Title: Quantum Spacetime

Date: Feb 14, 2008 02:00 PM

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Abstract: The principles of Quantum Mechanics and of Classical General Relativity imply Uncertainty Relations between the different spacetime coordinates of the events, which yield to a basic model of Quantum Minkowski Space, having the full (classical) Poincare\' group as group of symmetries.
The four dimensional Euclidean distance is a positive operator bounded below by a constant of order one in Planck units; the area operator and the four volume operator are normal operators - the latter being a Lorentz invariant operator with pure point spectrum - whose moduli are also bounded below by a constant of order one. While the spectrum of the 3 volume operator includes zero.
These findings are in perfect agreement with the physical intuition suggested by the Spacetime Uncertainty Relations which are implemented by the Algebra of Quantum Spacetime.
The formulations of interactions between quantum fields on Quantum Spacetime will be discussed. The various approaches to interactions, equivalent to one another on the Minkowski background, yield to different schemes on Quantum Spacetime, with the common feature of a breakdown of Lorentz invariance due to interactions. In particular one of these schemes will be discussed and motivated, which leads to fully Ultraviolet-Finite theories.
Quantum fields will depend on the quantum coordinates, but, in presence of Gravity, the commutators of the coordinates might in turn depend on the quantum fields, giving rise to a quantum texture where fields and spacetime coordinates cannot be separated. Possible deep physical consequences will be outlined.

QUANTUM SPACETIME

PERIMETER INSTITUTE

FEB. 14, 2008

I QM + CGR → STUR

→ QST SUGGESTING SCENARIO:

$$[q^\mu, q^\nu] = i\, Q^{\mu\nu}(q)$$

RELATED ISSUES:
- $\Lambda > 0$
- EQ OF CMB WITHOUT INFLATION
- DARK MATTER ?

BASIC MODEL: INDEP. OF q,

$$[Q^{\mu\nu}, q^\lambda] = 0 = Q_{\mu\nu} Q^{\mu\nu},$$

$$\left(\frac{1}{4} Q_{\mu\nu} (*Q)^{\mu\nu}\right)^2 = I;$$

IMPLEMENTS MINIMAL STUR

- POINCARE' COVARIANT
- EUCLIDEAN DISTANCE BETWEEN EVENTS

$$\sum_{\mu=0}^{3} (q^\mu - q'^\mu)^2 \geq 4$$

- EXTRADIMENSIONS, EFFECTIVE DISCRETIZATION

$\rightarrow$ QST SUGGESTING SCENARIO:

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IMPLEMENTS MINIMAL STUR

- POINCARÉ COVARIANT
- EUCLIDEAN DISTANCE BETWEEN EVENTS

$$\sum_{\mu=0}^{3} (q^\mu - q'^\mu)^2 \geq 4$$

- EXTRADIMENSIONS, EFFECTIVE DISCRETIZATION OF SPACETIME, ...

DFR '94, '95 ; D 01, 06

II QFT ON QST

- INTERACTION ALWAYS BREAKS LORENTZ INVARIANCE (?)

  DFR 95; BDFP 02

- TAKING THE QUANTUM NATURE OF $q - q'$ INTO ACCOUNT IN DEFINING (Q-) WICK PRODUCTS LEADS TO AN S MATRIX WHICH IS ULTRAVIOLET FINITE TERM BY TERM IN THE PERTURBATION EXPANSION, FOR ALL ϕ^n INTERACTIONS.

  S-MATRIX ELEMENTS FALL OFF AS POLIN. GAUSSIAN AT LARGE (TRANSPLANCKIAN) ENERGY-MOM. TRANSFERS

  BDFP 03

III GEOMETRY OF QST:

- $$\sum_{\mu=0}^{3} (q^\mu - q'^\mu)^2 \geq 2$$

- $$\sum_k |dq_j \wedge dq_l|^2 \geq 1,$$

  $$\sum_k |dq^0 \wedge dq^k|^2 \geq 1$$

- $dq^1 \wedge dq^2 \wedge dq^3$ is a NORMAL OPERATOR, SPECTRUM $= \mathbb{C}$

- $dq \wedge dq \wedge dq \wedge dq$ is a LORENTZ PSEUDOSCALAR NORMAL OPERATOR, PURE POINT SPECTRUM $=$

$\pm 2 + \mathbb{Z}\sqrt{5} + i\left(\mathbb{Z}\sqrt{5+2\sqrt{5}} + \mathbb{Z}\sqrt{5-2\sqrt{5}}\right)$.

dense in $\mathbb{R}$ ← K. FREDENHAGEN, S.D IN PREP.

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dense in $\mathbb{R}$ ← K. FREDENHAGEN, S.D. IN PREP

...

IF GRAVITATIONAL FORCES BETWEEN ELEMENTARY PARTICLES ARE NEGLECTED, QFT CAN BE BASED ON A CORE FIRST PRINCIPLE ON OBSERVABLES:

$$[A, B] = 0$$

IF A & B ARE LOCAL OBSERVABLES MEASURED IN SPACELIKE SEPARATED BD OPEN SETS.

$\Rightarrow$ SUPERSELECTION STRUCTURE, PARTICLE – ANTIPARTICLE SYMM OF S.Q.N., STATISTICS, SPIN–STATISTICS WITHOUT UNOBS. FIELDS, $\exists!$ COMPACT GLOBAL GAUGE GP, $\exists!$ ALL FIELD OP. NORMAL BOSE/FERM ...PLACING VACUUM TO ALL SUPERSELECTION SECTORS, ...

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$\exists$ [illegible] FIELD OP. NORMAL BOSE/FERM [illegible] VACUUM TO ALL SUPERSELECTION SECTORS, ...

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⇒ SUPERSELECTION STRUCTURE,
PARTICLE – ANTIPARTICLE SYMM
OF S.Q.N., STATISTICS,
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∃! COMPACT GLOBAL GAUGE GP,
∃! ALG FIELD OP. NORMAL Bose/Fermi
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BETWEEN ELEMENTARY PARTICLES ARE NEGLECTED, QFT CAN BE BASED ON A CORE FIRST PRINCIPLE ON OBSERVABLES:

$$[A, B] = 0$$

IF A & B ARE LOCAL OBSERVABLES MEASURED IN SPACELIKE SEPARATED 3D OPEN SETS.

$\Rightarrow$ SUPERSELECTION STRUCTURE, PARTICLE – ANTIPARTICLE SYMM OF S.Q.N., STATISTICS, SPIN–STATISTICS WITHOUT UNOBS. FIELDS, ...

$\exists!$ COMPACT GLOBAL GAUGE GP,

$\exists!$ ALG FIELD OP. NORMAL BOSE/FERMI RELATING VACUUM TO ALL SUPERSELECTION SECTORS, ...

BUT, IF GRAVITATION IS CONSIDERED,

POINTS IN SPACETIME:

LOCALIZATION OF EVENTS.

OPERATIONAL MEANING?

UNCERTAINTIES $\Delta x_0, \ldots, \Delta x_3$,
with inf = a, imply

TRANSFER OF ENERGY

$$E \gtrsim \frac{1}{a}$$

(QM: Heisenberg uncertainty principle)

...

E GENERATES A GRAVITATIONAL FIELD
which, should $\Delta x_0, \ldots, \Delta x_3$ be too small, WOULD PREVENT ANY

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(QM : Heisenberg uncertainty Principle)

E GENERATES A GRAVITATIONAL FIELD
which, should $\Delta x_0, \dots, \Delta x_3$ be too small, WOULD PREVENT ANY SIGNAL TO REACH A FAR DISTANT OBSERVER
(CLASSICAL GENERAL RELATIVITY)

3.

e.g. Δx_0 UNLIMITED, (STATIONARY)

$\Delta x_1 \sim \Delta x_2 \sim \Delta x_3 \sim a$,

spherical symmetry, THEN

$$a \gtrsim \text{Schwarzschild radius} = E \sim \frac{1}{a}$$

i.e. $a \gtrsim 1$ $(\hbar = G = c = 1)$

= PLANCK LENGTH, $\sim 1.6 \times 10^{-33}$ cm.

BUT IF $\Delta x_0 = \infty$ (stationary field),

$\Delta x_j, \Delta x_k$ ARBITRARILY SMALL BUT FIXED, $\Delta x_\ell \sim L \to \infty$,

THE CLASSICAL POTENTIAL GENERATED $\to 0$ as $L \to \infty$.

- A SINGLE COORDINATE CAN BE MEASURED WITH ARBITRARY PRECISION

- SPACE TIME UNCERTAINTY

5, 40 (1936)

- AMATI, CIAFALONI, VENEZIANO

- STUR: 90'S:

- A CAREFUL ANALYSIS SHOWS THAT AT LEAST

$$\Delta x_0 \cdot \sum_{j=1}^{3} \Delta x_j \gtrsim 1,$$

$$\sum_{1 \leq j < k \leq 3} \Delta x_j \cdot \Delta x_k \gtrsim 1.$$

ANY MODEL OF QST SHOULD IMPLEMENT AT LEAST... THESE STUR.

DFR, 1994 e 95

Comments :

- search for QST models
- quantum in the small, Minkowskian (flat) in the large (for the sake of elementary particle physics), NO COVARIANCE UNDER GENERAL TRANSFORMATIONS OF COORDINATES IS REQUIRED
- THE FULL POINCARÉ GROUP – gp of global motions – should act in the same way in the small and in the large – i.e. still act as symmetries of QST.

$$[q^\mu, q^\nu] = i\, Q^{\mu\nu}(g)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}(\psi)$$

$$F_g(\psi) = 0$$

ALL MUTUALLY COUPLED EQ.

CGR: GEOMETRY ~ DYNAMICS

QGR: ALGEBRA ~ DYNAMICS.

ARXIV 2001, 2005

BASIC MODEL OF Q.S.T:

$$[q^\mu, q^\nu] = i\, Q^{\mu\nu},$$

Q CENTRAL, $Q \cdot Q = 0$,

$$\left(\frac{1}{4} Q \cdot *Q\right)^2 = I$$

g

ALL MUTUALLY COUPLED EQ.

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$\Longrightarrow$ STUR.

...

RELATED TO NATURAL MEASURE OF NONCOMMUTATIVITY

$$[q_0, \ldots, q_3] \equiv$$

$$\varepsilon_{\mu\nu\lambda\rho}\, q^\mu q^\nu q^\lambda q^\rho \equiv$$

$$\det \begin{pmatrix} q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \end{pmatrix} =$$

$$= -\frac{1}{2}\, Q_{\mu\nu} (*Q)^{\mu\nu}$$

LORENTZ - PSEUDOSCALAR

$$Q_{\mu\nu} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix}$$

(CENTRAL!), QUANTUM CONDITIONS AMOUNT TO

$$e^2 = m^2,$$

$$(\vec{e} \cdot \vec{m})^2 = 1.$$

DEFINE MANIFOLD Σ (joint spectrum $Q_{\mu\nu}$):

$$\Sigma = \Sigma_+ \cup \Sigma_-, \quad \Sigma_+ \sim \Sigma_- \sim$$

$\sim L^{\uparrow}_{+}$ / boost along z $\sim SL(2, \mathbb{C})$/diag

$$Q_{\mu\nu} = \begin{pmatrix} \cdots & e_1 & 0 & m_3 & -m_2 \\ \cdots & -e_2 & -m_3 & 0 & m_1 \\ \cdots & -e_3 & m_2 & -m_1 & 0 \end{pmatrix}$$

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$$\sim L_+^\uparrow \Big/ \begin{matrix}\text{boost along } z \\ \text{rot around } z\end{matrix} \sim SL(2,\mathbb{C})/\text{diag}$$

$$\sim TS^2; \; \Sigma \sim \{ z \in \mathbb{C}^3 / z^4 + 1 = 0 \}.$$

$\mathcal{C}_0(\mathbb{R}^4)$ REPLACED BY

ENVELOPING C^*-ALGEBRA

$\mathcal{E}$ OF REGULAR REPS:

$$e^{i\alpha q} e^{i\beta q} = e^{\frac{i}{2}\alpha Q \beta} e^{i(\alpha+\beta)q};$$

$\mathcal{E}$ is gen. by $\int f(\alpha) e^{i\alpha q} d^4\alpha$,

linearly spanned by

$$\int f(Q, \alpha) e^{i\alpha q} d^4\alpha$$

$= C^*$ COMPLETION OF BANACH $*$ ALG OF TWISTED CONVOLUTION

$f : (\sigma, \alpha) \in \Sigma \times \mathbb{R}^4 \to \mathbb{C}$

$$(f \times g)(\sigma, \alpha) = \int f(\sigma, \alpha') g(\sigma, \alpha - \alpha') e^{\frac{i}{2}\alpha \sigma \alpha'} d^4\alpha',$$

$f \in \mathcal{C}_0(\Sigma, L^1)$.

$$e^{i\alpha q} e^{i\beta q} = e^{\frac{i}{2}\alpha Q \beta} e^{i(\alpha+\beta)q};$$

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$\dots = e^{\frac{i}{2}\dots} e^{i(\alpha+\beta)q}$;

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$f \in C_0(\Sigma, L^1)$.

DFR 1995. RESULT:

$$\mathcal{E} \simeq \mathcal{C}_0(\Sigma) \otimes \mathcal{K},$$

$\mathcal{K}$ = compact ops on separable ∞-dim H. S.

$\exists \; : \tau : \mathcal{G} \longrightarrow \mathrm{Aut}\,\mathcal{E}$ st

$$\tau_L(q) = L^{-1}q,$$

q^μ "AFFILIATED" TO $\mathcal{E}$,

FULFILLING SJUR.

STATES ON $\mathcal{E}$:

ω : [illegible] $\mathbb{C}$

LIN[illegible] S[illegible] NO[illegible]ALIZED,

[illegible] RE[illegible]LE POINTS.

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$$\omega : \mathcal{E} \longrightarrow \mathbb{C}$$

LIN, POS, NORMALIZED,

REPLACE POINTS.

FULL DISCUSSION IN DFR 1995. RESULT:

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LIN, <u>POS</u>, NORMALIZED,
REPLACE POINTS.

OPTIMALLY LOCALIZED STATES ω:

$$\sum_{\mu=0}^{3} (\Delta_\omega q^\mu)^2 = \min$$

($\exists$? min) DFR '95:

min $= 2$, reached iff:

1. ω carried by $\vec{e} = \pm \vec{m}$, i.e. $\sigma \in \Sigma_1 \simeq S^2 \times \{\pm 1\}$ $(\Sigma = T\Sigma_1 \simeq TS^2 \times \{\pm 1\})$

2. $\omega\left(\int f(\alpha) e^{i\alpha q} d^4\alpha\right) = \int f(\alpha) e^{-\frac{1}{2}|\alpha|^2} d^4\alpha$

(commutes [illegible] near 0).

Σ_1 = (improper) ROTATION ORBIT OF A SINGLE POINT

$\sigma_0 = (\vec{e}, \vec{m}) =$ ([illegible])

... of ω :

$$\sum_{\mu=0}^{3} (\Delta_\omega q^\mu)^2 = \min$$

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min = 2, reached iff:

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(centered at σ = 0).

Σ_1 = (unitary) ROTATION ORBIT OF A DIRECT POINT

$$\sigma_0 = (\vec{e}, \vec{m}) = (\vec{n}_1, \vec{n}_2)$$

CORRESPONDING IRREP:

$\mu = 0$

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min = 2, reached iff:

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$\sigma_0 \equiv (\vec{e}, \vec{m}) =$

... $= (\vec{n}_2, \vec{n}_1)$

CORRESPONDING IRREP:

$$q^\mu = \begin{pmatrix} Q \otimes I \\ P \otimes I \\ I \otimes Q \\ I \otimes P \end{pmatrix}$$

Q, P : SCHRÖDINGER OPS
1 - d. of f.
~ multipl by s , $-i\frac{d}{ds}$ on $L^2(\mathbb{R})$.

$$\sum_{\mu=0}^{3} (q^\mu)^2 = 2\,(H \otimes I + I \otimes H),$$

H = Hamiltonian of harmonic osc.
$\geq \frac{1}{2}$,

$$\Rightarrow \quad \sum_{\mu=0}^{3} (\Delta q^\mu)^2 \geq 2$$

$$\sum_{\mu=0} (\Delta_\omega g^{\mu}) = 2$$

$$\Updownarrow$$

$$\omega = \mu \circ \bar{\eta} \circ \rho$$

ρ = quotient map

$$\mathcal{E} \equiv C_0(\Sigma, \mathcal{K}) \to C(\Sigma_1, \mathcal{K})$$

$$\equiv \mathcal{E}_1$$

$\bar{\eta}$ ≡ UNIVERSAL LOCALIZATION CONDITIONAL EXPECTATION

$$\mathcal{E}_1 \longrightarrow C(\Sigma_1) \equiv Z(M(\mathcal{E}_1))$$

μ = ANY REGULAR PROBABILITY MEASURE ON Σ_1.

15'

$\eta \equiv \tilde{\eta} \circ \ell$:

$$\left\{ \sigma \in \Sigma \longrightarrow \int f(\sigma, \alpha) e^{i \alpha q_{\ell}} d^4\alpha \right\}$$

$$\longrightarrow \left\{ \sigma \in \Sigma_1 \to \int f(\sigma, \alpha) e^{-\frac{1}{4}|\alpha|^2} d^4\alpha \right\}$$

LARGE SCALE LIMIT

(compared to Planck):

$\times \longrightarrow$ ORDINARY CONVOLUTION

$= \mathcal{F}$(pointwise prod)

$\mathcal{E} \longrightarrow \mathcal{C}_0(\mathbb{R}^4 \times \Sigma)$

QST $\longrightarrow \mathbb{R}^4 \times \Sigma$

IF PROBED WITH OPTIMALLY LOCALIZED STATES:

$\longrightarrow \mathbb{R}^4 \times \Sigma_1 \sim \mathbb{R}^4 \times S^2 \times \{\pm\}$

CONNES-LOTT

OPS. OPT. LOC. STATES:

$(q^\mu) \rightarrow$ SCHROEDINGER OPS 2 dim

PHASE SPACE : CELLS OF VOLUME

$\sim \lambda_P^4$.

CALCULUS ON QST

(DFR '94, '95)

$$f(q) \equiv \int \check{f}(\alpha)\, e^{i\alpha q}\, d^4\alpha$$

$f(q)\, g(Q)$ span & densely, $f \in L^1$, $g \in \mathcal{C}_0(\Sigma)$.

$$f(q)\, g(q) = (f_1 \times f_2)(q),$$

$$(f_1 \times f_2)(Q, \alpha) =$$

$$= \int f_1(\alpha') f_2(\alpha - \alpha')\, e^{\frac{i}{2}\alpha Q \alpha'}\, d^4\alpha'$$

IN IRREPS: MOYAL. WARNING!

- $dA = \ldots \sum_\mu \frac{\partial}{\partial \alpha_\mu} \tau_\alpha(A)\, d\alpha_\mu \Big|_{\alpha=0}$;

- $\int f(q)\, d^4q = \mathrm{Tr}\, f(q) = \check{f}(0)$,

- $\int f(q)\, d^3q = \lim \mathrm{Tr}\, g_n(q)^* f(q)\, g_n(q)$

(DFR '94, '95)

$$f(q) \equiv \int \check{f}(\alpha) e^{i\alpha q} d^4\alpha$$

$f(q)\, g(Q)$ span $\mathcal{E}$ linearly,

$f \in L^1$, $g \in \mathcal{C}_0(\Sigma)$.

$$f(q)\, g(q) = (f_1 \times f_2)(q),$$

$$(f_1 \times f_2)(Q, \alpha) =$$

$$= \int f_1(\alpha') f_2(\alpha - \alpha') e^{\frac{i}{2} \alpha Q \alpha'} d^4\alpha'$$

IN IRREPS: MOYAL. WARNING!

- $$dA = \sum_\mu \frac{\partial}{\partial \alpha_\mu} \tau_\alpha(A)\, d\alpha_\mu \Big|_{\alpha=0};$$
- $$\int f(q)\, d^4q = \mathrm{Tr}\, f(q) = \check{f}(0),$$
- $$\int_{q_0=t} f(q)\, d^3q = \lim_n \mathrm{Tr}\, g_n(q)^* f(q)\, g_n(q)$$

$$= \int d\alpha_0 e^{i\alpha_0 t} \check{f}(\alpha_0, \vec{0})$$

$$f(q) \equiv \int \check{f}(\alpha)\, e^{i\alpha q}\, d^4\alpha$$

$f(q)\, g(Q)$ span $\mathcal{E}$ linearly,
$f \in L^1$, $g \in \mathcal{C}_0(\Sigma)$.

$$f(q)\, g(q) = (f_1 \times f_2)(q),$$

$$(f_1 \times f_2)(Q, \alpha) =$$

$$= \int f_1(\alpha') f_2(\alpha - \alpha')\, e^{\frac{i}{2}\alpha Q \alpha'}\, d^4\alpha'$$

IN IRREPS: MOYAL. WARNING!

- $dA = \sum_\mu \frac{\partial}{\partial \alpha_\mu} \tau_\alpha(A)\, d\alpha_\mu \Big|_{\alpha=0}$;
- $\int f(q)\, d^4q = \mathrm{Tr}\, f(q) = \check{f}(0)$,
- $\int_{q_0=t} f(q)\, d^3q = \lim_n \mathrm{Tr}\, g_n(q)^* f(q)\, g_n(q)$

$$= \int d\alpha_0\, e^{i\alpha_0 t}\, \check{f}(\alpha_0, \vec{0})$$

... EVENTS :

$$\mathcal{E} \otimes \mathcal{E} \otimes \ldots \otimes \mathcal{E} \quad n \text{ times}$$

$$q_j = I \otimes I \otimes \ldots \otimes q \otimes \ldots \otimes I$$

$\uparrow$ jth place.

REQUIRE :

$$[q_j^\mu, q_j^\nu] = [q_k^\mu, q_k^\nu]$$

i.e. $\otimes \longrightarrow \otimes_{\mathcal{Z}}$ of

$\mathcal{Z}$ — BIMODULES ,

$$\mathcal{Z} = Z(M(\mathcal{E})) = \mathcal{C}_B(\Sigma) ;$$

e.g. $A, B \in \mathcal{E}$, $f \in \mathcal{Z}$:

$$Af \otimes B = A \otimes fB$$

$\Longleftrightarrow$ (cf LATER)

$$dQ = 0.$$

$\Rightarrow$ • THE NORMALIZED BARYCENTER & DIFFERENCE OPERATORS

$$\bar{q} \equiv \frac{1}{\sqrt{n}} \Sigma q_j \,,$$

$$q_{jk} \equiv \frac{1}{\sqrt{2}} (q_j - q_k)$$

OBEY THE SAME CR as q^μ_j

$$[q^\mu_{jk}, q^\nu_{jk}] = i Q^{\mu\nu}$$

$$\Rightarrow \frac{1}{2} \sum_{\mu=0}^{3} (q^\mu_j - q^\mu_k)^2 \geq 2$$

MINIMAL EUCLIDEAN DISTANCE

• COMMUTE WITH BARYCENTER COORD:

$$\bar{q}^\mu \equiv \frac{1}{\sqrt{n}} \sum_{j=1}^{n} q^\mu_j \,,$$

$$[\bar{q}^\mu, q^{\lambda\rho}_{jk}] = 0$$

MINKOWSKIAN WICK PRODUCT:

SUBTRACTIONS S.T.

$$:\phi(x_1)\dots\phi(x_n): \xrightarrow[x_j - x_k \to 0]{}$$

$$\longrightarrow \; :\phi(x)^n:$$

MAKES SENSE;

ON QST $q_j - q_k \to 0$

$\boxed{\text{VIOLATES}}$ C.R.'s. BEST:

QUANTUM DIAGONAL MAP,

SETTING $\Sigma (q_j^\mu - q_k^\mu)^2 = 2$:

2 STEPS:

(1) $\bar{q}_\mu \longrightarrow q_\mu \otimes \underbrace{I \otimes \dots \otimes I}_{n \text{ times}}$

...

$q_j - q_k \longrightarrow I \otimes \underbrace{(q_j - q_k)}_{n\text{-fold } \otimes}$

DEFINES $\neq$ monom.

$$\mathcal{Y} : \mathcal{E}^{\otimes_2 n} \longrightarrow \mathcal{E}_{1}^{\otimes_{2_1} (n+1)}$$

MAKES SENSE;

ON QST $q_j - q_k \to 0$

$\boxed{\text{VIOLATES}}$ C.R.'s. BEST:

- QUANTUM DIAGONAL MAP, SETTING $\Sigma(q_j^\mu - q_k^\mu)^2 = 2$:

2 STEPS:

1 $\bar{q}_\mu \longrightarrow q_\mu \otimes \underbrace{I \otimes \ldots \otimes I}_{n \text{ times}}$

$q_j - q_k \longrightarrow I \otimes \underbrace{(q_j - q_k)}_{n\text{-fold } \otimes}$

DEFINES ≠ monom.

$\psi : \mathcal{E}^{\otimes_2 n} \longrightarrow \mathcal{E}_1^{\otimes_{z_1} (n+1)}$;

WITH $\eta = \bar{\eta} \circ \rho \; : \mathcal{E} \to \mathcal{C}(\Sigma_1)$

the localization map, s.t.

$\mu \circ \eta$ is the most general optimally loc. state, μ any prob measure on Σ_1, SET

$$E^{(n)} \equiv (I \otimes \underbrace{\eta \otimes \cdots \otimes \eta}_{n\text{-times}}) \circ \Delta$$

so that

$$E^{(n)} : f(q_1, \ldots, q_n) \longrightarrow$$

$$\longrightarrow E^{(n)}(f)\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} q_j\right).$$

$\in \mathcal{E}_1 = \mathcal{C}(\Sigma_1, \mathcal{K})$.

MAKES EACH $q_j - q_k$ AS SMALL AS C.R.'s ALLOW

DEPENDS ON LORENTZ FRAME

ROT, TRANS. COVARIANT.

EACH

LL AS C.R.'s ALLOW

LORENTZ FRAME

COVARIANT.

WITH $\eta = \bar{\eta} \circ \varphi : \mathcal{E} \to \mathcal{C}(\Sigma_1)$

the localization map, s.t.

$\mu \circ \eta$ is the most general optimally loc. state, μ any prob measure on Σ_1, SET

$$E^{(n)} \equiv (I \otimes \underbrace{\eta \otimes \cdots \otimes \eta}_{n\text{-times}}) \circ \varphi$$

so that

$$E^{(n)} : f(q_1, \ldots, q_n) \longrightarrow$$

$$\longrightarrow E^{(n)}(f)\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} q_j\right).$$

$\in \mathcal{E}_1 = \mathcal{C}(\Sigma_1, \mathcal{K})$.

MAKES EACH $q_j - q_k$ AS SMALL AS C.R.'s ALLOW

DE… LORENTZ FRAME

…R… …ARIANT.

QFT ON QST

FREE FIELDS:

$$\phi(q) \equiv \int e^{-ikq} \otimes \check{\phi}(k)\, d^4k$$

POINCARÉ COVARIANT,
LOCAL ALGEBRAS,

COMMUTATORS AT SPACELIKE SEP.

$\neq 0$ BUT VANISH AS FAST
AS GAUSSIANS (PLANCK LENGTH)
(DFR '95)

INTERACTION: VIOLATES

(NOT ONLY CAUSALITY AT PLANCK SCALE AS EXPECTED, BUT ALSO)

LORENTZ INVARIANCE.

SEVERAL (INEQ.) APPROACHES:

QFT ON QST

FREE FIELDS:

$$\phi(q) \equiv \int e^{-ikq} \otimes \check{\phi}(k)\, d^4k$$

POINCARÉ COVARIANT,
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(DFR '95)

INTERACTON: VIOLATES

(NOT ONLY CAUSALITY AT PLANCK SCALE AS EXPECTED, BUT ALSO)

LORENTZ INVARIANCE.

SEVERAL (INEQ.) APPROACHES:

1) YANG-FELDMAN EQ

& QUASIPLANAR WICK PRODUCT

(D. BAHNS, S.D., K. FREDENHAGEN, G. PIACITELLI, 2002, 2005, IN PREP.)

$$(\Box + m^2)\,\phi(q) = g :\phi^{n-1}(q):$$

- BUNDLE OF THEORIES ON Σ; REN. DEPENDS BADLY ON $\sigma \in \Sigma$;
- LORENTZ INV. BROKEN FOR ASYMPT. STATES (DISP. RELATIONS).

. FIRST POINT CURED BY

QUASIPLANAR WICK PRODUCT

BUT RESIDUAL REN. NEEDED

TO COMPARE WITH LIM. $\lambda \to 0$.

2) S-MATRIX APPROACHES

(DFR '95, BDFP '2002, '2003, IN PROGRESS):

DYSON-GELL'MANN-LOW:

$$S = T \exp i \int_{-\infty}^{\infty} H_I(t)\, dt$$

ϕ^n interaction:

$$\int_{q_0 = t} d^3 q \; g : \phi^n(q) :$$

still depends on $Q_{\mu\nu}$'s. Bundle on Σ

$\int_\Sigma d\mu(\sigma)$: NO INVARIANT FINITE MEASURE NOR MEAN EXISTS

FIRST CHOICE : GIVE UP LORENTZ INVARIANCE, SET

$$H_I(t) = \int_{\Sigma_1} d\sigma \int_{q_0 = t} d^3 q \; g : \phi^n(q) :$$

MILD REGULARIZATION

SUFFICIENT FOR ϕ^3 ...

$q_0 = t$

still depends on $Q_{\mu\nu}$'s. Bundle on Σ_1

$\int_\Sigma d\mu(\sigma)$: NO INVARIANT FINITE MEASURE NOR MEAN EXISTS

FIRST CHOICE : GIVE UP LORENTZ INVARIANCE, SET

$$H_I(t) = \int_{\Sigma_1} d\sigma \int_{q_0=t} d^3q \; g :\Phi^n(q):$$

MILD REGULARIZATION SUFFICIENT FOR Φ^3 (BOSONS)

SUBTLER CHOICE:

QUANTUM WICK PRODUCT

$$:\phi^n(q):_Q \equiv$$

$$\equiv E^{(n)}\big(:\underbrace{\phi(q)\otimes_2 \cdots \otimes_2 \phi(q)}_{n\text{-times}}:\big),$$

$$H_I(t) \equiv \int_{q_0=t} d^3q \; g :\phi^n(q):_Q$$

LIVES ON Σ_1 BUT DOES NOT DEPEND ON σ.

WITH ADIAB. SWITCHING $g \in \mathcal{S}(\mathbb{R})$

$$T\exp i\int_{-\infty}^{\infty} g(t)\, H_I(t)\, dt$$

VACUUM→VACUUM

IS ULTRAVIOLET FINITE

$$\equiv E^{(n)}\Big(:\phi(q) \underbrace{\otimes_2 \cdots \otimes_2}_{n\text{-times}} \phi(q): \Big),$$

$$H_I(t) \equiv \int_{q_0=t} d^3q \; g :\phi^n(q):_Q$$

LIVES ON Σ_1 BUT DOES NOT DEPEND ON σ.

WITH ADIAB. SWITCHING $g \in \mathcal{S}(\mathbb{R})$

$$T\exp i \int_{-\infty}^{\infty} g(t)\, H_I(t)\, dt \Big/ \text{VACUUM} \to \text{VACUUM}$$

IS $\boxed{\text{ULTRAVIOLET FINITE}}$

TERM BY TERM IN PERT. EXP.

BDFP 2003

SUBTLER CHOICE:

QUANTUM WICK PRODUCT

$$:\phi^n(q):_Q \equiv$$

$$\equiv E^{(n)}\Big(\underbrace{:\phi(q)\otimes_Z \cdots \otimes_Z \phi(q):}_{n\text{-times}}\Big),$$

$$H_I(t) \equiv \int_{q_0=t} d^3q \; g :\phi^n(q):_Q$$

LIVES ON Σ_1 BUT DOES NOT DEPEND ON σ.

WITH ADIAB. SWITCHING $g \in \mathcal{S}(\mathbb{R})$

$$T\exp i\int_{-\infty}^{\infty} g(t)\, H_I(t)\, dt \Big/ \text{VACUUM}\to\text{VACUUM}$$

IS $\boxed{\text{ULTRAVIOLET FINITE}}$

IN BOTH CASES

$$\int dt\, H_I(t) = \iint G(t, x_1, \dots x_m) \cdot$$

$$: \phi(x_1) \dots \phi(x_m) : d^4x_1 \dots d^4x_m\, dt$$

NON LOCAL INTERACTION.

S-MATRIX IS THE SAME AS FOR AN EFFECTIVE NON LOCAL THEORY ON ORDINARY MINKOWSKI SPACE.

$$S = T \exp i \int H_I(t)\, dt$$

TIME ORDERING REFERS TO "t's" NOT TO "x_j^0's" !!

$$(\phi_1 \times \phi_2)(x) = e^{\frac{i}{2} Q_{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu}} \phi(x)\phi(y) \Big|_{x=y}$$

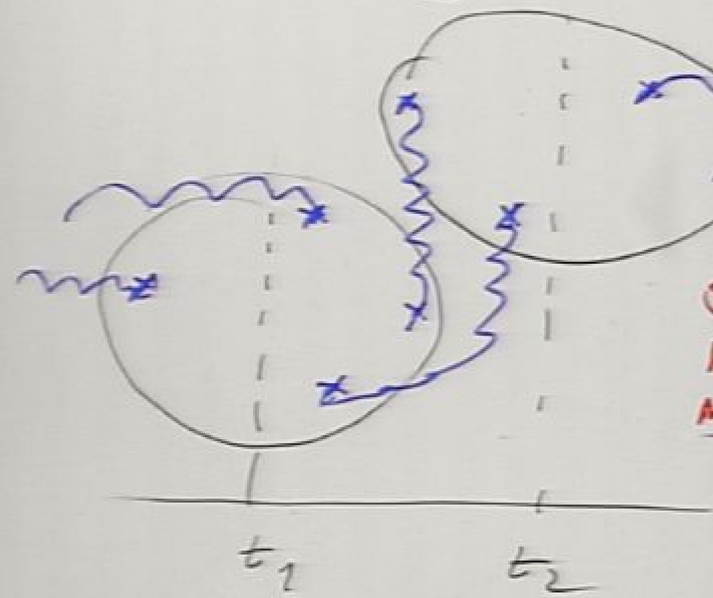
IS NOT LOCAL !!

MOD. FEYNMAN DIAGRAMS (DFR '95)

(G. PIACITELLI, JHEP 2004) LIAO SIBOLD DENK-SCHWEDA

NO VIOLATION OF UNITARITY!

$$H_I(t) = \int G(t, x_1, .., x_4) :\phi(x_1) .. \phi(x_4): dx_1 ..$$



G. PIACITELLI,
hep-th/0403055
NEW FEYNMAN RULES
(JHEP 2004)
cf DENK, SCHWEDA
hep-th/0306101

MUST TIME ORDER $t_1, t_2, \ldots$

$\boxed{\text{NOT}}$ THE TIME ARGUMENTS OF THE FIELD OPERATORS! THAT WOULD BE ALLOWED IF H_I WERE LOCAL

E.G. IF... WE COULD WRITE

$$(\phi * \phi)(x) = e^{\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}} \phi(x)\phi(y) \Big|_{x=y}$$

AS A LOCAL EXPANSION

$\mathcal{O}$ — algebra on $\mathbb{C}$;

$$da = I \otimes a - a \otimes I$$

UNIVERSAL DIFF. CALCULUS

(ALAIN CONNES, NCG)

ACCOMODATED IN

$$\Lambda(\mathcal{O}) = \bigoplus_{n=0}^{\infty} \mathcal{O}^{\otimes n}$$

viewing $\mathcal{O}^{\otimes n}$ as an $\mathcal{O}$-BIMODULE

$$a \cdot a_1 \otimes \cdots \otimes a_n = a a_1 \otimes \cdots \otimes a_n,$$
$$a_1 \otimes \cdots \otimes a_n \cdot b = a_1 \otimes \cdots \otimes a_n b,$$

$a, a_1, \dots, b \in \mathcal{O}$, $\Lambda(\mathcal{O})$ equipped with the $\mathcal{O}$-BIMODULE TENSOR PRODUCT IS AN ALGEBRA, WITH

$$\Omega(\mathcal{O}) \subset \Lambda(\mathcal{O})$$

$\mathcal{A}$ algebra on $\mathbb{C}$;

$$da = I \otimes a - a \otimes I$$

UNIVERSAL DIFF. CALCULUS

(ALAIN CONNES, NCG)

ACCOMODATED IN

$$\Lambda(\mathcal{A}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{A}^{\otimes n}$$

viewing $\mathcal{A}^{\otimes n}$ as an $\mathcal{A}$-BIMODULE

$$d \cdot a_1 \otimes \ldots \otimes a_m = d a_1 \otimes \ldots \otimes a_m,$$

$$a_1 \otimes \ldots \otimes a_m \cdot b = a_1 \otimes \ldots \otimes a_m b,$$

$a, a_1, \ldots, b \in \mathcal{A}$, $\Lambda(\mathcal{A})$ equipped with the $\mathcal{A}$-BIMODULE TENSOR PRODUCT IS AN ALGEBRA, WITH

$\mathfrak{A}$ algebra on $\mathbb{C}$;

$$da = I \otimes a - a \otimes I$$

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$a, a_1, \ldots, b \in \mathfrak{A}$, $\Lambda(\mathfrak{A})$ equipped with the $\mathfrak{A}$-BIMODULE TENSOR PRODUCT IS AN ALGEBRA, WITH

$\mathcal{A}$ algebra on $\mathbb{C}$;

$$da = I \otimes a - a \otimes I$$

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ACCOMODATED IN

$$\Lambda(\mathcal{A}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{A}^{\otimes n}$$

viewing $\mathcal{A}^{\otimes n}$ as an $\mathcal{A}$-BIMODULE

$$a \cdot a_1 \otimes \dots \otimes a_m = a a_1 \otimes \dots \otimes a_m,$$
$$a_1 \otimes \dots \otimes a_m \cdot b = a_1 \otimes \dots \otimes a_m b,$$

$a, a_1, \dots, b \in \mathcal{A}$, $\Lambda(\mathcal{A})$ equipped with the $\mathcal{A}$-BIMODULE TENSOR PRODUCT IS AN ALGEBRA, WITH

(ALAIN CONNES, NCG)

ACCOMODATED IN

$$\Lambda(\mathcal{O}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{O}^{\otimes n}$$

viewing $\mathcal{O}^{\otimes n}$ as an $\mathcal{O}$ – BIMODULE

$a \cdot a_1 \otimes \ldots \otimes a_n = a a_1 \otimes \ldots \otimes a_n$,

$a_1 \otimes \ldots \otimes a_n \cdot b = a_1 \otimes \ldots \otimes a_n b$,

$a, a_1, \ldots, b \in \mathcal{O}$, $\Lambda(\mathcal{O})$ equipped with the $\mathcal{O}$ – BIMODULE TENSOR PRODUCT IS AN ALGEBRA, WITH

$$\Omega(\mathcal{O}) \subset \Lambda(\mathcal{O})$$

$$(a_1\otimes a_2\otimes\cdots\otimes a_n)\cdot(b_1\otimes b_2\otimes\cdots\otimes b_m) =$$

$$= a_1\otimes a_2\otimes\cdots\otimes a_n b_1\otimes\cdots\otimes b_m$$

$$\in \Omega^{\otimes(n+m-1)}$$

IF, E.G. $A_1,\dots,A_n \in \Omega\otimes\Omega$,

$$A_1\cdot A_2\cdots A_n \in \Omega^{\otimes(n+1)}$$

IF, IN PARTICULAR, $a_1,\dots,a_n \in \Omega$,

$$da_1\cdot da_2\cdots da_n \in \Omega^{\otimes(n+1)}$$

$$(da = I\otimes a - a\otimes I).$$

e.g. $dq\wedge dq\wedge dq\wedge dq \in \mathcal{E}^{\otimes 5}$

AS THE SUBALGEBRA GEN.
BY a, db ; $a, b \in \mathfrak{A}$.

OTHER ALGEBRAIC STRUCTURE
IN $\Lambda(\mathfrak{A})$: DIRECT SUM OF
ALGEBRAS $\mathfrak{A} \otimes .. \otimes \mathfrak{A}$;
MAKES $\Lambda(\mathfrak{A})^{\sim}$ A C^*-ALGEBRA
IF $\mathfrak{A}$ IS. NATURAL NORM
AND SPECTRUM FOR FORMS:

FOR FORMS $a\, da_1 .. da_m$,
$a_1, .., a_m \in \mathfrak{A}$,

- PRODUCTS COMPUTED IN THE 1st ALG.;

- SPECTRA, NORMS, ... IN THE 2nd (C^*) ALG;

- SHUFFLING PRODUCT IN

$\mathcal{E} \sim C_0(\Sigma) \otimes \mathcal{K}(H)$

$M(\mathcal{E}) \equiv$ MULTIPLIERS $=$

$= \mathcal{C}_B(\Sigma) \otimes \mathcal{B}(H)$

$\mathcal{Z}(M(\mathcal{E})) \sim \mathcal{C}_B(\Sigma)$
$\equiv \mathcal{Z}$.

$\mathcal{E}$ is a $\mathcal{Z}$ – BIMODULE

$\mathcal{Z}$ – BIMODULE TENSOR PRODUCT:

$A, B \in \mathcal{E}$; $C \in \mathcal{Z}$:

$CA \otimes_{\mathcal{Z}} B = A \otimes_{\mathcal{Z}} CB$

i.e.

$dC = I \otimes C - C \otimes I = 0$
$C \in \mathcal{Z}$.

$C_0(\Sigma) \otimes \mathcal{K}(H)$

$M(\mathcal{E}) \equiv$ MULTIPLIERS $=$

$$= C_B(\Sigma) \otimes \mathcal{B}(H)$$

$$\mathcal{Z}(M(\mathcal{E})) \sim C_B(\Sigma)$$
$$\equiv \mathcal{Z}.$$

$\mathcal{E}$ IS a $\mathcal{Z}$ - BIMODULE

$\mathcal{Z}$ - BIMODULE TENSOR PRODUCT:

$A, B \in \mathcal{E}$; $C \in \mathcal{Z}$:

$$CA \otimes_{\mathcal{Z}} B = A \otimes_{\mathcal{Z}} CB$$

i.e.

$$dC = I \otimes C - C \otimes I = 0$$
$$C \in \mathcal{Z}.$$

SO THAT

$$d\,\mathcal{Q}_{\mu\nu} = 0$$

i.e. $I \otimes Q_{\mu\nu} = Q_{\mu\nu} \otimes I$;

WITH

$$q_1^\mu = q^\mu \otimes I, \quad q_2^\mu = I \otimes q^\mu,$$

$[q_1, q_2] = 0$ and

$$[q_1^\mu, q_1^\nu] = [q_2^\mu, q_2^\nu] = i\,Q^{\mu\nu};$$

$\frac{1}{\sqrt{2}}(q_2^\mu - q_1^\mu) \equiv \frac{1}{\sqrt{2}}\, dq^\mu$ fulfill

$$\left[\frac{1}{\sqrt{2}}\, dq^\mu, \frac{1}{\sqrt{2}}\, dq^\nu\right] = i\,Q^{\mu\nu}.$$

MAJOR RESULTS:

WITH

$$q_1^\mu = q^\mu \otimes I, \quad q_2^\mu = I \otimes q^\mu,$$

$[q_1, q_2] = 0$ and

$$[q_1^\mu, q_1^\nu] = [q_2^\mu, q_2^\nu] = i\,Q^{\mu\nu};$$

$$\frac{1}{\sqrt{2}}(q_2^\mu - q_1^\mu) \equiv \frac{1}{\sqrt{2}}\, dq^\mu \text{ fulfill}$$

$$\left[\frac{1}{\sqrt{2}} dq^\mu, \frac{1}{\sqrt{2}} dq^\nu\right] = i\,Q^{\mu\nu}.$$

MAJOR RESULTS:

...

$\mu = 0$

- $\displaystyle\sum_{1 \le j < k \le 3} | dq_j \wedge dq_k |^2 \ge 1,$

  $\displaystyle\sum_{j=1}^{3} | dq_0 \wedge dq_j |^2 \geq 1;$

- $\sigma \left(| dq^1 \wedge dq^2 \wedge dq^3 | \right) = [0, +\infty);$

- $dq \wedge dq \wedge dq \wedge dq$ IS A NORMAL OPERATOR WITH PURE POINT SPECTRUM;

$$\sigma_p (dq \wedge \ldots \wedge dq) = \pm 2 + \mathbb{Z}\, ab + i (\mathbb{Z} a + \mathbb{Z} b)$$

$$= \pm 2 + \mathbb{Z} \sqrt{5} + $$
$$+ i \left(\mathbb{Z} \sqrt{5 - 2\sqrt{5}} + \mathbb{Z} \sqrt{5 + 2\sqrt{5}} \right);$$

so that

$$| dq \wedge \ldots \wedge dq | \geq \sqrt{5} - 2 .$$

SPECTRUM [illegible] $\cong \Lambda_2(\varepsilon)$! $a^2 + b^2 = (ab)^2 = 5$

- $\sum_{\mu=0}^{3} |dq^\mu|^2 \geq 4$;

- $\sum_{1 \leq j < k \leq 3} |dq_j \wedge dq_k|^2 \geq 1$,

  $\sum_{j=1}^{3} |dq_0 \wedge dq_j|^2 \geq 1$;

- $\sigma\left(|dq^1 \wedge dq^2 \wedge dq^3|\right) = [0, +\infty)$;

- $dq \wedge dq \wedge dq \wedge dq$ IS A NORMAL OPERATOR WITH PURE POINT SPECTRUM;

$$\sigma_\rho(dq \wedge \dots \wedge dq) = \pm 2 + \mathbb{Z}\,ab + i(\mathbb{Z}a + \mathbb{Z}c)$$

$$= \pm 2 + \mathbb{Z}\sqrt{5} +$$

$$+ i\left(\mathbb{Z}\sqrt{5 - 2\sqrt{5}} + \mathbb{Z}\sqrt{5 + 2\sqrt{5}}\right);$$

so that

$$|dq \wedge \dots \wedge dq| \geq \sqrt{5} - 2$$

- $\sum_{\mu=0}^{3} |dq^\mu|^2 \geq 4$;

- $\sum_{1\leq j<k\leq 3} |dq_j \wedge dq_k|^2 \geq 1$,

  $\sum_{j=1}^{3} |dq_0 \wedge dq_j|^2 \geq 1$;

- $\sigma(|dq^1 \wedge dq^2 \wedge dq^3|) = [0, +\infty)$;

- $dq \wedge dq \wedge dq \wedge dq$ IS A

  NORMAL OPERATOR WITH

  PURE POINT SPECTRUM;

$\sigma_\varphi(dq \wedge \dots \wedge dq) = \pm 2 + \mathbb{Z}ab + i(\mathbb{Z}a + \mathbb{Z}b)$

$= \pm 2 + \mathbb{Z}\sqrt{5} +$

$+ i(\mathbb{Z}\sqrt{5-2\sqrt{5}} + \mathbb{Z}\sqrt{5+2\sqrt{5}})$;

so that

$|dq \wedge \dots \wedge dq| \geq \sqrt{5} - 2$.

NOTE $\Lambda_{\mathbb{Z}}(\mathcal{E})$ has

two alg. structures:

- that of Ω-bimod. tensor alg
- that of $\bigoplus_n \mathcal{E}^{\otimes_{\mathbb{Z}} n}$;

$dq \wedge .. \wedge dq$ has to be computed with the first prod; it produces elements whose norms spectra .. are evaluated in the (C*-completion of the) second.

Area: $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k$$

· that of $\bigoplus_n \mathcal{E}^{\otimes_{\mathcal{Z}} n}$;

$dq \wedge \ldots \wedge dq$ has to be computed with the first prod; it produces elements whose norms spectra .. are evaluated in the ((C^*-completion of the) second.

Area : $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$\text{s.a.} \quad -i\, Q^{jk}$$

· that of $\bigoplus_n \mathcal{E}^{\otimes_Z n}$;

$dq \wedge \ldots \wedge dq$ has to be computed with the first prod; it produces elements whose nonzero spectra .. are evaluated in the ((C^*-completion of the) second.

Area: $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$5. a \quad - i \; Q^{jk}$$

· [illegible] dq

· that of $\bigoplus_n \mathcal{E}^{\otimes_{\mathcal{Z}} n}$;

$dq \wedge \ldots \wedge dq$ has to be computed with the first prod; it produces elements whose many spectra .. are evaluated in the (C*-completion of the) second.

Area : $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$\text{s.a.} \quad -i\, Q^{jk}$$

with the first prod; it produces elements whose norm spectra ... are evaluated in the (C^*-completion of the) second.

Area:

$j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$\text{s.a.} \quad -i\, Q^{jk}$$

...

hence, as an operator on

$$\mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E},$$

$$|dq^j \wedge dq^k|^2 = (\ldots)^2 + (Q^{jk})^2$$

$$\geq (Q^{jk})^2,$$

and

$$\sum_{\substack{j,k,l \text{ cyclic} \\ \text{perm } \{1,2,3\}}} |dq^j \wedge dq^k|^2 \geq \vec{m}^2 \geq I;$$

similarly for the "timelike area"

$$\sum_{j=1}^{3} |dq^0 \wedge dq^j|^2 \geq \vec{e}^2 \geq I.$$

$$dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3 = \ldots \, dq^j \, dq^k \, dq^l =$$

$$|dq^j \wedge dq^k| = (\ldots)^2 + (Q^{jk})^2$$

$$\geq (Q^{jk})^2,$$

and

$$\sum_{\substack{j,k,l \text{ cyclic} \\ \text{perm of } 1,2,3}} |dq^j \wedge dq^k|^2 \geq \vec{m}^2 \geq I;$$

Similarly for the "timelike area"

$$\sum_{j=1}^{3} |dq^0 \wedge dq^j|^2 \geq \vec{e}^2 \geq I.$$

Space Volume

$$d\vec{q} \wedge d\vec{q} \wedge d\vec{q} = \varepsilon_{jkl}\, dq^j dq^k dq^l =$$

$$= \varepsilon_{jkl} \left([illegible] \right)$$

The Spacetime Volume Operator

$dq \wedge dq \wedge dq \wedge dq =$

$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$

$\varepsilon_{\mu\nu\lambda\rho} (I \otimes q^\mu - q^\mu \otimes I) \ldots (I \otimes q^\rho - q^\rho \otimes I)$

(products in $\Lambda_{\underline{Z}}[\mathcal{E}]$!).

$=$ (no pair of q's in same place) $+$ S. a.

A (two pairs of q's in same places) $+$ S. a.

(one pair of q's in same place) skew adjoint

B

$\Rightarrow \varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$

$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$

$= -\frac{1}{4}\, Q \wedge Q = -2\eta, \underline{\epsilon^{\underline{z}}},$

$\eta = \pm I$ on $\Sigma_{\pm}$.

The Spacetime Volume Operator

$$dq \wedge dq \wedge dq \wedge dq =$$

$$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$$

$$= \varepsilon_{\mu\nu\lambda\rho} (I \otimes q^\mu - q^\mu \otimes I) \dots (I \otimes q^\rho - q^\rho \otimes I)$$

(products in $\Lambda_{\mathcal{Z}}(\mathcal{E})$!).

= (no pair of q's in same place) + S. a.

A (two pairs of q's in same places) + S a.

(one pair of q's in same place) skew adjoint

B

$$\to \varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$$

$$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$$

$$= -\frac{1}{4}\, Q \wedge Q = -2\eta, \underline{\epsilon \tfrac{\lambda}{2}},$$

$$\eta = \pm I \text{ on } \Sigma_\pm .$$

$dq \wedge dq \wedge dq \wedge dq =$

$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$

$= \varepsilon_{\mu\nu\lambda\rho} \left(I \otimes q^\mu - q^\mu \otimes I\right) \dots \left(I \otimes q^\rho - q^\rho \otimes I\right)$

(products in $\Lambda_{Z}[\mathcal{E}]$!!).

$\doteq$ (no pair of q's in same place) + s.a.

$A \cong$ (two pairs of q's in same places) + s.a.

(one pair of q's in same place) skew adjoint

$\cong$

$\to \varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$

$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$

$= -\frac{1}{4}\, Q \wedge Q = -2\eta, \in Z,$

$= \pm I$ on $\Sigma_\pm$.

$$dq \wedge dq \wedge dq \wedge dq =$$

$$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$$

$$= \varepsilon_{\mu\nu\lambda\rho} (I \otimes q^\mu - q^\mu \otimes I) \ldots (I \otimes q^\rho - q^\rho \otimes I)$$

(product in $\Lambda_{\underline{z}}[E]$!).

$=$ (no pair of q's in same place) $+$ S. Q.

A $\parallel\!\!\parallel$ (two pairs of q's in same places) $+$ S. Q.

$\parallel\!\!\parallel$ (one pair of q's in same place) skew adjoint

B

$$\to \varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$$

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$= \epsilon_{\mu\nu\lambda\rho} (I\otimes q^\mu - q^\mu \otimes I) \dots (I\otimes q^\rho - q^\rho \otimes I)$

(products in $\Lambda_{\mathcal{Z}}(\mathcal{E})$!!).

$=$ (no pair of q's in same place) + S.Q.

A $\equiv$ (two pairs of q's in same places) + Σ a.

$\equiv$ (one pair of q's in same place) skew adjoint

B

$\longrightarrow \epsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$

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$$= \varepsilon_{\mu\nu\lambda\rho} (I \otimes q^\mu - q^\mu \otimes I) \ldots (I \otimes q^\rho - q^\rho \otimes I)$$

(product in $\Lambda_{\mathcal{Z}}(\mathcal{E})$!).

= (no pair of q's in same place) + S. a.

A (two pairs of q's in same places) + S. a.

(one pair of q's in same place) skew adjoint

B

$$\longrightarrow \varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$$

$$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$$

$$= -\frac{1}{4}\, Q \wedge Q \quad = -2\gamma, \ \underline{\in \mathcal{Z}},$$

$$\gamma = \pm I \ \text{ on } \Sigma_\pm .$$

$dq \wedge dq \wedge dq \wedge dq = A - 2\eta + i B,$

$$A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i = \sum_{j=1}^{5} (-1)^j A_j ,$$

$$B = \frac{1}{2} \sum_{i<j} (-1)^{i-j} Q \wedge q_i \wedge q_j = \frac{1}{2} \sum (-1)^{i-j} B_{ij} \ldots$$

Compute:

$$[q_k^\mu, B_{ij}] = \delta_{ik} \underbrace{Q^{\mu} \wedge Q \wedge q_j} - \delta_{jk} Q^{\mu} \wedge Q \wedge q_i$$

$$\varepsilon_{\nu\lambda\rho\sigma} Q^{\mu\nu} Q^{\lambda\rho} q_j^\sigma =$$

$$= \quad 2 \left(Q^{\mu\nu} (*Q)_{\nu\sigma} \right) q_j^\sigma$$

But antisymmetry + Centrality of $Q \Rightarrow$

$$Q^{\mu\nu} (*Q)_{\nu\sigma} = \frac{1}{4} Q^{\nu\rho} (*Q)_{\nu\rho} \cdot \delta_{\mu\sigma}$$

$$= \eta \, \delta_{\mu\sigma} ; \quad \text{hence}$$

$$[q_k^\mu, B_{ij}] = \eta (\delta_{ik} q_j^\mu - \delta_{jk} q_i^\mu)$$

$$dq \wedge dq \wedge dq \wedge dq = A - 2\eta + iB,$$

$$A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i \equiv \sum_{j=1}^{5} (-1)^j A_j,$$

$$B = \frac{1}{2} \sum_{i<j} (-1)^{i-j}\, Q \wedge q_i \wedge q_j \equiv \frac{1}{2} \sum (-1)^{i-j} B_{ij}.$$

Compute:

$$[q_k^\mu, B_{ij}] = \delta_{ik} \underbrace{Q^{\mu\cdot} \wedge Q \wedge q_j}_{} - \delta_{jk} Q^{\mu\cdot} \wedge Q \wedge q_i$$

$$\varepsilon_{\nu\lambda\rho\sigma}\, Q^{\mu\nu} Q^{\lambda\rho} q_j^\sigma =$$

$$= \quad 2\left(Q^{\mu\nu}(*Q)_{\nu\sigma}\right) q_j^\sigma$$

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$$[q_k^\mu, B_{ij}] = \eta\left(\delta_{ik} q_j^\mu - \delta_{jk} q_i^\mu\right).$$

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Thus $\mathrm{Ad}\, B_{ij}$ act on q_k's as ($\mathfrak{z}\cdot$) Lie Algebra gen of $SO(5)$; by $*$,

$$(\mathrm{Ad}\, B)\left(\sum_{j=1}^{5} q_j\right) = 0$$

hence $\mathrm{Ad}\, B$ acts as a generator of 1 par sbgp in $\mathcal{N} \equiv$ stabilizer in $SO(5)$ of $(1,1,1,1,1)$.

Now $A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i =$

$$= \det \begin{pmatrix} I & q_1^0 & q_1^1 & \cdots & q_1^3 \\ I & q_2^0 & q_2^1 & \cdots & q_2^3 \\ - & - & - & - & - \\ I & q_5^0 & q_5^1 & \cdots & q_5^3 \end{pmatrix} =$$

$= \det R \cdot (\quad)$

hence Ad's acts as a generator

of 1par sbgp in $\mathcal{N}$ $\equiv$ stabilizer

in $SO(5)$ of $(1,1,1,1,1)$.

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$= \det R \cdot \left(\quad \right)$

and, if $R \in \mathcal{N}$:

hence $A \wedge B$ acts as a generator

of 1 par sbgp in $\mathcal{N}$ $\equiv$ stabilizer

in $SO(5)$ of $(1,1,1,1,1)$.

Now $A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i =$

$$= \det \begin{pmatrix} I & q_1^0 & q_1^1 & \cdots & q_1^3 \\ I & q_2^0 & q_2^1 & \cdots & q_2^3 \\ - & - & - & - & - \\ I & q_5^0 & q_5^1 & \cdots & q_5^3 \end{pmatrix} =$$

$$= \det R \cdot \left(\quad \right)$$

and, if $R \in \mathcal{N}$:

...

in $SO(5)$ of $(1,1,1,1,1)$.

Now $A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i =$

$$= \det \begin{pmatrix} I & q_1^0 & q_1^1 & \cdots & q_1^3 \\ I & q_2^0 & q_2^1 & \cdots & q_2^3 \\ - & - & - & - & - \\ I & q_5^0 & q_5^1 & \cdots & q_5^3 \end{pmatrix} =$$

$$= \det R \cdot \left(\quad \right)$$

and, if $R \in N$:

28'

$$= \det \begin{pmatrix} I & q'^{0}_{1} & q'^{1}_{1} & \cdots & q'^{3}_{1} \\ I & q'^{0}_{2} & q'^{1}_{2} & \cdots & q'^{3}_{2} \\ - & - & - & - & - \\ I & q'^{0}_{5} & q'^{1}_{5} & \cdots & q'^{3}_{5} \end{pmatrix}$$

if $\quad q'^{a}_{j} = R^{a}{}_{m}\, q^{m}_{j},$

$R \in N,$

and for any generator D of a 1-param. subgp in N,

$\mathrm{AdD}(A) = 0.$

25'

$$= \det \begin{pmatrix} I & q_1^{'0} & q_1^{'1} & \cdots & q_1^{'3} \\ I & q_2^{'0} & q_2^{'1} & \cdots & q_2^{'3} \\ - & - & - & - & - \\ I & q_5^{'0} & q_5^{'1} & \cdots & q_5^{'3} \end{pmatrix},$$

$$q_j^{'k} = R^{jk} q_k,$$

$$R \in N,$$

and for any generator D of a 1-par subgp in N,

$$A d D J(A) = 0.$$

$$= \det \begin{pmatrix} I & q_1'^0 & q_1'^1 & \cdots & q_1'^3 \\ I & q_2'^0 & q_2'^1 & \cdots & q_2'^3 \\ - & - & - & - & - \\ I & q_5'^0 & q_5'^1 & \cdots & q_5'^3 \end{pmatrix},$$

if

$$q_j'^\mu = R^{jk} q_k^\mu,$$

$$R \in N,$$

and for any generator D

of a 1-par sgp in N,

$$(A, D)(A) = 0.$$

$$= \det \begin{pmatrix} I & q_1'^0 & q_1'^1 & \cdots & q_1'^3 \\ I & q_2'^0 & q_2'^1 & \cdots & q_2'^3 \\ - & - & - & - & - \\ I & q_5'^0 & q_5'^1 & \cdots & q_5'^3 \end{pmatrix},$$

if

$$q_j'^\mu = R^{jk} q_k^\mu,$$

$$R \in N,$$

and for any generator D

of a 1-par sgp in N,

$$A \wedge D(A) = 0.$$

Since $[A, B] = 0$ and

$$dq\wedge dq\wedge dq\wedge dq = A - 2z + iB$$

is NORMAL (z is central!), and

$$|dq\wedge dq\wedge dq\wedge dq|^2 = (A-2z)^2 + B^2$$
$$\geq (A-2z)^2 .$$

Now as a field of operators on Σ, by LORENTZ invariance, $dq\wedge dq\wedge dq\wedge dq$ is CONSTANT and of opposite signs on $\Sigma_\pm$.
It suffices to compute at $\sigma \in \Sigma$, $\sigma = (\vec{e}, \vec{m})$, $\vec{e} = \vec{m} = (1,0,0)$ as before: q^j_r act on $H^{\otimes 5} \otimes H^{\otimes 5}$; if we let q_j, p_j denote Schroedinger's q, p acting on the j-th place in $H^{\otimes 5}$, we have

Hence $[A, B] = 0$ and

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...

$$q^j_\mu = \begin{pmatrix} q_j \otimes I \\ p_j \otimes I \\ I \otimes q_j \\ I \otimes p_j \end{pmatrix},$$

and

$$A = \det \begin{pmatrix} I & q_1 \otimes I & p_1 \otimes I & I \otimes q_1 & I \otimes p_1 \\ I & q_2 \otimes I & p_2 \otimes I & I \otimes q_2 & I \otimes p_2 \\ - & - & - & - & - \\ I & q_5 \otimes I & p_5 \otimes I & I \otimes q_5 & I \otimes p_5 \end{pmatrix}$$

$$= \frac{1}{4} \sum_1^5 \varepsilon_{ijklm} M_{jk} \otimes M_{lm},$$

with $M_{jk} = q_j p_k - q_k p_j$ gen. of rotations in (j,k) plane in $SO(5)$,

$$[M_{jk}, M_{lm}] = i\left(\delta_{jl} M_{km} - \delta_{jm} M_{kl} + \right.$$

UNCHANGED IF WE ROTATE TO ANOTHER ORTHONORMAL BASIS $\xi_1, \dots \xi_5$ IN $\mathbb{R}^5$: choosing

$$\xi_5 = \frac{1}{\sqrt{5}}(1,1,1,1,1)$$

we get $\longrightarrow$ 30'

$$A = \frac{1}{4}\sqrt{5}\sum_1^4 \varepsilon_{ijkl} M'_{ij} \otimes M'_{kl} .$$

Now with

$$\vec{B} \equiv (M'_{23}, M'_{31}, M'_{12}),$$

$$\vec{D} \equiv (M'_{14}, M'_{24}, M'_{34}),$$

we have that

$$\vec{L}^{(\pm)} \equiv \frac{1}{2}(\vec{B} \pm \vec{D})$$

are mutually commuting generators of SU(2) and

$$A = 2\sqrt{5}\left(\vec{L}^{(+)} \otimes \vec{L}^{(+)} - \vec{L}^{(-)} \otimes \vec{L}^{(-)}\right)$$

30'

$$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \to \sqrt{5}\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix},$$

$$A = \sqrt{5}\,\det\begin{pmatrix} 0 & q_1'\otimes I & p_1'\otimes I & I\otimes q_1' & I\otimes p_1' \\ 0 & q_2'\otimes I & p_2'\otimes I & I\otimes q_2' & I\otimes p_2' \\ 0 & - & - & - & \\ 0 & - & - & - & \\ 1 & q_5'\otimes I & p_5'\otimes I & I\otimes q_5' & I\otimes p_5' \end{pmatrix}$$

$$\det\begin{pmatrix} q_1'\otimes I & p_1'\otimes I & I\otimes q_1' & I\otimes p_1' \\ - & - & - & - \\ - & - & - & - \\ q_4'\otimes I & p_4'\otimes I & I\otimes q_4' & I\otimes p_4' \end{pmatrix}$$

$\sqrt{5}$ · det (minors with the first 2 ~~rows~~ columns)

- det (

30'

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \sqrt{5} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$A = \sqrt{5} \det \begin{pmatrix} 0 & q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \times p_1' \\ 0 & q_2' \otimes I & p_2' \otimes I & I \otimes q_2' & I \otimes p_2' \\ 0 & - - & & - & \\ 0 & - & - & - & - \\ I & q_5' \otimes I & p_5' \otimes I & I \otimes q_5' & I \otimes p_5' \end{pmatrix}$$

$$= \sqrt{5} \det \begin{pmatrix} q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \otimes p_1' \\ - - & - & - & - \\ - - & - & - & - \\ q_4' \otimes I & p_4' \otimes I & I \otimes q_4' & I \otimes p_4' \end{pmatrix}$$

$= \sqrt{5}$. det (minors with the first 2 ~~rows~~ columns)

- det (...

in $\mathbb{R}^5$: choosing

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$$A = \frac{1}{4}\sqrt{5}\sum_{1}^{4} \varepsilon_{ijkl} M'_{ij} \otimes M'_{kl}.$$

Now with

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are mutually commuting generators of SU(2) and

$$A = 2\sqrt{5}\left(\vec{L}^{(+)} \otimes \vec{L}^{(+)} - \vec{L}^{(-)} \otimes \vec{L}^{(-)}\right)$$

$$\vec{J} \otimes \vec{J} = \frac{1}{2}\{ (\vec{J} \otimes I + I \otimes \vec{J})^2 - \vec{J}^2 \otimes I - I \otimes \vec{J}^2 \}$$

by Clebsch-Gordon has eigenvalues

$$\frac{1}{2} \left(s(s+1) - u(u+1) - v(v+1) \right),$$

$$u, v \in \frac{1}{2} \mathbb{N}_0 = \{ 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots \}$$

$$s = u+v, \ u+v-1, \ \dots \ |u-v|;$$

keeping track of the fact that eigenvalues $s^+(s^++1)$ of $\vec{L}^{(+)2}$ and $s^-(s^-+1)$ of $\vec{L}^{(-)2}$ arising from reps of $SO(4)$ must be with s^+, s^- simultaneously integers or half integers, we see that

$$|dq \wedge dq \wedge dq \wedge dq| \geq |A - 2q| \geq \sqrt{5} - 2.$$

$$\sum_k |dq_j \wedge dq_k|^2 \geq 1,$$

$$\sum_k |dq^0 \wedge dq^k|^2 \geq 1$$

- $dq^1 \wedge dq^2 \wedge dq^3$ is a NORMAL OPERATOR, SPECTRUM $= \mathbb{C}$

- $dq \wedge dq \wedge dq \wedge dq$ is a LORENTZ PSEUDOSCALAR, NORMAL OPERATOR, PURE POINT SPECTRUM $=$

$$\pm 2 + \mathbb{Z}\sqrt{5} + i\left(\mathbb{Z}\sqrt{5-2\sqrt{5}} + \mathbb{Z}\sqrt{5+2\sqrt{5}}\right).$$

dense in $\mathbb{R}$ ← K. FREDENHAGEN, S.D. IN PREP.

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