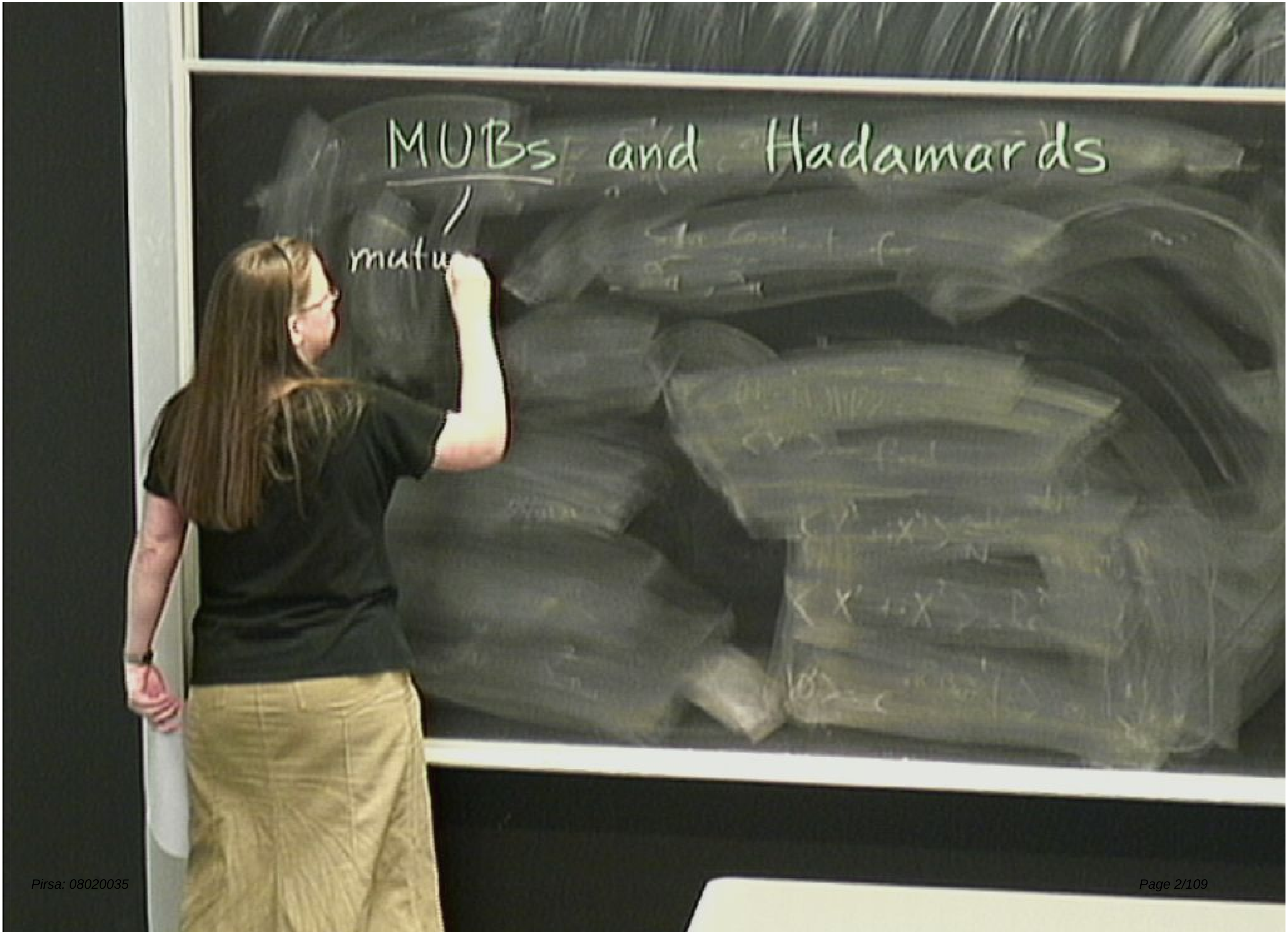
Title: MUBs and Hadamards

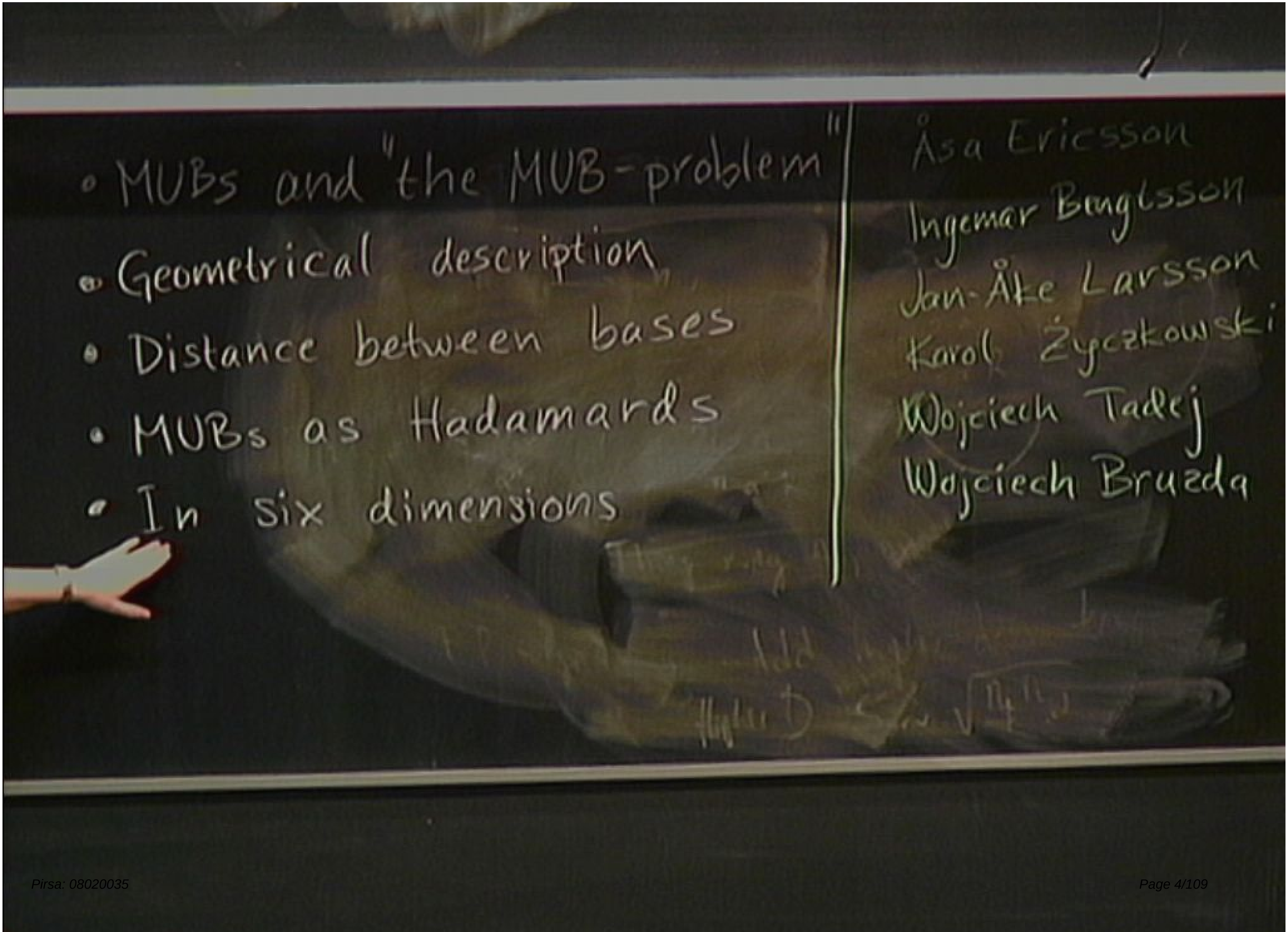
Date: Feb 12, 2008 04:00 PM

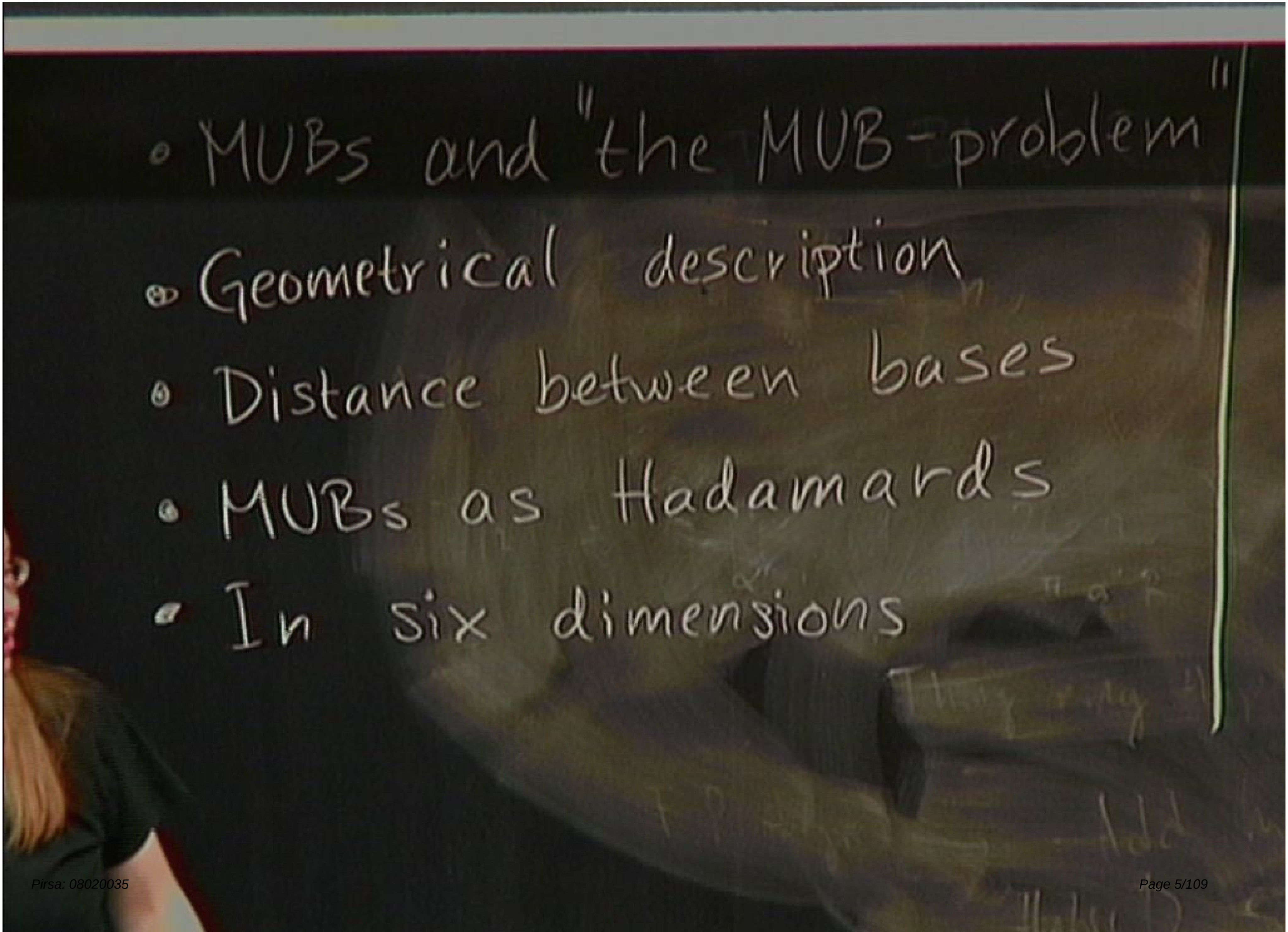
URL: http://pirsa.org/08020035

Abstract: Mutually unbiased bases (MUBs) have attracted a lot of attention the last years. These bases are interesting for their potential use within quantum information processing and when trying to understand quantum state space. A central question is if there exists complete sets of N+1 MUBs in N-dimensional Hilbert space, as these are desired for quantum state tomography. Despite a lot of effort they are only known in prime power dimensions.
I will describe in geometrical terms how a complete set of MUBs would sit in the set of density matrices and present a distance between basesÂ–a measure of unbiasedness. Then I will explain the relation between MUBs and Hadamard matrices, and report on a search for MUB-sets in dimension N=6. In this case no sets of more than three MUBs are found, but there are several inequivalent triplets.


MUBs and Hadamards


MUBs and Hadamards
mutually unbiased bases


• MUBs and "the MUB-problem"
• Geometrical description
• Distance between bases
• MUBs as Hadamards
• In six dimensions
Åsa Ericsson
Ingemar Bengtsson
Jan-Åke Larsson
Karol Życzkowski
Wojciech Tadej
Wojciech Bruzda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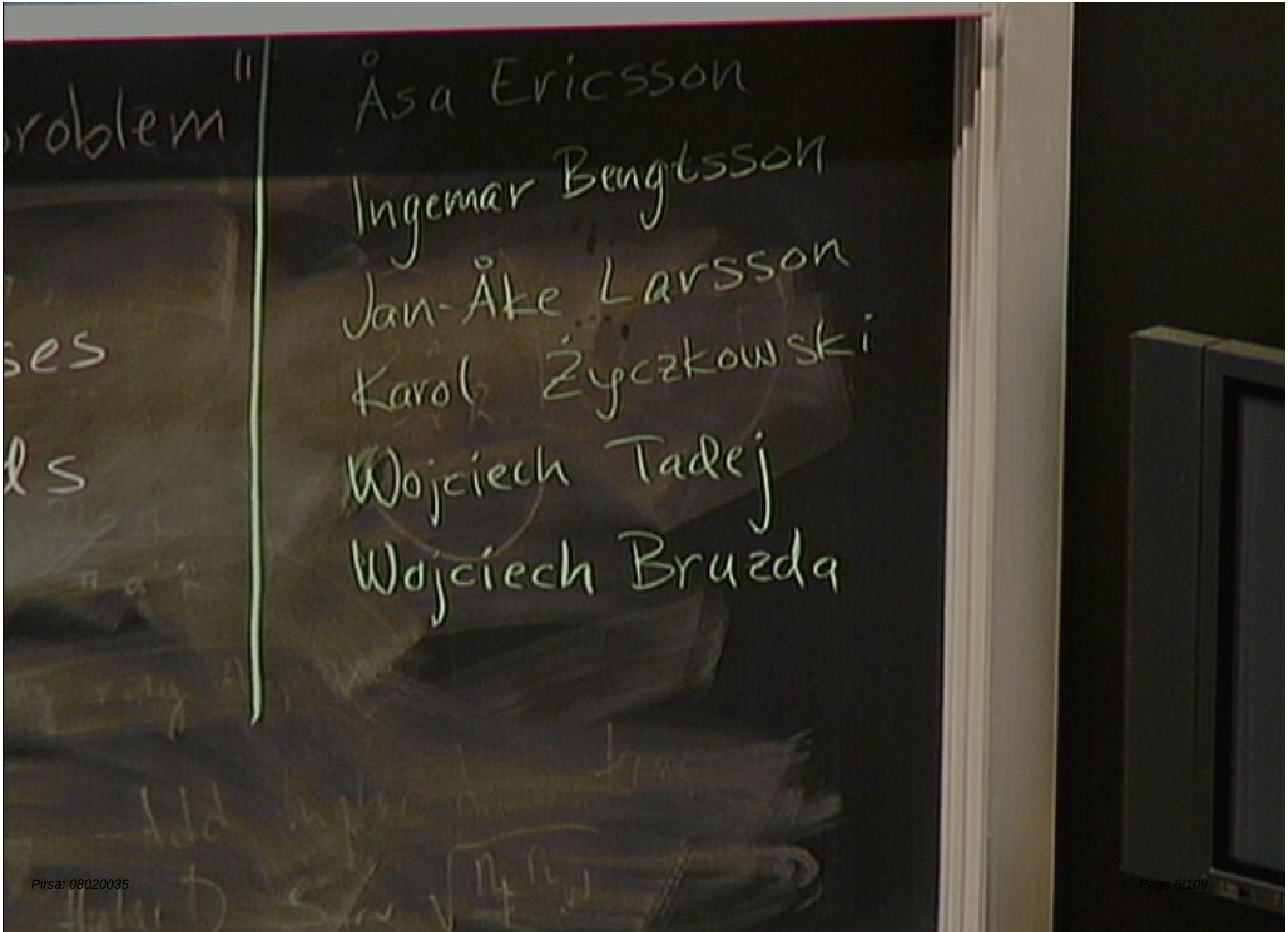
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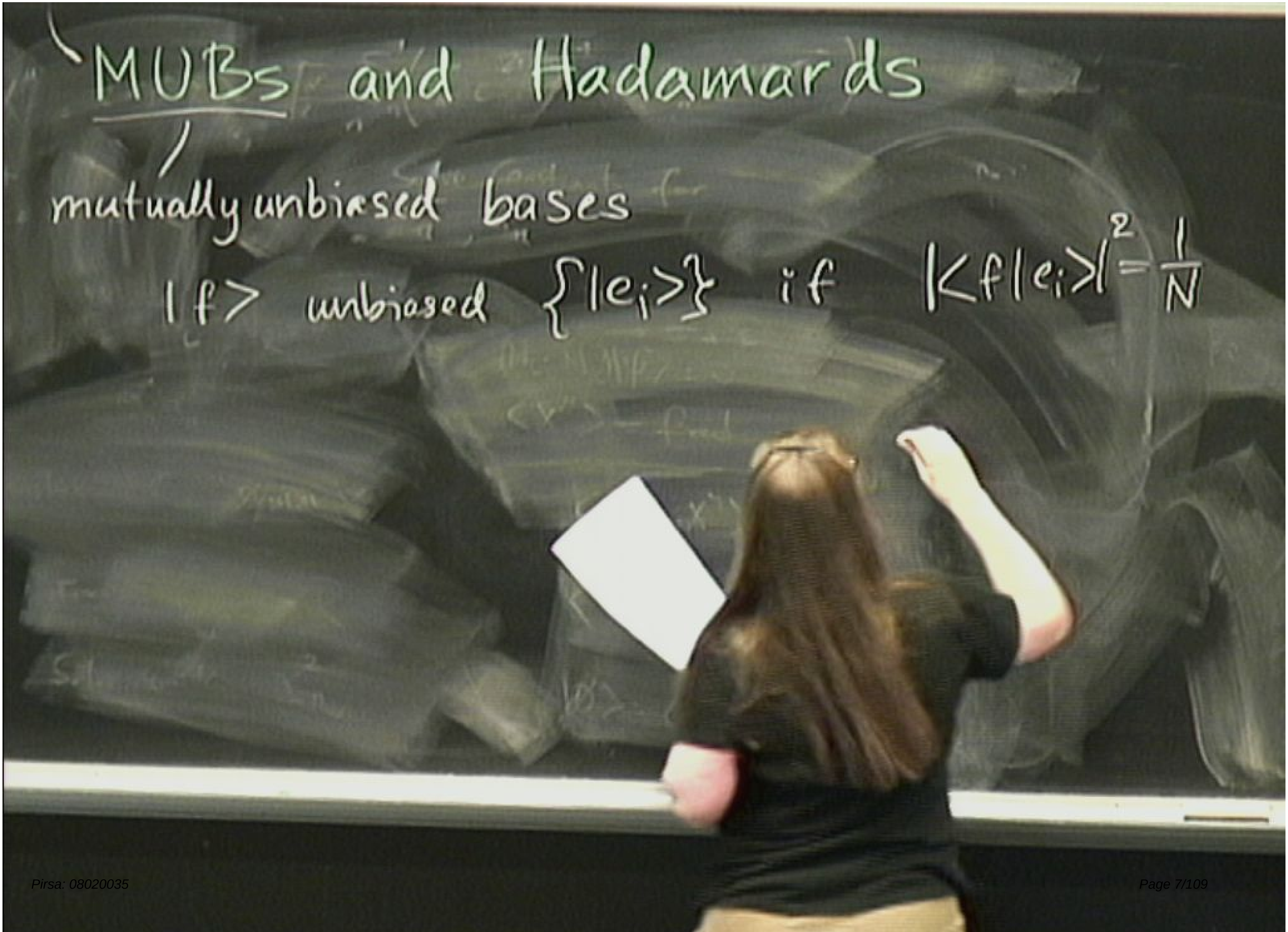
MUBs and Hadamards
mutually unbiased bases
$|f\rangle$ unbiased $\{|e_i\rangle\}$ if $|\langle f|e_i\rangle|^2 = \frac{1}{N}$

MUBs and Hadamards

$N = \dim$

mutually unbiased bases

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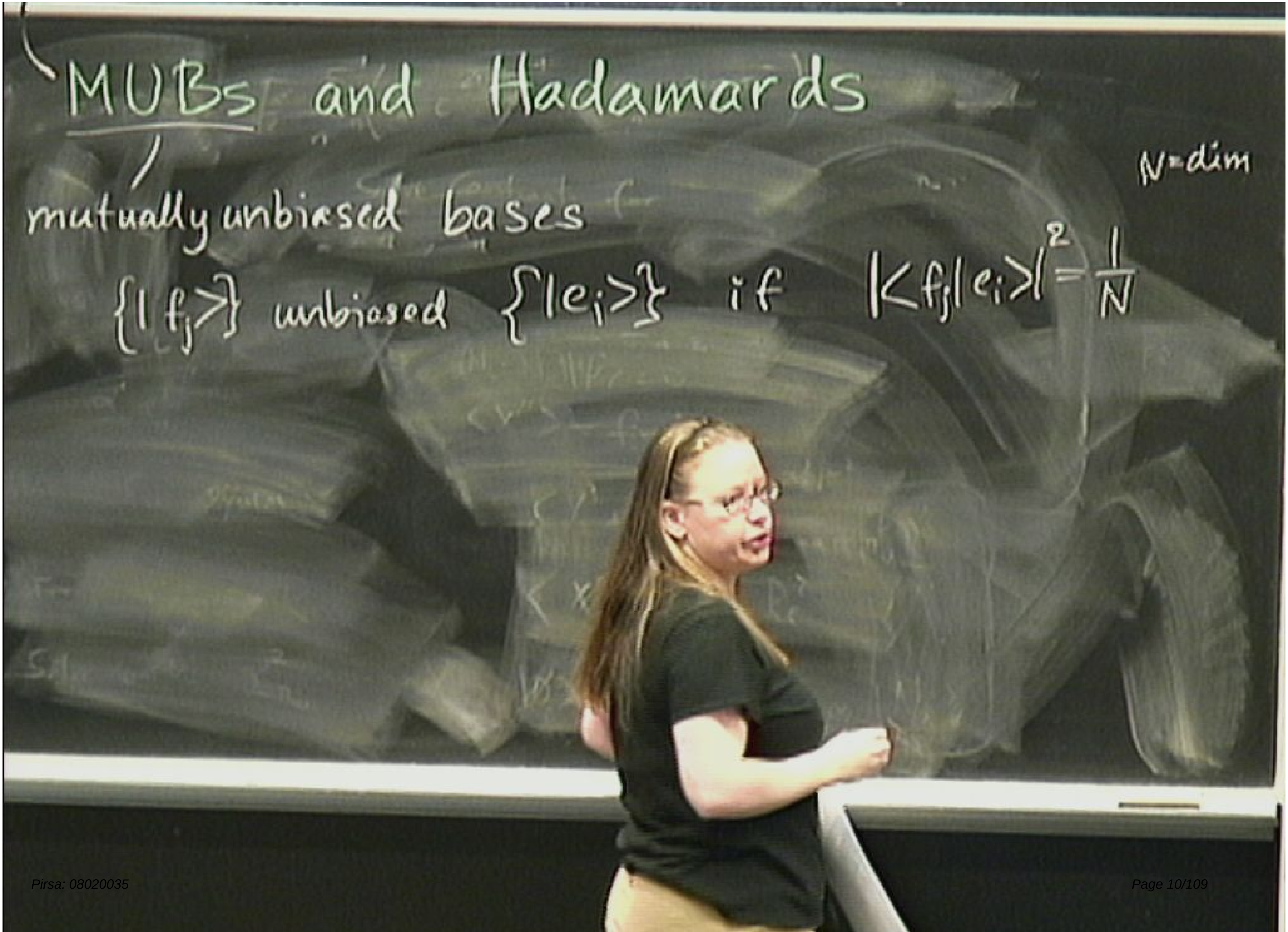
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mutually unbiased bases

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all i, j

A set of MUBs

MUBs and Hadamards

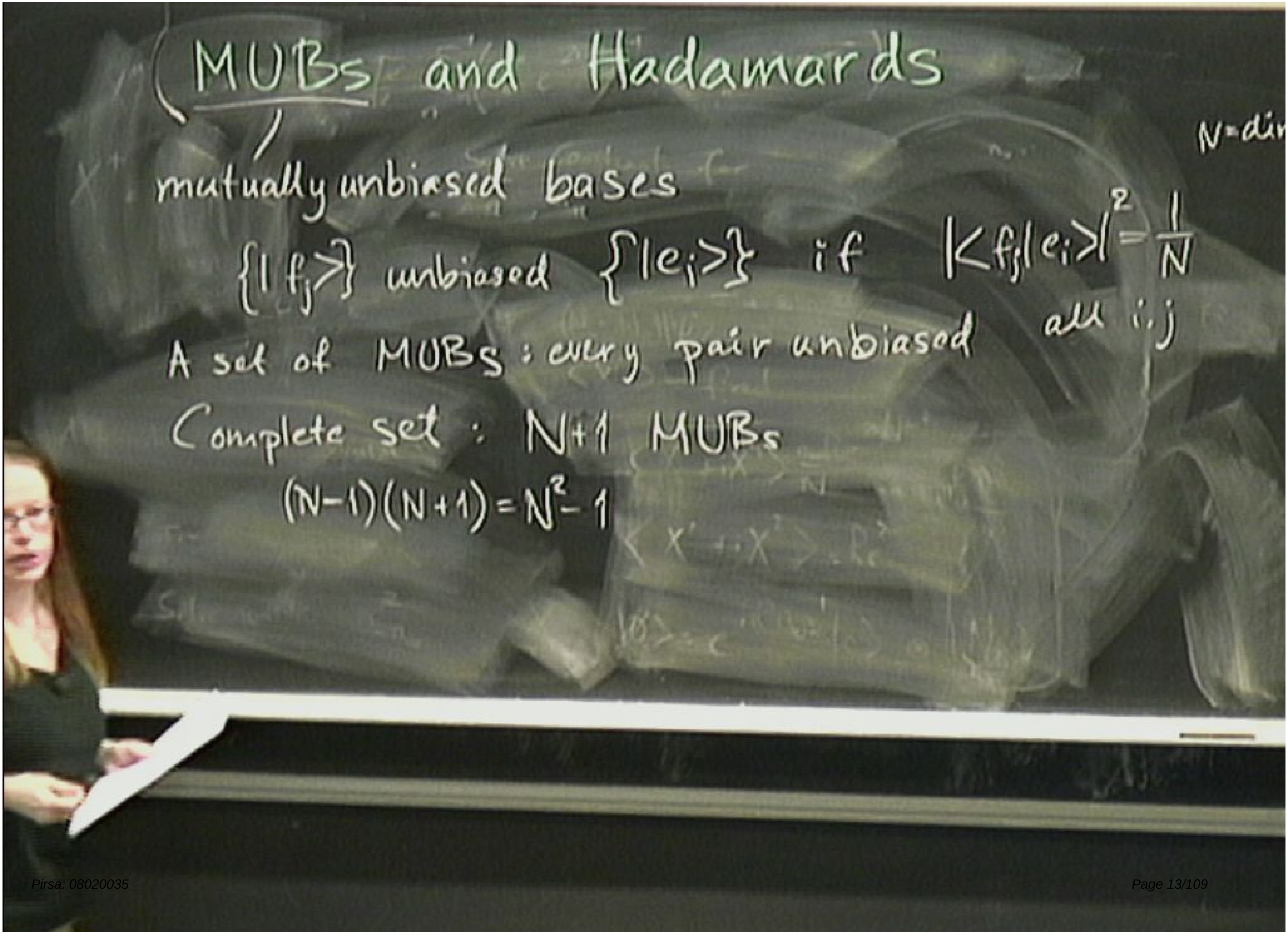
$N = d$

mutually unbiased bases

$\{|f_j\rangle\}$ unbiased $\{|e_i\rangle\}$ if $|\langle f_j|e_i\rangle|^2 = \frac{1}{N}$ all i, j

A set of MUBs: every pair unbiased

Complete set: $N+1$ MUBs


MUBs and Hadamards
N=di
mutually unbiased bases
$\{|f_j\rangle\}$ unbiased $\{|e_i\rangle\}$ if $|\langle f_j|e_i\rangle|^2 = \frac{1}{N}$
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A set of MUBs: every pair unbiased
Complete set: N+1 MUBs
$(N-1)(N+1) = N^2 - 1$

MUBs and Hadamards

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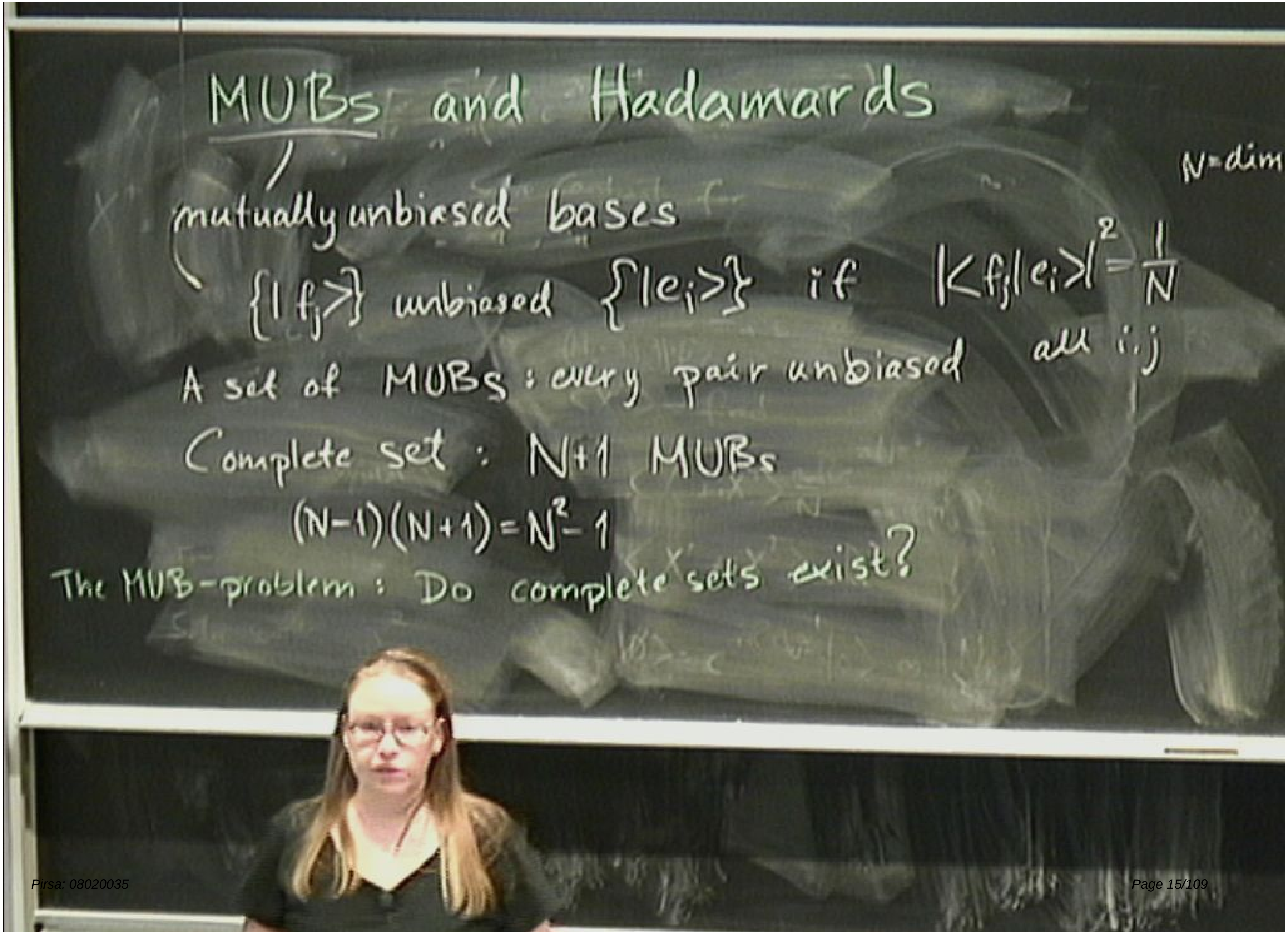
The MUB-problem:

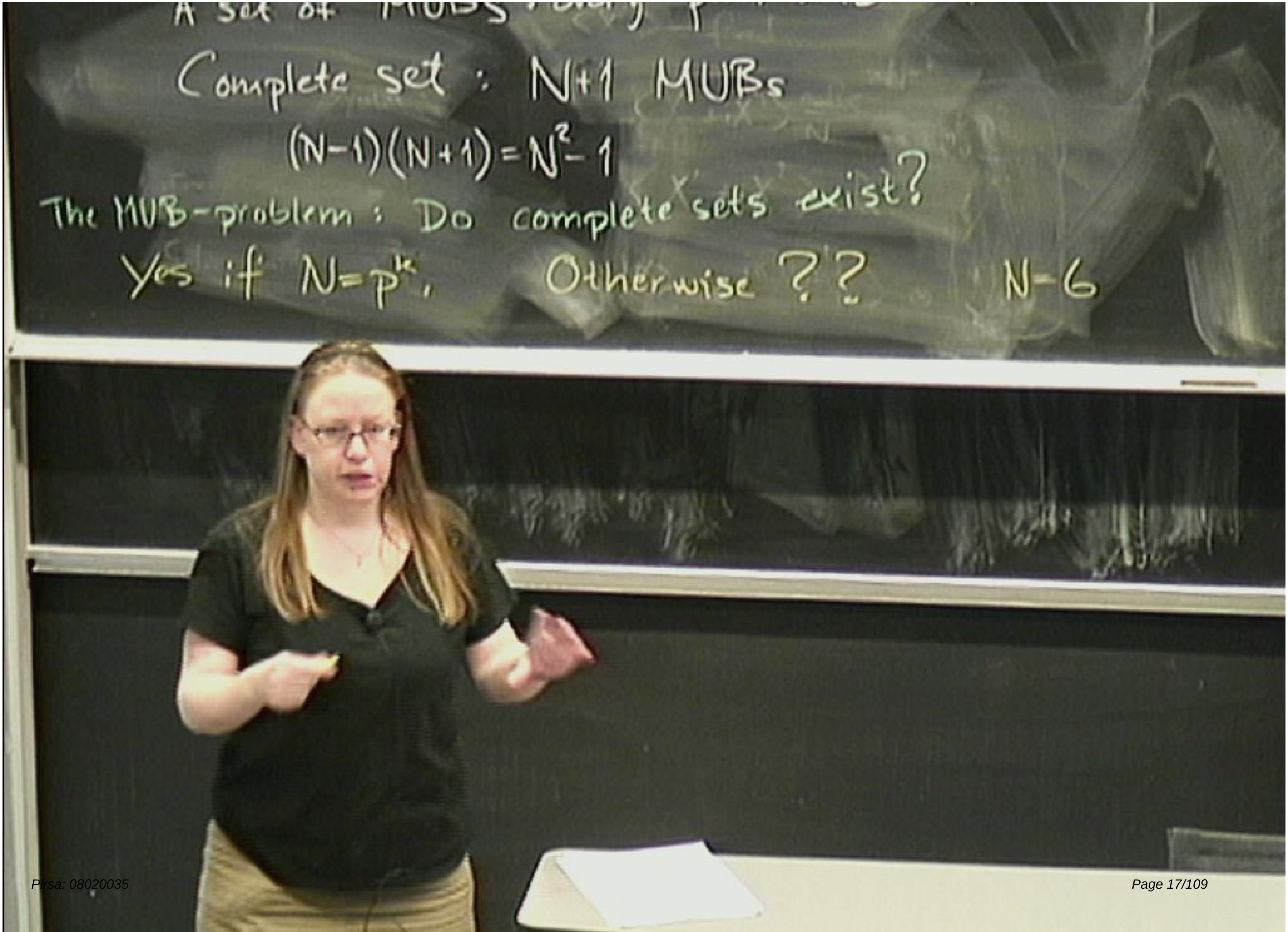

MUBs and Hadamards
N = dim
mutually unbiased bases
$\{|f_j\rangle\}$ unbiased $\{|e_i\rangle\}$ if $|\langle f_j|e_i\rangle|^2 = \frac{1}{N}$
all i, j
A set of MUBs: every pair unbiased
Complete set: N+1 MUBs
$(N-1)(N+1) = N^2 - 1$
The MUB-problem: Do complete sets exist?

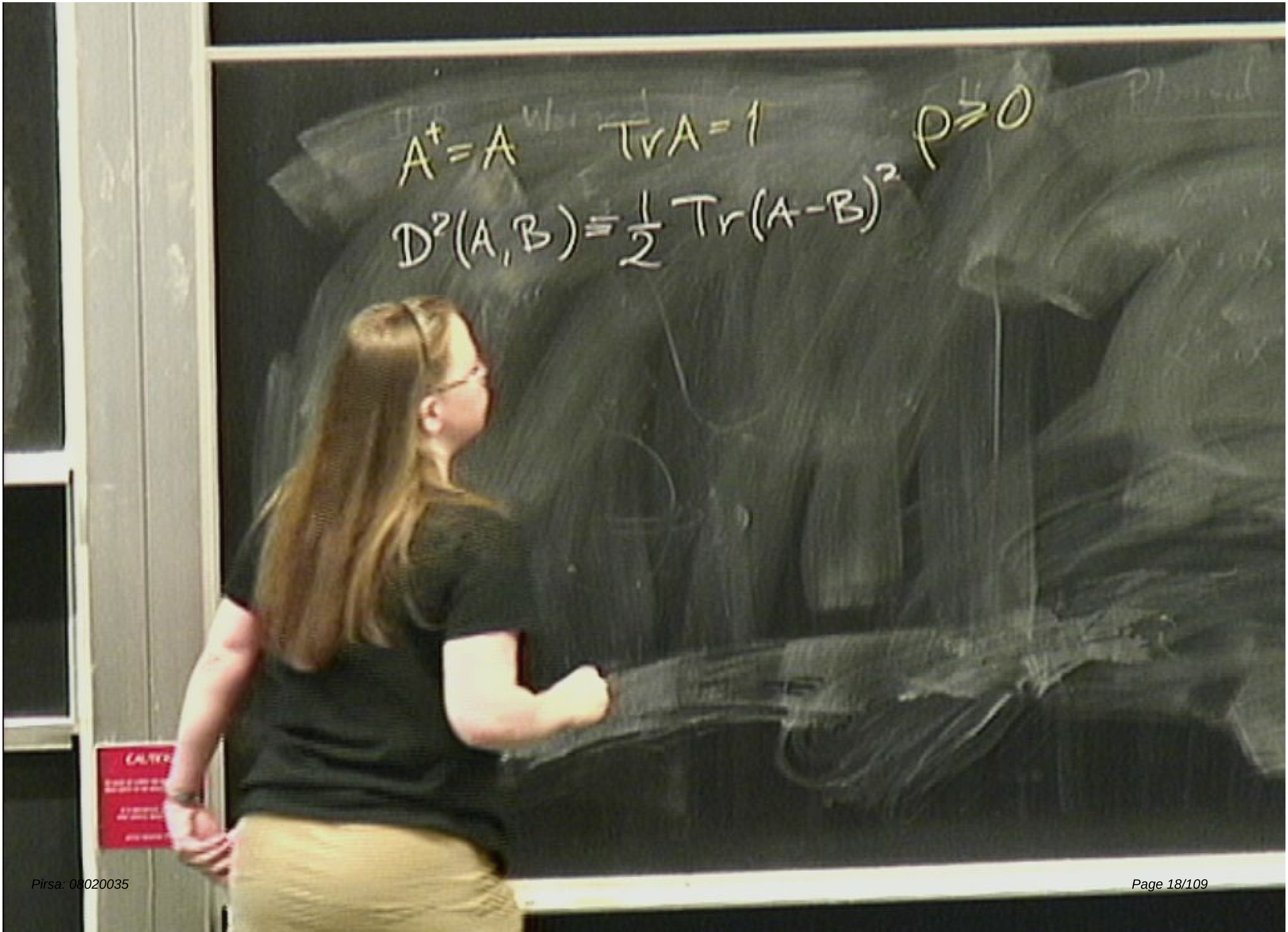
Complete set : $N+1$ MUBs

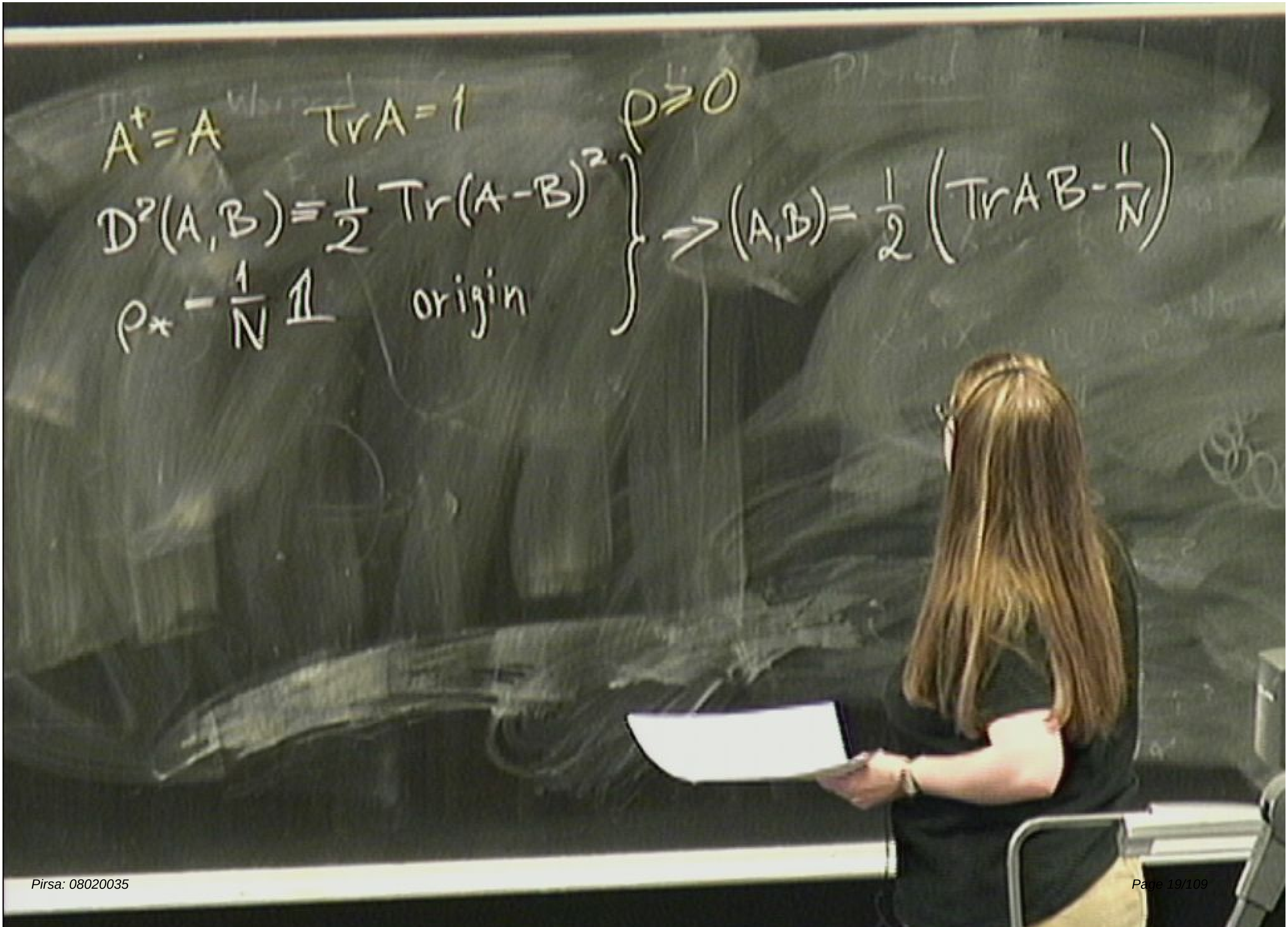
$$(N-1)(N+1) = N^2 - 1$$

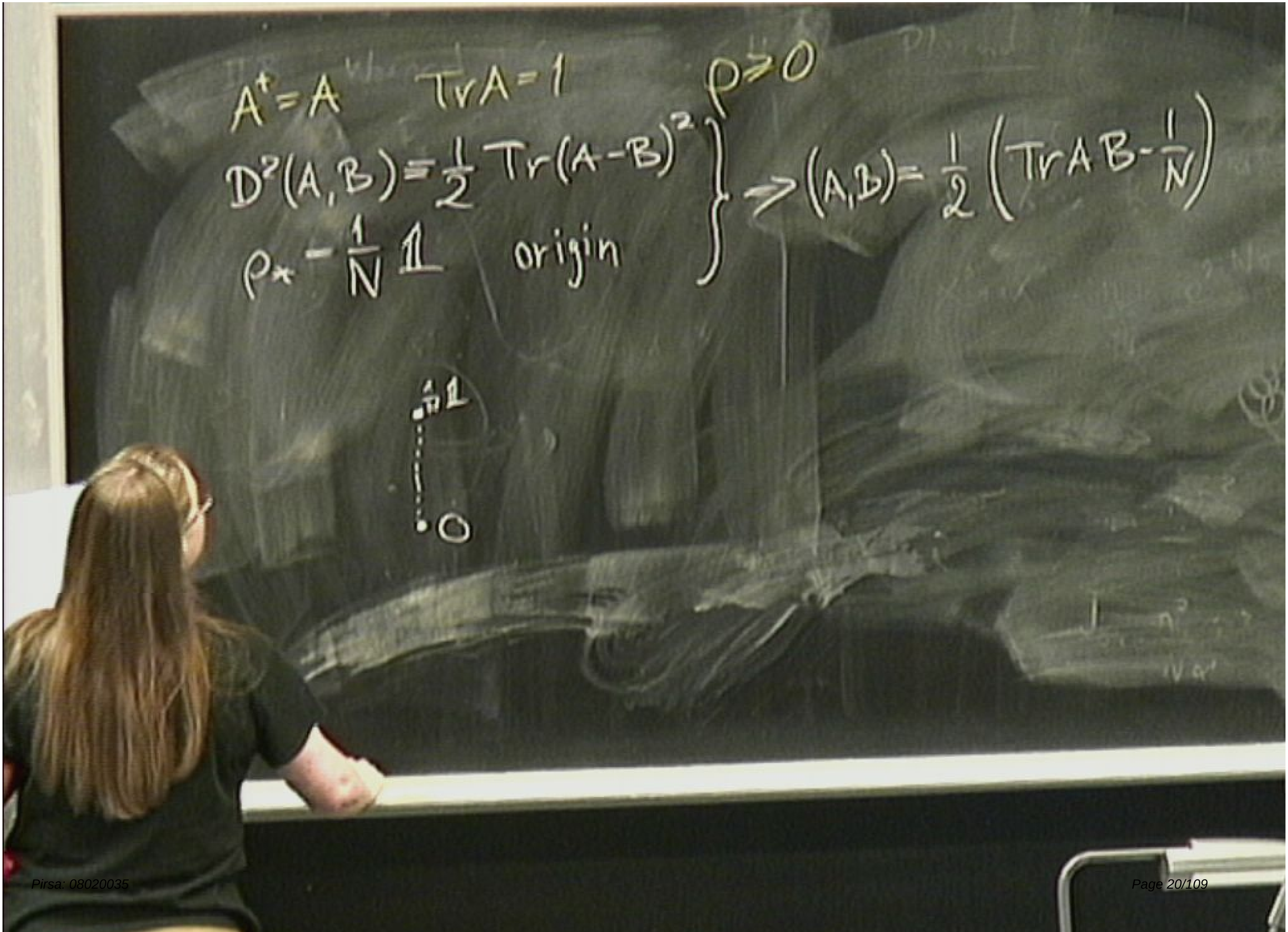
The MUB-problem : Do complete sets exist?

Yes if $N = p^k$. Otherwise ??


Complete set : N+1 MUBs
(N-1)(N+1) = N² - 1
The MUB-problem : Do complete sets exist?
Yes if N=pᵏ, Otherwise ??
N=6


$A^\dagger = A$ $\mathrm{Tr}A = 1$ $\rho \geq 0$
$D^2(A,B) = \frac{1}{2}\mathrm{Tr}(A-B)^2$


$A^{\dagger} = A$ $\mathrm{Tr}A = 1$ $\rho \geq 0$
$D^2(A,B) = \frac{1}{2}\mathrm{Tr}(A-B)^2$
$\rho_* - \frac{1}{N}\mathbb{1}$ origin
$\Rightarrow (A,B) = \frac{1}{2}\left(\mathrm{Tr}AB - \frac{1}{N}\right)$



$A^\dagger = A \quad \mathrm{Tr}A = 1 \quad \rho \geq 0$
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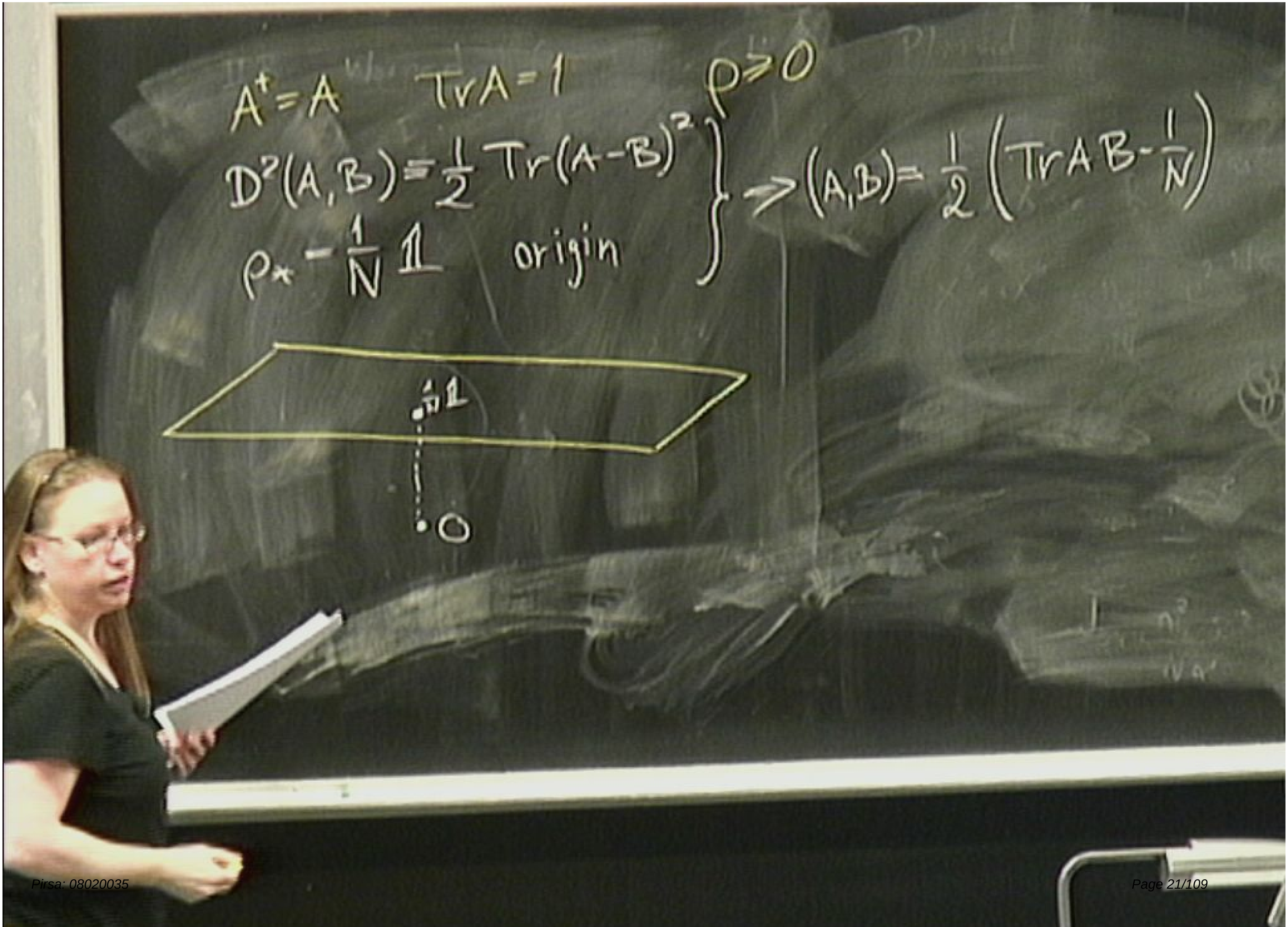
$\rho_* = \frac{1}{N}\mathbb{1}$ origin
$\frac{1}{N}\mathbb{1}$
0



$A^\dagger = A$ $\mathrm{Tr}A = 1$ $\rho \geq 0$
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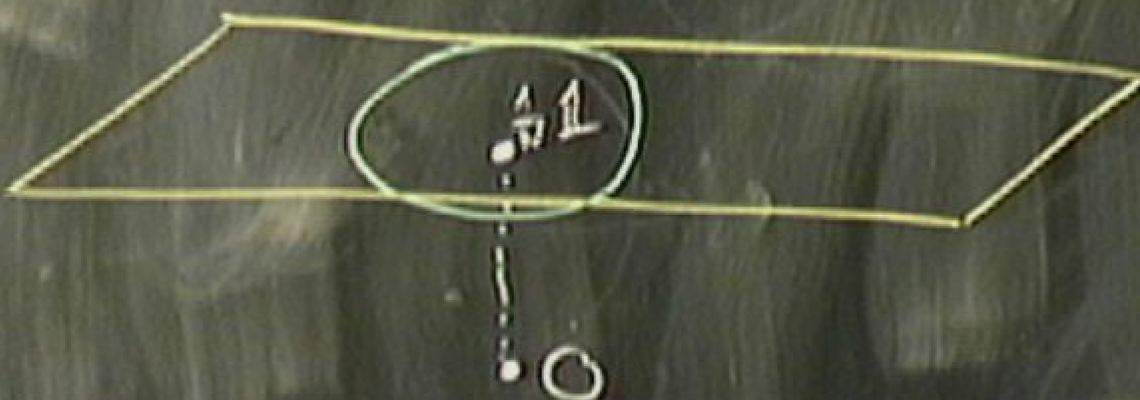
$\frac{1}{N}\mathbb{1}$
0

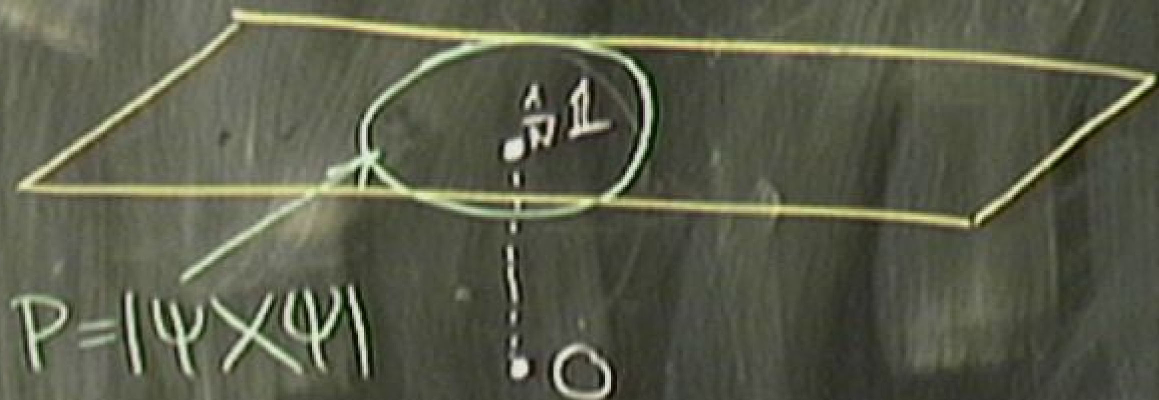
$A^{\dagger} = A \qquad \mathrm{Tr}A = 1 \qquad \rho \geq 0$

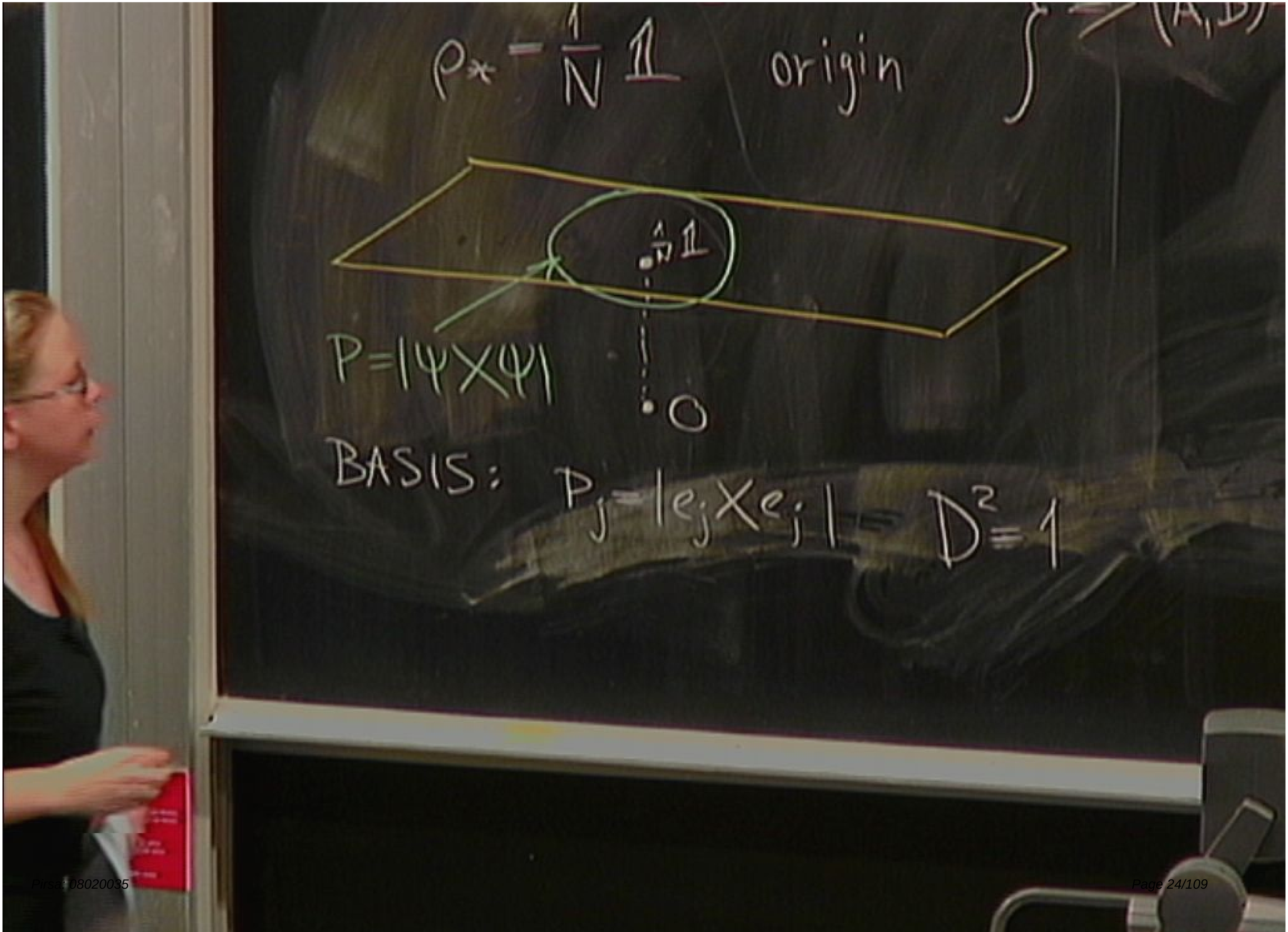
$$\left.\begin{array}{l} D^2(A,B) = \frac{1}{2}\,\mathrm{Tr}(A-B)^2 \\ \rho_* = \frac{1}{N}\mathbb{1} \quad \text{origin} \end{array}\right\} \Rightarrow (A,B) = \frac{1}{2}\Big(\mathrm{Tr}AB - \frac{1}{N}$$

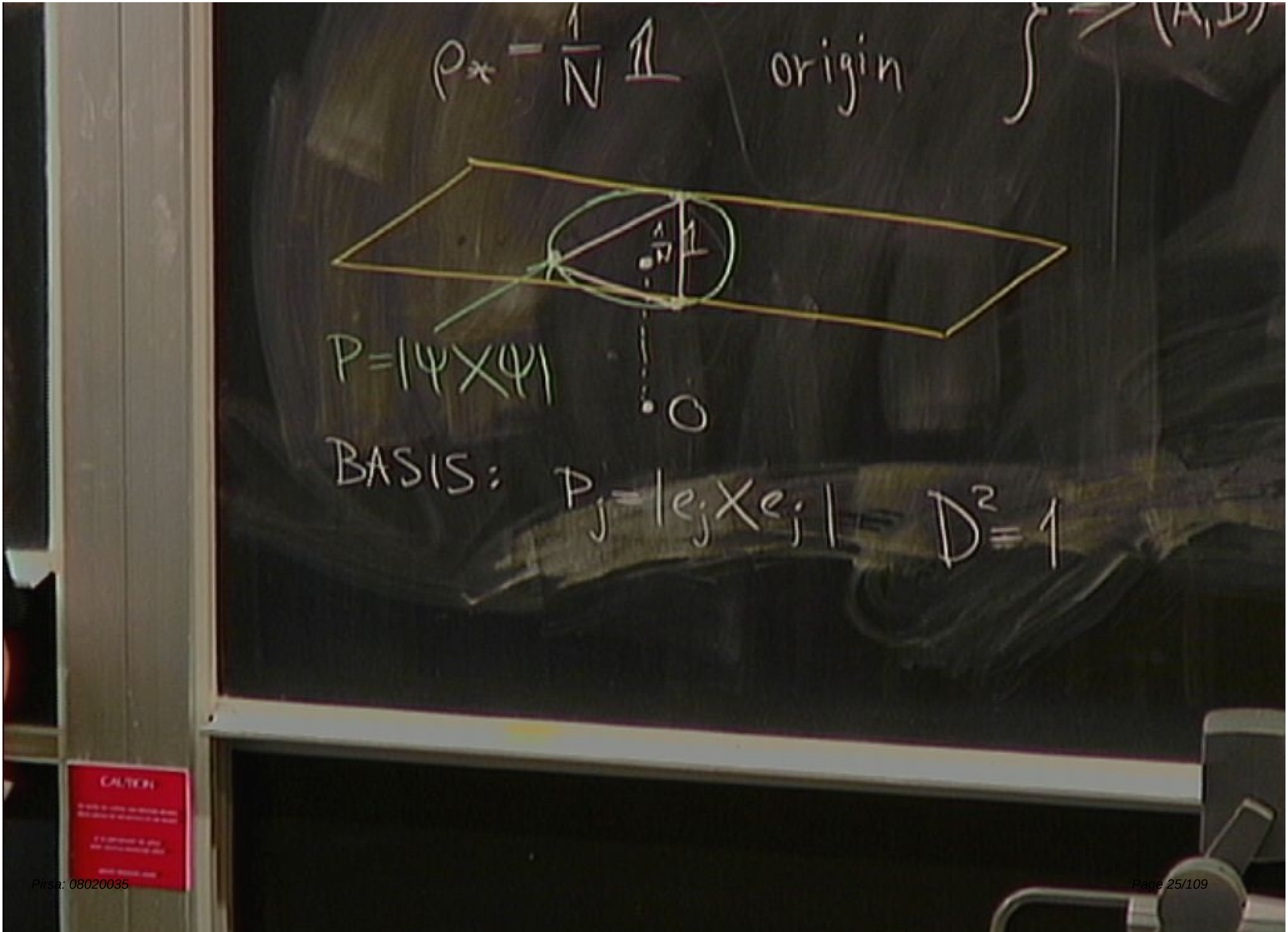


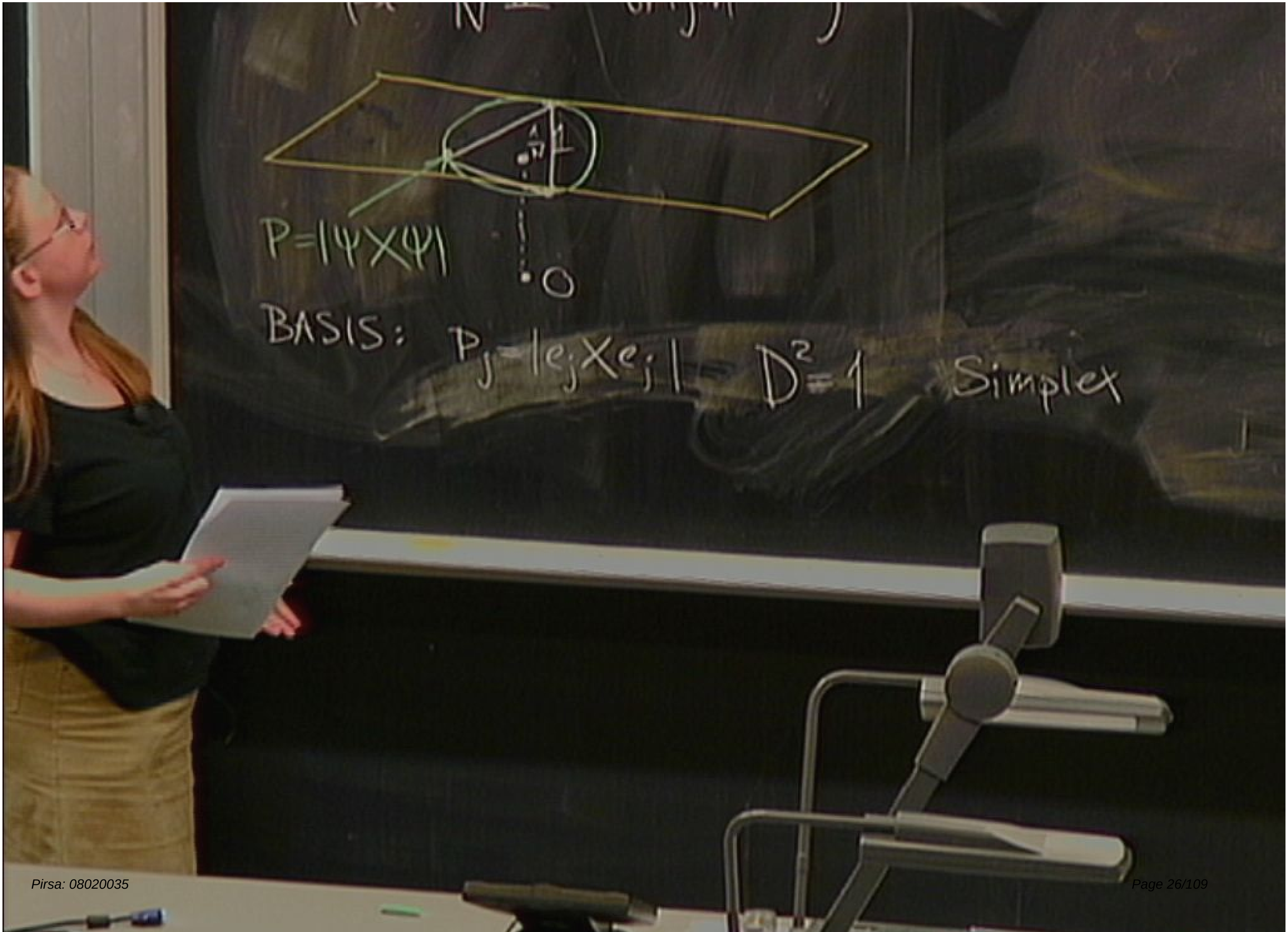
$A^{\dagger} = A \qquad \mathrm{Tr}A = 1 \qquad \rho \geq 0$

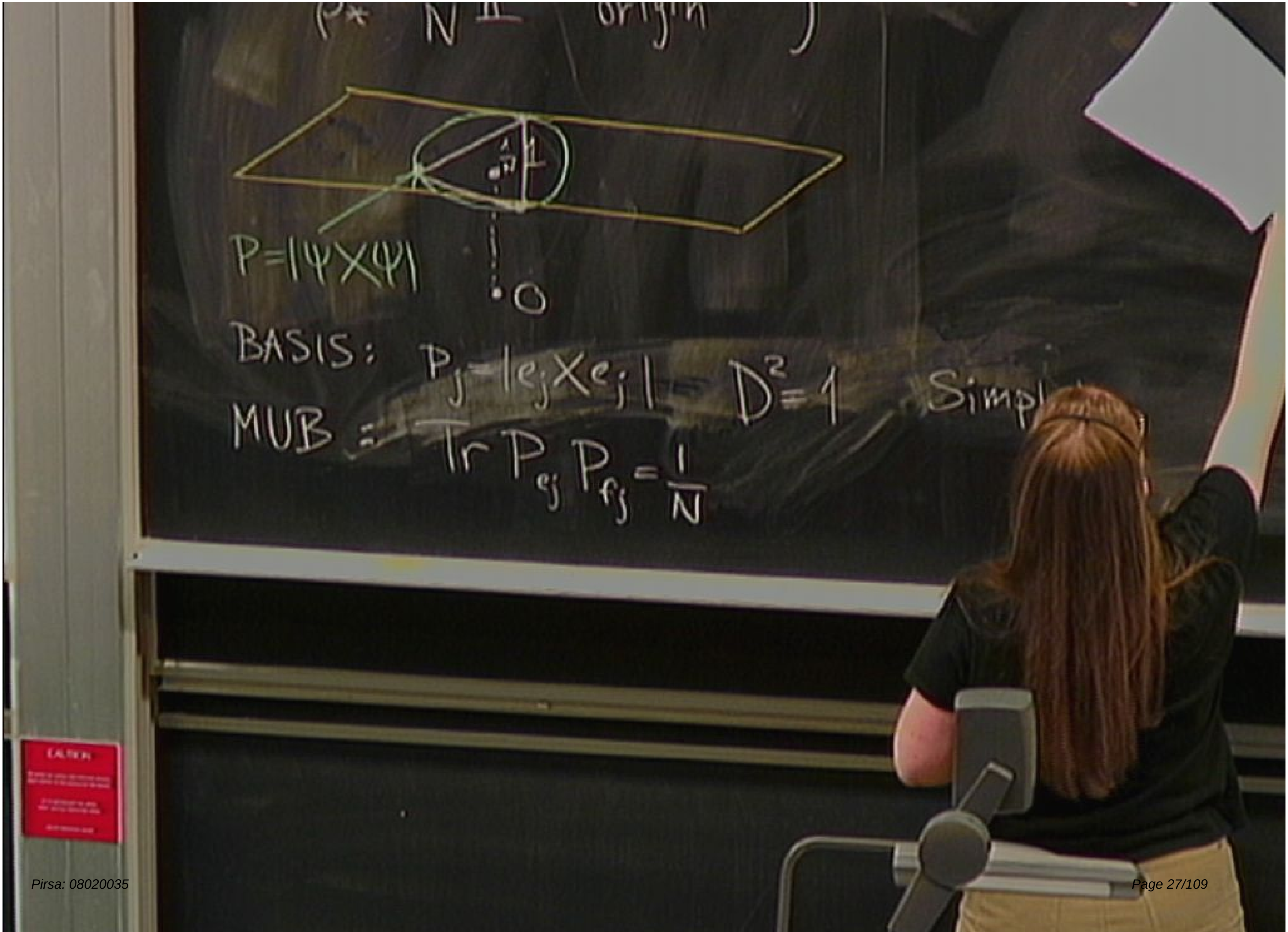
$$\left.\begin{array}{l} D^2(A,B) = \frac{1}{2}\mathrm{Tr}(A-B)^2 \\ \rho_* = \frac{1}{N}\mathbb{1} \quad \text{origin} \end{array}\right\} \Rightarrow (A,B) = \frac{1}{2}\big(\mathrm{Tr}$$

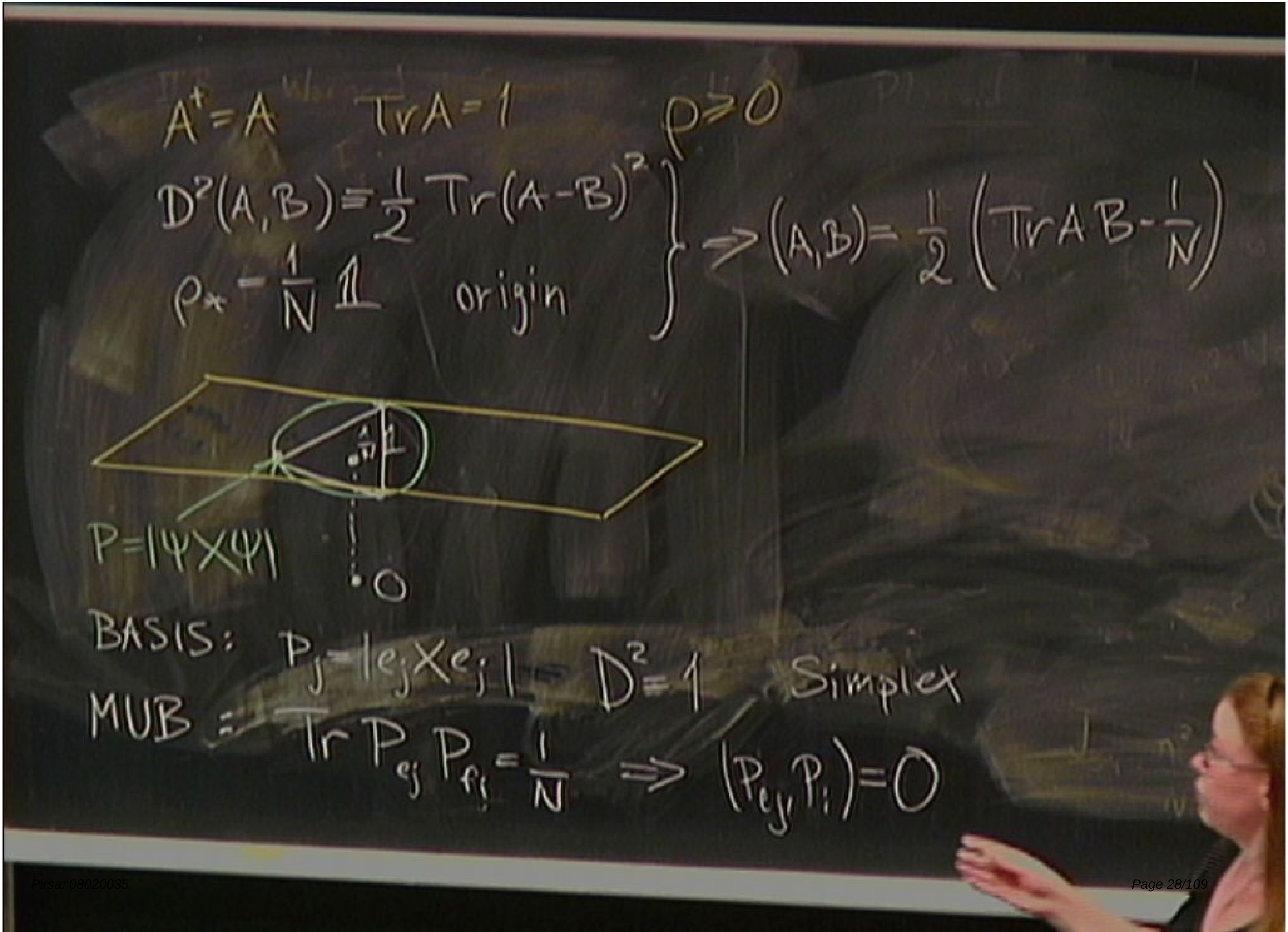



ρ* = 1/N 𝟙 origin
1/N 𝟙
P=|ψXψ|
O
BASIS: P_j = |e_j X e_j| D² = 1


ρ* = 1/N 𝟙 origin
(A,B)
1/N 𝟙
P=|ψ⟩⟨ψ|
O
BASIS: P_j = |e_j⟩⟨e_j| D² = 1
CAUTION


P=|ψXψ|
0
BASIS: Pj=|ejXej|
D2=1
Simplex


P=|Ψ⟩⟨Ψ|
O
BASIS: P_j = |e_j⟩⟨e_j| D² = 1   Simpl
MUB: Tr P_{e_j} P_{f_j} = 1/N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$\rho_* = \frac{1}{N}\mathbb{1}$ origin
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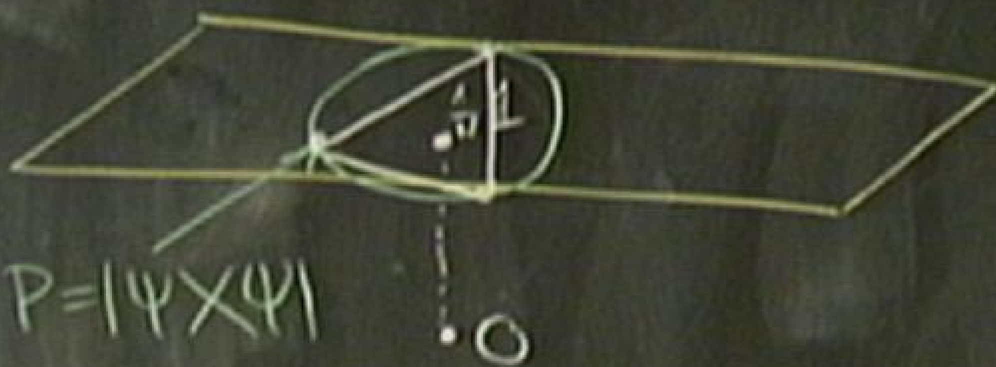
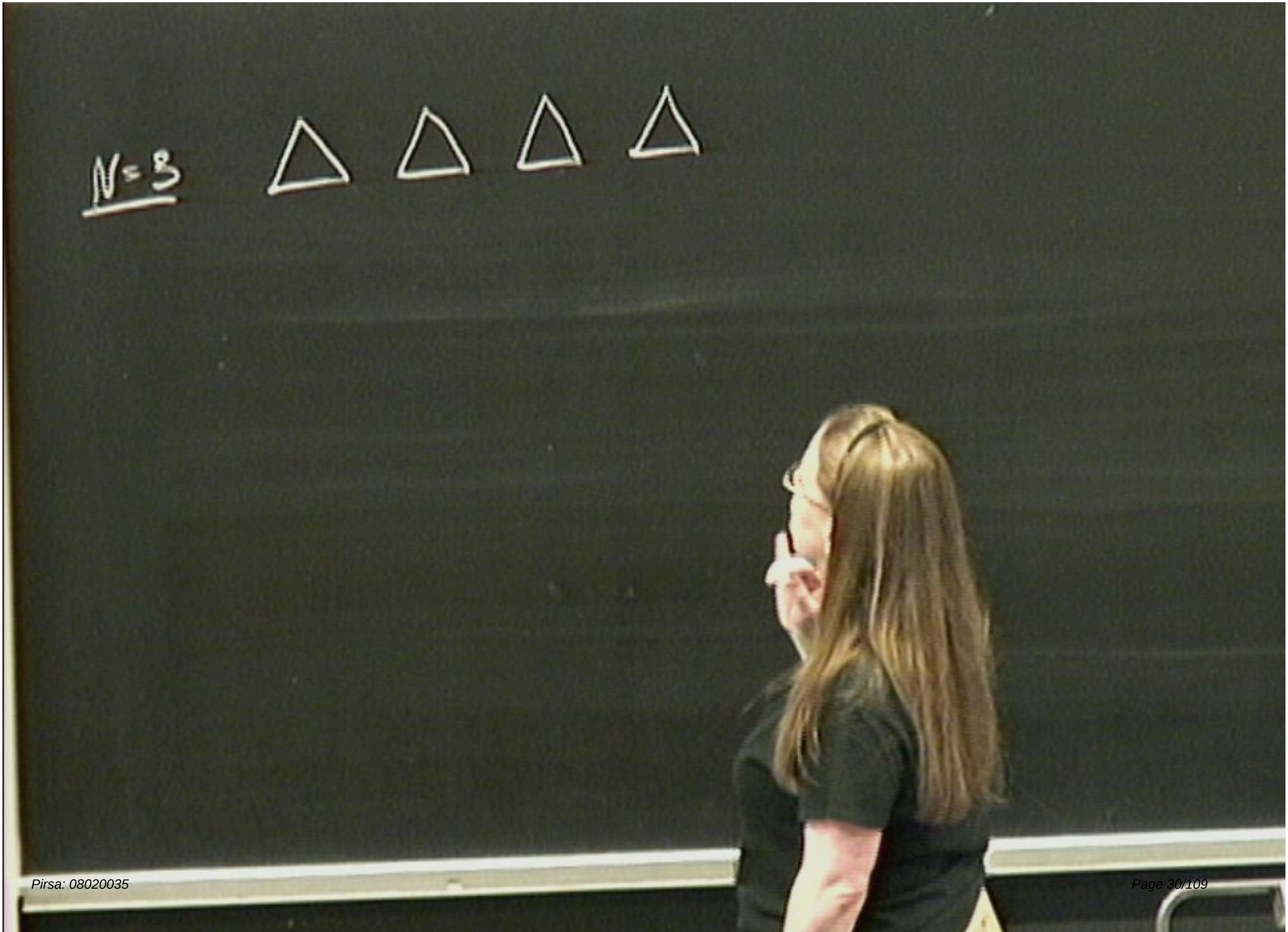
$\frac{1}{N}\mathbb{1}$
$P = |\psi\rangle\langle\psi|$
O
BASIS: $P_j = |e_j\rangle\langle e_j|$ — $D^2 = 1$ Simplex
MUB: $\mathrm{Tr}P_{e_j}P_{f_i} = \frac{1}{N} \Rightarrow (P_{e_j}, P_i) = 0$

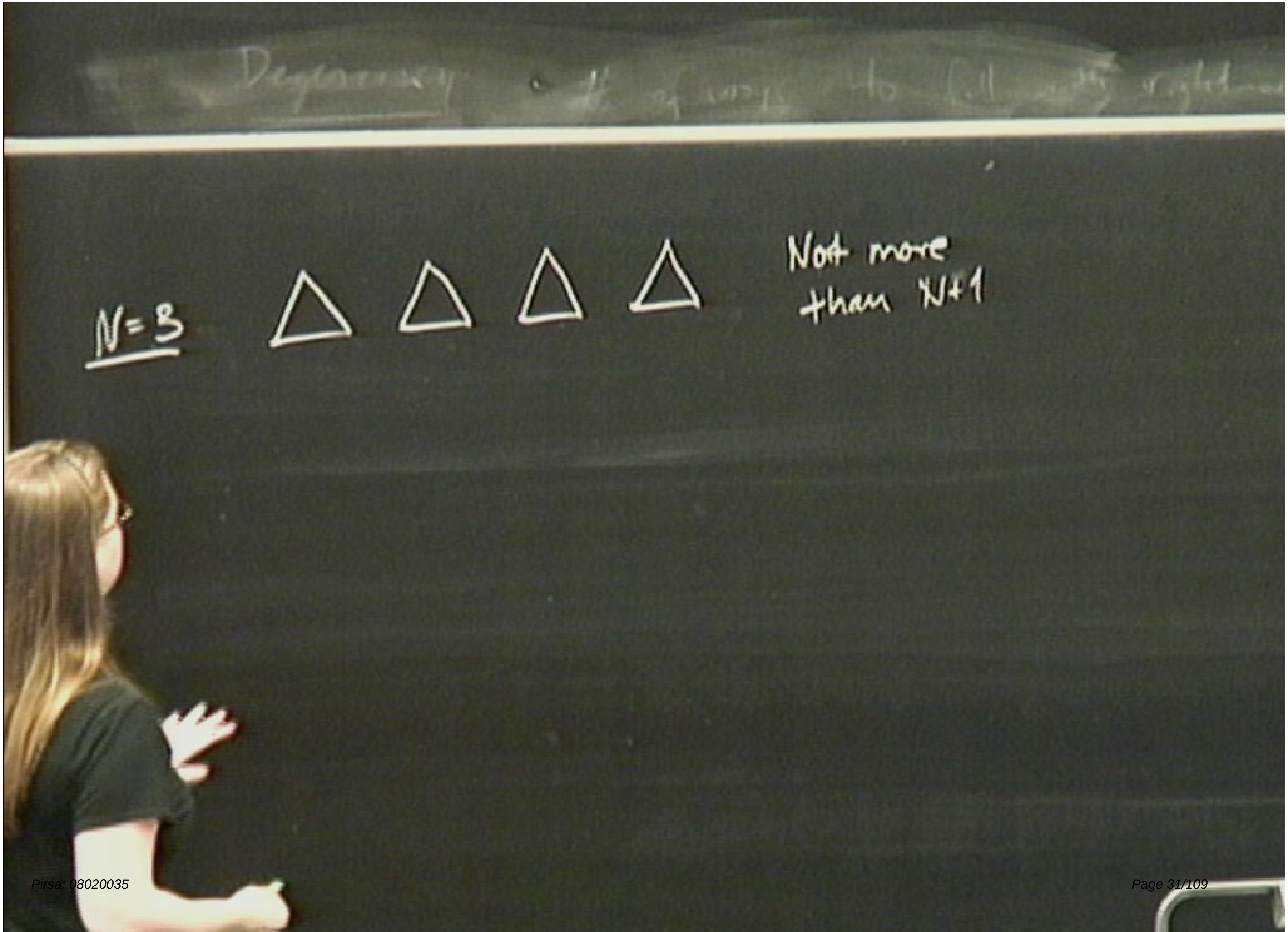
$$\left.\begin{array}{l} D^2(A,B) \equiv \frac{1}{2}\,\mathrm{Tr}(A-B)^2 \\ \rho_* = \frac{1}{N}\,\mathbb{1} \quad \text{origin} \end{array}\right\} \Rightarrow (A,B) = \frac{1}{2}\left(\mathrm{Tr}\,AB - \frac{1}{N}\right)$$

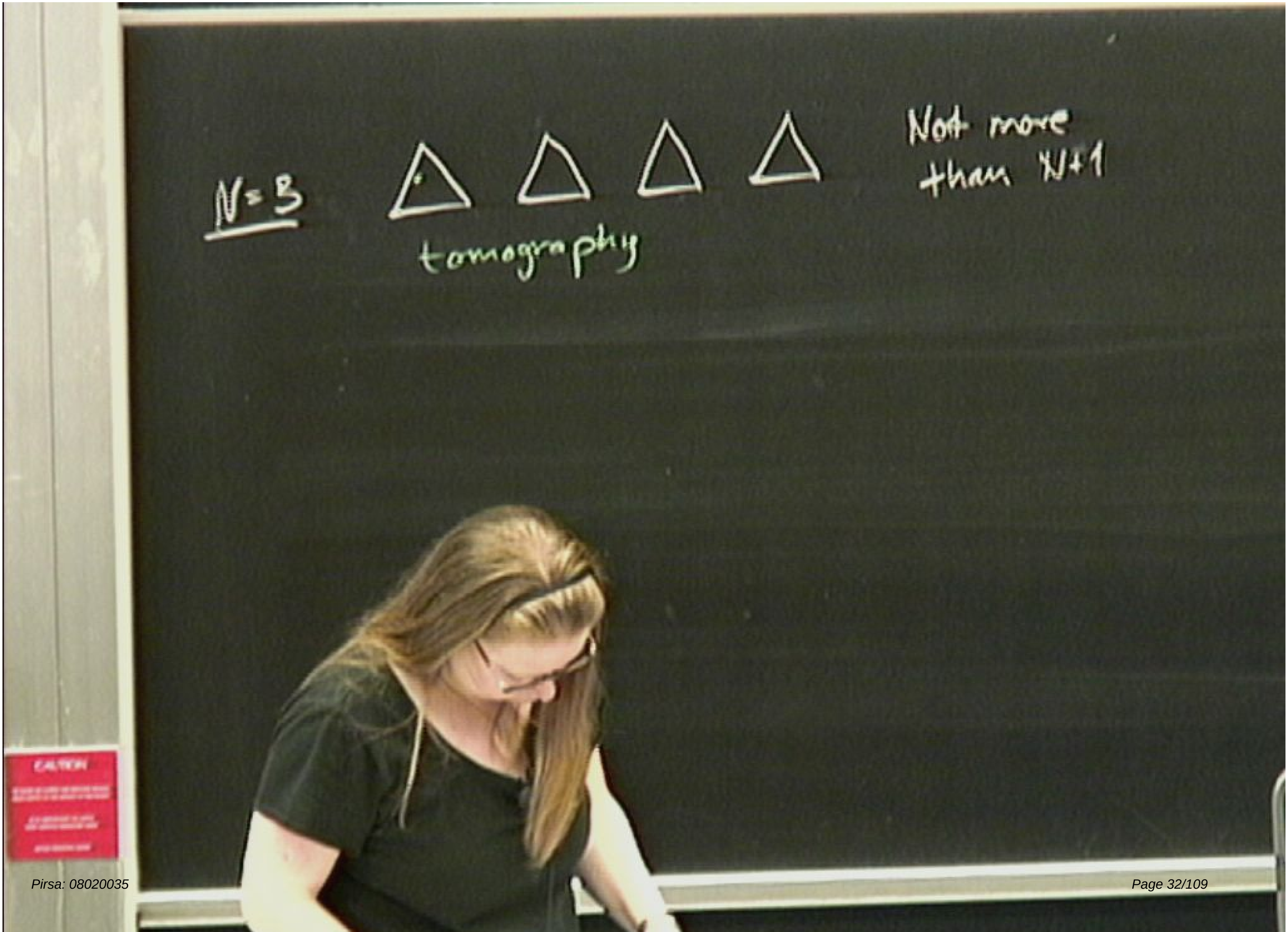


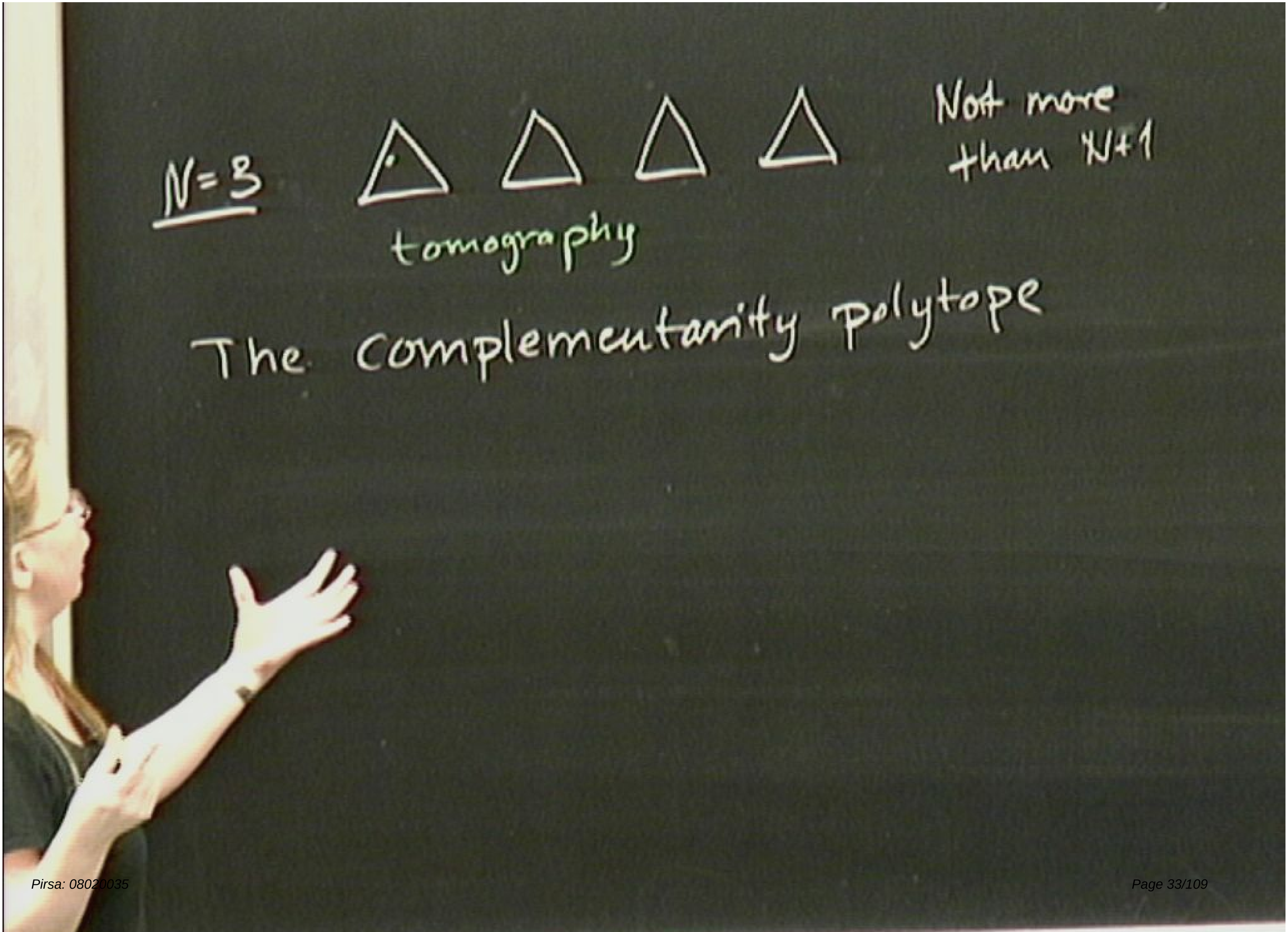
BASIS: $P_j = |e_j\rangle\langle e_j| \quad D^2 = 1$ Simplex

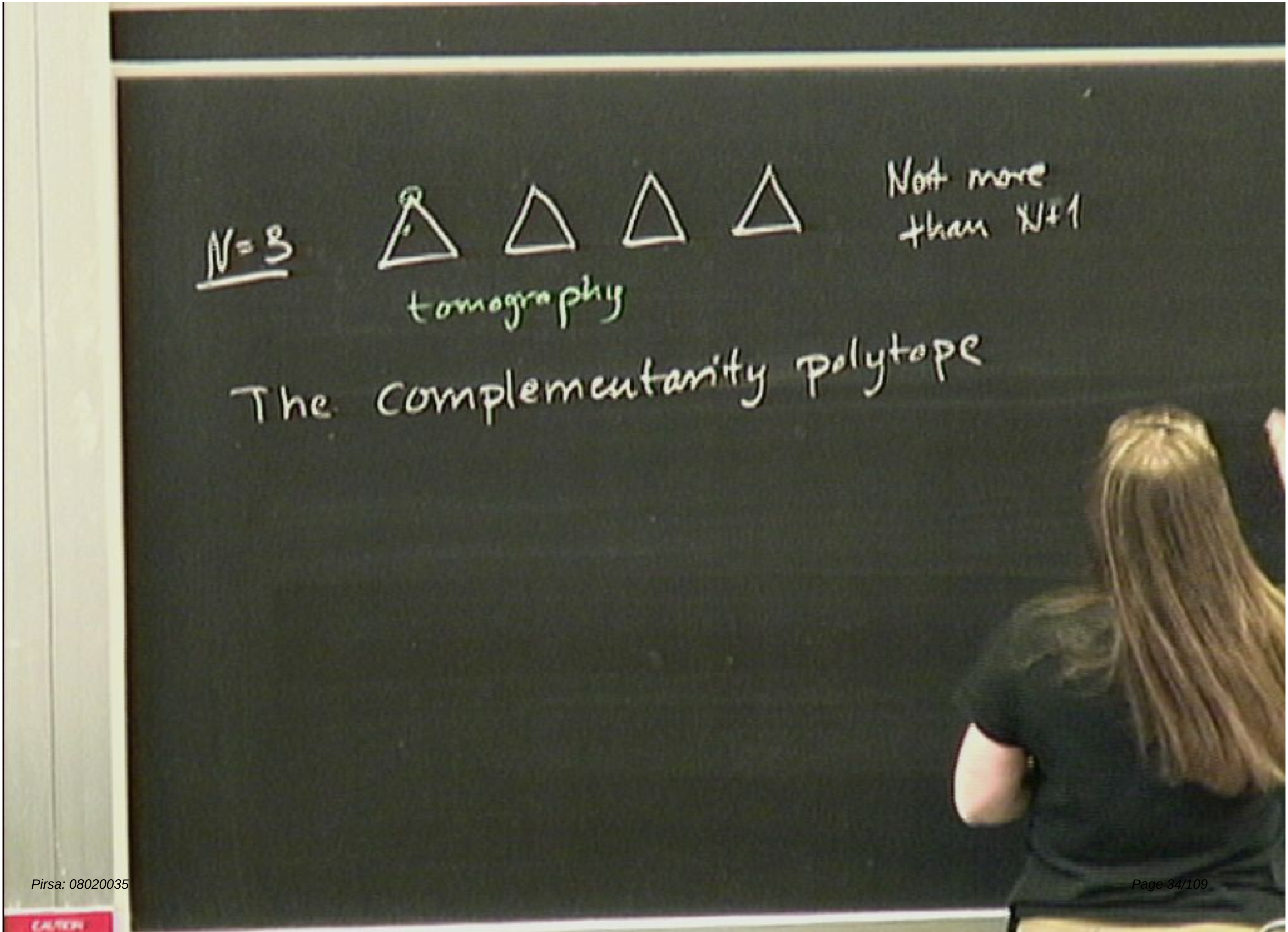
MUB = $\mathrm{Tr}\,P_{e_j} P_{f_j} = \frac{1}{N} \Rightarrow (P_{e_j}, P_i) = 0$ Orthogonal


N=3

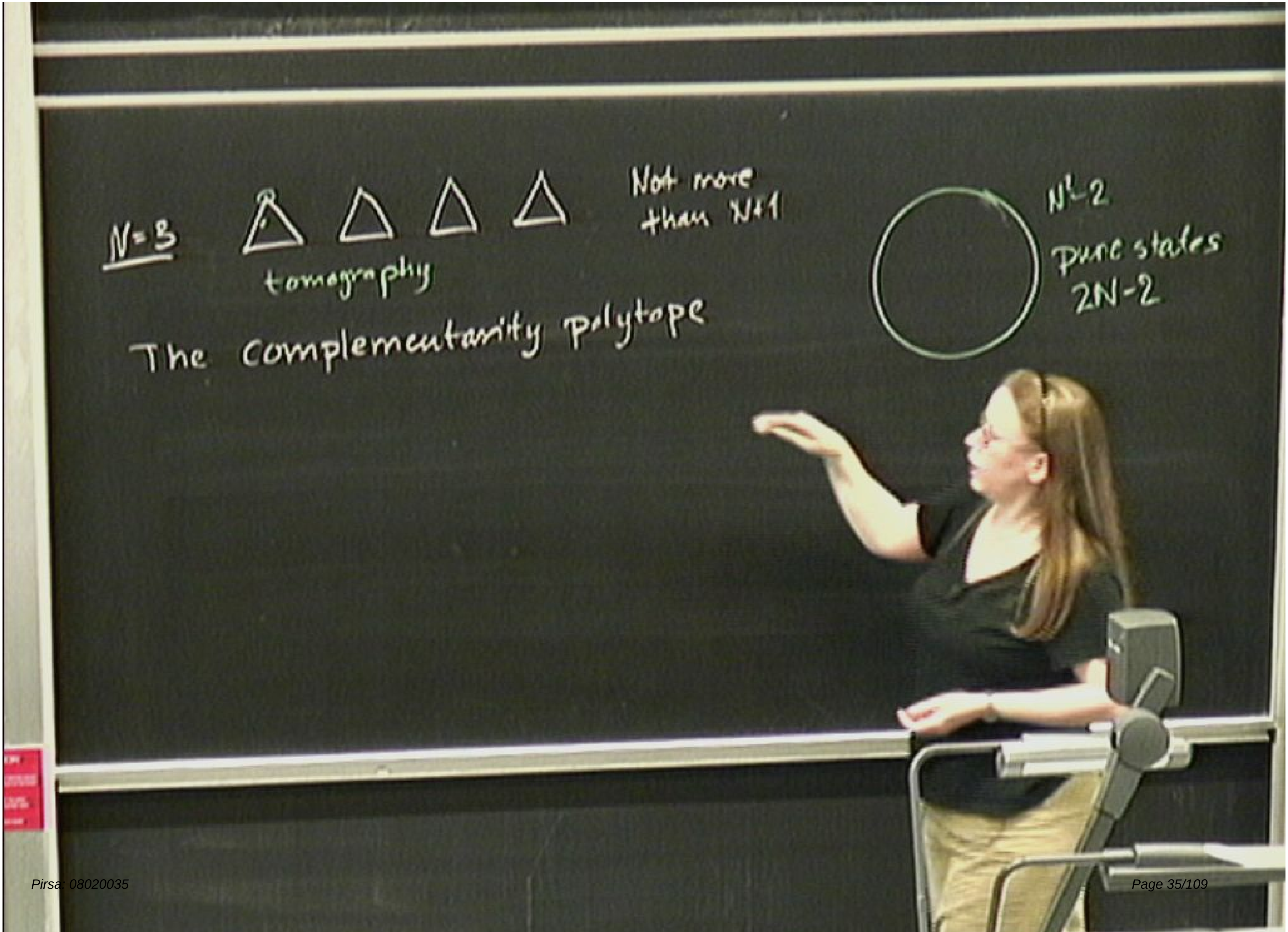

N=3
Not more than N+1


N=3
tomography
Not more
than N+1


N=3
Not more than N+1
tomography
The complementarity polytope



N=3
Not more
than N+1
tomography
The complementarity polytope


N=3
Not more
than N+1
tomography
N^2-2
pure states
$2N-2$
The complementarity polytope

$\underline{N=3}$

Not more than $N+1$

tomography

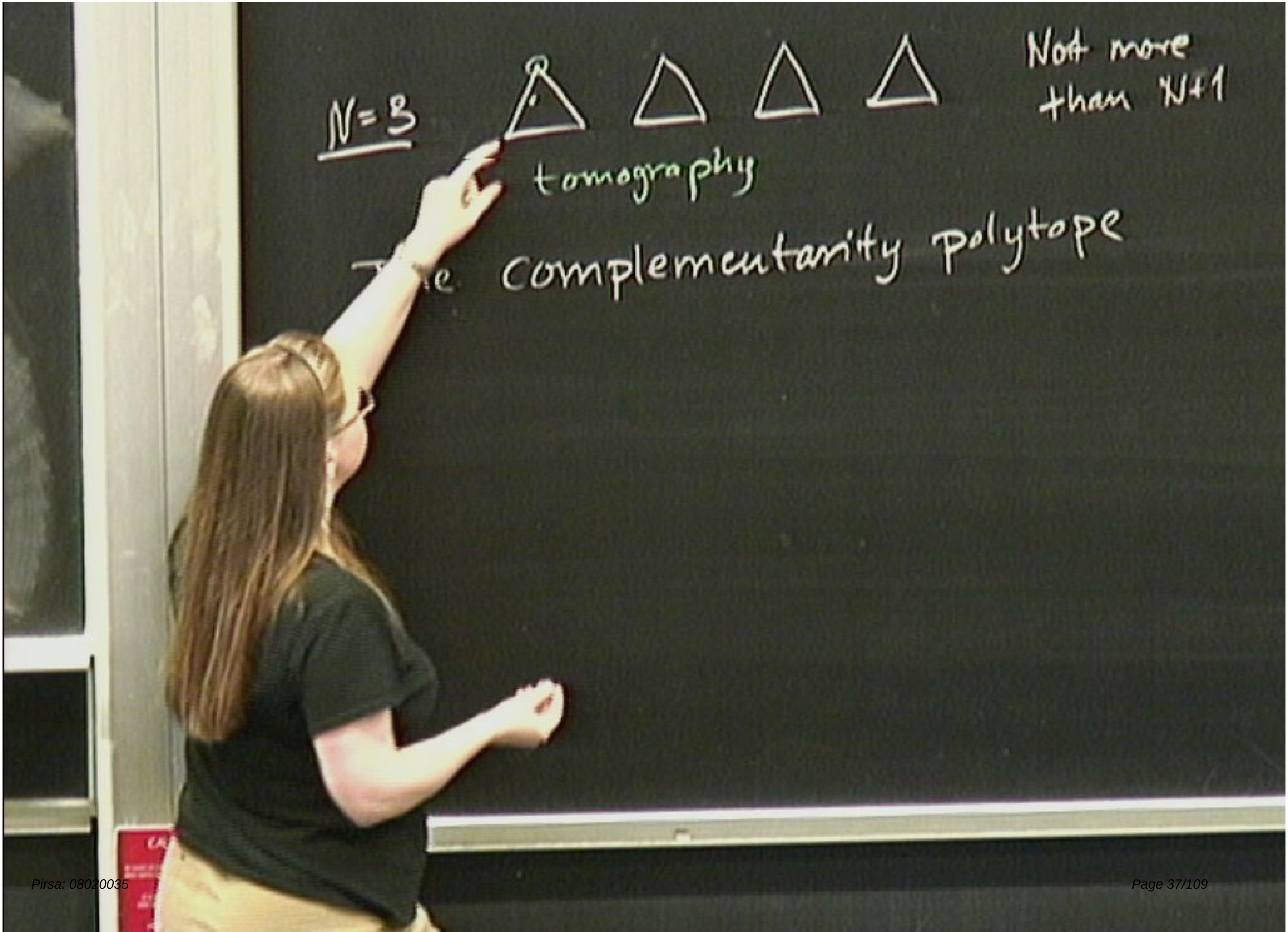
N^2-2

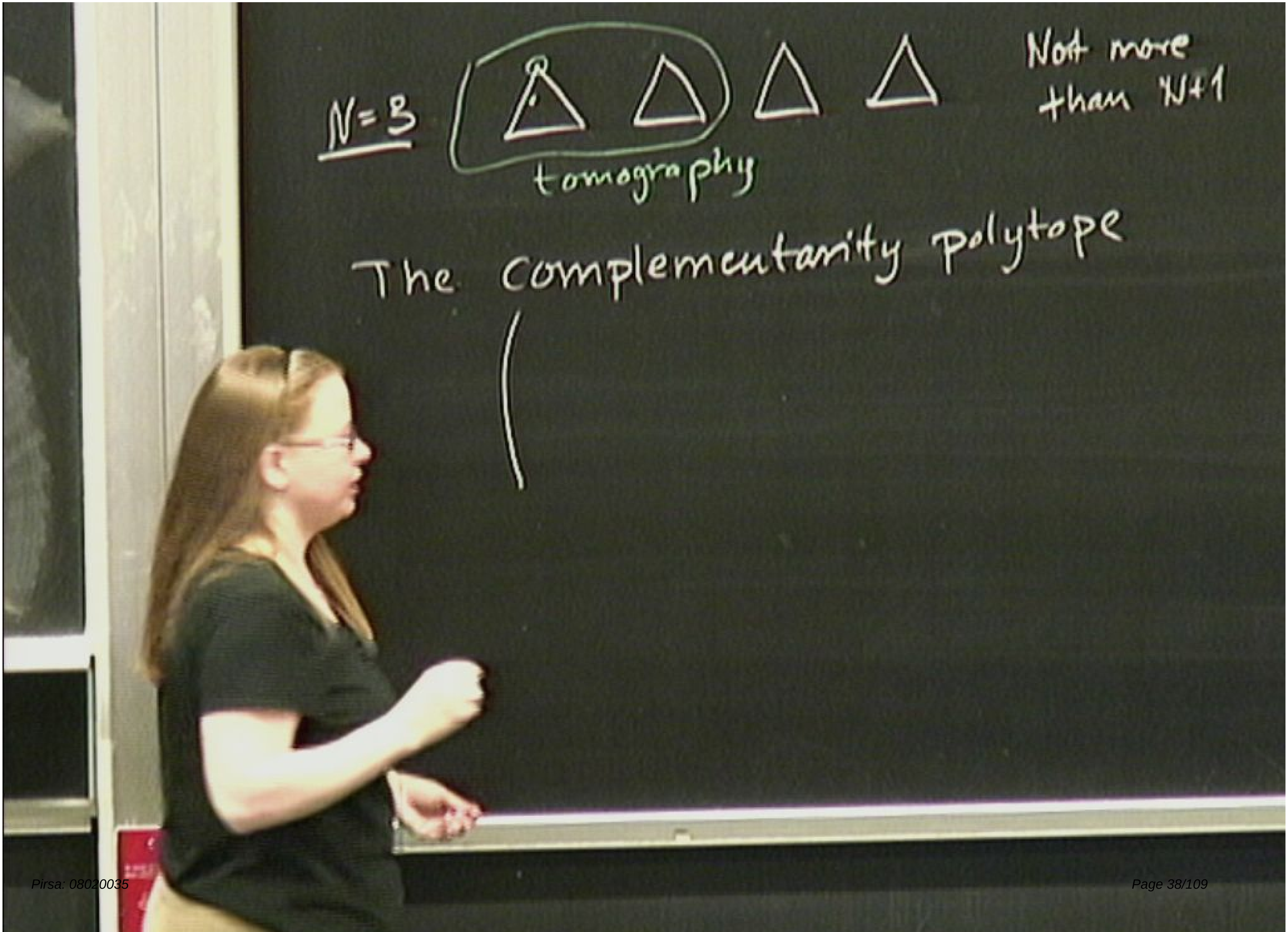
pure states

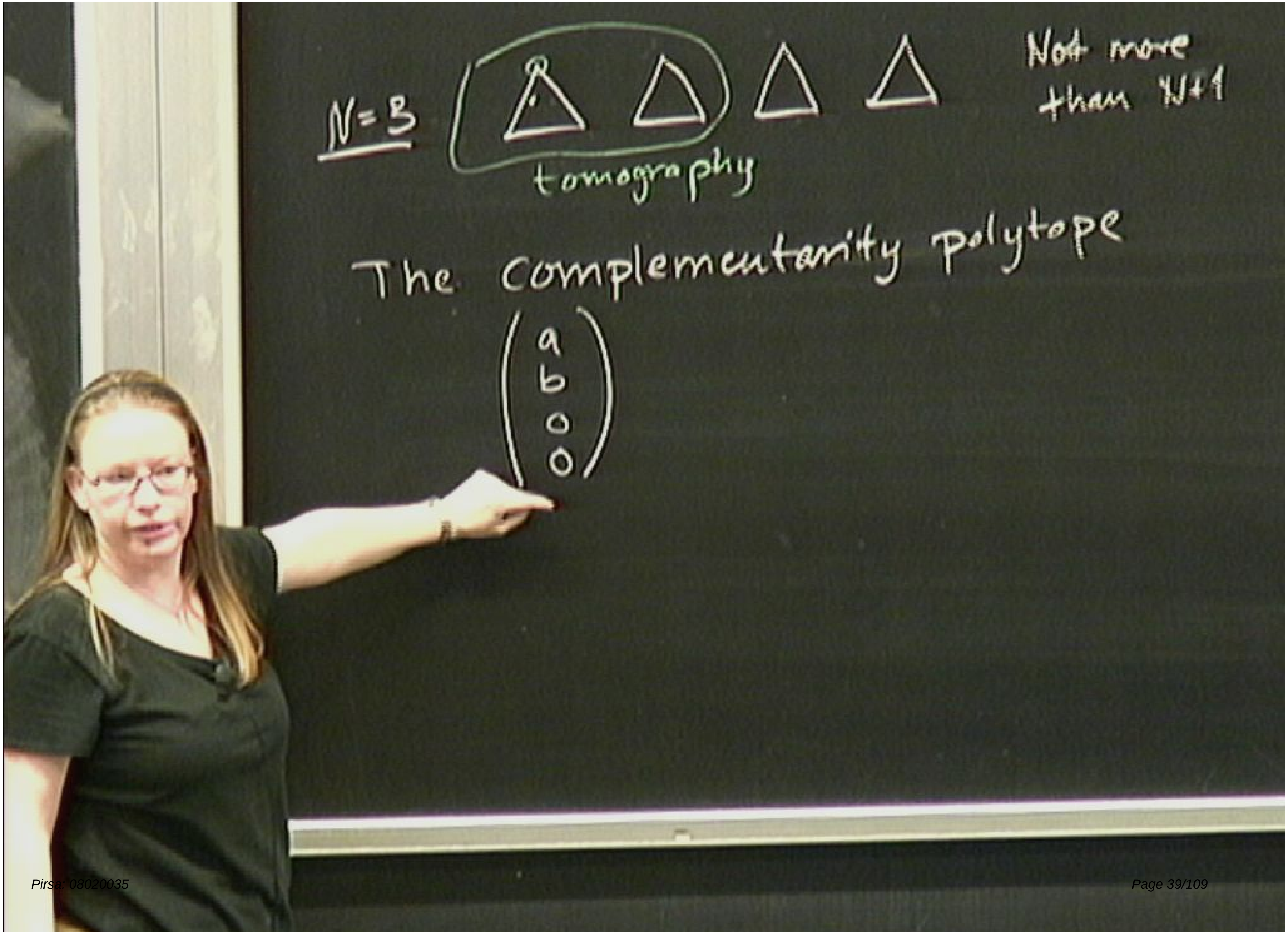
$2N-2$

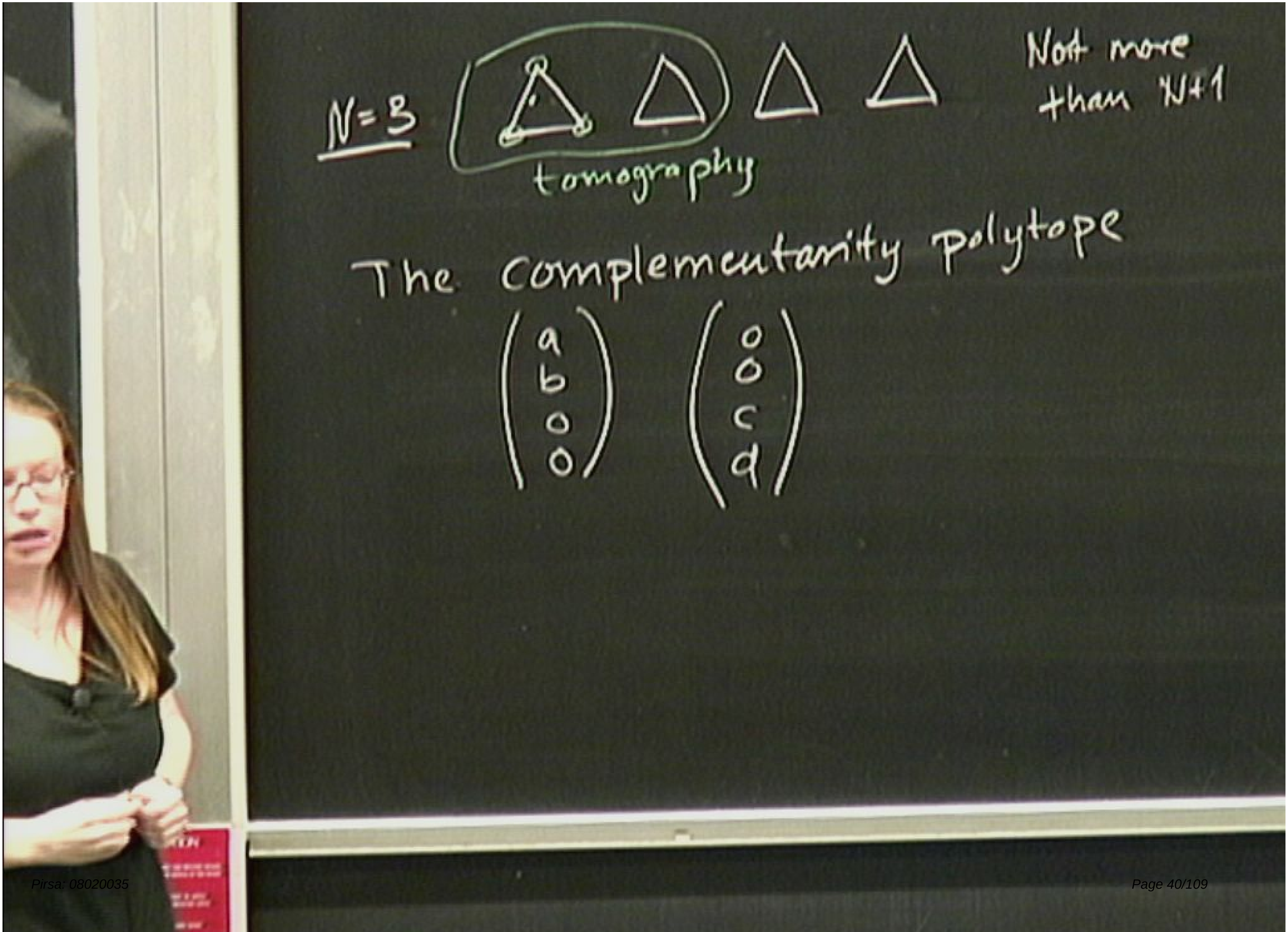
real

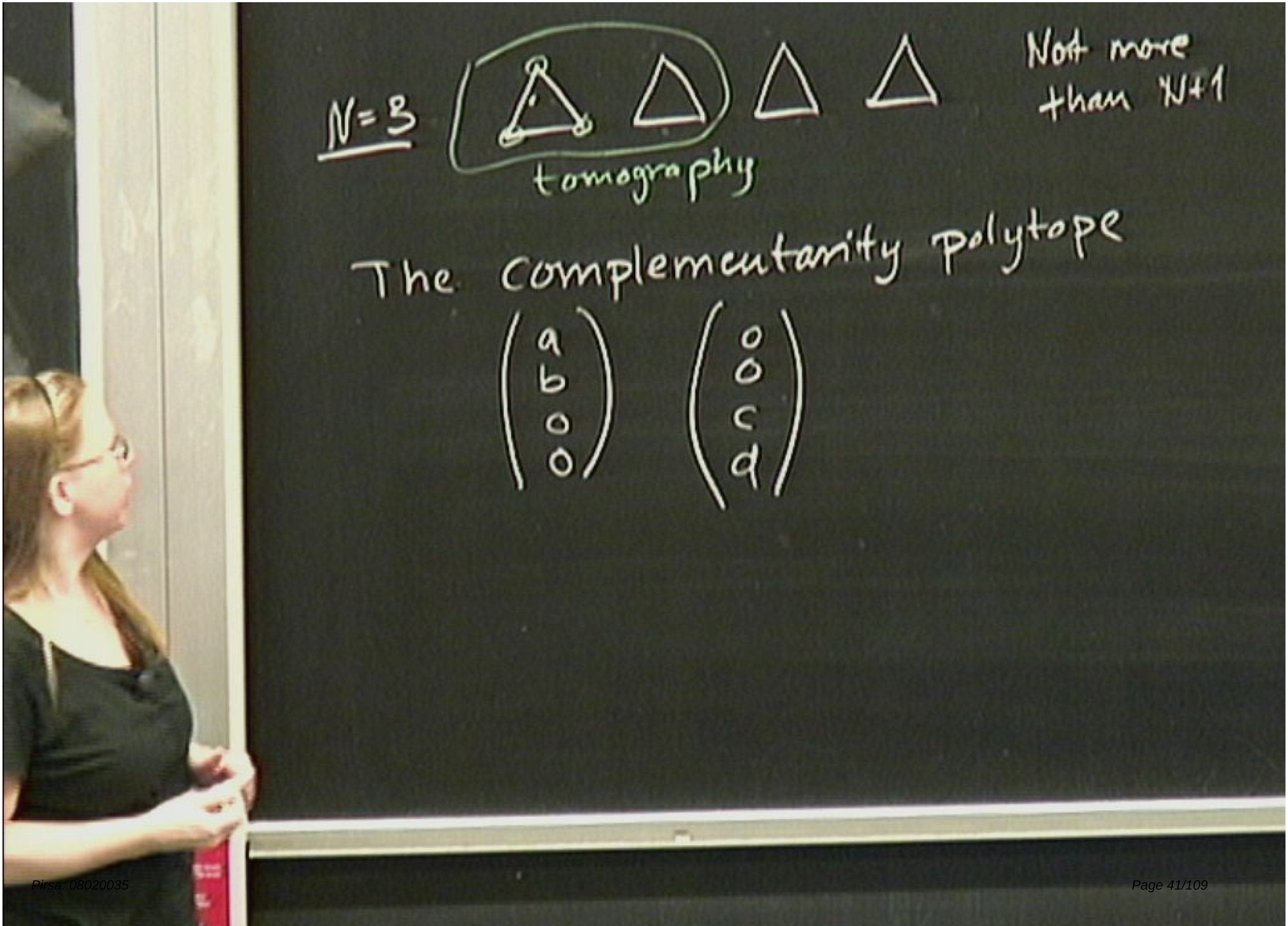
The complementarity polytope

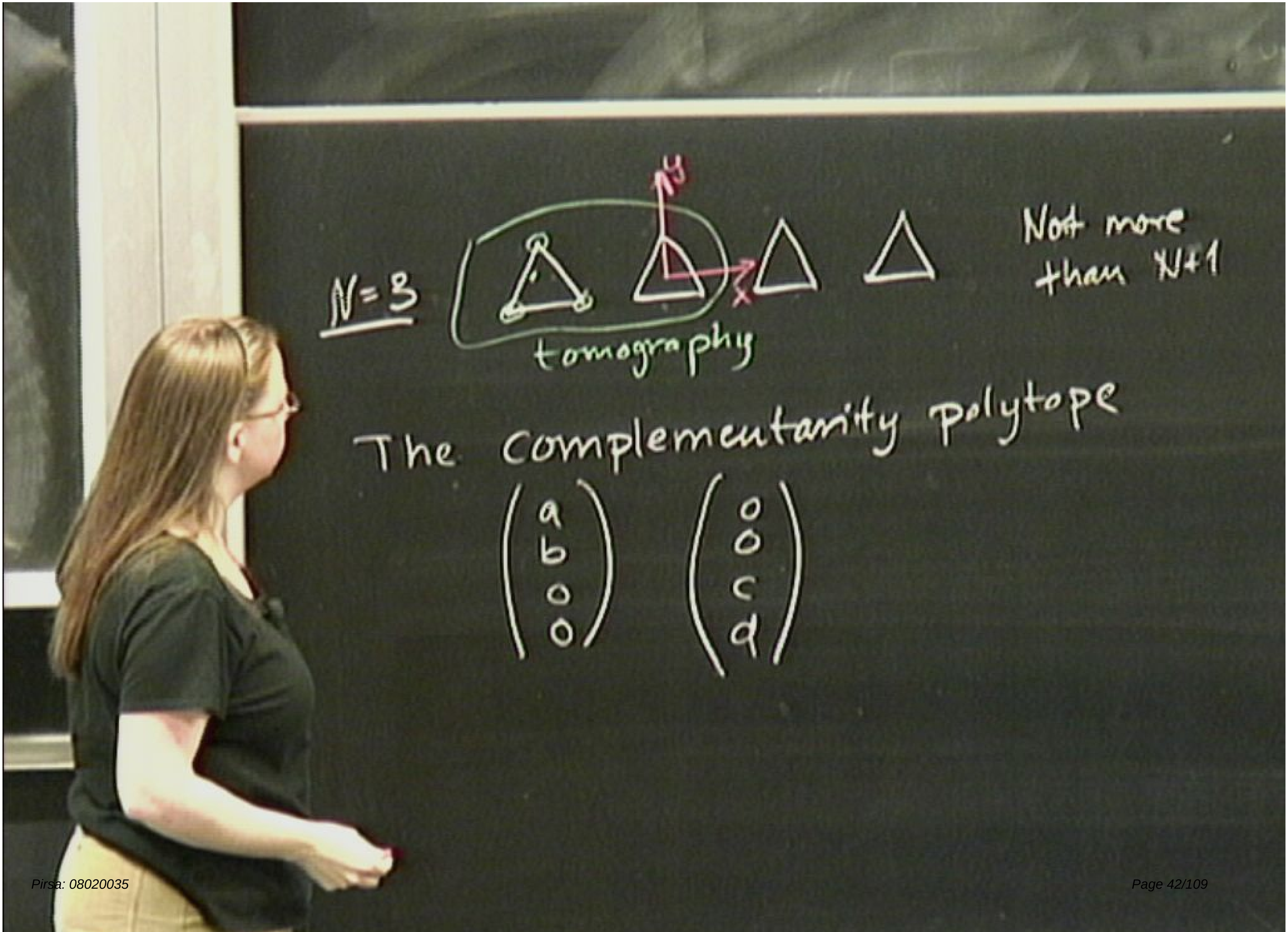

N=3
Not more than N+1
tomography
complementarity polytope

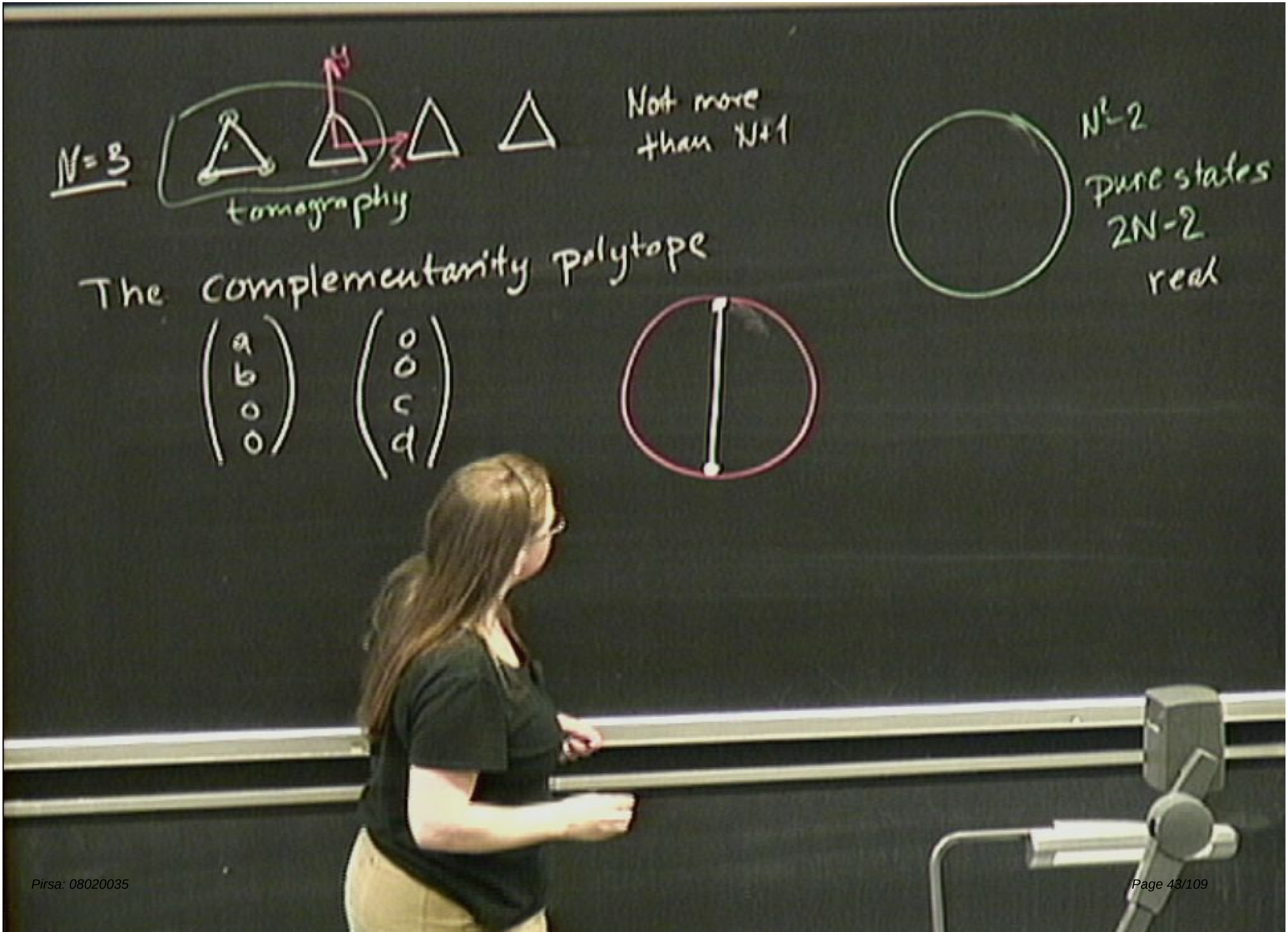

N=3
tomography
Not more than N+1
The complementarity polytope

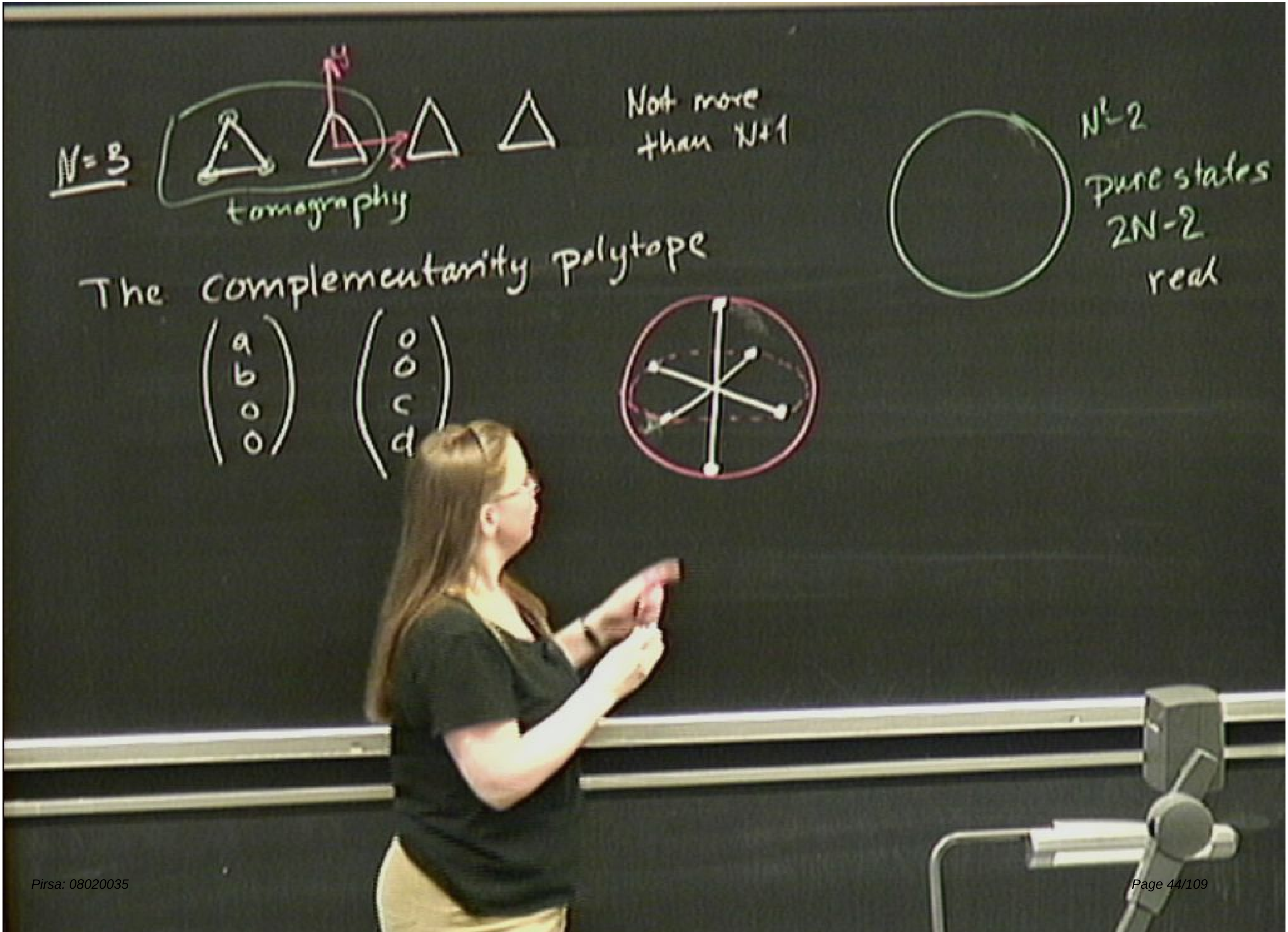

N=3
Not more
than N+1
tomography
The Complementarity polytope
a
b
0
0

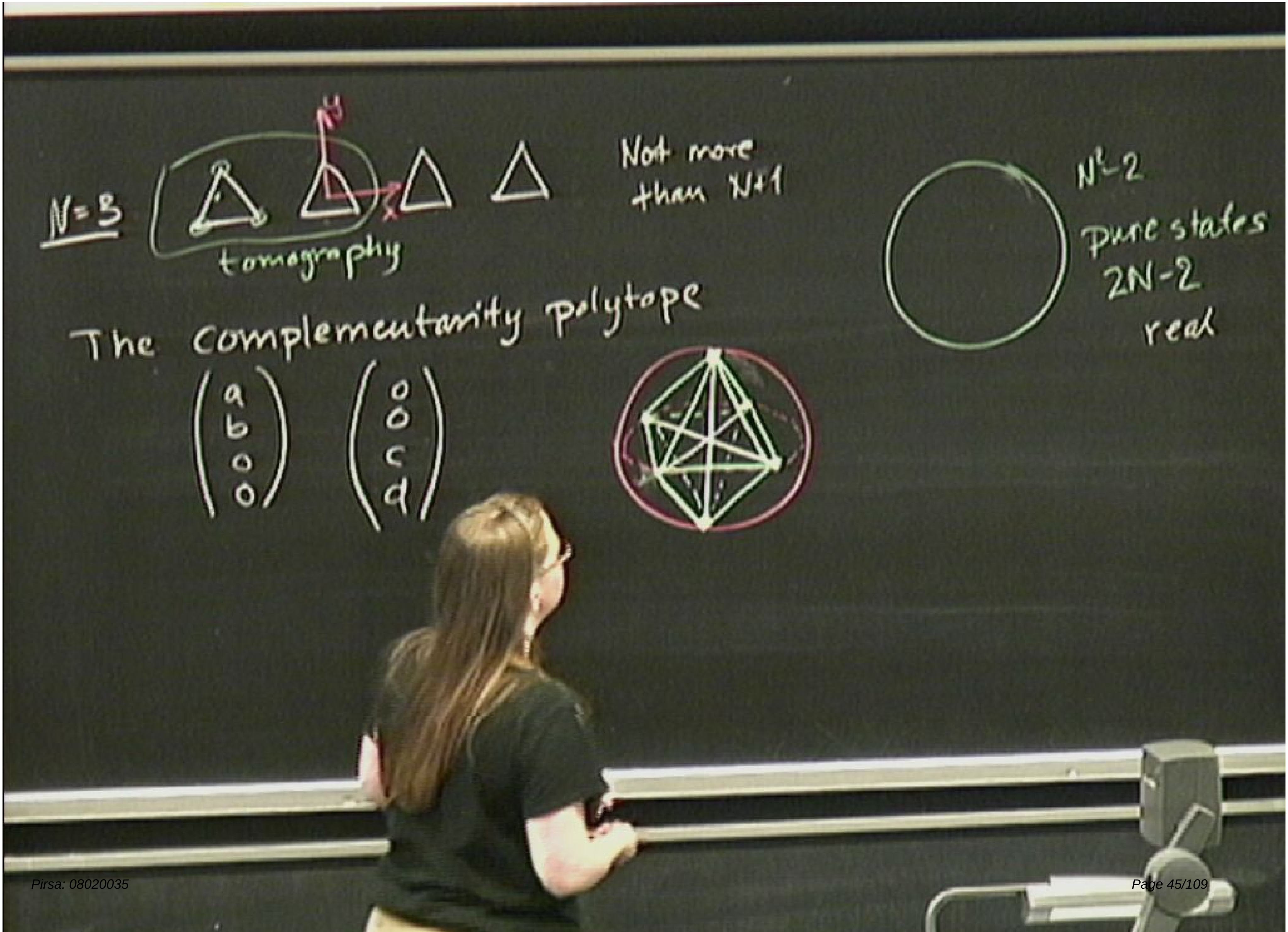

N=3
tomography
Not more than N+1
The complementarity polytope
a b 0 0
0 0 c d

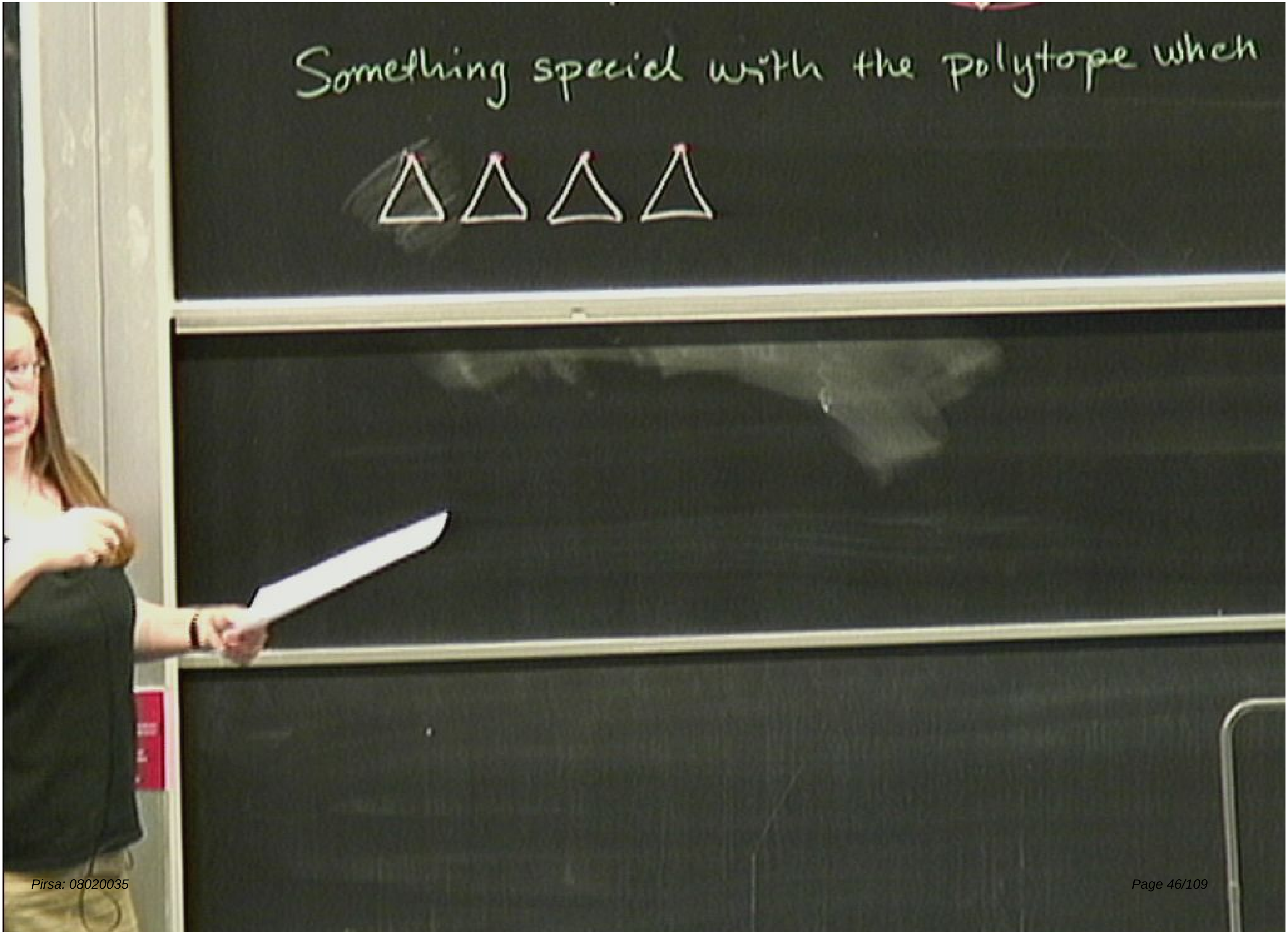

N=3
Not more than N+1
tomography
The complementarity polytope
a b 0 0
0 0 c d


N = 3
y
x
Not more
than N+1
tomography
The complementarity polytope
a
b
0
0
0
0
c
d


N=3
tomography
Not more than N+1
N²-2
pure states
2N-2
real
The complementarity polytope
a b 0 0
0 0 c d

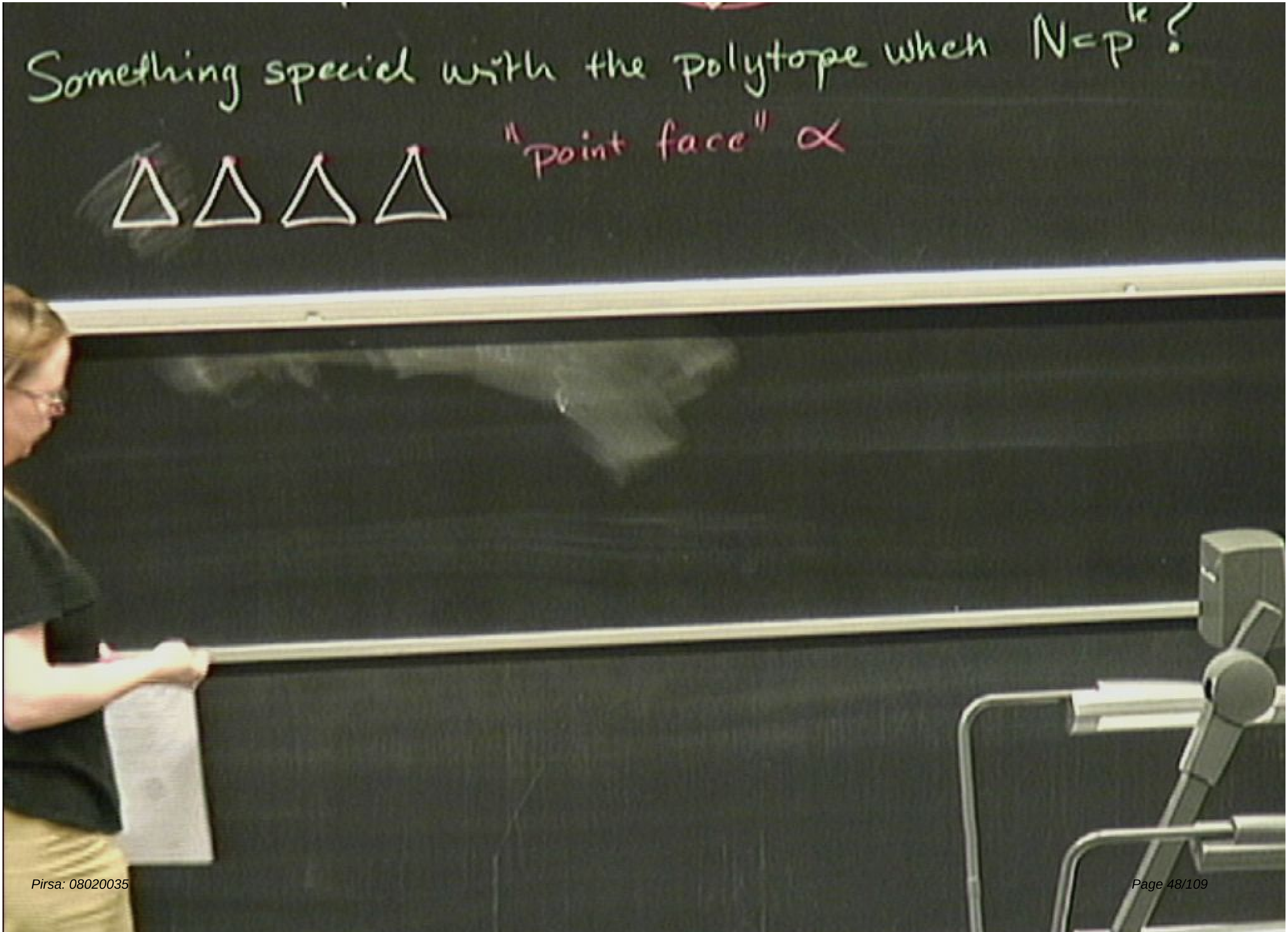

N = 3
tomography
Not more than N+1
N²-2
pure states 2N-2 real
The complementarity polytope
(a b 0 0)
(0 0 c d)


N=3
tomography
Not more than N+1
N²-2
pure states
2N-2
real
The complementarity polytope
a b 0 0
0 0 c d


Something special with the polytope which

Something special with the polytope when $N=p^k$?

"point face" α


Something special with the polytope when $N=p^k$?
"point face" α

Something special with the polytope when N=

"point face" α

$$A_\alpha = \rho_* - \sum_{p \in \alpha} (p - \rho_*)$$

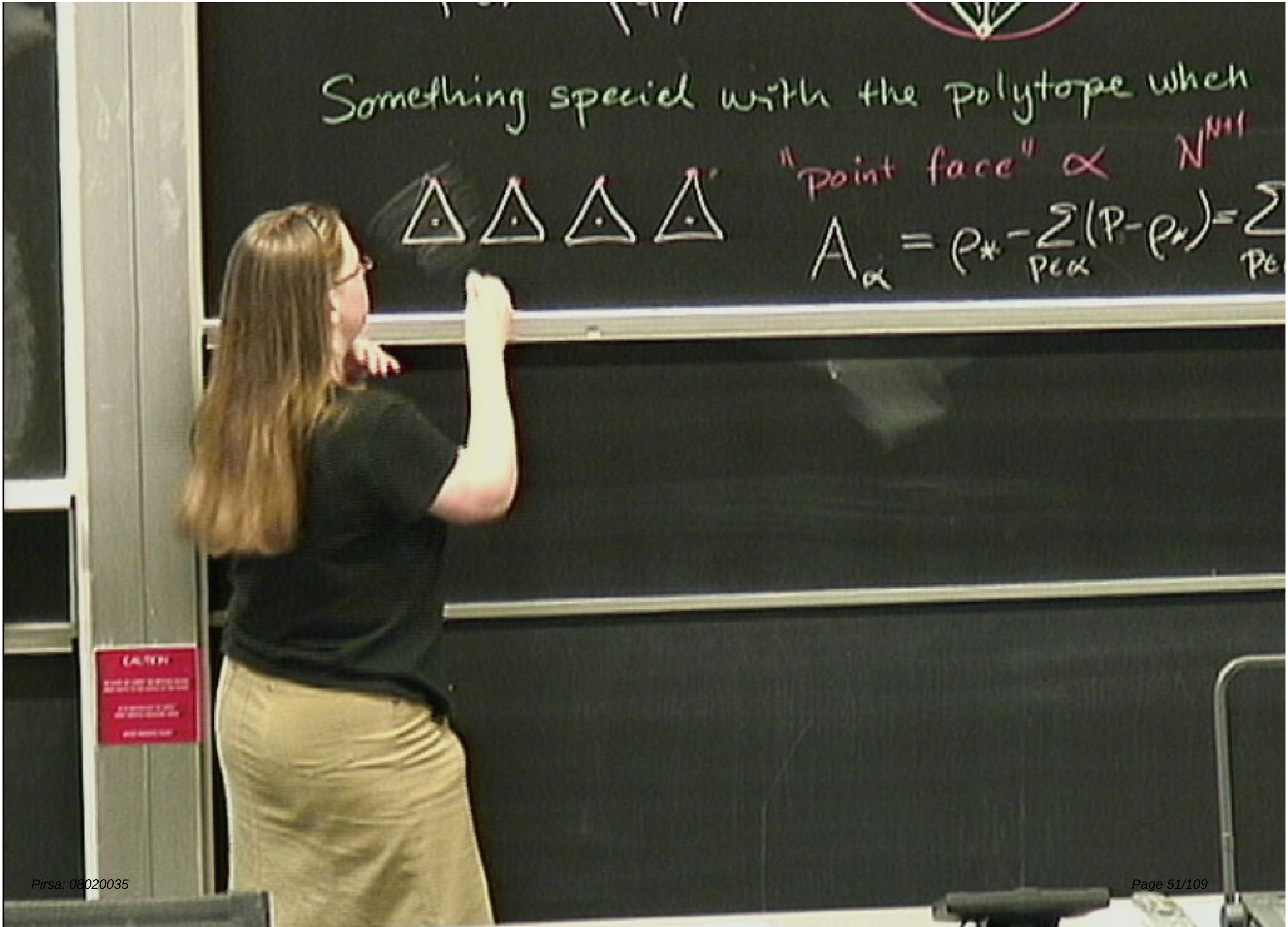
$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$$

Something special with the polytope when $N = p^k$?

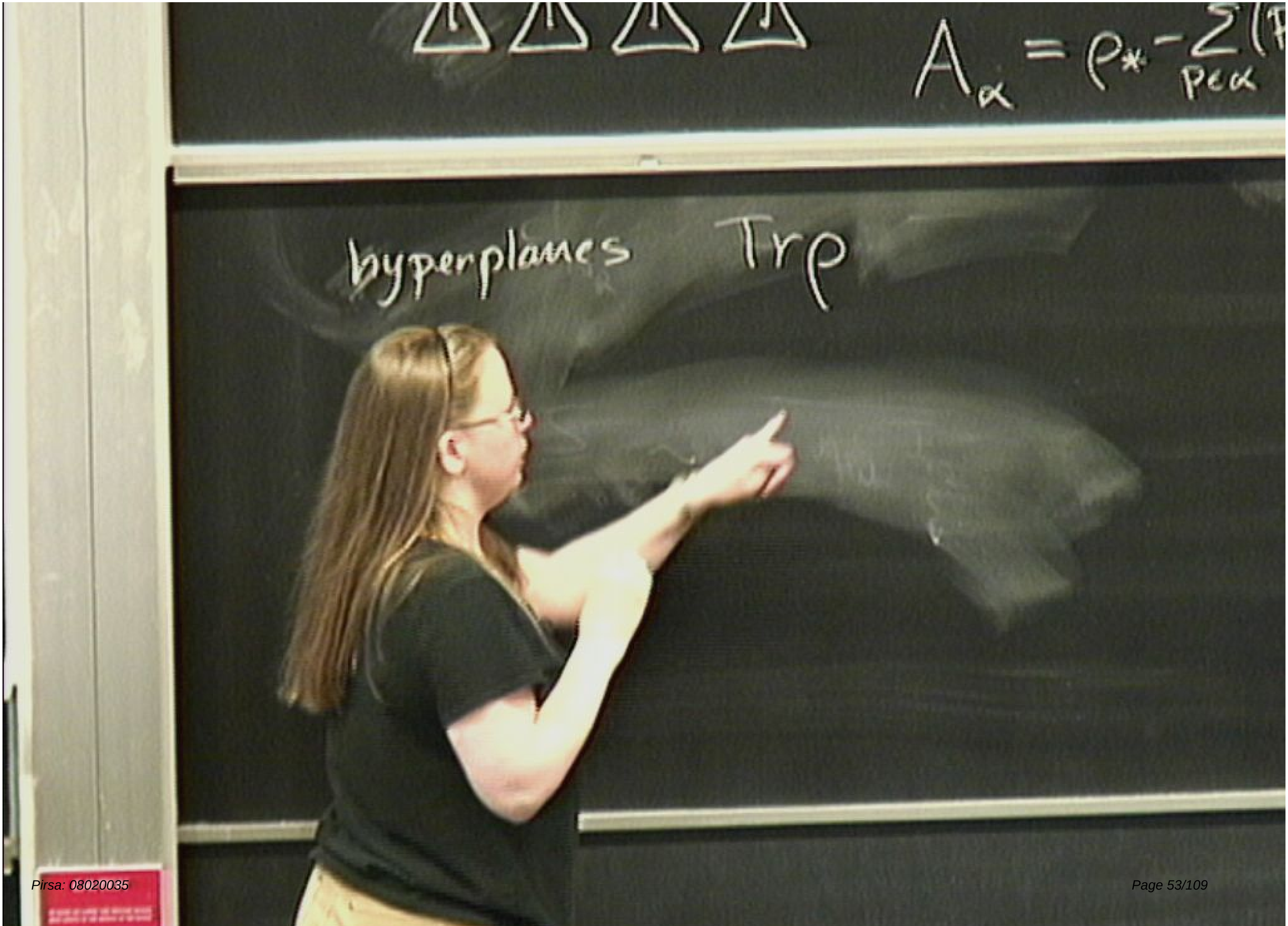
"point face" α $\quad N^{N+1}$

$\triangle \; \triangle \; \triangle \; \triangle$

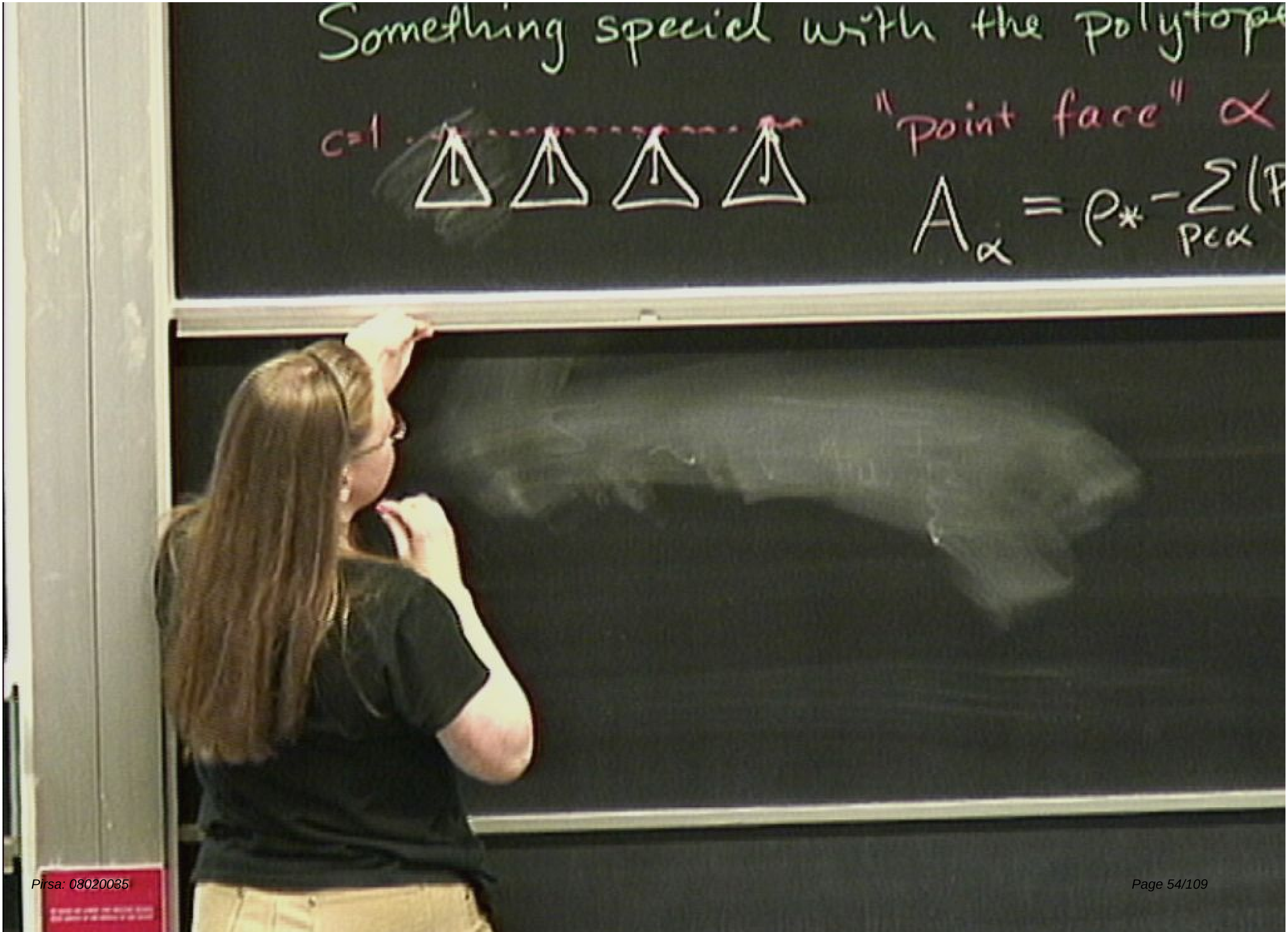
$$A_\alpha = \rho_* - \sum_{P \in \alpha} (P - \rho_*) = \sum_{P \in \alpha} P - \mathbb{1}$$

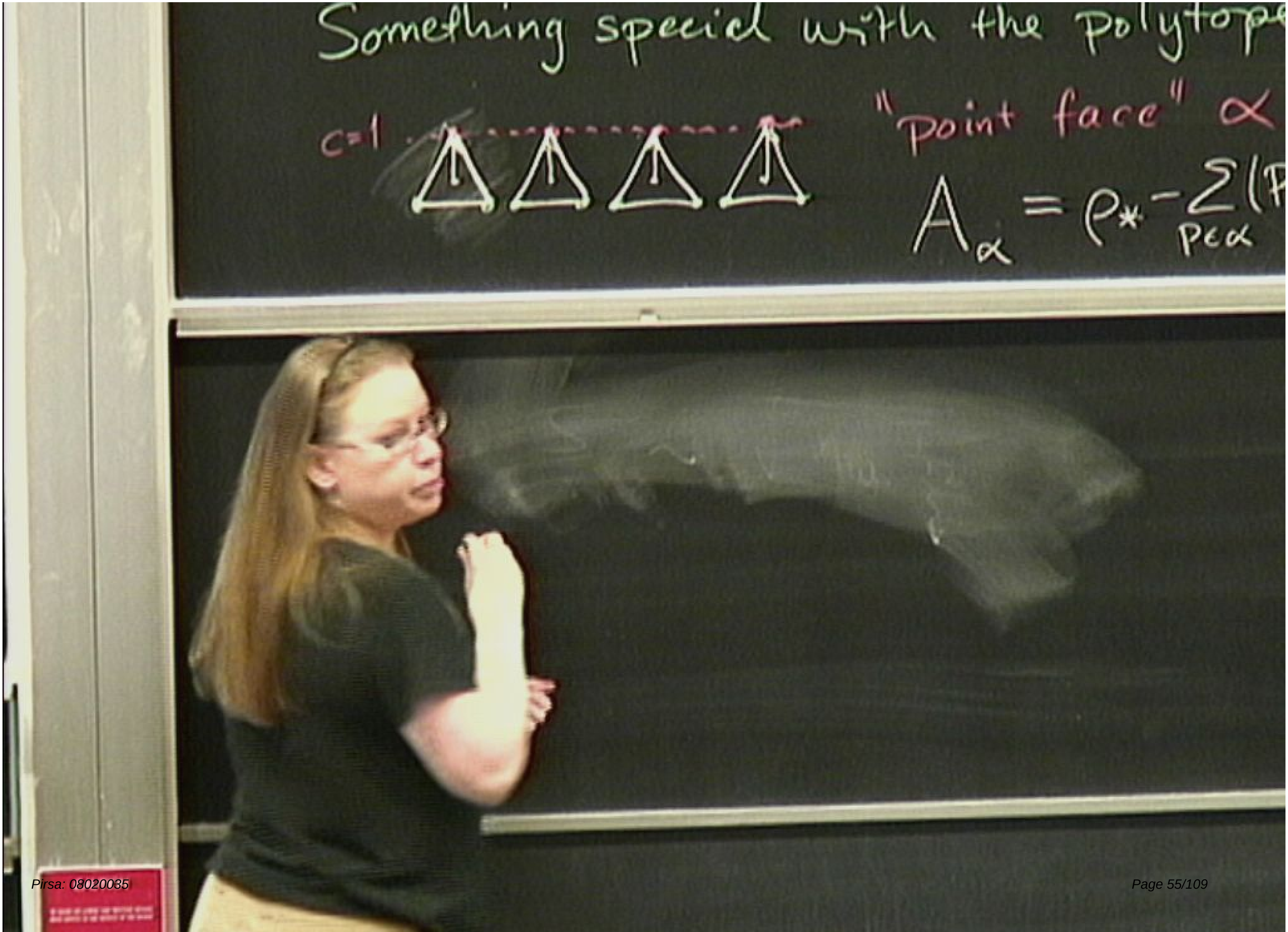

Something special with the polytope when
"point face" α N^{N+1}
A_α = ρ* − Σ_{p∈α} (p − ρ*) = Σ

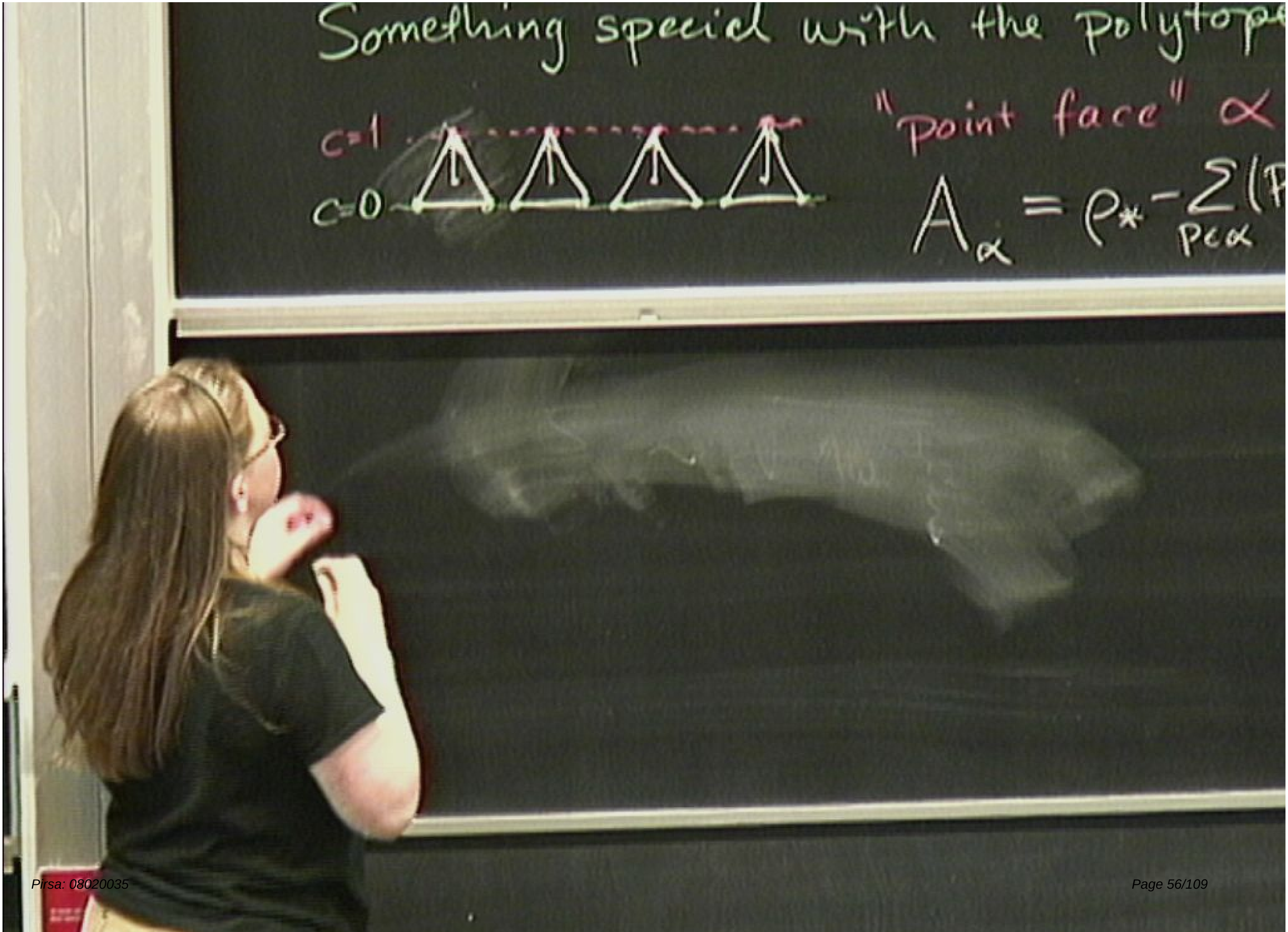

"point
A_α

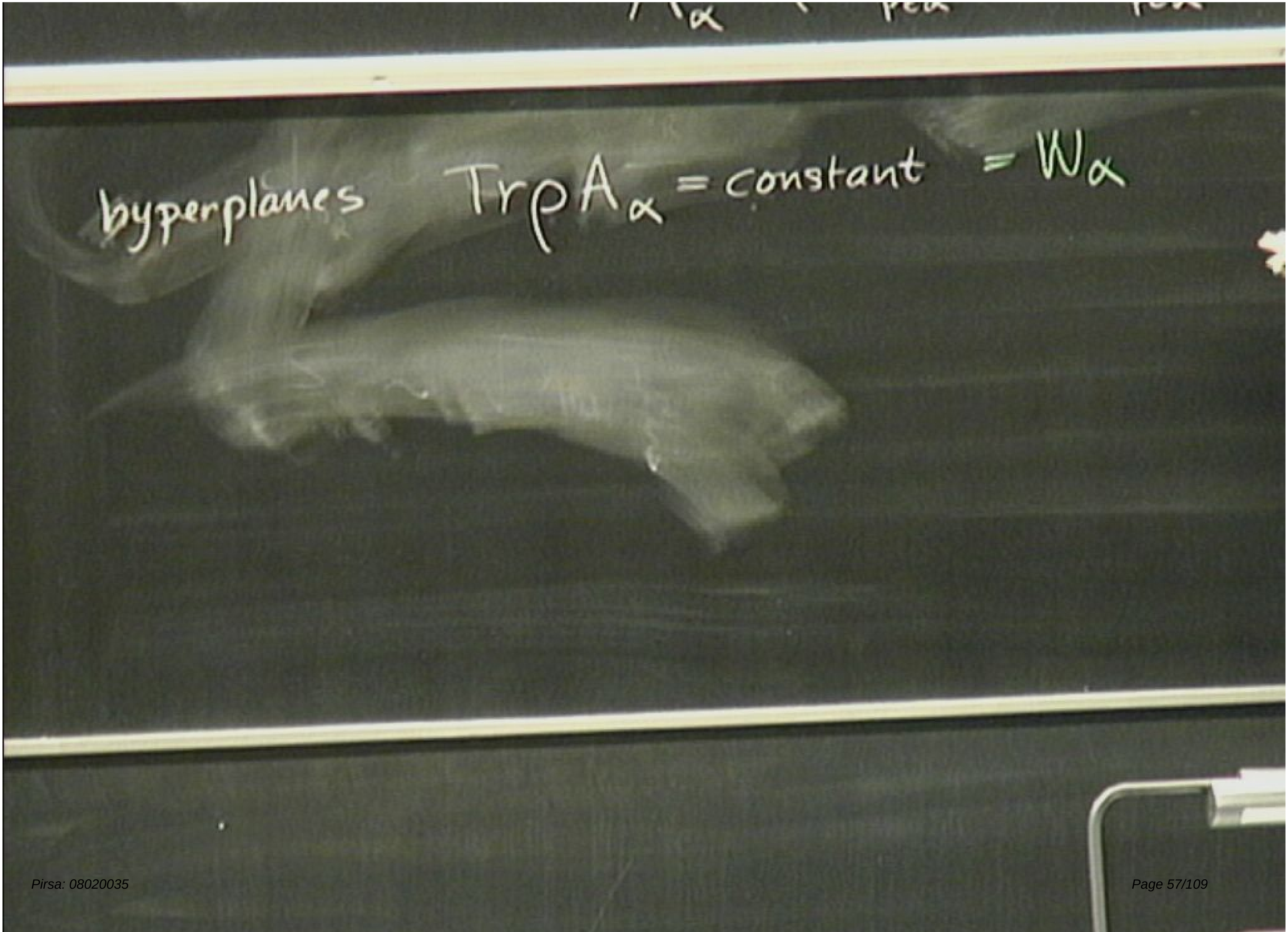


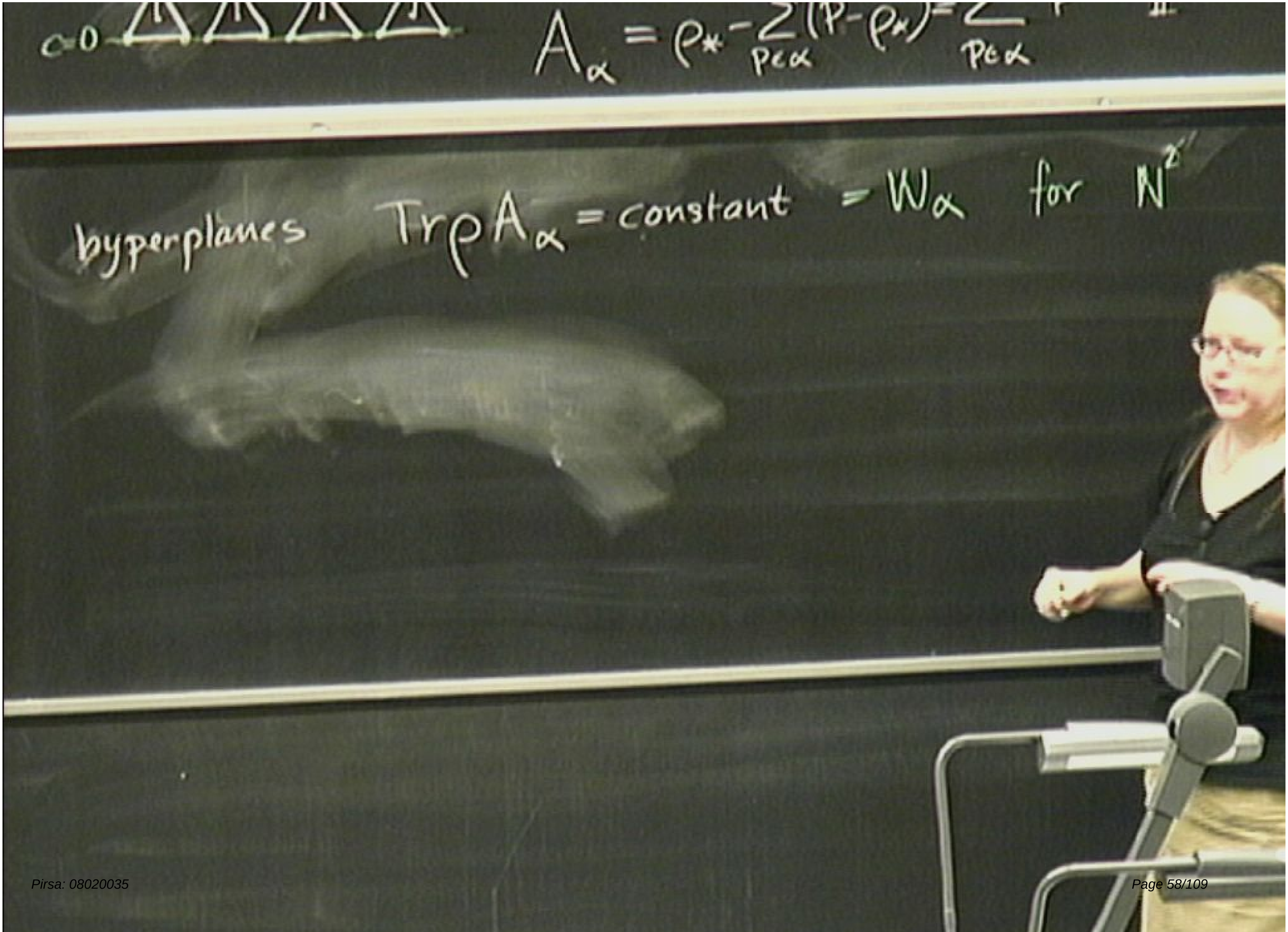
$A_\alpha = \rho_* - \sum_{p \in \alpha}$
hyperplanes
Trρ


Something special with the polytope
c=1
"point face" α
A_α = ρ_* − Σ_{p∈α} (P


Something special with the polytop
c=1
"point face" α
A_α = ρ* - Σ
p∈α


Something special with the polytope
c=1
c=0
"point face" α
$A_\alpha = \rho_* - \sum_{p \in \alpha}$


hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant} = W_\alpha$

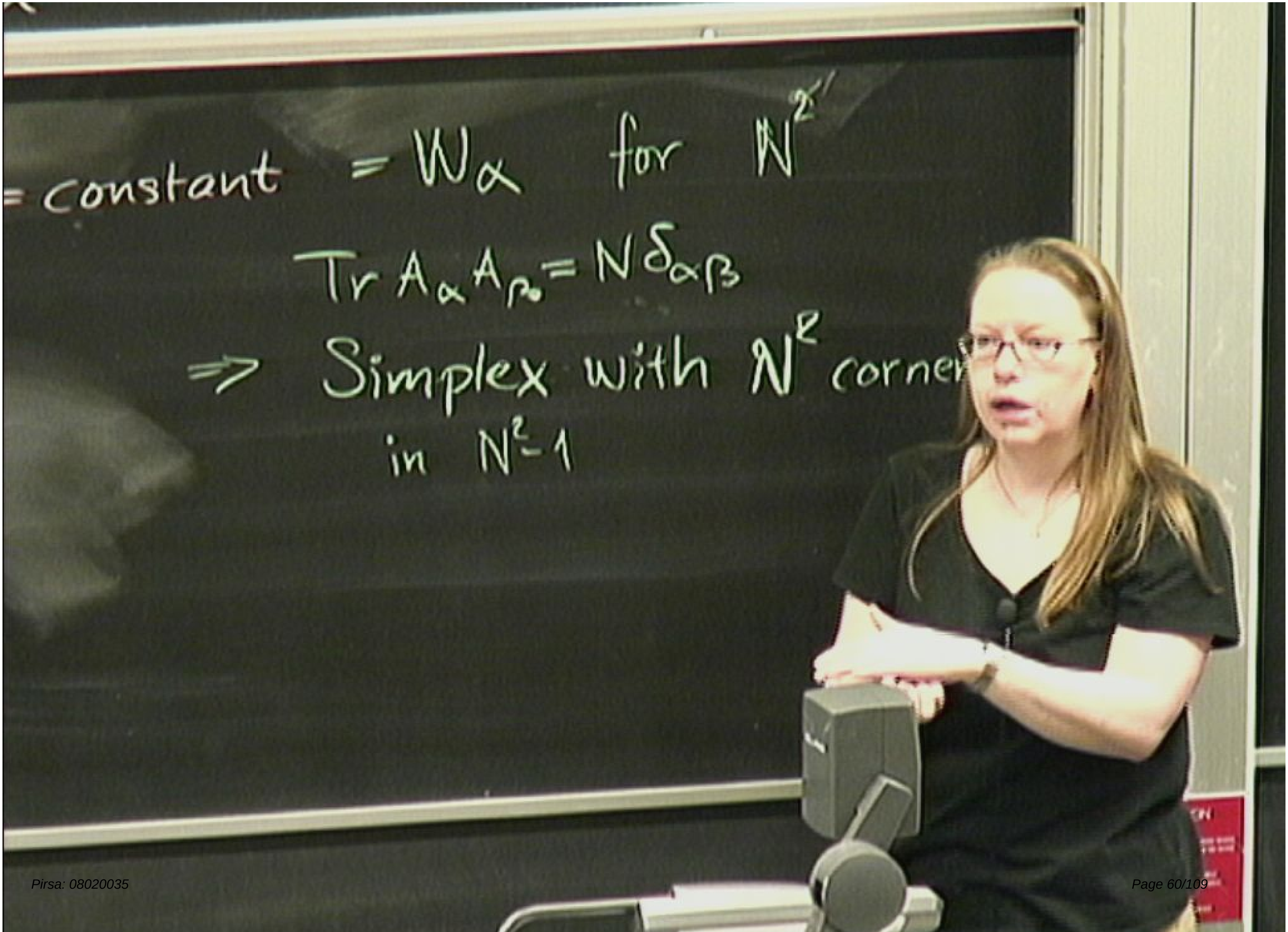

$A_\alpha = \rho_* - \sum_{p\in\alpha}(p - \rho_*) = \sum_{p\in\alpha}$
hyperplanes $\mathrm{Tr}\,\rho A_\alpha$ = constant = W_α for N^2

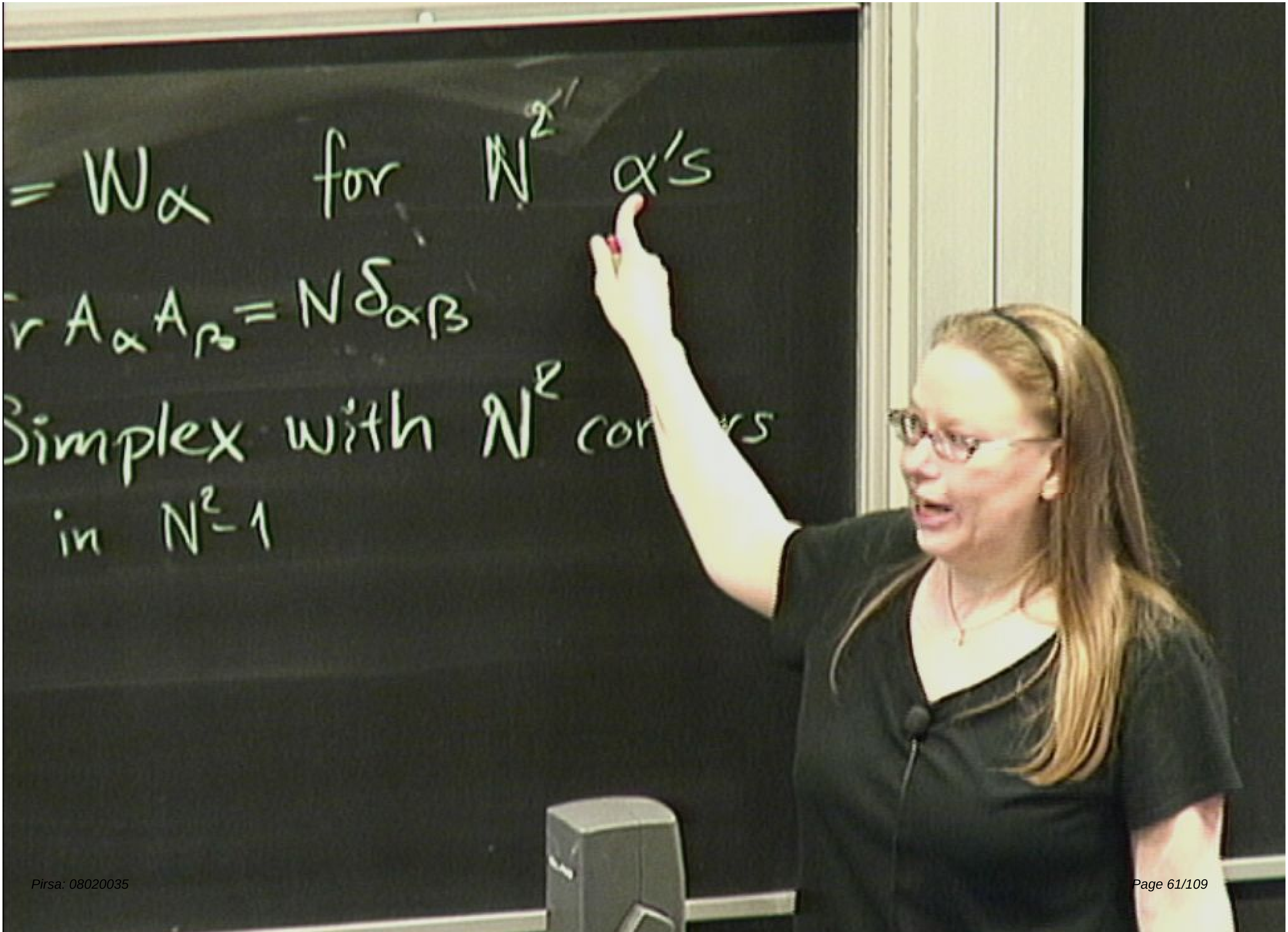
$c=0$

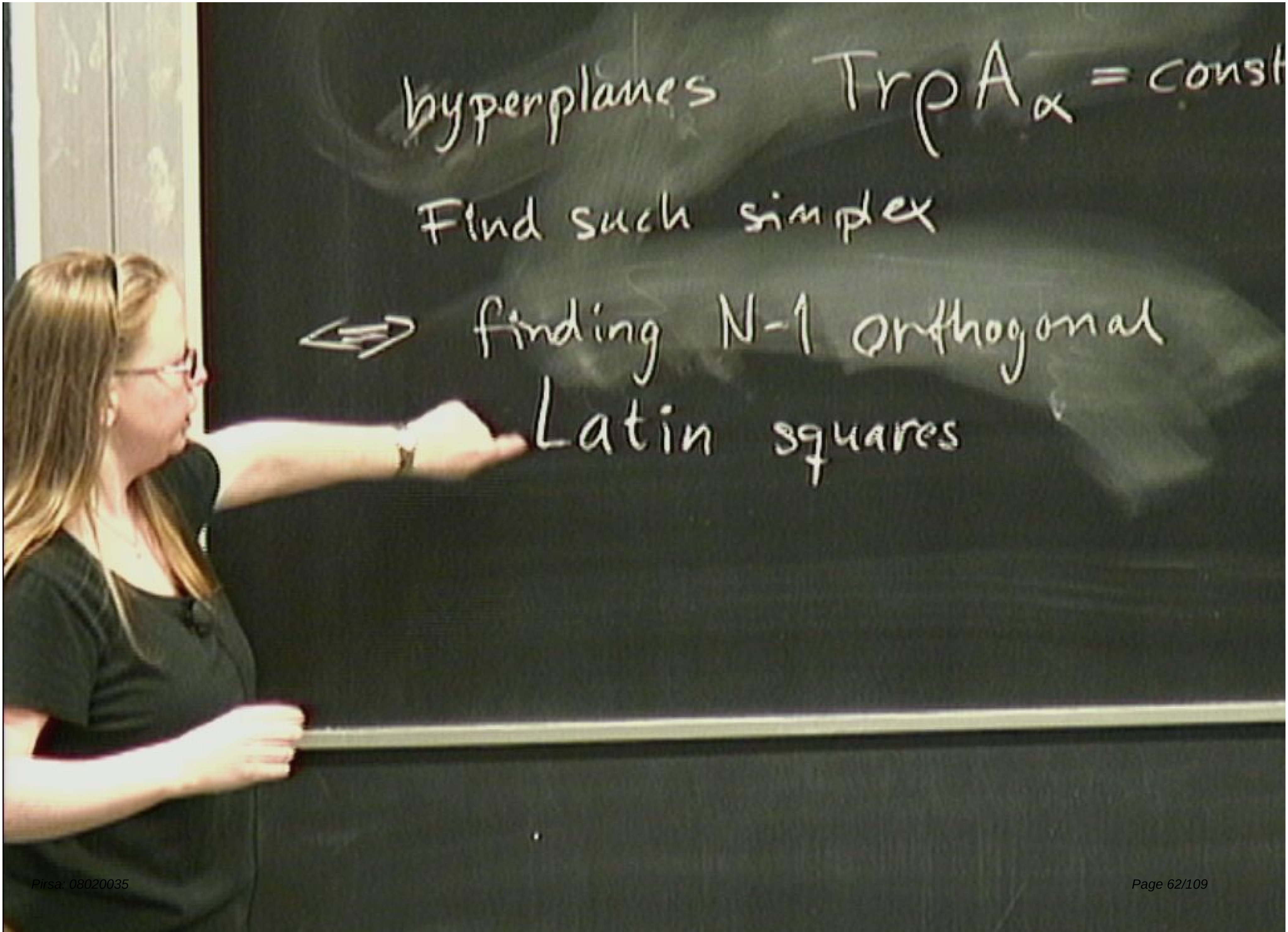
$$A_\alpha = \rho_* - \sum_{p\in\alpha} (P - \rho_*) = \sum_{p\in\alpha}$$

hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant} = W_\alpha$ for N^2

$$\mathrm{Tr}\, A_\alpha A_\beta = N \delta_{\alpha\beta}$$


= constant = W_α for N^2
$\mathrm{Tr}\, A_\alpha A_\beta = N\delta_{\alpha\beta}$
$\Rightarrow$ Simplex with N^2 corner
in N^2-1


$= W_\alpha$ for N^2 α's
$r\, A_\alpha A_\beta = N \delta_{\alpha\beta}$
Simplex with N^2 cor rs
in N^2-1


hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \mathrm{const}$
Find such simplex
$\Leftrightarrow$ finding N-1 orthogonal
Latin squares

hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant}$ $= W_{\alpha}$

$\mathrm{Tr}\, A_\alpha A$

Find such simplex

$\Rightarrow$ Simpl

$\Leftrightarrow$ finding N-1 orthogonal

in N

Latin squares

1	2	3
3	1	2
2	3	1

hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant}$

Find such simplex $\Rightarrow$

$\Longleftrightarrow$ finding N-1 orthogonal Latin squares

1	2	3
3	1	2
2	3	1

$\Longleftrightarrow$ finding finite affine plane of ord

hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant} = W_\alpha$

$\mathrm{Tr}\, A_\alpha A$

Find such simplex

$\Rightarrow$ Simple

$\Longleftrightarrow$ finding $N-1$ orthogonal Latin squares

in N

1	2	3
3	1	2
2	3	1

$\Longleftrightarrow$ finding finite affine plane of order N

hyperplanes $\mathrm{Tr}\,\rho A_\alpha = \text{constant} = W_\alpha$

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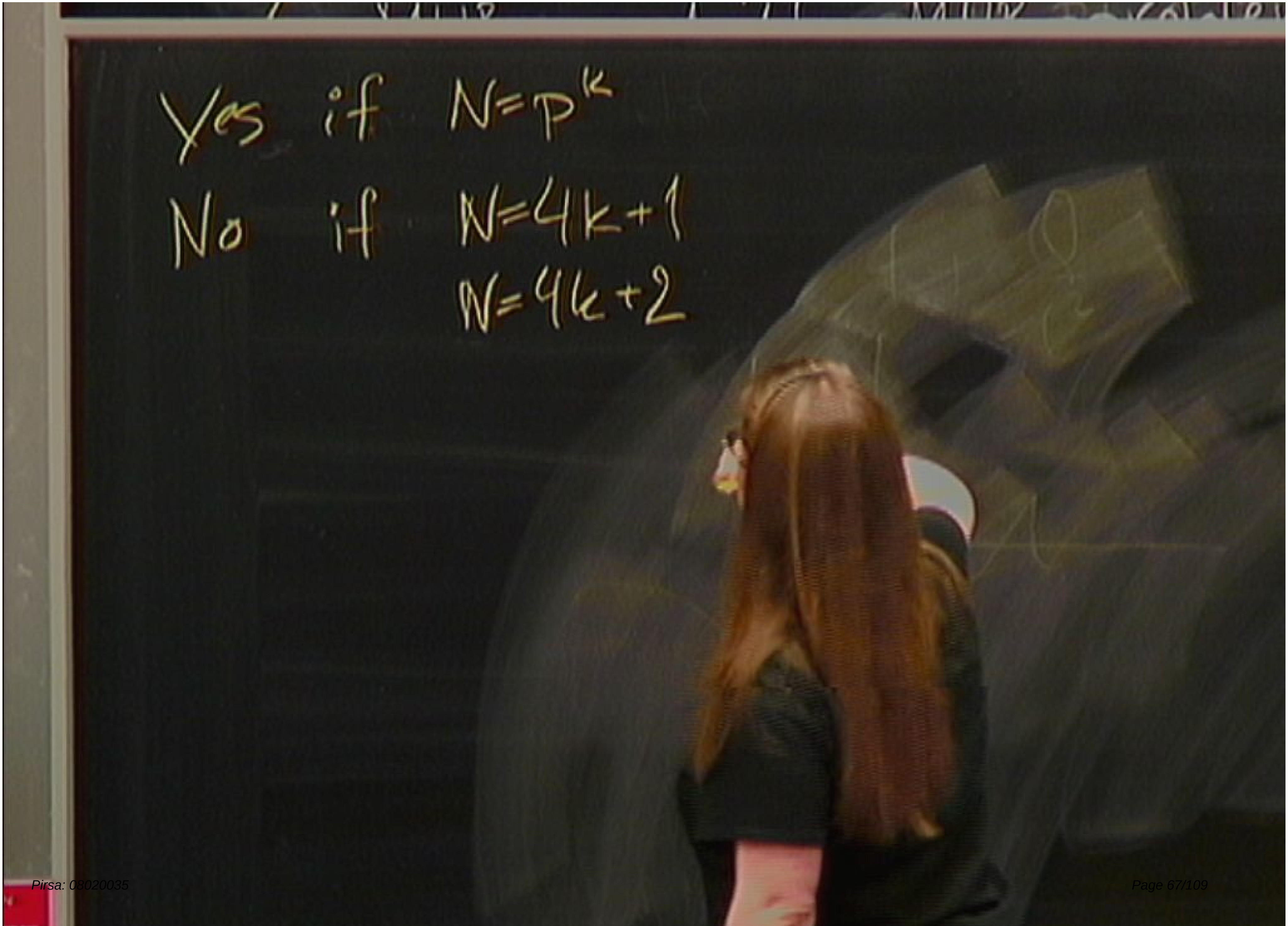
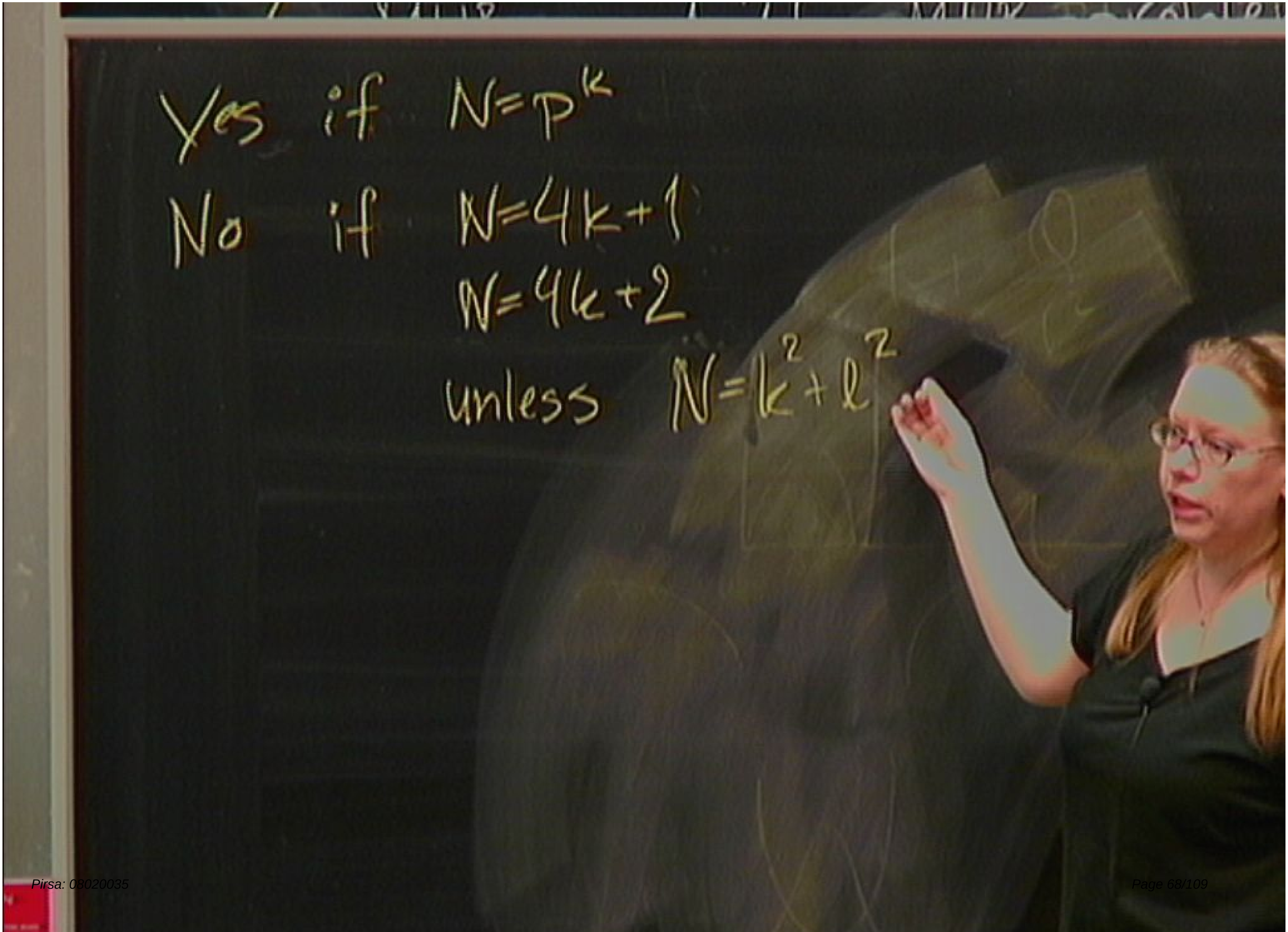
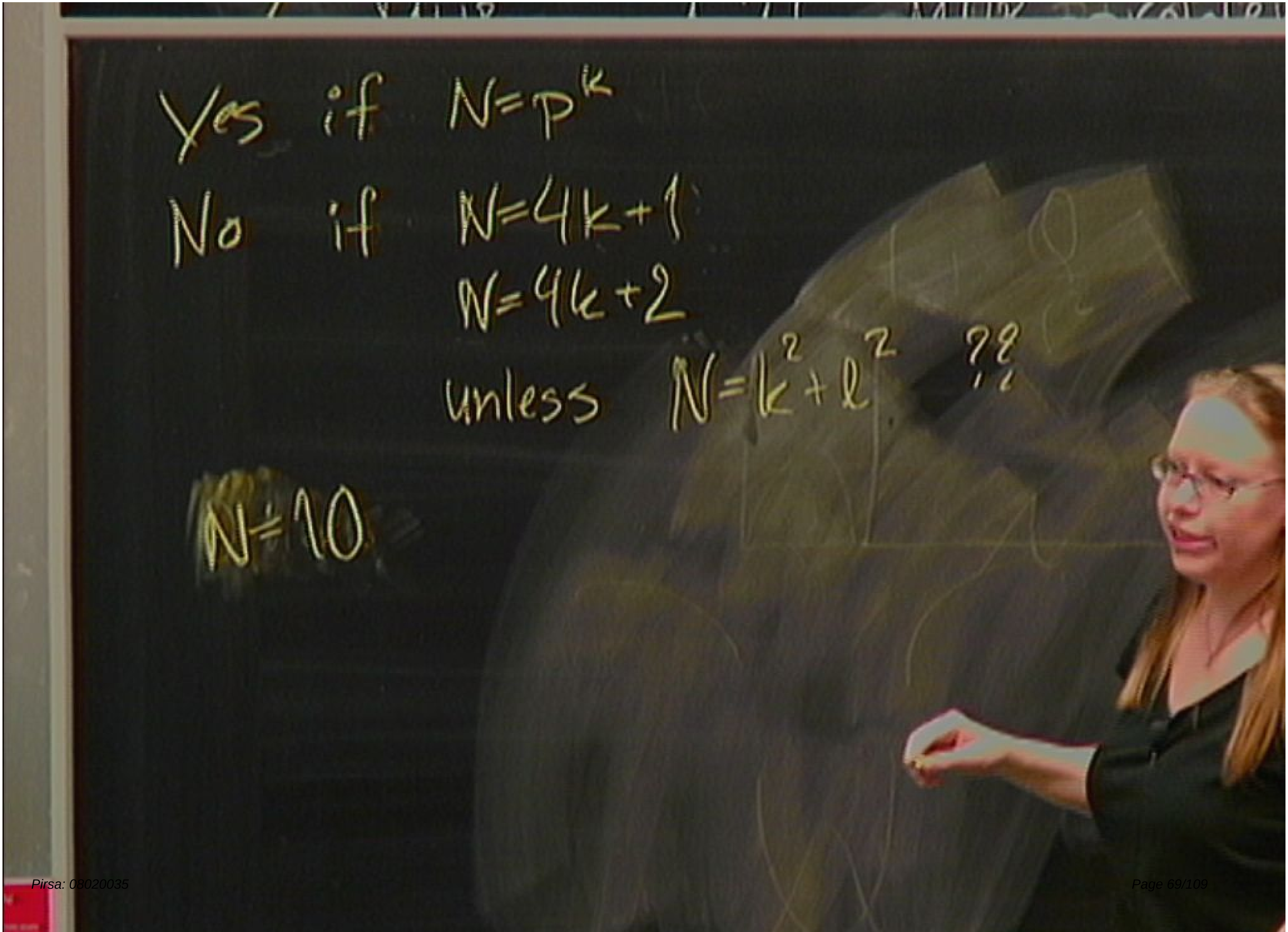
Yes if $N=p^k$

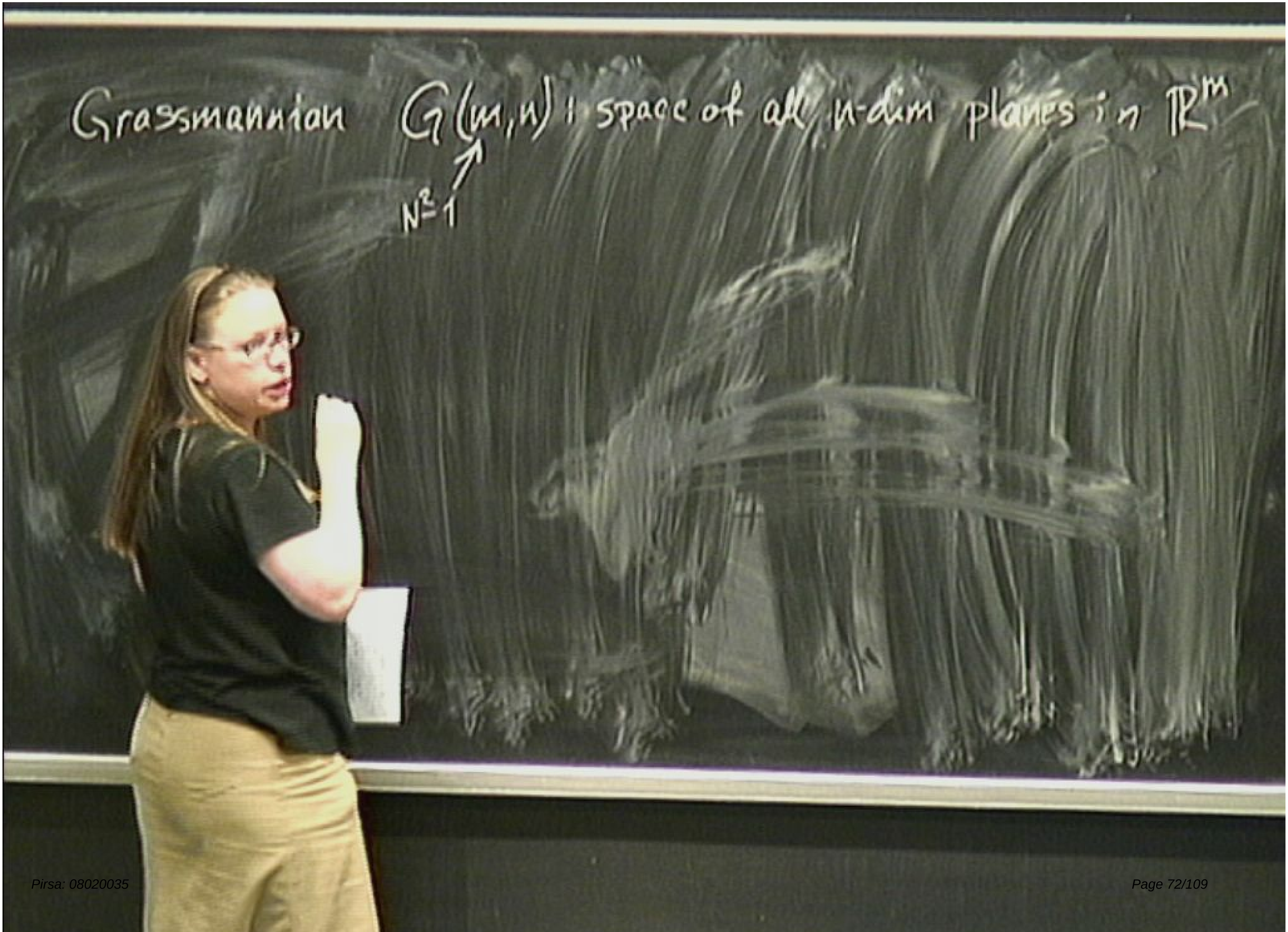

Yes if N=p^k
No if N=4k+1
N=4k+2


Yes if N=p^k
No if N=4k+1
N=4k+2
unless N=k^2+l^2


Yes if N=p^k
No if N=4k+1
N=4k+2
unless N=k^2+l^2 ??
N≠10


Grassmannian G(m,n)

Grassmannian $G(m,n)$: space of all n-dim planes in $\mathbb{R}^m$


Grassmannian G(m,n): space of all n-dim planes in ℝ^m
N²-1

Grassmannian $G(\underset{N^2-1}{m}, \underset{N-1}{n})$: space of all n-dim planes in $\mathbb{R}^m$

Grassmannian $G(m,n)$: space of all n-dim planes in $\mathbb{R}^m$

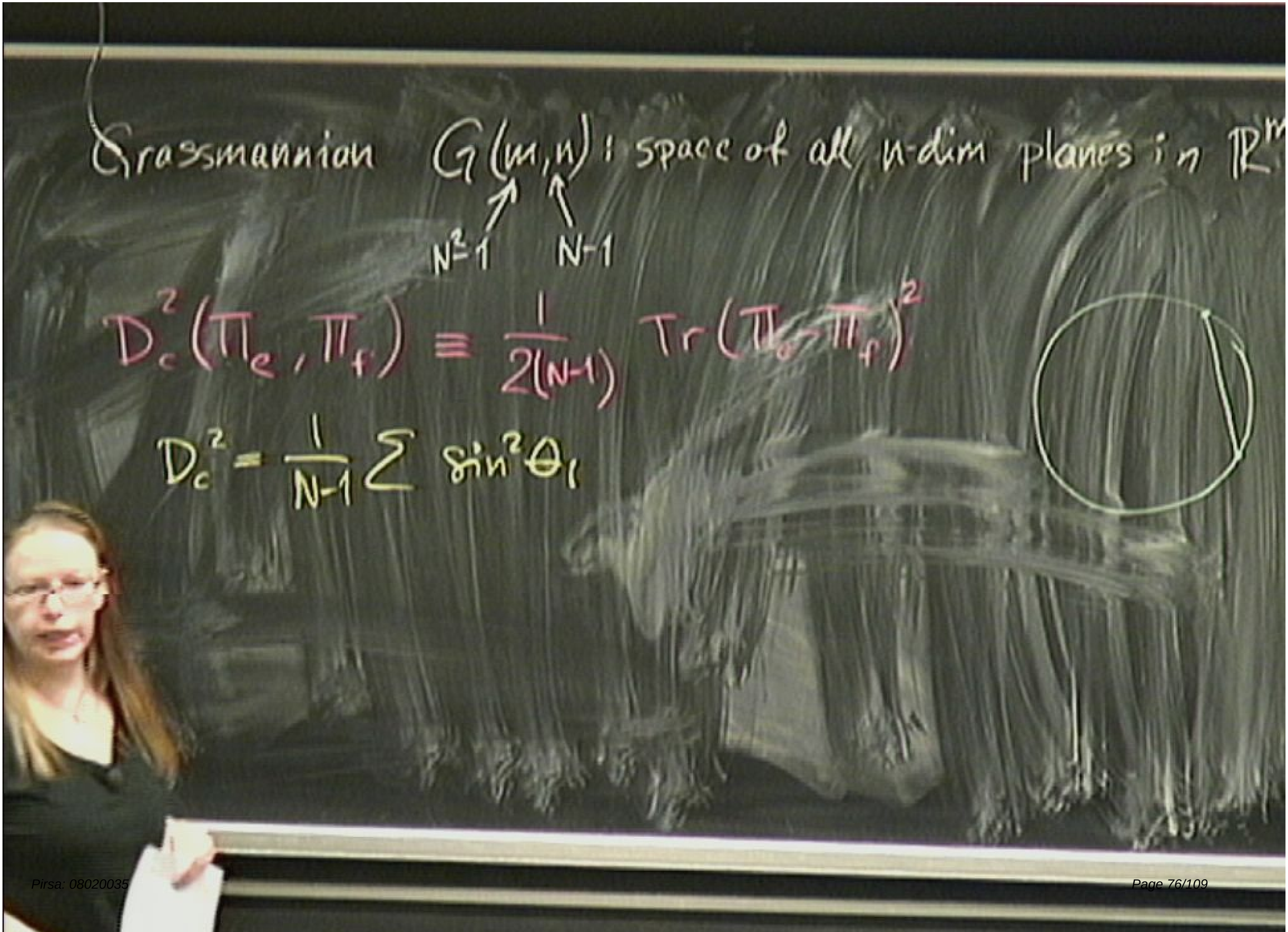
$$G(\underset{N^2-1}{m},\underset{N-1}{n})$$

$$D_c^2(\Pi_e, \Pi_f)$$

Grassmannian $G(m,n)$: space of all n-dim planes in $\mathbb{R}^m$

$m = N^2-1$, $n = N-1$

$$D_c^2(\Pi_e, \Pi_f) \equiv \frac{1}{2(N-1)} \operatorname{Tr}(\Pi_e - \Pi_f)^2$$


Grassmannian $G(m,n)$: space of all n-dim planes in $\mathbb{R}^m$
N^2-1
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$$D_c^2\left(\{|e_i\rangle, |f_j\rangle\}\right) =$$

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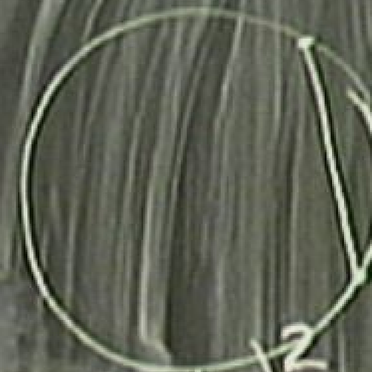
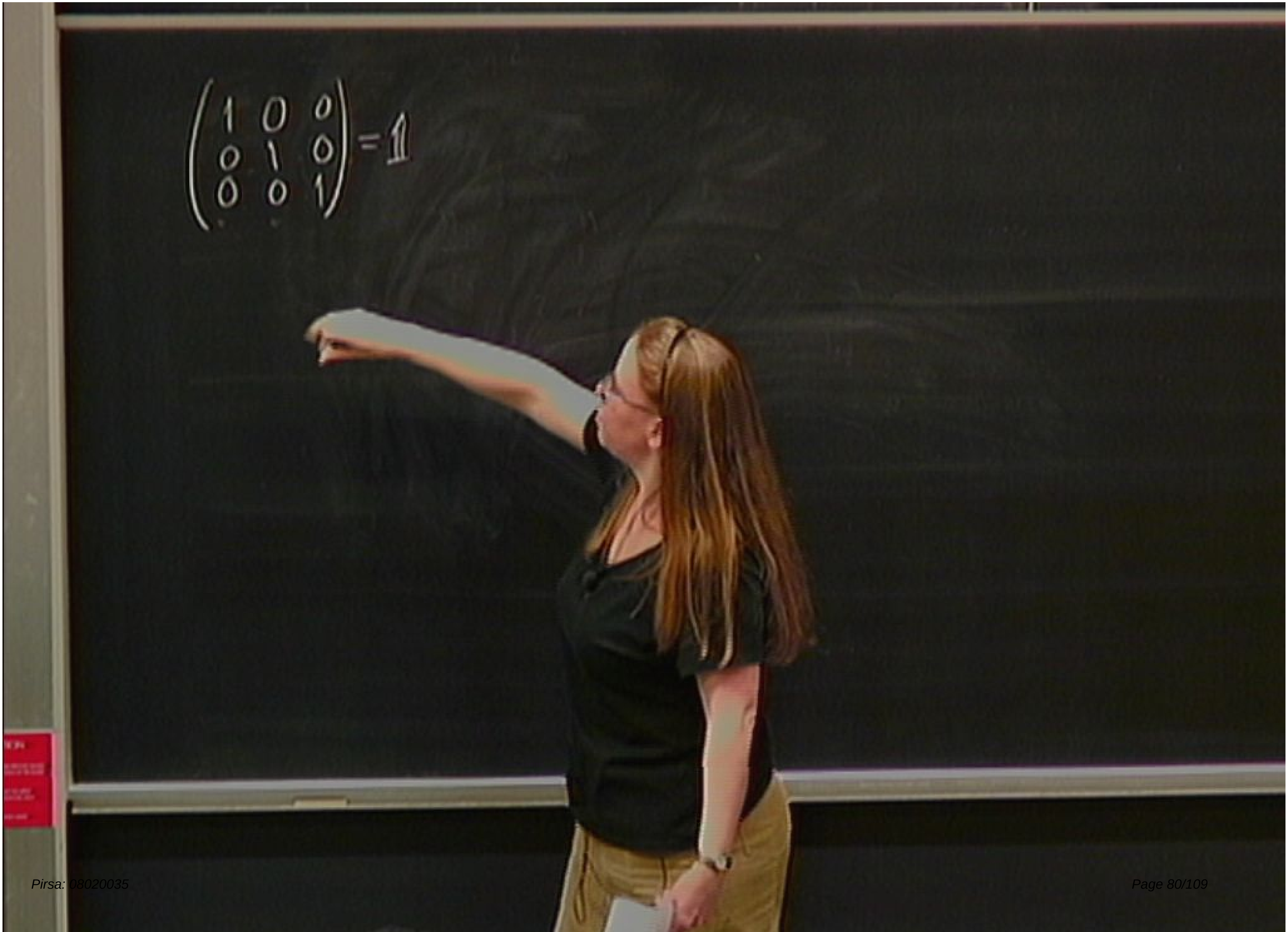


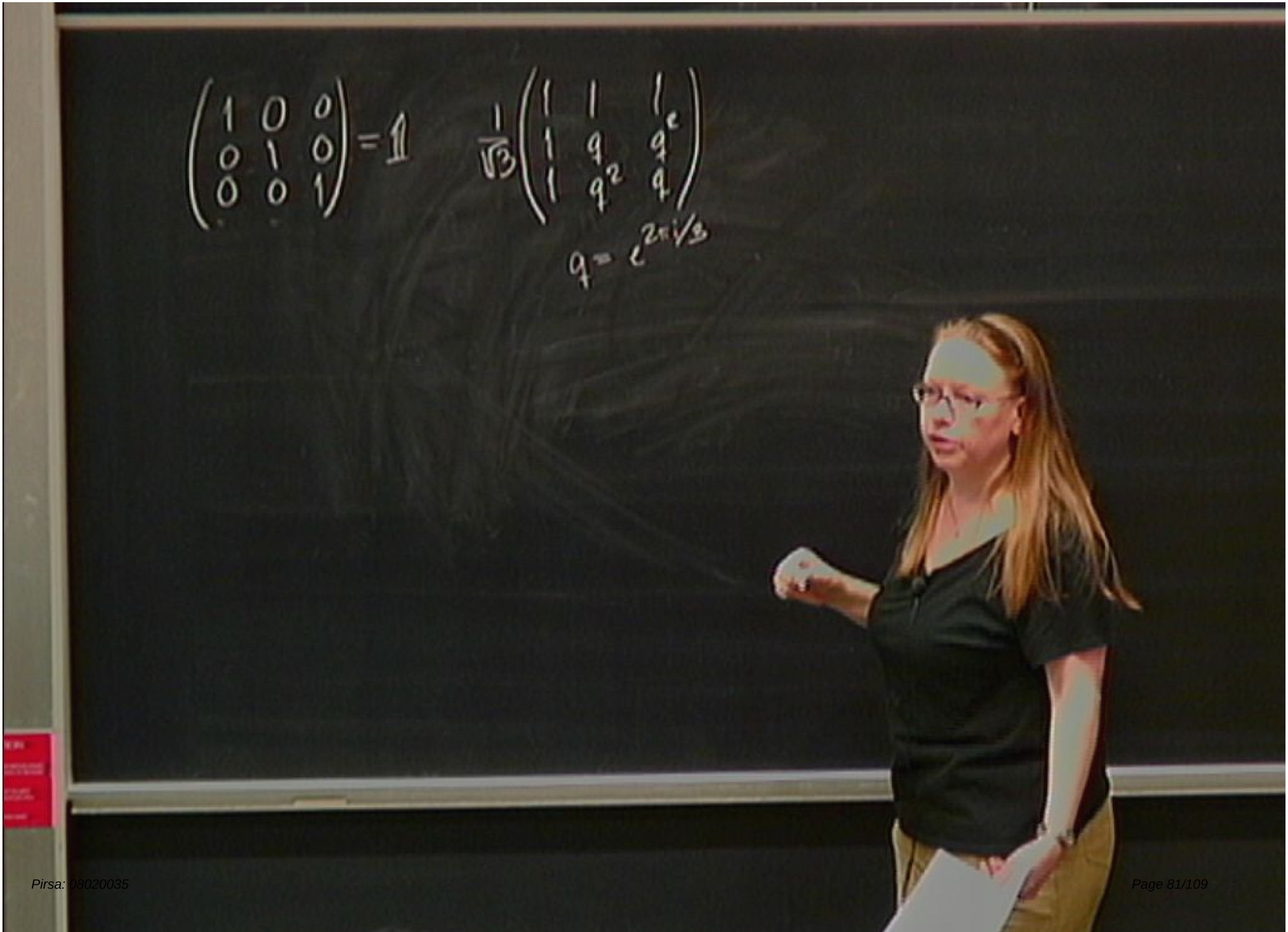
$$D_c^2 = \frac{1}{N-1} \sum_i \sin^2 \theta_i$$

$$D_c^2(\{|e_i\rangle\}, \{|f_j\rangle\}) = 1 - \frac{1}{N-1} \sum_i \sum_j \left(|\langle e_i | f_j \rangle|^2 - \frac{1}{N} \right)^2$$

$$0 \le D_c^2 \le 1$$

MUB


$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$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$q = e^{2\pi i/3}$

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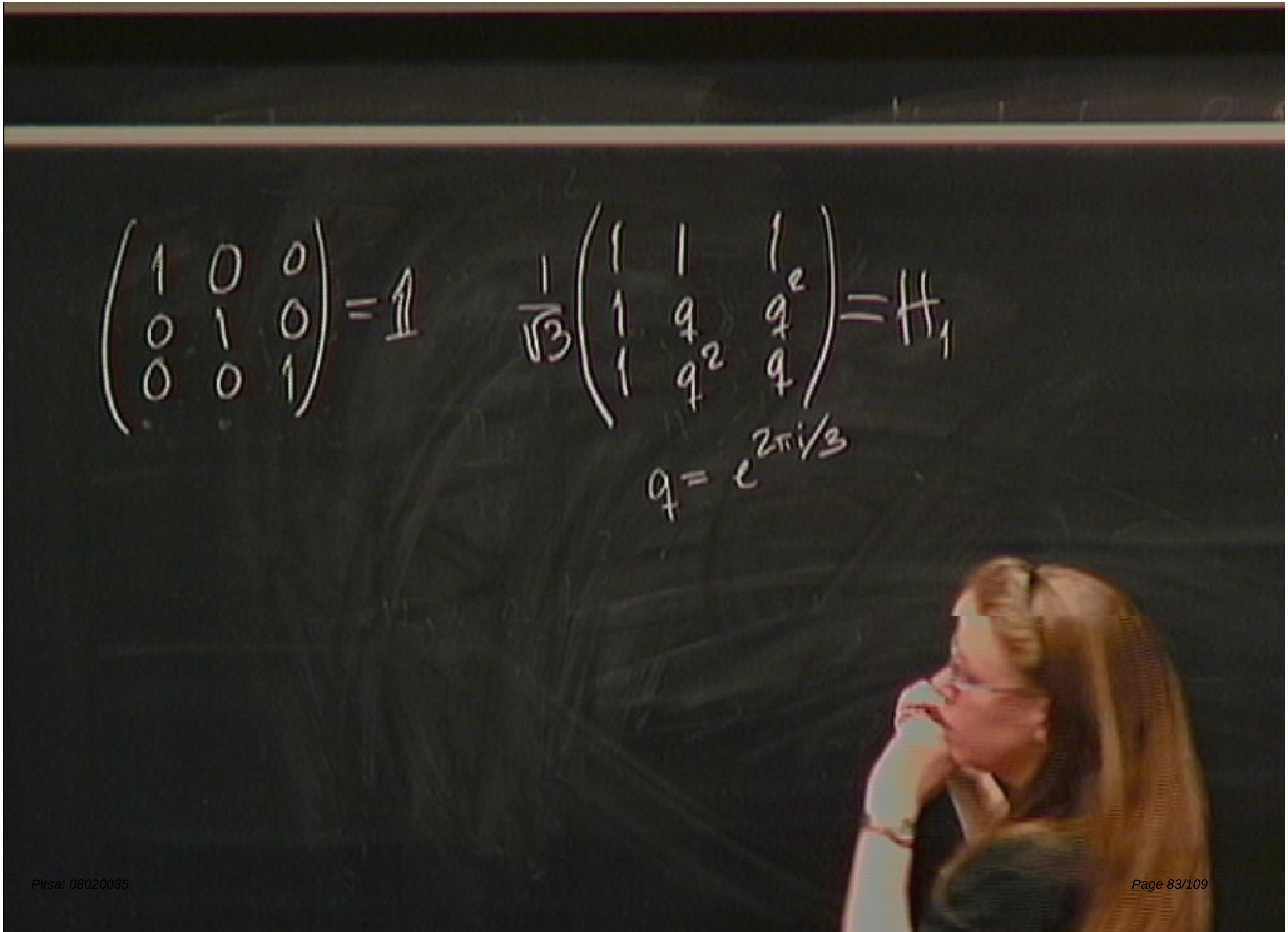
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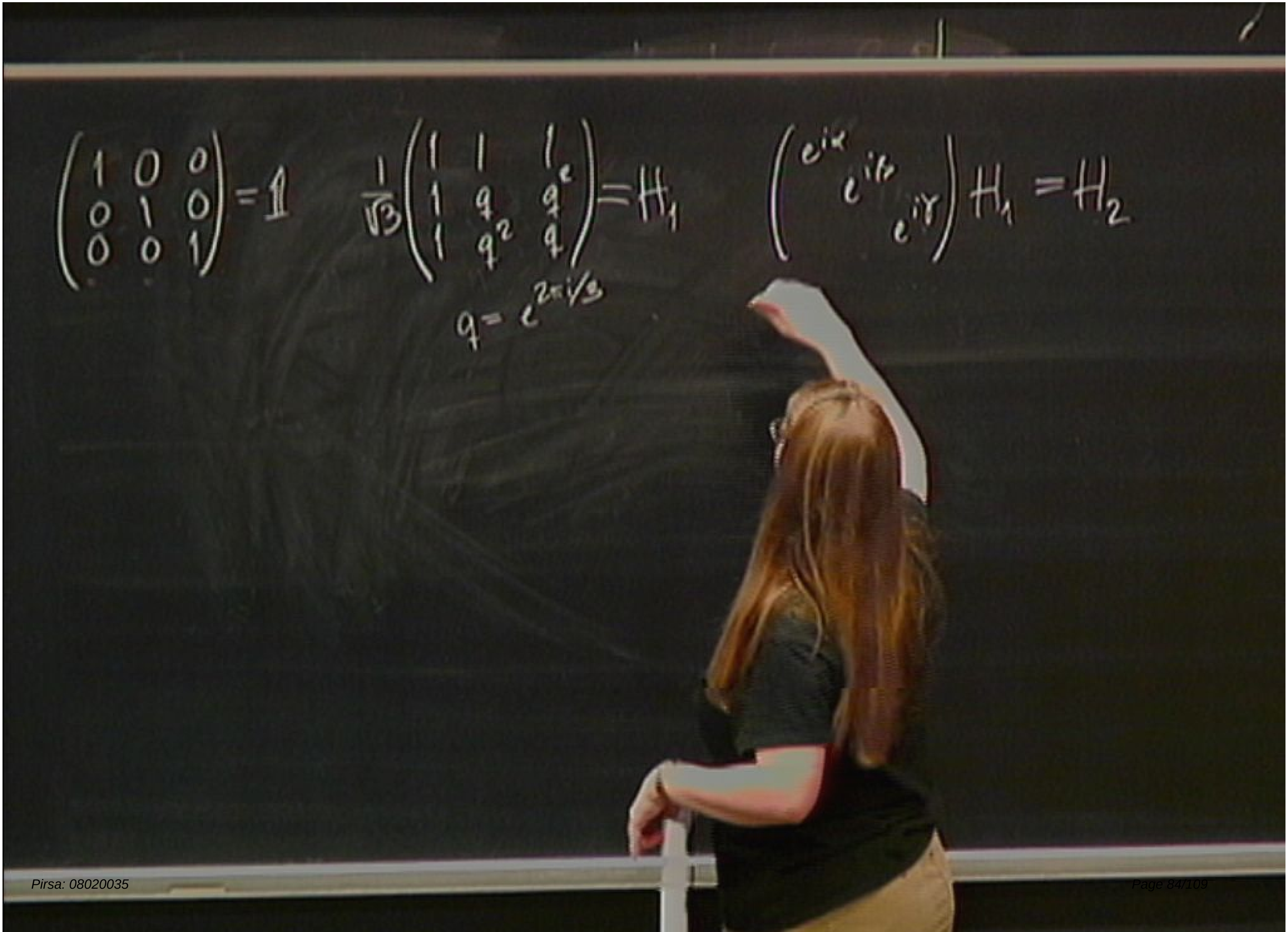
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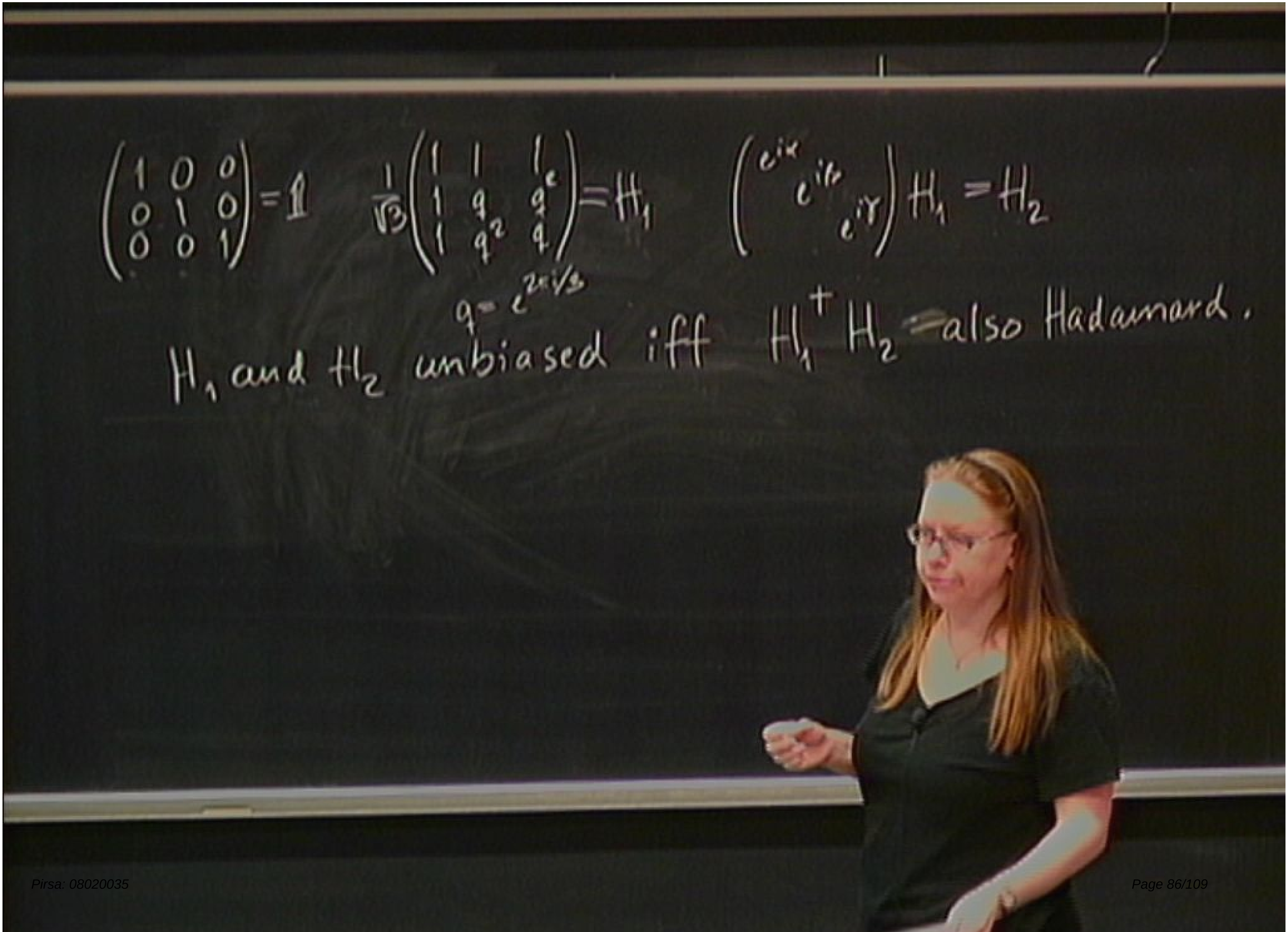
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HADAMARD MATRICES $N=6$ (TADEJ & ŻYCZKOWSKI)

FOURIER $F(\phi_1, \phi_2) = q^{kl}\, q^{\phi_r}$ $\quad (q = e^{2\pi i/6})$

CYCLIC C with $\pm 1, \pm i, d = \frac{1-\sqrt{3}}{2} \pm i\sqrt{\frac{\sqrt{3}}{2}}$

DITA $D(x)$ $\quad \pm 1, \pm i, q^{x}$

SPECTRAL S $\quad w = e^{2\pi i/3}$ $\quad$ (isolated)

HERMITIAN $B(\theta)$ "non-affine" (connects C & D)

SYMMETRIC $M(\gamma)$ $\quad$ (F & D)

(and $\cdot^{*}$ $\cdot^{T}$ $\cdot^{\dagger}$)

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CYCLIC	C	with $\pm 1, \pm i, \ d = \frac{1-\sqrt{3}}{2} \pm i\sqrt{\frac{\sqrt{3}}{2}}$	
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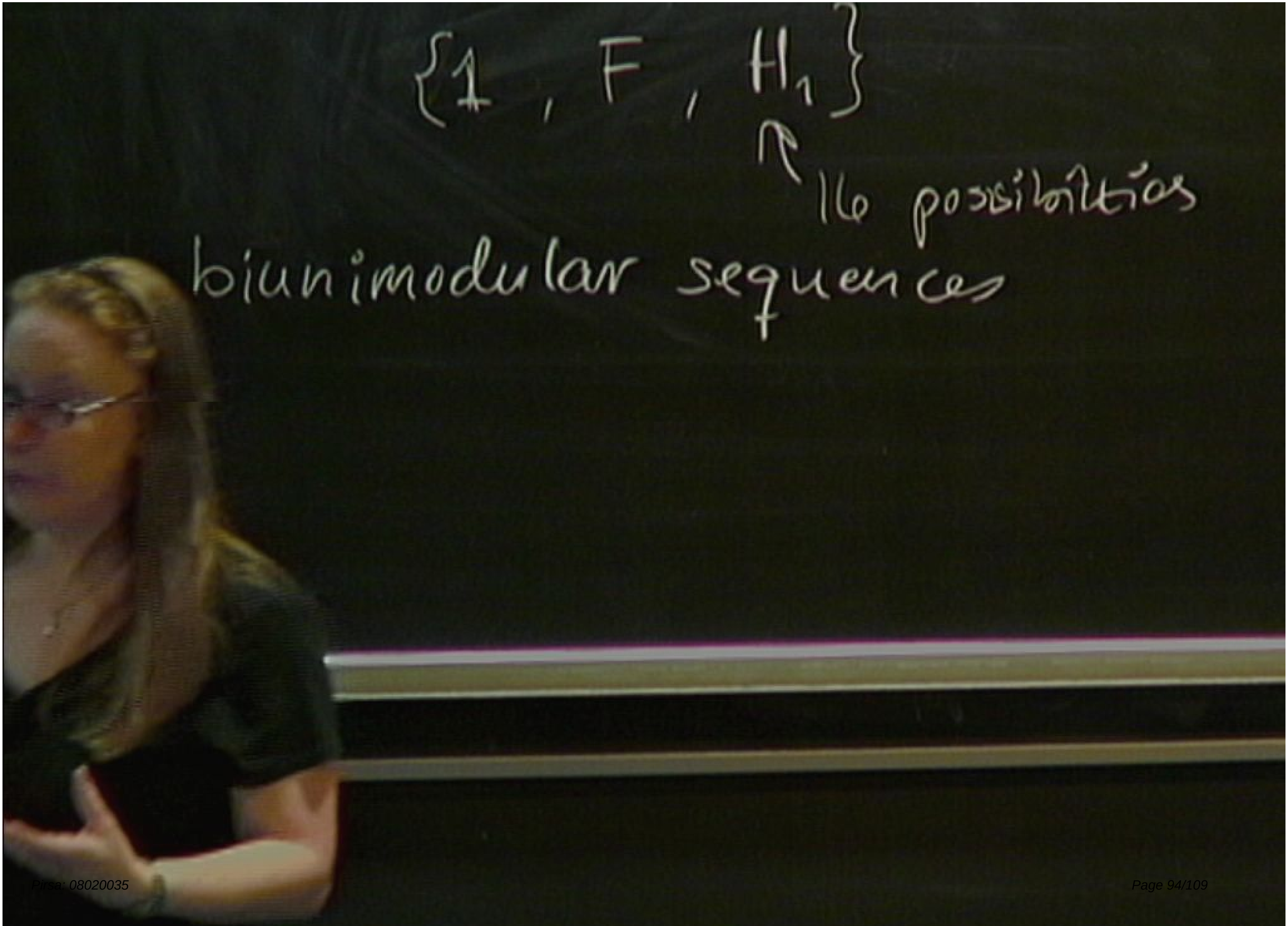
(and .


{1, F, H1}
16 possibilities


{1 , F , H1}
16 possibilities
biunimodular sequences

H_1 and H_2 unbiased iff $H_1 \; H_2 = a$

$$\{1, \cancel{F}, H_1\}$$

16 possibilities

biunimodular sequences

MUB-triplets

[With Fourier] $\{ \underbrace{\mathbb{1} \ , \ F}_{\text{48 unbiased vectors}} \ , \ \mathrm{diag}UF \}$

diagUF: or any choice of 16 bases: Fouriers F and cyclic C

No more than 3 bases! GRASSL (2004)

[Other possibilities] $\{\mathbb{1}\ , F(\tfrac{1}{6}, \tfrac{1}{12})\ , \tilde{D}(\tfrac{1}{8})\} \sim \{\mathbb{1}\ , F^T(\tfrac{1}{6}, \tfrac{1}{12})\ , \tilde{\tilde{F}}(c_1, 0)\}$

$\sim \{\mathbb{1}\ , D(-\tfrac{1}{8})\ , \tilde{\tilde{F}}(c_1, 0)\}$ $\qquad \tan c_1 = 1/\sqrt{2}$

$\{\mathbb{1}\ , D(0)\ , \tilde{\tilde{F}}(\tfrac{9}{24} + c_2, 0)\}$ $\qquad \tan c_2 = -2$

$\sim \{\mathbb{1}\ , \tilde{F}^T(\tfrac{9}{24} + c_2, 0)\ , \tilde{\tilde{F}}^T(\tfrac{9}{24} + c_2, 0)\}$

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$\tan c_2 = -2$

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SUMMARY

* MUB-polytope — affine planes
* Grassman distance/unbiasedness
* $\{\mathbb{1}, H_1, \ldots, H_n\}$ MUBs $\Leftrightarrow$

  H_k and $H_k^+ H_l$ Hadamards
* Fourier-MUBs/biunimodular sequences
* Inequivalent new triplets, $N=6$

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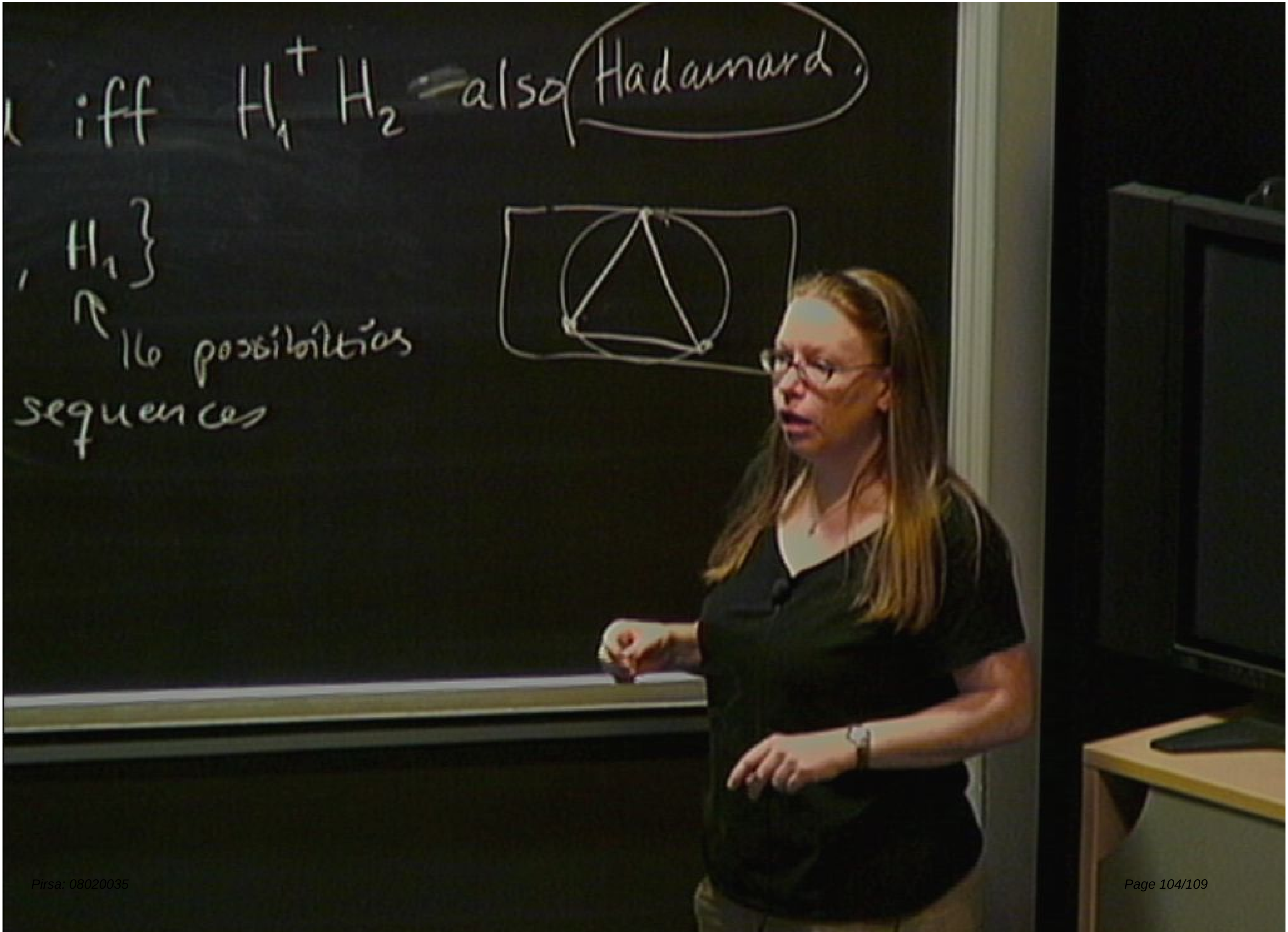
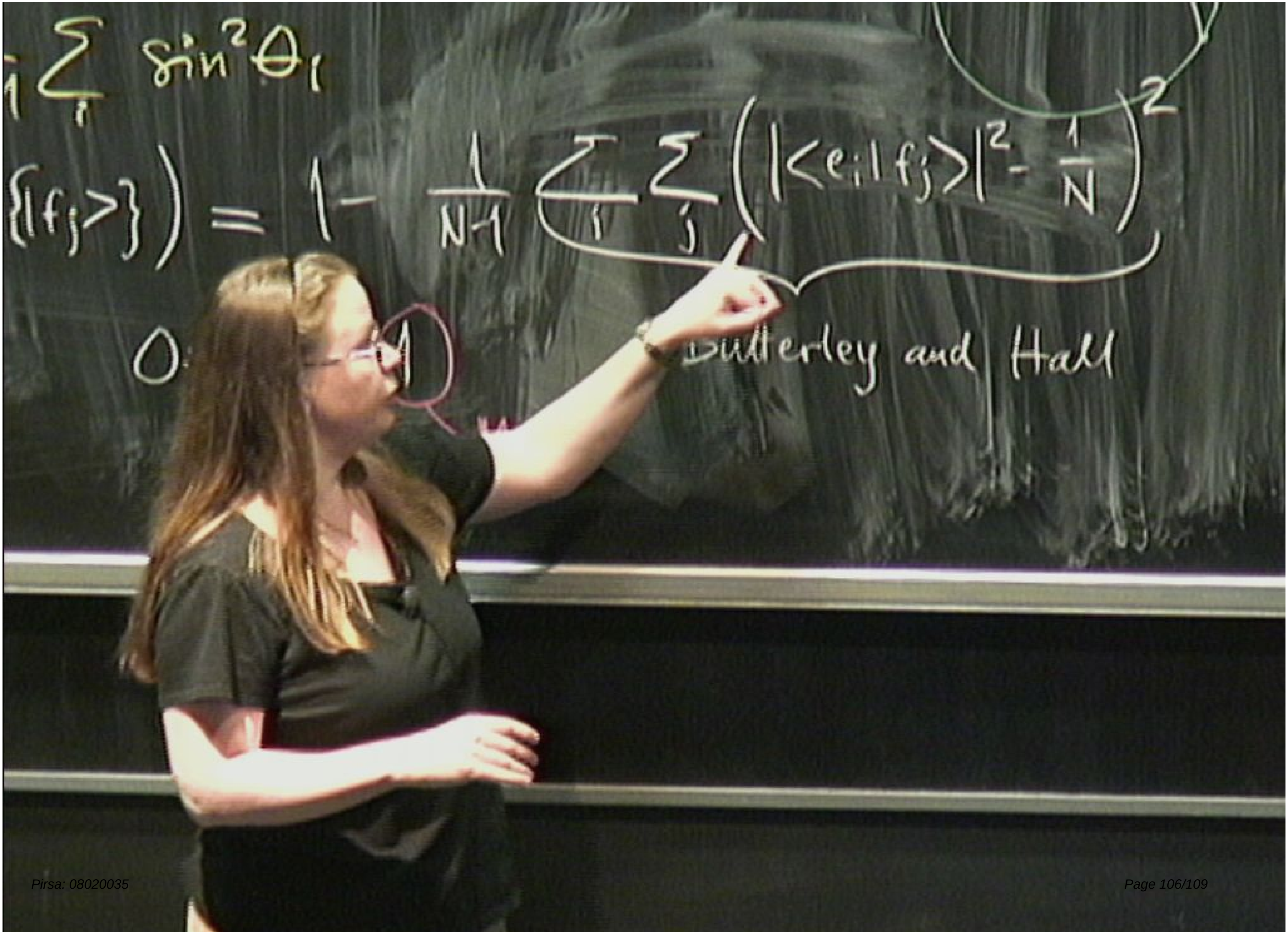
see also http://chaos.if.uj.edu.pl/hadamard

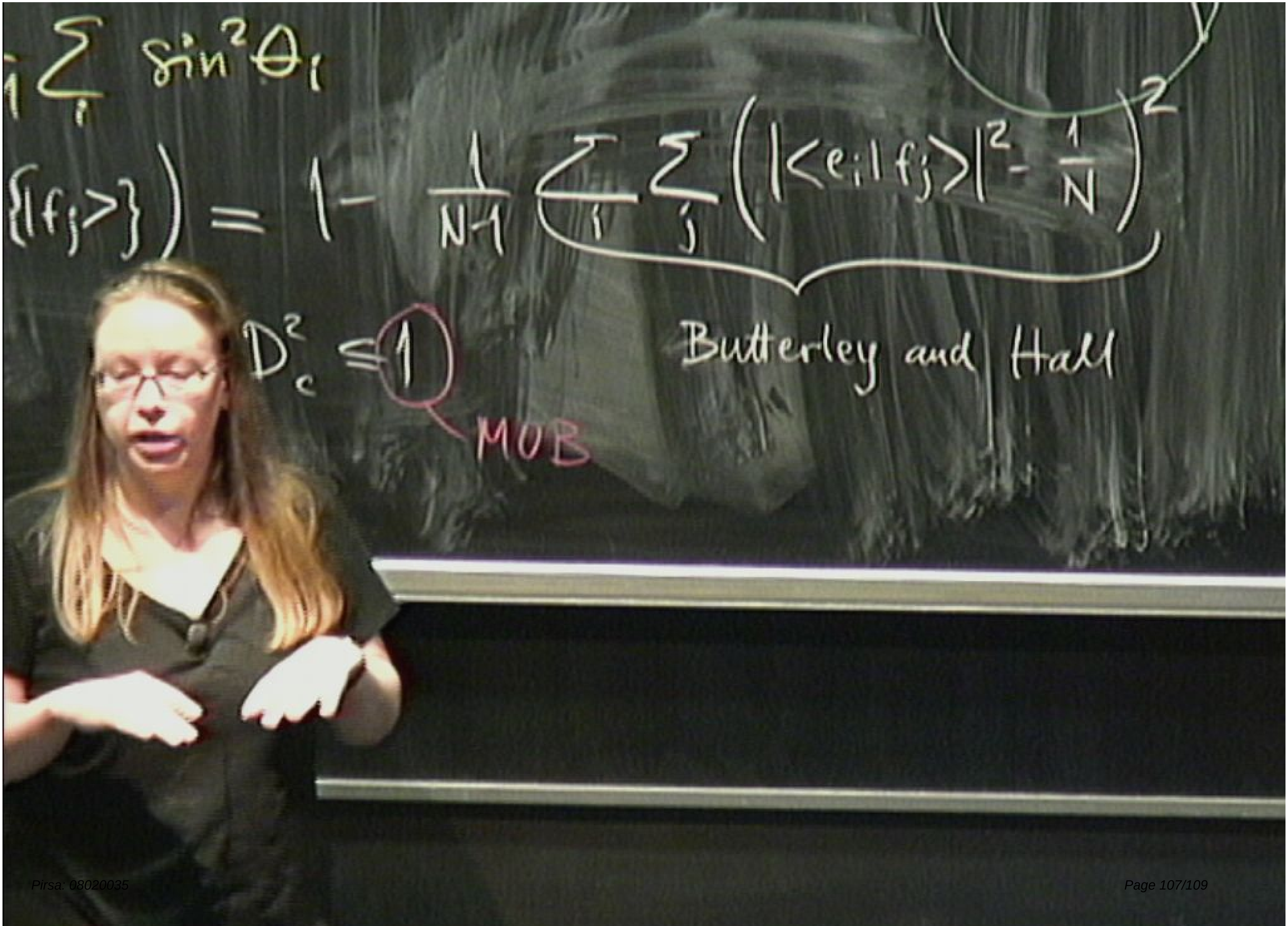
BJÖRCK &

BUTTERLEY & HALL Phys. Lett. A 369 (2007) 5


iff $H_1^{+} H_2$ also Hadamard.
H_1 }
16 possibilities
sequences


2(N-1)
sin²θ
{|fj>}) = 1/(N-1) Σi Σj (|<ei|fj>|² - 1/N)²


$\sum_i \sin^2\theta_i$
$\{|f_j\rangle\}) = 1 - \frac{1}{N-1} \sum_i \sum_j \left(|\langle e_i|f_j\rangle|^2 - \frac{1}{N} \right)^2$
utterley and Hall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$D_c^2 \leq 1$
MOB
Butterley and Hall


sin²θᵢ
1 − 1/(N−1) Σᵢ Σⱼ (|<eᵢ|fⱼ>|² − 1/N)²
0 ≤ D² ≤ 1
MOB
Butterley and Hall
7
4 0


$0 \le D_c^2 \le 1$
MOB
Butterley and Hall
7
4 0
7