

Title: Introduction of bosonic fields into causal set theory

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Abstract: The purpose of this talk is to describe bosonic fields and their Lagrangians in the causal set context. Spin-0 fields are defined to be real-valued functions on a causal set. Gauge fields are viewed as $SU(n)$ -valued functions on the set of pairs of elements of a causal set, and gravity is viewed as the causal relation itself. The purpose of this talk is to come up with expressions for the Lagrangian densities of these fields in such a way that they approximate the Lagrangian densities expected from regular Quantum Field Theory on a differentiable manifold in the special case where the causal set is a random sprinkling of points in the manifold. I will then conjecture that that same expression is appropriate for an arbitrary causal set.

BOSONIC FIELDS AND THEIR LAGRANGIANS IN CAUSAL SET THEORY

Roman Sverdlov, University of Michigan

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QFT in causal sets = extrapolation

Example of extrapolation: trigonometric functions of non-reals

Geometry \rightarrow Taylor series a theorem \rightarrow Taylor series a definition \rightarrow no geometry

Coordinates \rightarrow Theorem: $L = \dots \rightarrow$ Definition: $L = \dots \rightarrow$ No coordinates

Trigonometry:

- Problem: compute the sin of an angle without a device to measure angles and/or a ruler to measure lengths. NOTE: angles and lengths DO exist but are unknown.
- Solution: Taylor Series
- Definition: neither angles nor lengths exist. Taylor series is definition of trigonometric function.

Key: Unknown \rightarrow Non-existent

GETTING RID OF COORDINATES IN QFT:

a) Given:

- Lorentzian manifold, which both has coordinates and is a continuum
- p_1, \dots, p_n are Poisson distribution of points in bounded region of that space
- $\varphi, A^{\mu\nu}$, and $g_{\mu\nu}$ are differentiable functions on Minkowski space that vary very slowly compared to the density of scattered points
- Causal relations between the points
- $\varphi(p_1) = \dots, \varphi(p_2) = \dots, \dots, \varphi(p_n) = \dots$
- $\exp \int_{\Gamma(p_i, p_j)} \Gamma^\mu A^\nu_{\mu\nu} dx^\mu = U(p_i, p_j)$ where $I(p_i, p_j)$ is a geodesic connecting p_i and p_j

Unknowns: $p_i^\mu, A^{\mu\nu}(p_i)$ and $g_{\mu\nu}(p_i)$

Find: Lagrangian densities associated with each of the above fields around p_1, \dots, p_n

b) Solution: some expressions that don't explicitly involve coordinates

c) Definition: expressions in part b are actually definitions of Lagrangian densities for causal sets.

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GETTING RID OF COORDINATES IN QFT:

a) Given:

- Lorentzian manifold, which both has coordinates and is a continuum
- p_1, \dots, p_n are Poisson distribution of points in bounded region of that space
- φ , A^μ , and $g_{\mu\nu}$ are differentiable functions on Minkowski space that vary very slowly compared to the density of scattered points
- Causal relations between the points
- $\varphi(p_1), \dots, \varphi(p_2), \dots, \varphi(p_n)$
- $\exp \int_{l(p_i, p_j)} A^\mu dx^\mu = U(p_i, p_j)$ where $l(p_i, p_j)$ is a geodesic connecting p_i and p_j

Unknowns: p_i^μ , $A^\mu(p_i)$ and $g_{\mu\nu}(p_i)$

Find: Lagrangian densities associated with each of the above fields around p_1, \dots, p_n

b) Solution: some expressions that don't explicitly involve coordinates

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SCALAR FIELD

SIMPLER PROBLEM: EXTRA ASSUMPTIONS

- (vii) We are in a FLAT Minkowski space
- (vii) φ is linear

$$S = \{P_1, \dots, P_n\}$$

We are interested in evaluating $L(\varphi; p)$ where $p \in S$

Find $q \in S$ such that $\#(S \cap \alpha(p, q)) > M$ where $\alpha(p, q) = \{r \mid p < r < q\}$ is Alexandroff set based on p and q .

Select coordinate system where t axis passes through p and q with origin lying between these two points.

$$\varphi(x) = a^t x_t + b$$

$$\alpha(p, q) = \{t \mid p < t < q\}$$

$$A_t(q^t - p^t)^2 = \frac{1}{\rho} \#(S \cap \alpha(p, q))$$

$$a_t = \frac{\varphi(q) - \varphi(p)}{q^t - p^t} = \frac{(\varphi(q) - \varphi(p))(\rho t)^{1/d}}{(\#(S \cap \alpha(p, q)))^{1/d}} \quad (1)$$

$$\begin{aligned} \frac{1}{\rho^2} \sum_{r, s \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 &= \int_{r, s \in S \cap \alpha(p, q)} d^d r d^d s a^t a^t (r_t - s_t)(r_t - s_t) \\ &= \frac{(\#(S \cap \alpha(p, q)))^{2+2/d}}{\rho^{2+2/d}} (B_d(a_t)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2) \end{aligned}$$

$$B_d(a_t)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2 = \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 \quad (2)$$

$$\left(1 + \frac{B_d}{C_d}\right) (1)^2 = \frac{1}{C_d} (2) \Rightarrow$$

$$\Rightarrow a^t a_t = \left(1 + \frac{B_d}{C_d}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho t)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} = \frac{1}{C_d} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2$$

$$L(\varphi; p) = \frac{1}{2} \left(1 + \frac{B_d}{C_d}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho t)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} = \frac{1}{2C_d} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 = \frac{m^2 \varphi^2(p)}{2}$$



$$e_i = -\frac{1}{2}(\omega v_i - \omega \phi^2) \quad \int \dots$$



$$\alpha(p, q) = \{r \mid p \leq r \leq q\}$$

$$e. = -\frac{1}{2} m (\dot{x}_m - \omega t)^2 \quad S \quad | \quad \dots$$



$$\psi = a^m x_m + b$$

$$\alpha(p, q) = \{r \mid p \leq r \leq q\}$$

$$\# \alpha(p, q) > M$$



$$\psi = a^m x_m + b$$

$$\mathcal{L}(\psi) = \frac{1}{2} (a^m a_m) + \frac{m^2}{2} \psi^2$$

$$\alpha(p, q) = \{r \mid p \leq r \leq q\}$$

$$a^0 = \frac{\psi(q) - \psi(p)}{\tau(p, q)}$$

$$\# \alpha(p, q) > M$$

$$V(\alpha(p, q)) = K_d (\tau(p, q))^d$$

$e. - \dots$



$$\psi = a^{\mu} x_{\mu} + b$$

$$\alpha(\psi) = \frac{1}{2} (a^{\mu} a_{\mu}) + \frac{m^2}{2} \psi^2$$

$$\alpha(p, q) = \{t \mid p \leq t \leq q\}$$

$$a^0 = \frac{\psi(q) - \psi(p)}{\tau(p, q)}$$

$$\# \alpha(p, q) > M$$

$$\sqrt{\alpha(p, q)} = K_d (\tau(p, q))^d$$

$$p \# \alpha(p, q)$$

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Find $q \in S$ such that $\#(S \cap \alpha(p, q)) > M$ where $\alpha(p, q) = \{r \mid p < r < q\}$ is Alexandroff set based on p and q .

Select coordinate system where t axis passes through p and q with origin lying between these two points.

$$\varphi(x) = a^t x_t + b$$

$$\alpha(p, q) = \{r \mid p < r < q\}$$

$$A_d(q^2 - p^2)^d = \frac{1}{\rho} \#(S \cap \alpha(p, q))$$

$$a_t = \frac{\varphi(q) - \varphi(p)}{q^2 - p^2} = \frac{\varphi(q) - \varphi(p)}{\#(S \cap \alpha(p, q))^{1/d}} \quad (1)$$

$$\frac{1}{\rho^2} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 = \int_C \alpha^t \alpha^t (r_t - s_t)(r_t - s_t) \\ = \frac{\#(S \cap \alpha(p, q))}{\rho^{2+2/d}} (B_d(a_t)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2)$$

$$B_d(a_t)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2 = \frac{\rho^d}{\#(S \cap \alpha(p, q))^{2+2/d}} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 \quad (2)$$

$$\left(1 + \frac{B_d}{C_d}\right) a_t^2 = \frac{1}{C_d} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2$$

$$\Rightarrow a^t a_t = \left(1 + \frac{B_d}{C_d}\right) \frac{\rho^d}{\#(S \cap \alpha(p, q))^{2+2/d}} \frac{1}{C_d} \frac{\rho^{2/d}}{\rho^2} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2$$

$$L(\varphi; p) = \frac{1}{2} \left(1 + \frac{B_d}{C_d}\right) \frac{\rho^d}{\#(S \cap \alpha(p, q))^{2+2/d}} \frac{1}{2C_d} \frac{\rho^{2/d}}{\rho^2} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(s))^2 = \frac{m^2 \sigma^2(p)}{2}$$



$$\psi = a^\mu x_\mu + b$$

$$\mathcal{L}(\psi) = \frac{1}{2} (a^\mu a_\mu) + \frac{m^2}{2} \psi^2$$

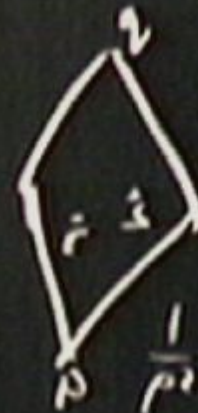
$$\alpha(p, \mathcal{Q}) = \{r \mid p \leq r \leq \mathcal{Q}\}$$

$$a^0 = \frac{\psi(\mathcal{Q}) - \psi(p)}{\tau(p, \mathcal{Q})}$$

$$\# \alpha(p, \mathcal{Q}) > M$$

$$\sqrt{\# \alpha(p, \mathcal{Q})} = K_d \left(\tau(p, \mathcal{Q}) \right)^d$$

$$a^\mu a_\mu$$



$$\frac{1}{\tau^2} \sum (\psi(r) - \psi(p))$$

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$$a_a = \frac{\varphi(q) - \varphi(p)}{q^2 - p^2} = \frac{(\varphi(q) - \varphi(p))(\rho t)^{1/d}}{(\#(S \cap \alpha(p, q)))^{1/d}} \quad (1)$$

$$\begin{aligned} \frac{1}{\rho^2} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(x))^2 &= \int_{r \in S \cap \alpha(p, q)} d^d r d^d s a^d a^d (r_a - s_a)(r_a - s_a) \\ &= \frac{(\#(S \cap \alpha(p, q)))^{2+2/d}}{\rho^{2+2/d}} (B_d(a_a)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2) \end{aligned}$$

$$B_d(a_a)^2 + C_d \sum_{i=1}^{d-1} (a_i)^2 = \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(x))^2 \quad (2)$$

$$\left(1 + \frac{B_d}{C_d}\right)(1)^2 = \frac{1}{C_d}(2) \Rightarrow$$

$$\Rightarrow a^d a_a = \left(1 + \frac{B_d}{C_d}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho t)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} = \frac{1}{C_d} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(x))^2$$

$$L(\varphi; p) = \frac{1}{2} \left(1 + \frac{B_d}{C_d}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho t)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} = \frac{1}{2} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r \in S \cap \alpha(p, q)} (\varphi(r) - \varphi(x))^2 = \frac{m^2 \varphi^2(p)}{2}$$

$$a_s = \frac{\varphi(q) - \varphi(p)}{q^s - p^s} = \frac{(\varphi(q) - \varphi(p)) (\rho A)^{1/d}}{(\#(S \cap \alpha(p, q)))^{1/d}} \quad (1)$$

$$\begin{aligned} \frac{1}{\rho^2} \sum_{r, s \in \mathbb{N}(p, q)} (\varphi(r) - \varphi(s))^2 &= \int d^d r d^d s \omega^d a^2 (r_s - s_s) (r_s - s_s) \\ &= \frac{(\#(S \cap \alpha(p, q)))^{2+2/d}}{\rho^{2+2/d}} (B_d(a_0))^2 + C_d \sum_{s=1}^{d-1} (a_s)^2 \end{aligned}$$

$$B_d(a_s)^2 + C_d \sum (a_s)^2 = \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in \mathbb{N}(p, q)} (\varphi(r) - \varphi(s))^2 \quad (2)$$

$$\left(1 + \frac{B}{C}\right)^2 - \frac{1}{C} (2) \Rightarrow$$

$$\Rightarrow a^s a_s = \left(1 + \frac{B_s}{C_s}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho A)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} - \frac{1}{C_s} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in \mathbb{N}(p, q)} (\varphi(r) - \varphi(s))^2$$

$$L(\varphi, \rho) = \frac{1}{2} \left(1 + \frac{B_s}{C_s}\right) \frac{(\varphi(q) - \varphi(p))^2 (\rho A)^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} - \frac{1}{2C_s} \frac{\rho^{2/d}}{(\#(S \cap \alpha(p, q)))^{2+2/d}} \sum_{r, s \in \mathbb{N}(p, q)} (\varphi(r) - \varphi(s))^2 - \frac{m^2 \varphi^2(p)}{2}$$

LINEAR FIELD IN FLAT SPACE \rightarrow GENERAL CASE

Field F is linear and space is flat $\Rightarrow L(F; p) = \tilde{L}(F; p, q)$ whenever $\#\alpha(p, q) > M$ where \tilde{L} is some expression that doesn't involve Lorentz indices

Relax assumption of flat space and linearity and replace them with assumptions of slowly varying curvature and slowly varying derivatives of fields.

GOAL: find point q such that F is approximately linear and space is approximately flat on $\alpha(p, q)$.

NOTE: small Lorentzian distance between p and q (measured through $\#\alpha(p, q)$ or other means) doesn't help us because the Lorentzian distance is arbitrarily small in vicinity of a light cone

SOLUTION

Choose $N \gg M$. Find point $q > p$ such that

$$(i) \#(S \cap \alpha(p, q)) > N$$

$$(ii) L(F, \alpha(r_1, s_1)) = L(F, \alpha(r_2, s_2)) \text{ whenever } p \prec r_1 \prec s_1 \prec q, p \prec r_2 \prec s_2 \prec q, \# \alpha(r_1, s_1) > M \text{ and } \# \alpha(r_2, s_2) > M$$

$$err_{M,N}(\alpha(p, q))$$

$$= \frac{1}{2} (\max\{L(F; r, s) \mid p \prec r \prec s \prec q \text{ and } \# \alpha(r, s) > M\} - \min\{L(F; r, s) \mid p \prec r \prec s \prec q \text{ and } \# \alpha(r, s) > M\})$$

Let $q_{M,N}(p, F)$ be a point such that

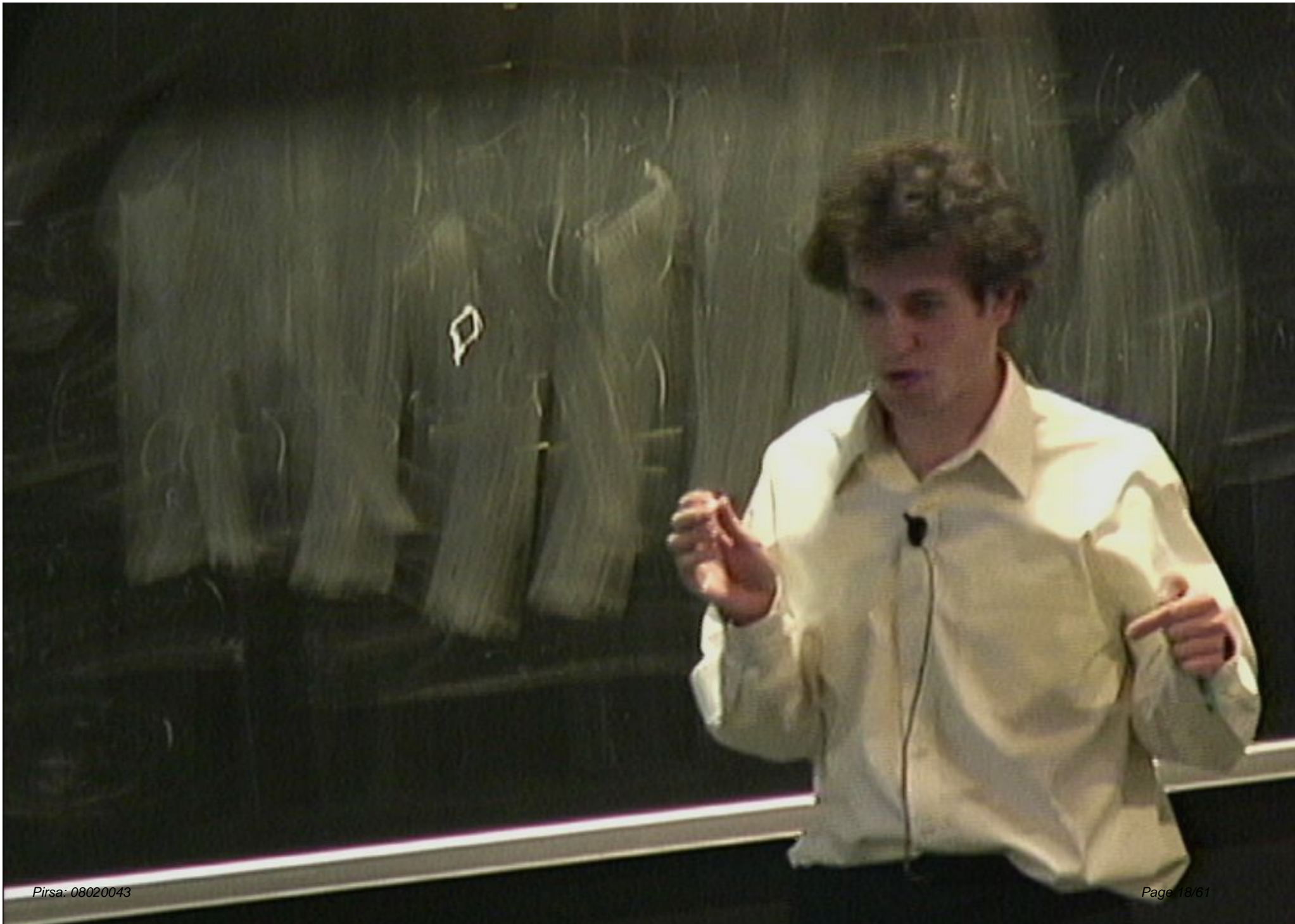
$$(i) \#(S \cap \alpha(p, q_{M,N}(F, p))) > N$$

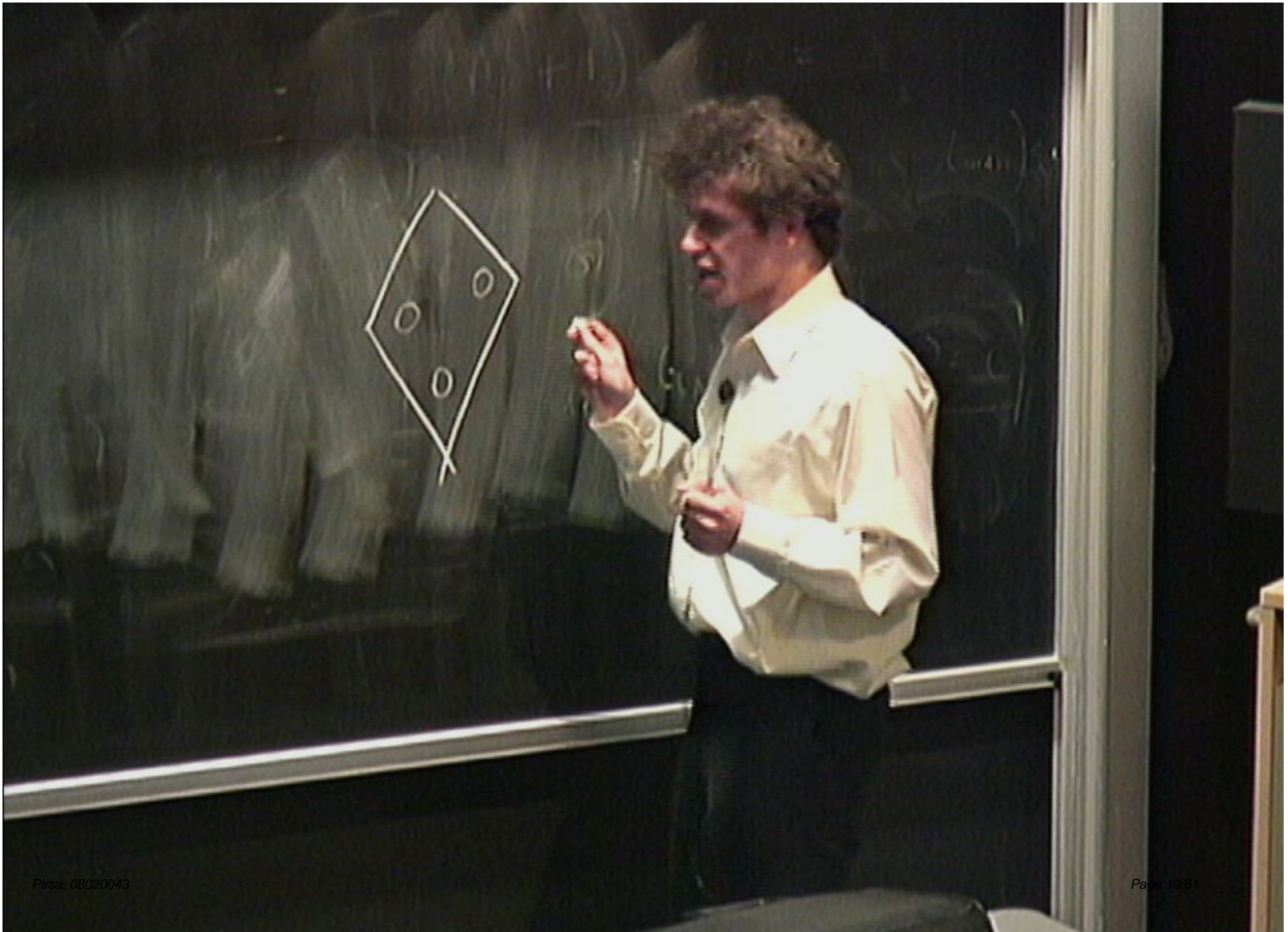
$$(ii) err_{M,N}(p, q) \geq err_{M,N}(p, q(F, p)) \text{ for any other } q \text{ satisfying } \# \alpha(p, q) > N$$

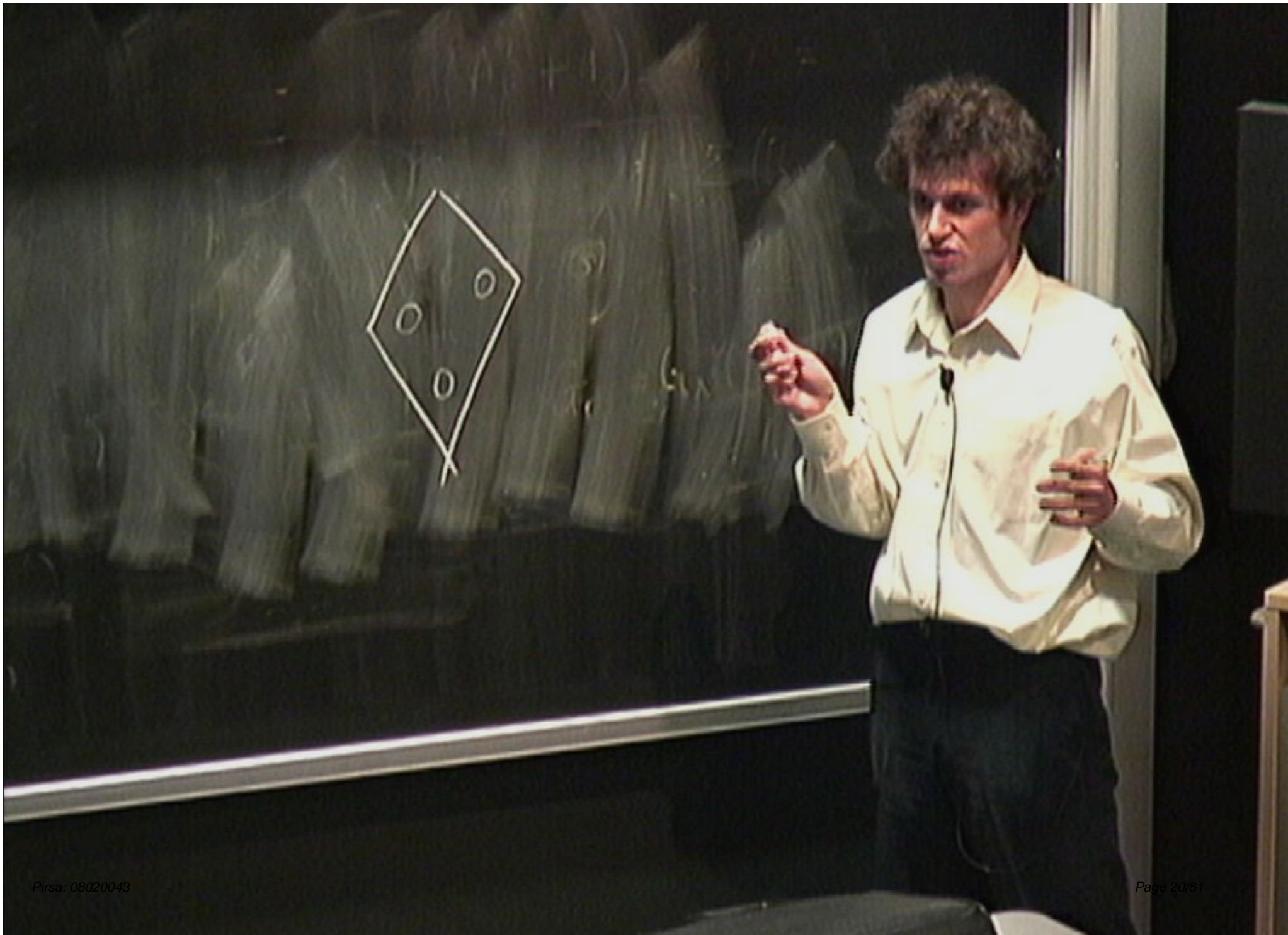
If there is more than one choice for the above, then it is undefined, so we throw such cases out of path integration, but this would be a set of measure 0.

FINAL ANSWER: $L_{M,N}(F, p) = \tilde{L}_{M,N}(F, p, q(F, p))$ given that curvature of spacetime as well as fields and their derivatives vary slowly enough

DEFINITION: $L'_{M,N}(F, p) = \tilde{L}'_{M,N}(F, p, q(F, p)) = \lim_{M,N \rightarrow \infty} L(F, p, q)$; For discrete case equate S with entire set of points.

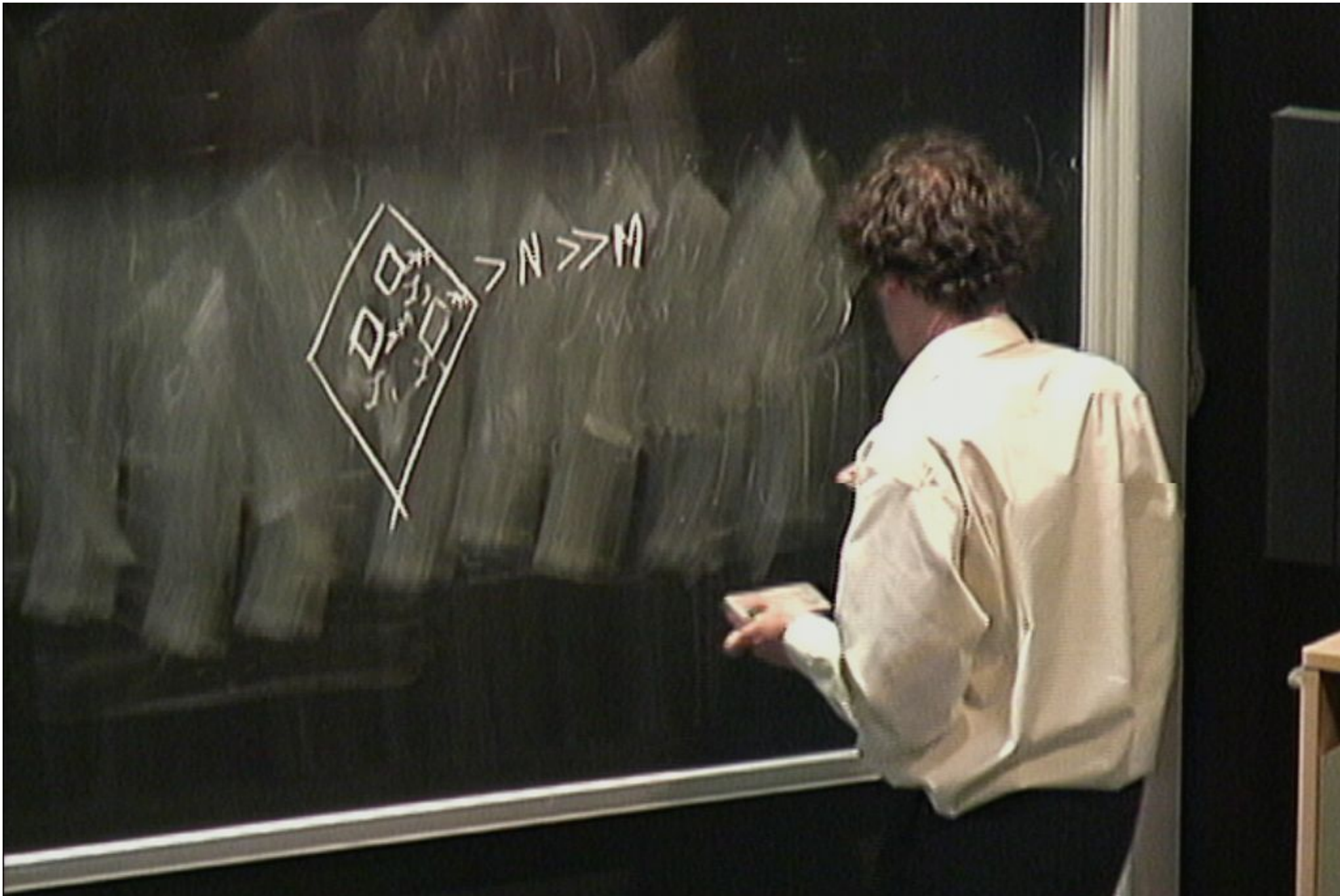








$\max \xi_{j,1}, \dots, \xi_{j,n}$
 $-\min \xi_{j,1}, \dots, \xi_{j,n}$





$\rightarrow N \Rightarrow M$

$err(\alpha(p, \dots)) =$
 $\{ \max\{L, D, \dots\}$
 $\{ \min\{L, D, \dots\}$



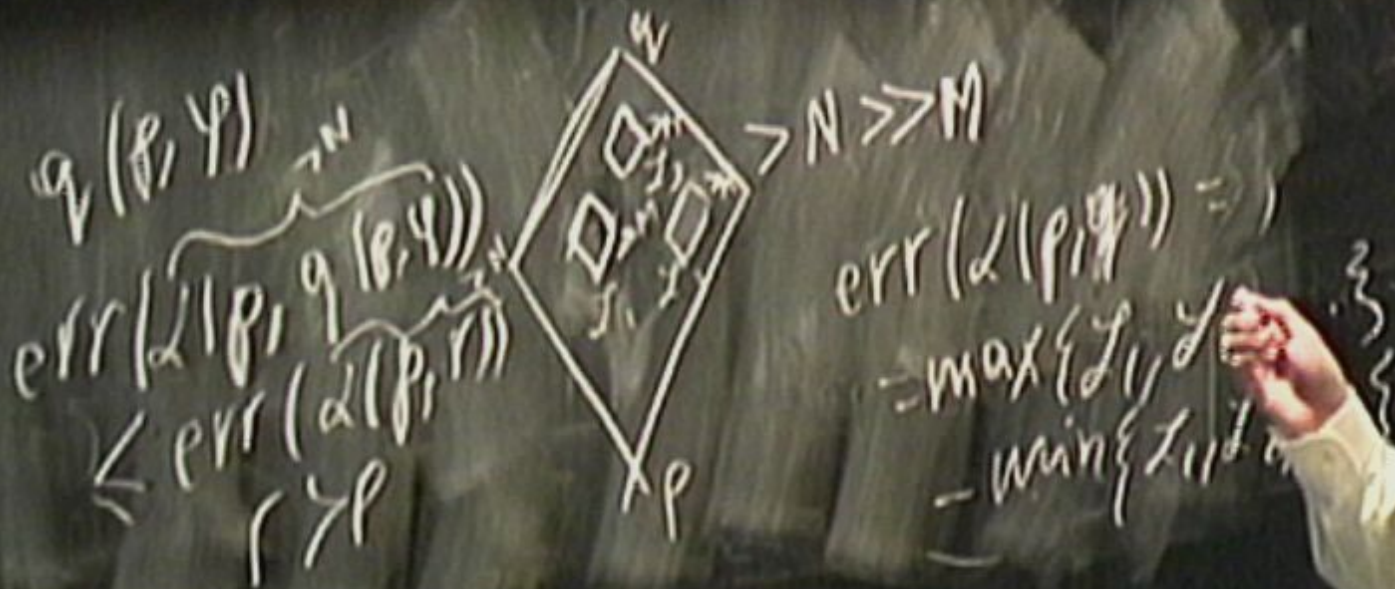


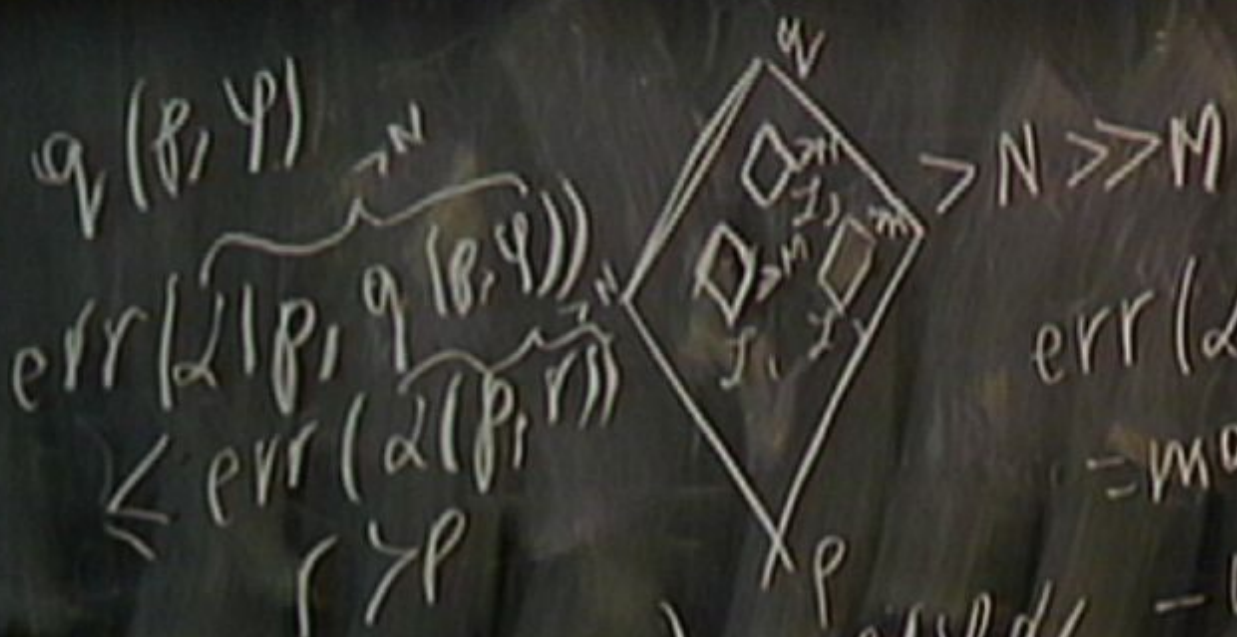
$N \Rightarrow M$

$err(\alpha(p)) =$

$= \max\{z_1, z_2, \dots\}$

$= \min\{z_1, z_2, \dots\}$





$\text{err}(L(p, \psi)) =$
 $= \max\{L_1, L_2, \dots\}$
 $= \min\{L_1, L_2, \dots\}$

$L(\psi, p) \approx L(\psi, d(p, \alpha(p, \psi)))$

GAUGE FIELD

Given: $\exp \int_{(p,q)} T^a A^a dx^\mu = U(p,q)$

Unknowns: p^a and A^a

Suppose we know that for some GIVEN q the following is true:

- a) $\#(S \cap \alpha(p,q)) > M$
- b) Both A^a and $B(r,s)$ are linear on $\alpha(p,q)$

$$U(a,b)U(b,c)U(c,a) = 1 + \frac{1}{2} T^a F^a_{\mu\nu} (a^\mu - c^\mu)(b^\nu - c^\nu)$$

$$\text{Tr}(U(a,b)U(b,c)U(c,a) - 1)^2 = \frac{C_2}{4} F^a_{\mu\nu} F^a_{\mu\nu} (a^\mu - c^\mu)(b^\nu - c^\nu)(a^\mu - c^\mu)(b^\nu - c^\nu)$$

Choose coordinate system so that t axis passes through p and q and origin lies in the middle between two points

$$\frac{1}{\rho^{4+2d}} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(p,q)U(q,r)U(r,p) - 1)^2) \\ = \frac{C_2}{4} \int_{\alpha(p,q)} d^d r F^a_{\mu\nu} F^a_{\mu\nu} (p^\mu - r^\mu)(q^\nu - r^\nu)(p^\mu - r^\mu)(q^\nu - r^\nu) = \frac{k_1}{\rho^{4+2d}} \sum (F^a_{\mu\nu}(p))^2$$

$$\sum (F^a_{\mu\nu}(p))^2 = \frac{\rho^{4+d}}{k_1} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(p,q)U(q,r)U(r,p) - 1)^2) \quad (1)$$

$$\frac{1}{\rho^{4+2d}} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(a,b)U(b,c)U(c,a) - 1)^2) \\ = \frac{C_2}{4} \int_{\alpha(p,q)} d^d a d^d b d^d c F^a_{\mu\nu} F^a_{\mu\nu} (p^\mu - r^\mu)(q^\nu - r^\nu)(p^\mu - r^\mu)(q^\nu - r^\nu)$$

$$= \frac{1}{\rho^{4+2d}} (k_2 \sum_{a,b,c \in \alpha(p,q)} (F^a_{\mu\nu}(p))^2 + k_3 \sum_{a,b,c \in \alpha(p,q)} (F^a_{\mu\nu}(p))^2)$$

$$k_2 \sum_{a,b,c \in \alpha(p,q)} (F^a_{\mu\nu}(p))^2 + k_3 \sum_{a,b,c \in \alpha(p,q)} (F^a_{\mu\nu}(p))^2 = \rho^{4+d} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(p,q)U(q,r)U(r,p) - 1)^2) \quad (2)$$

$$\frac{1}{k_2} (2) - (1 + \frac{k_3}{k_2})(1) \Rightarrow F^a_{\mu\nu} F^a_{\mu\nu} = \frac{\rho^{4+d}}{k_2} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(p,q)U(q,r)U(r,p) - 1)^2)$$

$$-(1 + \frac{k_3}{k_2}) \frac{\rho^{4+d}}{k_2} \sum_{a,b,c \in \alpha(p,q)} \text{Tr}((U(p,q)U(q,r)U(r,p) - 1)^2)$$

$$U: S^2 \rightarrow \text{SU}(N)$$

$$U = \exp \int T^a A_\mu^a$$

The chalkboard contains several other mathematical expressions and diagrams:

- Top right: (ψ, ψ) with a bracket and $\rightarrow N$
- Middle right: (ρ, ρ) and $(\alpha(\rho, \rho))$ with a bracket and $\rightarrow N$
- Bottom right: (ψ, ρ) with a bracket and $\rightarrow N$
- Far right: A diagram showing a square with a diagonal line and a small square inside, possibly representing a Lie algebra structure.

$$U: S^2 \rightarrow SU(n)$$

$$U = \exp \int T^a A^a_{\mu} dx^{\mu} \quad \underbrace{g(\mathcal{P}, \Psi)}_{\rightarrow N}$$

$$U(a, b) U(b, c) U(c, a) \quad \underbrace{\text{err}(\alpha | \mathcal{P}, g(\mathcal{P}, \Psi))}_{\rightarrow N}$$

$$\quad \underbrace{\text{err}(\alpha | \mathcal{P}, r)}_{r \geq \rho}$$

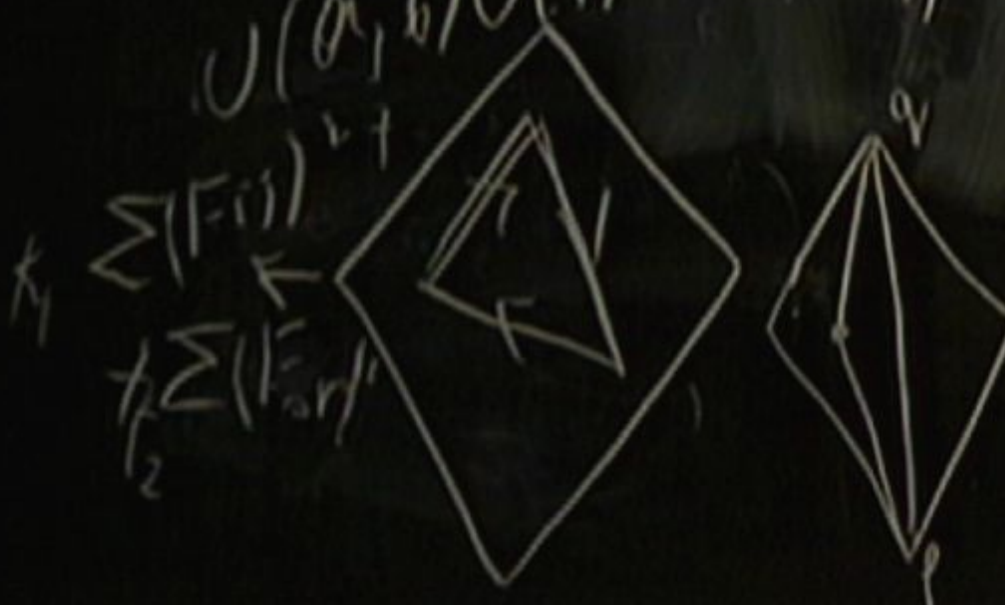


$$L(\Psi, \mathcal{P}) \rightarrow$$

$$U: S^2 \rightarrow SU(n) \quad q(p, \psi) \rightarrow N$$

$$U = \exp \int T^a A_\mu^a dx^\mu \quad \text{err}(\alpha(p, q(p, \psi)))$$

$$U(a, b)U(b, c)U(c, a) \rightarrow \text{err}(\alpha(p, r))$$



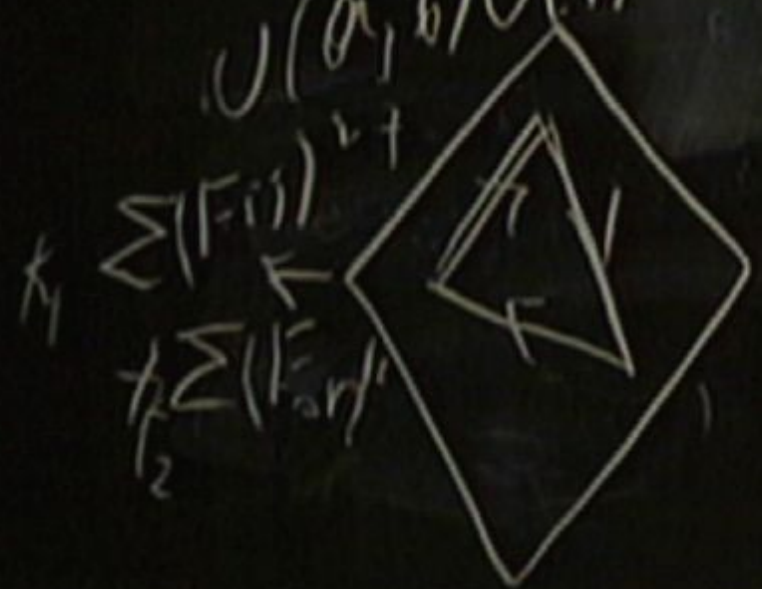
$$\int (\psi, p)$$

$$\rightarrow \sum (F_{\alpha\beta})^2$$

$$U: S^2 \rightarrow SU(n) \quad q(p, \psi)$$

$$U = \exp \int T^a A_\mu^a dx^\mu \quad \text{err}(\alpha(p, q(p, \psi)))$$

$$U(a, b)U(b, c)U(c, a) \rightarrow \text{err}(\alpha(p, r))$$



$$\mathcal{L}(\psi, p) \approx \mathcal{L}(\psi)$$

$$\rightarrow \sum(F_{orb})^2$$

$$a^H a_m$$



$$\varphi^2$$
$$\langle \varphi^2 \rangle$$
$$P$$

$$\frac{1}{P^2} \sum (\varphi(r) - \varphi(s))$$

$$= K_d \left(r(P, \varphi) \right)^d$$





$$\frac{v(d(p, q))}{K_d(d(p, q))} \rightarrow 1$$





$$\frac{v(\alpha(p, q))}{K_d(\alpha(p, q))} \rightarrow 1 = \text{corr}(\alpha(p, q))$$

d/p



$$\frac{V(d(p, q))}{K_d(d(p, q))} = \text{corr}(d(p, q))$$

The equation above is crossed out with a large 'X' drawn over it.



$$\text{covr}(P, Q) = \tau^2$$

$$\frac{V(\alpha(P, Q))}{K_d(\tau(P, Q))^2} = \text{corr}(\alpha(P, Q))$$



$$\text{covr}(P, Q) = \tau^2 (E P_{00} + F P)$$

$$\text{corr}(\alpha(P, Q))$$

$$\frac{V(\alpha(P, Q))}{K_d (\tau(P, Q))^2} - 1 = \text{X}$$



$$\text{covr}(p, q) = \tau^2 (E p_{00} + F p)$$

$$\frac{V(\alpha(p, q))}{K_d (\tau(p, q))^2} \rightarrow 1 = \text{corr}(\alpha(p, q))$$





$$\text{covr}(p, q) = \tau^2 (E p_{00} + F p)$$

$$\frac{V(\alpha(p, q))}{K_d (\tau(p, q))^2} = \text{corr}(\alpha(p, q))$$





$$l(\gamma) = \int \sqrt{(dx^0)^2 + \sum (dx^i)^2}$$



$$\begin{aligned} R(x) &= \int \sqrt{(dx^0)^2 - \sum (dx^i)^2} \\ &\leq \int |dx^0| = \int dx^0 \\ &= \tau(P, q) \end{aligned}$$



$$\begin{aligned}
 R(x) &= \int \sqrt{(dx^0)^2 + \sum (dx^i)^2} \\
 &\leq \int |dx^0| = \int dx^0 \\
 &= \tau(p, q)
 \end{aligned}$$

$$\max_{\gamma} \int_p^q \sqrt{\dot{\gamma}^2} < r_1 < r_2 < \dots < r_n < r_{n+1}$$

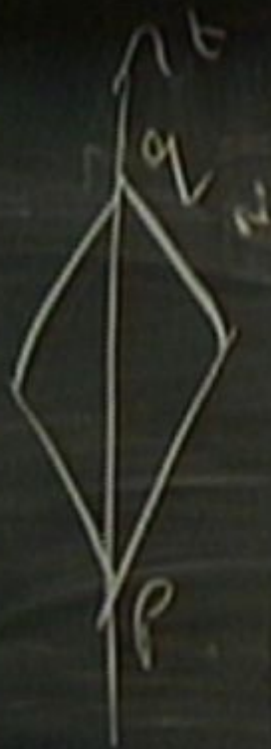
max $\sum p_i x_i$
s.t. $x_1 \leq r_1$
 $x_2 \leq r_2$
 $x_3 \leq r_3$
 $x_4 \leq r_4$
 $x_5 \leq r_5$



$$\begin{aligned}
 P(x) &= \int \sqrt{(dx)^2 + \sum (dx_i)^2} \\
 &\leq \int |dx| = \int dx = x \\
 &= C(p, q)
 \end{aligned}$$

$$\begin{aligned}
 \max_{r_1/p} &< r_1 < r_2 < \\
 &r_1 < r_2 < r_3 < r_4
 \end{aligned}$$

$$r_1 < r_2 < r_3 < r_4$$



$$\text{covr}(P, q) = \tau^2 (E R_{00} + FR)$$

$$\frac{V(\alpha(P, q))}{K_d (\tau(P, q))^2} = \text{corr}(\alpha(P, q))$$



$$\text{corr}(P, q) = \tau^2 (E P_{00} + F P)$$

$$\frac{V(\alpha(P, q))}{K_d (\tau(P, q))^2} = \text{corr}(\alpha(P, q))$$



$$\text{corr}(P, q, v) = \tau^2 (E R_{00} + F R)$$

$$\text{corr}(\alpha(P, q, v))$$

$$\frac{V(\alpha(P, q, v))}{K_d (\tau(P, q, v))^2} \rightarrow 1 = \text{crossed out}$$

$$\text{corr}(p, q) = \frac{\# \alpha(p, q)}{k, (\max\{n \mid \exists r_1, \dots, r_n (p < r_1 < \dots < r_n < q)\})^d - 1} = (ER_{\text{ave}} + FRG_{\text{ave}}) \chi(p^* - q^*) \chi(p^* - q^*)$$

whenever $\# \alpha(p, q) \geq M_1$,

Suppose $\# \alpha(p, q) \geq M_2 \gg M_1$, and again assume that t axis passes through p and q with origin in the middle: $p = (-\frac{\tau}{2}, 0, 0, 0)$ and $q = (\frac{\tau}{2}, 0, 0, 0)$

$$\text{corr}(p, q) = (ER_{\text{ave}} + FR) \tau^2$$

$$\frac{1}{\rho^2} \sum_{\substack{p, q \text{ indep. } \# \alpha(p, q) \geq M_1}} \text{corr}(\alpha(r, s)) = \int_{\substack{p, q \text{ indep.}}} d^d r d^d s (ER_{\text{ave}} + FRG_{\text{ave}}) \chi(r^* - s^*) \chi(r^* - s^*) = \tau^{2d+2} (GR_{\text{ave}} + HR)$$

$$R = \frac{1}{\frac{\tau}{2} - \frac{\tau}{2}} \left(\frac{\text{corr}(\alpha(p, q))}{E \tau^2} - \frac{\sum_{\substack{p, q \text{ indep. } \# \alpha(p, q) \geq M_1}} \text{corr}(\alpha(r, s))}{F \tau^{2d+2}} \right)$$

$$= \frac{1}{\frac{\tau}{2} - \frac{\tau}{2}} \left(\frac{\rho^{2/d} \text{corr}(\alpha(p, q))}{E (\max\{n \mid \exists r_1, \dots, r_n (p < r_1 < \dots < r_n < q)\})^2} - \frac{\rho^{2+2/d} \sum_{\substack{p, q \text{ indep. } \# \alpha(p, q) \geq M_1}} \text{corr}(\alpha(r, s))}{F (\max\{n \mid \exists r_1, \dots, r_n (p < r_1 < \dots < r_n < q)\})^{2d+2}} \right)$$



$$\text{corr}(P, V) = \tau^2 (E R_{00} + FR)$$

$$\frac{V(\alpha(P, V))}{K_d (\tau(P, V))} = \text{corr}(\alpha(P, V))$$





$$\text{corr}(P, q) = \tau^2 (E_{P_{00}} + FR)$$

$$\frac{V(\alpha(P, q))}{K_d (\tau(P, q))^\alpha}$$

$$V(\alpha(P, q))$$



$\int p^2 dx$
tunnel



$$\text{corr}(p, \psi) = \tau^2 (E_{p,00} + FR)$$

$$\text{corr}(\alpha(p, \psi))$$

$$\frac{\psi}{\psi} = \text{circle with a cross through it}$$

$\int e^{i\alpha} \text{junk}$

$$\text{corr}(P, V) = \tau^2 (EP)$$

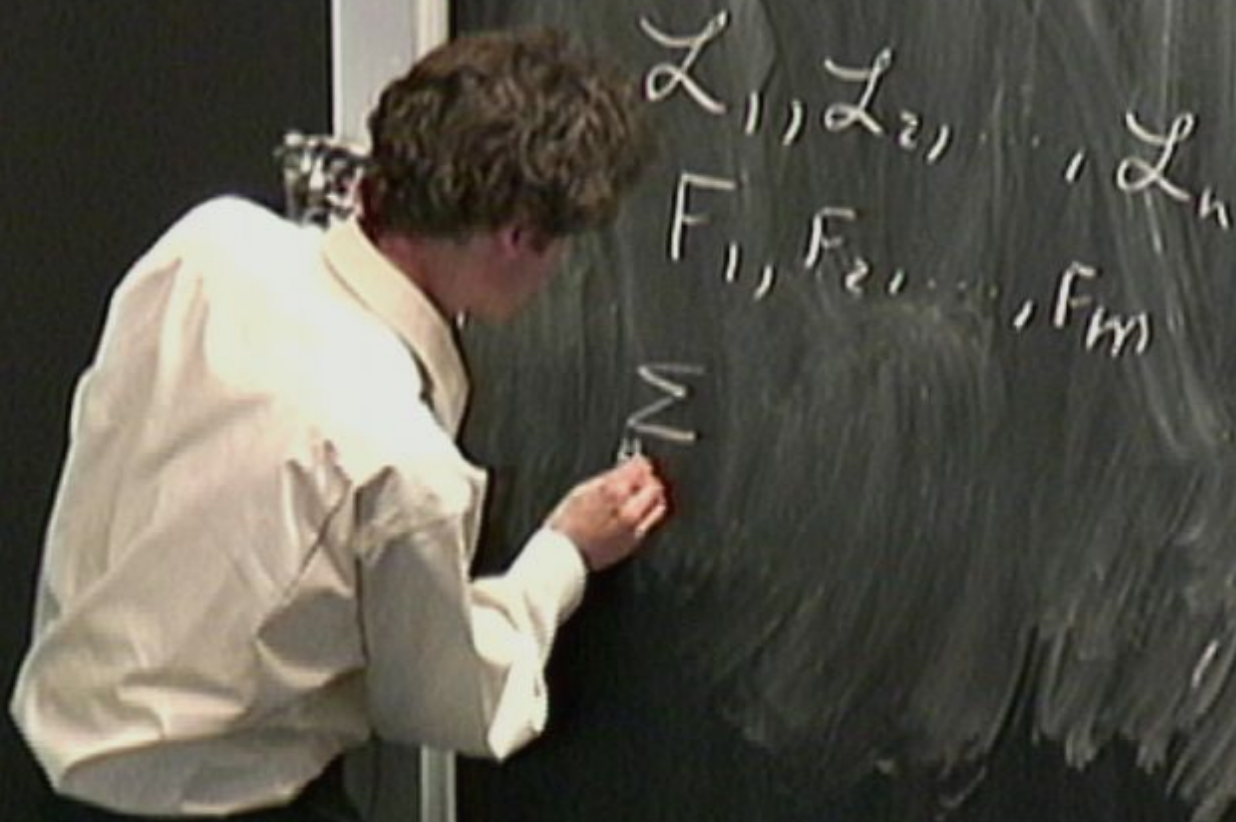


$$\frac{V(\alpha(P, q))}{K_d(\tau(P, q))} = \text{corr}(d)$$

K_4



Z_1, Z_2, \dots, Z_n

A man with dark, curly hair, wearing a white long-sleeved shirt and dark trousers, is standing in profile facing a chalkboard. He is holding a piece of chalk and appears to be in the process of writing. The chalkboard is mostly covered with dark, horizontal chalk strokes, but some text is visible in the upper left quadrant. The text consists of two rows of mathematical symbols: the first row contains Z_1, Z_2, \dots, Z_n and the second row contains F_1, F_2, \dots, F_m . Below these, the letter 'M' is written vertically. The lighting is somewhat dim, and the background is dark.

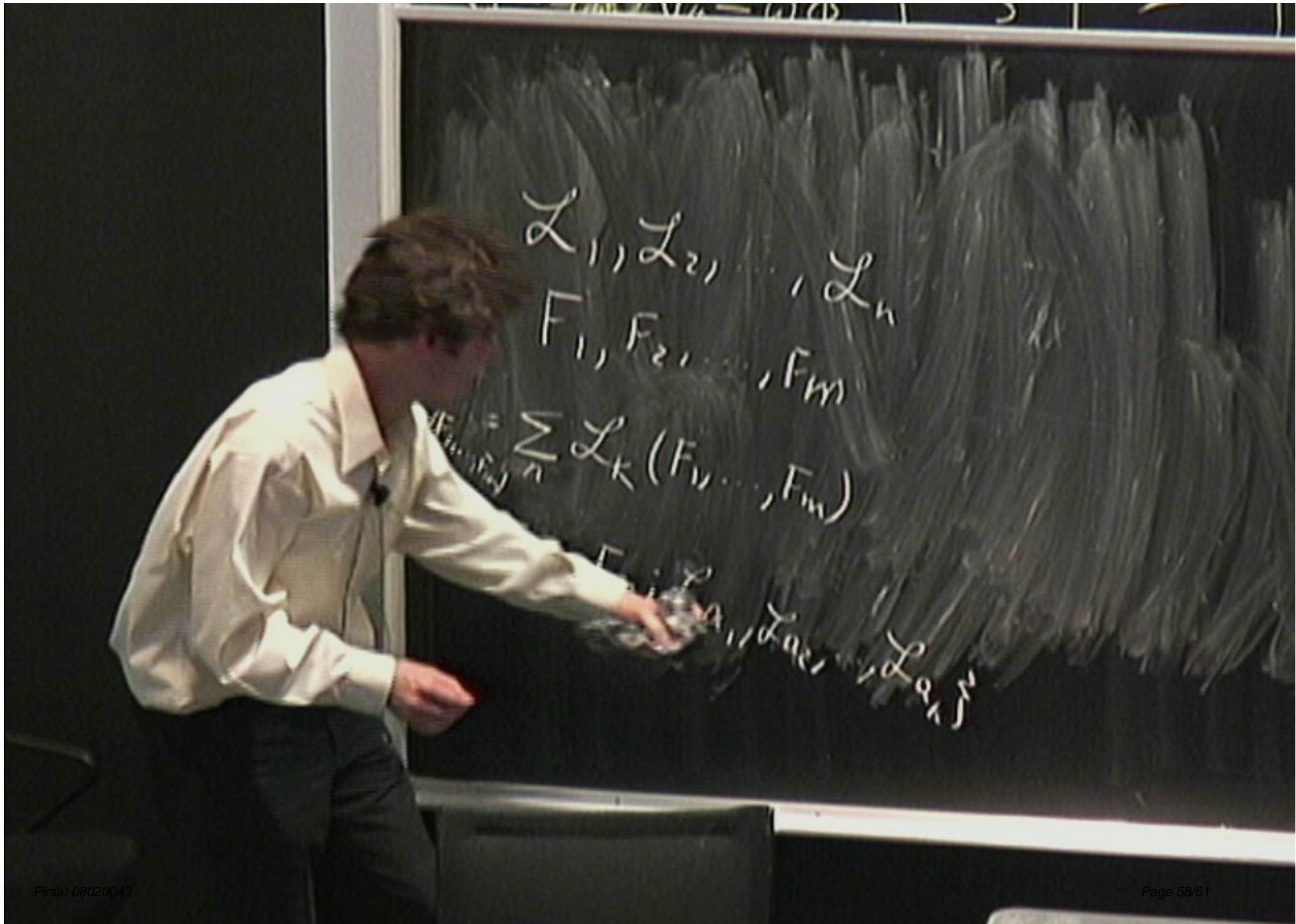
Z_1, Z_2, \dots, Z_n
 F_1, F_2, \dots, F_m
M

$$\begin{aligned} & \mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n \\ & F_1, F_2, \dots, F_m \\ & S = \sum_k \mathcal{L}_k(F_1, \dots, F_m) \end{aligned}$$

Z_1, Z_2, \dots, Z_n
 F_1, F_2, \dots, F_m

$$S(F_1, \dots, F_m) = \sum_k L_k(F_1, \dots, F_m)$$

$S(F_1, \dots, F_m)$



L_1, L_2, \dots, L_n
 F_1, F_2, \dots, F_m

$$S(F_1, \dots, F_m) = \sum_k L_k(F_1, \dots, F_m)$$

$S(F_1, \dots, F_m; L_1, L_2, \dots, L_n)$

$\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$
 F_1, F_2, \dots, F_m

$$S(F_1, \dots, F_m) = \sum_k \mathcal{L}_k(F_1, \dots, F_m)$$

$$S(F_1, \dots, F_m; \mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n)$$

$$= \sum_{\alpha_k} \mathcal{L}_{\alpha_k}(F_1, \dots, F_m)$$

$\alpha^1 \alpha_m$



$\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$
 F_1, F_2, \dots, F_m

$$S(F_1, \dots, F_m) = \sum_k \mathcal{L}_k(F_1, \dots, F_m)$$

$$S(F_1, \dots, F_m; \mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n)$$

$$= \sum_{\alpha_k} \mathcal{L}_{\alpha_k}(F_1, \dots, F_m)$$

$\alpha^1 \alpha_m$

