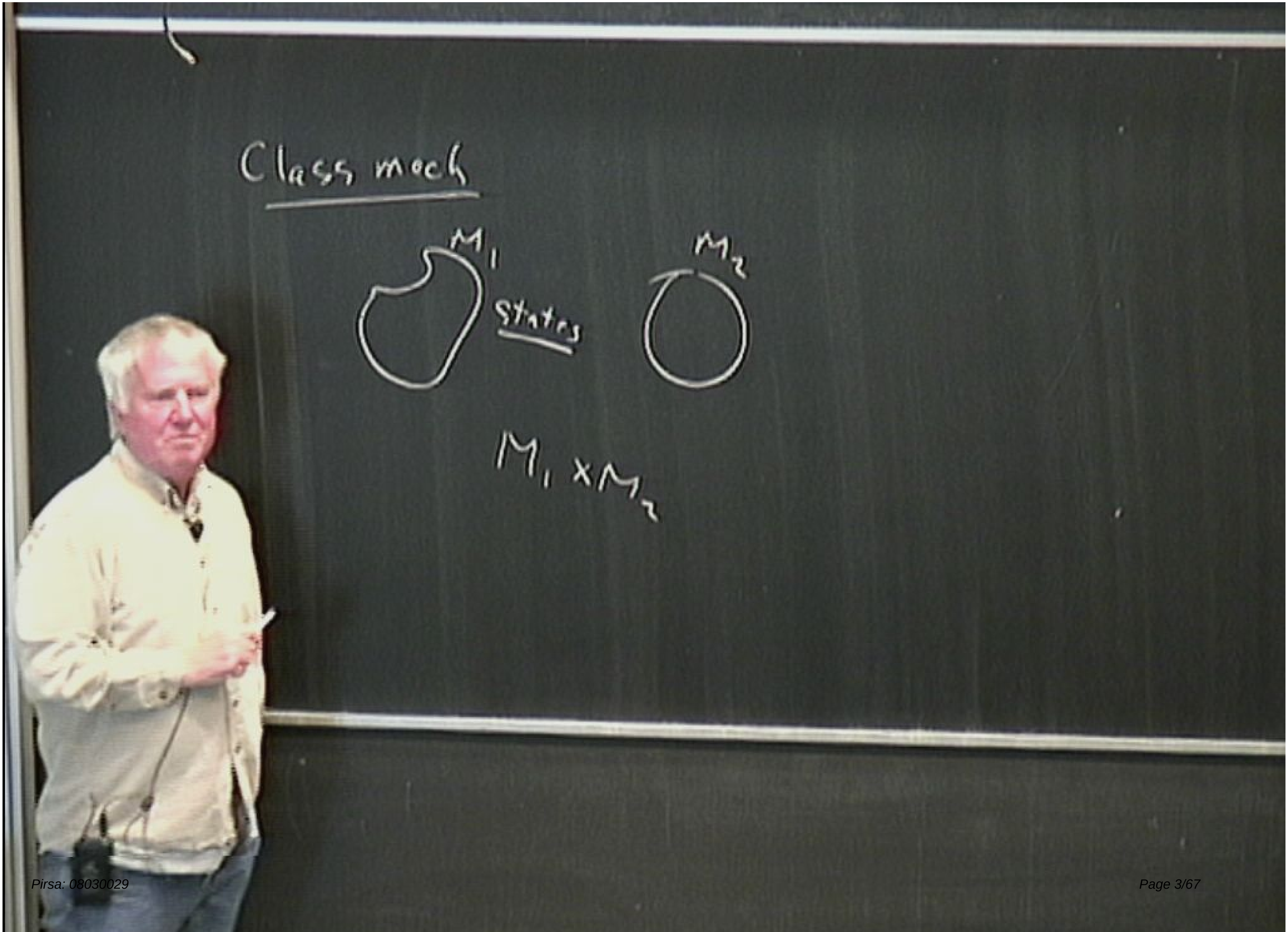
Title: Combining An Infinite Number of Quantum Systems

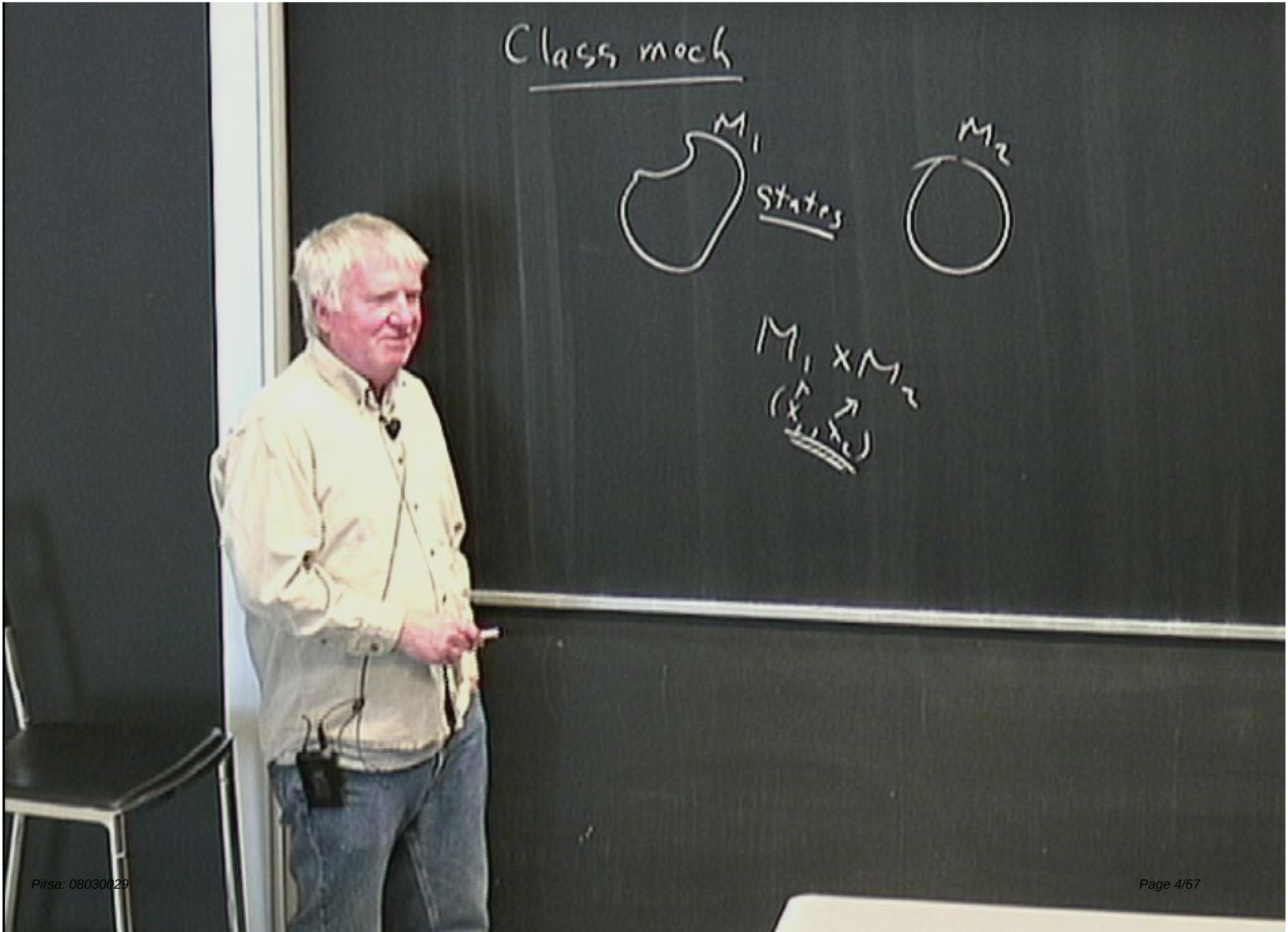
Date: Mar 13, 2008 11:00 AM

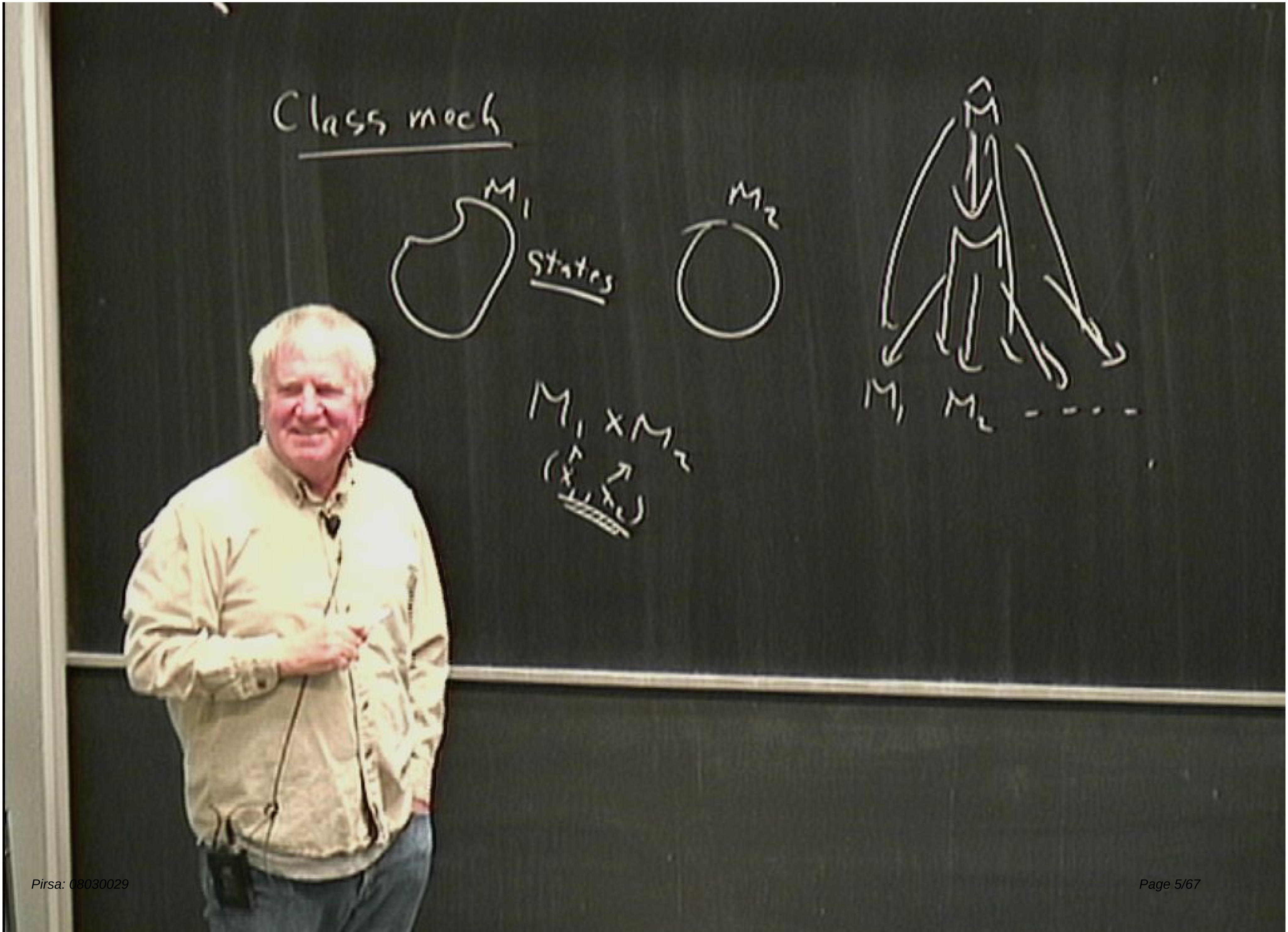
URL: http://pirsa.org/08030029

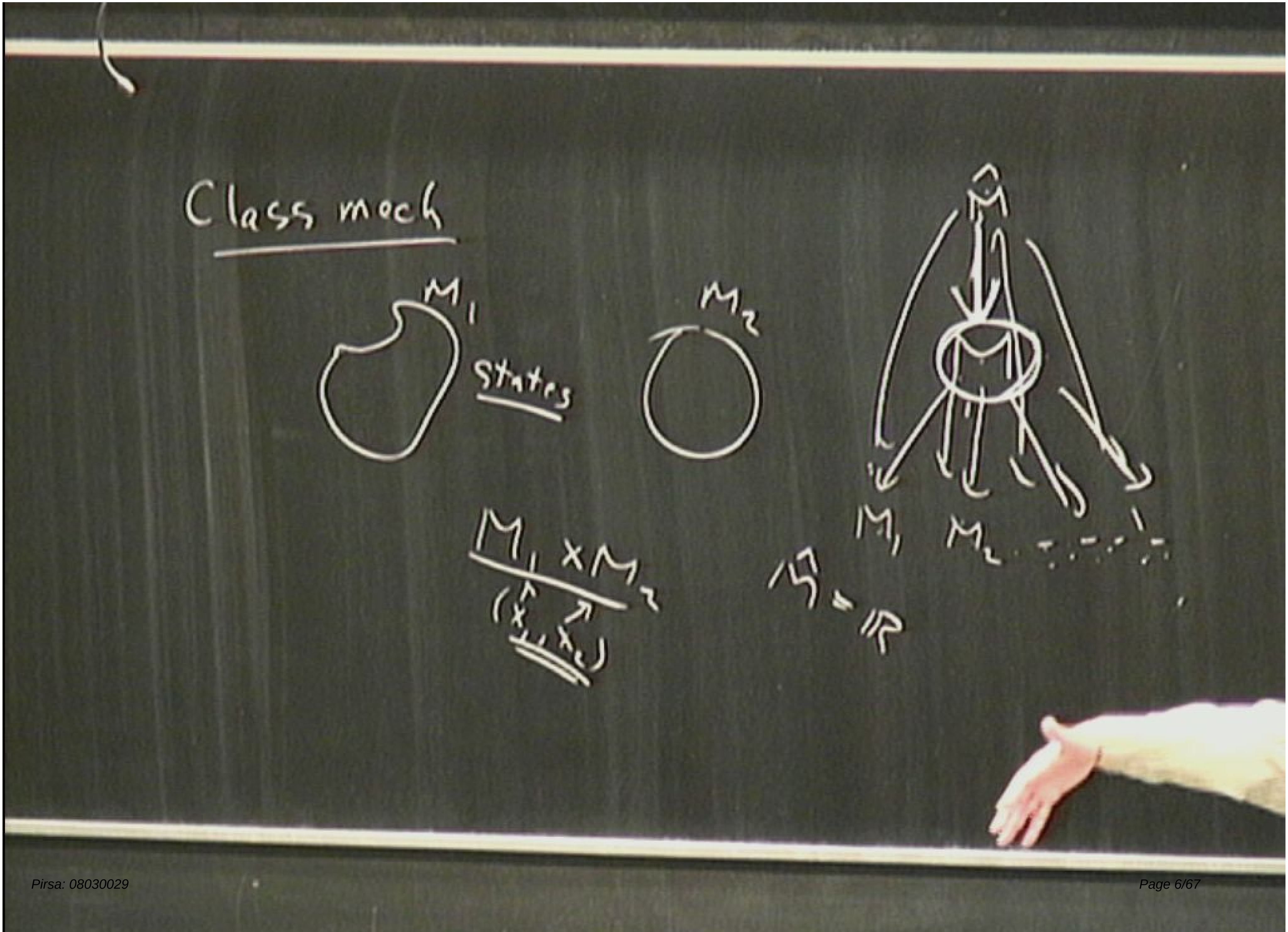
Abstract: A single classical system is characterized by its manifold of states; and to combine several systems, we take the product of manifolds. A single quantum system is characterized by its Hilbert space of states; and to combine several systems, we take the tensor product of Hilbert spaces. But what if we choose to combine an infinite number of systems? A naive attempt to describe such combinations fails, for there is apparently no natural notion of an infinite product of manifolds; nor of an infinite tensor product of Hilbert spaces. But, at least in the quantum case, the situation is not as hopeless as it might appear. We argue that there does indeed exist a natural mathematical framework for combinations of infinite numbers of quantum systems.

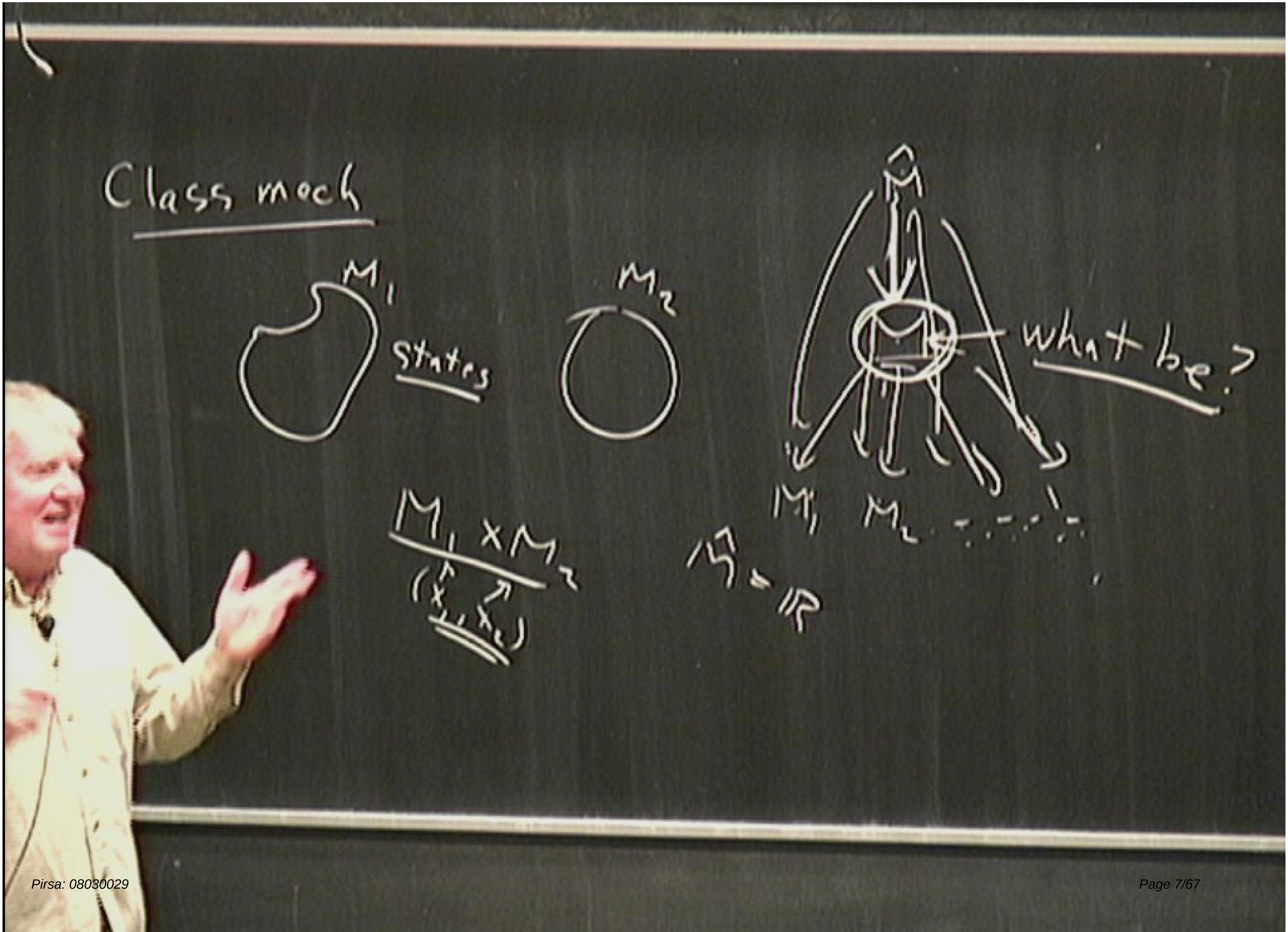

Class mech
M
States

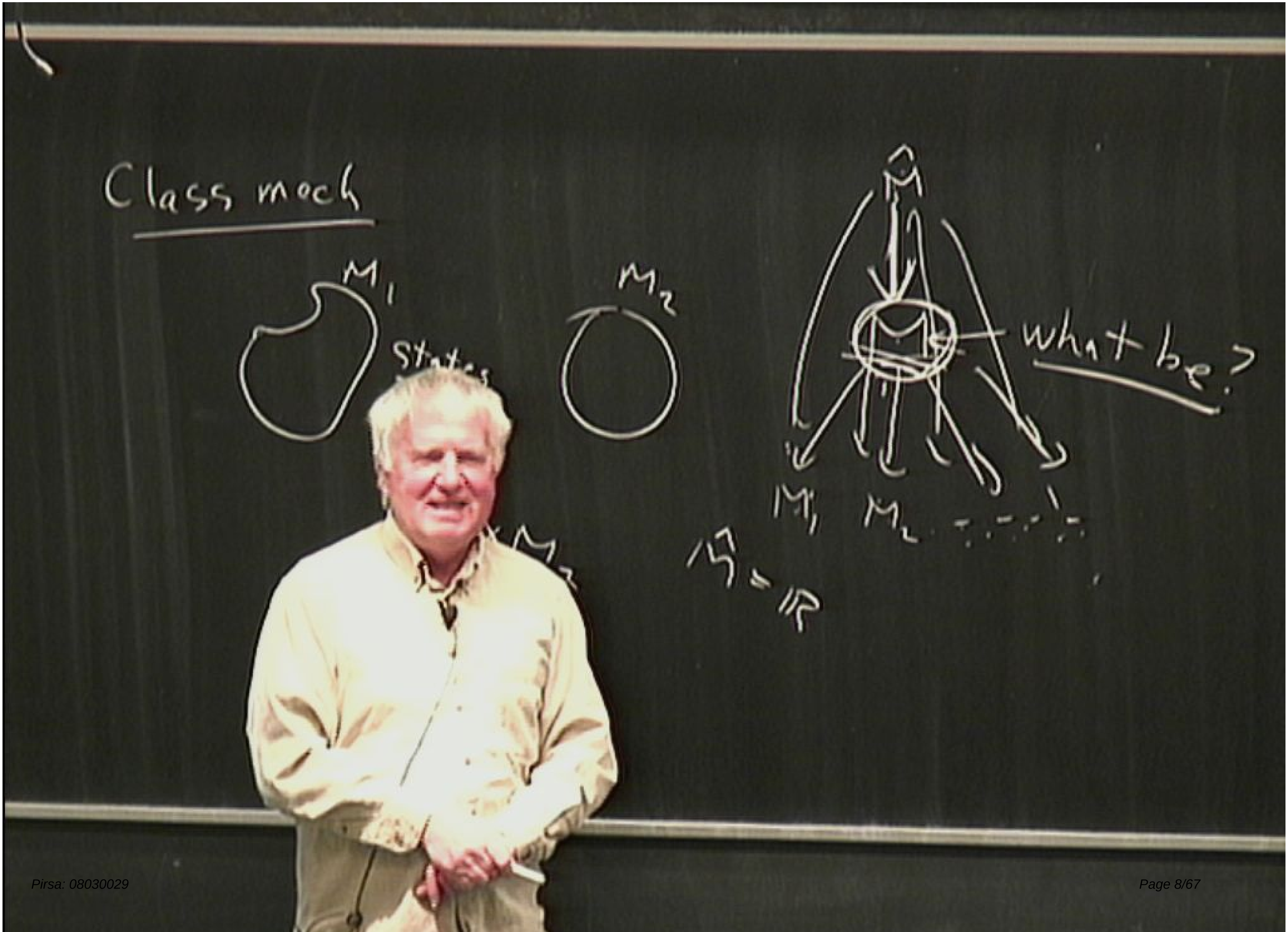

Class mech
M_1
states
M_2
$M_1 \times M_2$

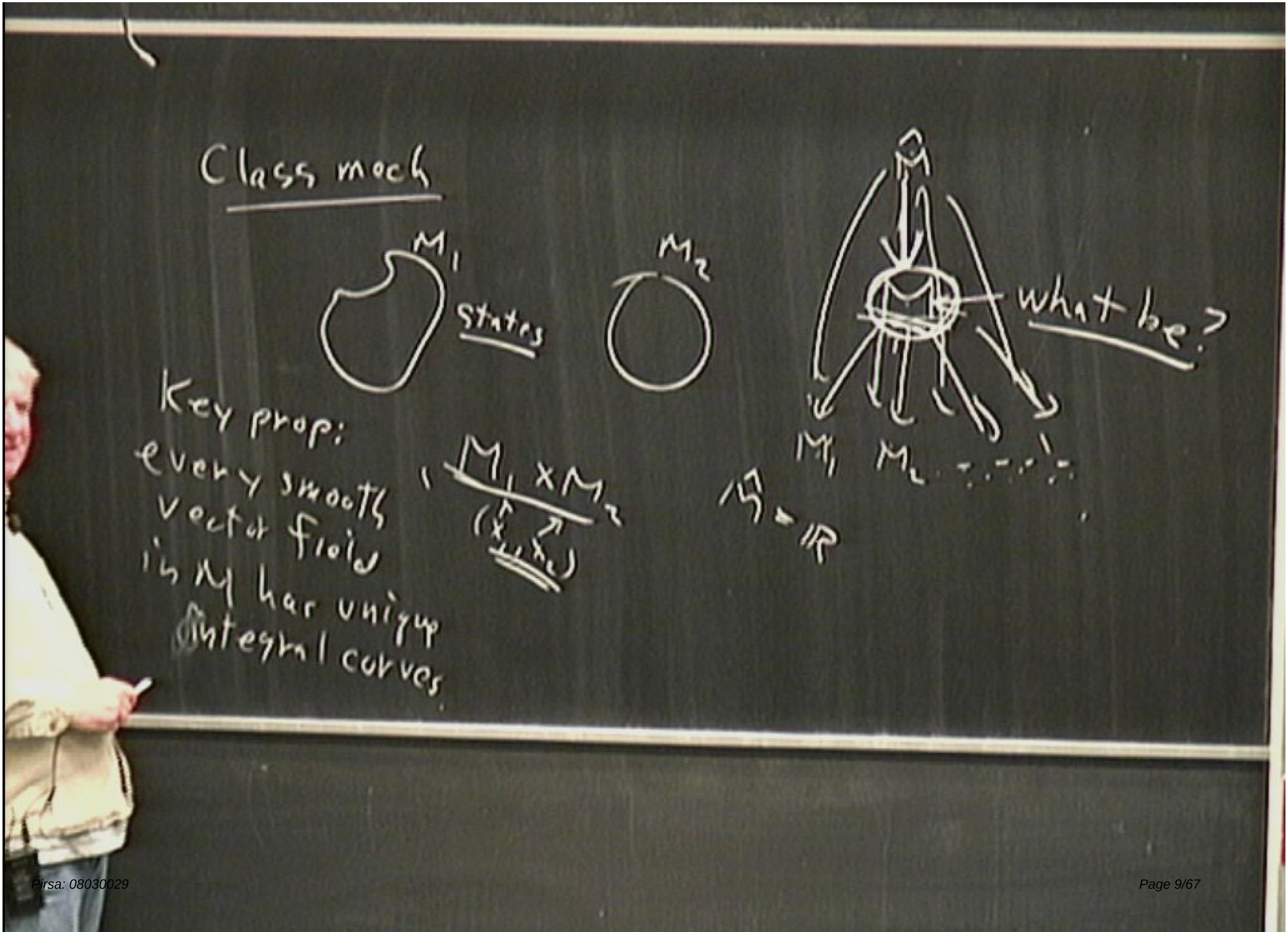

Class mech
M1
states
M2
M1 x M2
(x1, x2)

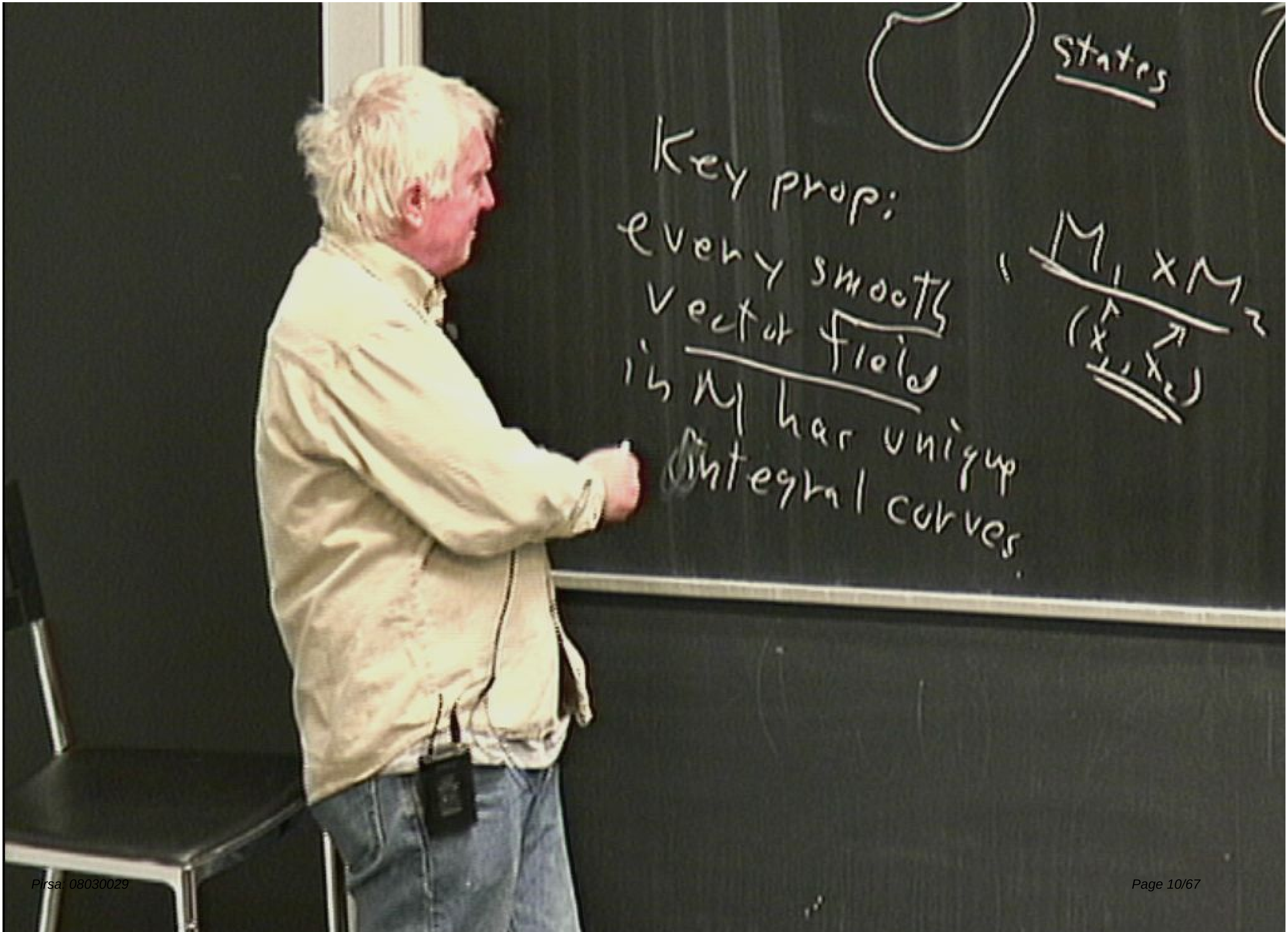

Class mech
M_1
States
M_2
$M_1 \times M_2$
(x_1, x_2)
M
M_1 M_2 - - - -

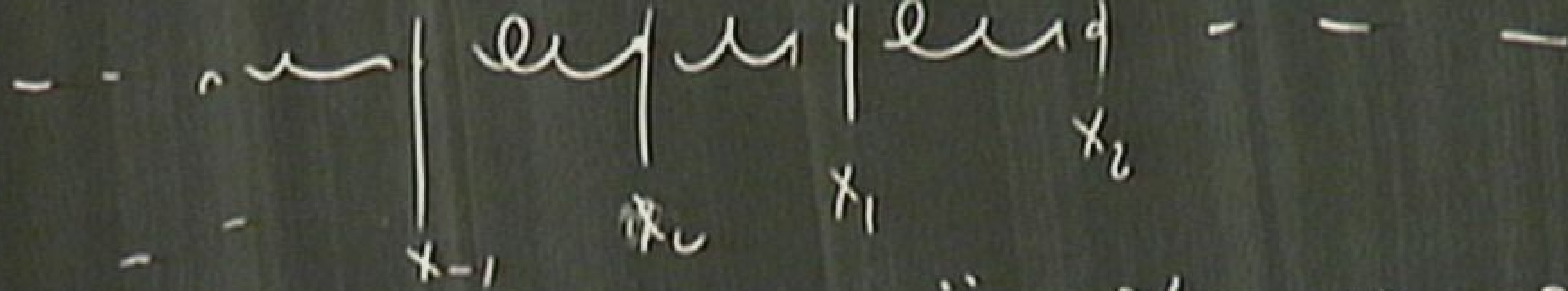

Class mech
M1
states
M2
M1 x M2
(x1, x2)
M1 M2


Class mech
M1
states
M2
what be?
M1 M2
M1 x M2
(x1, x2)

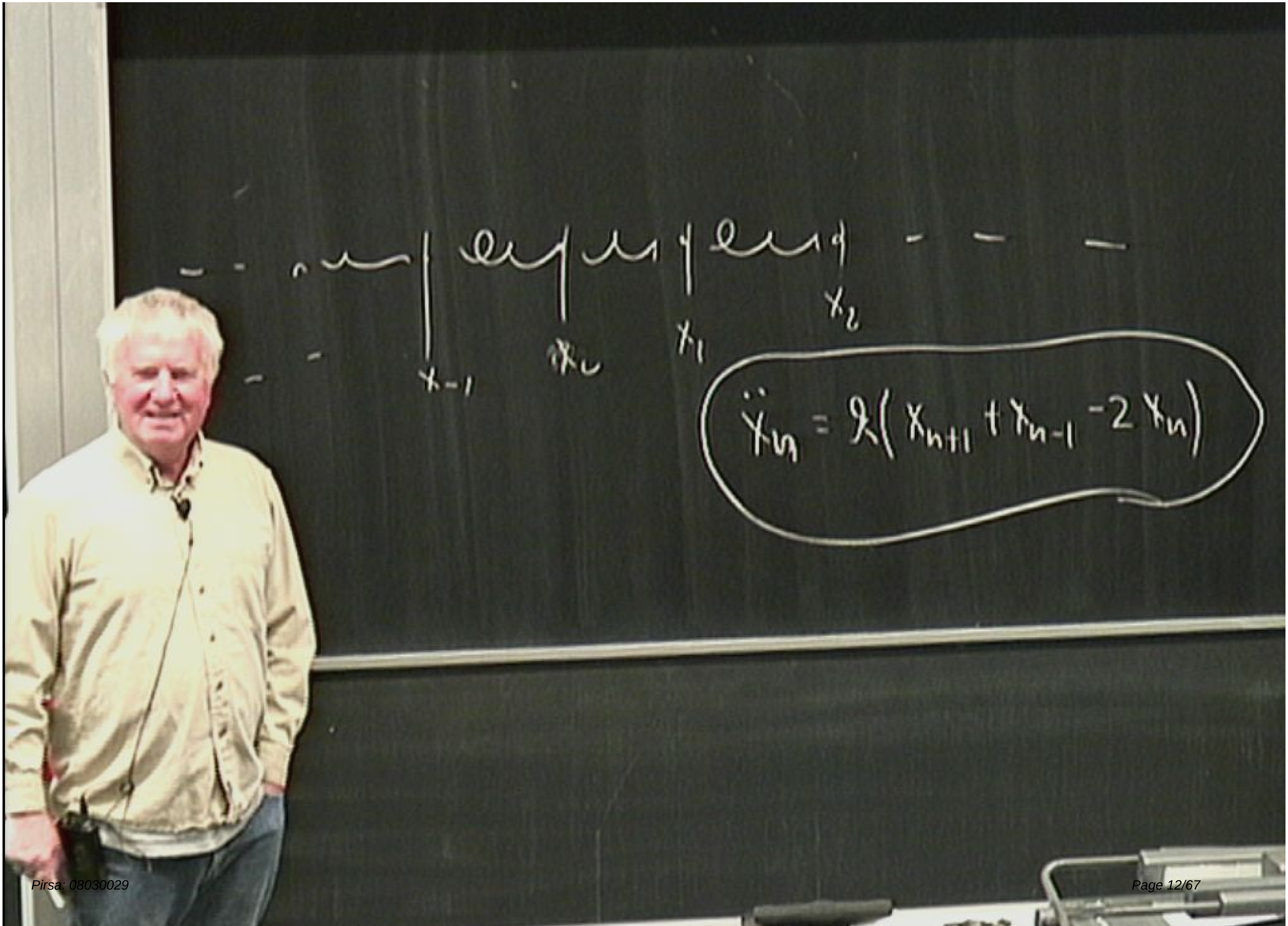

Class mech
M1
States
M2
what be?
M1 M2

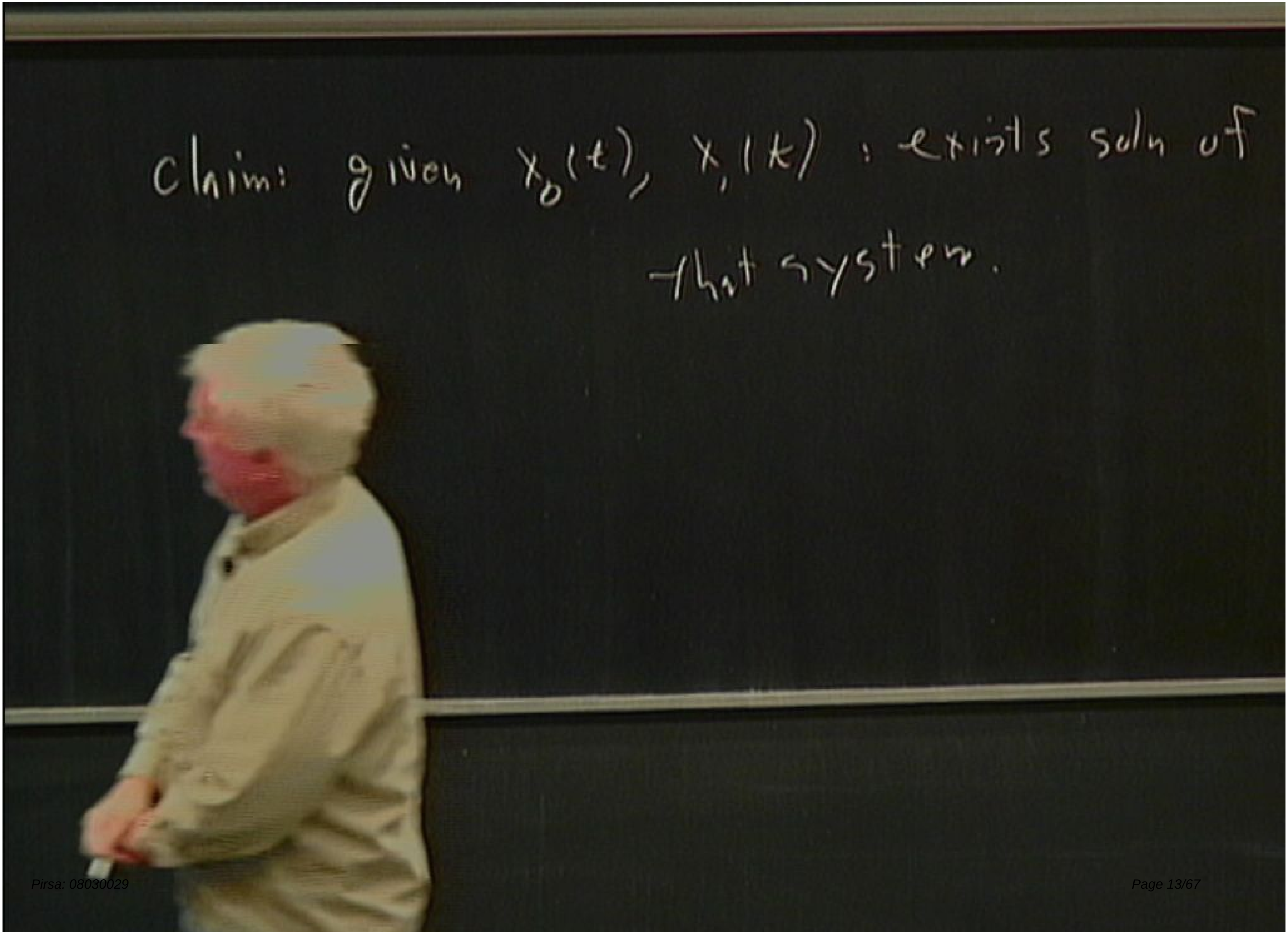

Class mech
M_1
states
M_2
$\hat{M}$
what be?
M_1 M_2
Key prop:
every smooth vector field in M has unique integral curves
$\dfrac{M_1 \times M_2}{(x_1, x_2)}$
$\mathbb{R}$

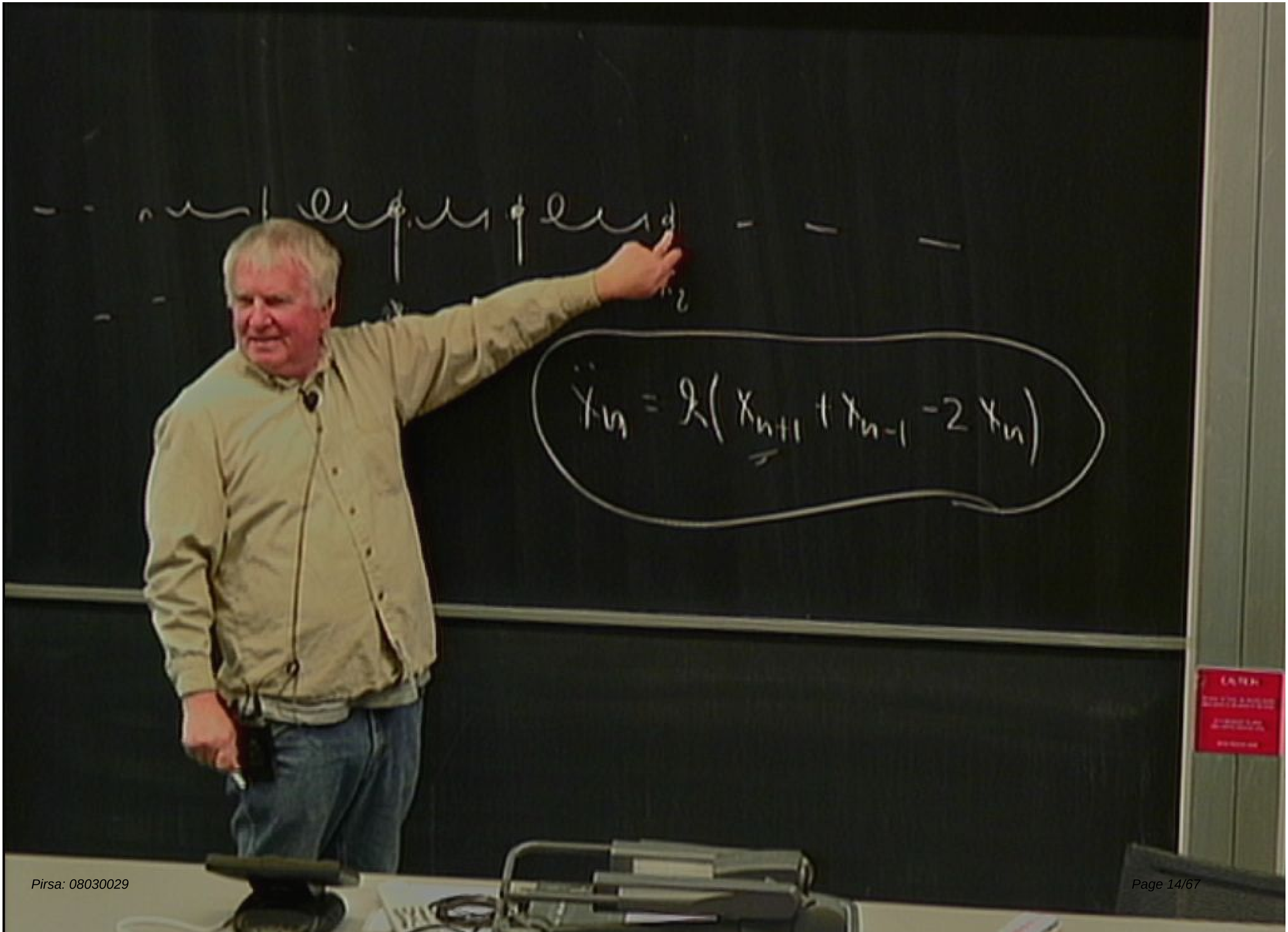

states
Key prop:
every smooth
vector field
in M har unique
integral curves
M1 x M2
(x1, x2)

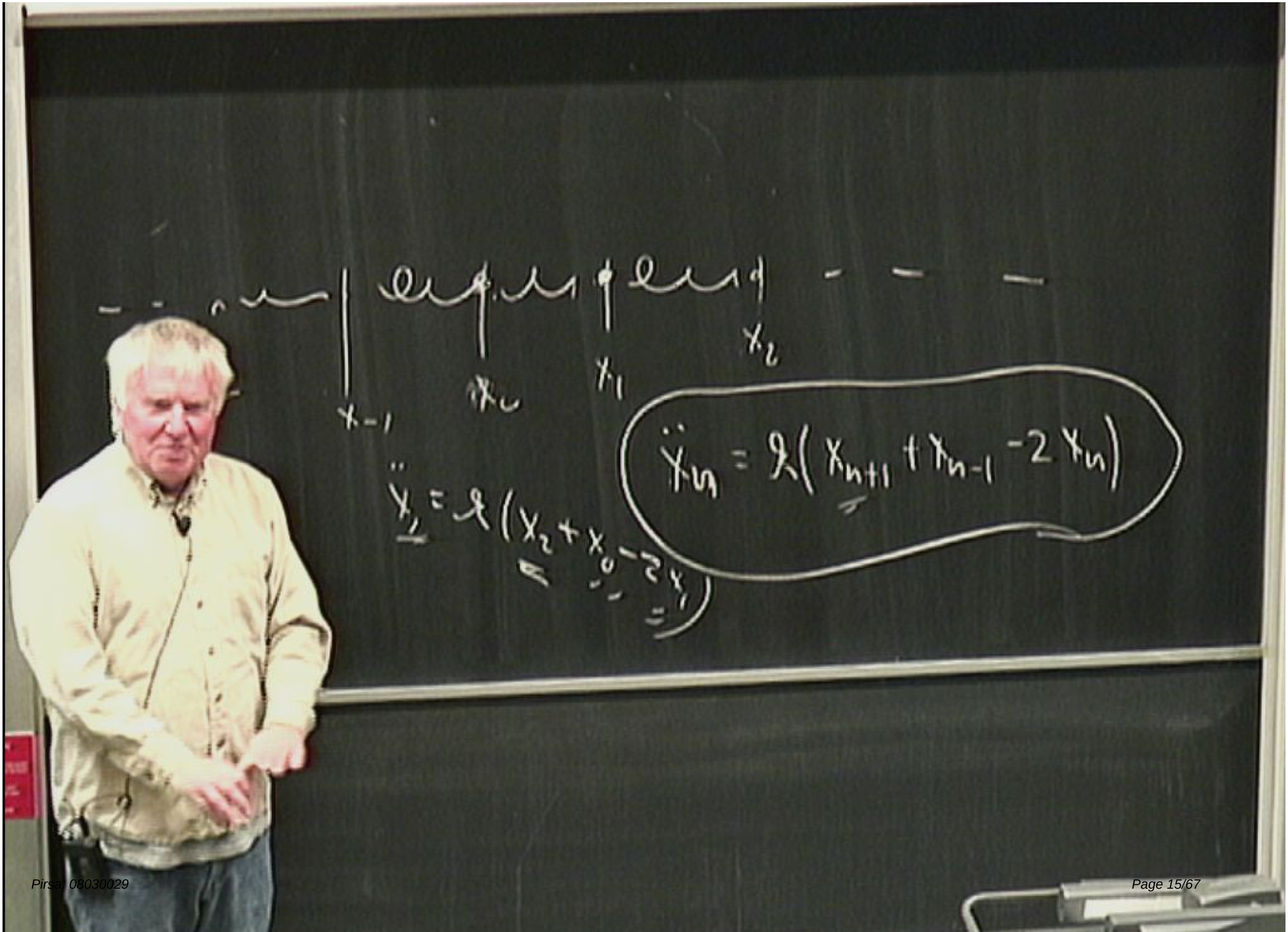


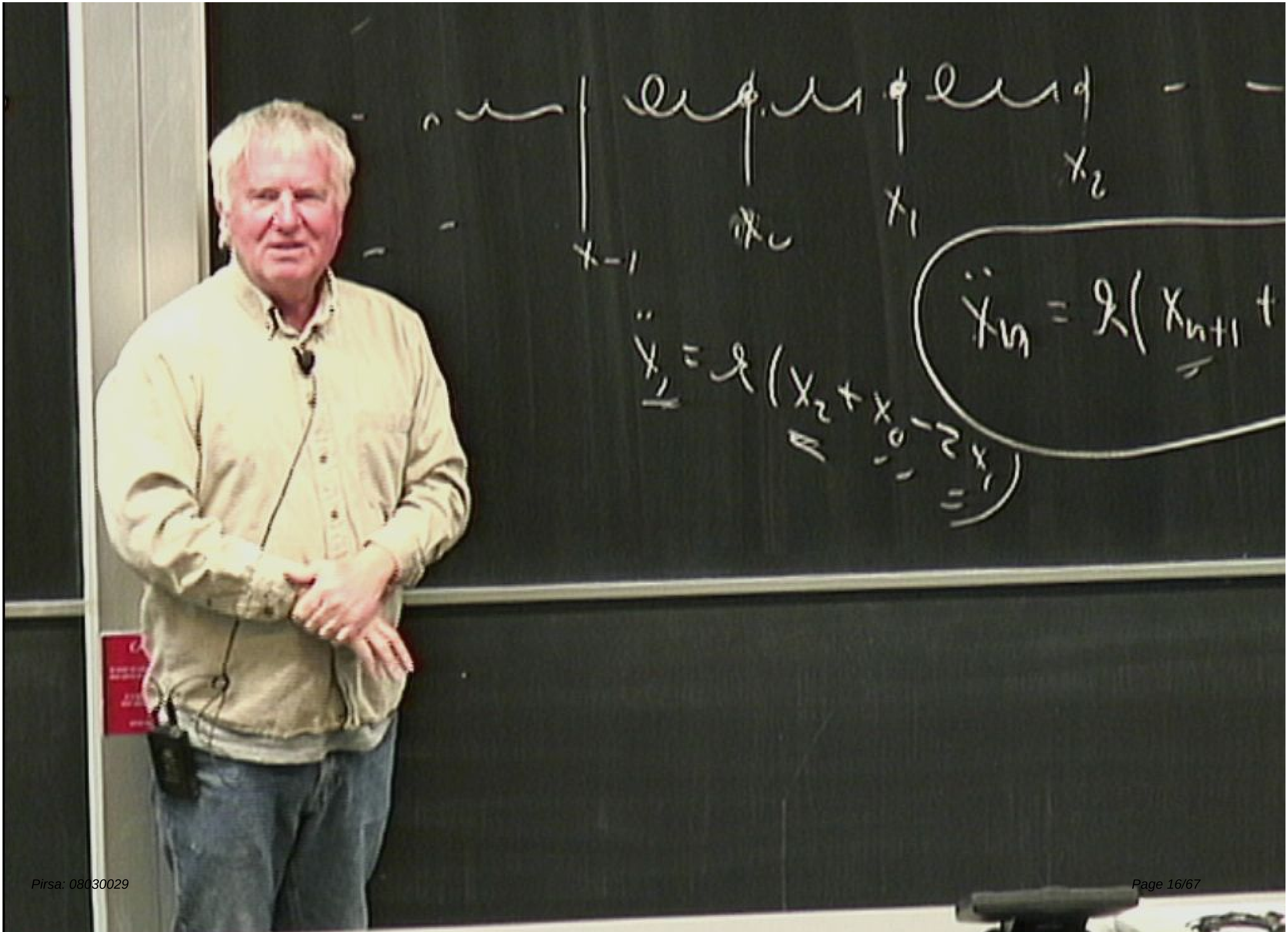
$$\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)$$


x_{-1}
x_0
x_1
x_2
$\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)$

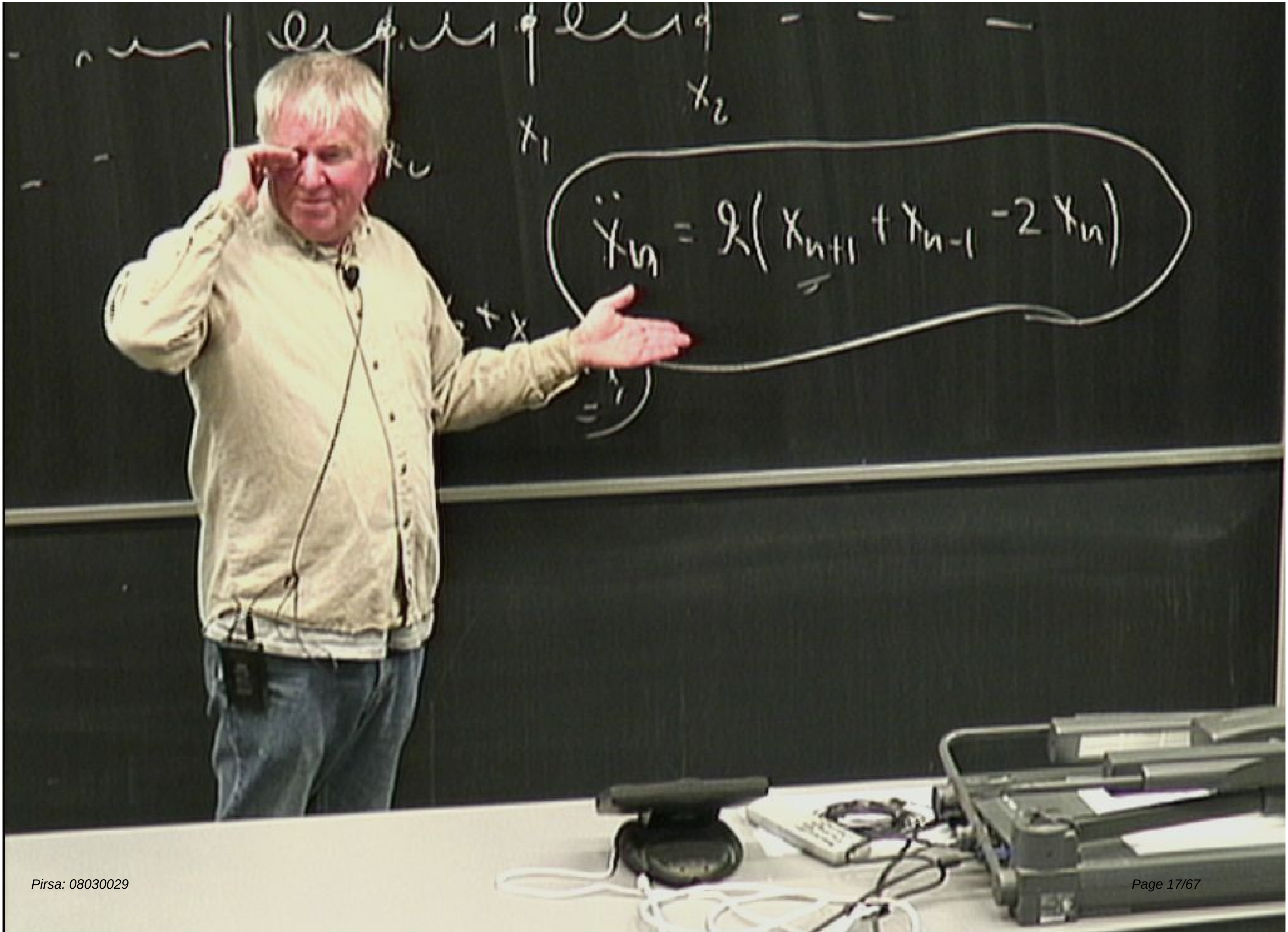

Claim: given x0(t), x1(t) : exists soln of
that system.

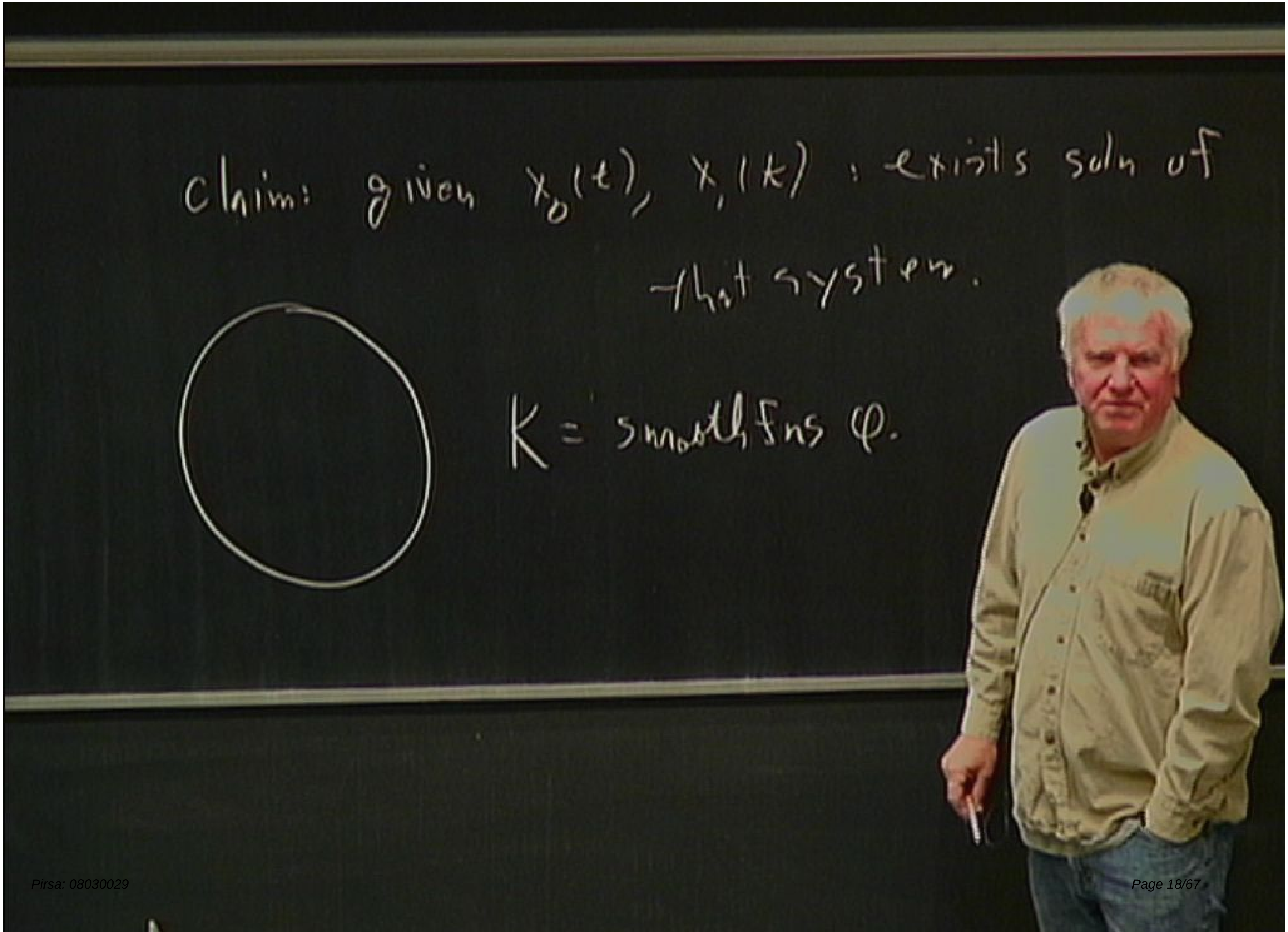

$\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)$

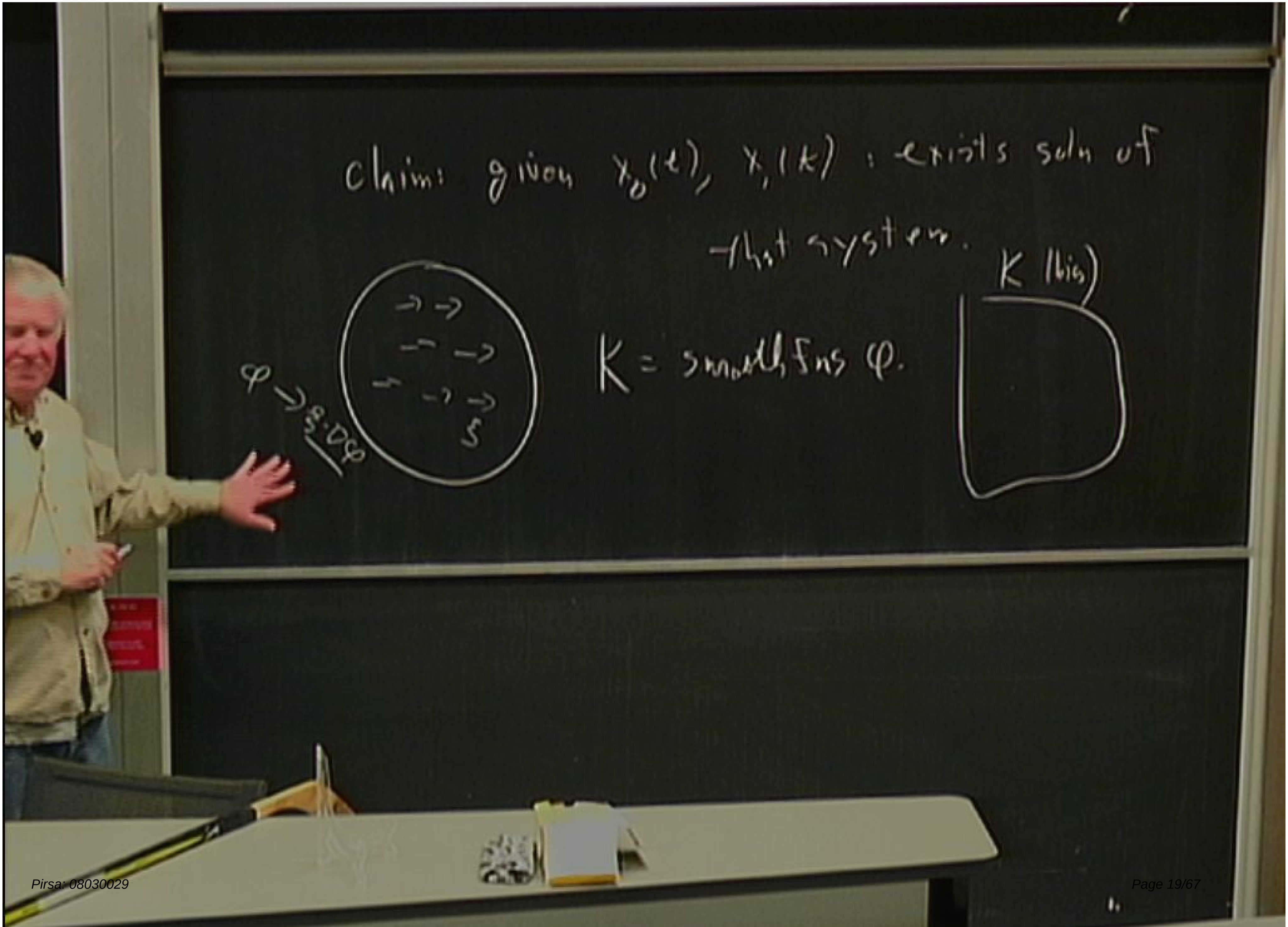

x_{-1}
x_0
x_1
x_2
ẍ_n = k(x_{n+1} + x_{n-1} - 2x_n)
ẍ_1 = k(x_2 + x_0 - 2x_1)

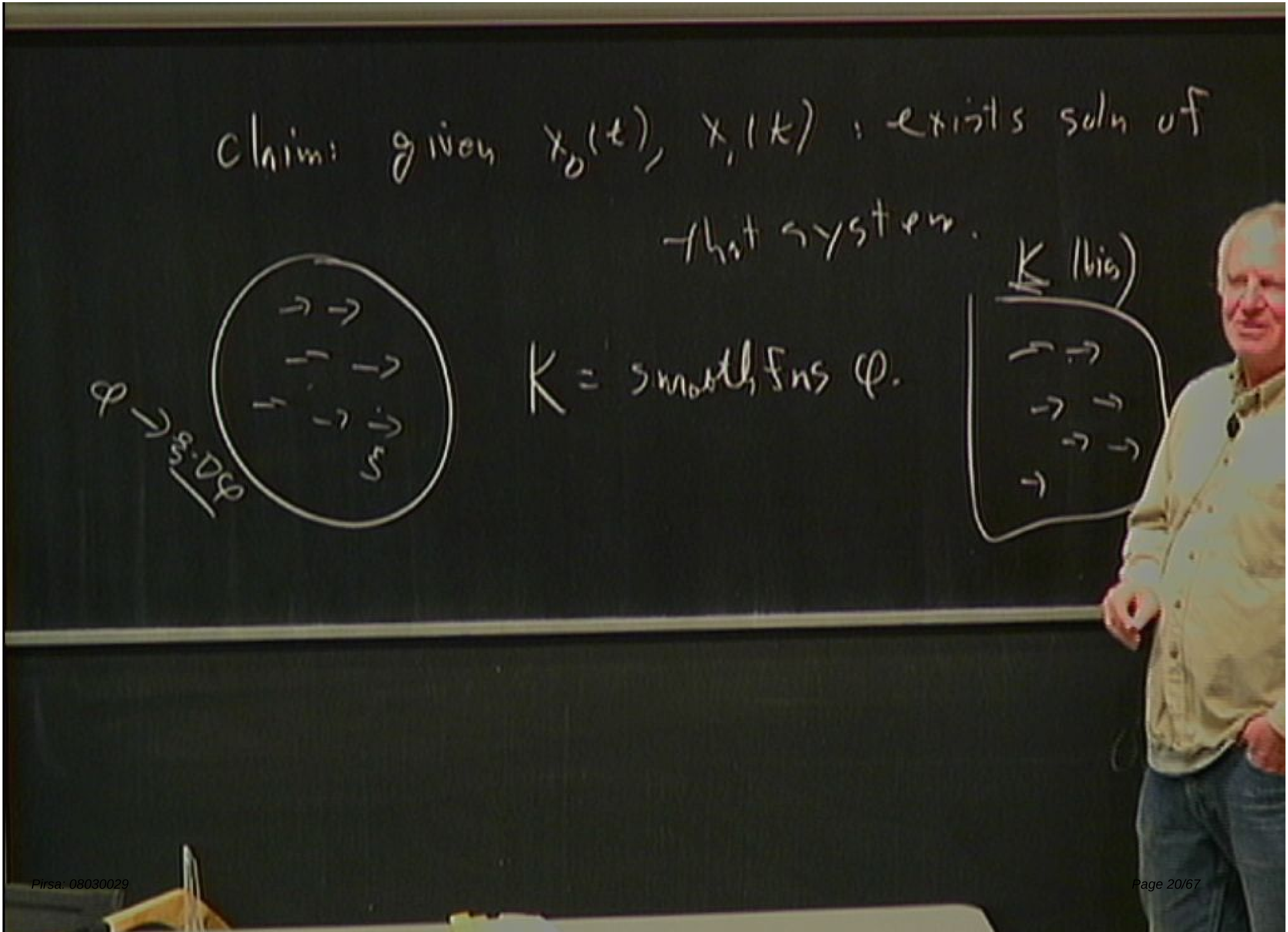


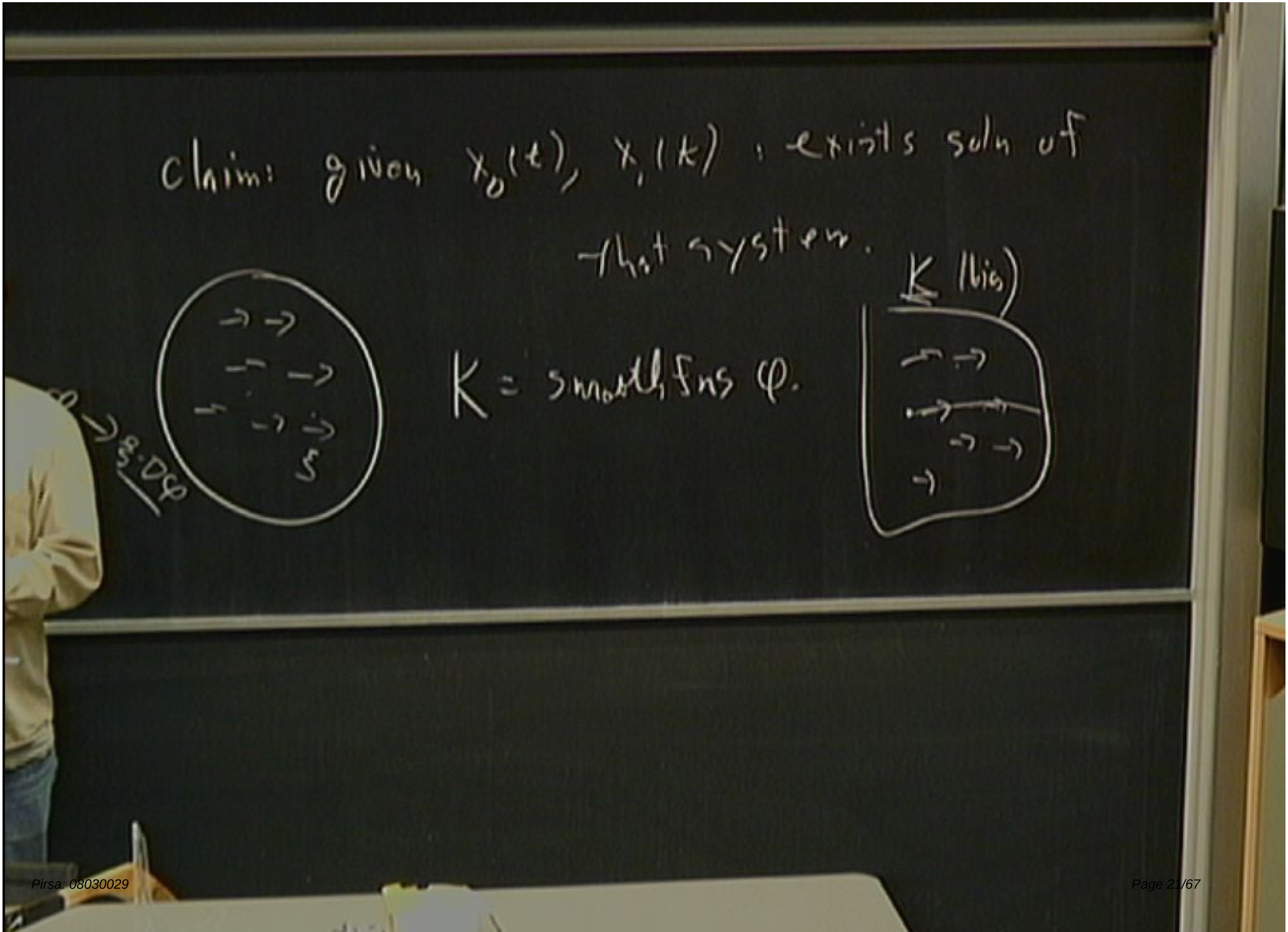
x_{-1} x_0 x_1 x_2
$\ddot{x}_n = k(x_{n+1} +$
$\ddot{x}_1 = k(x_2 + x_0 - 2x_1)$

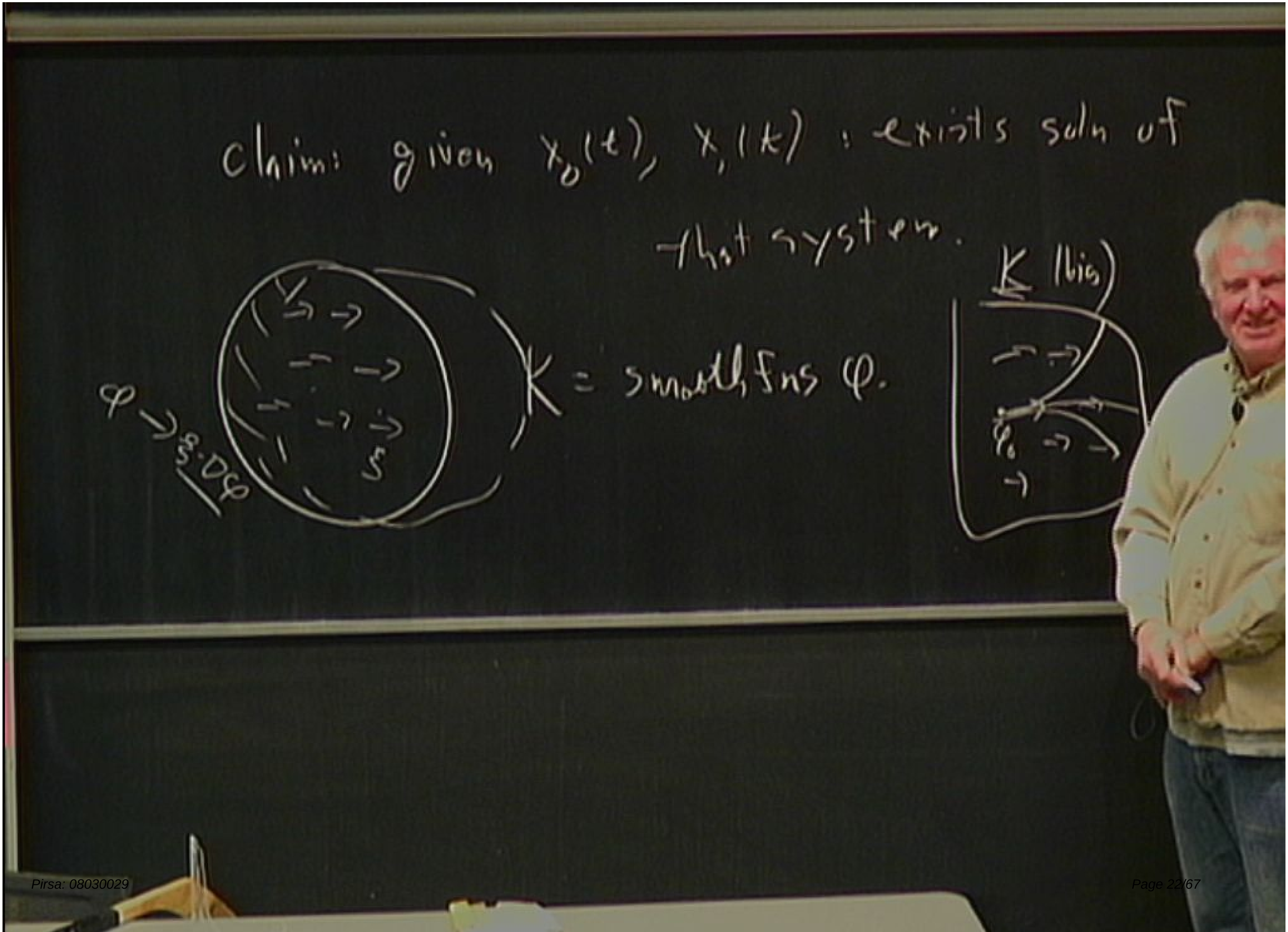

x_1
x_2
$\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)$


claim: given $x_0(t)$, $x_1(k)$: exists soln of
that system.
K = smooth fns φ.


Claim: given x0(t), x1(k) : exists soln of that system.
K (big)
K = smooth fns φ.
φ → g·Dφ
S

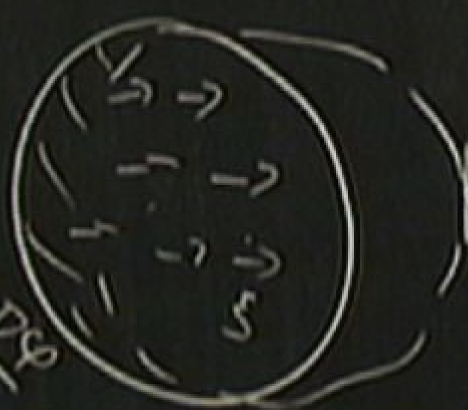

claim: given $x_0(t)$, $x_1(k)$: exists soln of that system.
ξ
$\varphi \to \xi \cdot D\varphi$
K = smooth fns φ.
K (big)


Claim: given $x_0(t)$, $x_1(k)$, exists soln of
that system.
K = smooth fns φ.
K (this)
g·Dφ
S


Claim: given $x_0(t)$, $x_1(t)$: exists soln of that system.
K = smooth fns φ.
$\varphi \to \varphi + \xi\cdot D\varphi$
K (big)
φ_0

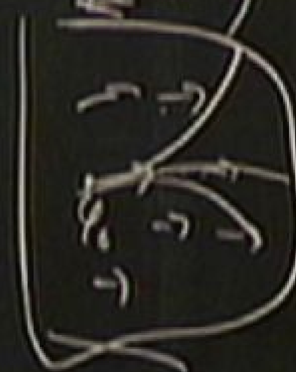
Claim: given $x_0(t)$, $x_1(k)$: exists soln of that system.

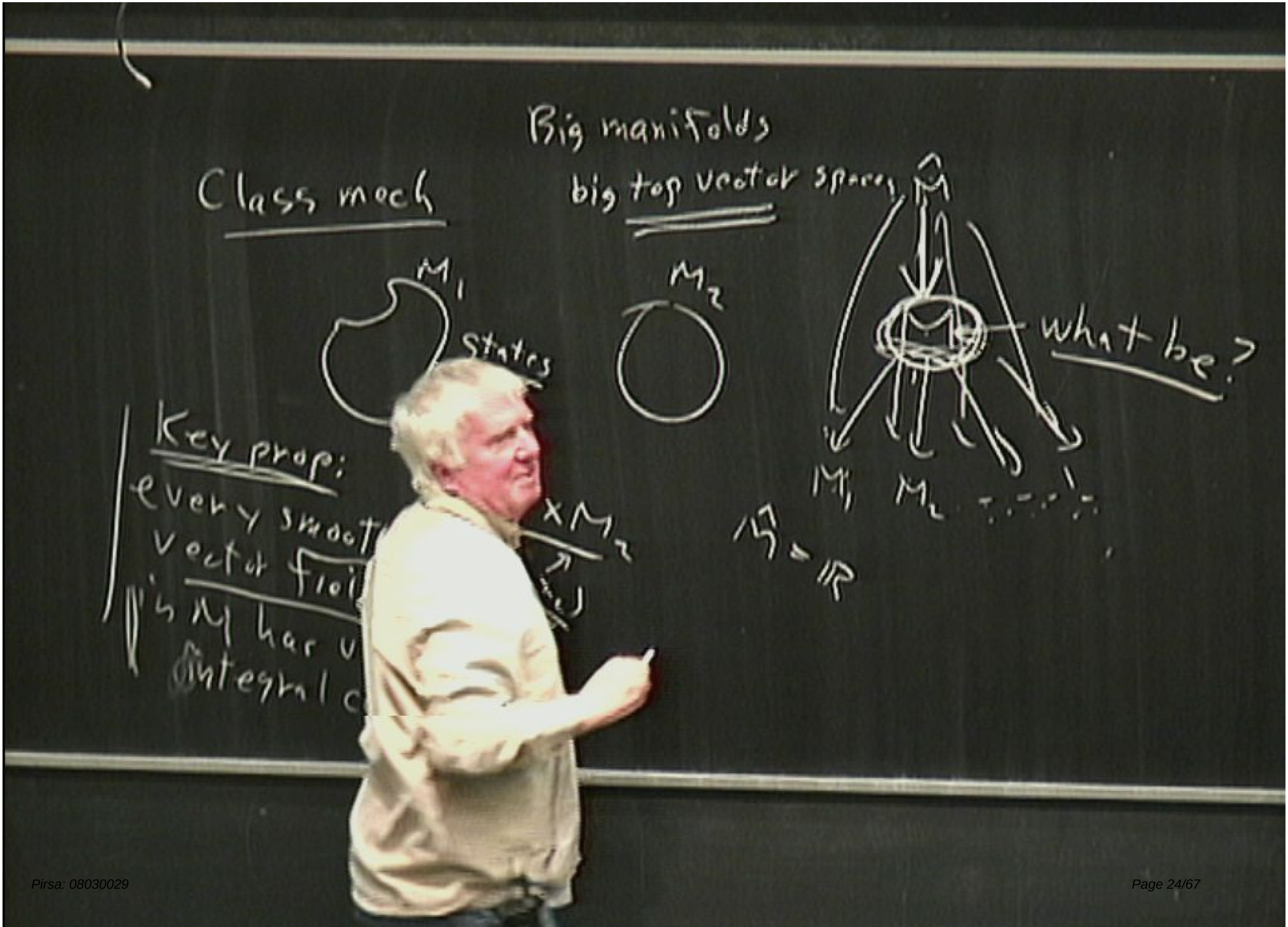
$\varphi \to g\cdot D\varphi$

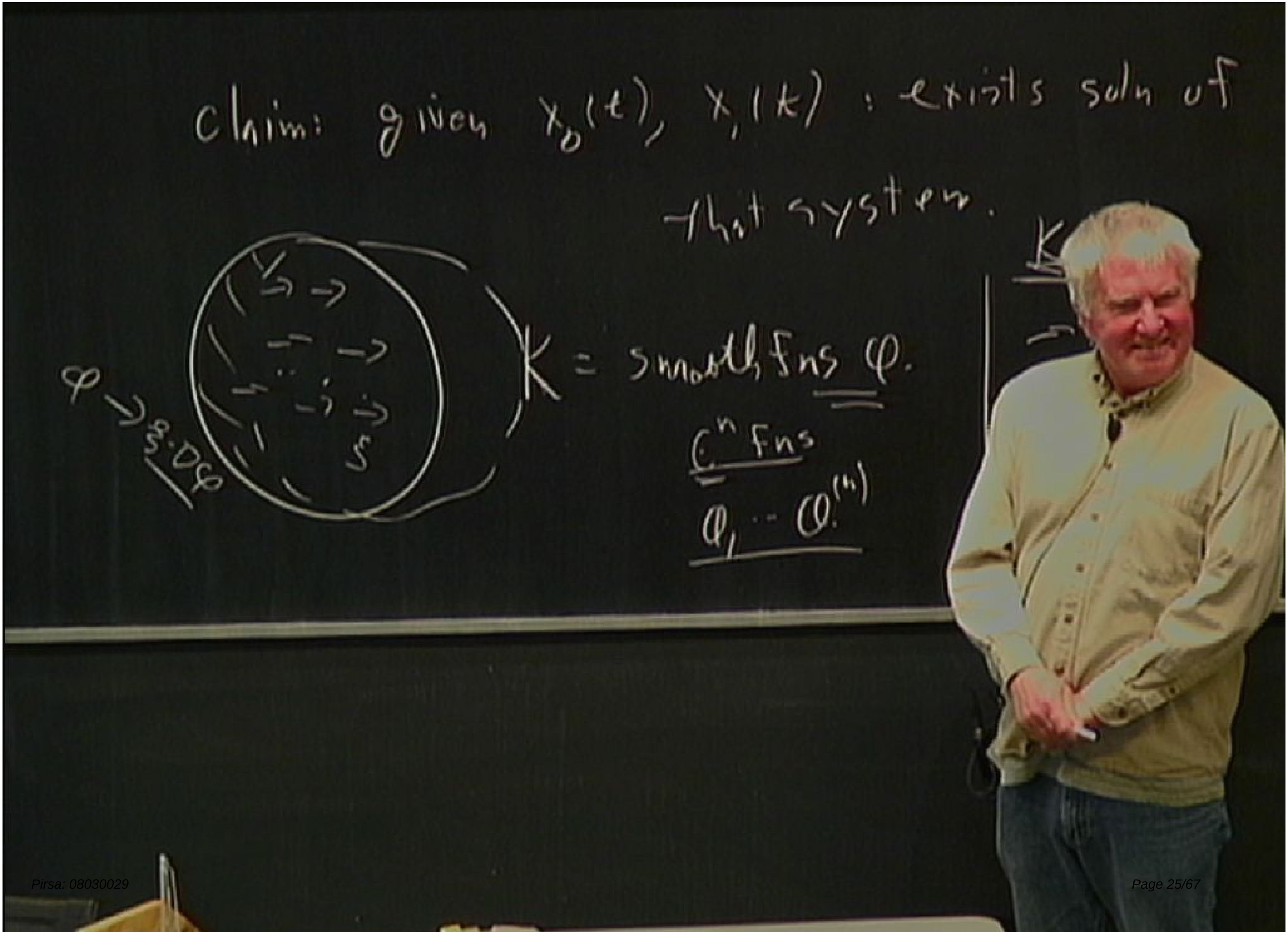


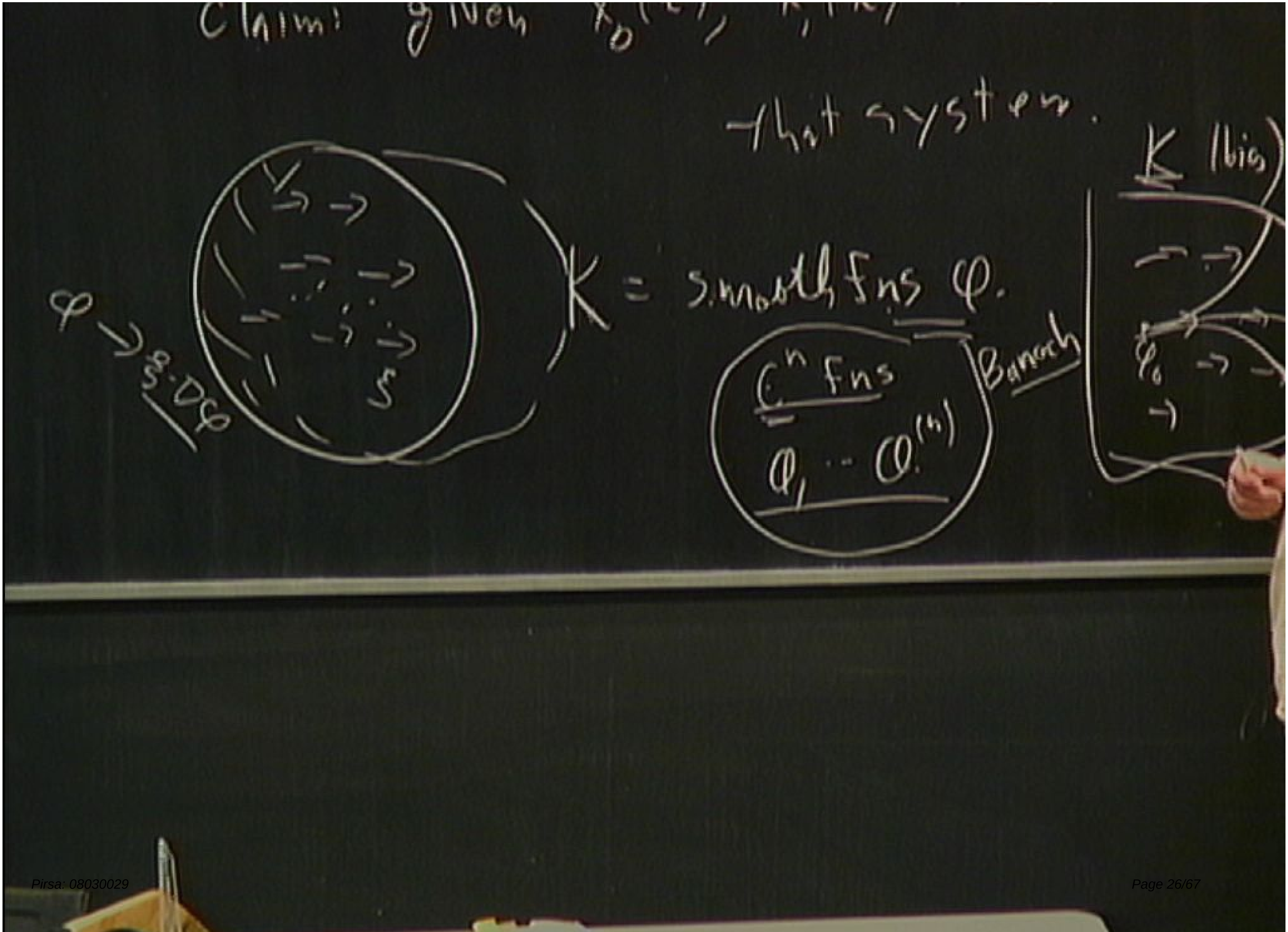
K = smooth fns φ.

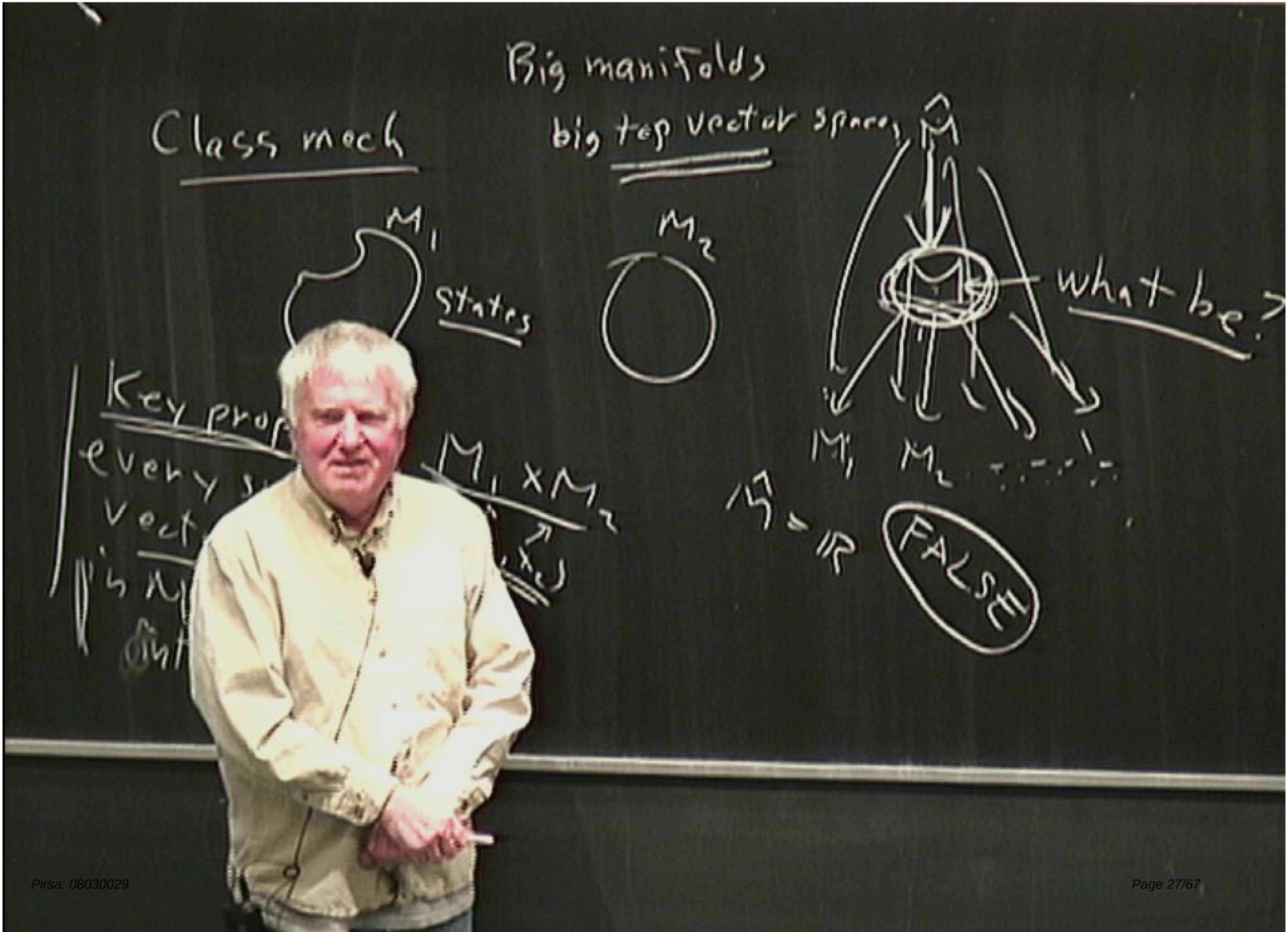
K (big)

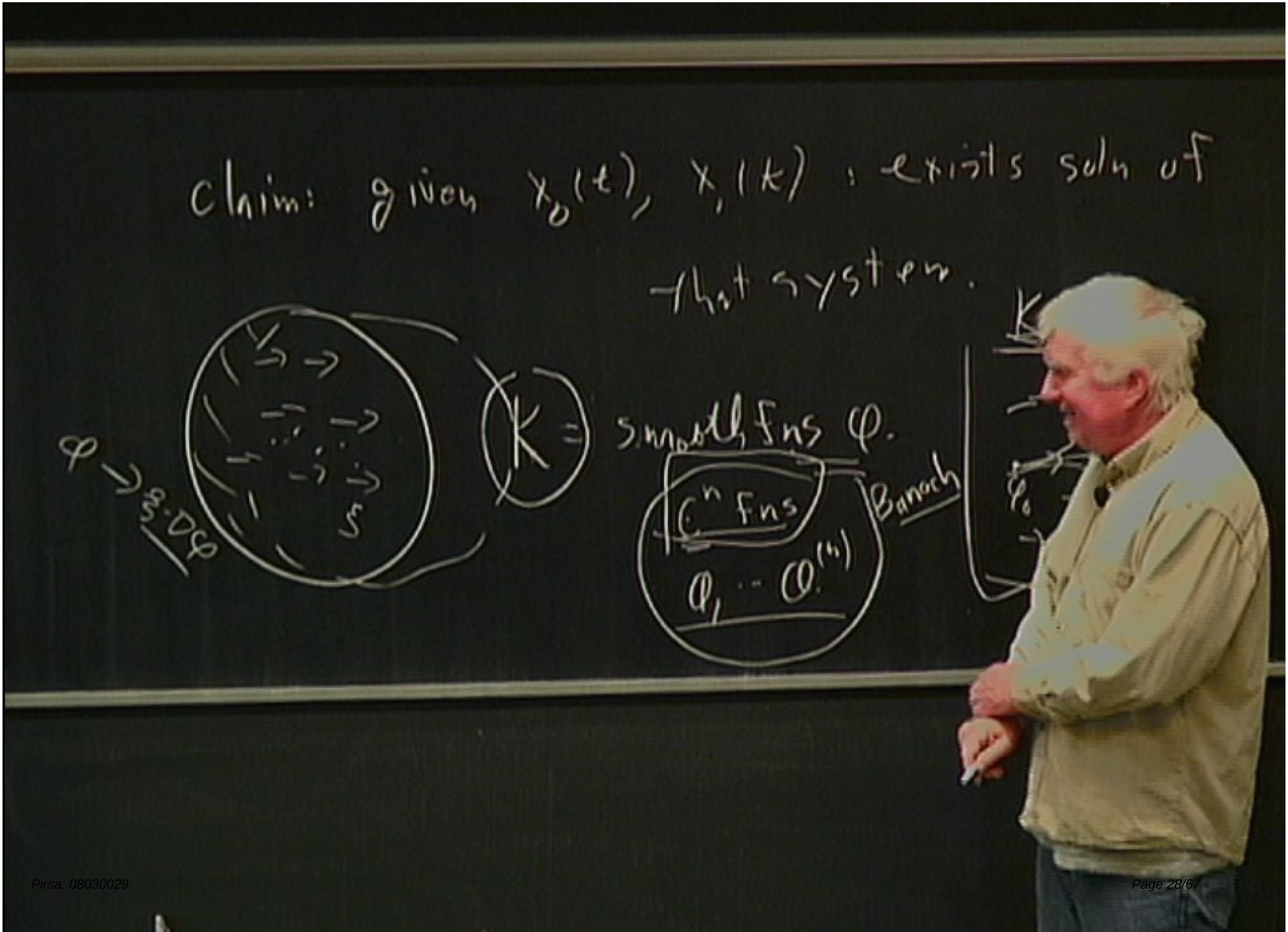



Big manifolds
Class mech
big top vector space
M_1
M_2
states
what be?
Key prop:
every smoot
vector fiel
in M has
integral c
M_1 M_2


Claim: given $x_0(t)$, $x_1(t)$: exists soln of
that system.
K = smooth fns φ.
C^n fns
$\varphi_1 \cdots \varphi^{(n)}$
$\varphi \to \xi \cdot D\varphi$
ξ


Claim: given
that system.
K = smooth fns φ.
Cⁿ fns
φ, ... φ⁽ⁿ⁾
Banach
K (big)
φ₀


Big manifolds
Class mech
big top vector space
M1
states
M2
what be?
Key prop
M1 × M2
M1
M2
FALSE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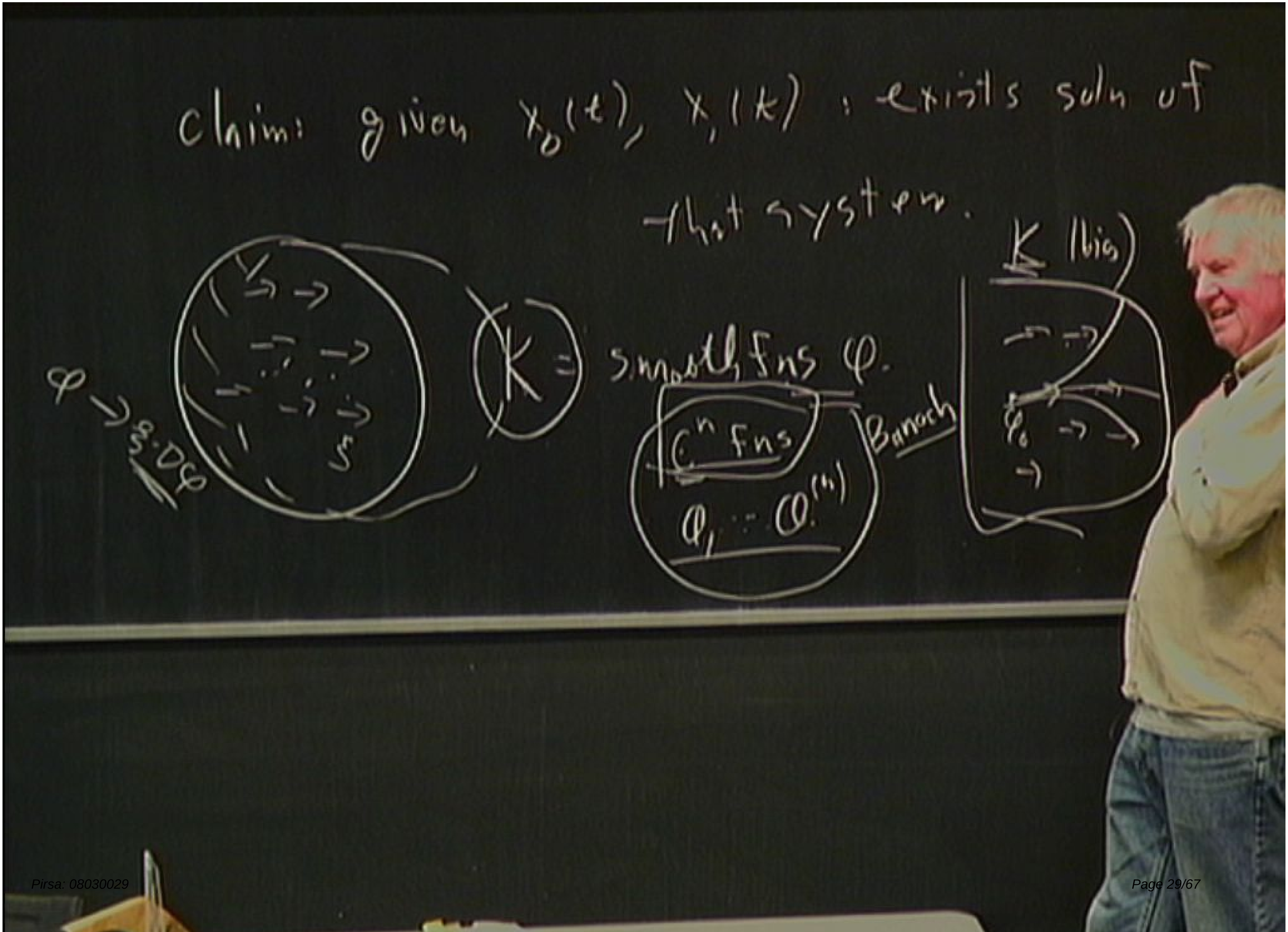
K = smooth fns φ.
C^n fns
Banach
φ_1 ... φ^(n)
φ → ξ·Dφ
K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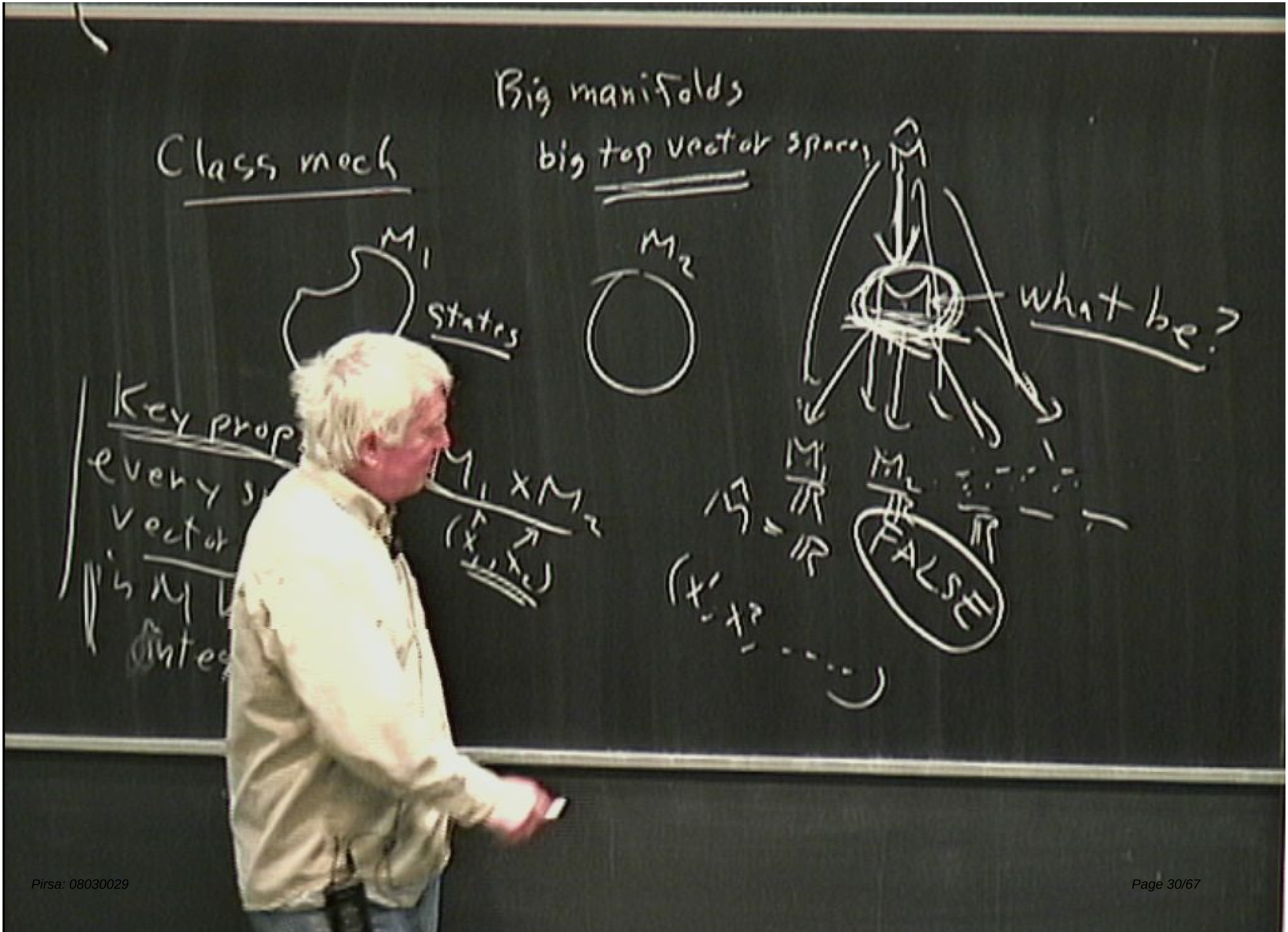
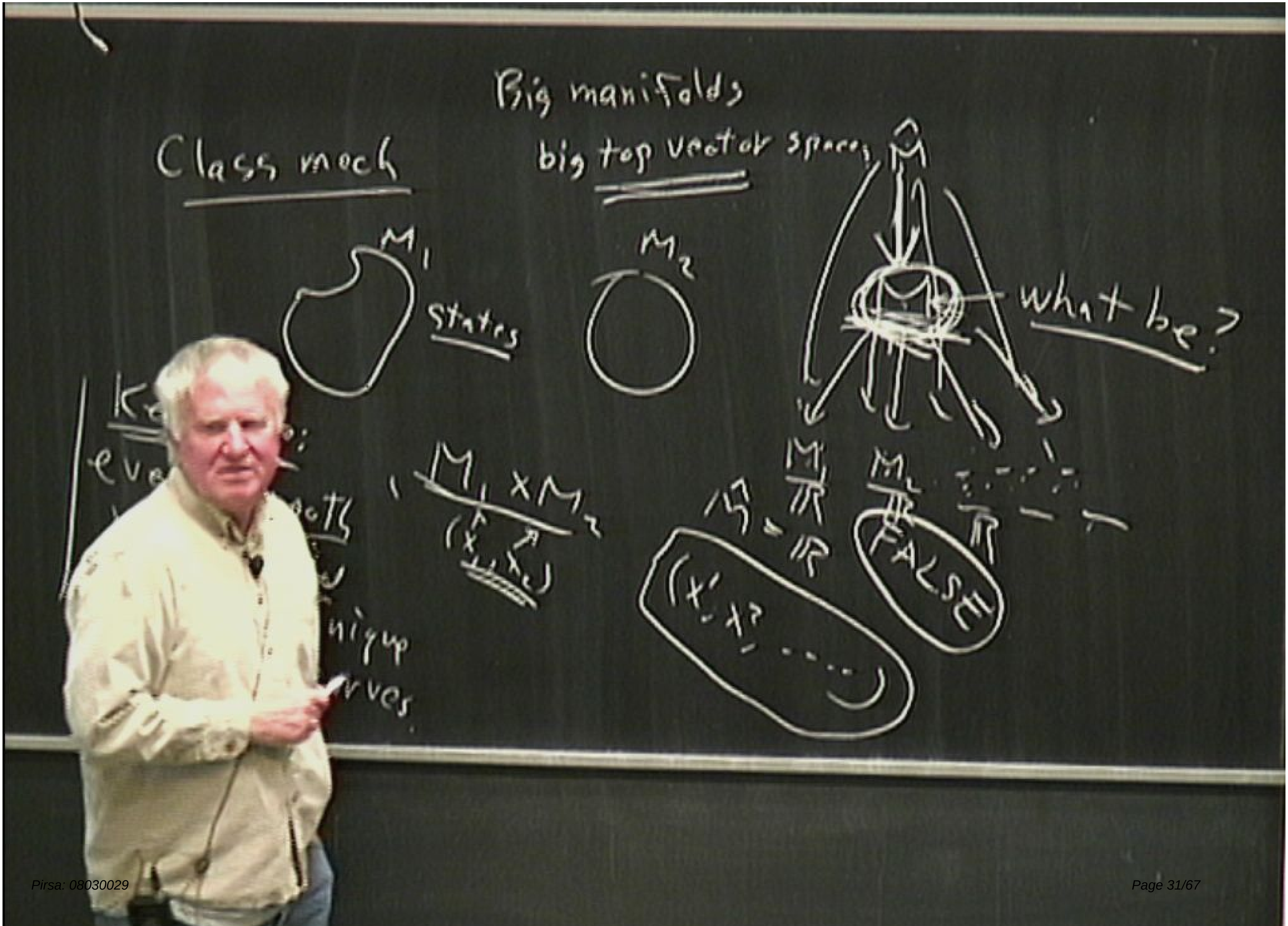
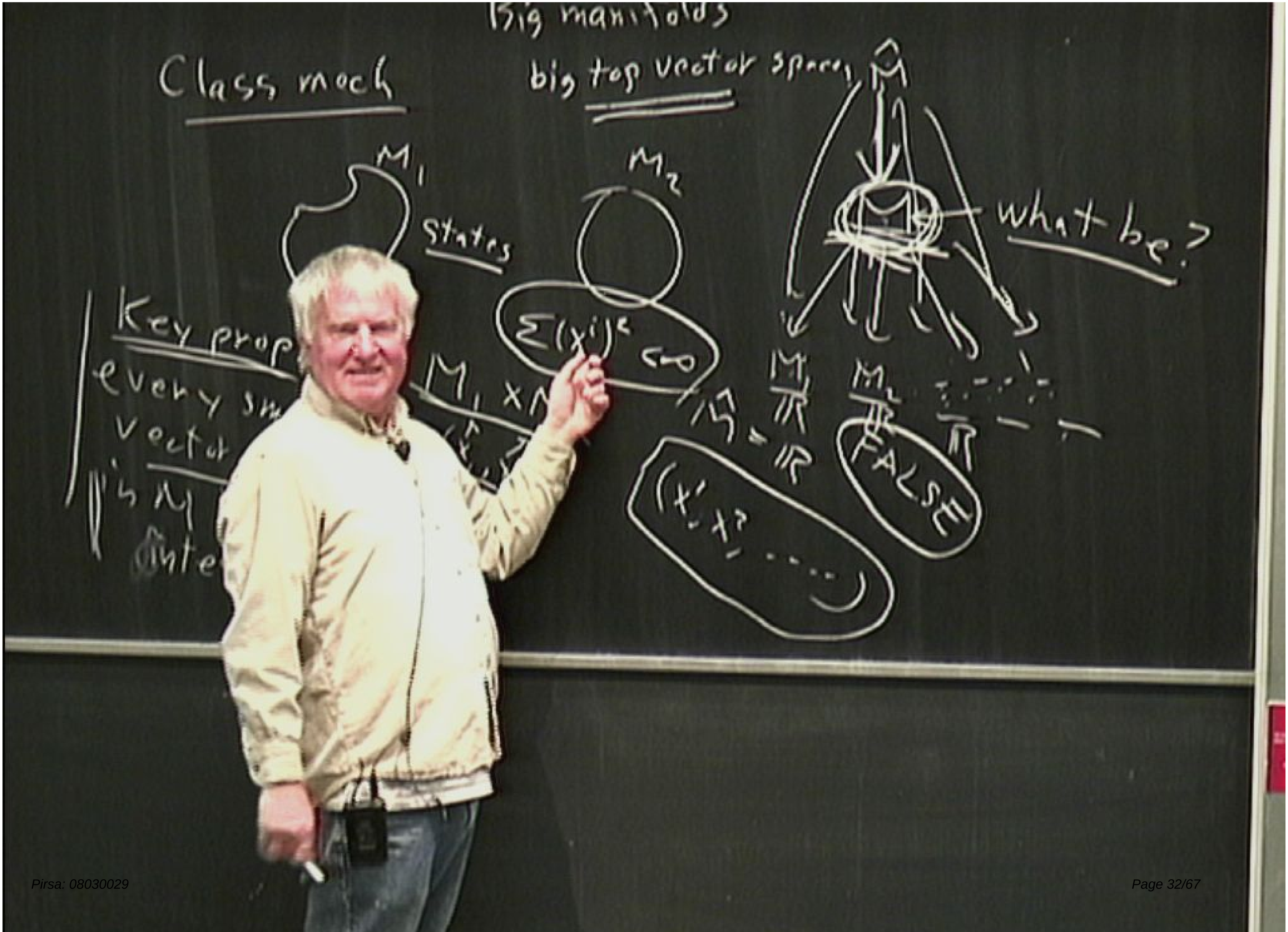
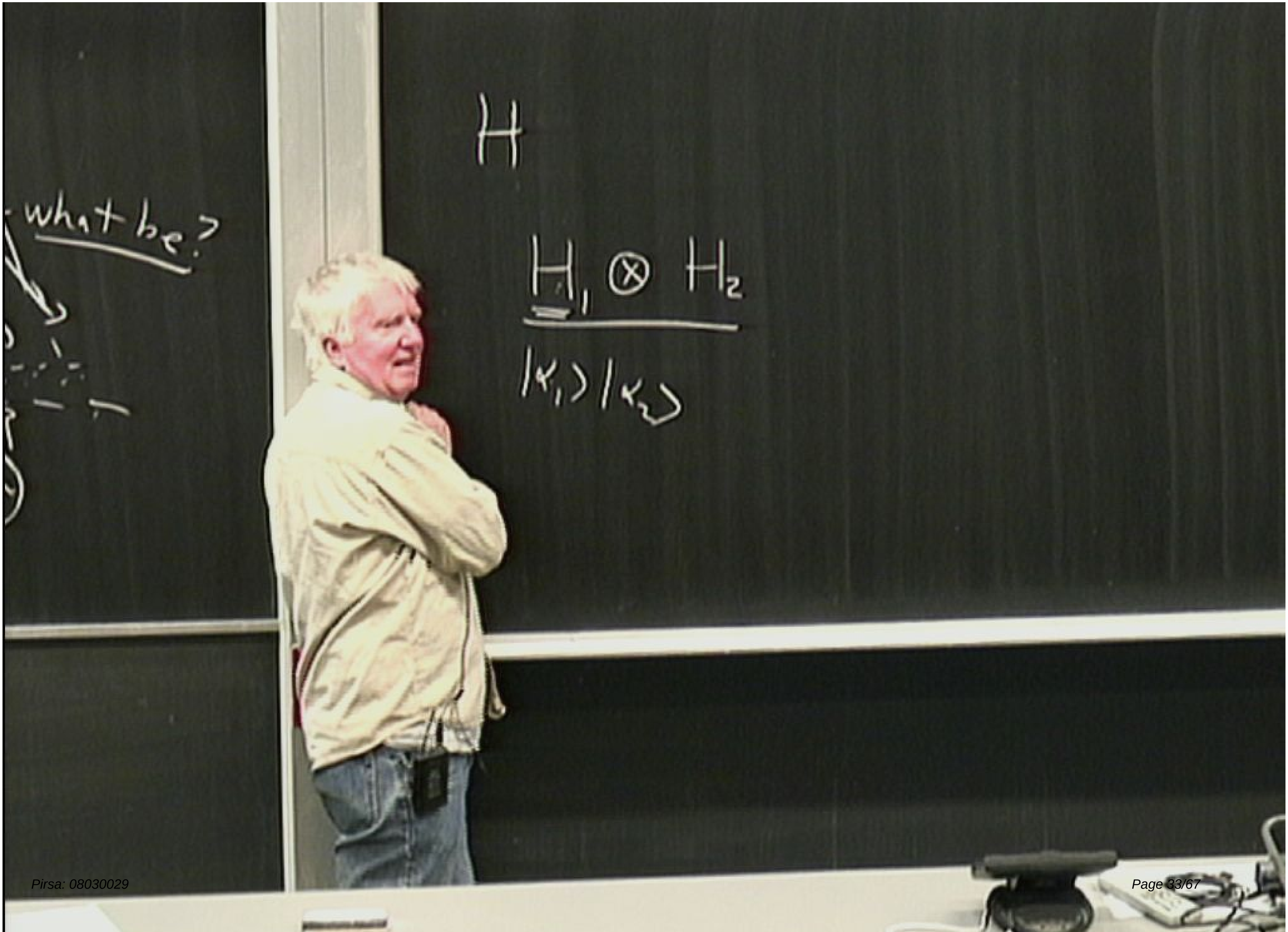
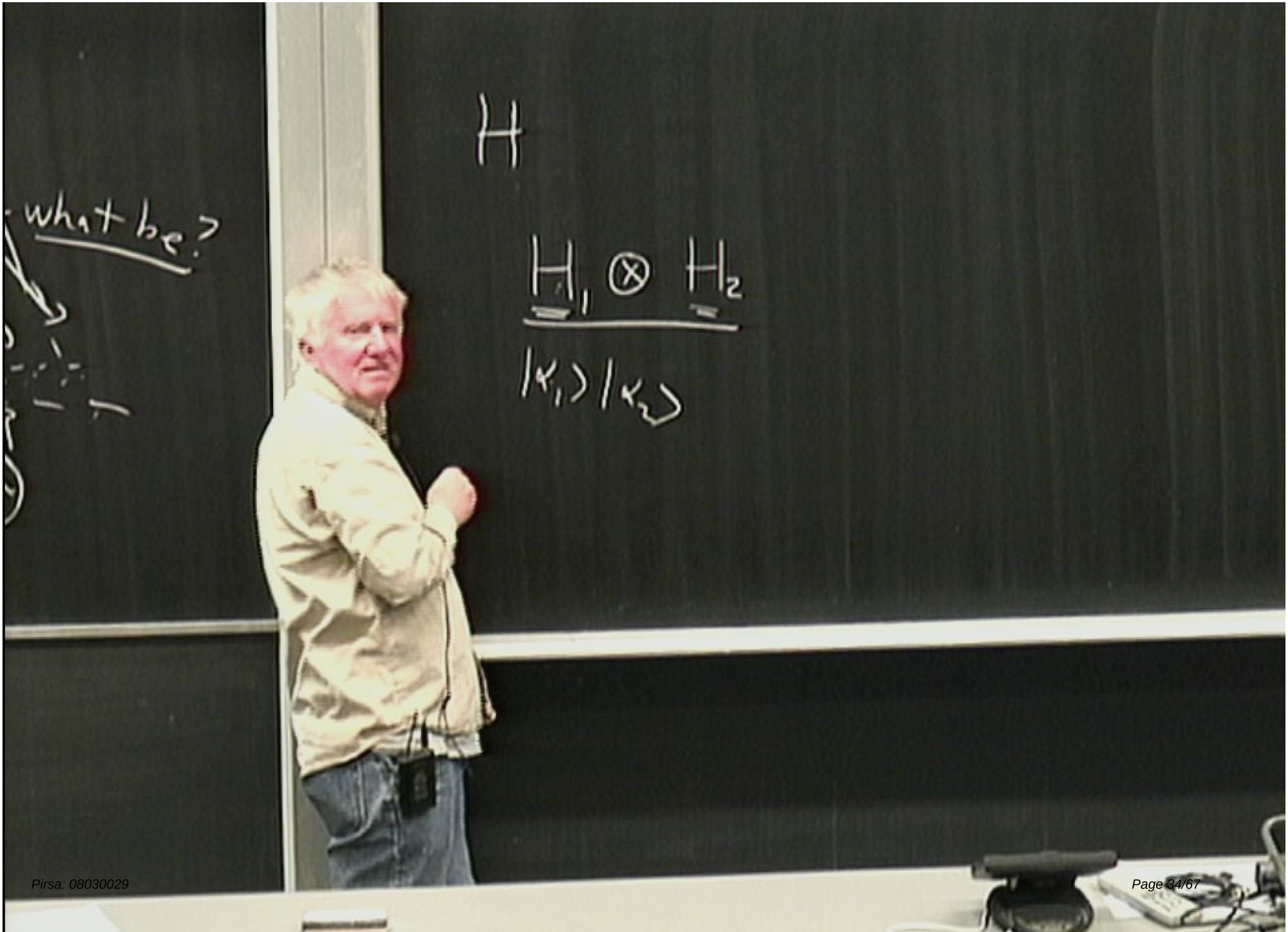
C^n fns
$\varphi_1 \cdots \varphi^{(n)}$
Banach
K (big)
φ_0

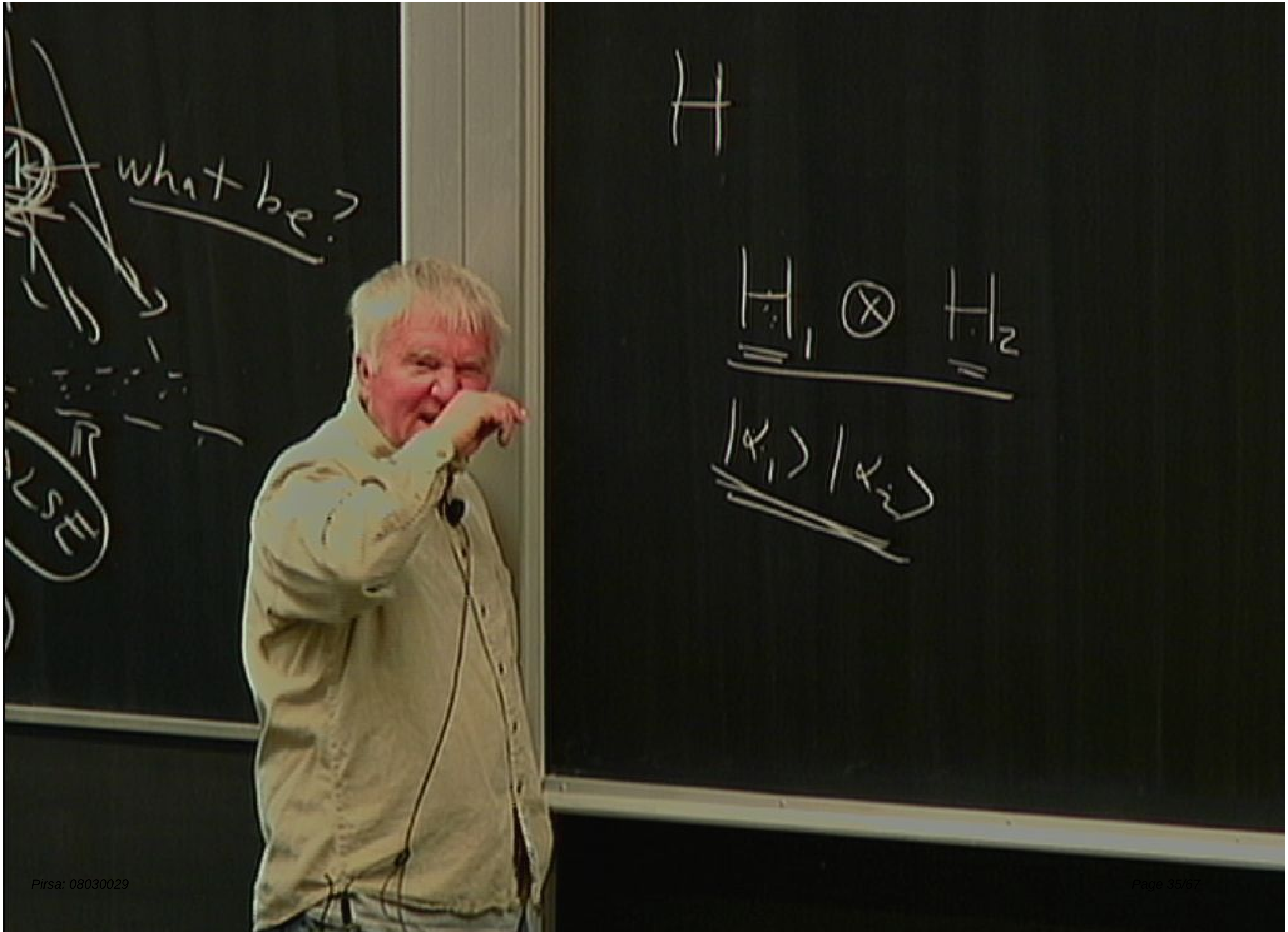

Big manifolds
Class mech
big top vector spaces
M1
M2
states
what be?
Key prop
FALSE

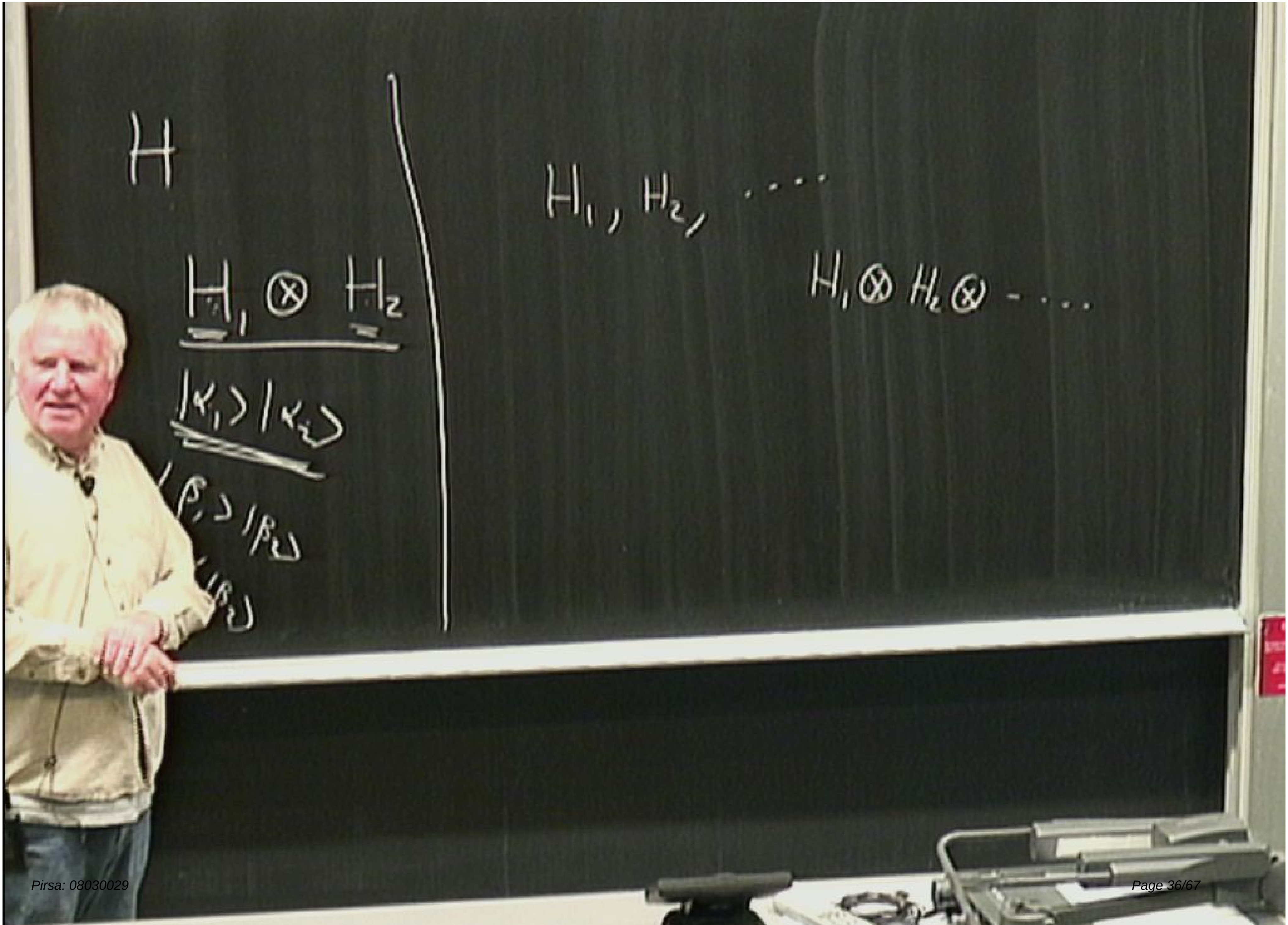

Big manifolds
Class mech
big top vector spaces
M1
M2
states
what be?
FALSE


Big manifolds
Class mech
big top vector spaces
M1
M2
states
what be?
Key prop
FALSE


what be?
H
$H_1 \otimes H_2$
$|\alpha_1\rangle |\alpha_2\rangle$


what be?
H
H₁ ⊗ H₂
|α₁⟩|α₂⟩


what be?
H
H1 ⊗ H2
|α1⟩ |α2⟩


H
$H_1 \otimes H_2$
$|\alpha_1\rangle |\alpha_2\rangle$
$|\beta_1\rangle |\beta_2\rangle$
$H_1, H_2, \ldots$
$H_1 \otimes H_2 \otimes \ldots$

H

$\underline{\underline{H_1} \otimes \underline{H_2}}$

$\underline{\underline{|\alpha_1\rangle |\alpha_2\rangle}}$

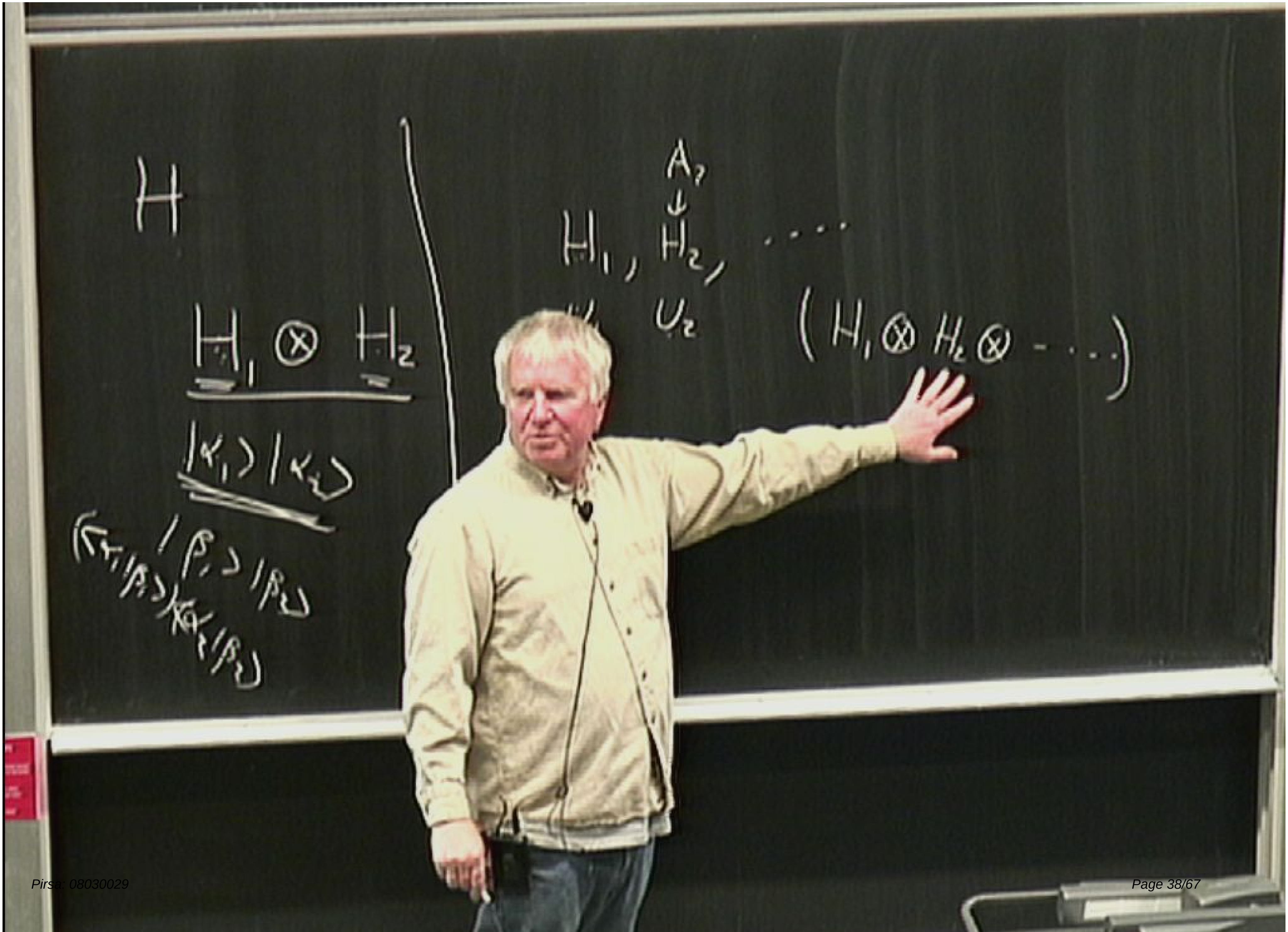
$|\beta_1\rangle |\beta_2\rangle$

$\langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle$

$H_1, H_2, \ldots\ldots$

$U_1 \quad U_2$

$(H_1 \otimes H_2 \otimes \ldots\ldots)$


H
H1 ⊗ H2
|α1⟩|α2⟩
|β1⟩|β2⟩
A?
H1, H2, ...
U2
(H1 ⊗ H2 ⊗ ...)

H

$\underline{H_1 \otimes H_2}$

$|\alpha_1\rangle |\alpha_2\rangle$

$|\beta_1\rangle |\beta_2\rangle$

$\langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle$

$A_?$

$\downarrow$

$H_1, \; H_2, \; \ldots$

$U_1 \quad U_2$

$\mathcal{H} = \underbrace{(H_1 \otimes H_2 \otimes \ldots)}$

$$|\alpha_1\rangle\,|\alpha_2\rangle\,|\alpha_3\rangle \cdots \in \mathcal{H}$$

$$\underset{H_1}{\Uparrow} \qquad \underset{H_2}{\Uparrow}$$

$$K = (\alpha_1, \overset{\in H_2}{\alpha_2}, \ldots\ldots)$$

$$\sum \left(1 - \|\alpha_i\|\right) \text{ Converge}$$

$|\alpha_1\rangle\, |\alpha_2\rangle\, |\alpha_3\rangle \cdots\cdots \in \mathcal{H}$

$\Uparrow \quad \Uparrow$

$H_1 \quad H_2$

"Converse"
(indep of order)

$\underline{K} = (\alpha_1, \alpha_2^{\in H_2} \cdots\cdots)$

$\sum (1 - \|\alpha_i\|)$ converges

$\Downarrow$

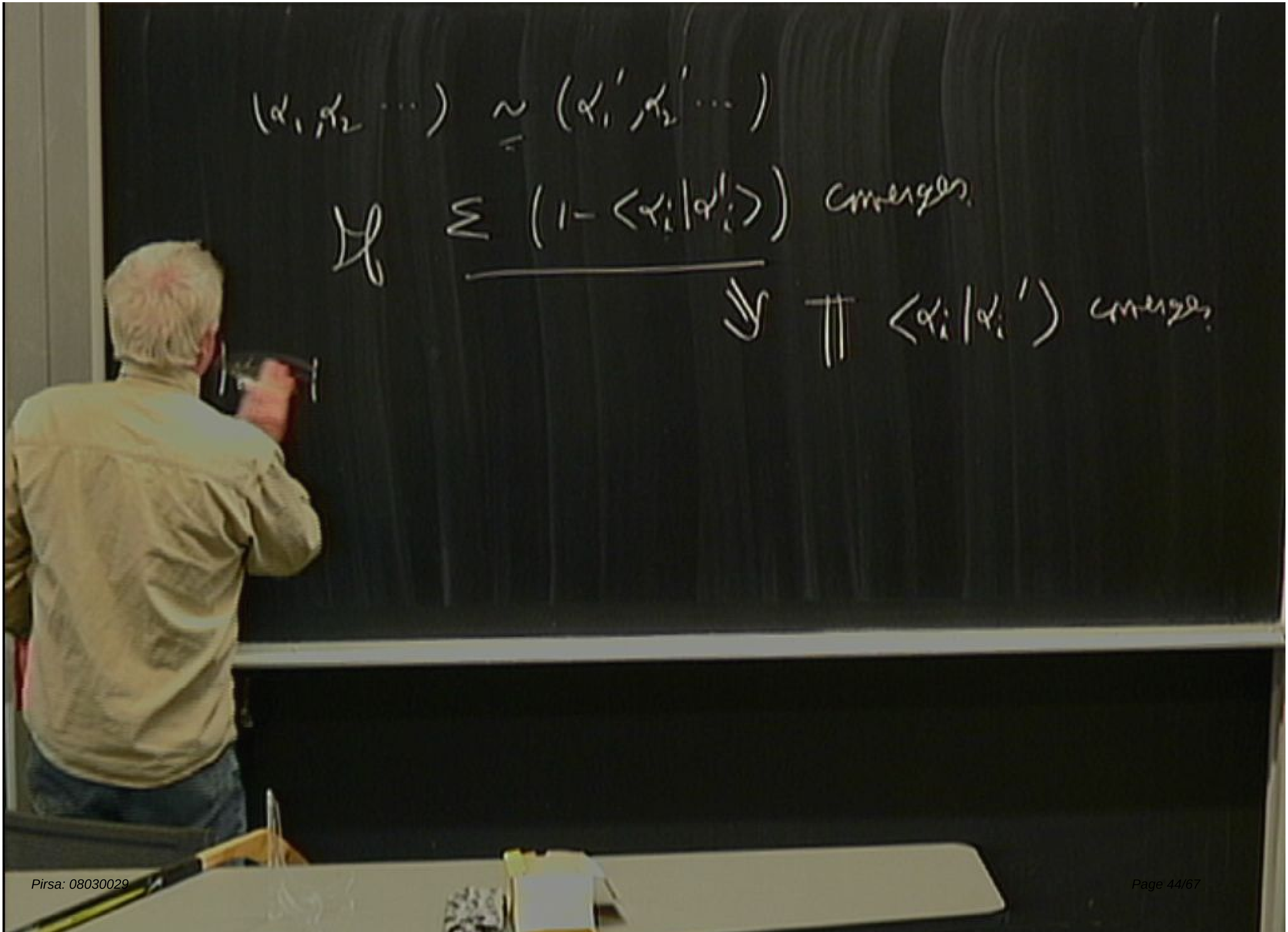
$\prod \|\alpha_i\|$ converges

$$(\alpha_1, \alpha_2 \cdots) \sim (\alpha_1', \alpha_2' \cdots)$$

$$\text{iff} \quad \sum \left(1 - \langle \alpha_i | \alpha_i' \rangle\right) \text{ converges.}$$

$$(\alpha_1, \alpha_2 \dots) \underset{=}{\sim} (\alpha_1', \alpha_2' \dots)$$

$$\text{iff} \quad \underline{\sum \left(1 - \langle \alpha_i | \alpha_i' \rangle\right)} \quad \text{converges.}$$


$(\alpha_1, \alpha_2 \cdots) \approx (\alpha_1', \alpha_2' \cdots)$
iff $\sum (1 - \langle \alpha_i | \alpha_i' \rangle)$ converges.
$\Downarrow$ $\prod \langle \alpha_i | \alpha_i' \rangle$ converges.

$(\alpha_1, \alpha_2 \cdots) \cong (\alpha_1', \alpha_2' \cdots)$

iff $\underline{\sum \left(1 - \langle \alpha_i | \alpha_i' \rangle\right)}$ converges.

$\Downarrow$ $\prod \langle \alpha_i | \alpha_i' \rangle$ converges.

unit

$(\alpha_1, \alpha_2, \ldots)$

$(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4, \ldots)$

$(\alpha_1, \alpha_2 \cdots) \underset{=}{\sim} (\alpha_1', \alpha_2' \cdots)$

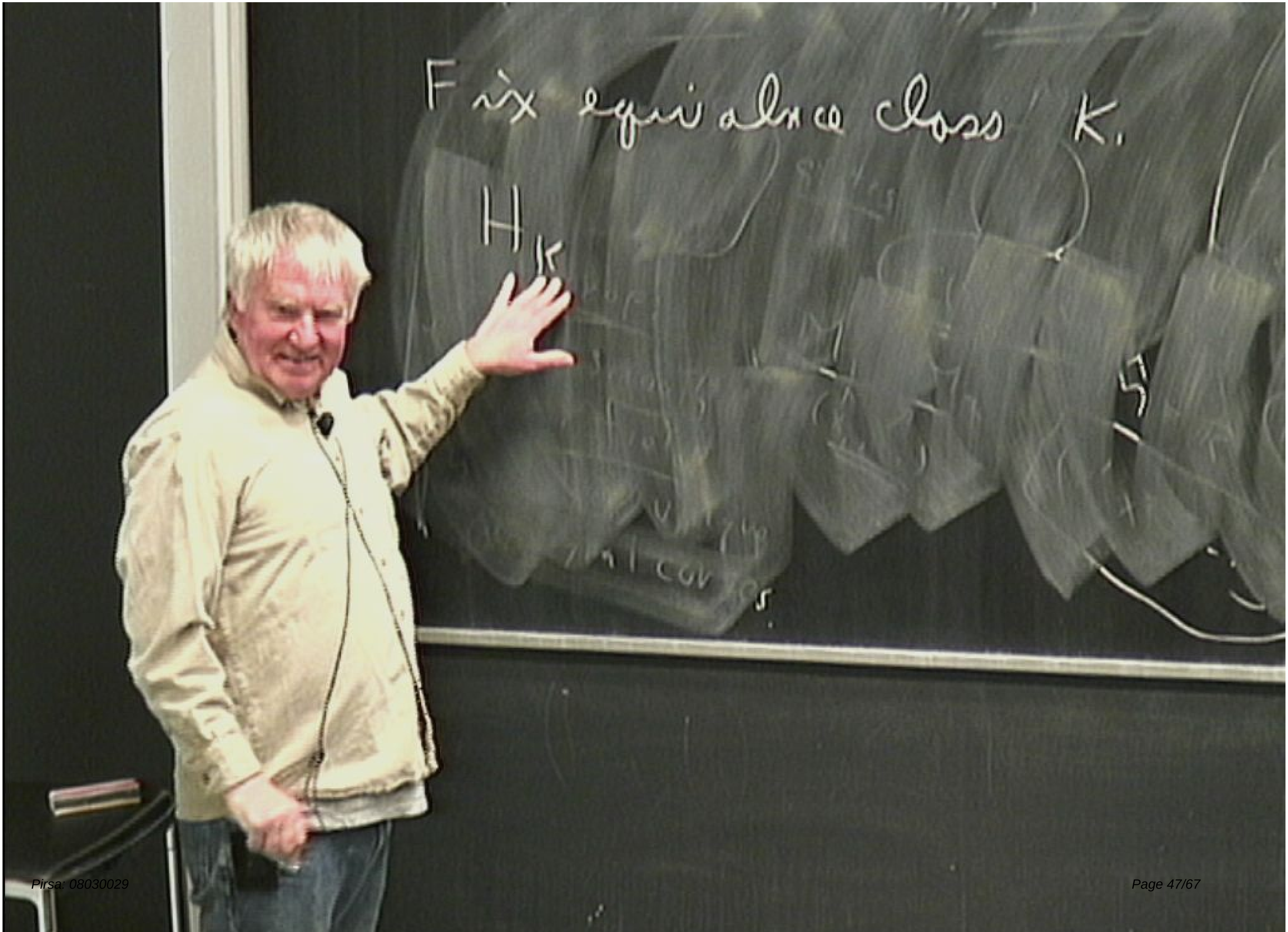
$\forall\ \underline{\sum (1 - \langle \alpha_i | \alpha_i' \rangle)}$ converges.

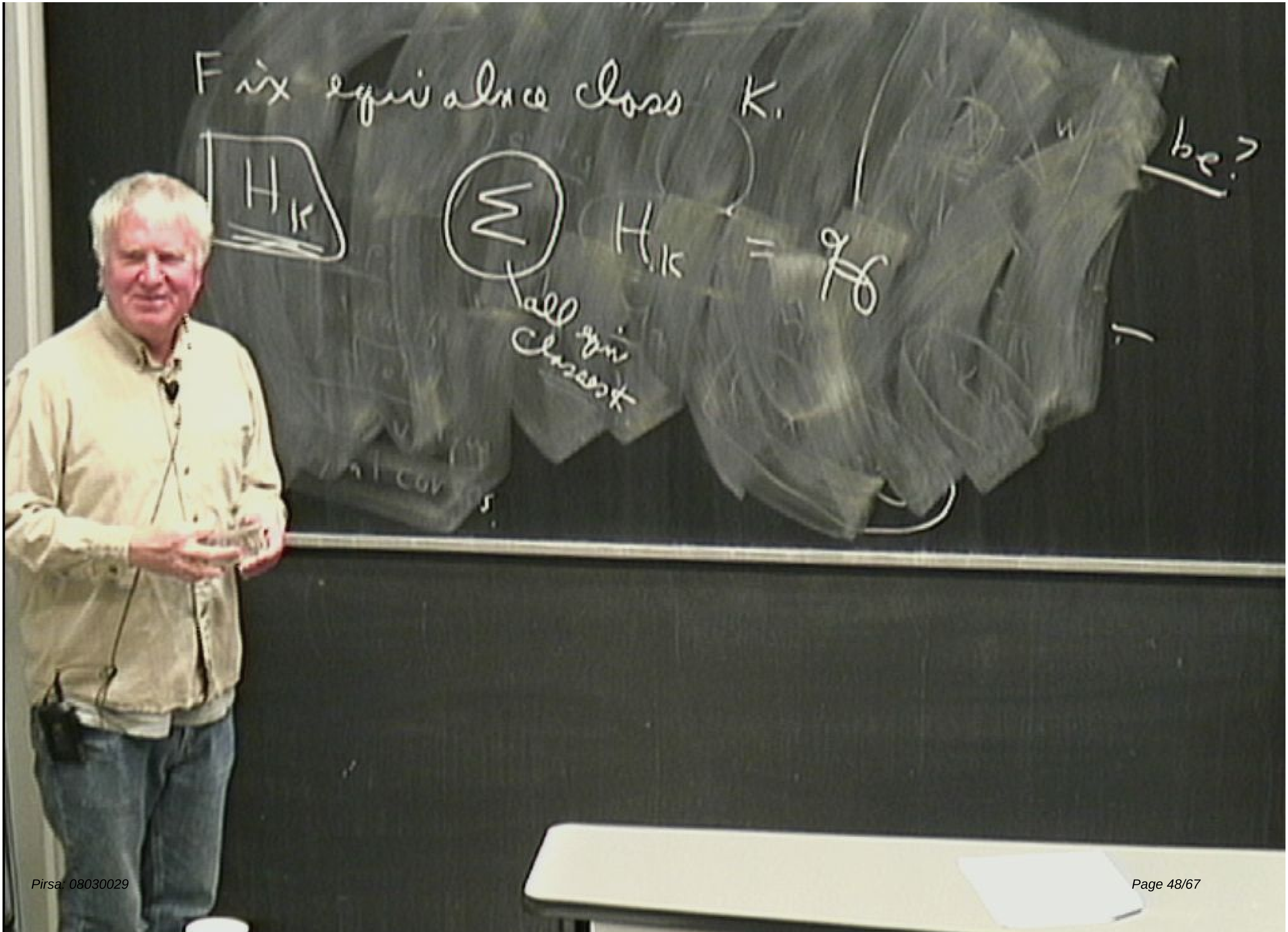
$\Rightarrow \underline{\underline{\prod \langle \alpha_i | \alpha_i' \rangle}}$ converges.

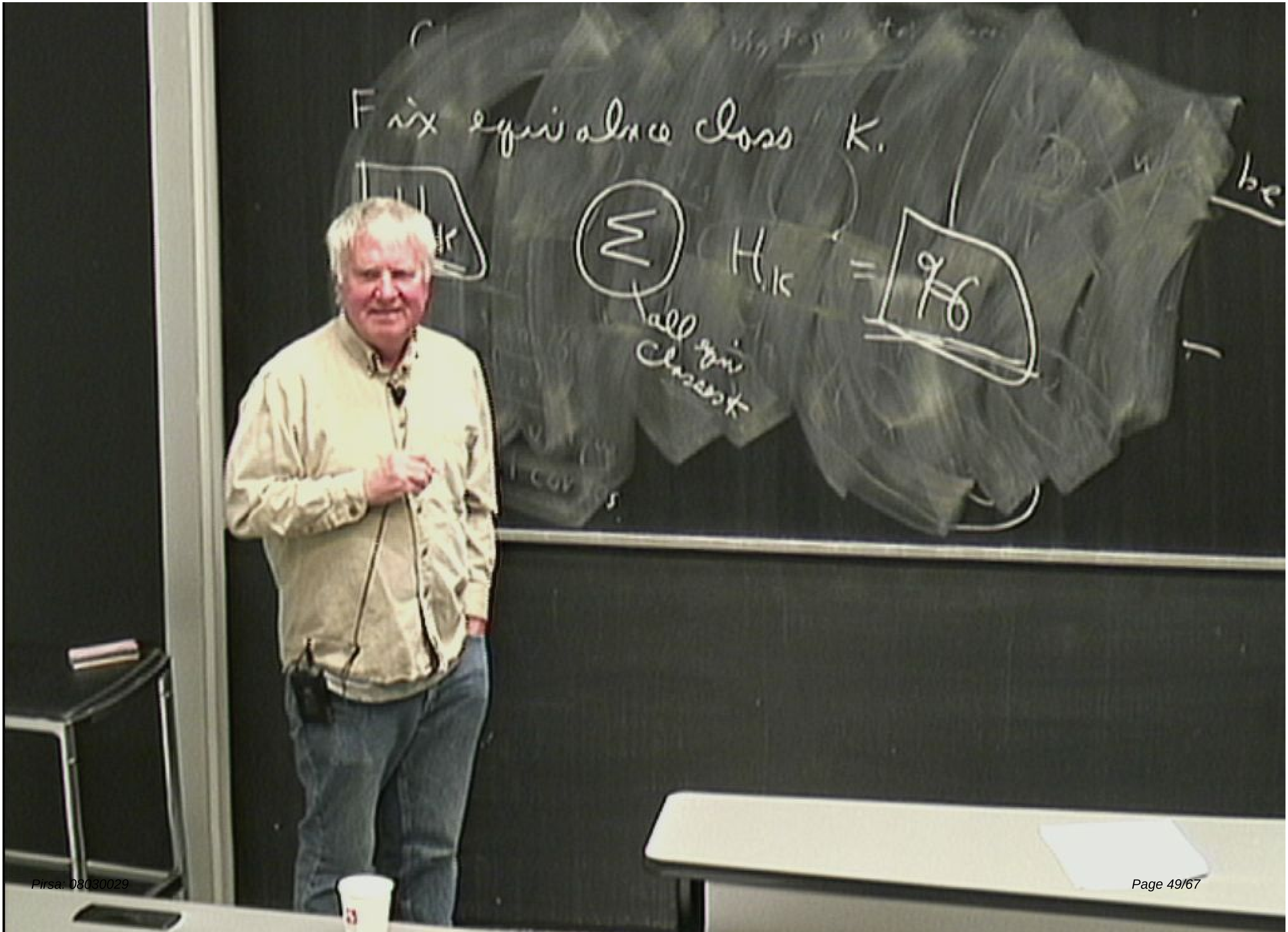
unit

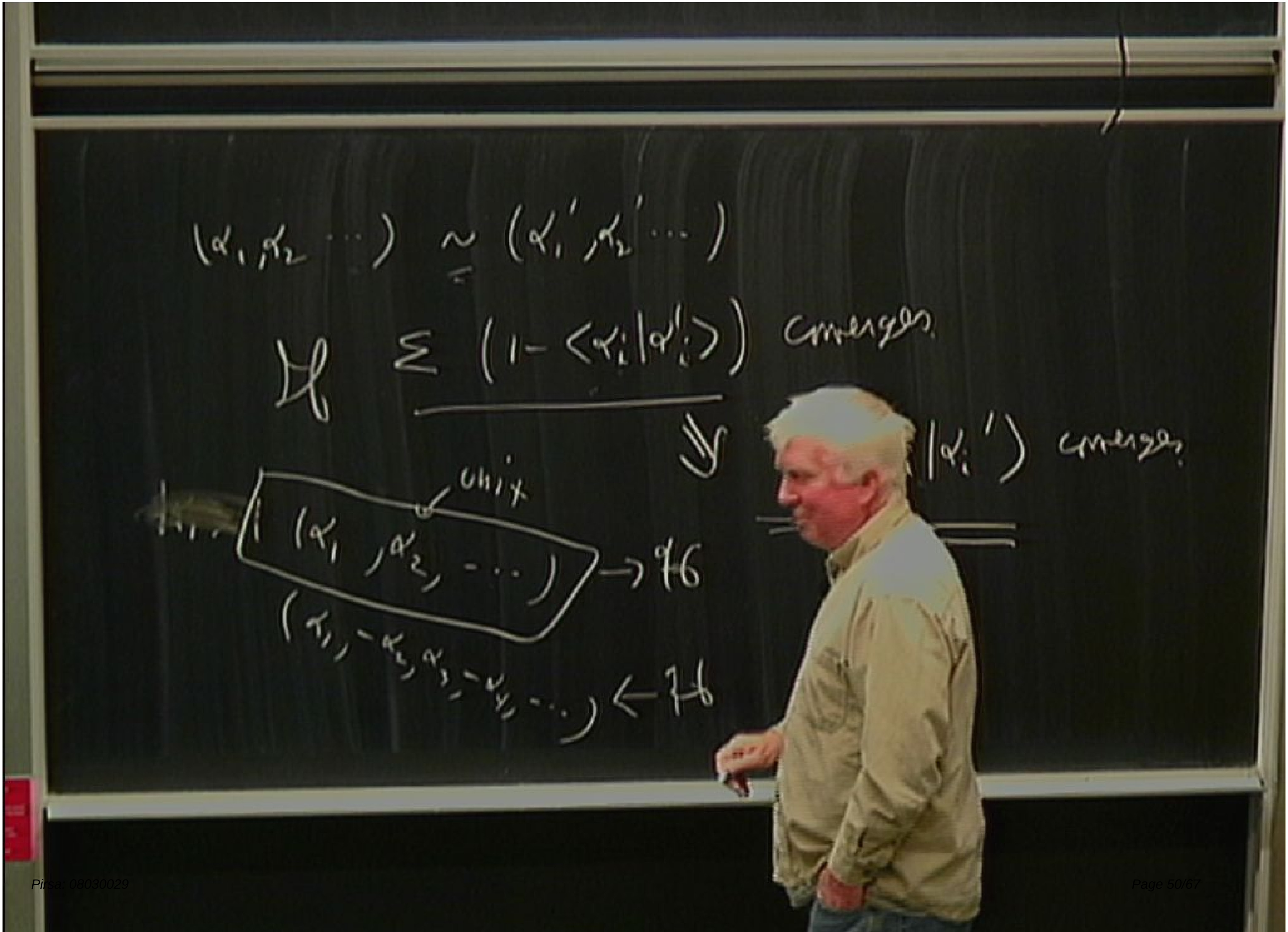
$(\alpha_1, \alpha_2, \cdots)$

$(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4, \cdots)$


Fix equivalence class K.
H_K


Fix equivalence class K.
H_K
Σ H_K = ℋ
be?


Fix equivalence class K.
H_K


$(\alpha_1, \alpha_2 \cdots) \sim (\alpha_1', \alpha_2' \cdots)$
iff $\sum (1 - \langle \alpha_i | \alpha_i' \rangle)$ converges
$\Downarrow$
$|\alpha_i'\rangle)$ converges
unit
$(\alpha_1, \alpha_2, \ldots) \to \mathcal{H}$
$(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4, \ldots) \leftarrow \mathcal{H}$

H

$\underline{H_1 \otimes H_2}$

$\underline{|\alpha_1\rangle|\alpha_2\rangle}$

$|\beta_1\rangle|\beta_2\rangle$

$\langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle$

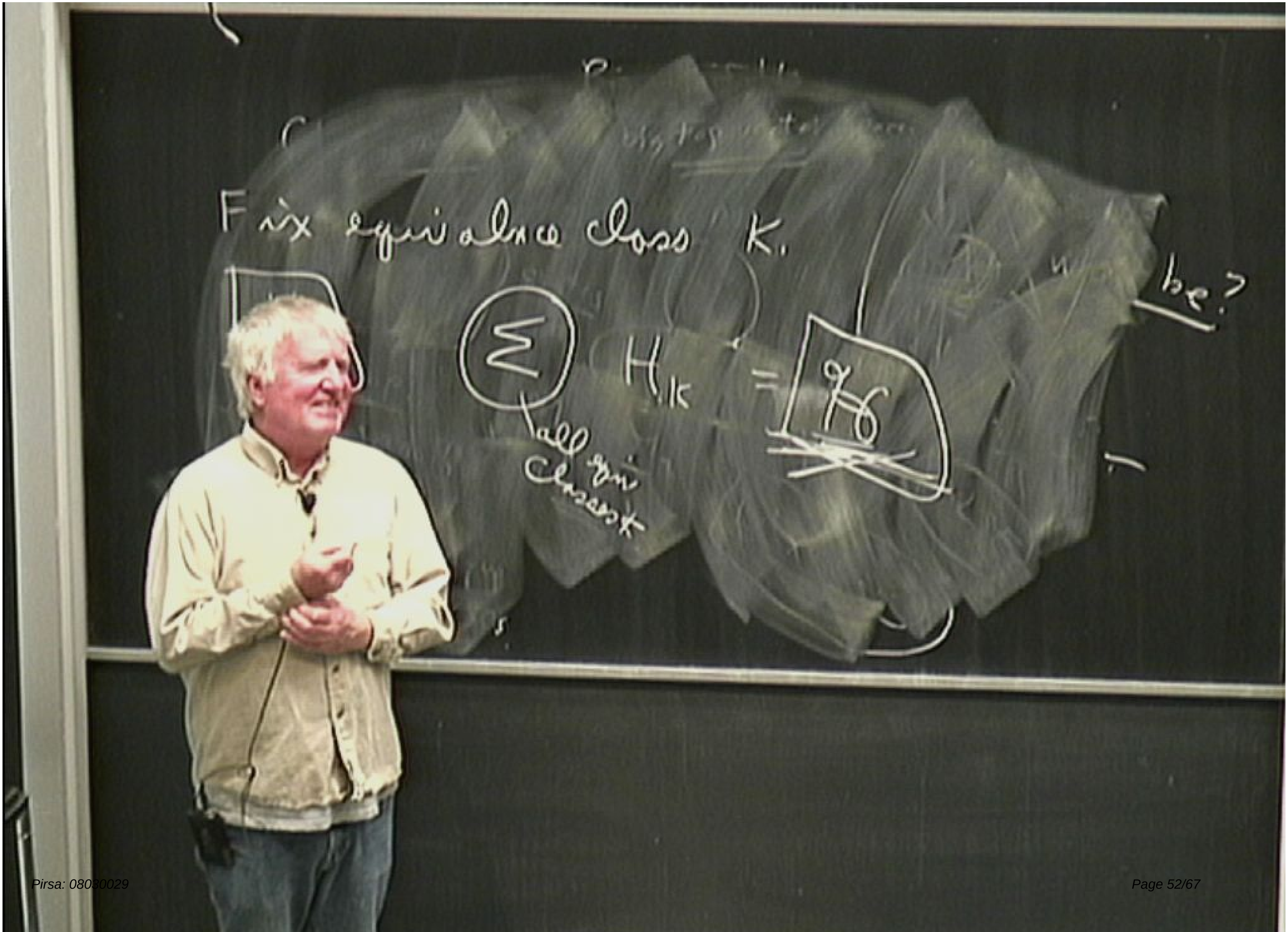
$A_?$

$\downarrow$

$H_1, H_2, \ldots$

$U_1 \quad U_2 \quad \mathcal{H} = \underbrace{(H_1 \otimes H_2 \otimes \ldots)}$

$|\alpha_1\rangle, |\alpha_2\rangle, \ldots$


Fix equivalence class K.
be?

H

$\underline{H_1 \otimes H_2}$

$\underline{|\alpha_1\rangle|\alpha_2\rangle}$

$|\beta_1\rangle|\beta_2\rangle$

$|\beta_1\rangle|\alpha_2\rangle$

$|\alpha_2\rangle|\beta_2\rangle$

$A_?$

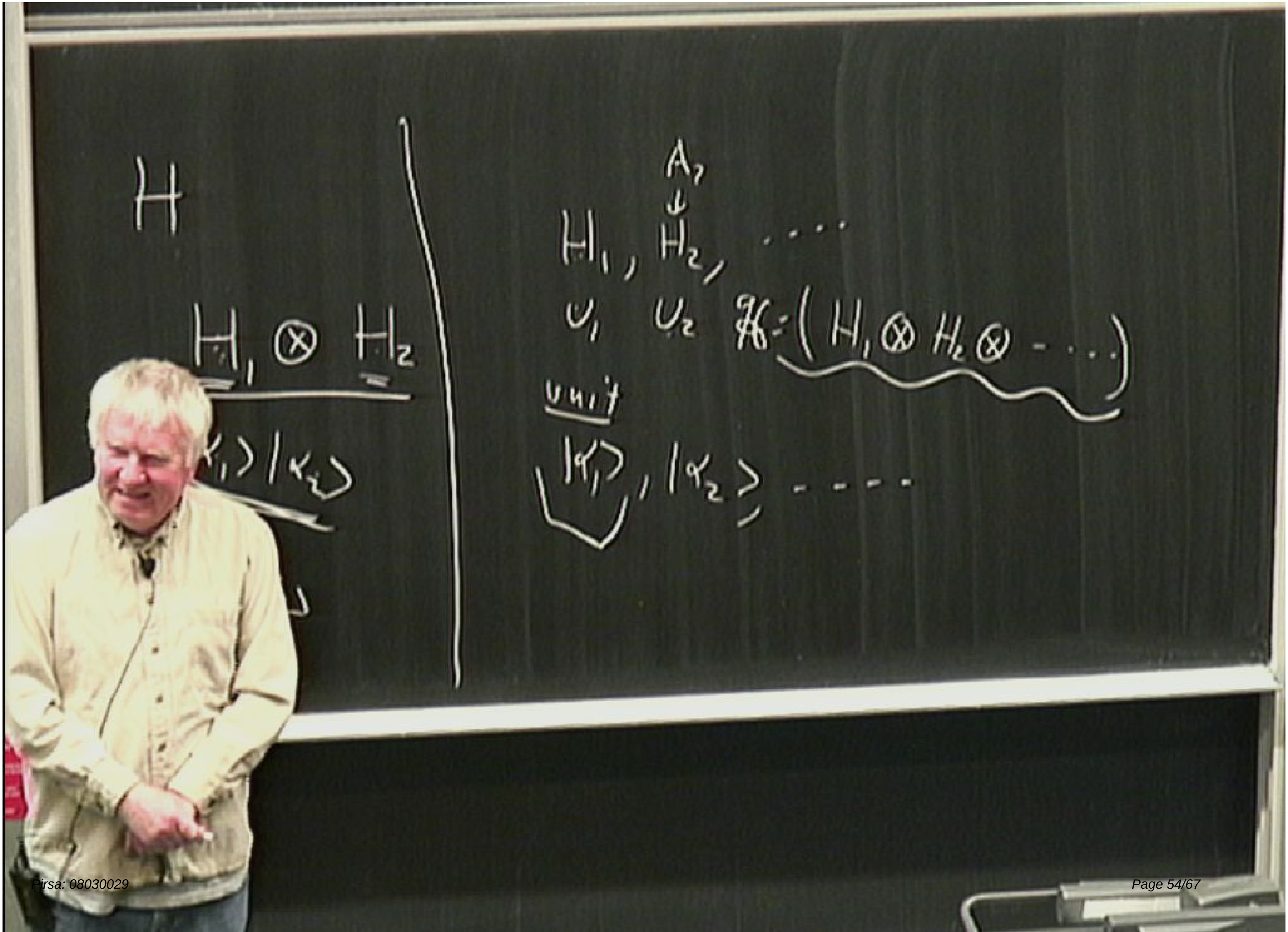
$\downarrow$

$H_1, H_2, \ldots$

$U_1 \quad U_2 \quad \mathcal{H} = (H_1 \otimes H_2 \otimes \ldots)$

unit

$|\alpha_1\rangle, |\alpha_2\rangle, \ldots$


H
$H_1 \otimes H_2$
$|\alpha_1\rangle|\alpha_2\rangle$
A?
$H_1, H_2, \ldots$
$U_1 \quad U_2 \quad \mathcal{H} = (H_1 \otimes H_2 \otimes \ldots)$
unit
$|\alpha_1\rangle, |\alpha_2\rangle \ldots$

H

$H_1 \otimes H_2$

$|\alpha_1\rangle |\alpha_2\rangle$

$|\beta_1\rangle |\beta_2\rangle$

$|\beta_1\rangle \cancel{|\alpha_2\rangle} |\beta_2\rangle$

$A_?$

$\downarrow$

$H_1, H_2, \ldots$

$U_1 \quad U_2 \quad \mathcal{H} = (H_1 \otimes H_2 \otimes \cdots)$

unit

$|\alpha_1\rangle, |\alpha_2\rangle, \ldots$

H

$\underline{H_1 \otimes H_2}$

$|\alpha_1\rangle|\alpha_2\rangle$

$|\beta_1\rangle|\beta_2\rangle$

$\langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle$

$A_?$

$\downarrow$

$H_1, H_2, \ldots$

$U_1 \quad U_2 \quad \mathcal{H} = (H_1 \otimes H_2 \otimes \ldots)$

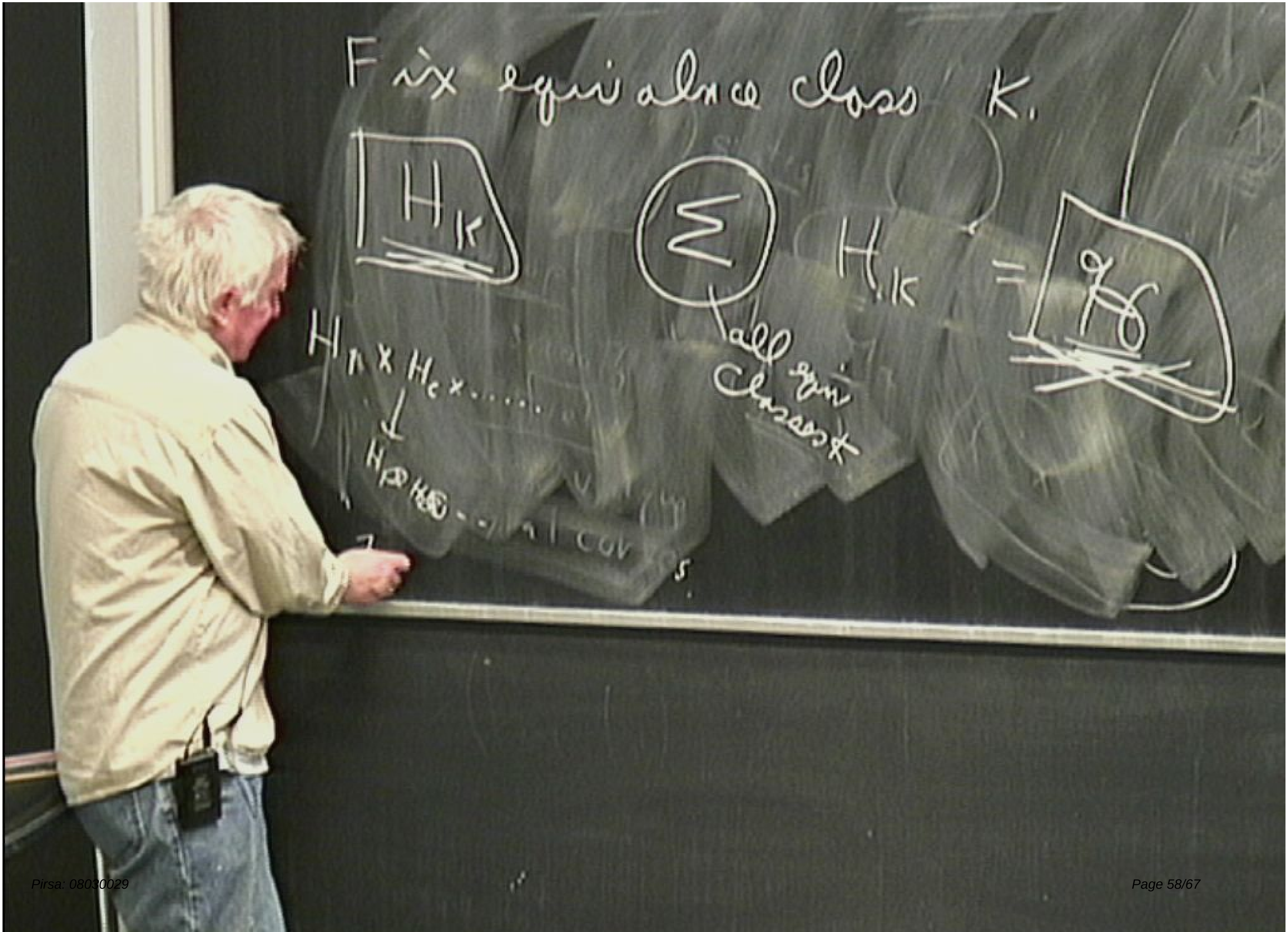
unit

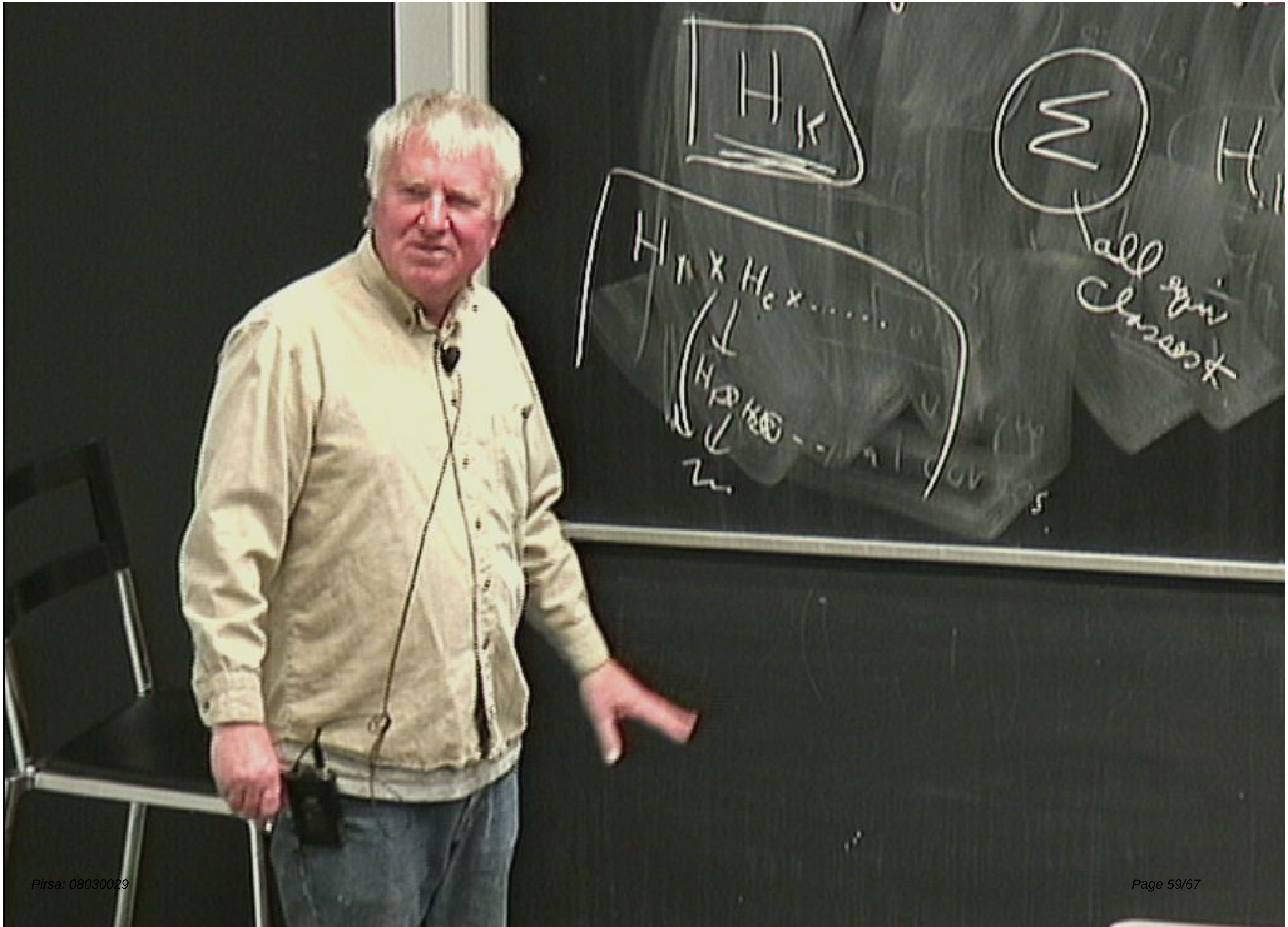
$|\alpha_1\rangle, |\alpha_2\rangle \ldots$

$\boxed{\text{Actual system}}$

Fix equivalence class K.

$$\boxed{H_K} \qquad \sum_{\substack{\text{all equiv} \\ \text{classes } K}} H_K = \boxed{\mathcal{H}}$$


Fix equivalnce class K.
H_K
Σ H_K = ℋ
all equiv classes K
H_n × H_c × ......



$K \sim K'$

$|\kappa_1\rangle|\kappa_2\rangle \ldots \quad |\kappa_1'\rangle|\kappa_2'\rangle \ldots$

syst

ns φ.

Ban

ns

$\varphi \ldots \varphi^{(n)}$

Fix equivalence class K.

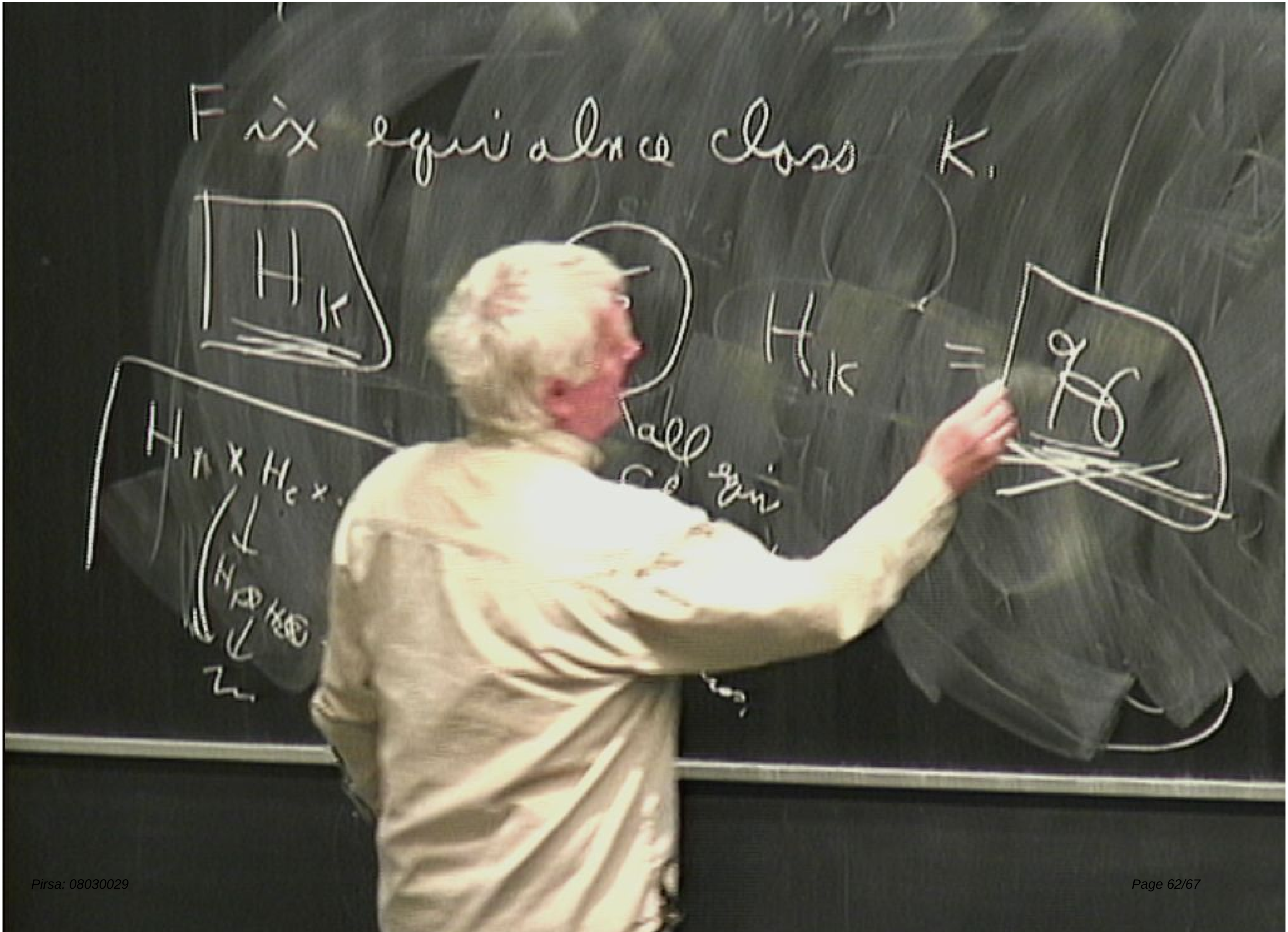
$$\boxed{H_K} \qquad \sum_{\text{all equiv classes } K} H_K = \boxed{\mathcal{H}}$$

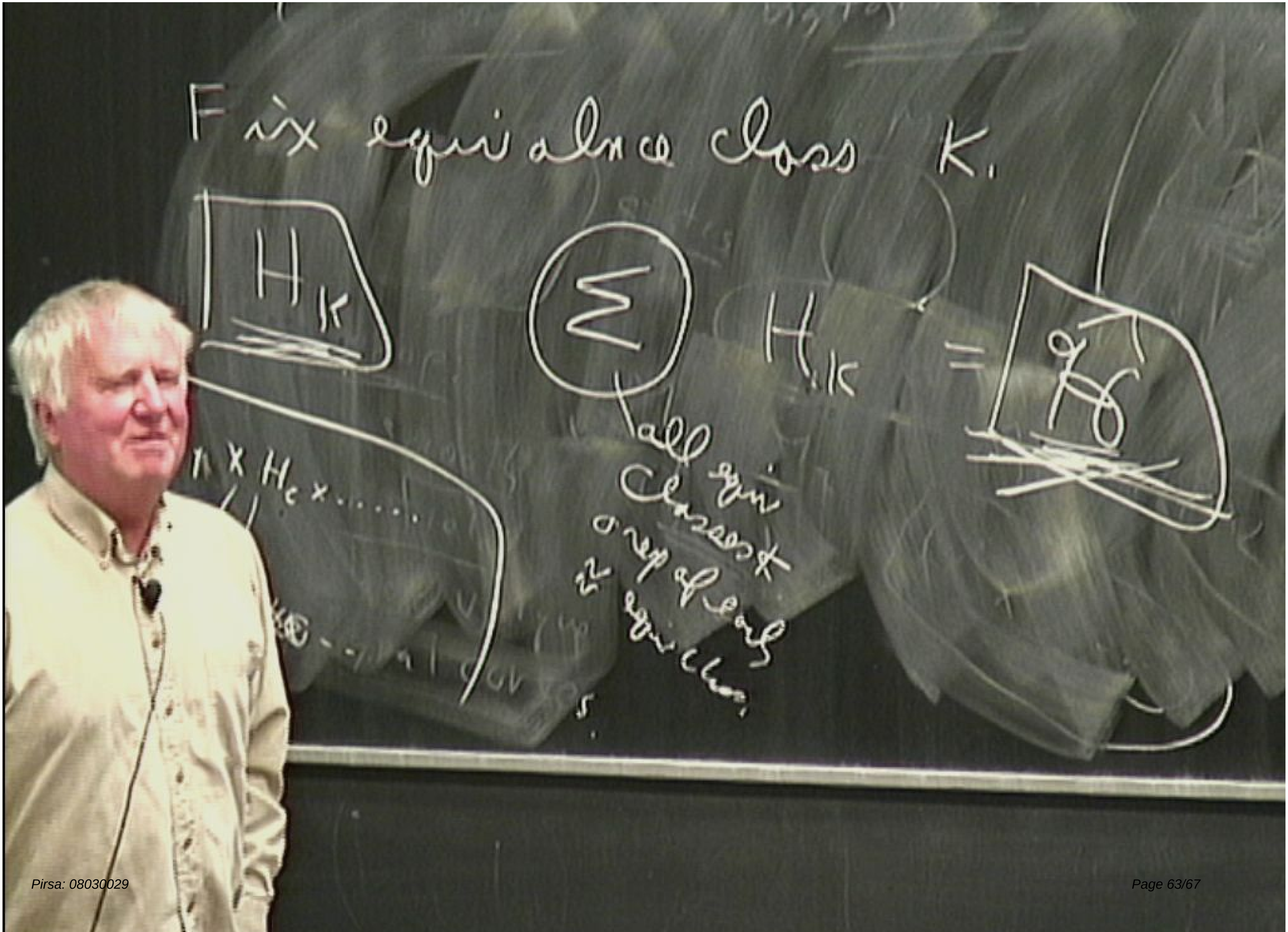
be?

$$H_n \times H_e \times \ldots$$

$$\downarrow$$

$$H_n \otimes H_e \ldots$$


Fix equivalence class K.
H_K
H_K =


Fix equivalence class K.
H_K
Σ H_K = ℋ
all equiv classes
1 rep of each
equiv class

Fix equivalence class K.

$$\boxed{H_K} \quad \bigoplus_{K} H_{K} = \boxed{\mathcal{H}}$$

all equiv classes K

one rep of each equiv class

$$\boxed{H_K \times H_L \times \ldots\ldots}$$

$$\downarrow$$

$$H_K \otimes H_L \otimes \ldots$$

$$\downarrow$$

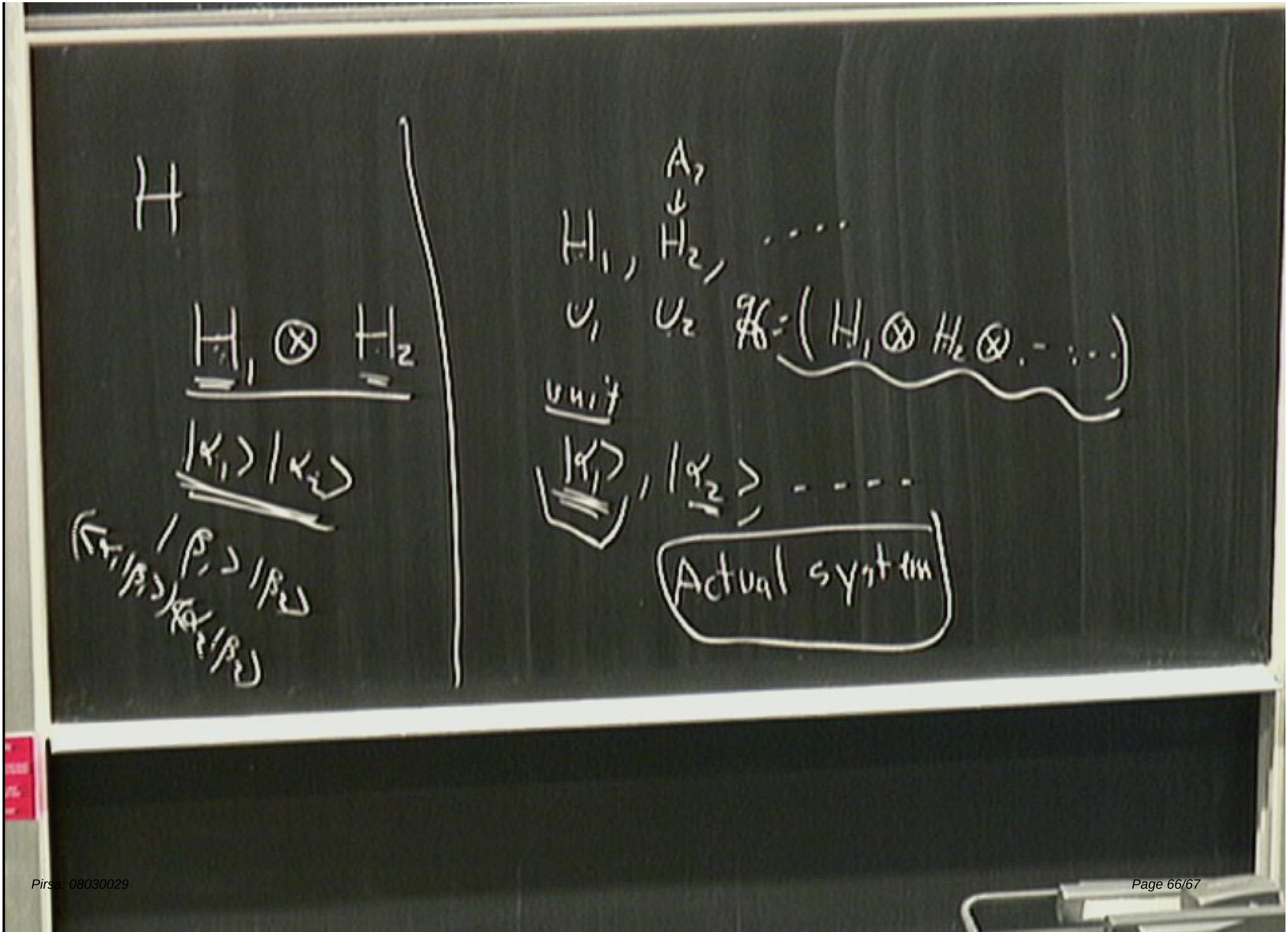
Fix equivalence class K.

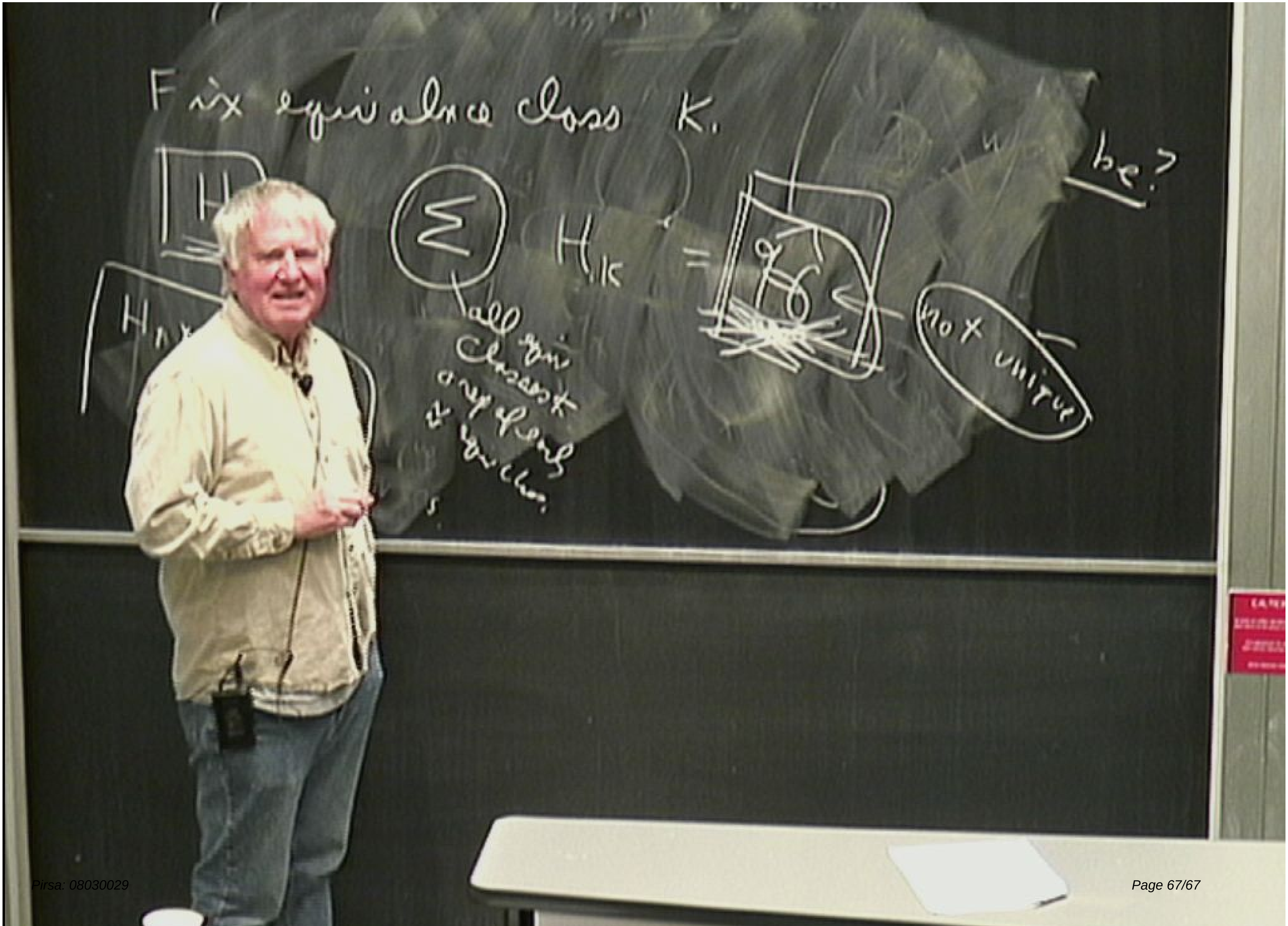
$$\boxed{H_K} \qquad \Sigma H_K = \boxed{\mathcal{H}}$$

not unique

all equi classes of reps of the equi class.

$$H_n \times H_c \times \ldots\ldots$$
$$\downarrow$$
$$H_n \otimes H_c \ldots$$
$$\downarrow$$
$$\mathbb{Z}$$


H
$H_1 \otimes H_2$
$|\alpha_1\rangle|\alpha_2\rangle$
$|\beta_1\rangle|\beta_2\rangle$
$\langle\alpha_1|\beta_1\rangle\langle\alpha_2|\beta_2\rangle$
$A_?$
$H_1, H_2, \ldots$
$U_1 \quad U_2 \quad \mathcal{H} = (H_1 \otimes H_2 \otimes \ldots)$
unit
$|\alpha_1\rangle, |\alpha_2\rangle, \ldots$
Actual system


Fix equivalence class K.
be?
not unique