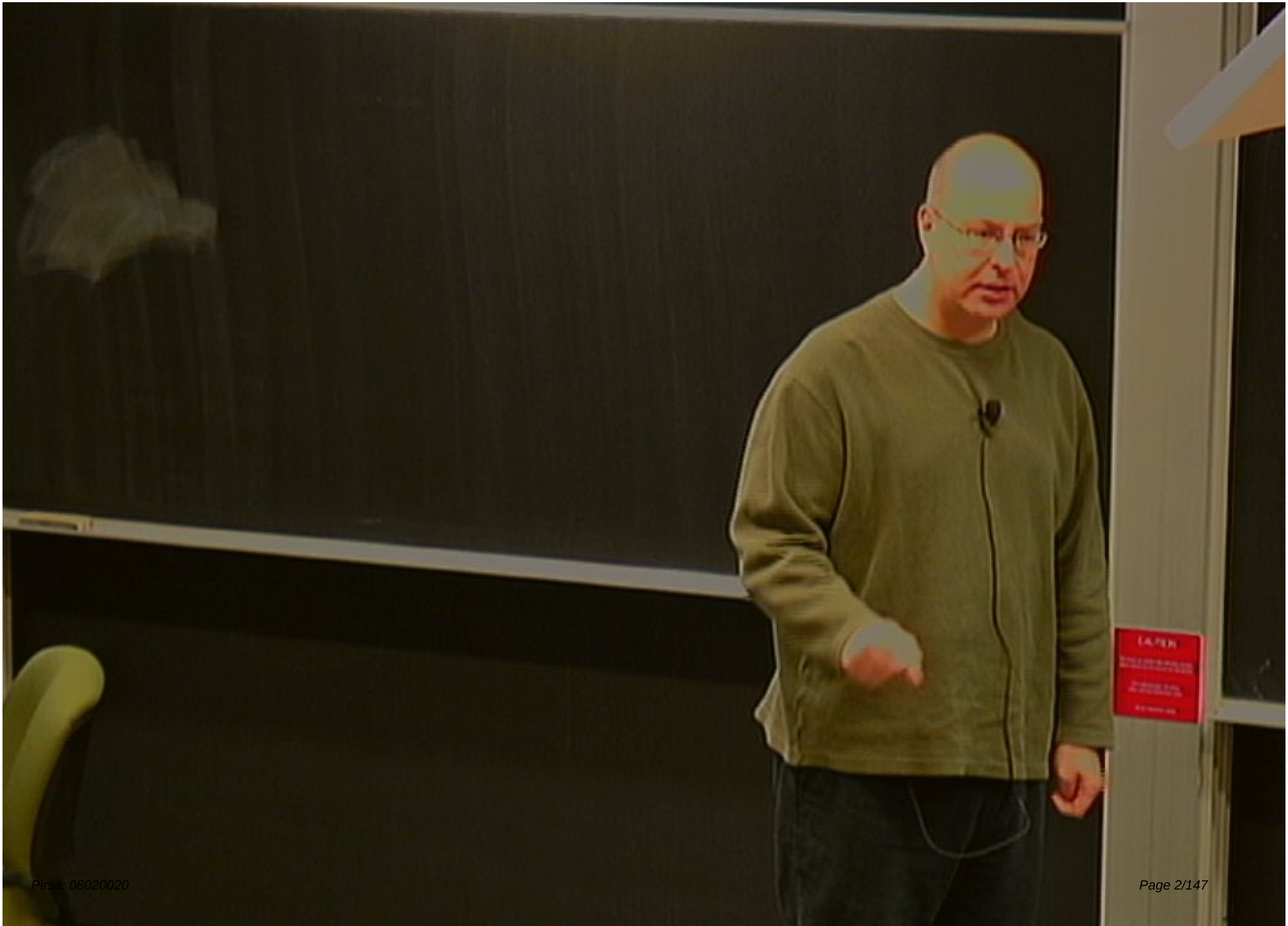
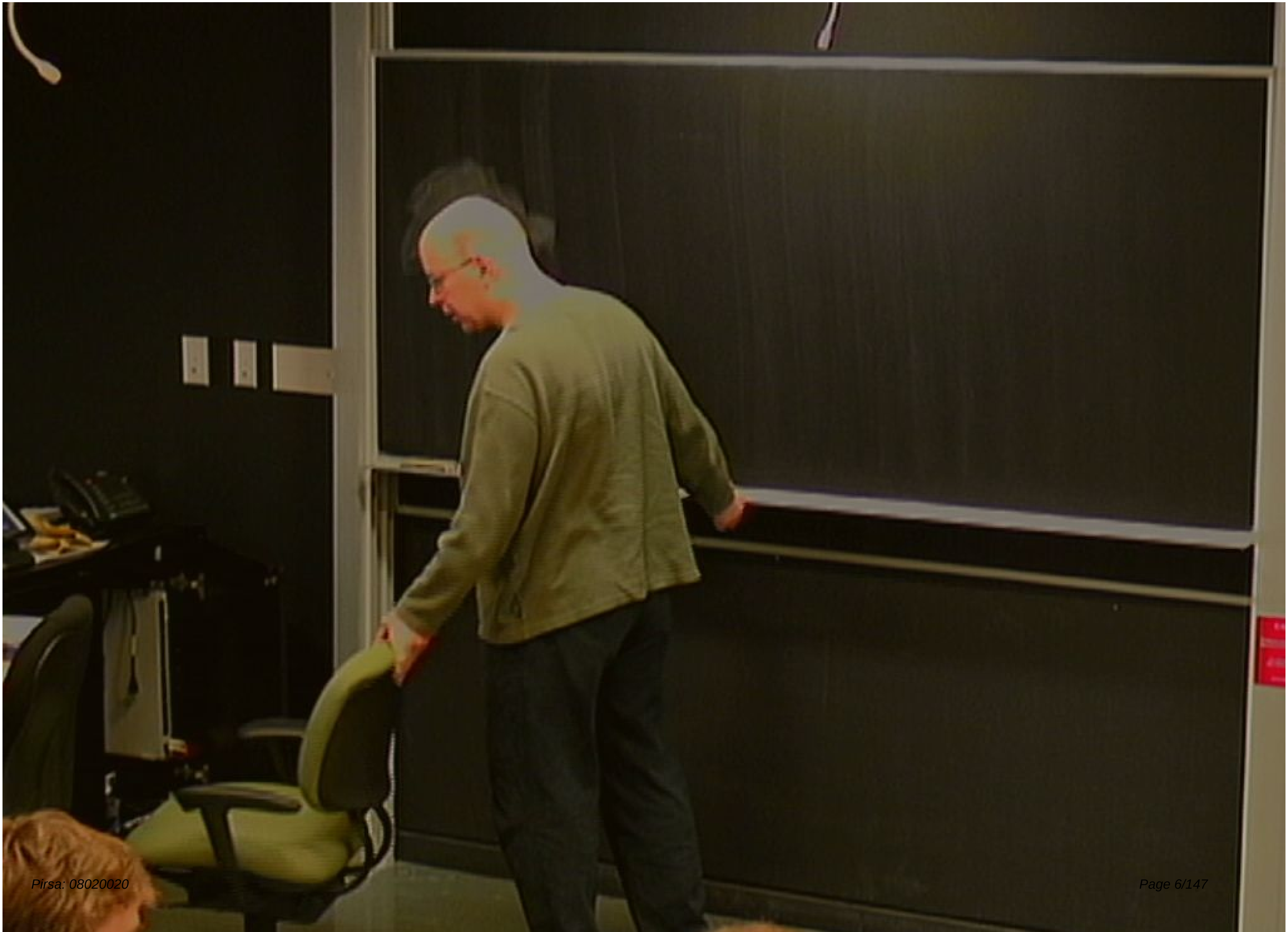
Title: Advanced General Relativity - Lecture 5B

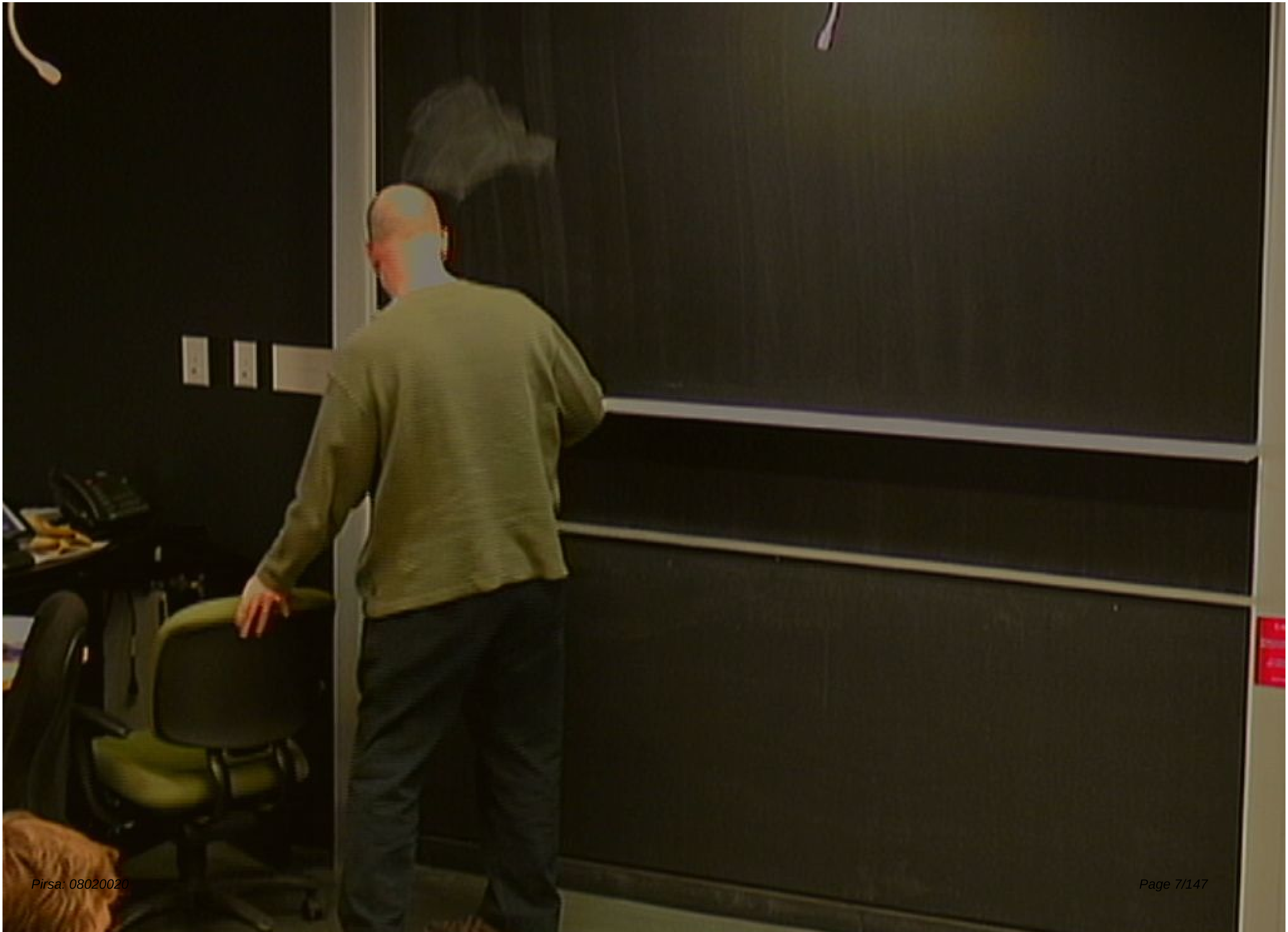
Date: Feb 06, 2008 04:00 PM

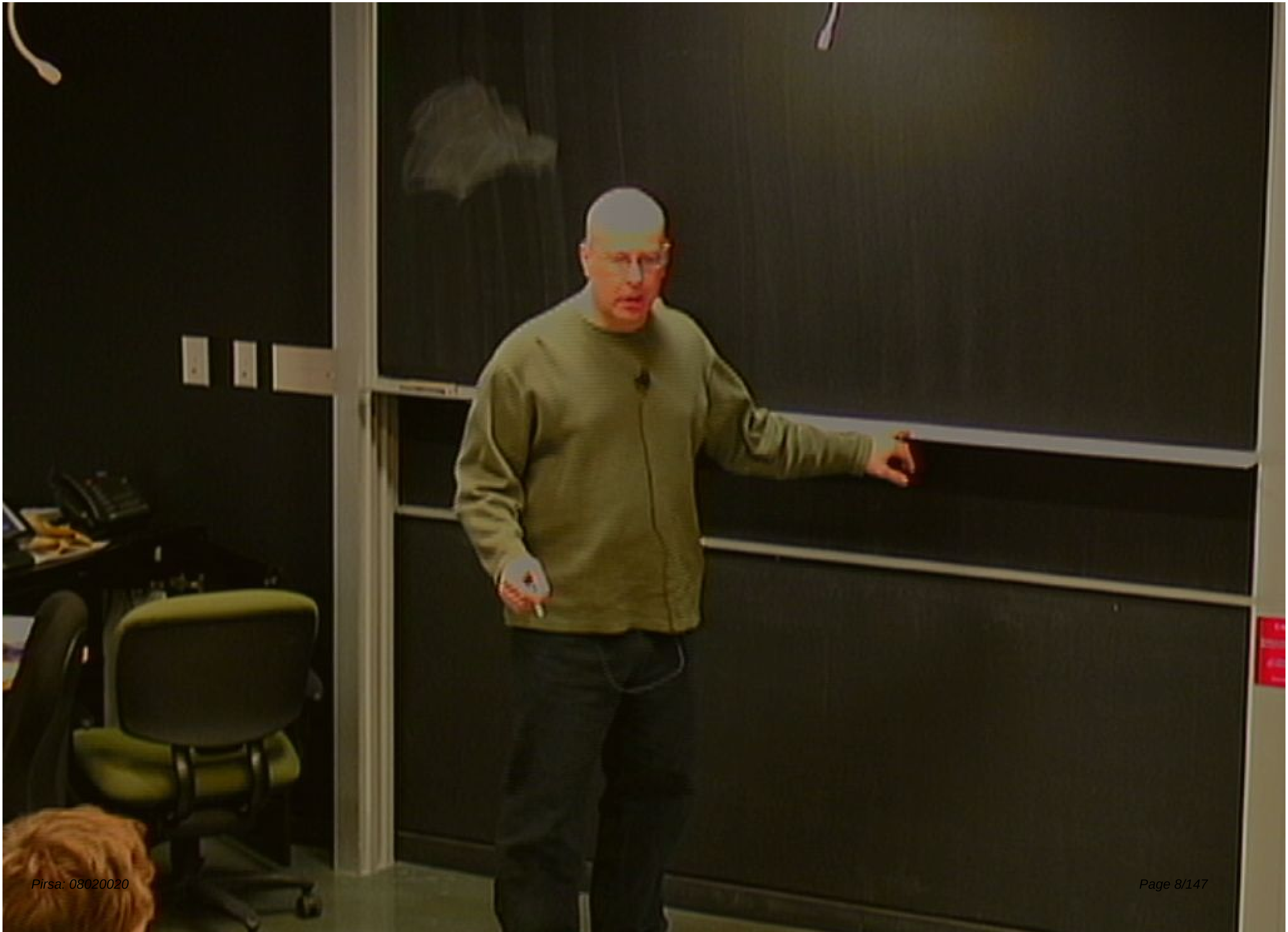
URL: http://pirsa.org/08020020

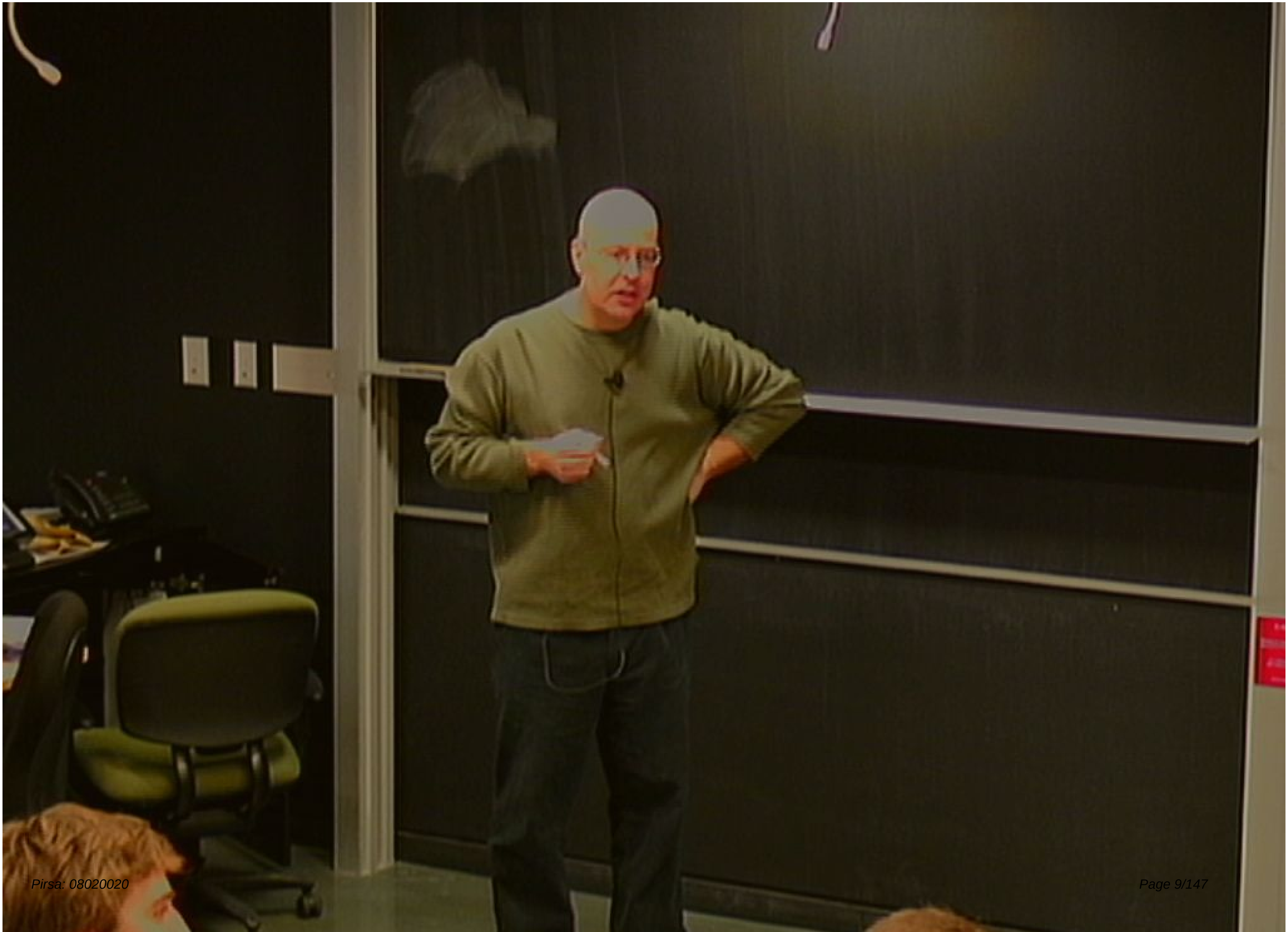
Abstract: Advanced General Relativity

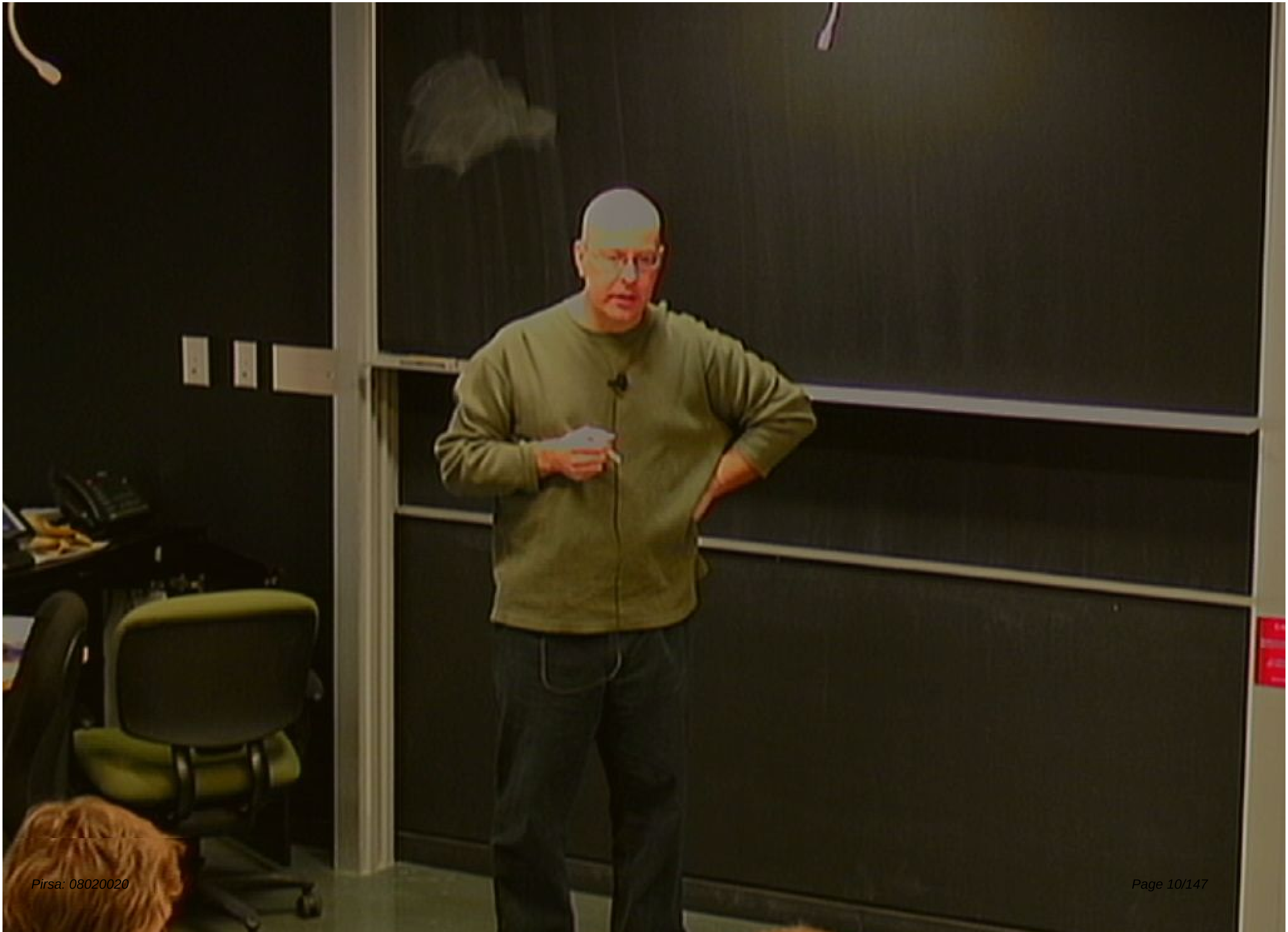


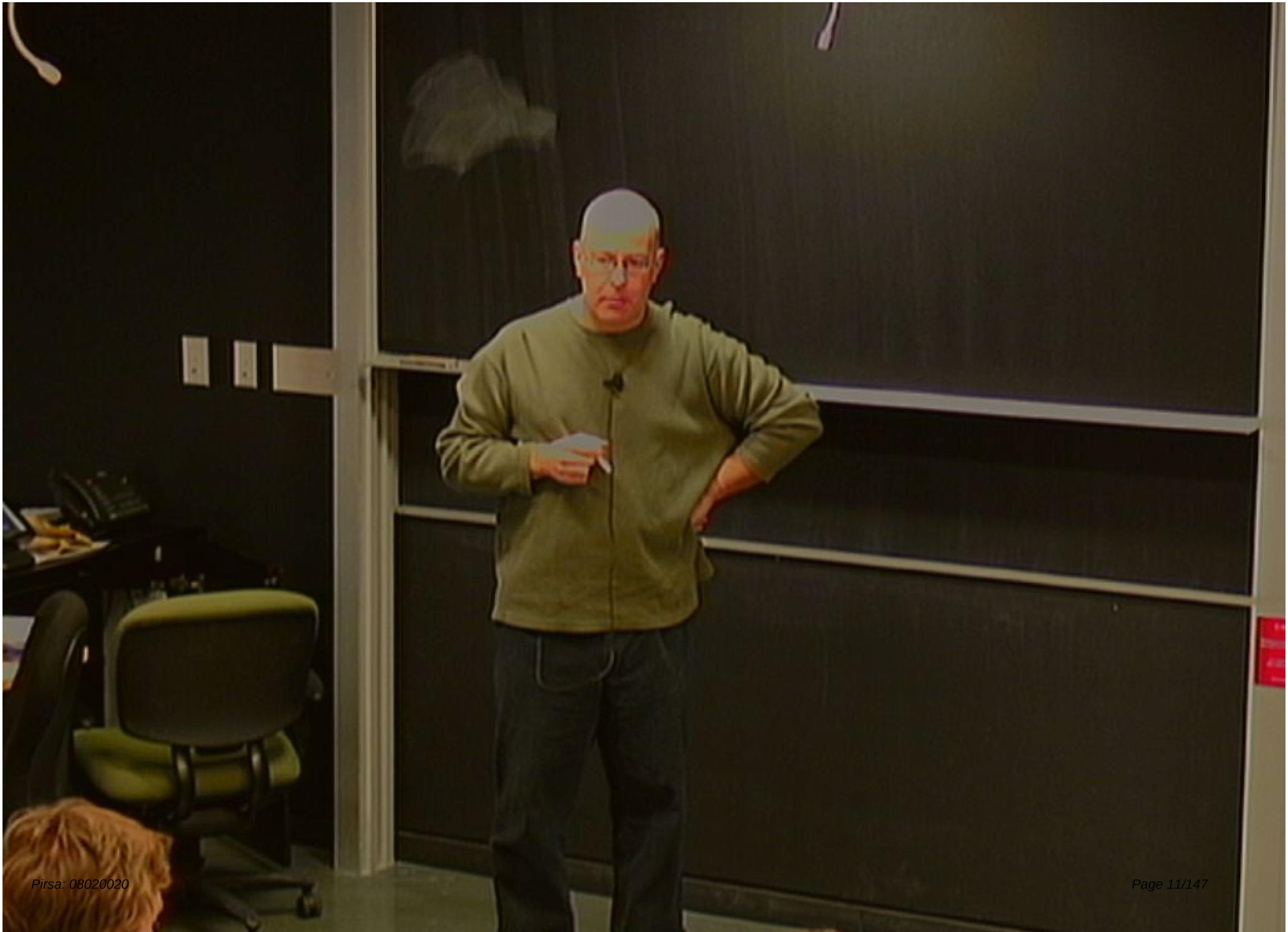


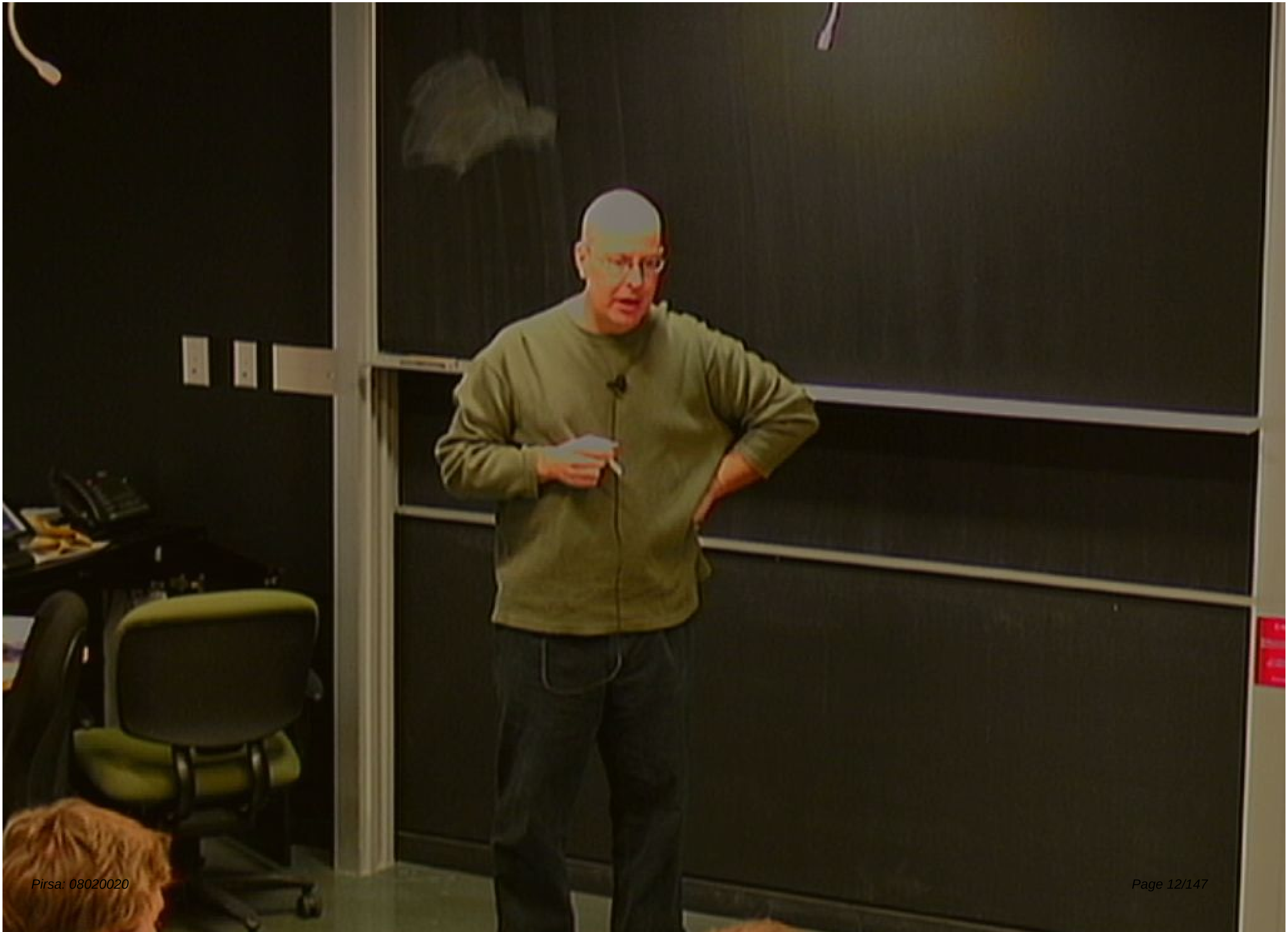


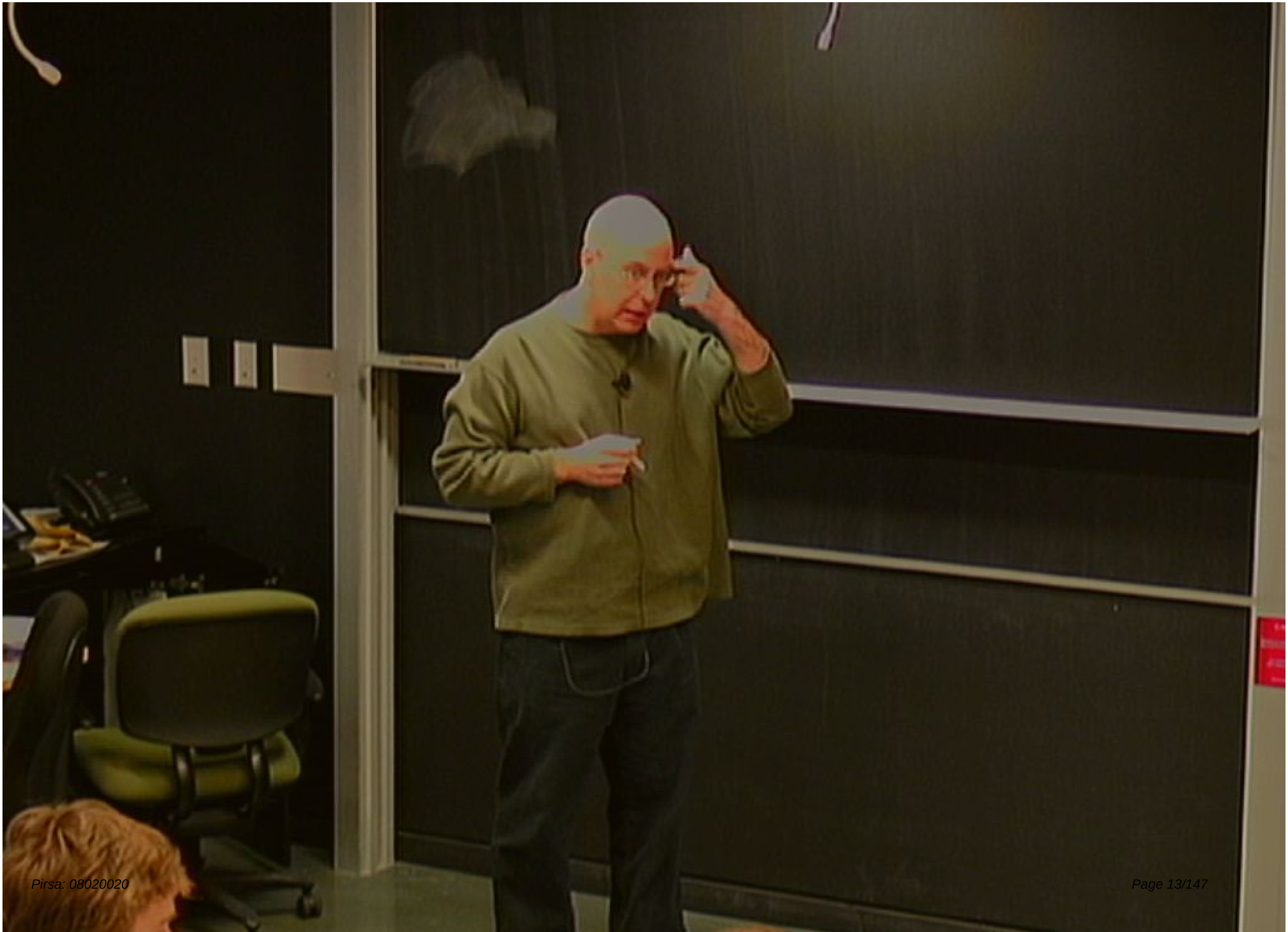


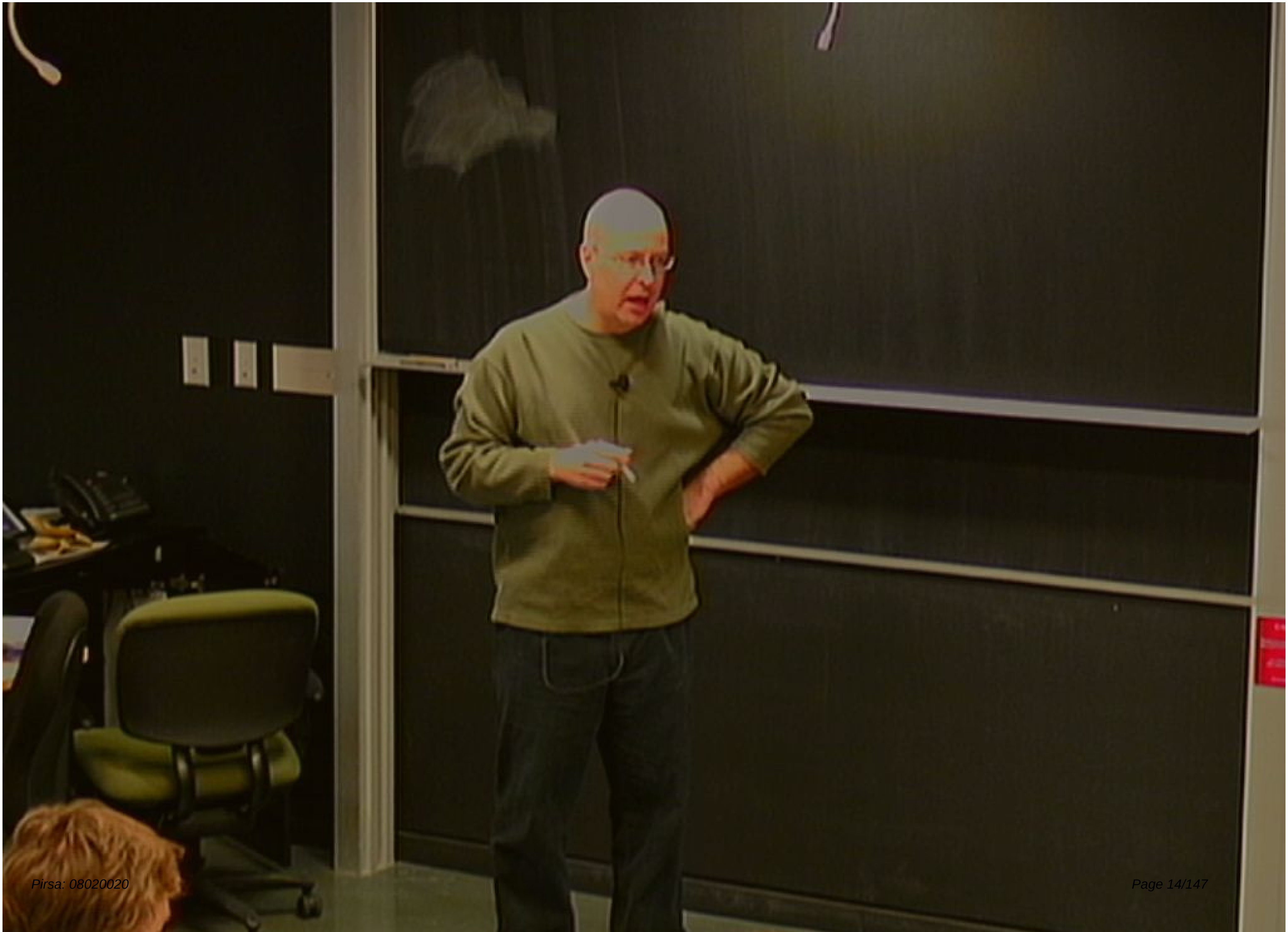


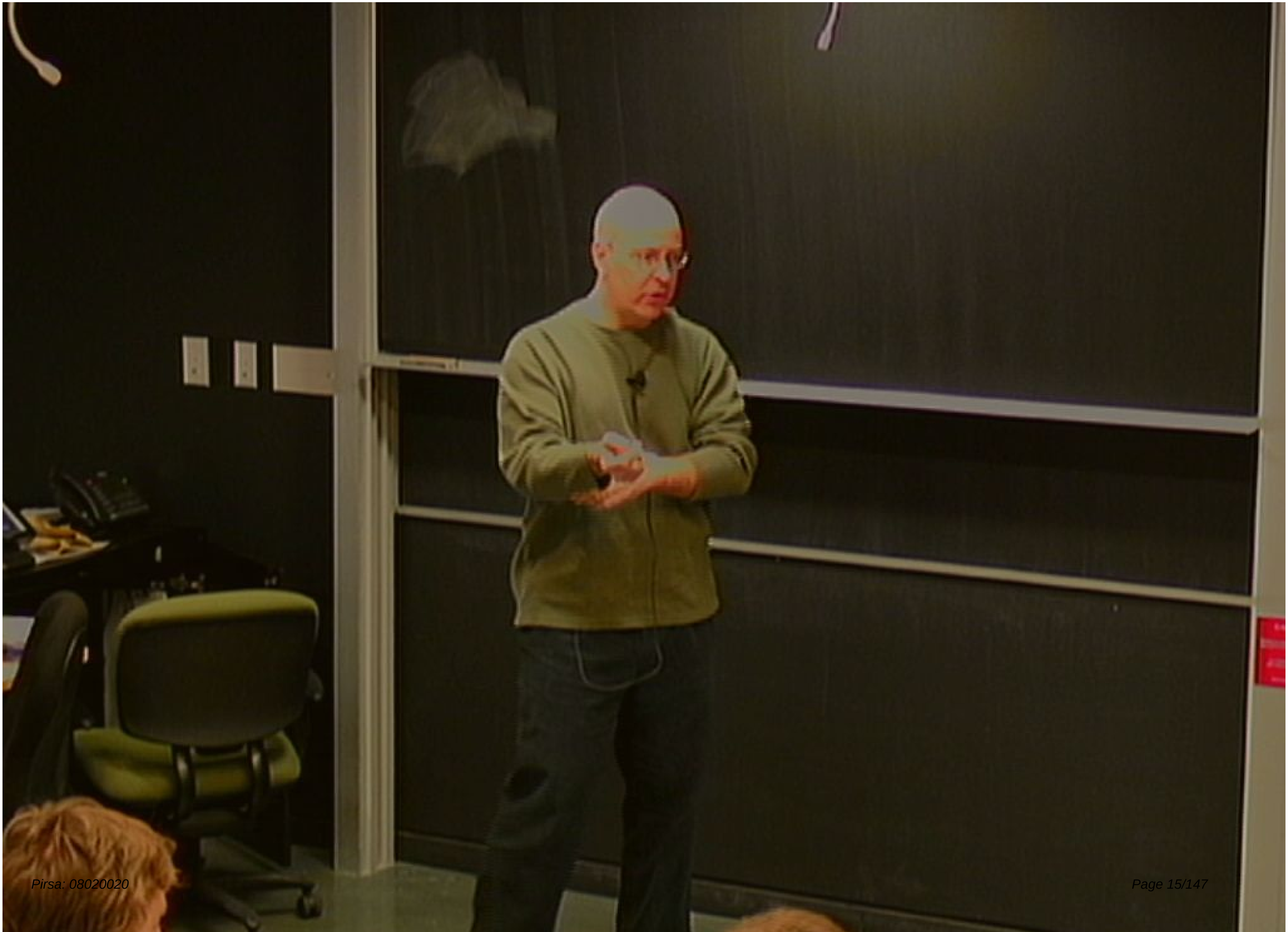


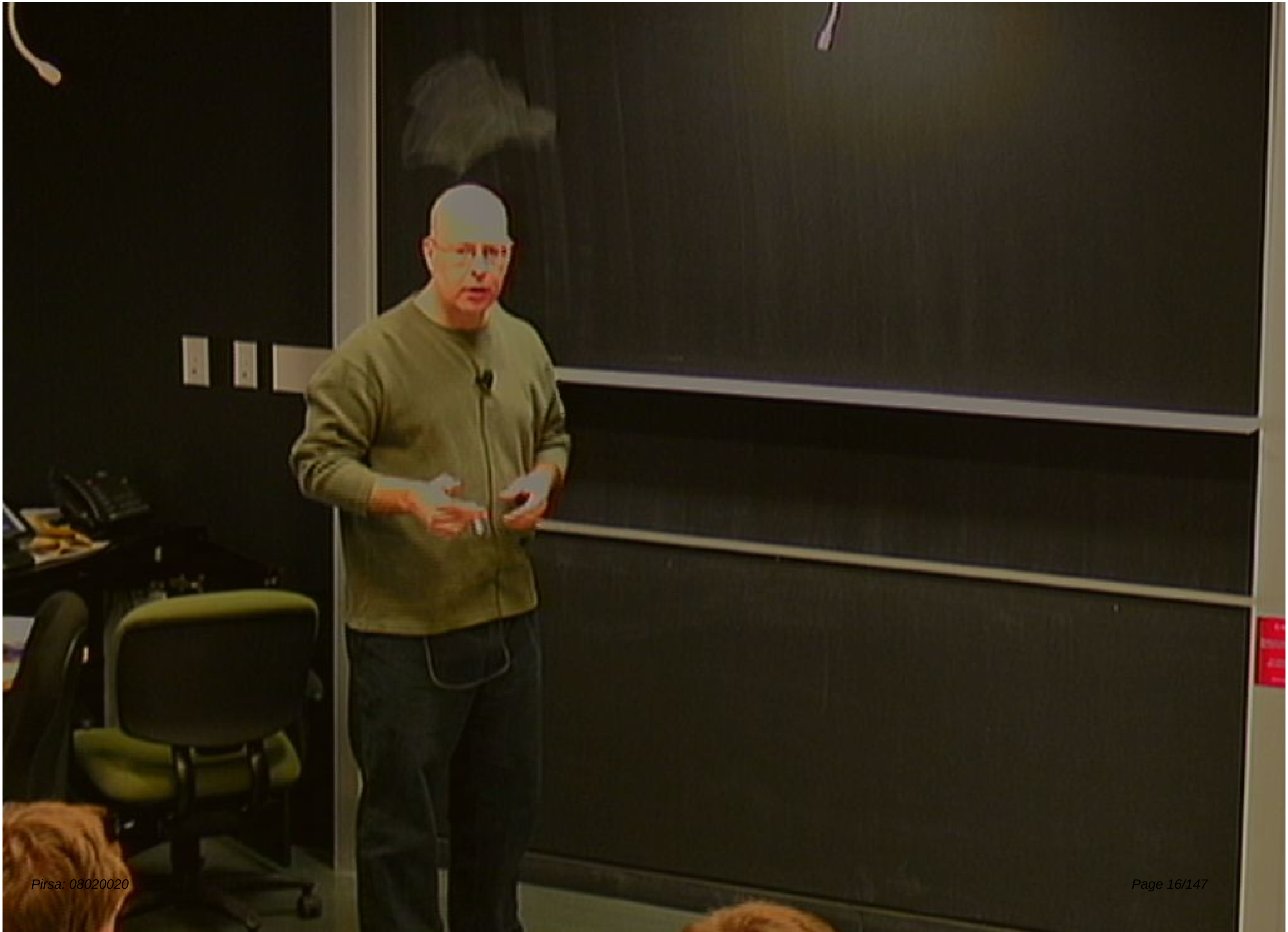


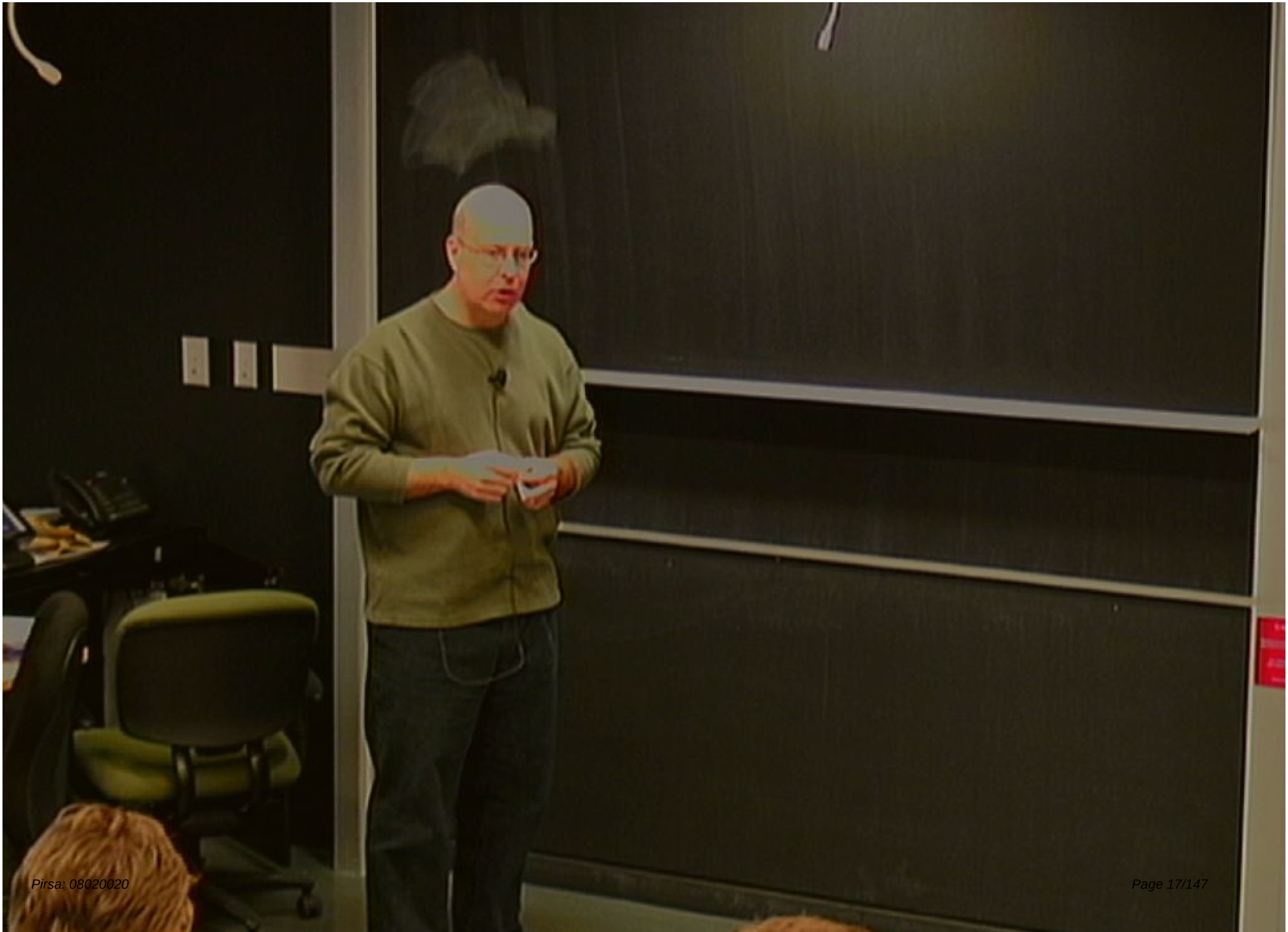


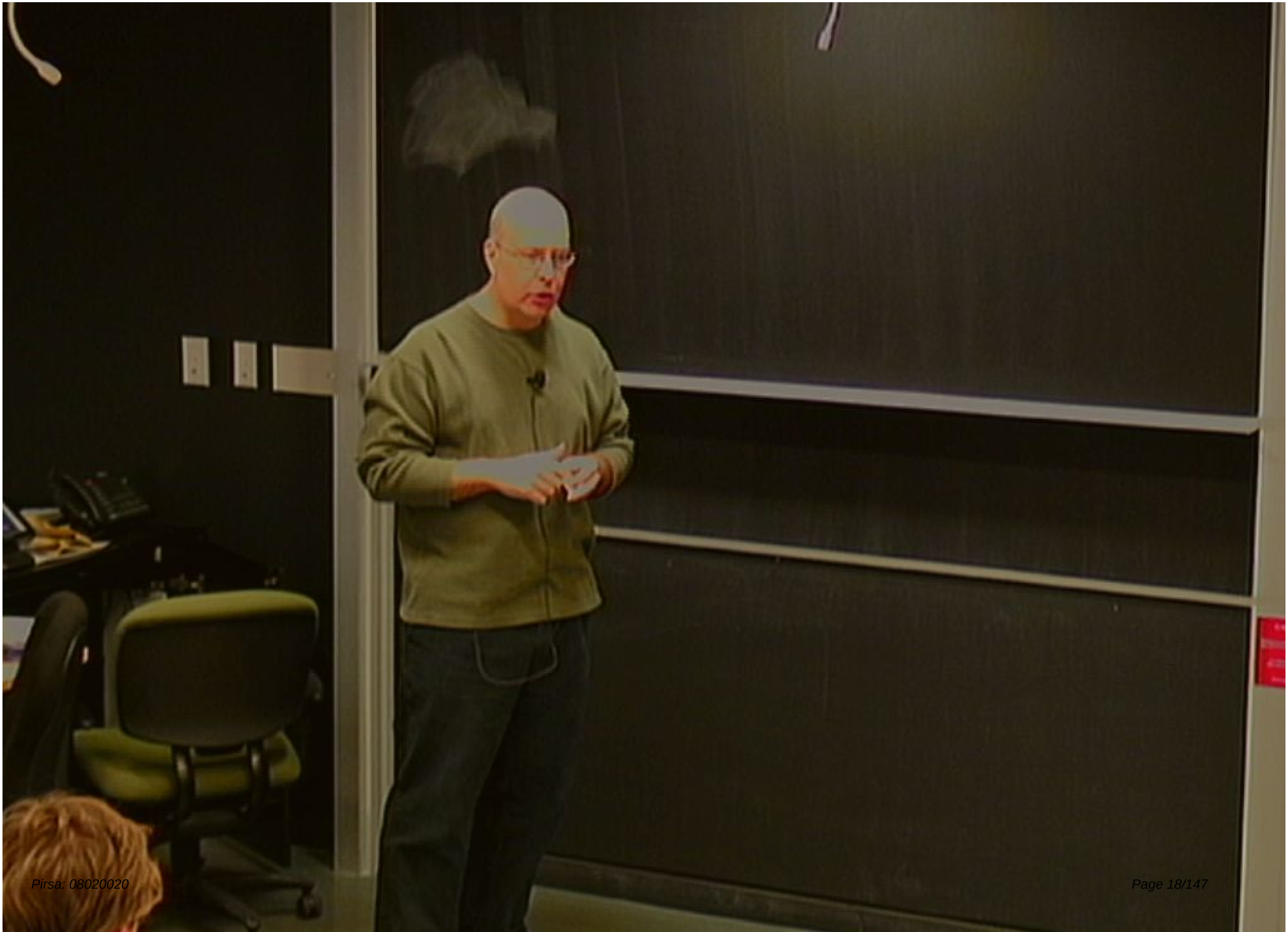


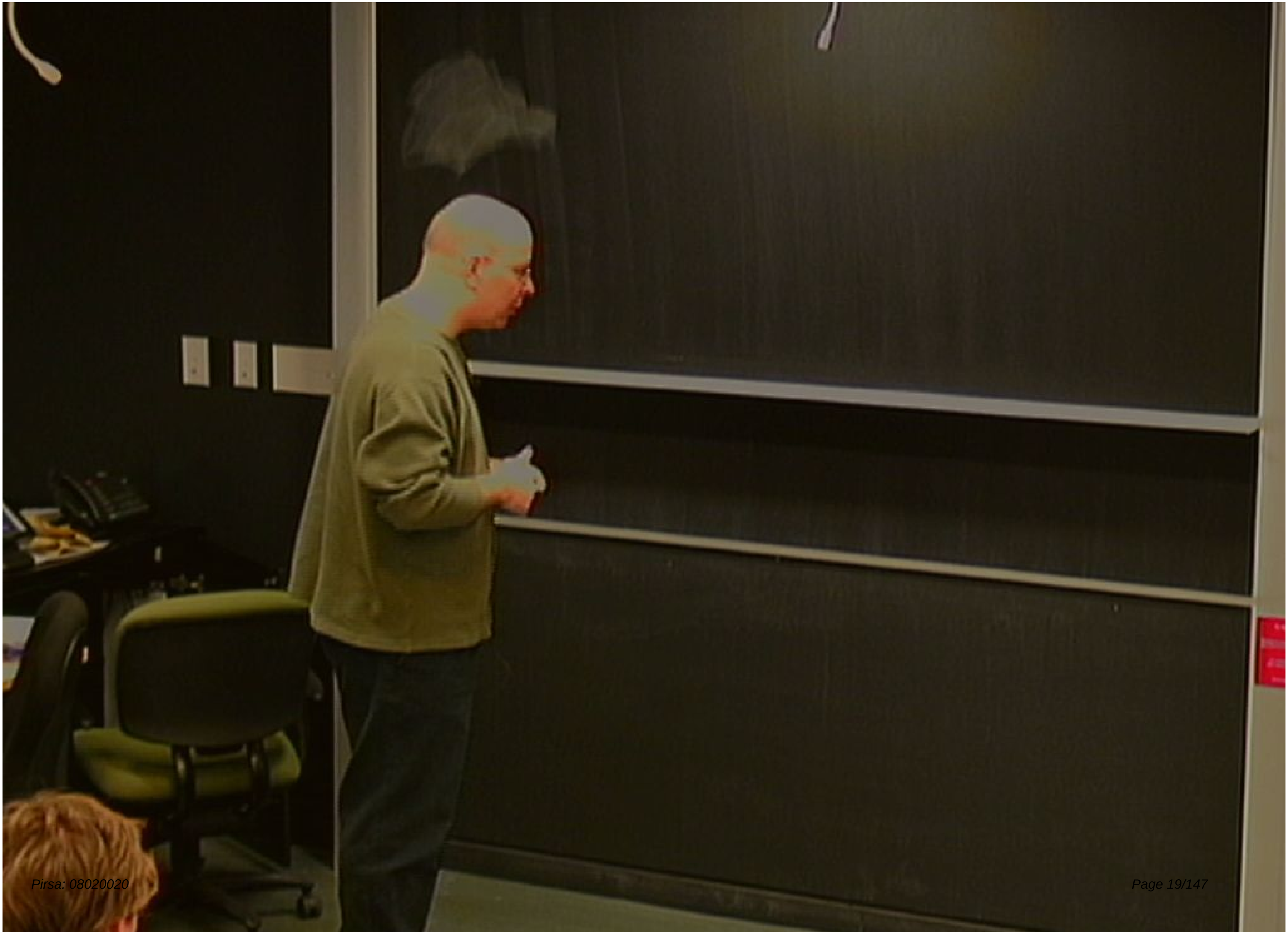


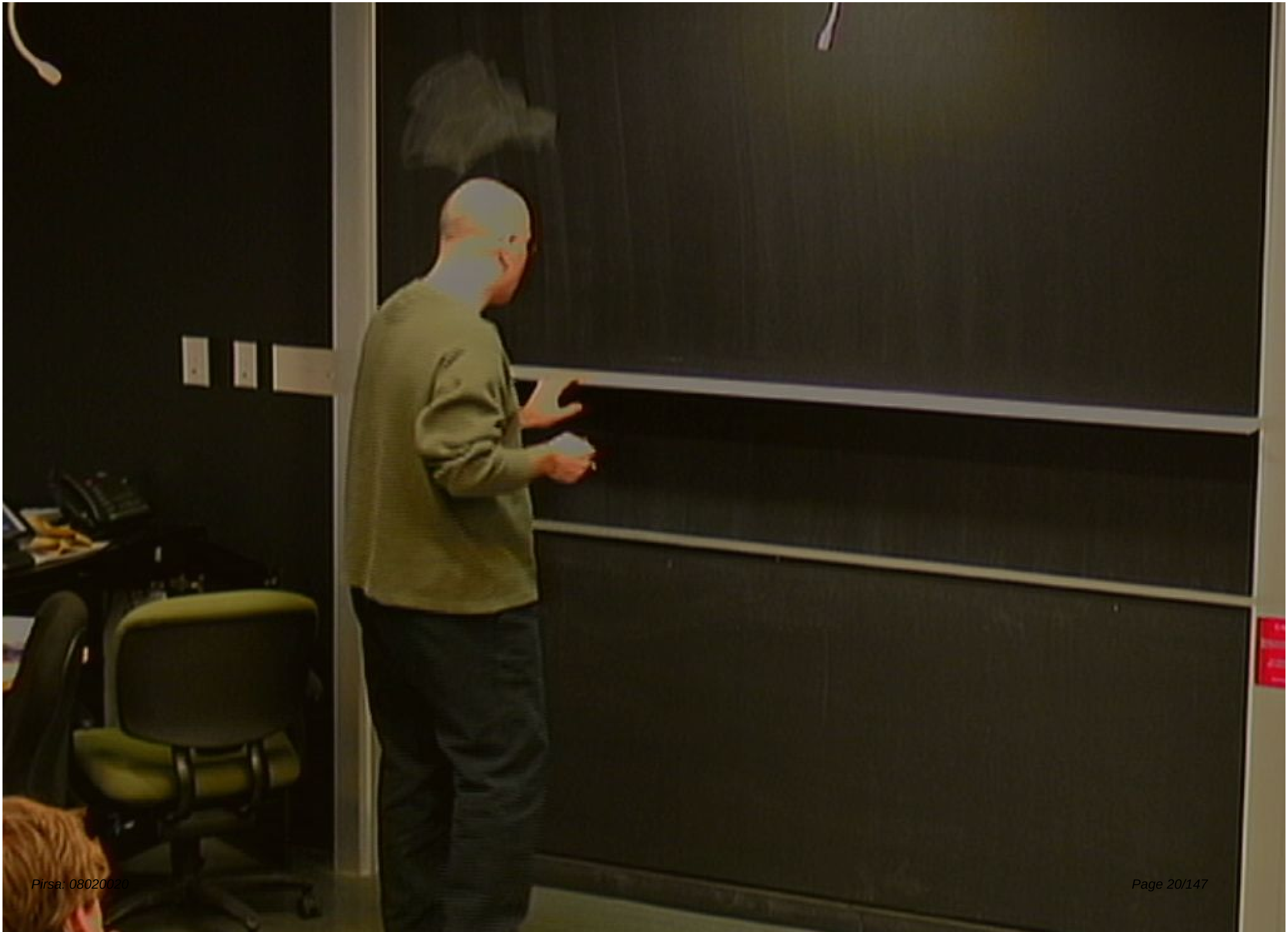


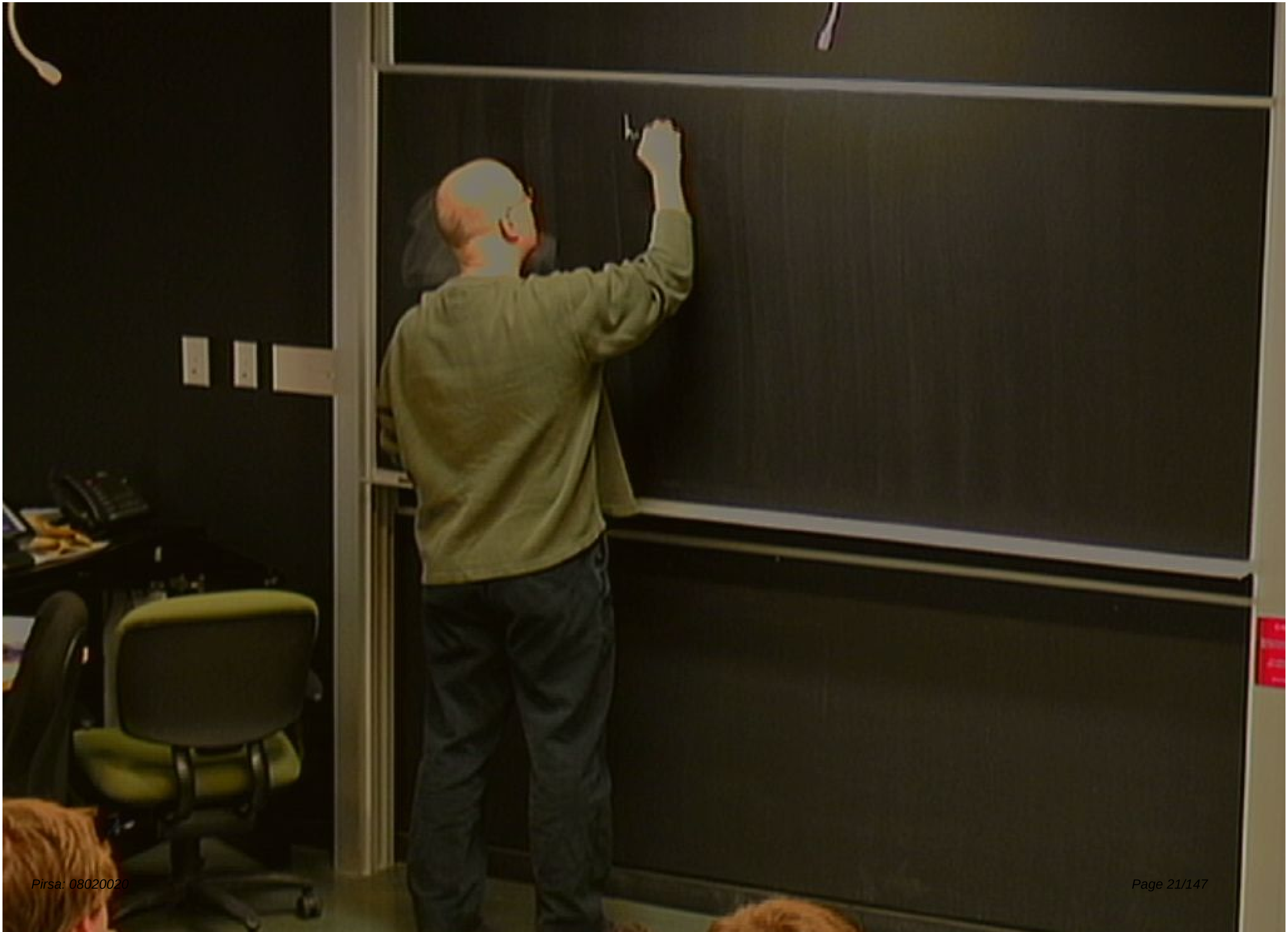


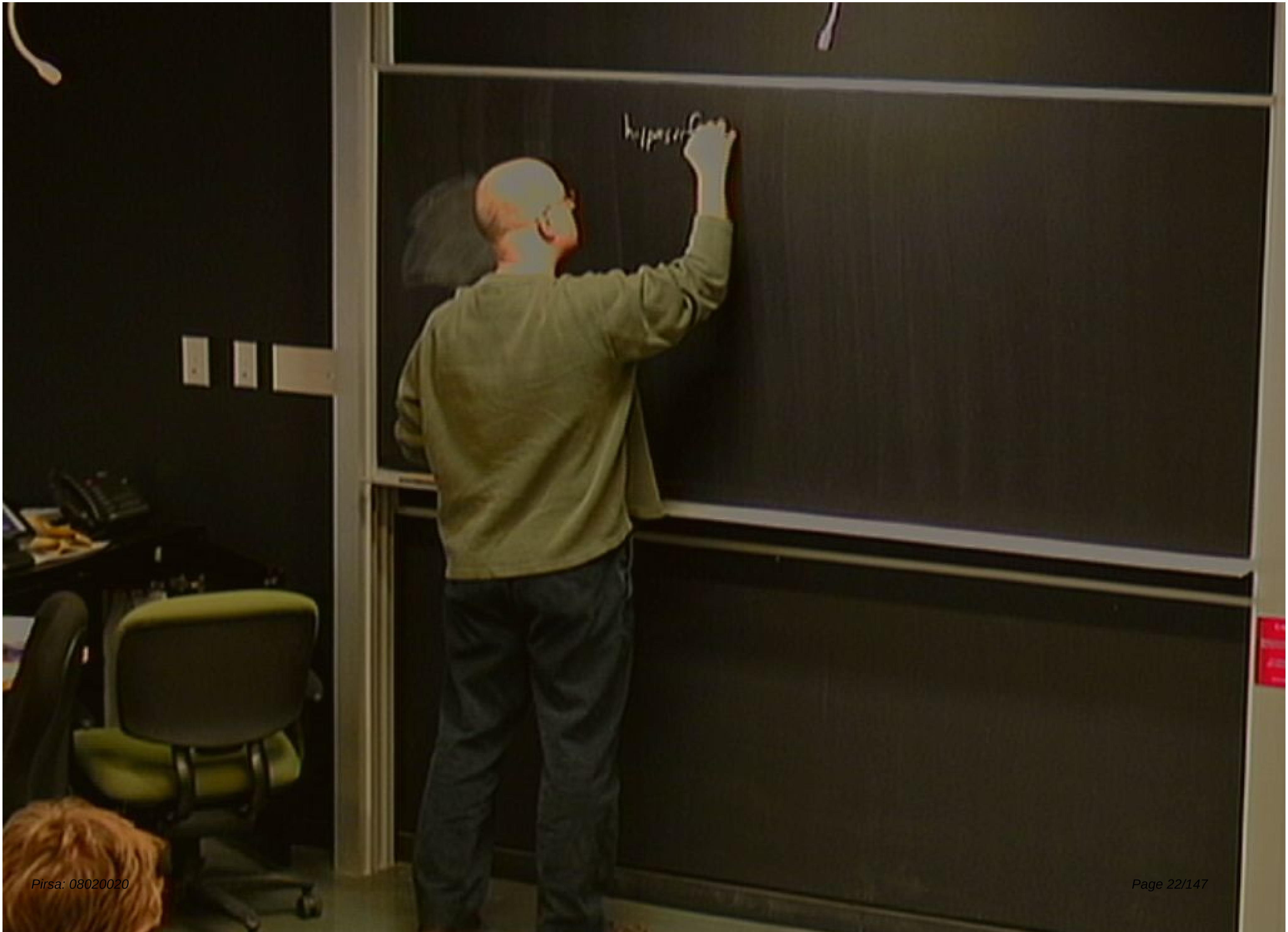


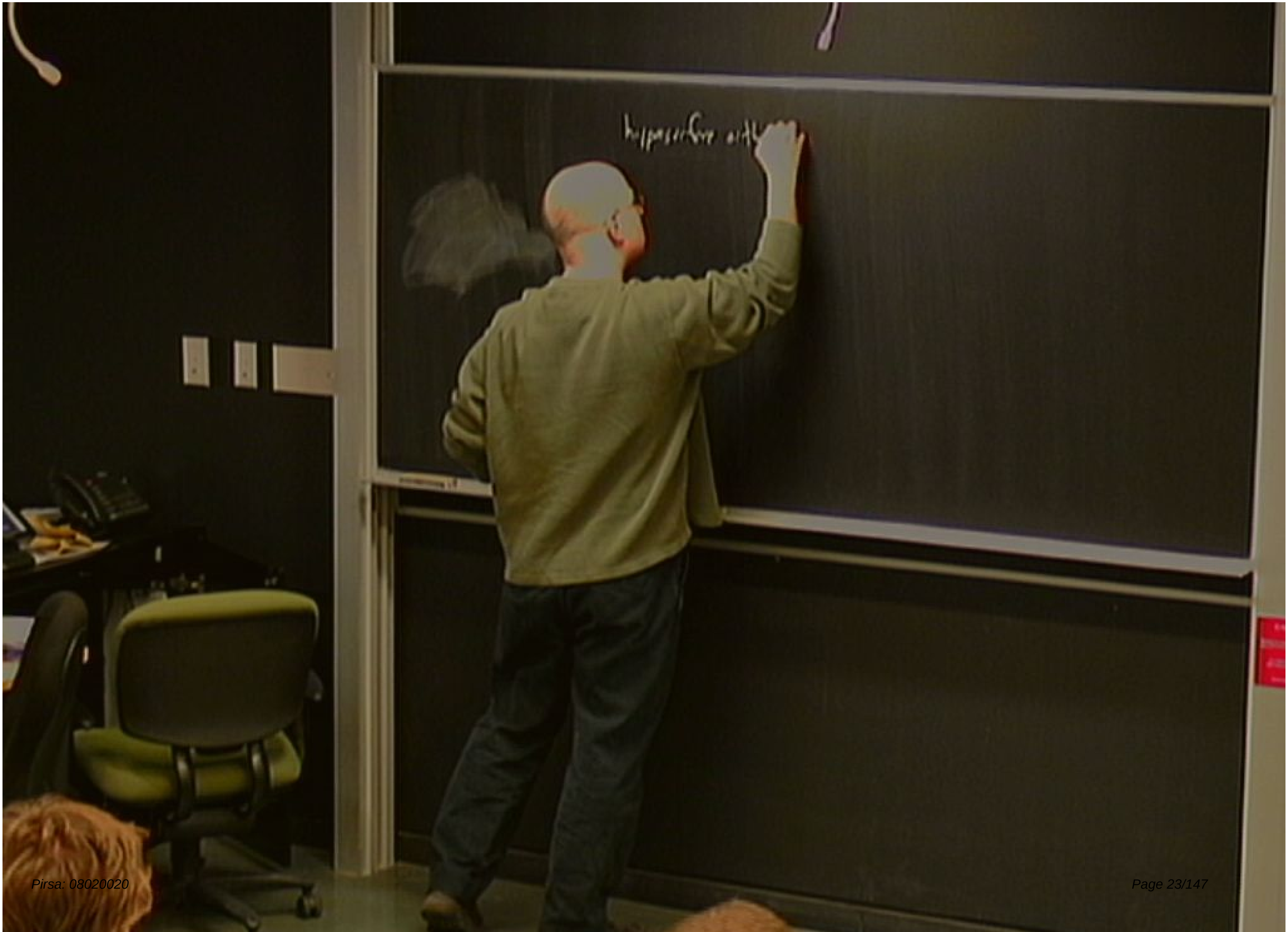








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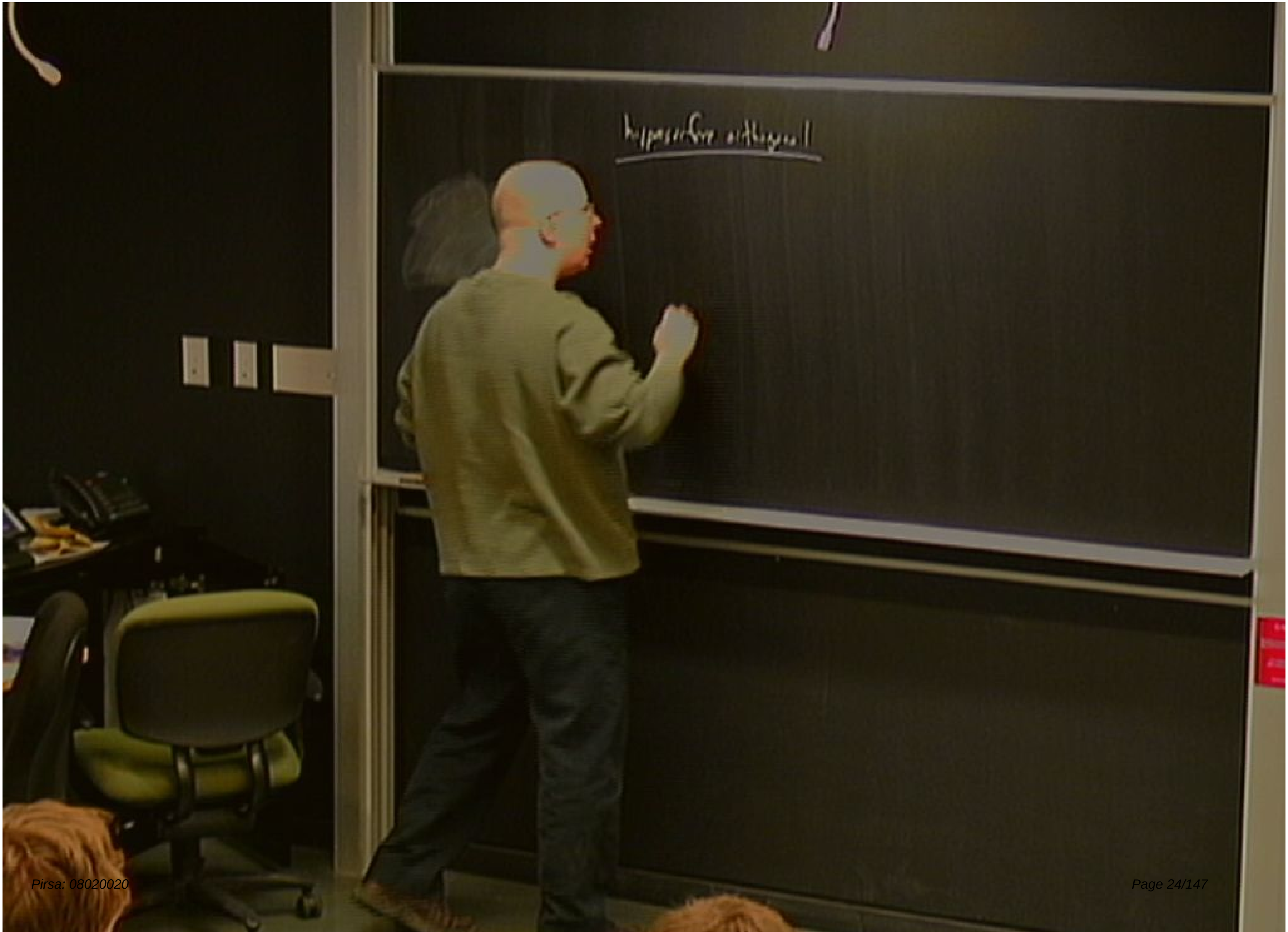
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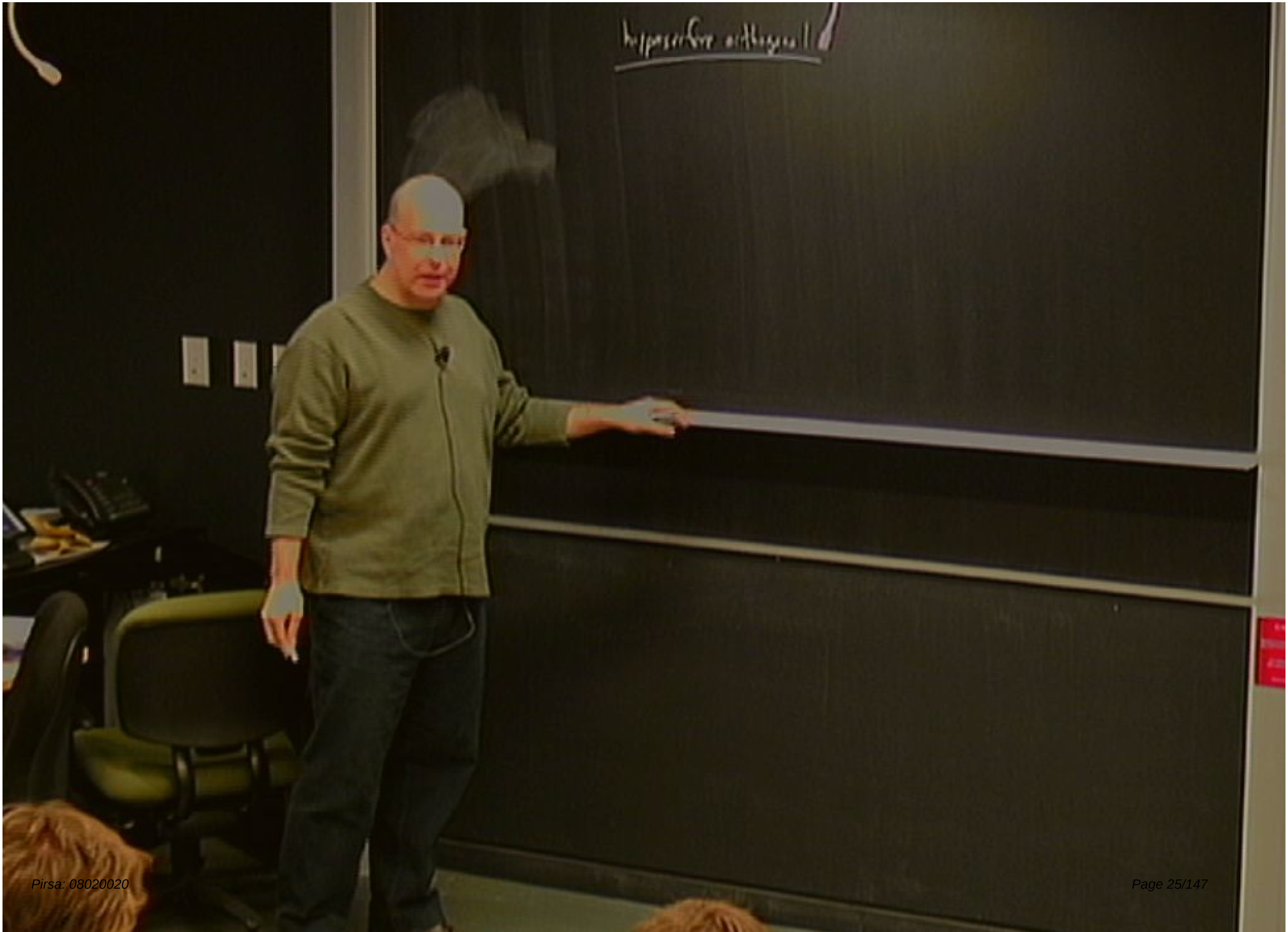
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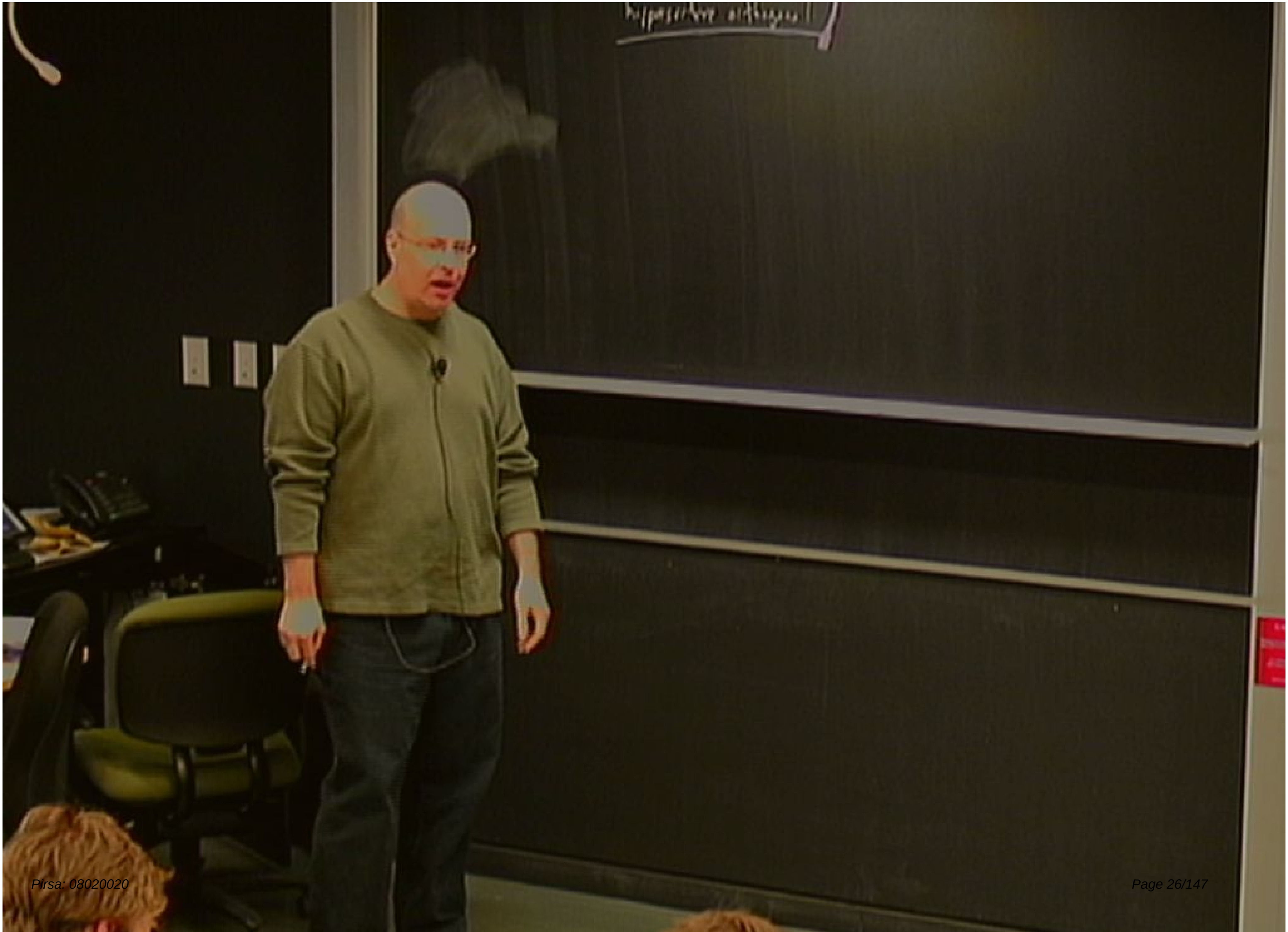

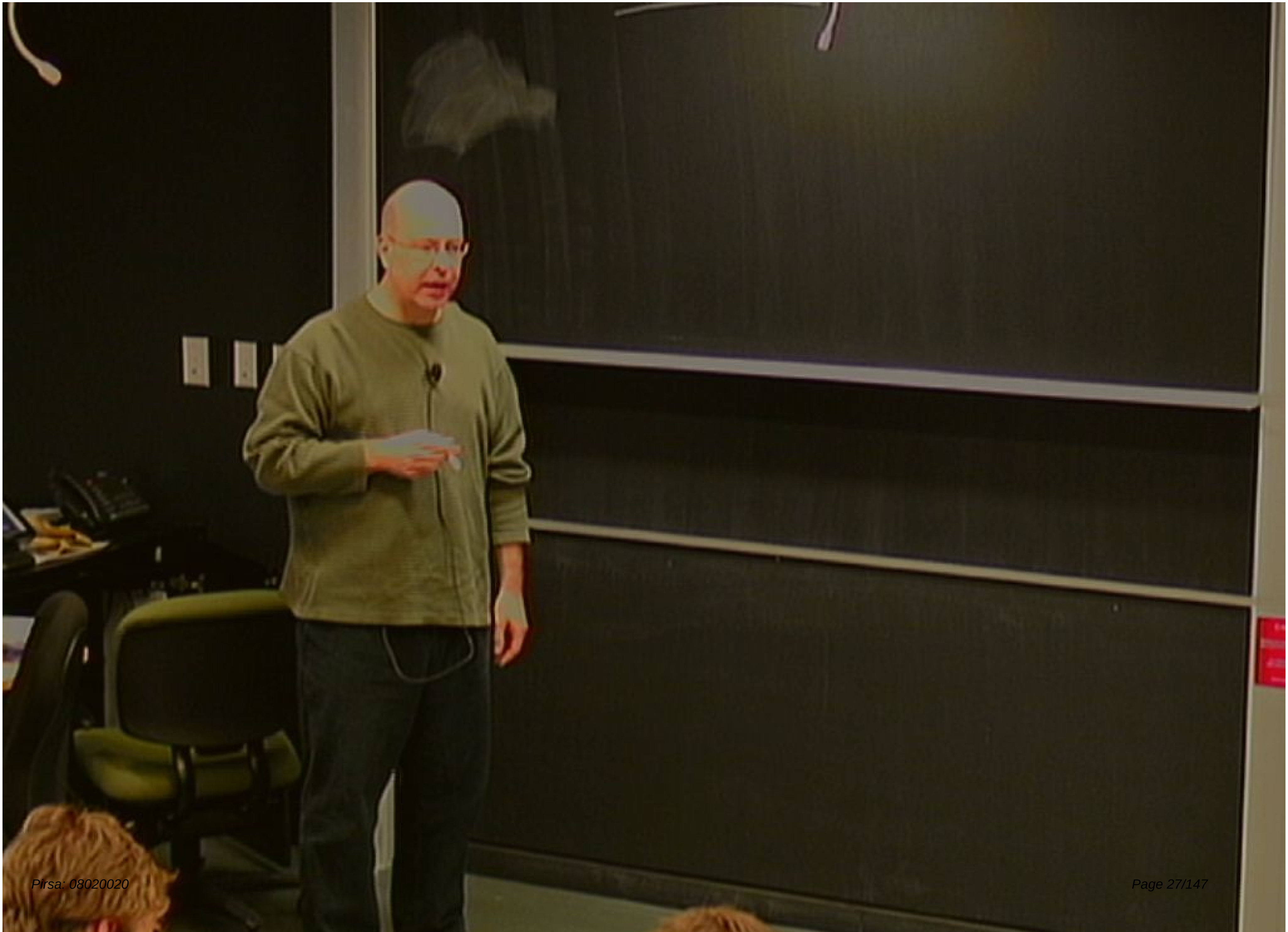

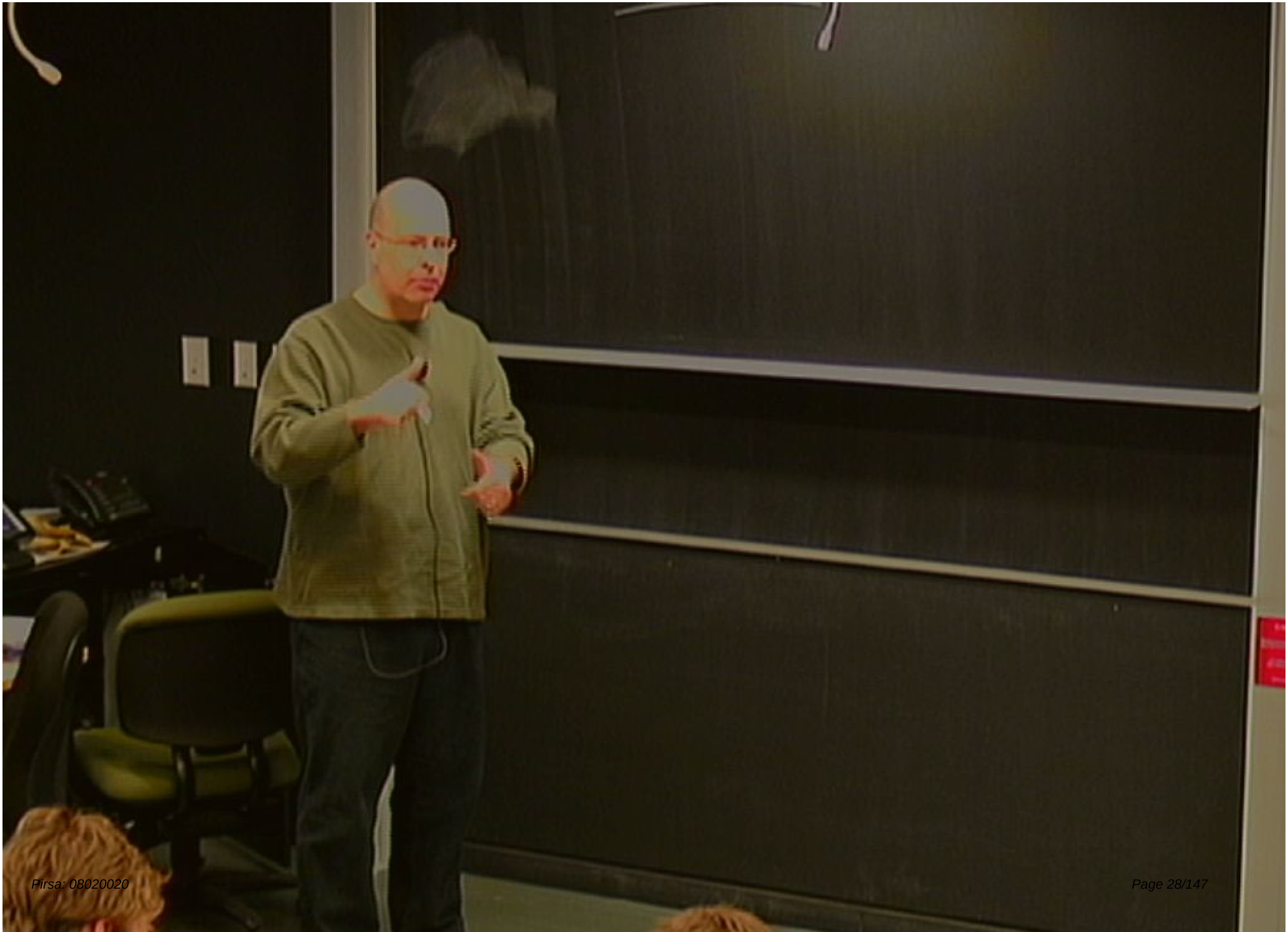

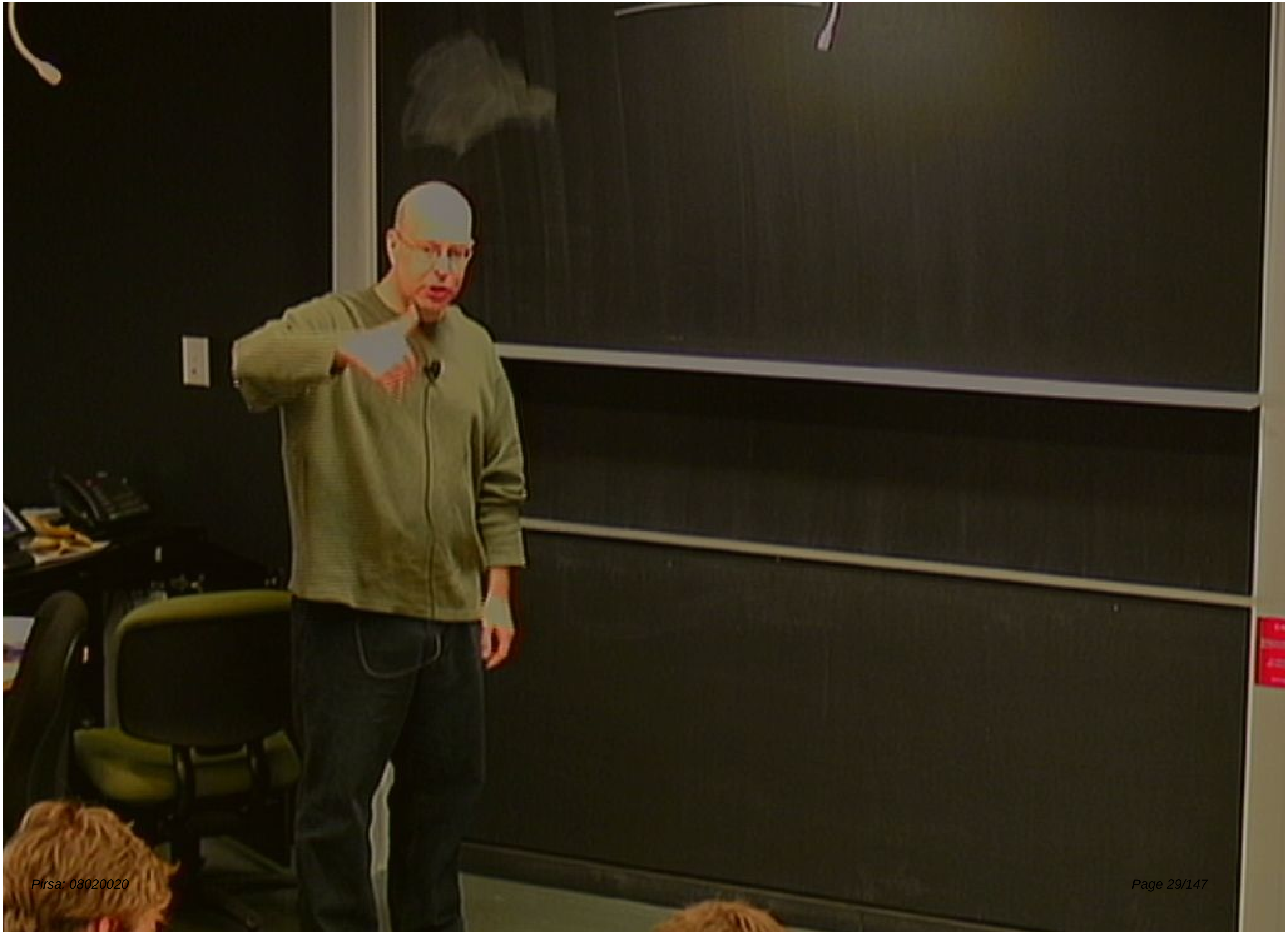

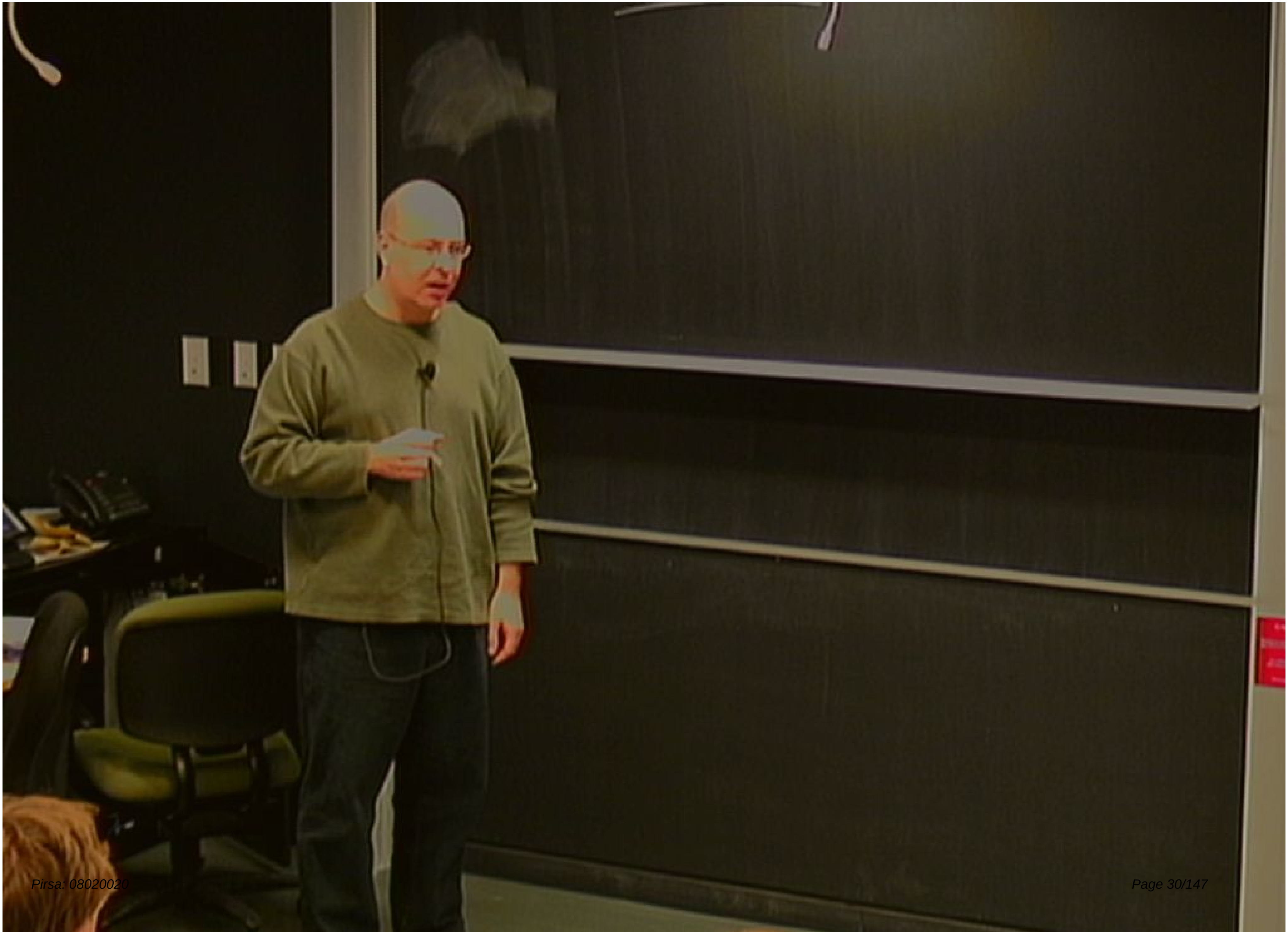

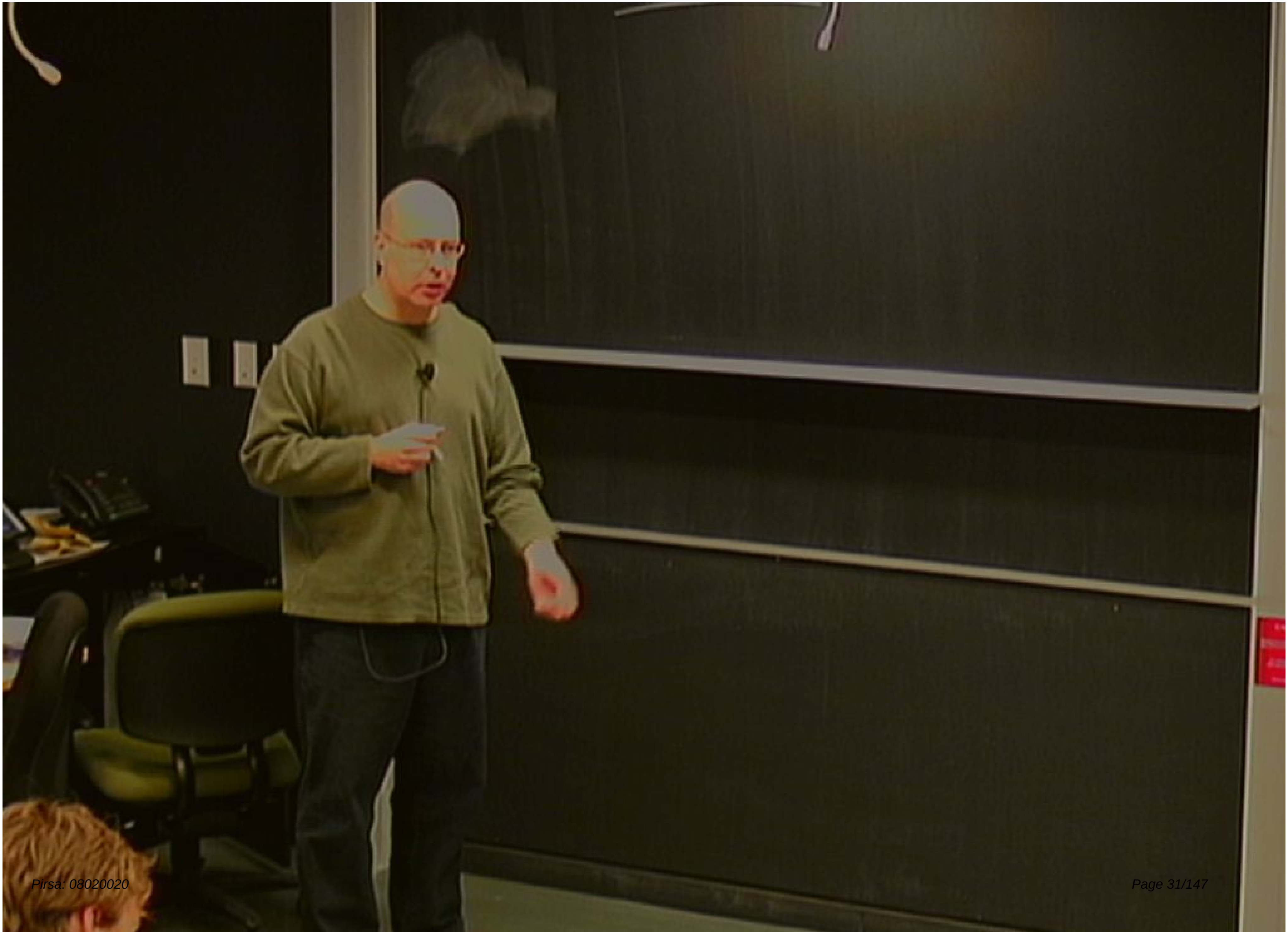

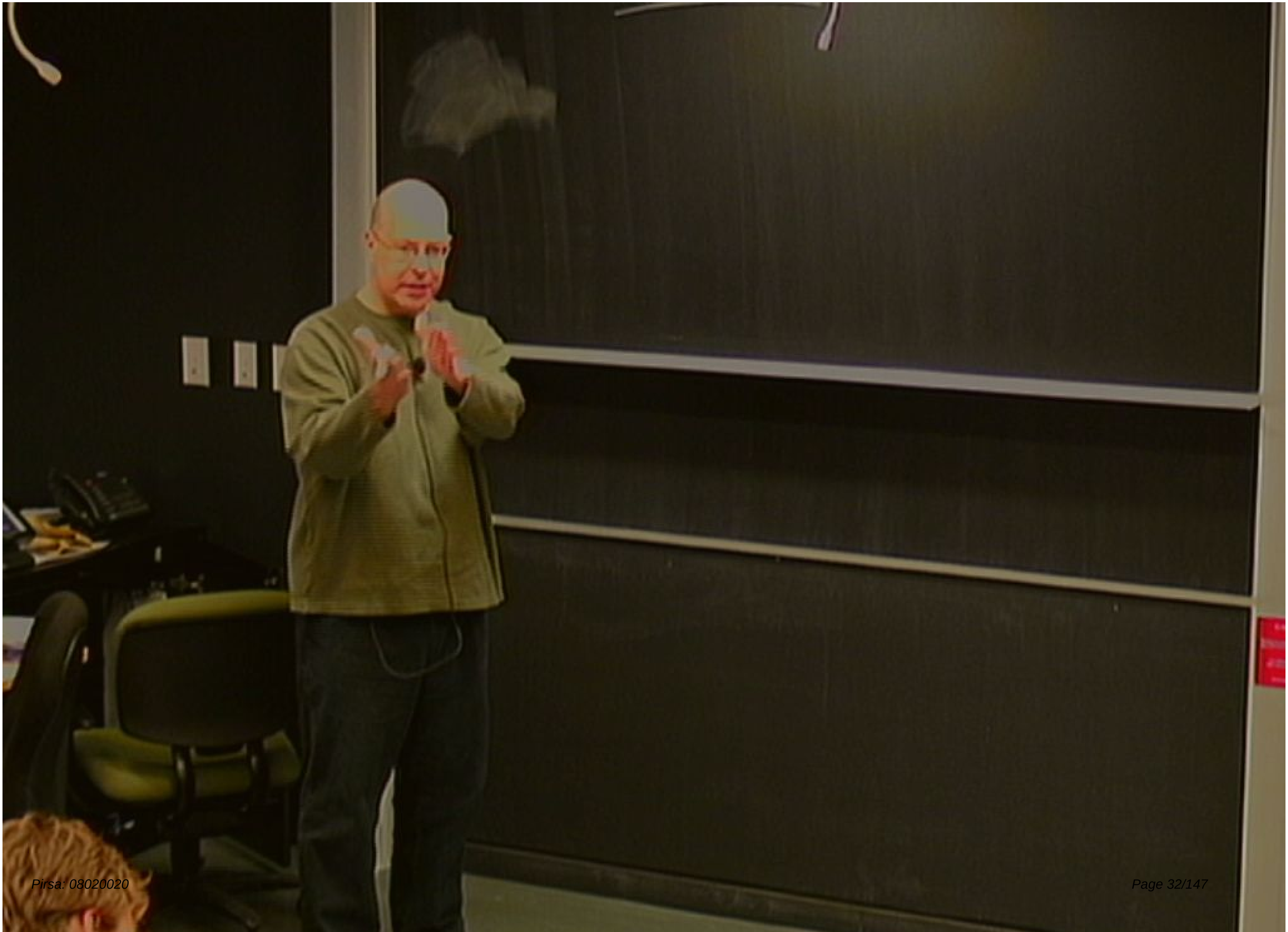

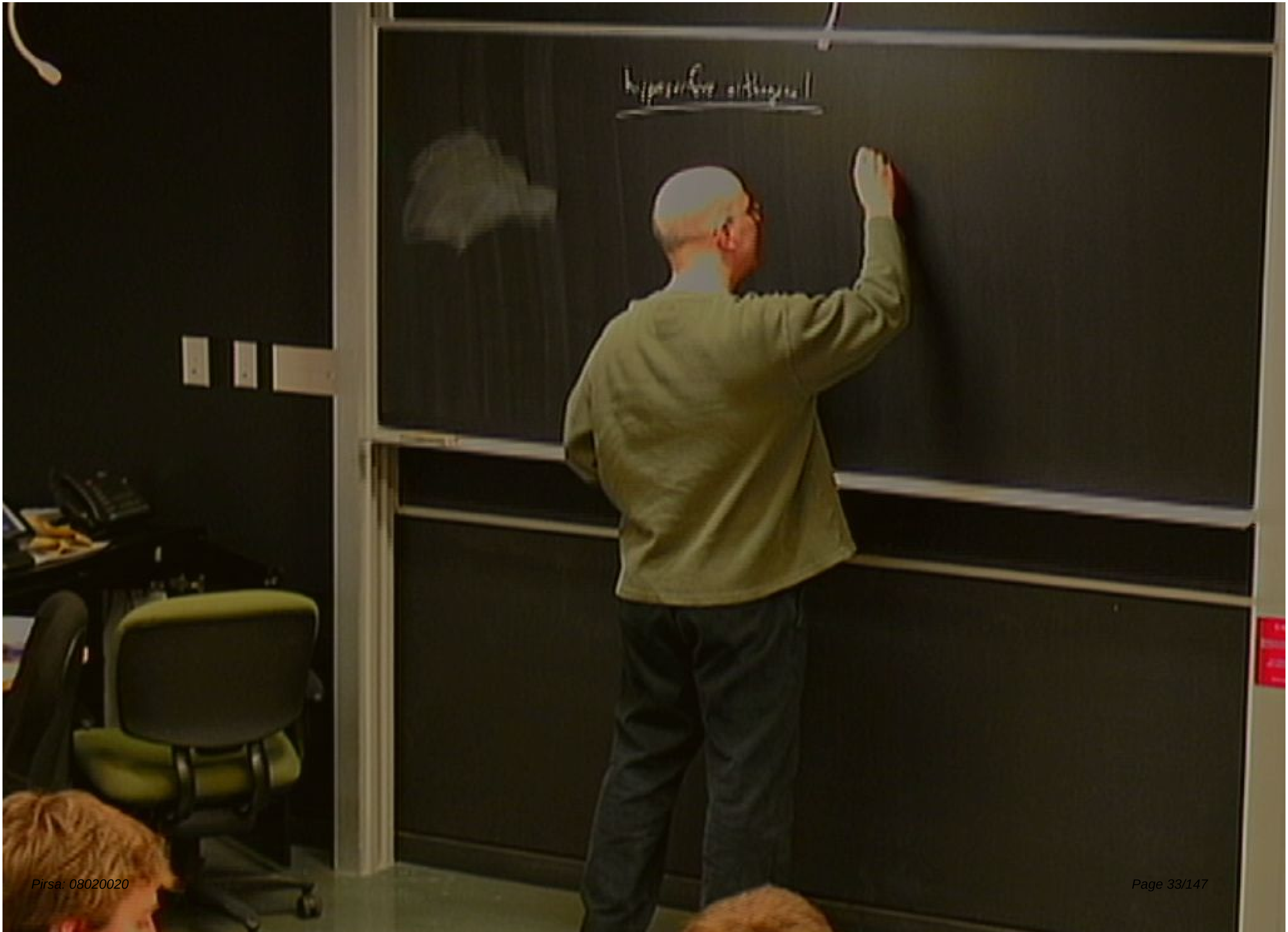
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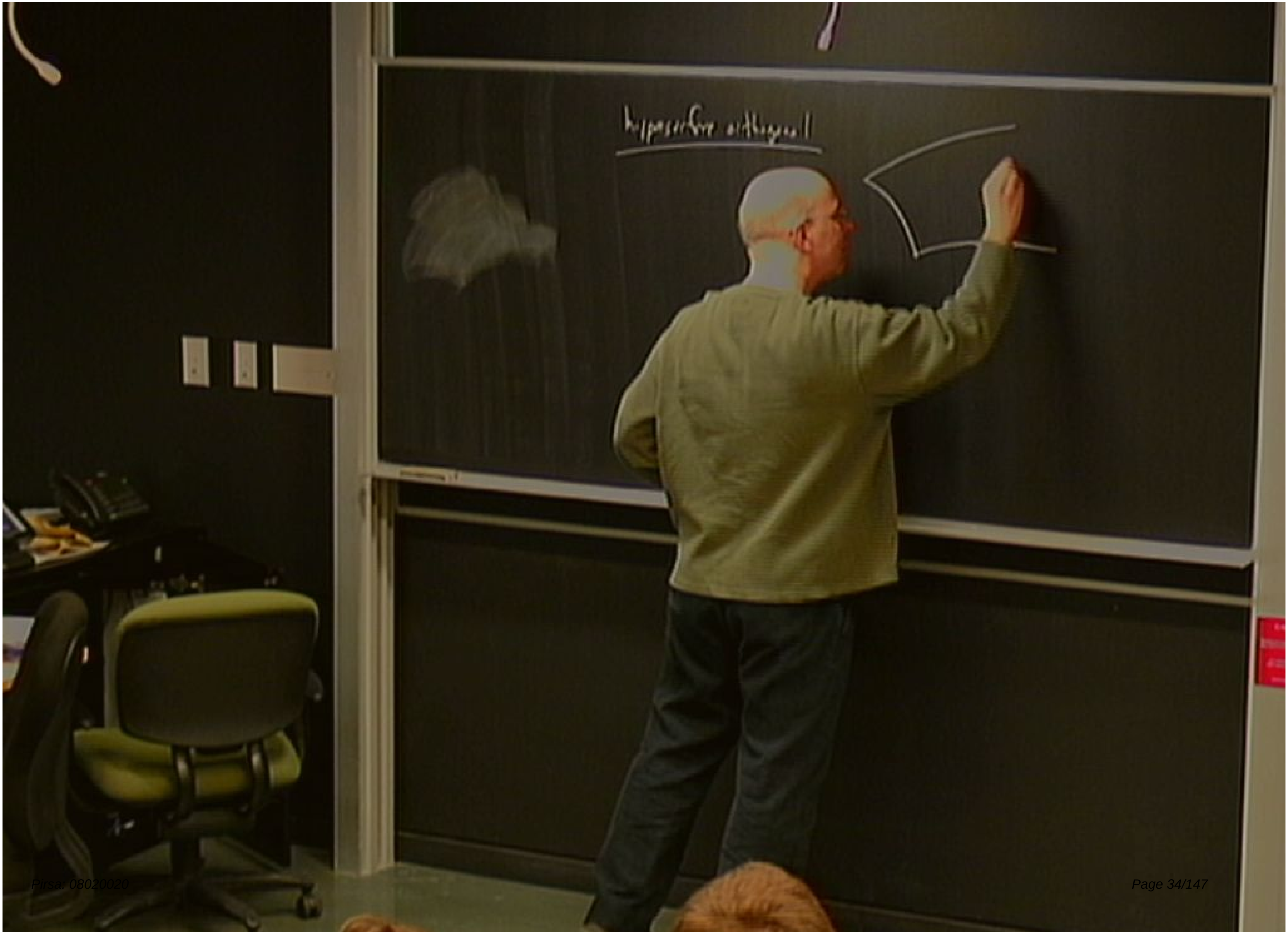
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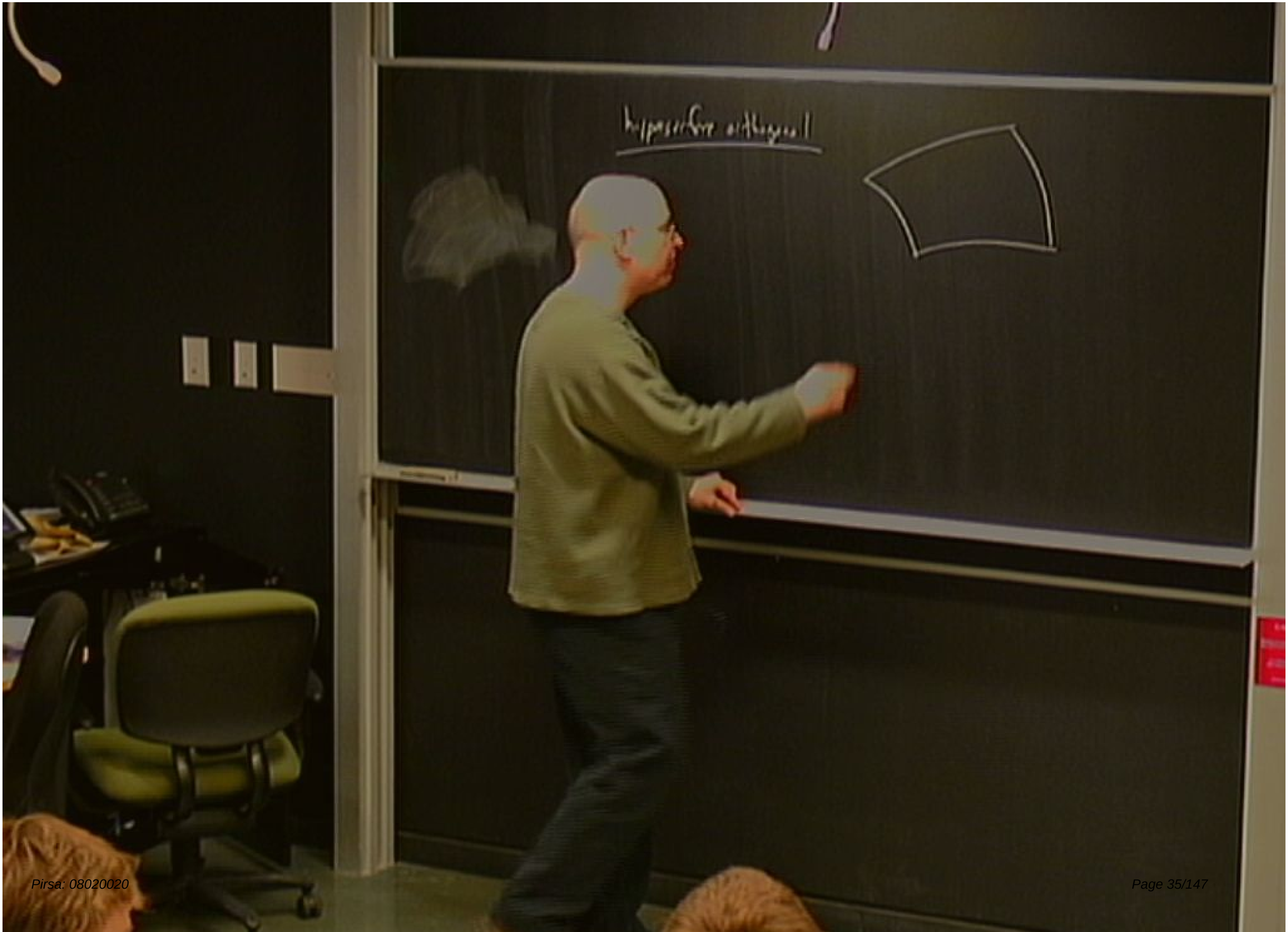
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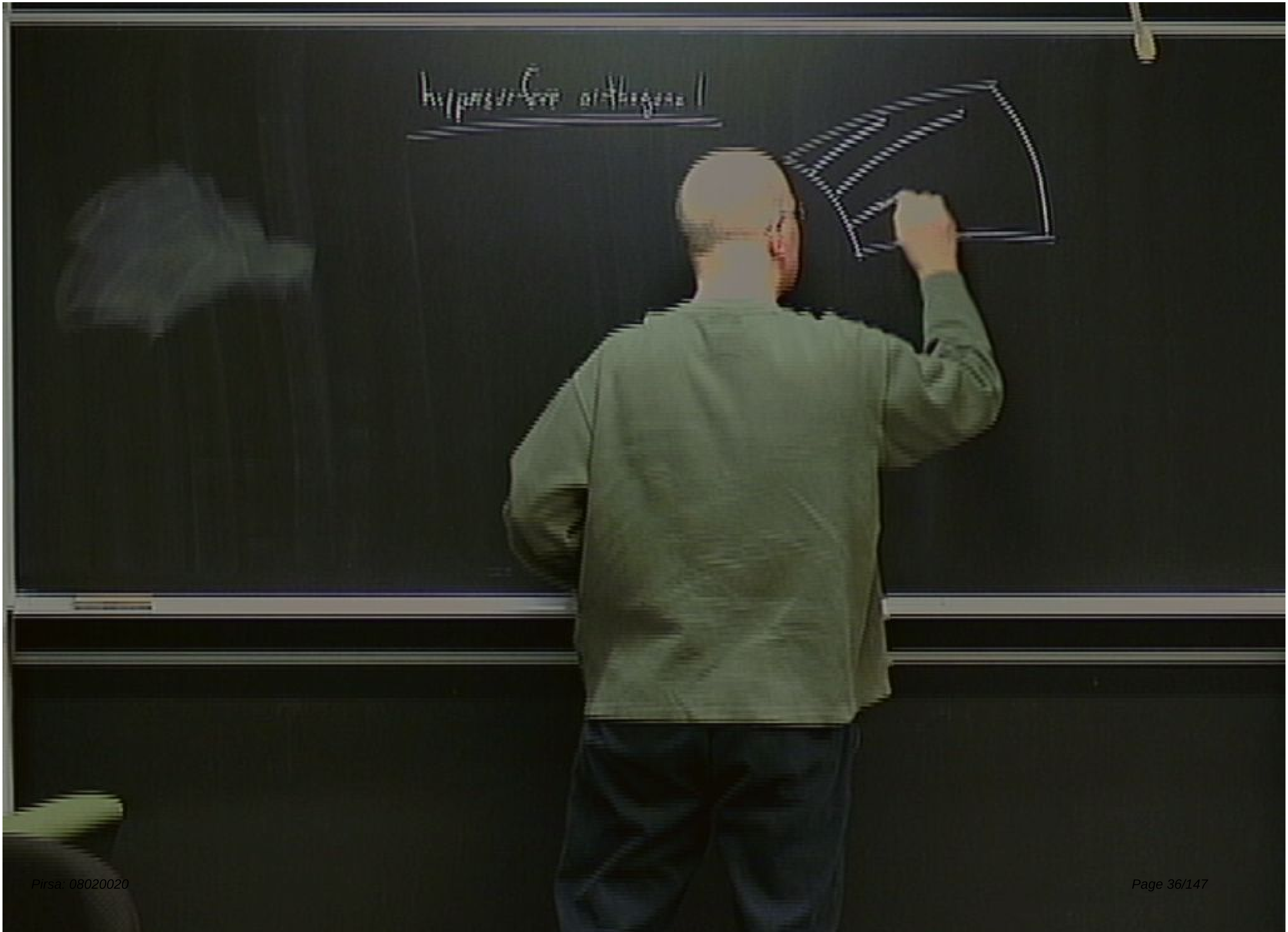
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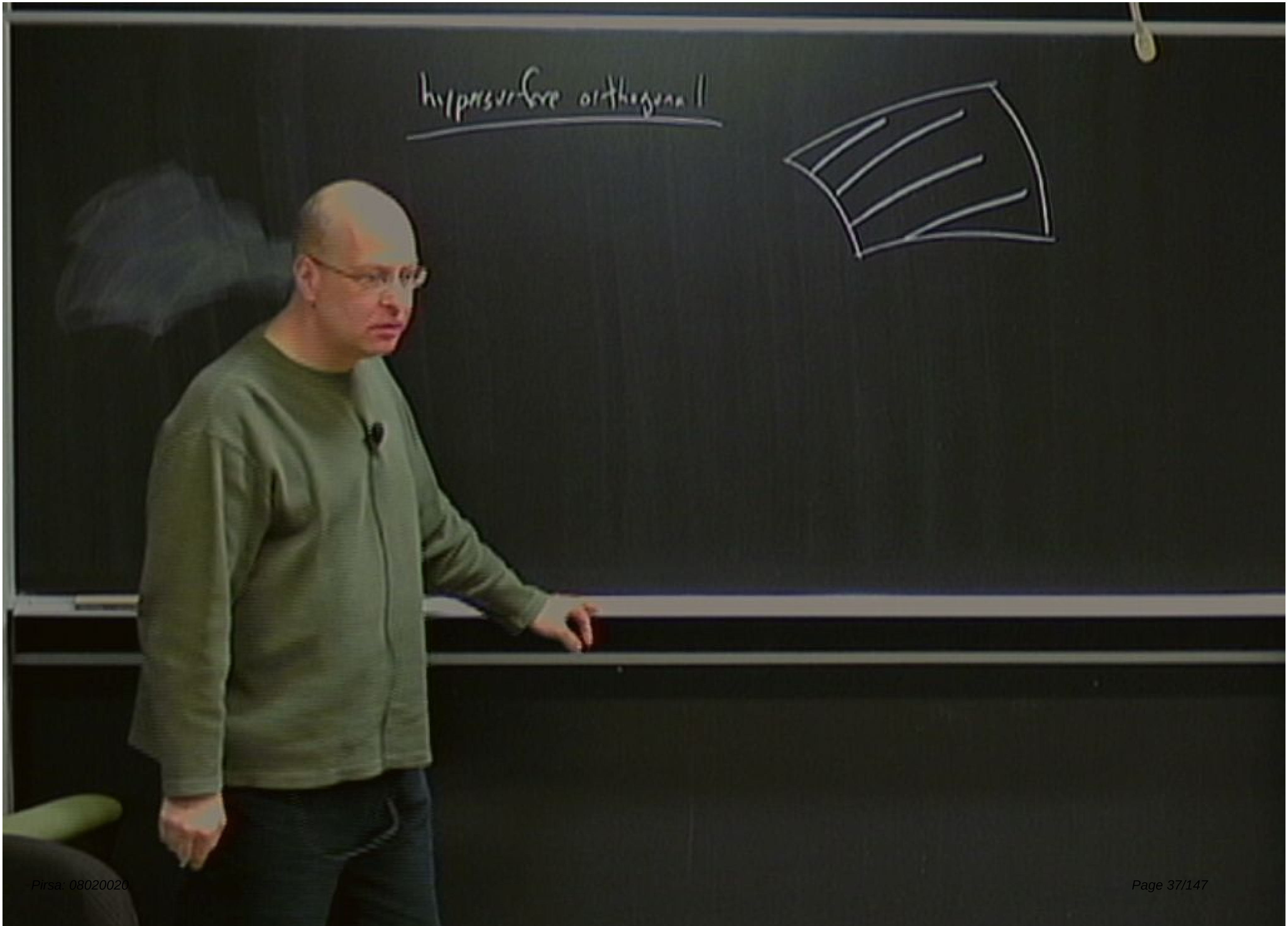
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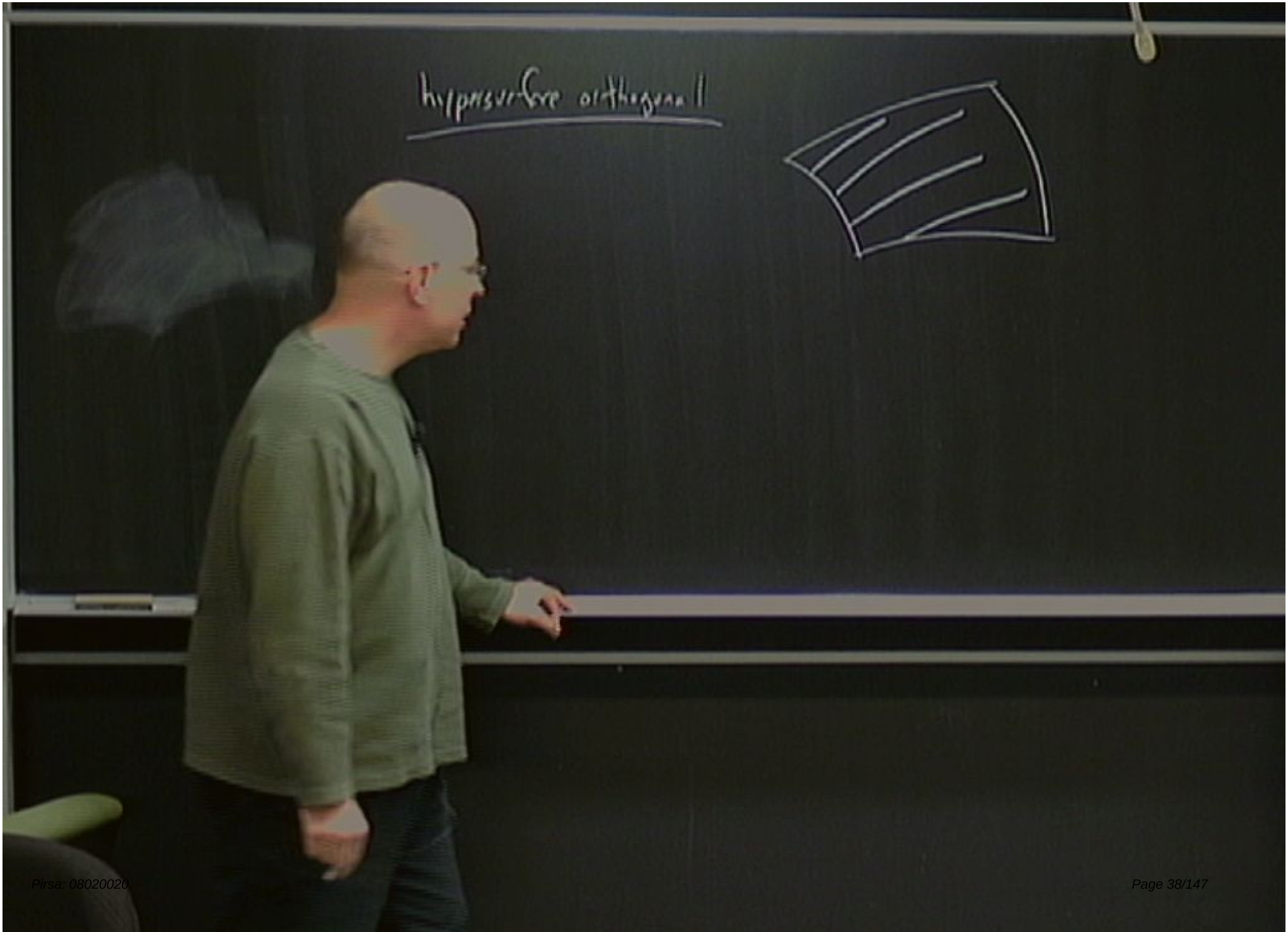
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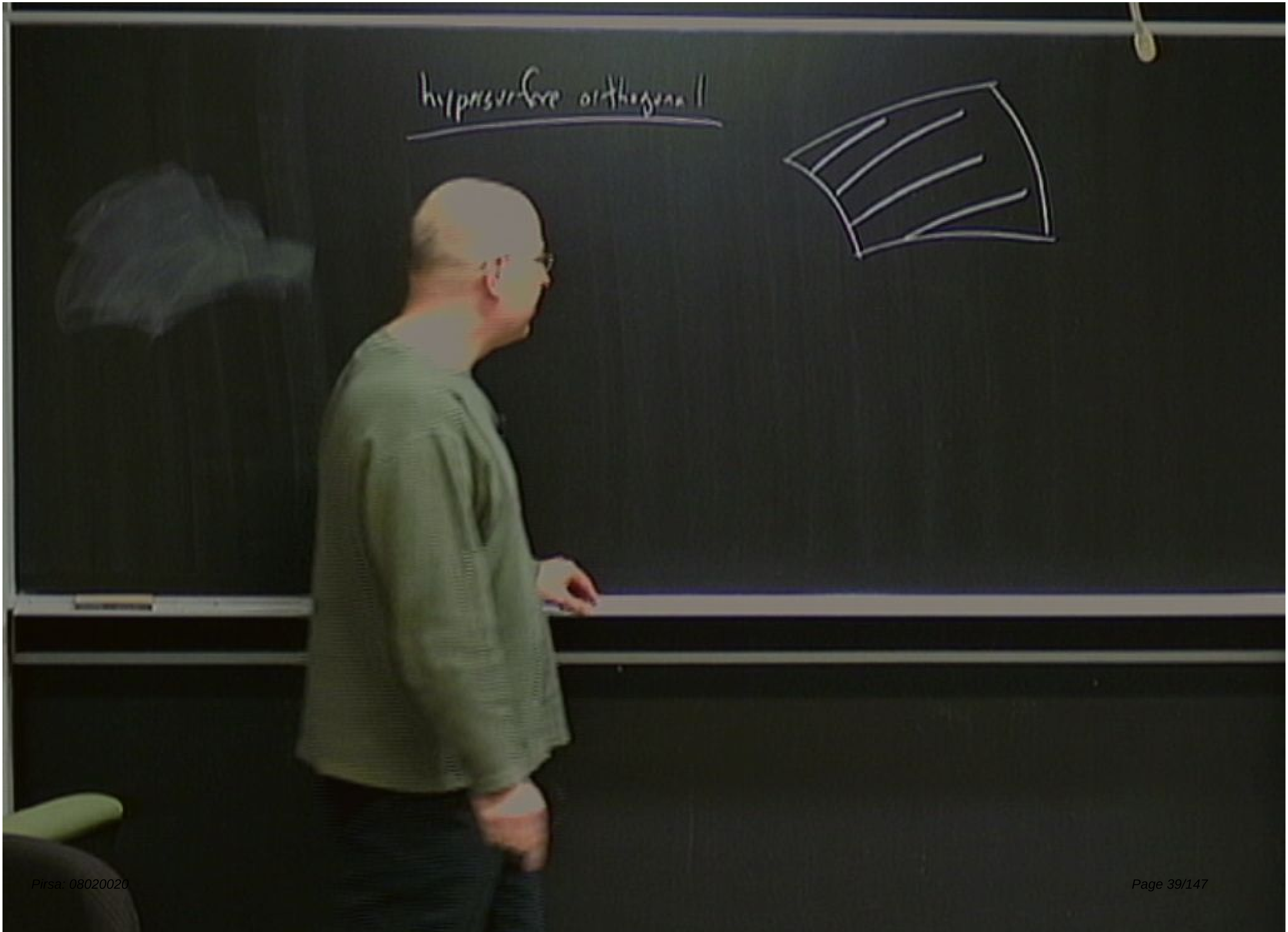
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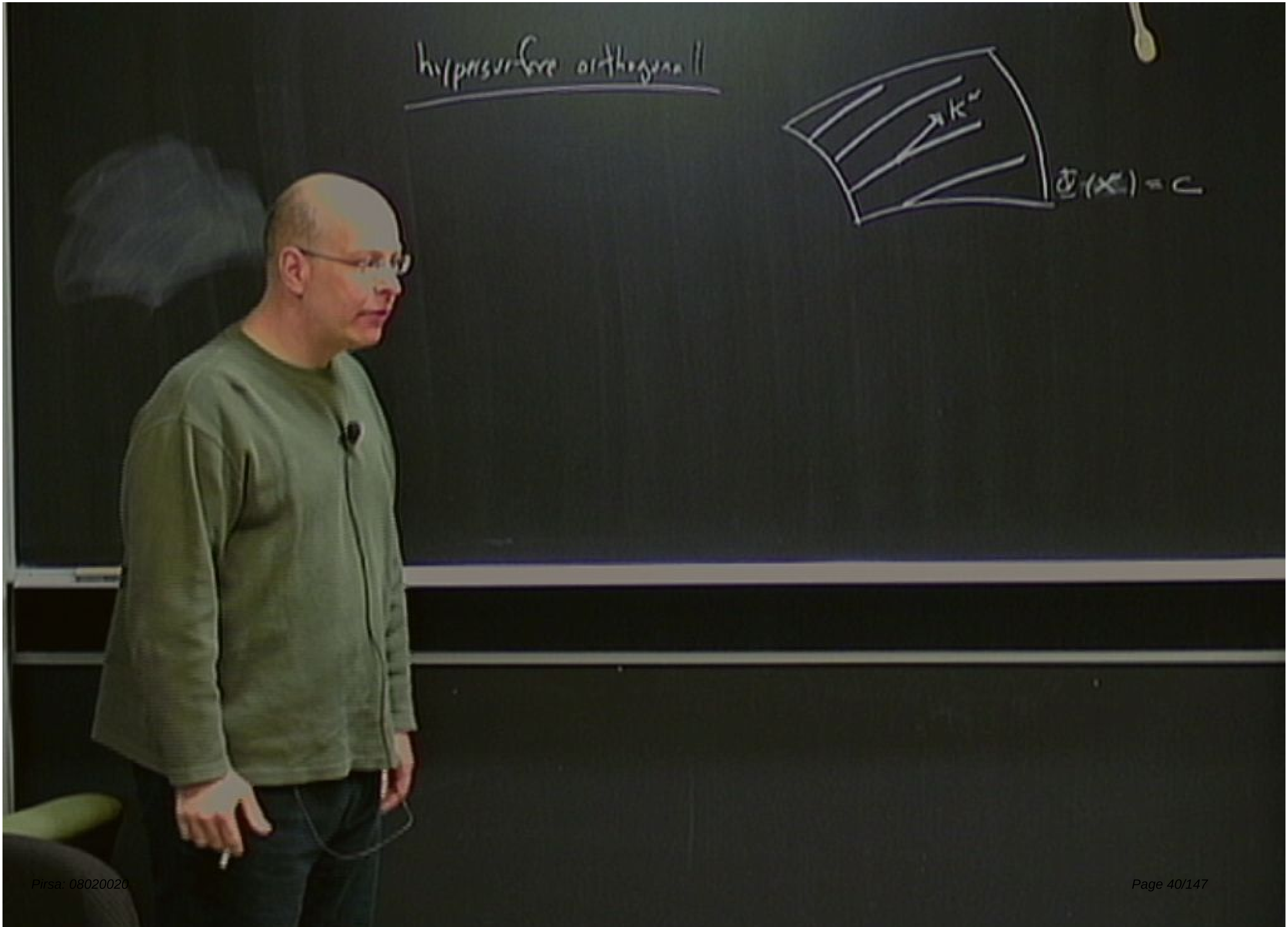
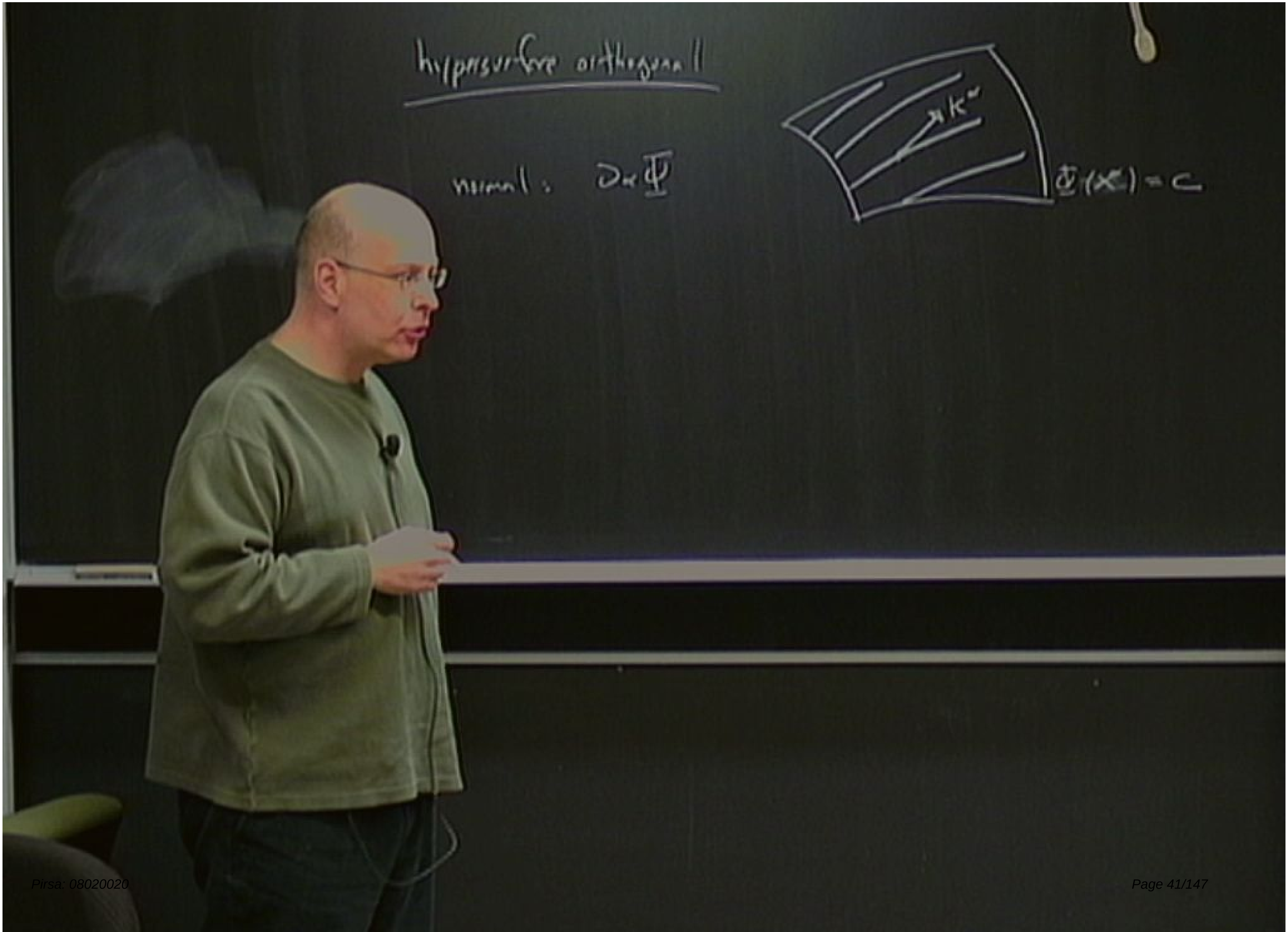
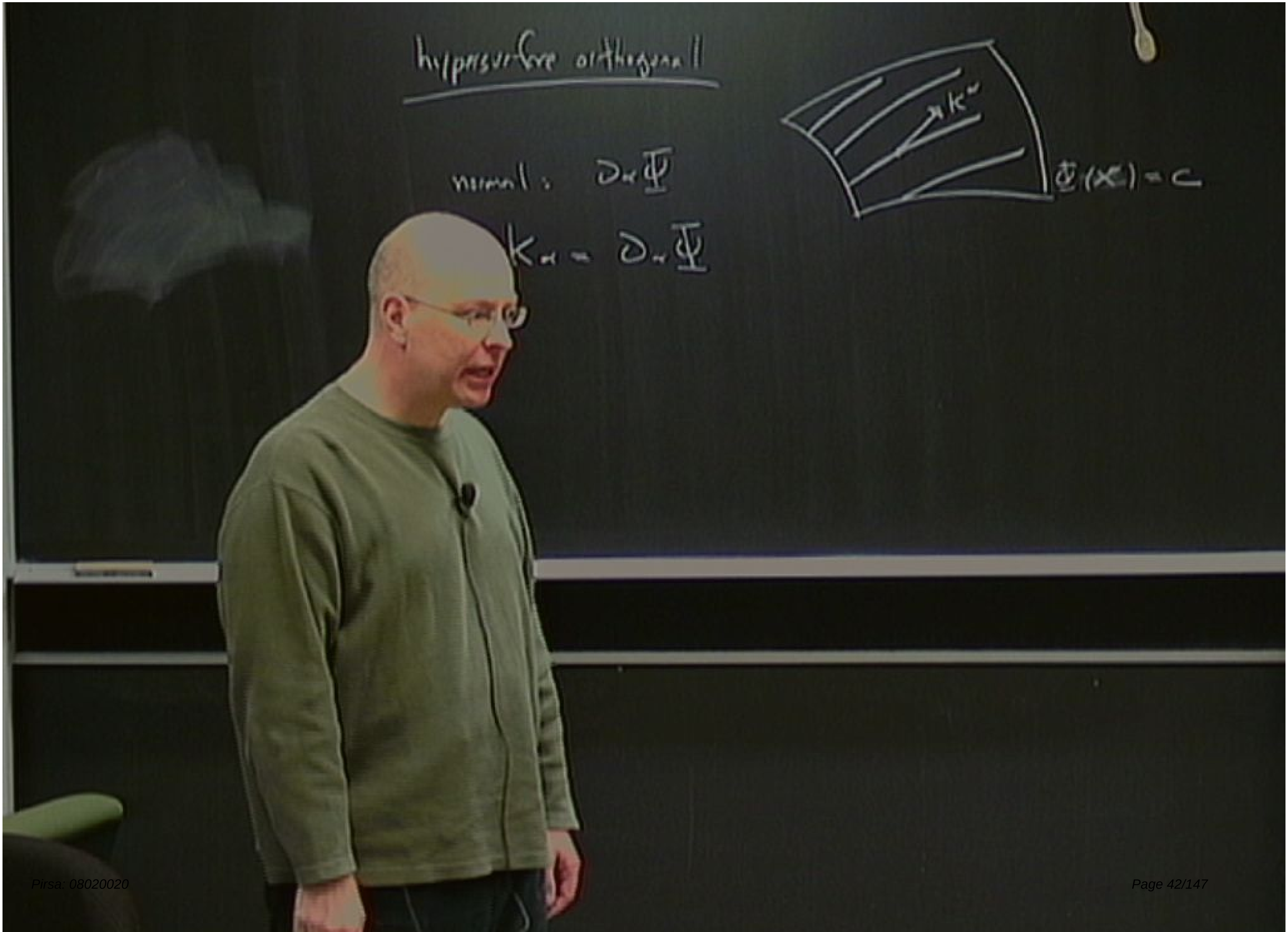
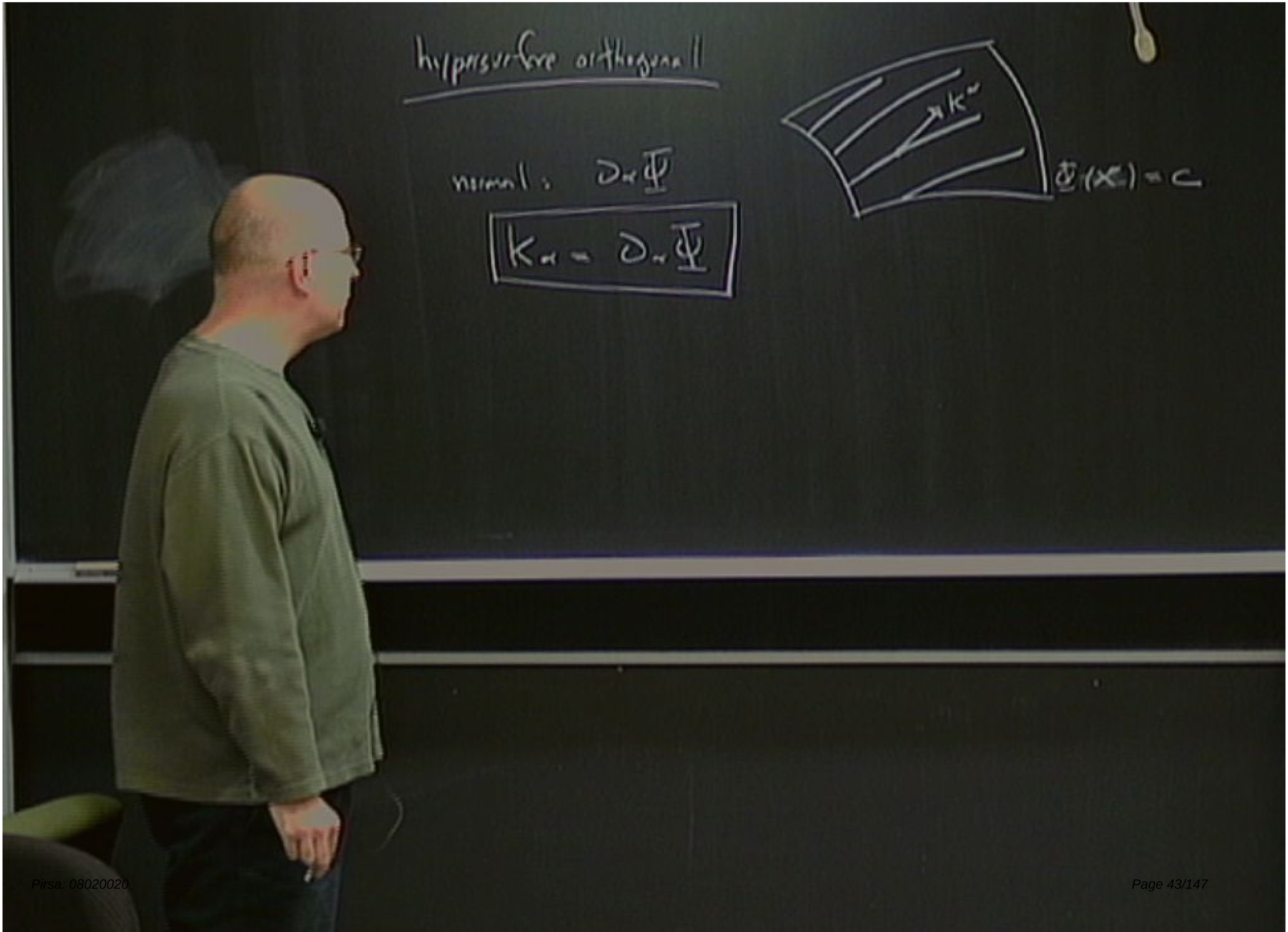
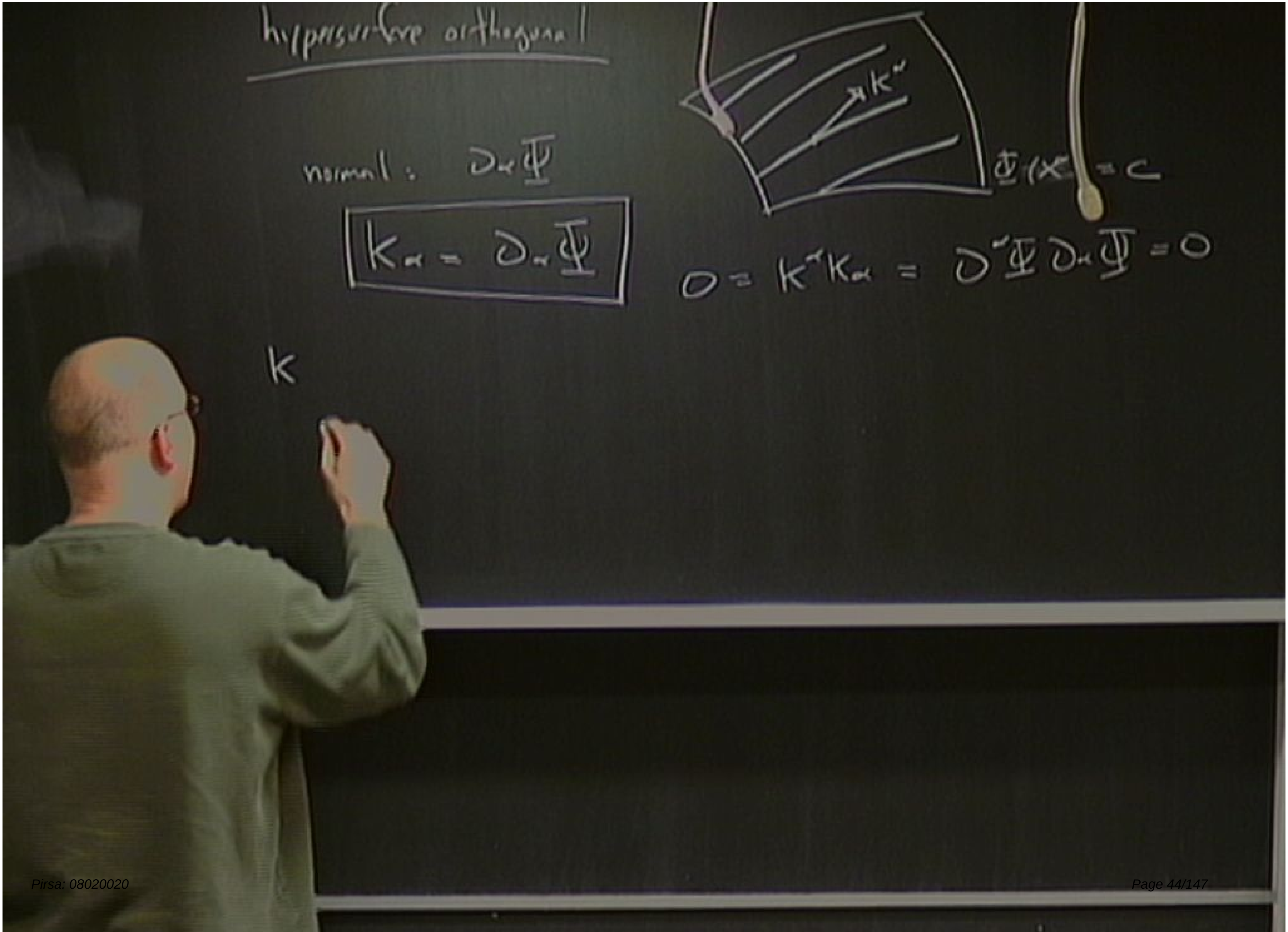
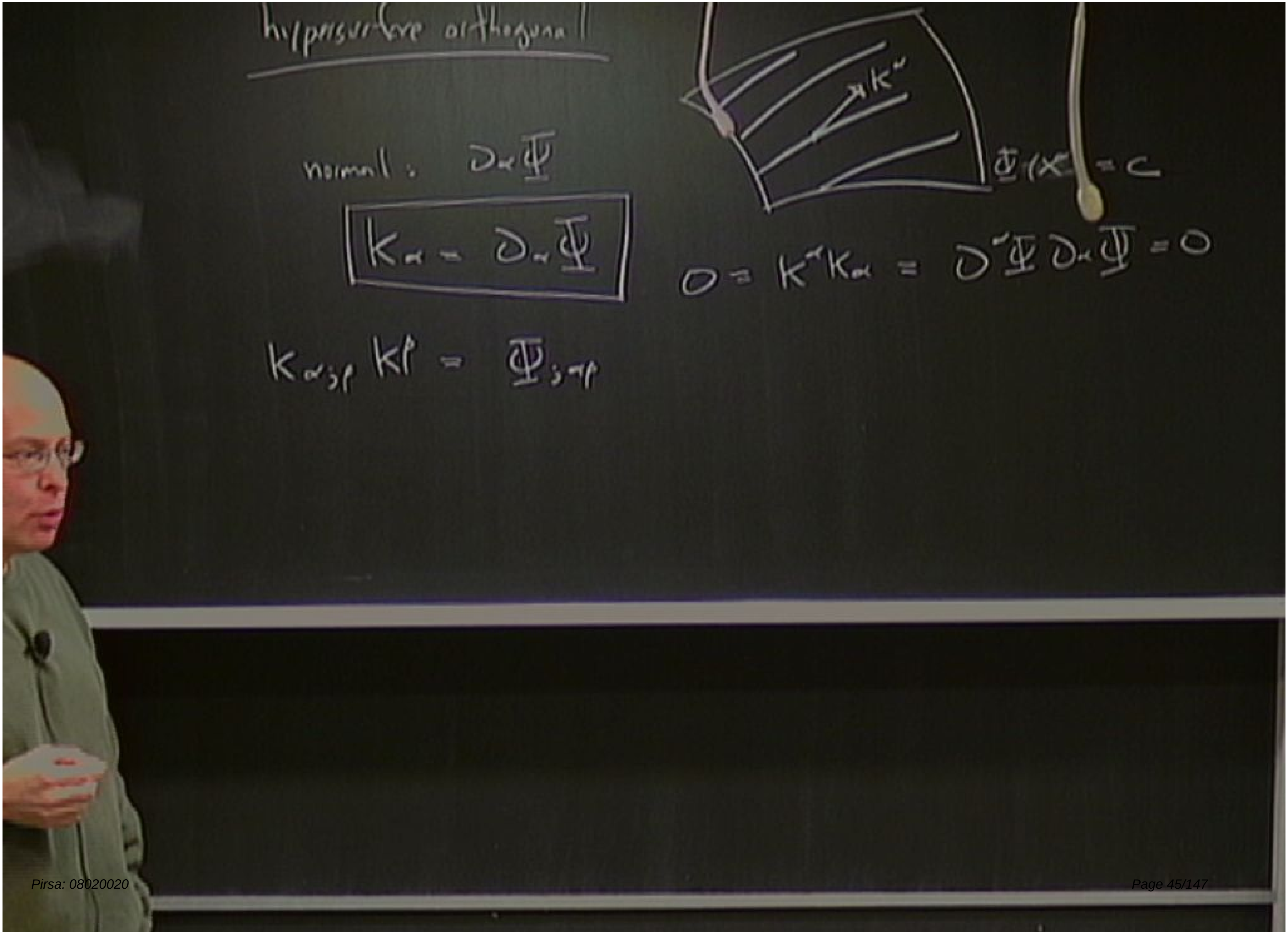

hypersurface orthogonal
k^μ
Φ(x) = c

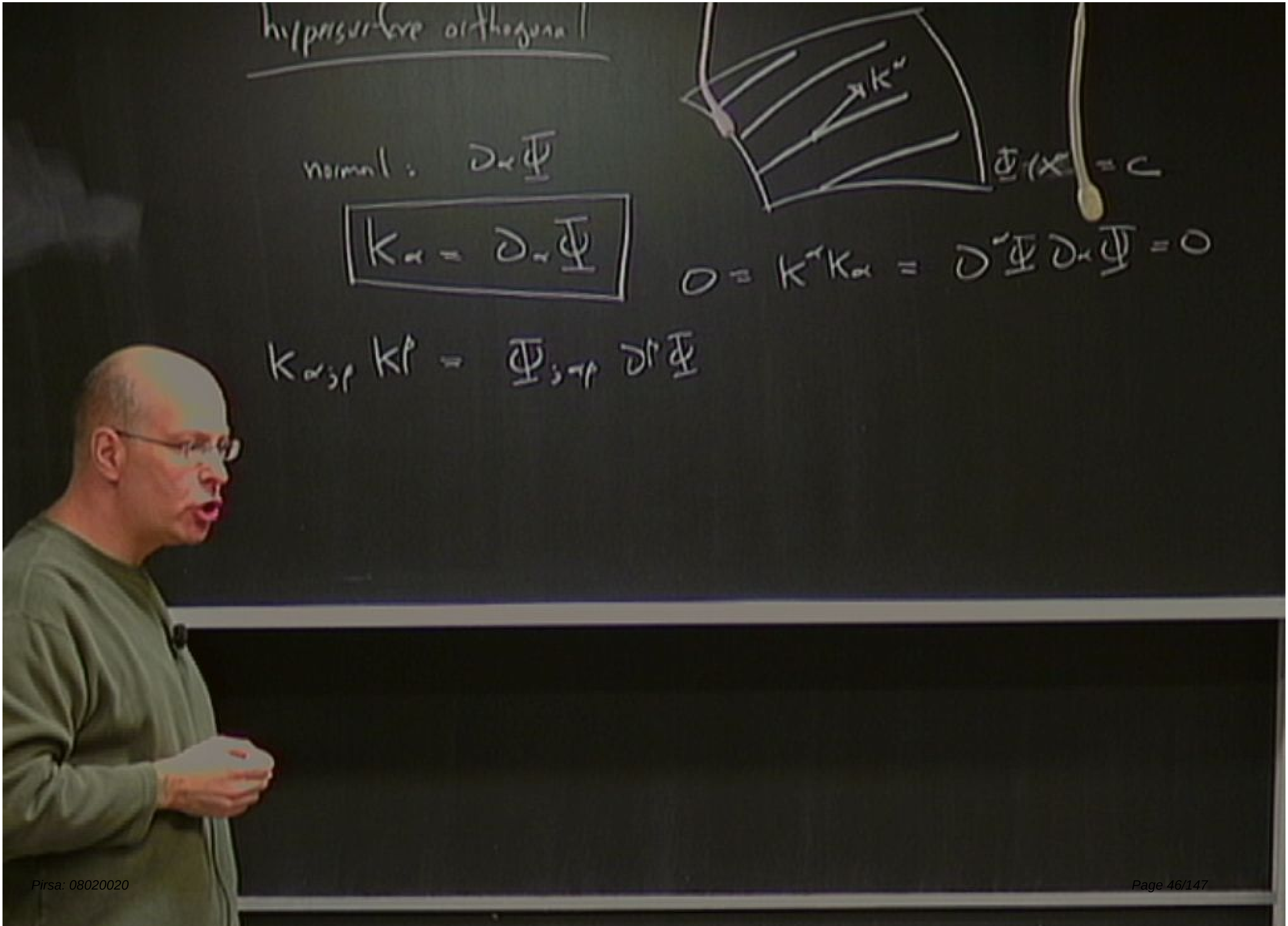

hypersurface orthogonal
normal: $\partial_\alpha \Phi$
k^α
$\Phi(x^\alpha) = c$

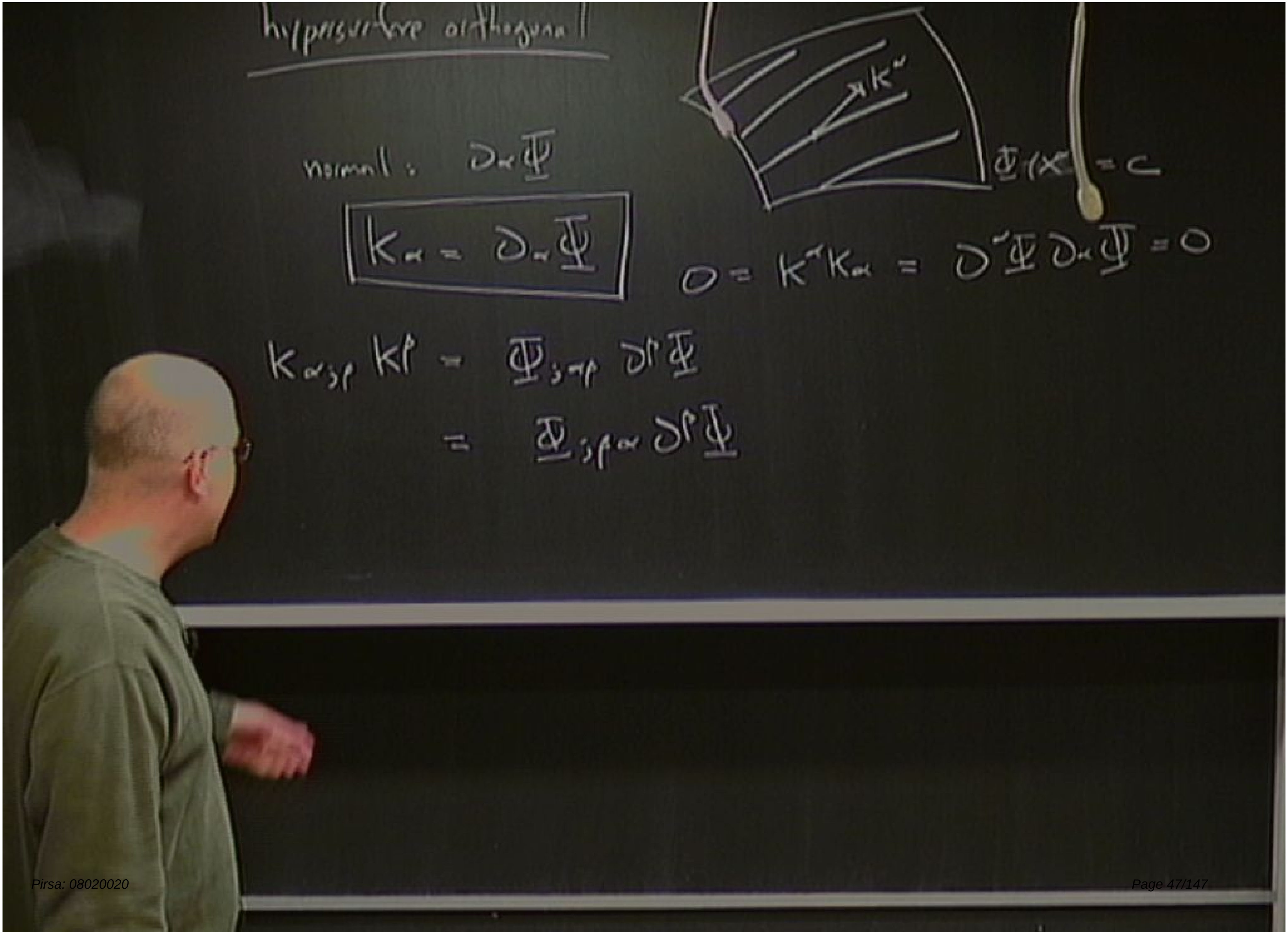

hypersurface orthogonal
normal: DαΦ
kα = DαΦ
kα
Φ(x) = c

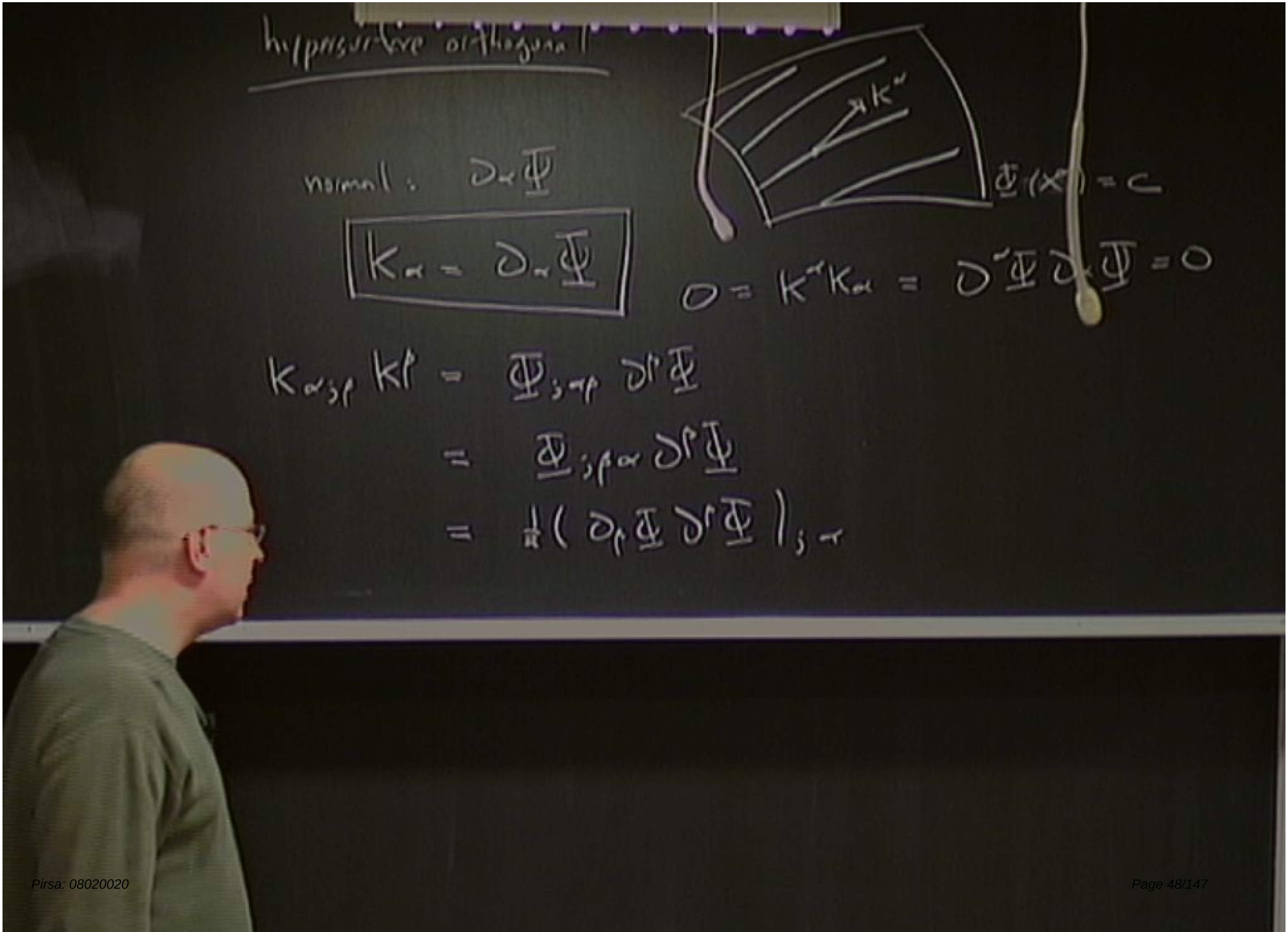

hypersurface orthogonal
normal: ∂αΦ
kα = ∂αΦ
k^α
Φ(x^α) = c

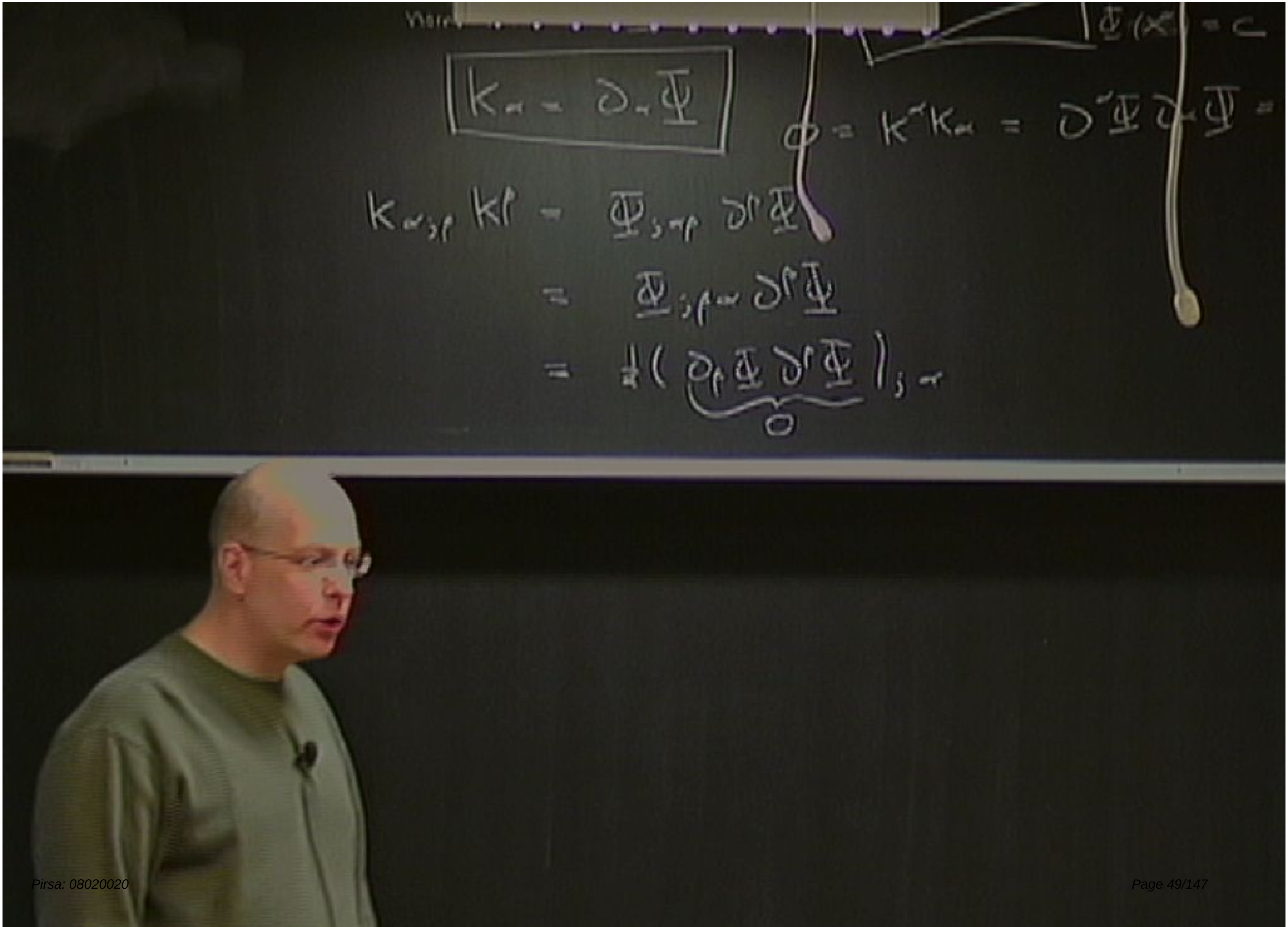

hypersurface orthogonal
normal : ∂α Φ
kα = ∂α Φ
k^α
Φ(x^α) = c
0 = k^α kα = ∂^α Φ ∂α Φ = 0
k

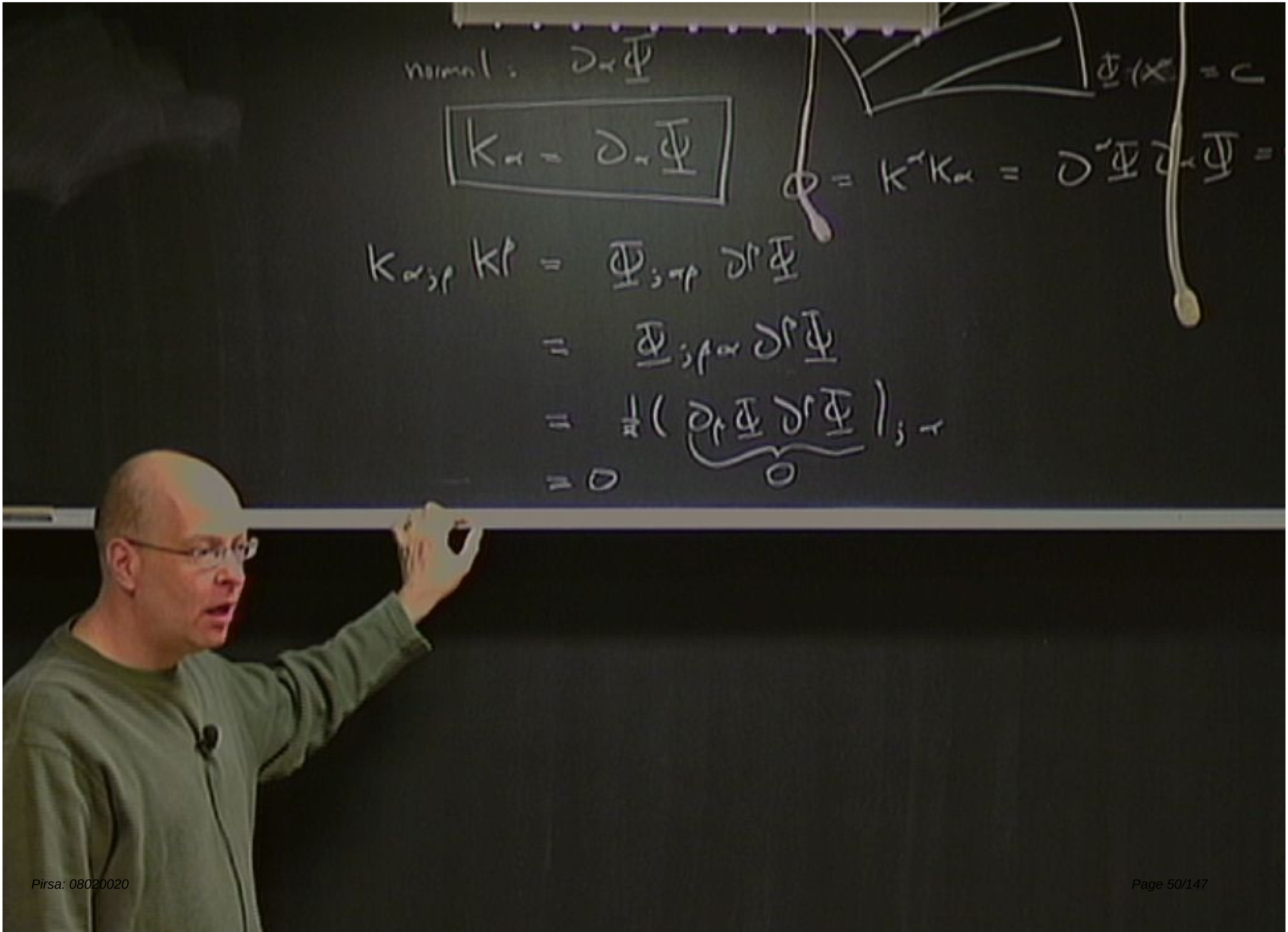

hypersurface orthogonal
normal: ∂α Φ
Kα = ∂α Φ
K^α
Φ(x^α) = c
0 = K^α Kα = ∂^α Φ ∂α Φ = 0
Kα;β K^β = Φ;αβ

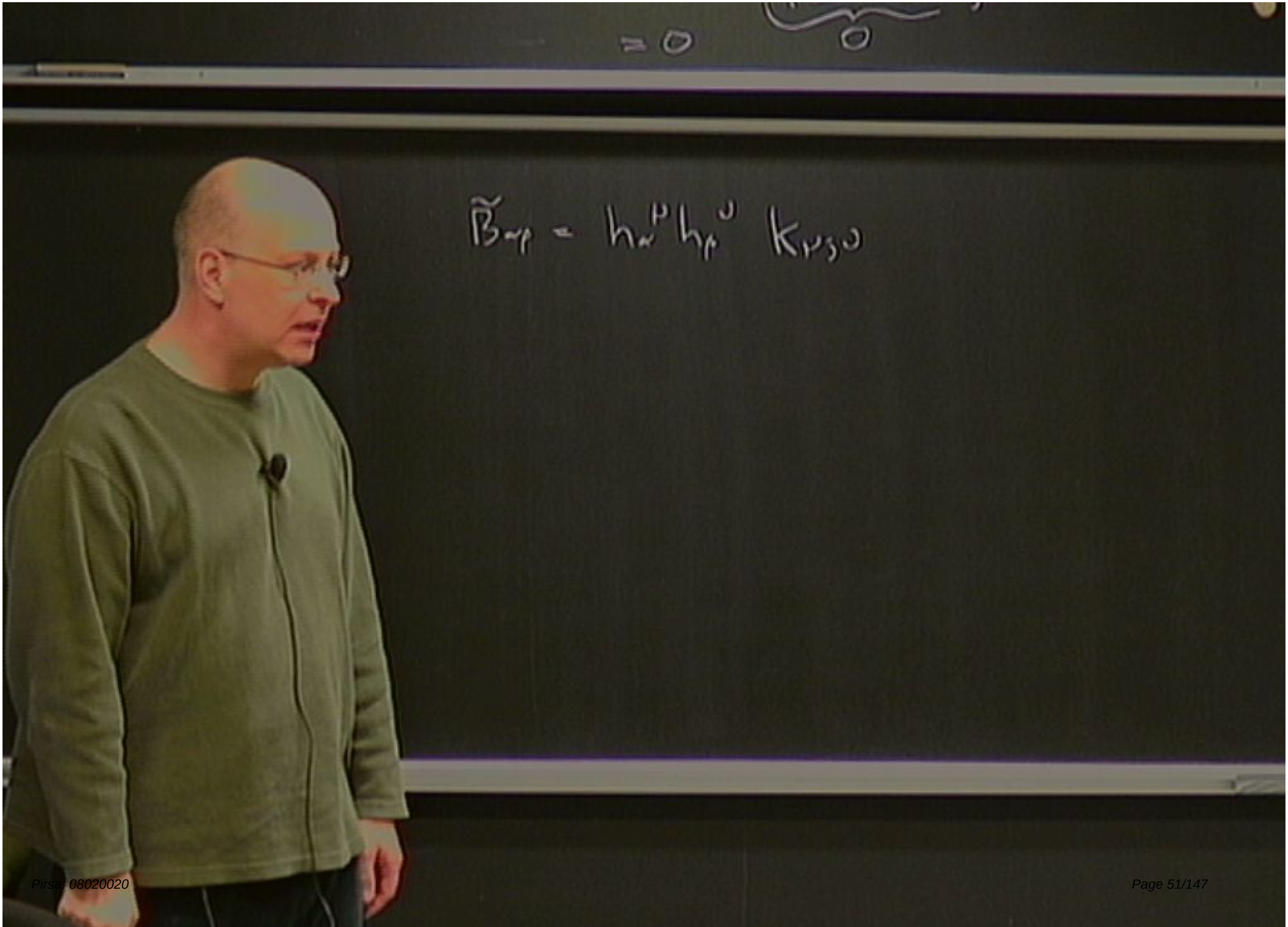

hypersurface orthogonal
normal: ∂_α Φ
K_α = ∂_α Φ
K^α
Φ(x) = c
0 = K^α K_α = ∂^α Φ ∂_α Φ = 0
K_{α;β} K^β = Φ_{;αβ} ∂^β Φ

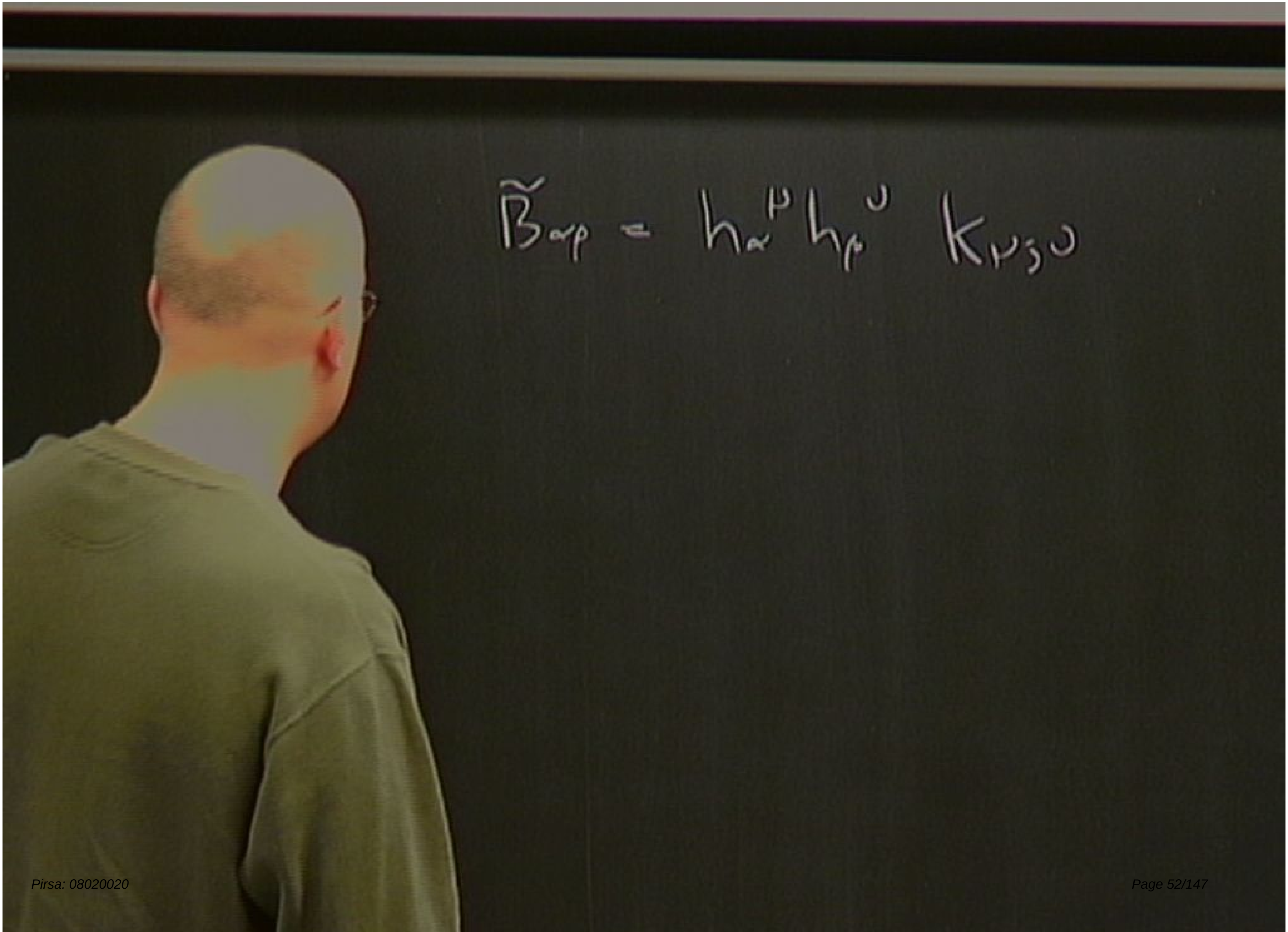

hypersurface orthogonal
K^α
normal: $\partial_\alpha \Phi$
$\Phi(x) = c$
$K_\alpha = \partial_\alpha \Phi$
$0 = K^\alpha K_\alpha = \partial^\alpha \Phi \, \partial_\alpha \Phi = 0$
$K_{\alpha;\beta} K^\beta = \Phi_{;\alpha\beta} \, \partial^\beta \Phi$
$= \Phi_{;\beta\alpha} \, \partial^\beta \Phi$

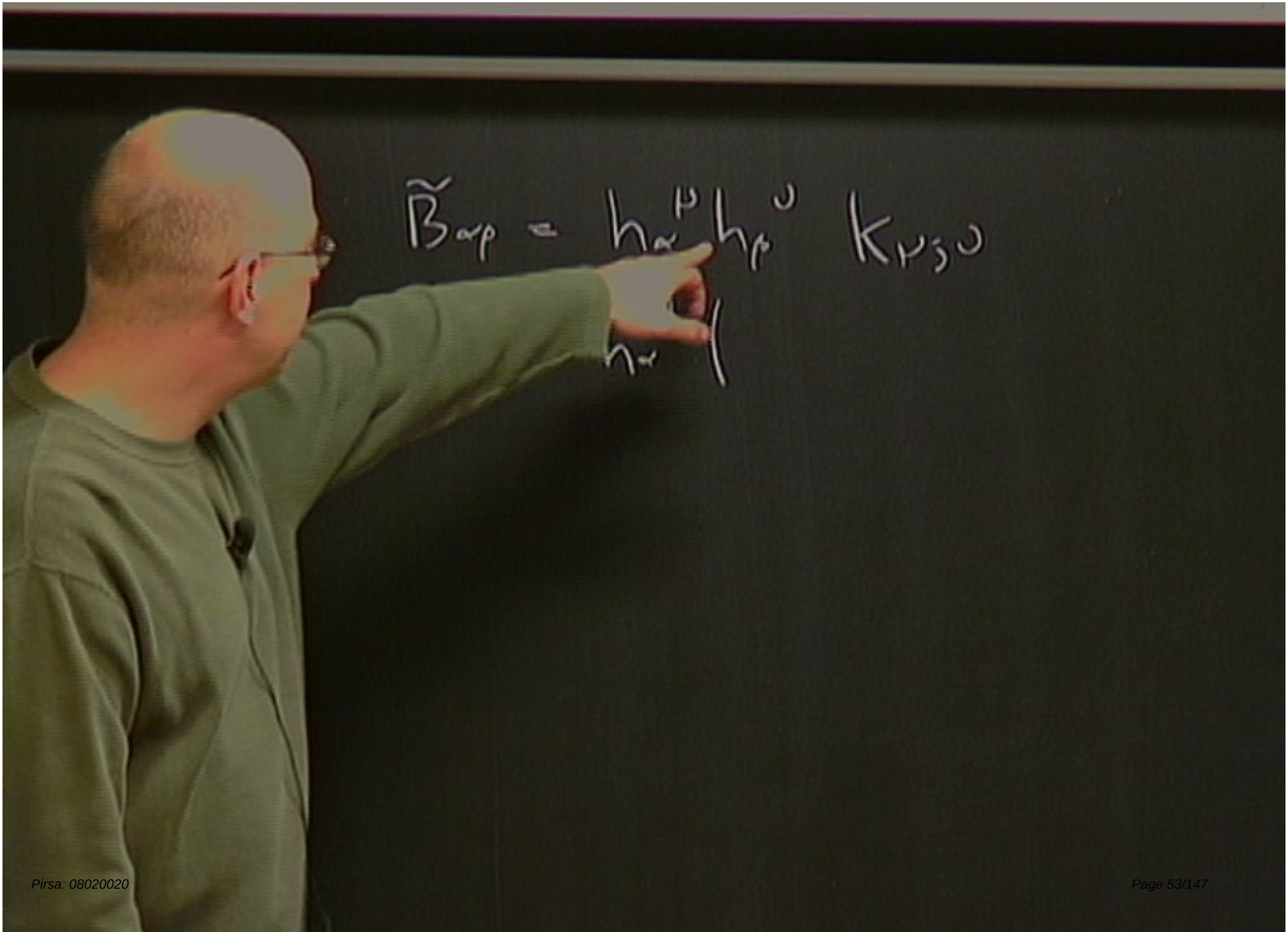

hypersurface orthogonal
normal : ∂_α Φ
k_α = ∂_α Φ
k^α
Φ(x^α) = c
0 = k^α k_α = ∂^α Φ ∂_α Φ = 0
k_{α;β} k^β = Φ_{;αβ} ∂^β Φ
= Φ_{;βα} ∂^β Φ
= ½ (∂_β Φ ∂^β Φ)_{;α}


Kα = ∂α Φ
Φ(x) = c
φ = K^α Kα = ∂^α Φ ∂α Φ =
Kα;β K^β = Φ;αβ ∂^β Φ
= Φ;βα ∂^β Φ
= ½ (∂β Φ ∂^β Φ);α


normal : ∂_α Φ
K_α = ∂_α Φ
Φ(x) = c
Φ = K^α K_α = ∂^α Φ ∂_α Φ =
K_α;β K^β = Φ;αβ ∂^β Φ
= Φ;βα ∂^β Φ
= ½ (∂_β Φ ∂^β Φ);α
0
= 0


= 0
0
$\tilde{B}_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu K_{\mu 3 \nu}$


$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$


$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu\rho\nu}$
h_α (

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu \, k_{\mu;\nu}$$

$$= h_\alpha{}^\mu \left(g_\beta{}^\nu + k_\beta N^\nu + N_\beta k^\nu \right) k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu \, k_{\mu;\nu}$$

$$= h_\alpha{}^\mu \left(g_\beta{}^\nu + k_\beta N^\nu + \cancel{N_\beta k^\nu} \right) k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu \, k_{\mu;\nu}$$

$$= h_\alpha{}^\mu \left(g_\beta{}^\nu + k_\beta N^\nu + \cancel{N_\beta k^\nu} \right) k_{\mu;\nu}$$

$$= h_\alpha{}^\mu \left(k_{\mu;\beta} + k_\beta \, k_{\mu;\nu} N^\nu \right)$$

$$= h_\alpha{}^\mu \left(g_\beta{}^\nu + k_\beta N^\nu + \cancel{N_\beta k^\nu} \right) k_{\mu;\nu}$$

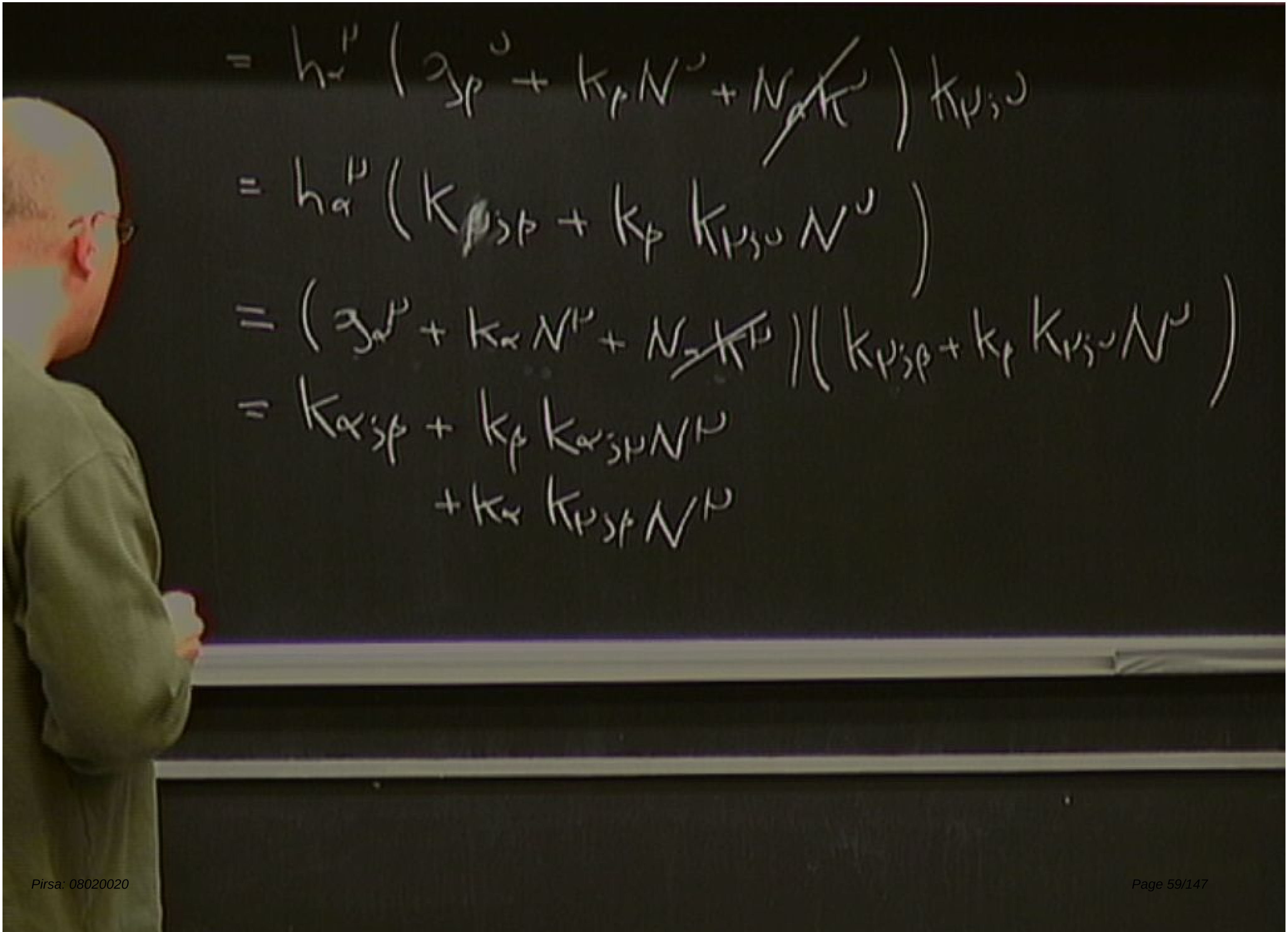
$$= h_\alpha{}^\mu \left(k_{\mu;\beta} + k_\beta \, k_{\mu;\nu} N^\nu \right)$$

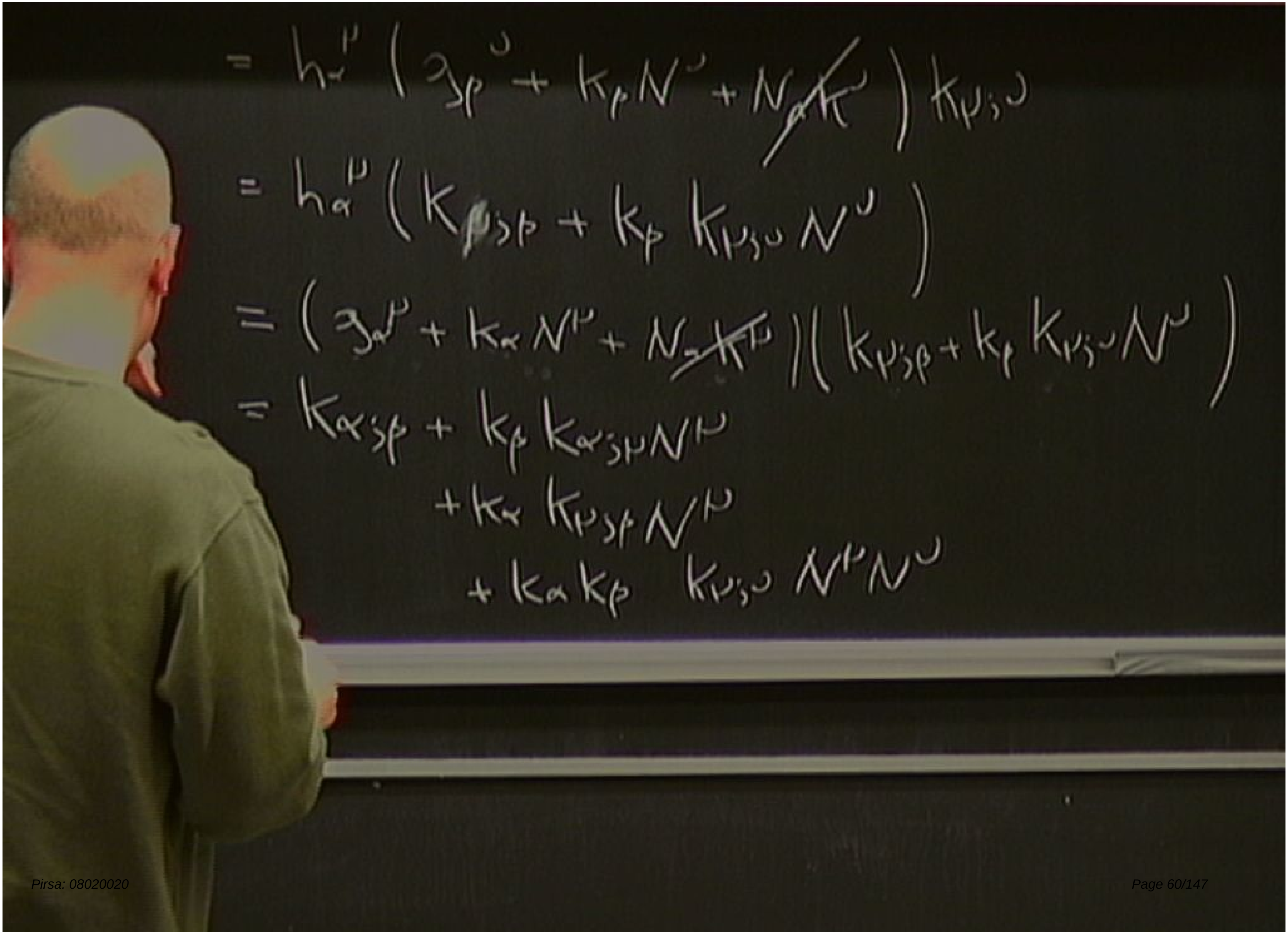
$$= \left(g_\alpha{}^\mu + k_\alpha N^\mu + N_\alpha k^\mu \right)\left(k_{\mu;\beta} + k_\beta \, k_{\mu;\nu} N^\nu \right)$$

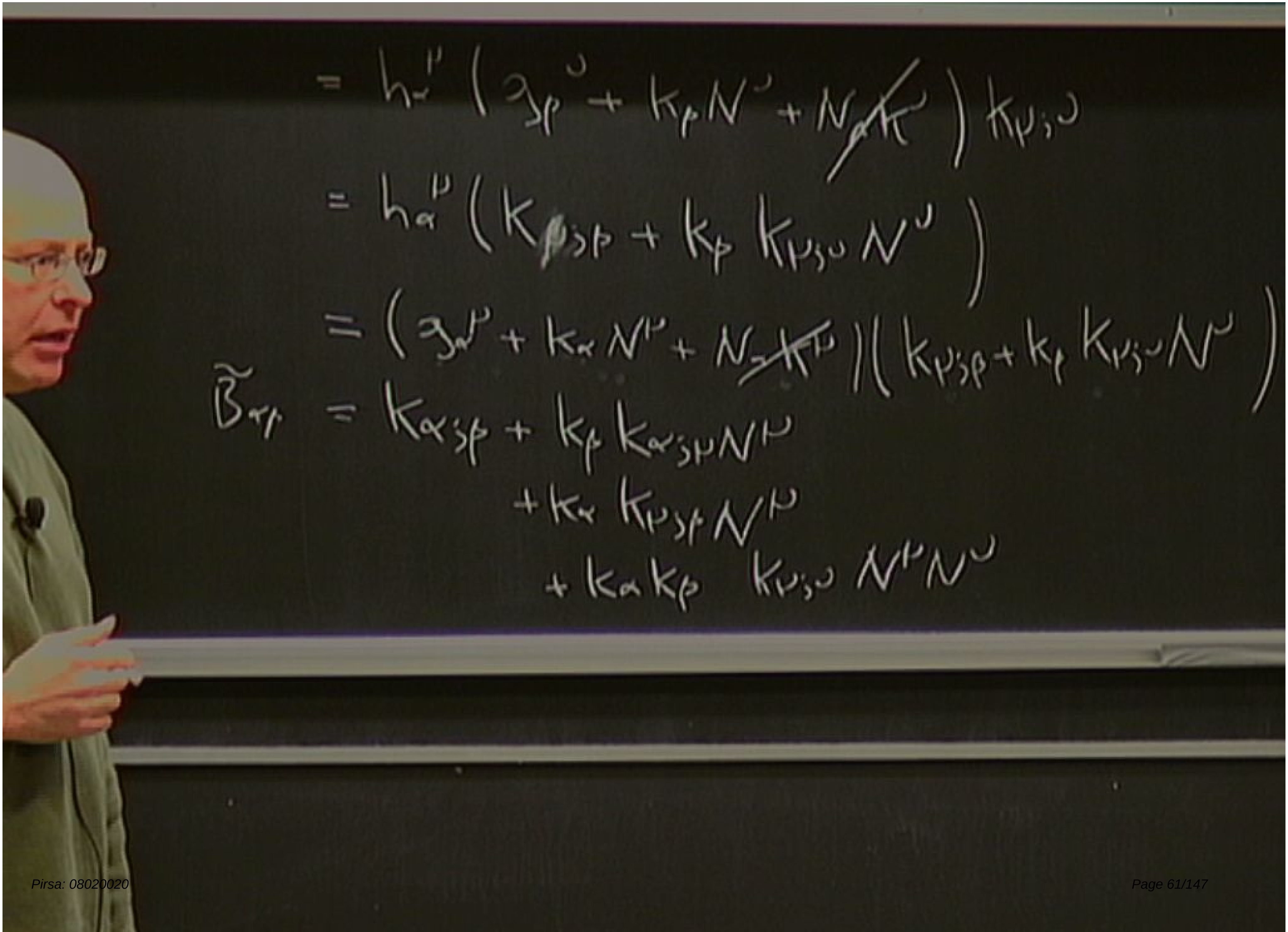
$$= h_\alpha{}^\mu \left(g_\beta{}^\nu + k_\beta N^\nu + \cancel{N_\beta k^\nu} \right) k_{\mu;\nu}$$

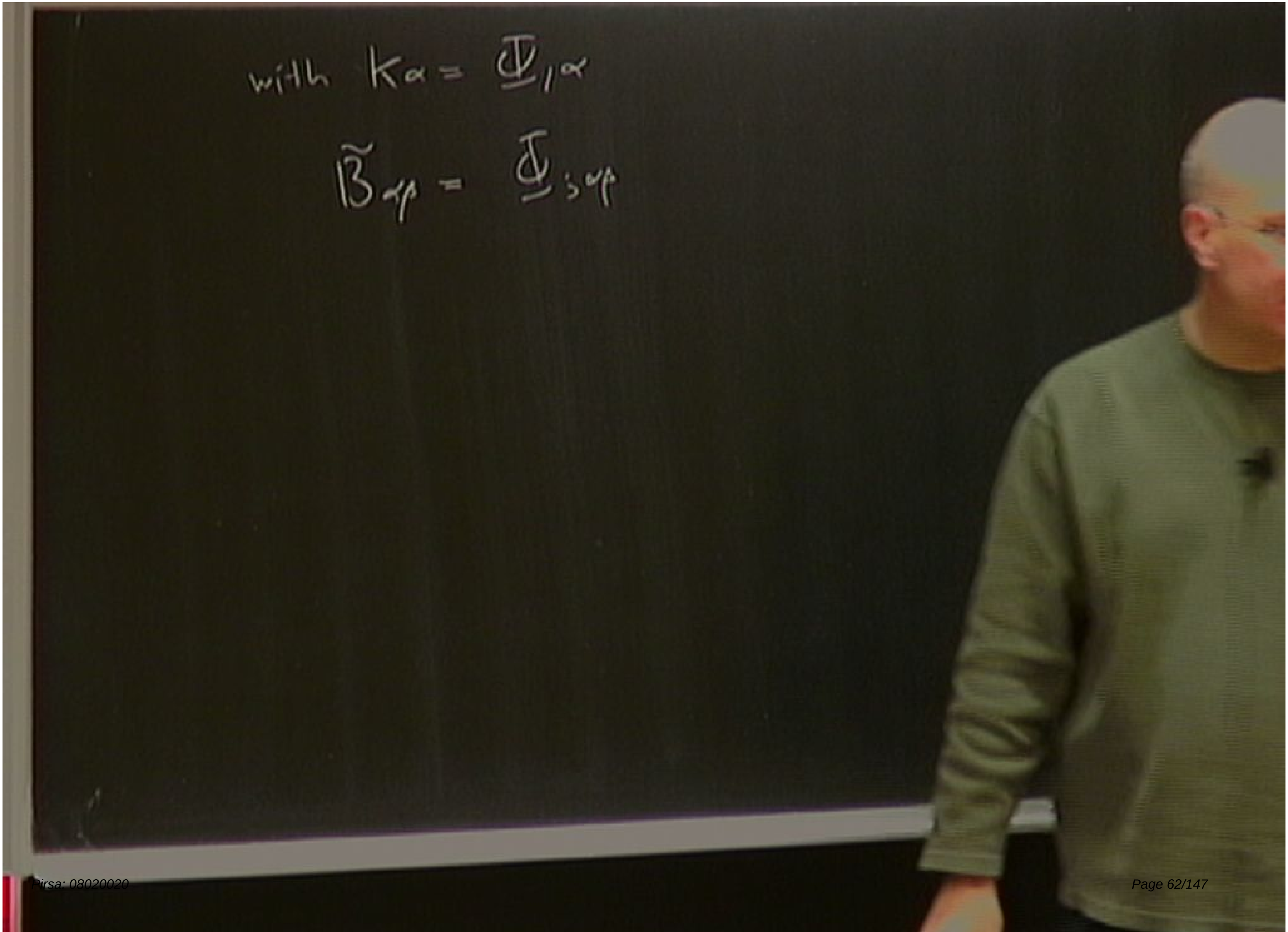
$$= h_\alpha{}^\mu \left(k_{\mu;\beta} + k_\beta\, k_{\mu;\nu} N^\nu \right)$$

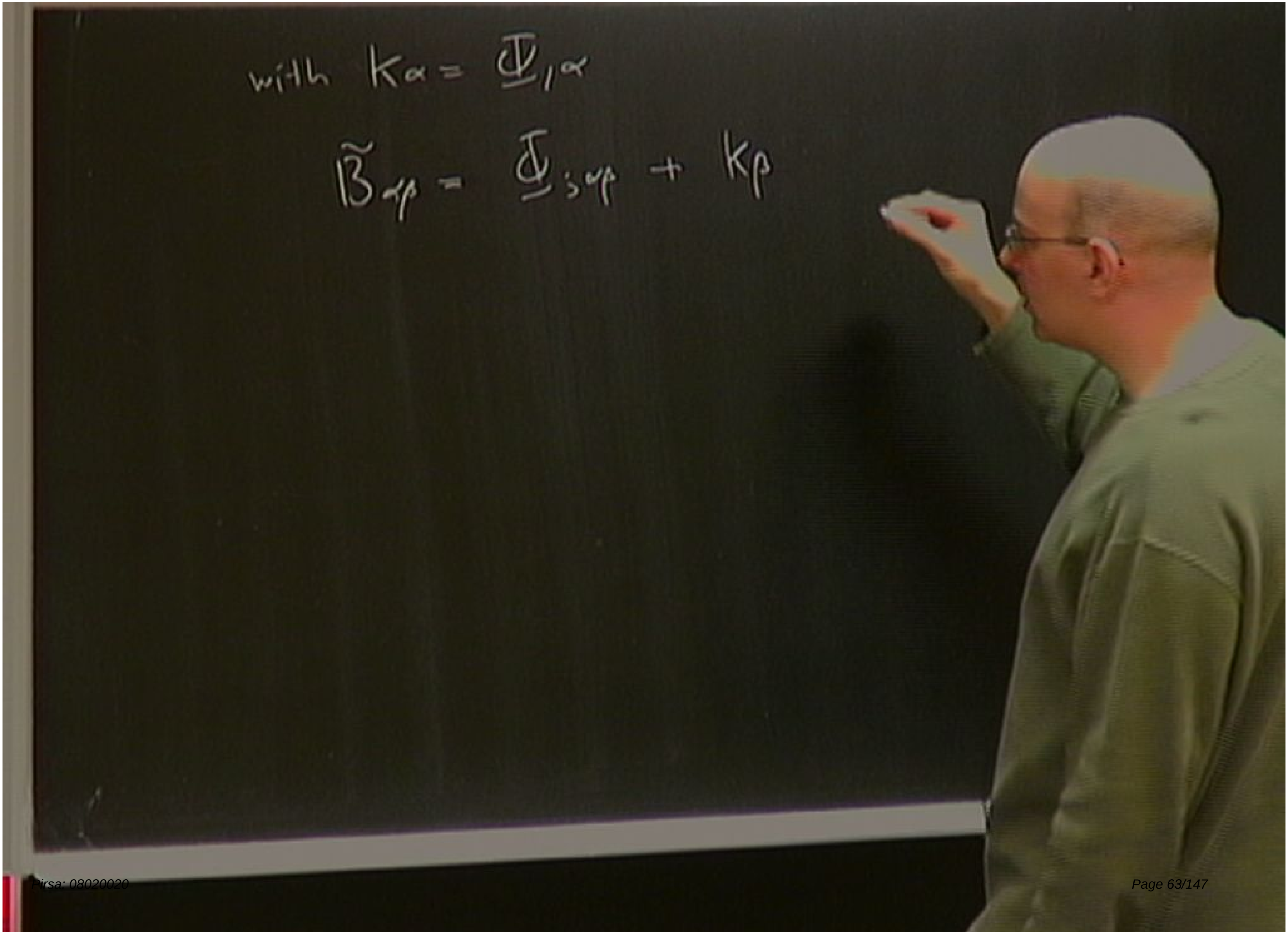
$$= \left(g_\alpha{}^\mu + k_\alpha N^\mu + \cancel{N_\alpha k^\mu} \right)\left(k_{\mu;\beta} + k_\beta\, k_{\mu;\nu} N^\nu \right)$$


$= h_\alpha{}^\mu \left(g_\rho{}^\nu + k_\rho N^\nu + N_\rho k^\nu \right) k_{\mu;\nu}$
$= h_\alpha{}^\mu \left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$= \left(g_\alpha{}^\mu + k_\alpha N^\mu + N_\alpha k^\mu \right)\left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$= k_{\alpha;\rho} + k_\rho k_{\alpha;\mu} N^\mu$
$+ k_\alpha k_{\mu;\rho} N^\mu$


$= h_\alpha{}^\mu \left(g_\rho{}^\nu + k_\rho N^\nu + N_\rho k^\nu \right) k_{\mu;\nu}$
$= h_\alpha{}^\mu \left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$= \left(g_\alpha{}^\mu + k_\alpha N^\mu + N_\alpha k^\mu \right)\left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$= k_{\alpha;\rho} + k_\rho k_{\alpha;\mu} N^\mu + k_\alpha k_{\mu;\rho} N^\mu + k_\alpha k_\rho k_{\mu;\nu} N^\mu N^\nu$


$= h_\alpha{}^\mu \left(g_\rho{}^\nu + k_\rho N^\nu + \cancel{N_\rho k^\nu} \right) k_{\mu;\nu}$
$= h_\alpha{}^\mu \left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$= \left(g_\alpha{}^\mu + k_\alpha N^\mu + \cancel{N_\alpha k^\mu} \right)\left(k_{\mu;\rho} + k_\rho k_{\mu;\nu} N^\nu \right)$
$\tilde{B}_{\alpha\rho} = k_{\alpha;\rho} + k_\rho k_{\alpha;\mu} N^\mu$
$+ k_\alpha k_{\mu;\rho} N^\mu$
$+ k_\alpha k_\rho k_{\mu;\nu} N^\mu N^\nu$


with $K_\alpha = \underline{\Phi}_{,\alpha}$
$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta}$

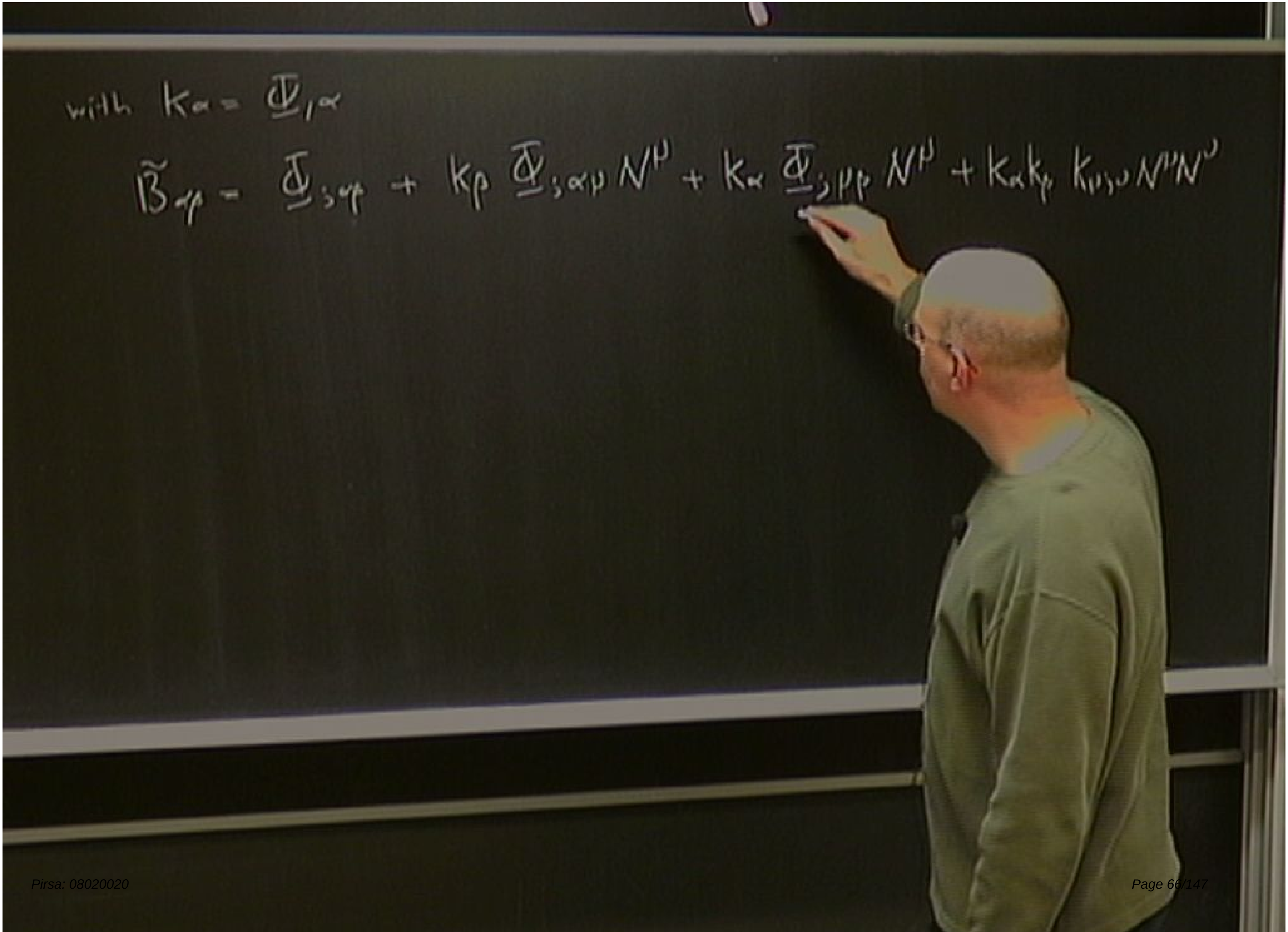

with $K_\alpha = \underline{\Phi}_{,\alpha}$
$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + K_\beta$

with $K_\alpha = \underline{\Phi}_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + K_\beta \, \underline{\Phi}_{;\alpha\mu} N^\mu + K_\alpha \ldots$$

with $k_\alpha = \underline{\Phi}_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + k_\beta \underline{\Phi}_{;\alpha\mu} N^\mu + k_\alpha \underline{\Phi}_{;\mu\beta} N^\mu$$



with $k_\alpha = \underline{\Phi}_{,\alpha}$
$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + k_\beta \underline{\Phi}_{;\alpha\mu} N^\mu + k_\alpha \underline{\Phi}_{;\mu\beta} N^\mu + k_\alpha k_\beta k_{\mu;\nu} N^\mu N^\nu$

with $k_\alpha = \underline{\Phi}_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + k_\beta \underline{\Phi}_{;\alpha\mu} N^\mu + k_\alpha \underbrace{\underline{\Phi}_{;\mu\beta}}_{\underline{\Phi}_{;\alpha\beta}} N^\mu + k_\alpha k_\beta k_{\mu;\nu} N^\mu N^\nu$$

with $k_\alpha = \underline{\Phi}_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + k_\beta\, \underline{\Phi}_{;\alpha\mu} N^\mu + k_\alpha \underbrace{\underline{\Phi}_{;\mu\beta}}_{\underline{\Phi}_{;\alpha\beta}} N^\mu + k_\alpha k_\beta\, k_{\mu;\nu} N^\mu N^\nu$$

$$= \tilde{B}_{(\alpha\beta)}$$

with $K_\alpha = \underline{\Phi}_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \underline{\Phi}_{;\alpha\beta} + K_\beta \underline{\Phi}_{;\alpha\mu} N^\mu + K_\alpha \underbrace{\underline{\Phi}_{;\mu\beta}}_{\underline{\Phi}_{;\alpha\beta}} N^\mu + K_\alpha K_\beta K_{\mu;\nu} N^\mu N^\nu$$

$$= \tilde{B}_{(\alpha\beta)}$$

$$\Rightarrow \tilde{B}_{[\alpha\beta]} = \omega_{\alpha\beta} = 0$$

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$$\tilde{B}_{\alpha\beta} = \Phi_{;\alpha\beta} + k_\beta \Phi_{;\alpha\mu} N^\mu + k_\alpha \underbrace{\Phi_{;\mu\beta}}_{\Phi_{;\alpha\beta}} N^\mu + k_\alpha k_\beta k_{\mu;\nu} N^\mu N^\nu$$

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hypersurfrace orthogonal $\longrightarrow$ $\omega_{\alpha\beta} = 0$

with $k_\alpha = \Phi_{,\alpha}$

$$\tilde{B}_{\alpha\beta} = \Phi_{;\alpha\beta} + k_\beta \Phi_{;\alpha\mu} N^\mu + k_\alpha \underbrace{\Phi_{;\mu\beta}}_{\Phi_{;\alpha\beta}} N^\mu + k_\alpha k_\beta k_{\mu;\nu} N^\mu N^\nu$$

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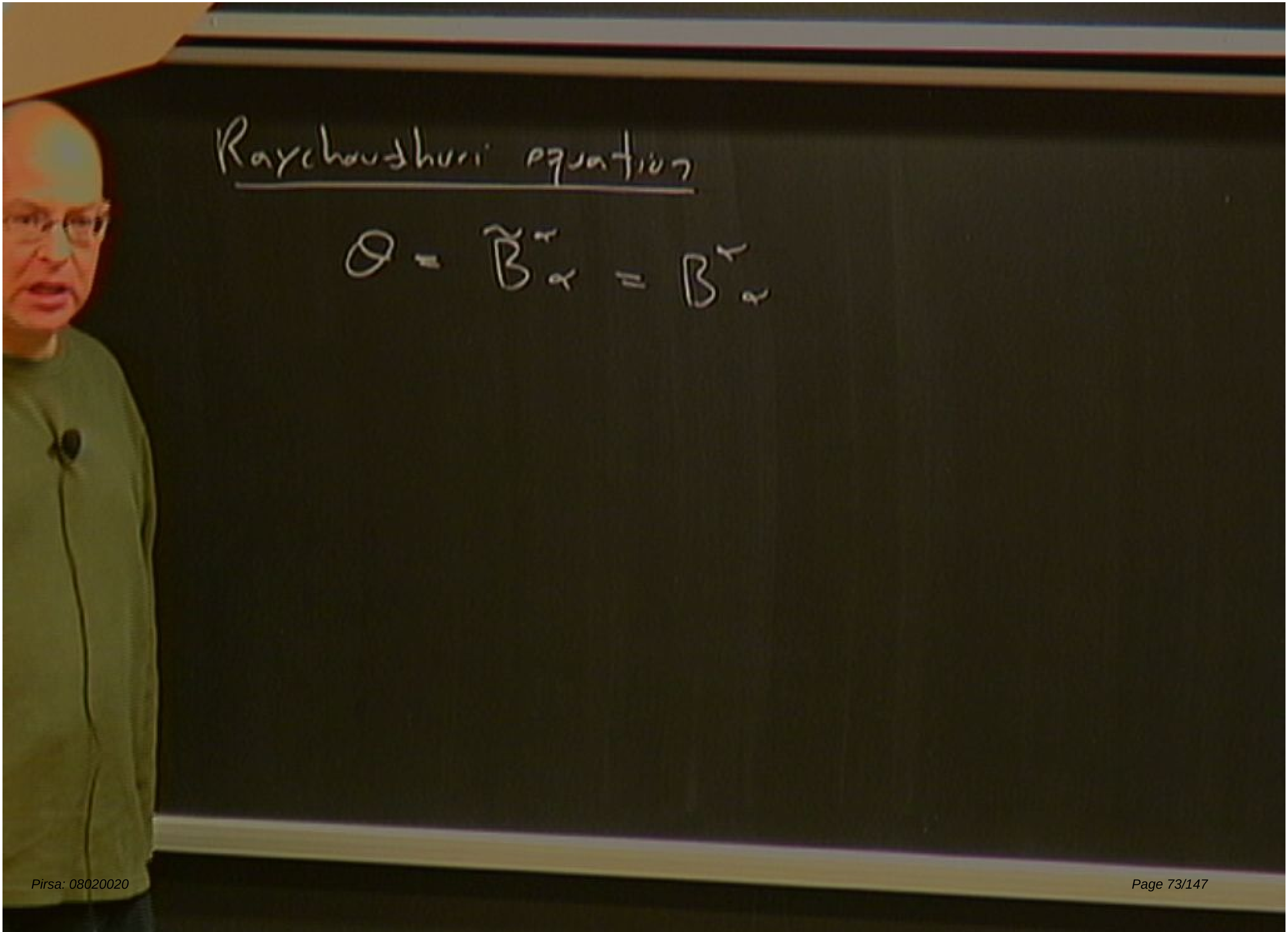
$$\boxed{\text{hypersurface orthogonal} \rightleftarrows \omega_{\alpha\beta} = 0}$$

Raychaudhuri equation


Raychaudhuri equation
$\Theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha}$

Raychaudhuri equation

$$\Theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha} - k^{\alpha}{}_{;\alpha}$$

Raychaudhuri equation

$$\theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha} - k^{\alpha}{}_{;\alpha}$$

$$\frac{d\theta}{d\lambda} =$$

Raychaudhuri equation

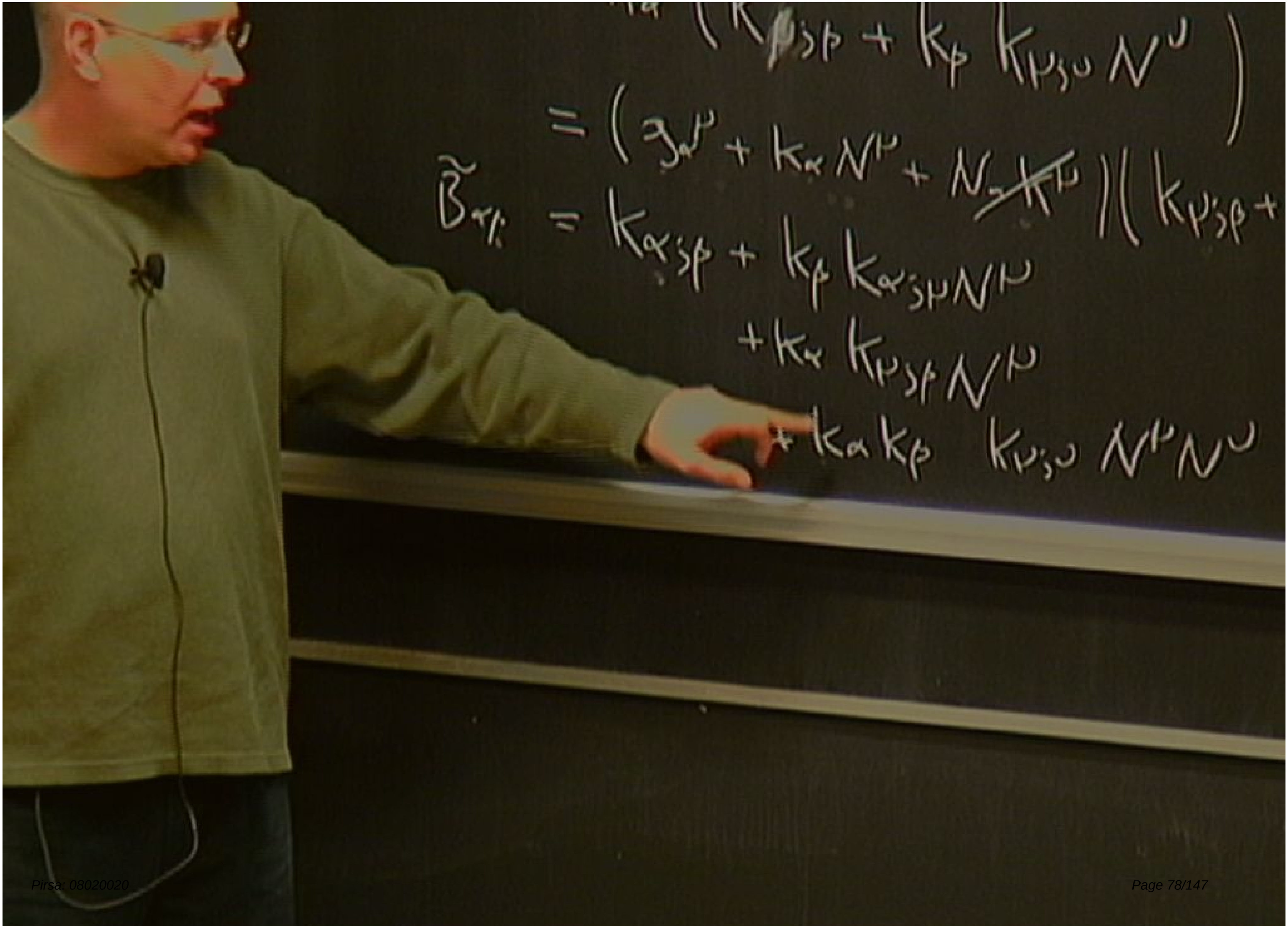
$$\Theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha} - k^{\alpha}{}_{;\alpha}$$

$$\frac{d\Theta}{d\lambda} = - B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$$

Raychoudhuri equation

$$\Theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha} = k^{\alpha}{}_{;\alpha}$$

$$\frac{d\Theta}{d\lambda} = -B^{\alpha\beta}B_{\beta\alpha} - R_{\alpha\beta}k^{\alpha}k^{\beta} \quad \text{(same as timelike case)}$$


$(K_{\mu;\beta} + k_\beta K_{\mu;\nu} N^\nu)$
$= (g_\alpha^\mu + k_\alpha N^\mu + N_\alpha k^\mu)(K_{\mu;\beta} +$
$\tilde{B}_{\alpha\beta} = K_{\alpha;\beta} + k_\beta K_{\alpha;\mu} N^\mu$
$+ k_\alpha K_{\mu;\beta} N^\mu$
$+ k_\alpha k_\beta K_{\mu;\nu} N^\mu N^\nu$

Raychaudhuri equation

$$\theta = \tilde{B}^{\alpha}{}_{\alpha} = B^{\alpha}{}_{\alpha} - k^{\alpha}{}_{;\alpha}$$

$$\frac{d\theta}{d\lambda} = - B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$$ (same as timelike case

$$= - \tilde{B}^{\alpha\beta} \tilde{B}_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$$

$\theta = B^\alpha{}_\alpha = \tilde{B}^\alpha{}_\alpha = k^\alpha{}_{;\alpha}$

$$\frac{d\theta}{d\lambda} = -B^{\alpha\beta}B_{\beta\alpha} - R_{\alpha\beta}k^\alpha k^\beta \quad \text{(same as timelike case}$$

$$= -\tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}k^\alpha k^\beta$$

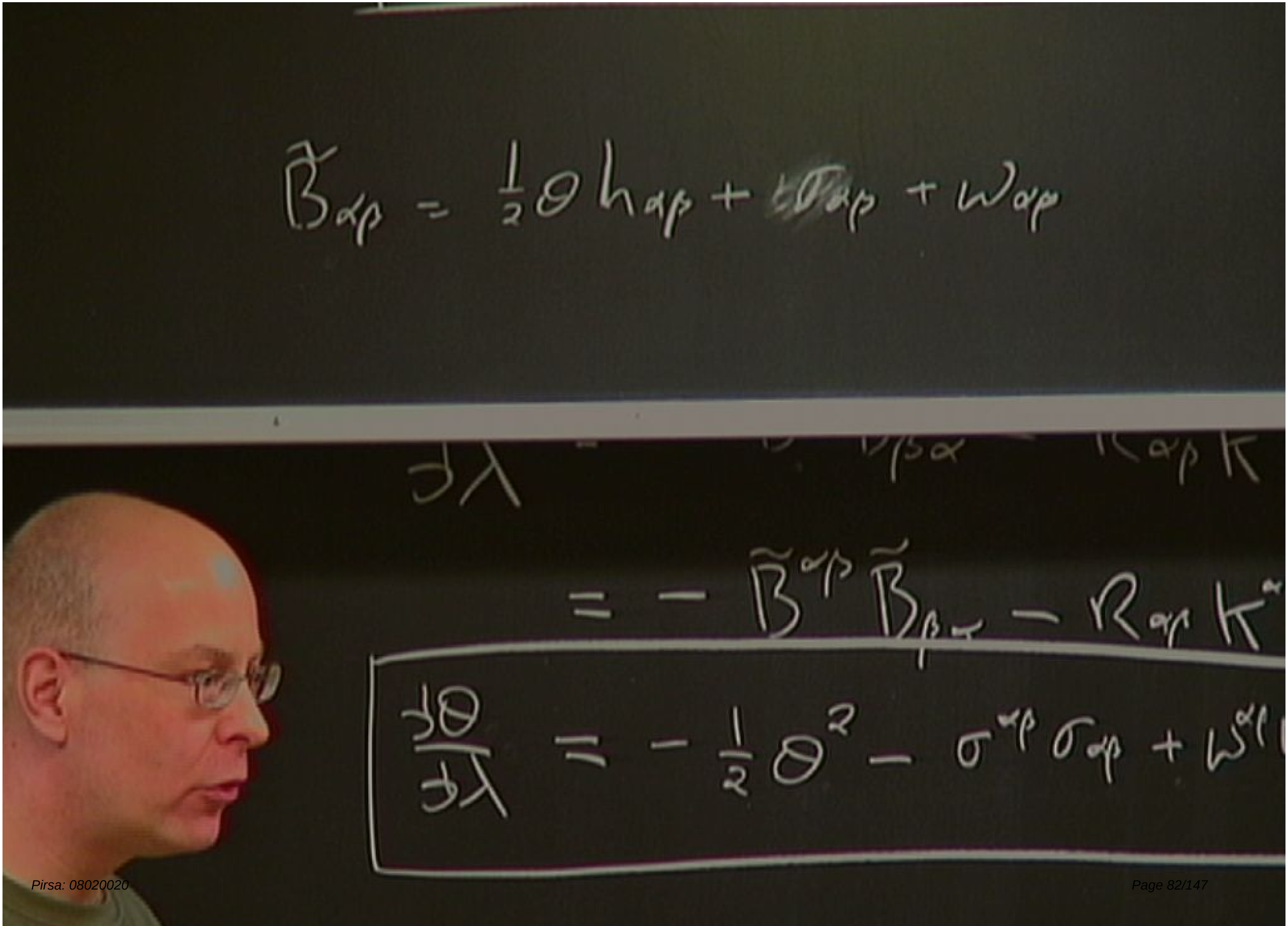
$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2$$

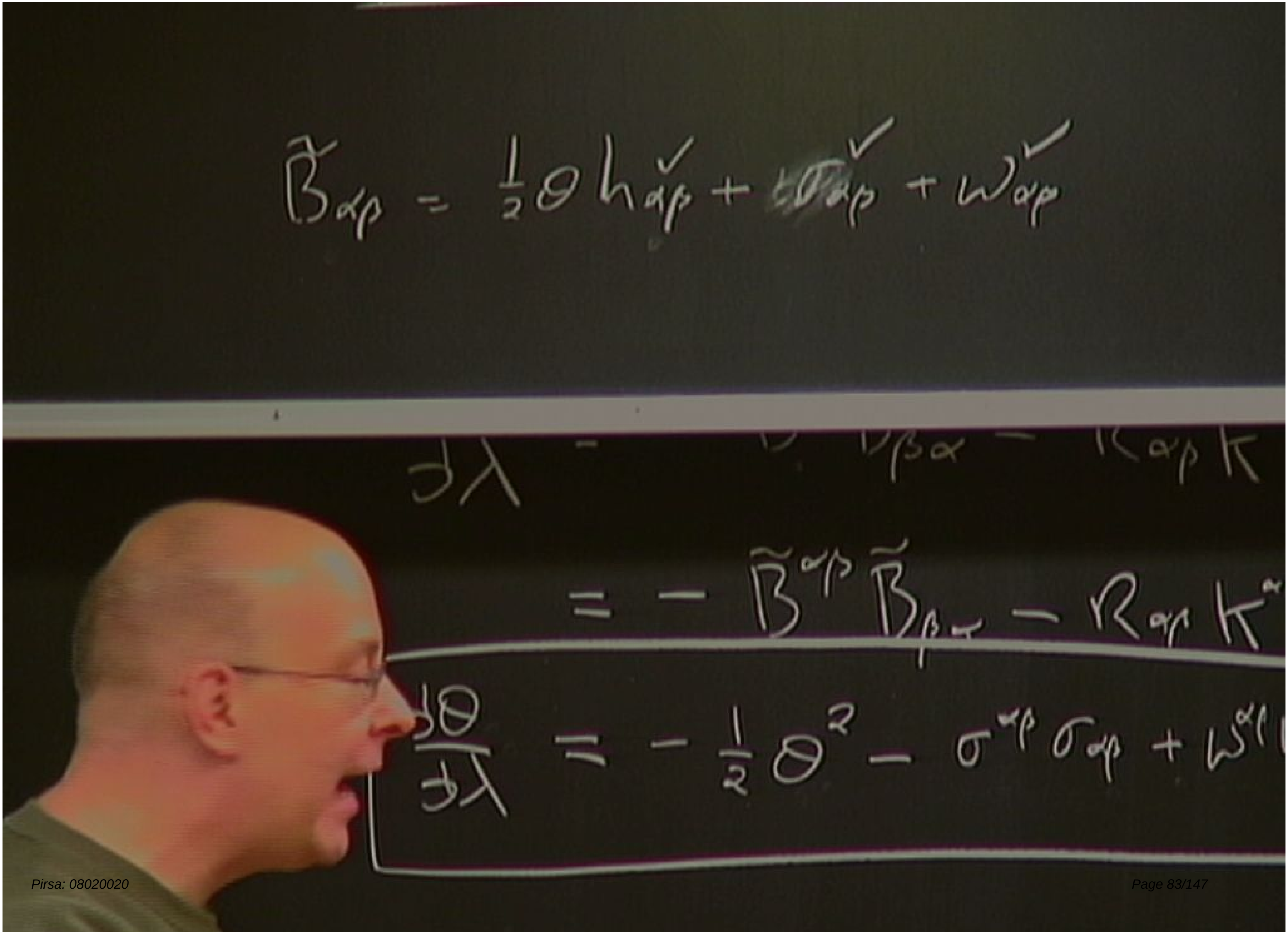
$$\theta = B^{\alpha}{}_{\alpha} = \tilde{B}^{\alpha}{}_{\alpha} = k^{\alpha}{}_{;\alpha}$$

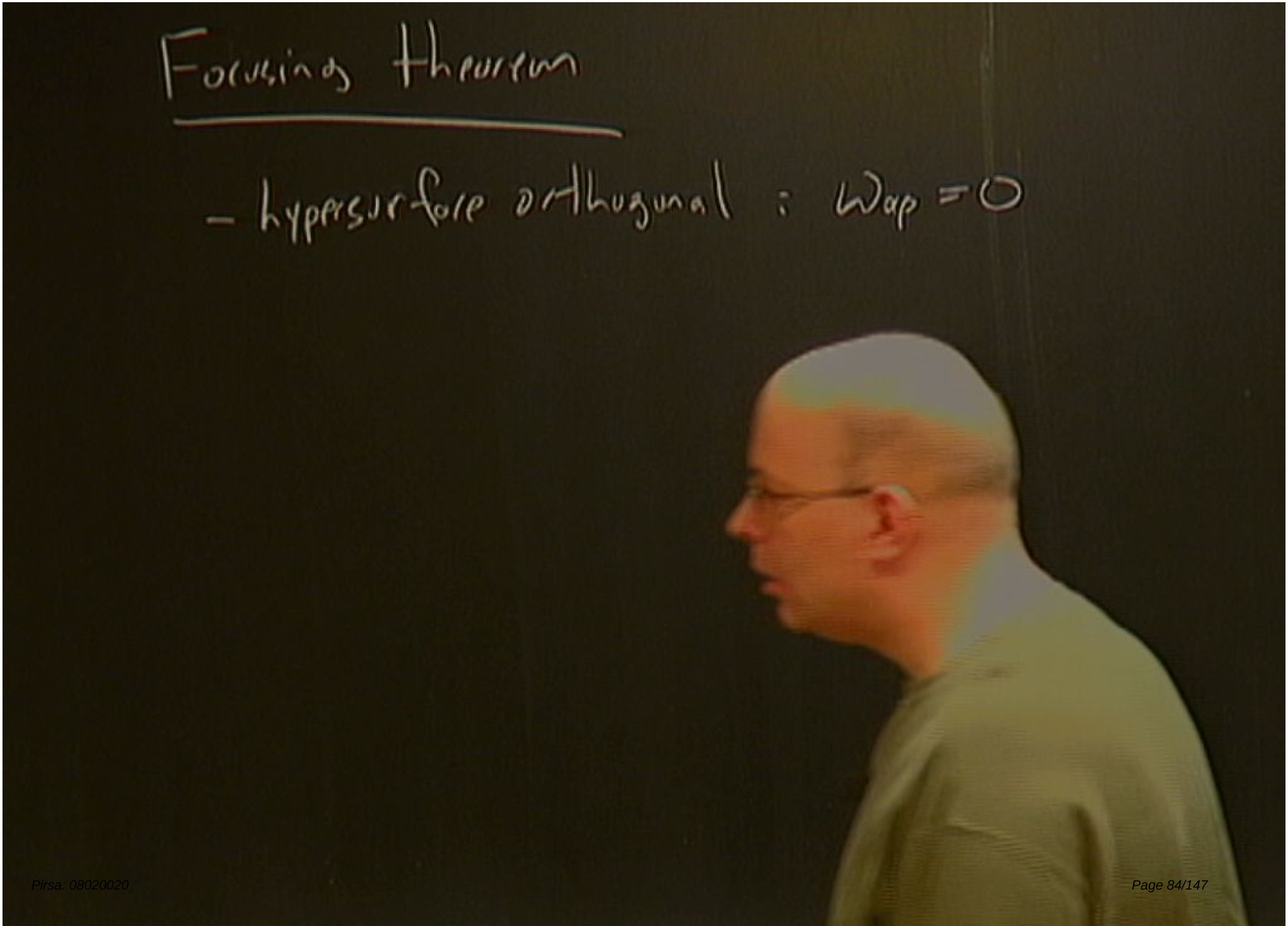
$$\frac{d\theta}{d\lambda} = -B^{\alpha\beta}B_{\beta\alpha} - R_{\alpha\beta}k^{\alpha}k^{\beta}$$ (same as timelike case)

$$= -\tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}k^{\alpha}k^{\beta}$$

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^{\alpha}k^{\beta}}$$


$\tilde{B}_{\alpha\beta} = \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$= - \tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}K^{\alpha}$
$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}$


$\tilde{B}_{\alpha\beta} = \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$= - \tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}K^{\alpha}$
$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}$


Focusing theorem
– hypersurface orthogonal : $\omega_{ab} = 0$

Focusing theorem

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- null energy condition : $R_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0$

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Focusing theorem

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~~$\frac{1}{2} T g_{\alpha\beta}$~~

$$\boxed{T_{\alpha\beta} k^\alpha k^\beta \geq 0}$$

Focusing theorem

- hypersurface orthogonal : $\omega_{\alpha\beta} = 0$
- null energy condition : $R_{\alpha\beta} k^\alpha k^\beta \geq 0$

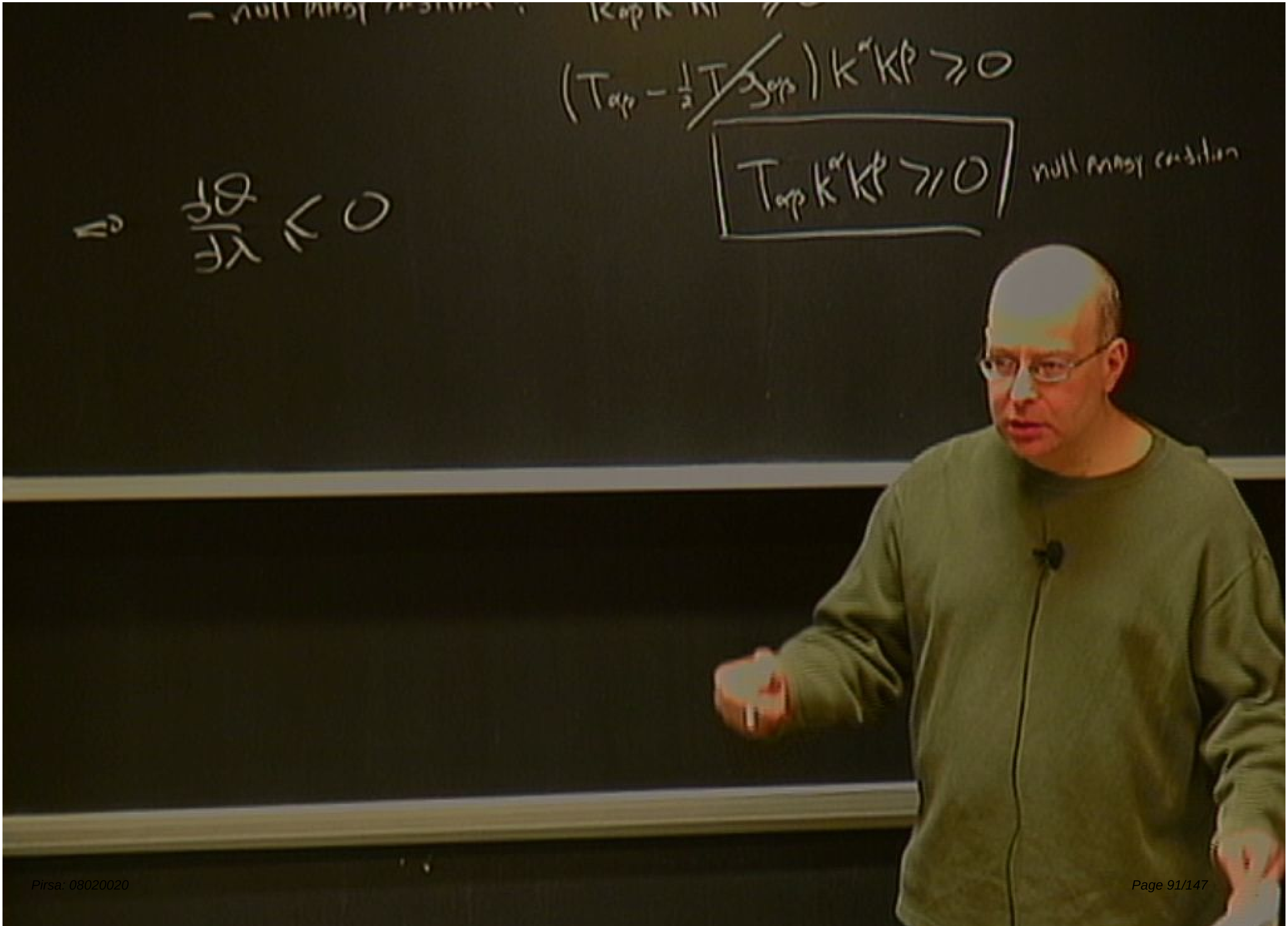
$$\left(T_{\alpha\beta} - \tfrac{1}{2} T g_{\alpha\beta}\right) k^\alpha k^\beta \geq 0$$

$$\boxed{T_{\alpha\beta} k^\alpha k^\beta \geq 0}$$ null energy condition

$\ldots_{\beta} k^{\alpha} k^{\beta} \geqslant 0$

$\ldots - \frac{1}{2} \not{T} g_{\alpha\beta} \big) k^{\alpha} k^{\beta} \geqslant 0$

$$\boxed{T_{\alpha\beta} k^{\alpha} k^{\beta} \geqslant 0}$$ null energy condition


$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) k^\alpha k^\beta \geq 0$
$T_{\alpha\beta} k^\alpha k^\beta \geq 0$
null energy condition
$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$

$\Rightarrow \quad \frac{d\theta}{d\lambda} \leq 0$

$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) k$

$T_{\alpha\beta} k^{\alpha}$

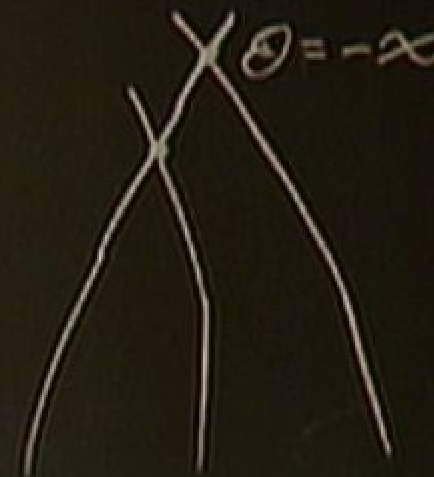
$$T_{\alpha\beta} k^\alpha k^\beta \geq 0$$ null

Initially contracting congruence will form caustics.

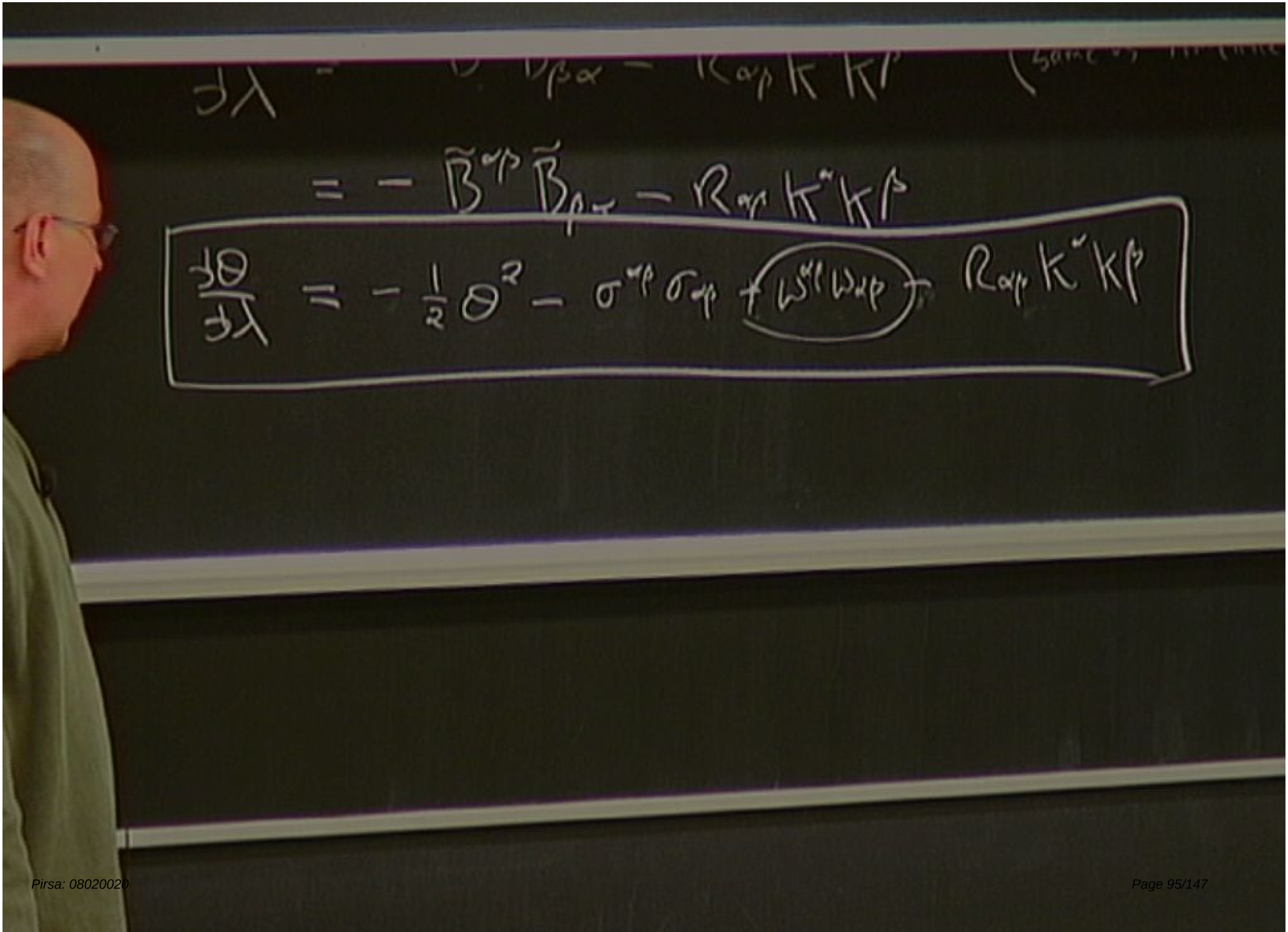
O

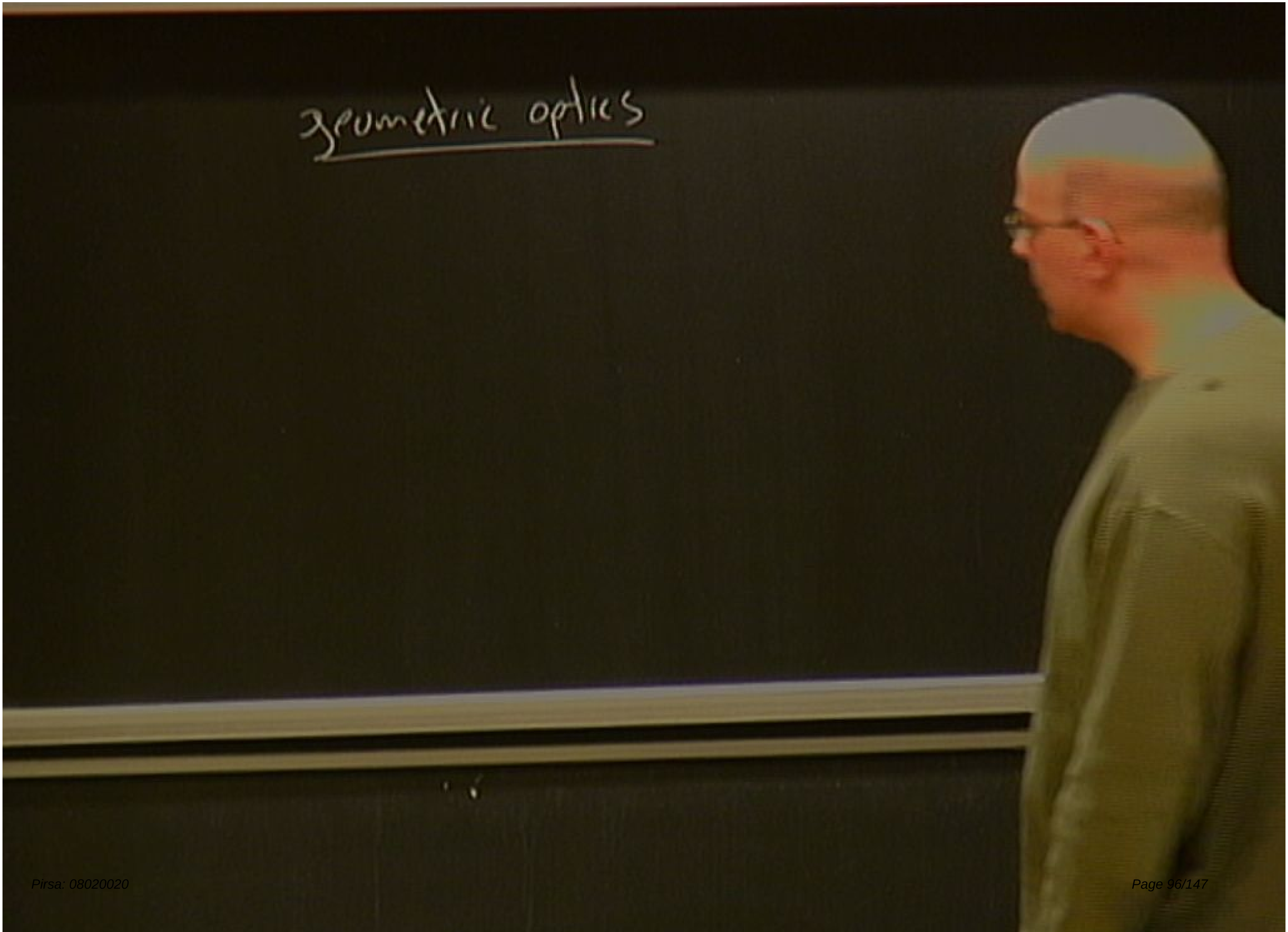
$$T_{\alpha\beta}k^\alpha k^\beta \geq 0$$

null energy condition



Initially contracting congruence will form caustics.


$= - \tilde{B}^{\alpha\beta}\tilde{B}_{\beta\alpha} - R_{\alpha\beta}k^\alpha k^\beta$
$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta$

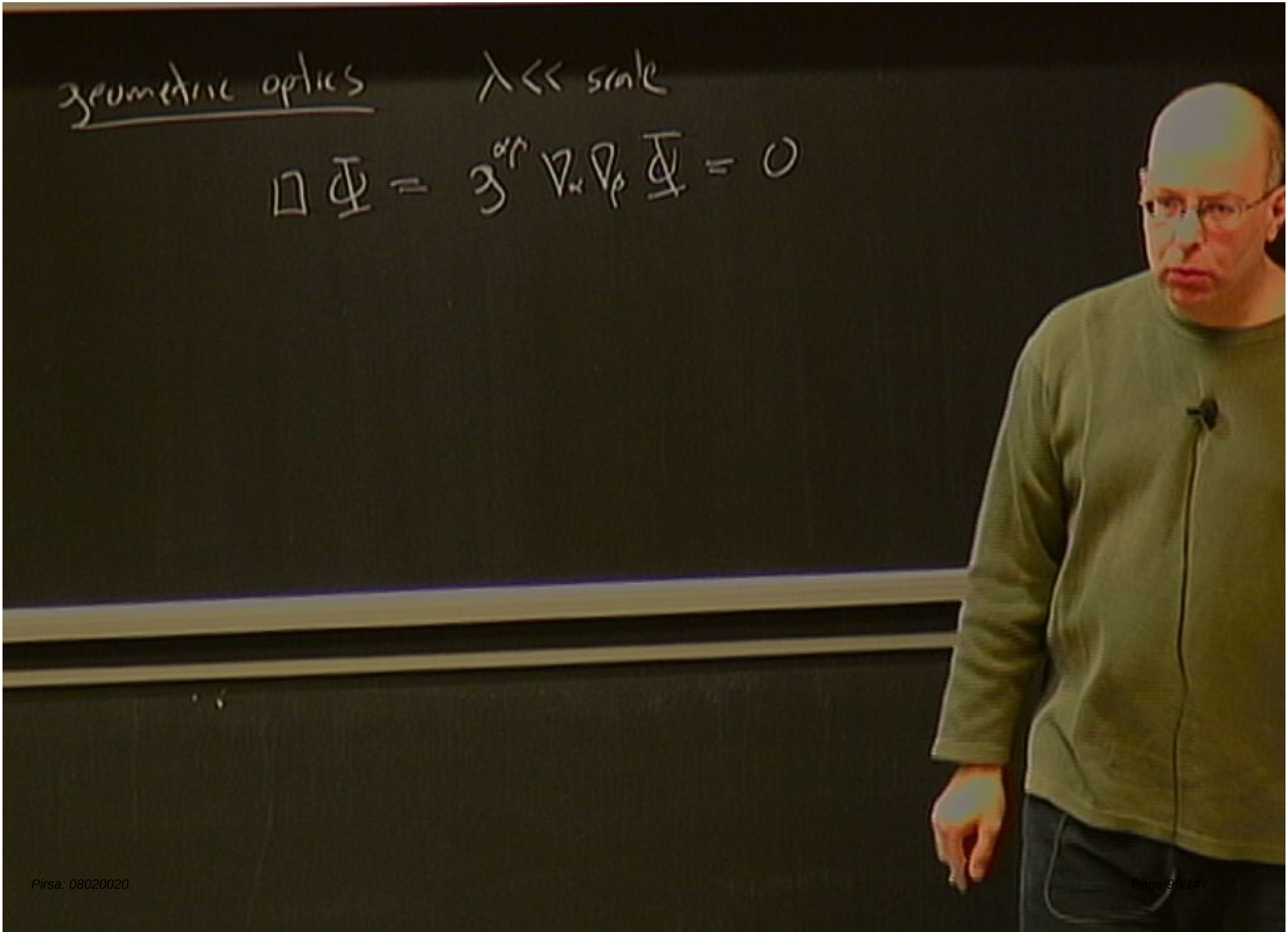

geometric optics

geometric optics

$\lambda \ll$ scale

geometric optics $\lambda \ll$ scale

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi$$


geometric optics
λ << scale
□ Φ = g^{αβ} ∇_α ∇_β Φ = 0

geometric optics $\lambda \ll$ scale

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$$

$$\Phi = A$$

geometric optics $\lambda << \text{scale}$

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$$

$$\Phi = A e^{iS}$$

geometric optics $\lambda << \text{scale}$

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$$

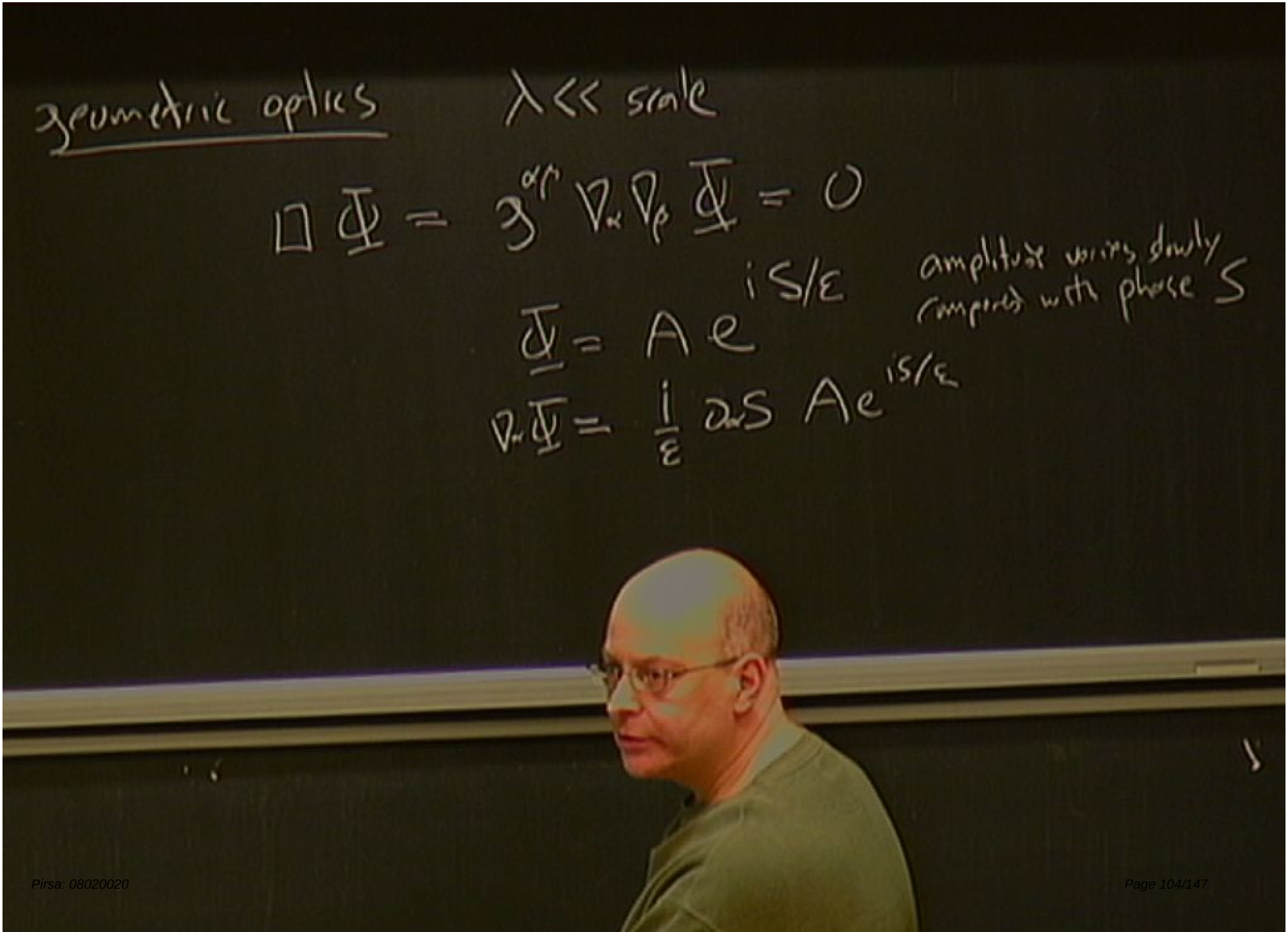
$$\Phi = A e^{iS/\varepsilon}$$

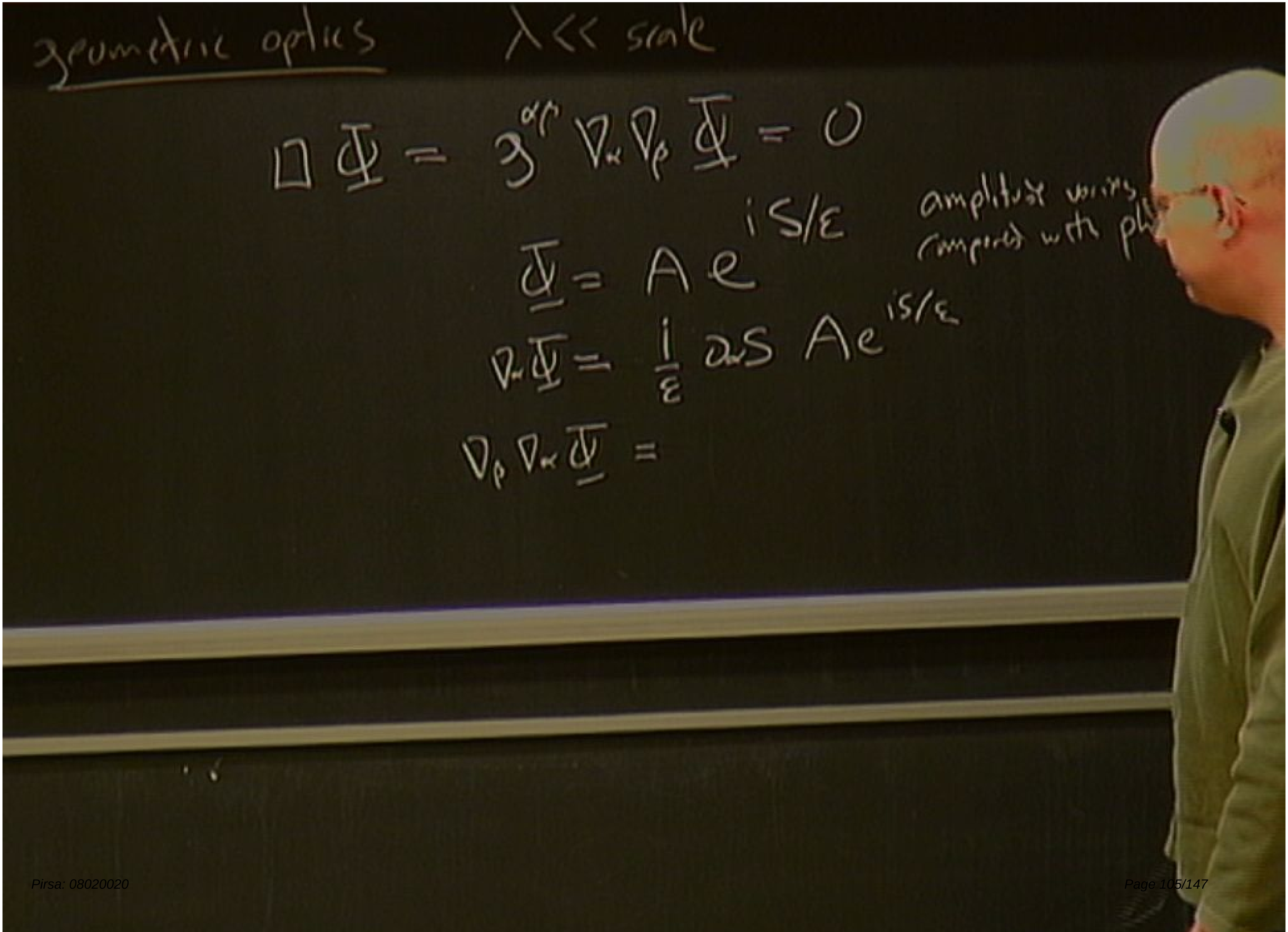
geometric optics $\quad \lambda \ll$ scale

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$$

$$\Phi = A e^{iS/\varepsilon}$$

amplitude varies slowly compared with phase S


geometric optics
λ << scale
□Φ = g^{αβ} ∇_α ∇_β Φ = 0
Φ = A e^{iS/ε}
amplitude varies slowly compared with phase S
∇_α Φ = (i/ε) ∇_α S A e^{iS/ε}


geometric optics
λ << scale
□Φ = g^αβ ∇_α ∇_β Φ = 0
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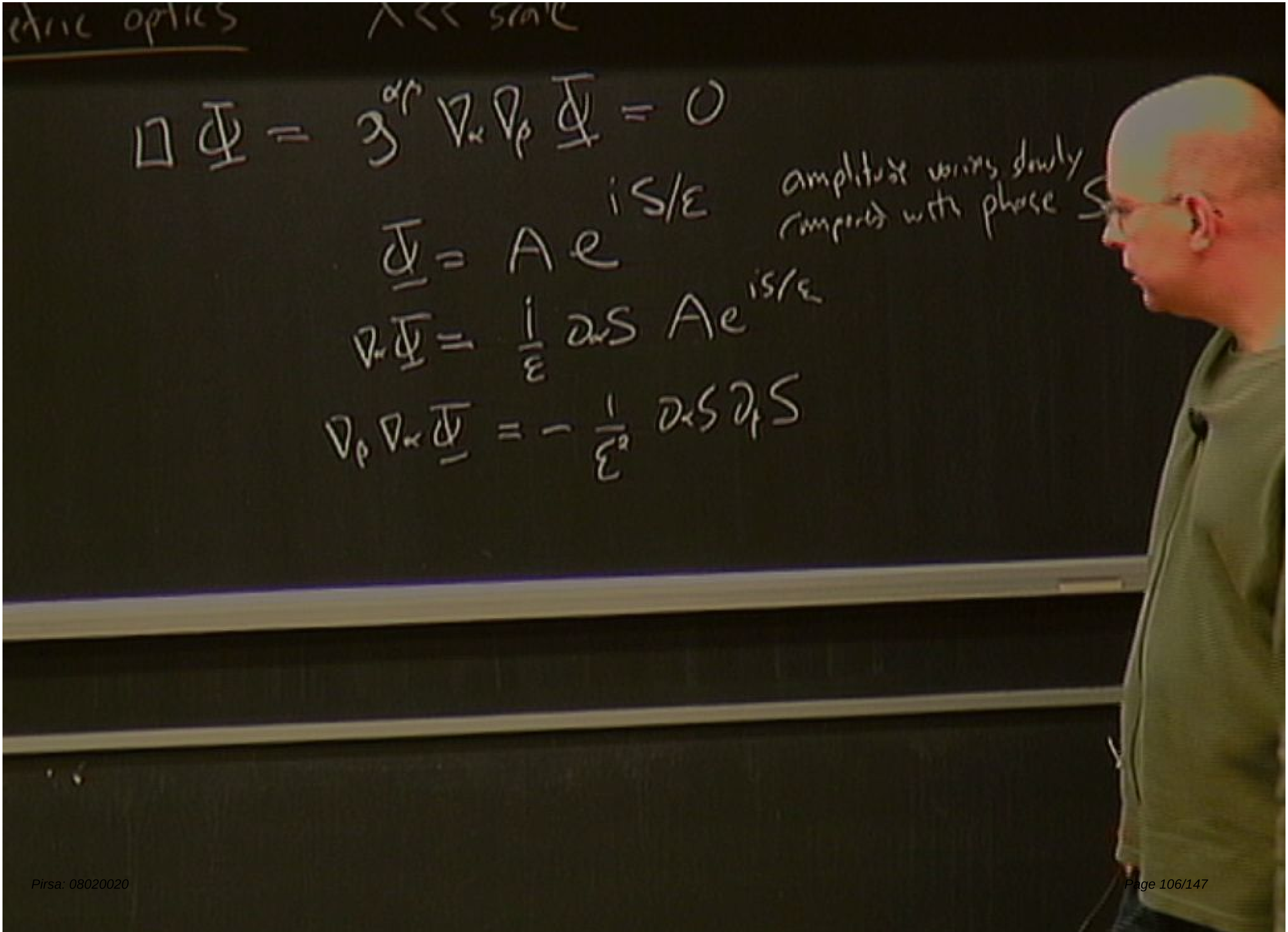
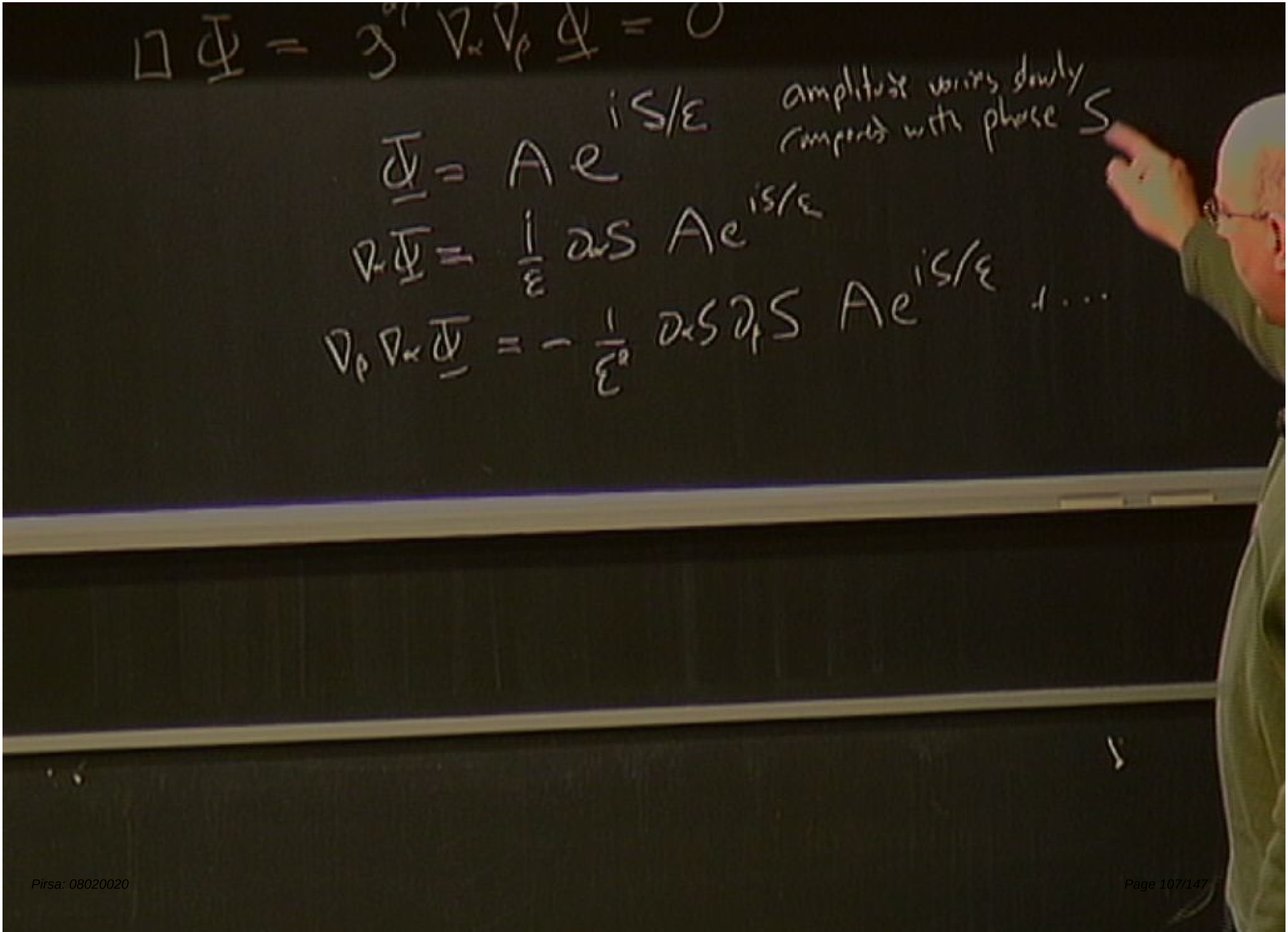
compared with ph
∇_α Φ = (i/ε) ∂_α S A e^{iS/ε}
∇_β ∇_α Φ =


etric optics
λ << scale
□Φ = g^{αβ} ∇_α ∇_β Φ = 0
Φ = A e^{iS/ε}
amplitude varies slowly compared with phase S
∇_α Φ = (i/ε) ∂_α S A e^{iS/ε}
∇_β ∇_α Φ = − (1/ε²) ∂_α S ∂_β S


$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$
$\Phi = A e^{iS/\varepsilon}$
amplitude varies slowly compared with phase S
$\nabla_\alpha \Phi = \frac{i}{\varepsilon} \partial_\alpha S \, A e^{iS/\varepsilon}$
$\nabla_\beta \nabla_\alpha \Phi = -\frac{1}{\varepsilon^2} \partial_\alpha S \, \partial_\beta S \, A e^{iS/\varepsilon} + \ldots$

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$$

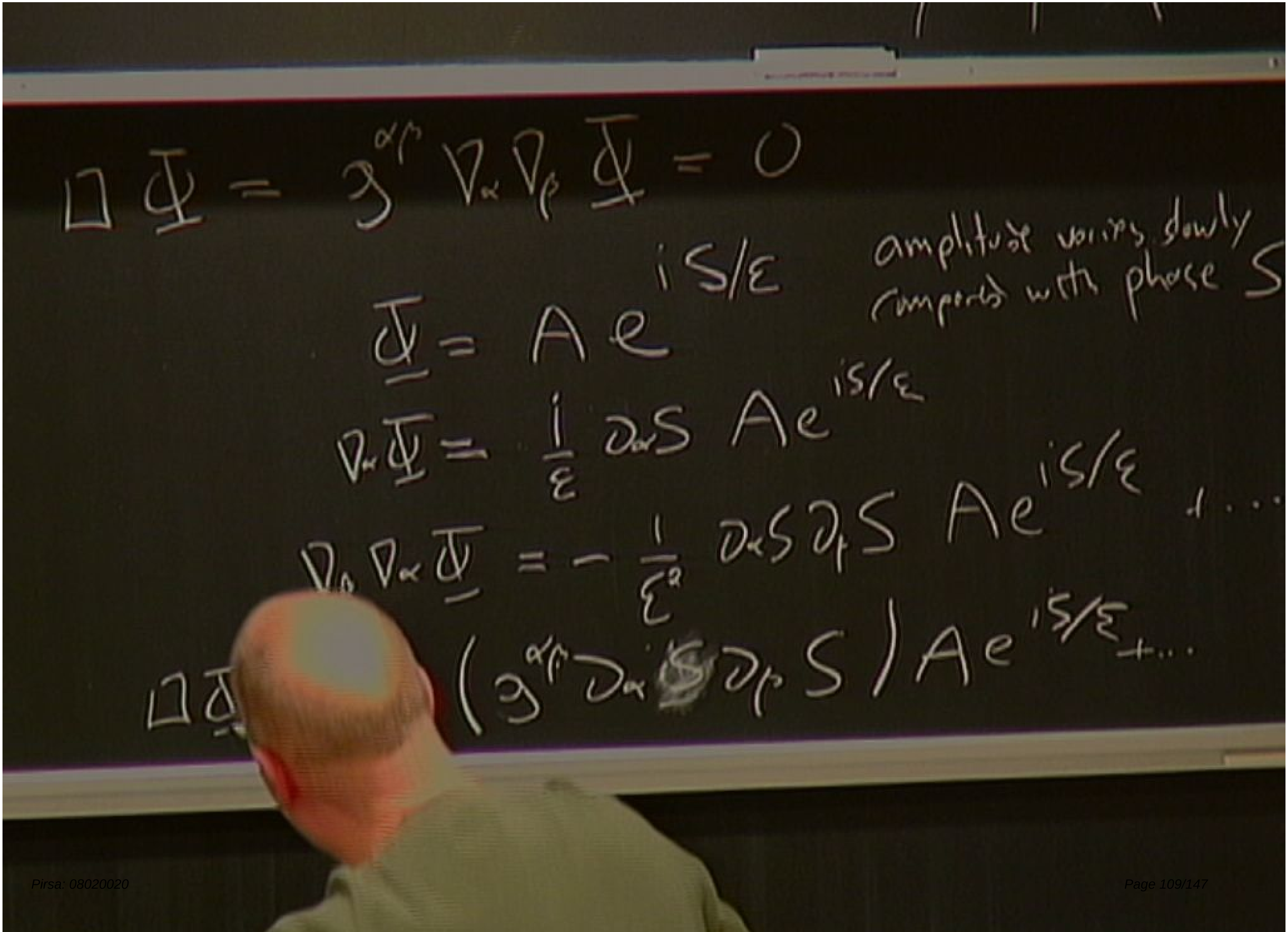
$$\Phi = A e^{iS/\varepsilon}$$

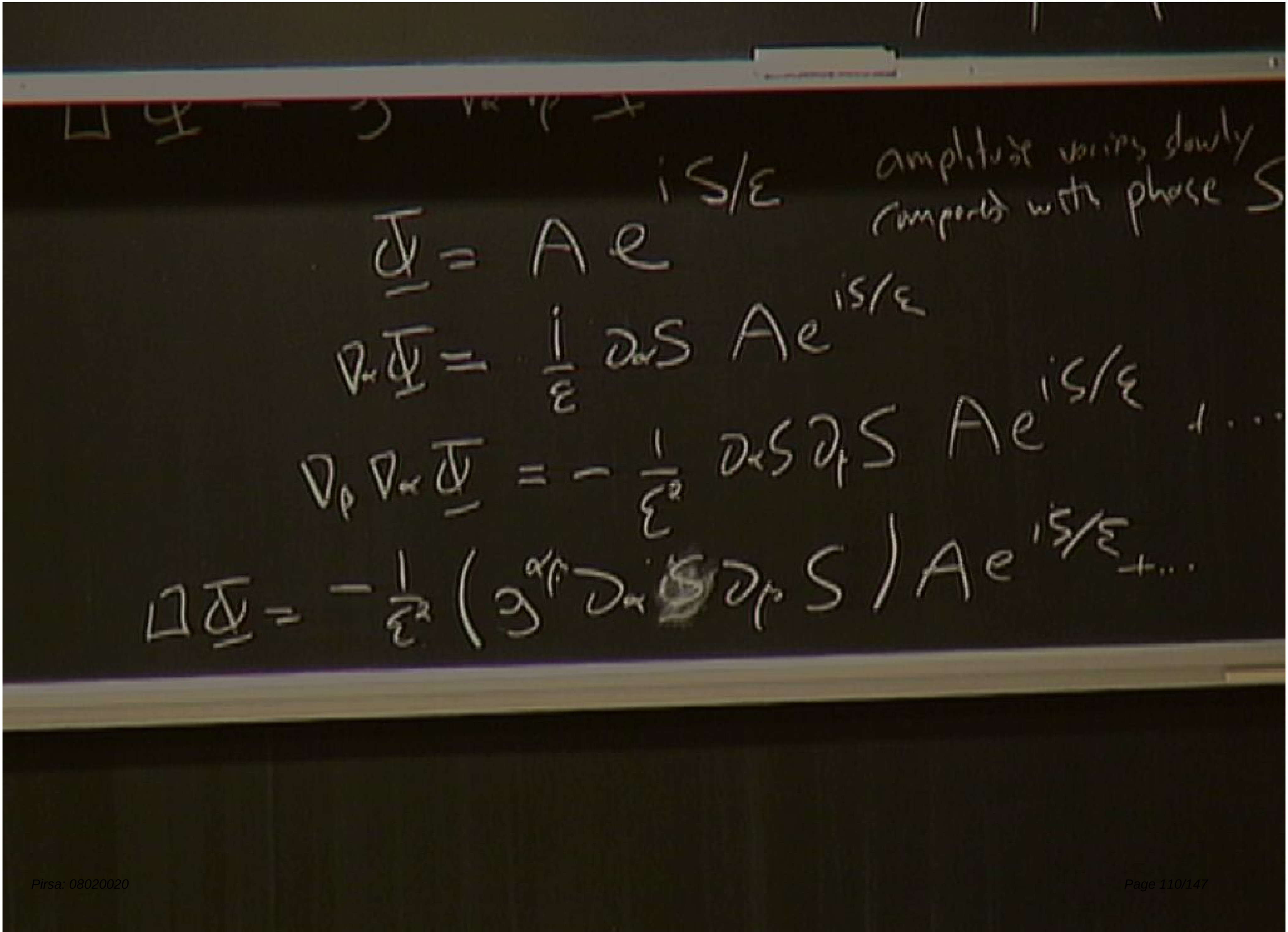
amplitude varies slowly compared with phase S

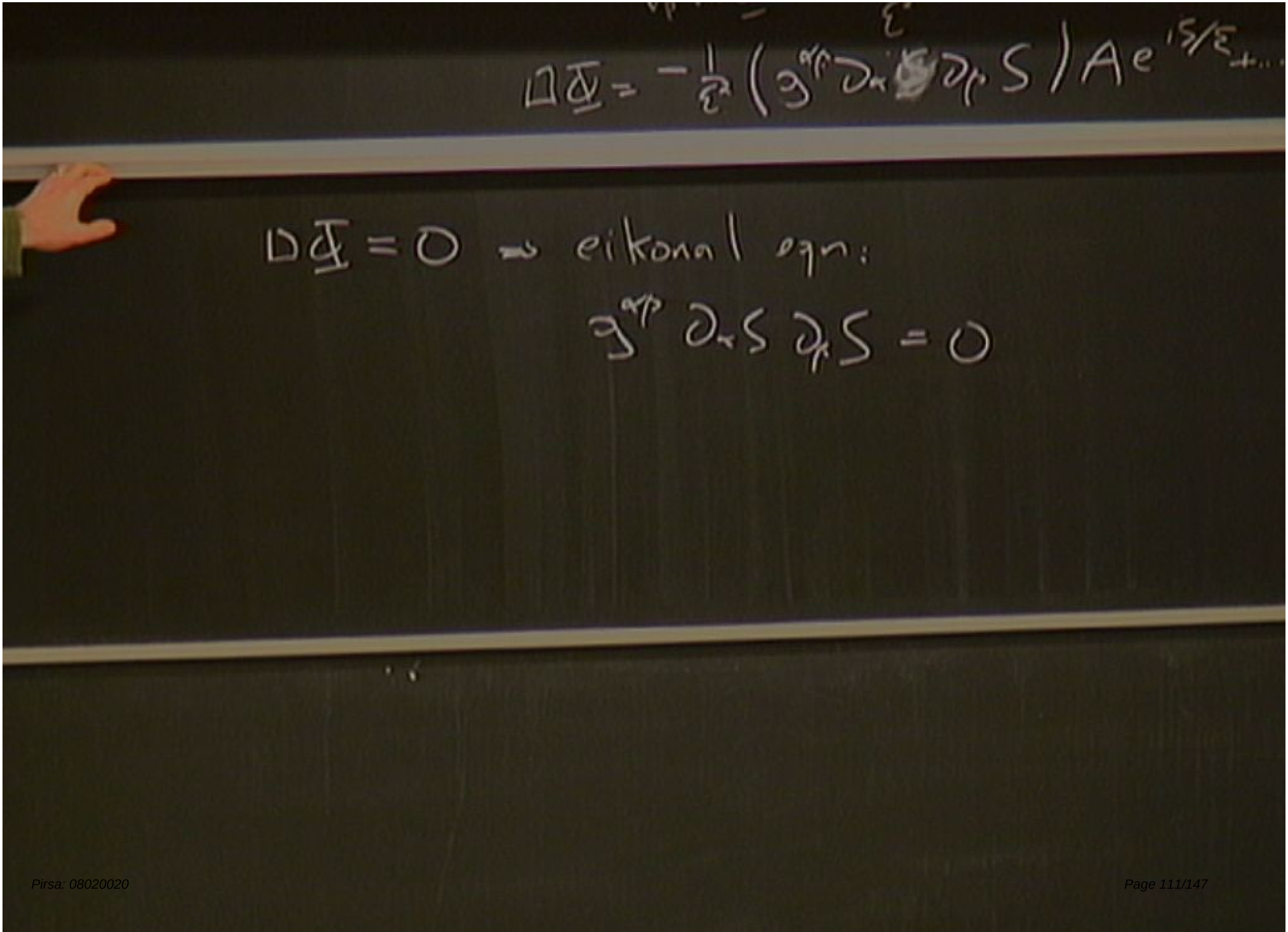
$$\nabla_\alpha \Phi = \frac{i}{\varepsilon} \partial_\alpha S \, A e^{iS/\varepsilon}$$

$$\nabla_\beta \nabla_\alpha \Phi = -\frac{1}{\varepsilon^2} \partial_\alpha S \, \partial_\beta S \, A e^{iS/\varepsilon} + \ldots$$

$$\Box \Phi = -\frac{1}{\varepsilon^2} \left(g^{\alpha\beta} \partial_\alpha S \, \partial_\beta S \right)$$


$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = 0$
amplitude varies slowly
compared with phase S
$\Phi = A e^{iS/\varepsilon}$
$\nabla_\alpha \Phi = \frac{i}{\varepsilon} \partial_\alpha S \, A e^{iS/\varepsilon}$
$\nabla_\beta \nabla_\alpha \Phi = -\frac{1}{\varepsilon^2} \partial_\alpha S \, \partial_\beta S \, A e^{iS/\varepsilon} + \ldots$
$\Box \Phi$
$(g^{\alpha\beta} \partial_\alpha S \, \partial_\beta S) A e^{iS/\varepsilon} + \ldots$


amplitude varies slowly compared with phase S
Φ = A e^{iS/ε}
∇_α Φ = (i/ε) ∂_α S A e^{iS/ε}
∇_β ∇_α Φ = − (1/ε²) ∂_α S ∂_β S A e^{iS/ε} + ...
□Φ = − (1/ε²) (g^{αβ} ∂_α S ∂_β S) A e^{iS/ε} + ...


$\Box\Phi = -\frac{1}{\epsilon^2}\left(g^{\alpha\beta}\partial_\alpha S \partial_\beta S\right) A e^{iS/\epsilon} + \dots$
$\Box\Phi = 0 \Rightarrow$ eikonal eqn:
$g^{\alpha\beta}\partial_\alpha S \partial_\beta S = 0$

$\Box \Phi = 0 \Rightarrow$ eikonal eqn:

$$g^{\alpha\beta} \partial_\alpha S \, \partial_\beta S = 0$$

$$k_\alpha \equiv \partial_\alpha S = \text{null vector}$$

$$k^\alpha{}_{;\beta} k^\beta = 0$$

$\Box \Phi = 0 \Rightarrow$ eikonal eqn:

$$g^{\alpha\beta} \partial_\alpha S \, \partial_\beta S = 0$$

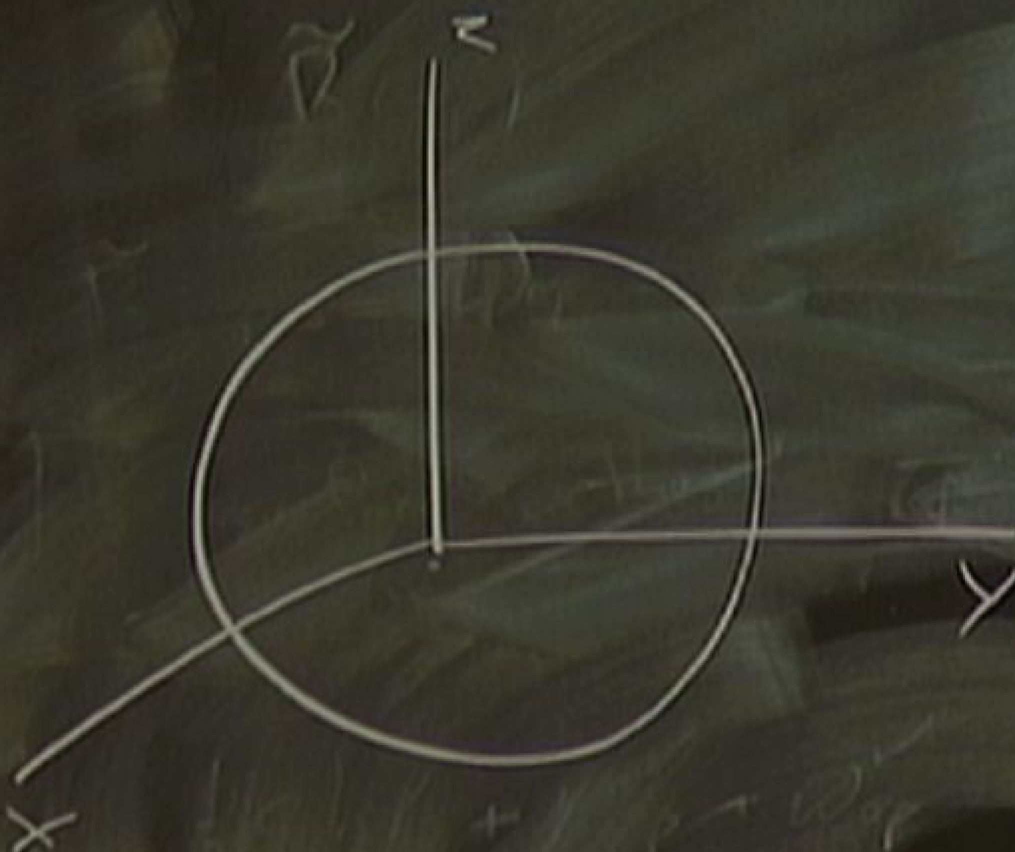
$k_\alpha \equiv \partial_\alpha S$ = null vector

$$k^\alpha{}_{;\beta} k^\beta = 0$$

$\rightarrow$ hypersurface orthogonal !

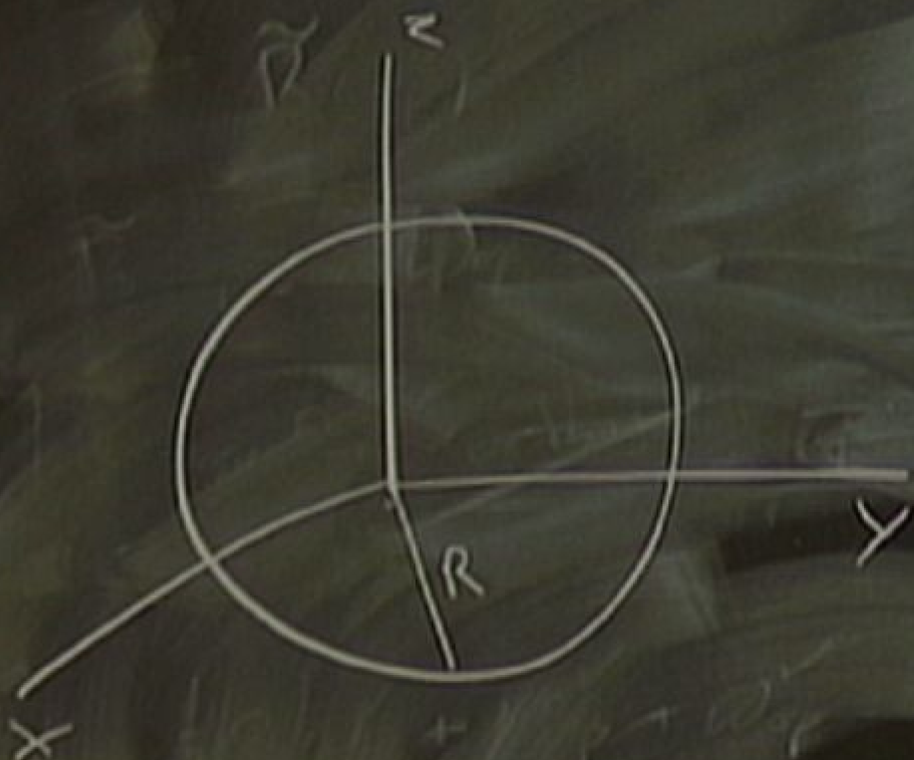
3- HYPERSURFACES

2D sphere in 3D flat space



3- HYPERSURFACES

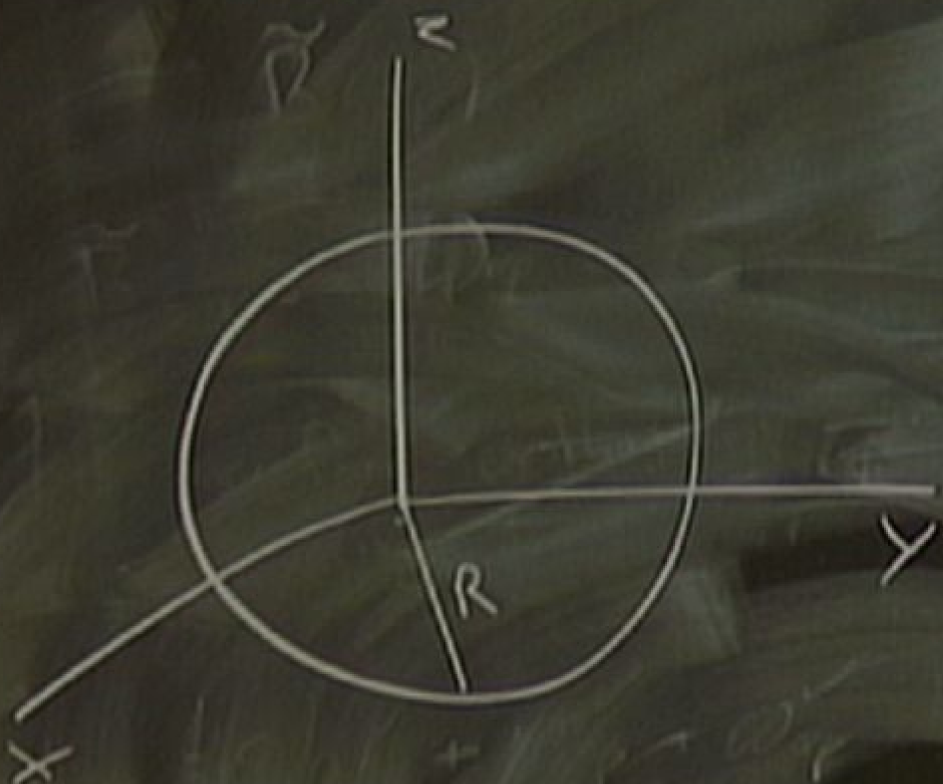
2D sphere in 3D flat space



$$\Phi(x, y, z) = c$$

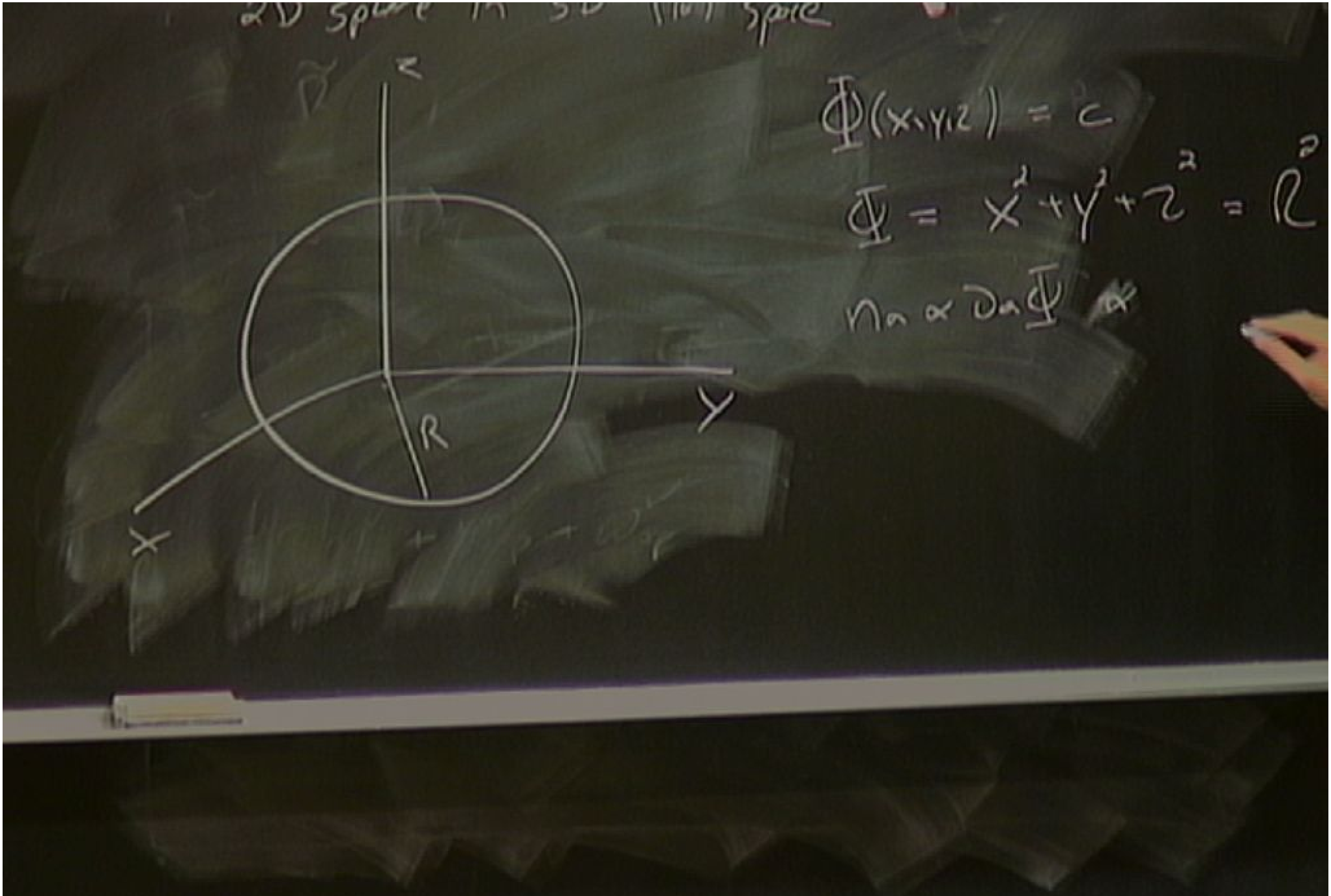
3- HYPERSURFACES

2D sphere in 3D flat space

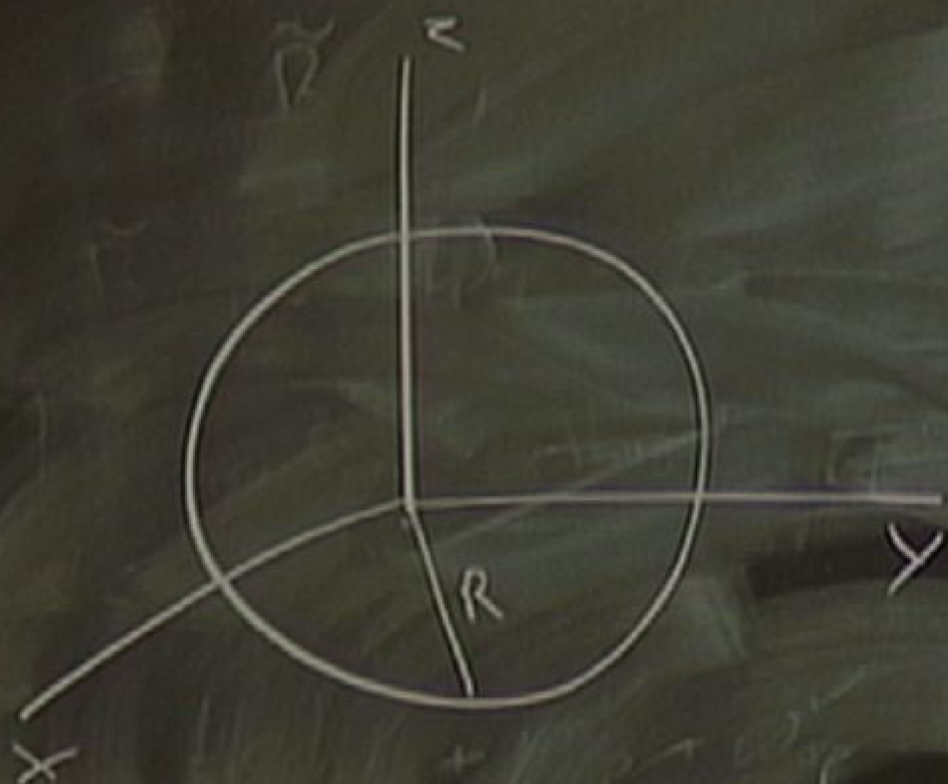


$$\Phi(x,y,z) = c$$

$$\Phi = x^2 + y^2 + z^2 = R^2$$


Φ(x,y,z) = c
Φ = x² + y² + z² = R²
n_a ∝ ∂_a Φ
z
R
x
y

2D sphere in 3D flat space

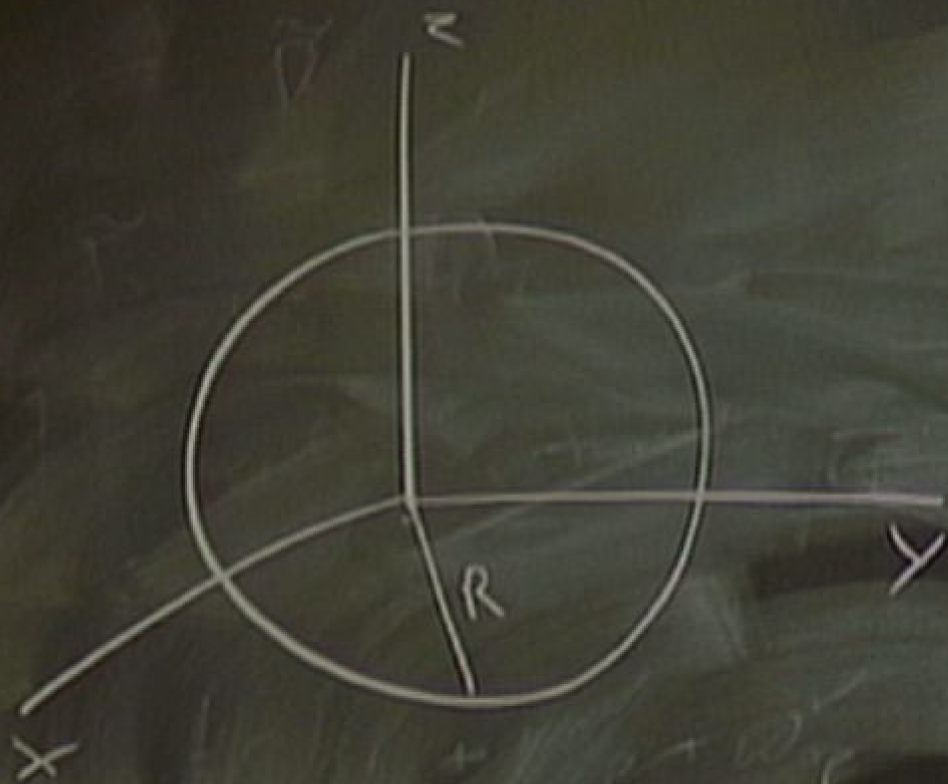


$$\Phi(x,y,z) = c$$

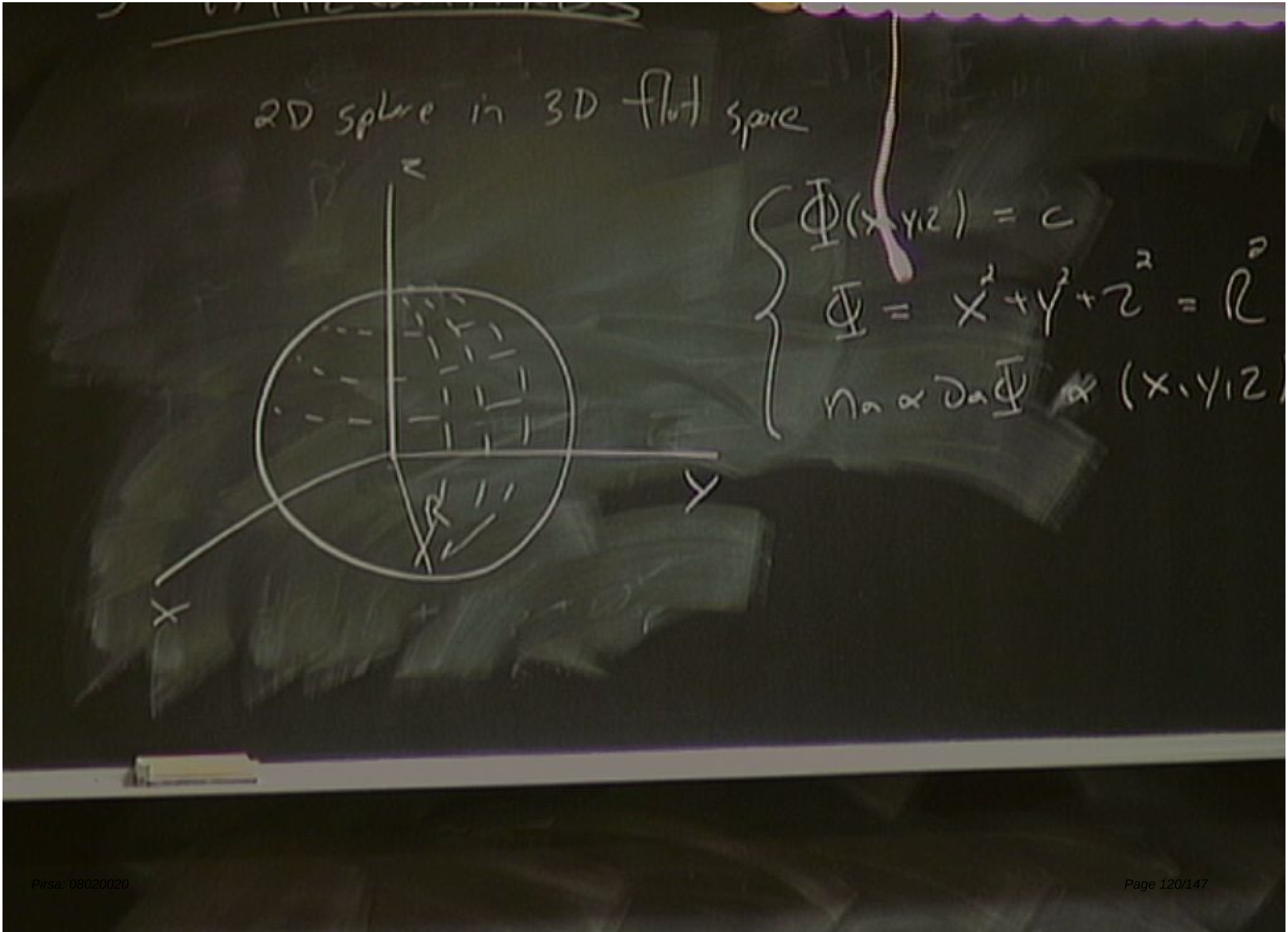
$$\Phi = x^2 + y^2 + z^2 = R^2$$

$$n_a \propto \partial_a \Phi \propto (x,y,z)$$

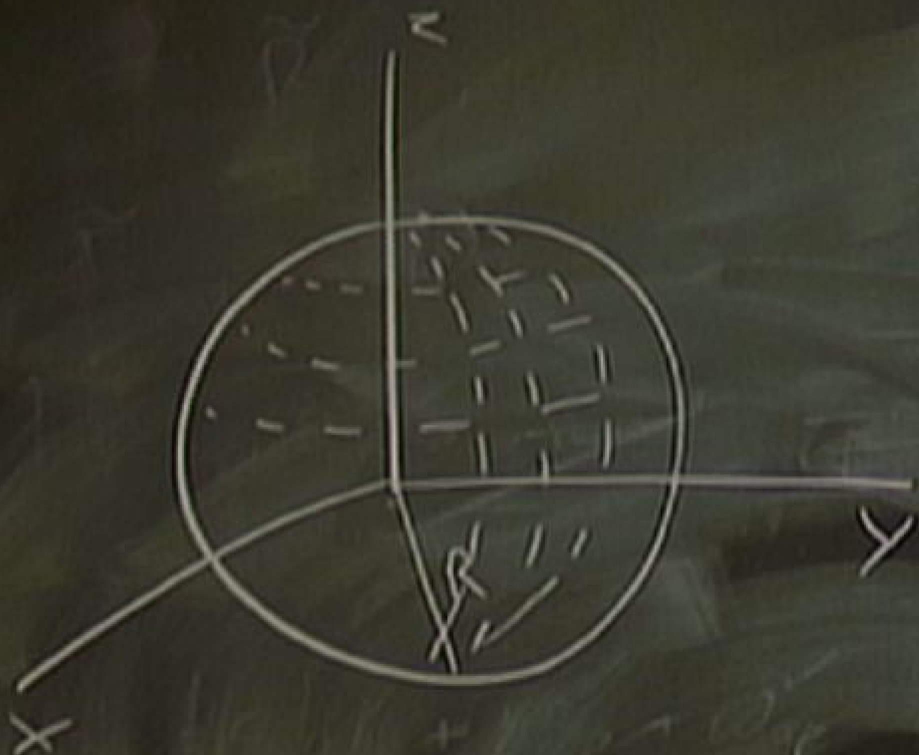
2D sphere in 3D flat space



$$\begin{cases} \Phi(x,y,z) = c \\ \Phi = x^2 + y^2 + z^2 = R^2 \\ n_a \propto \partial_a \Phi \propto (x,y,z) \end{cases}$$

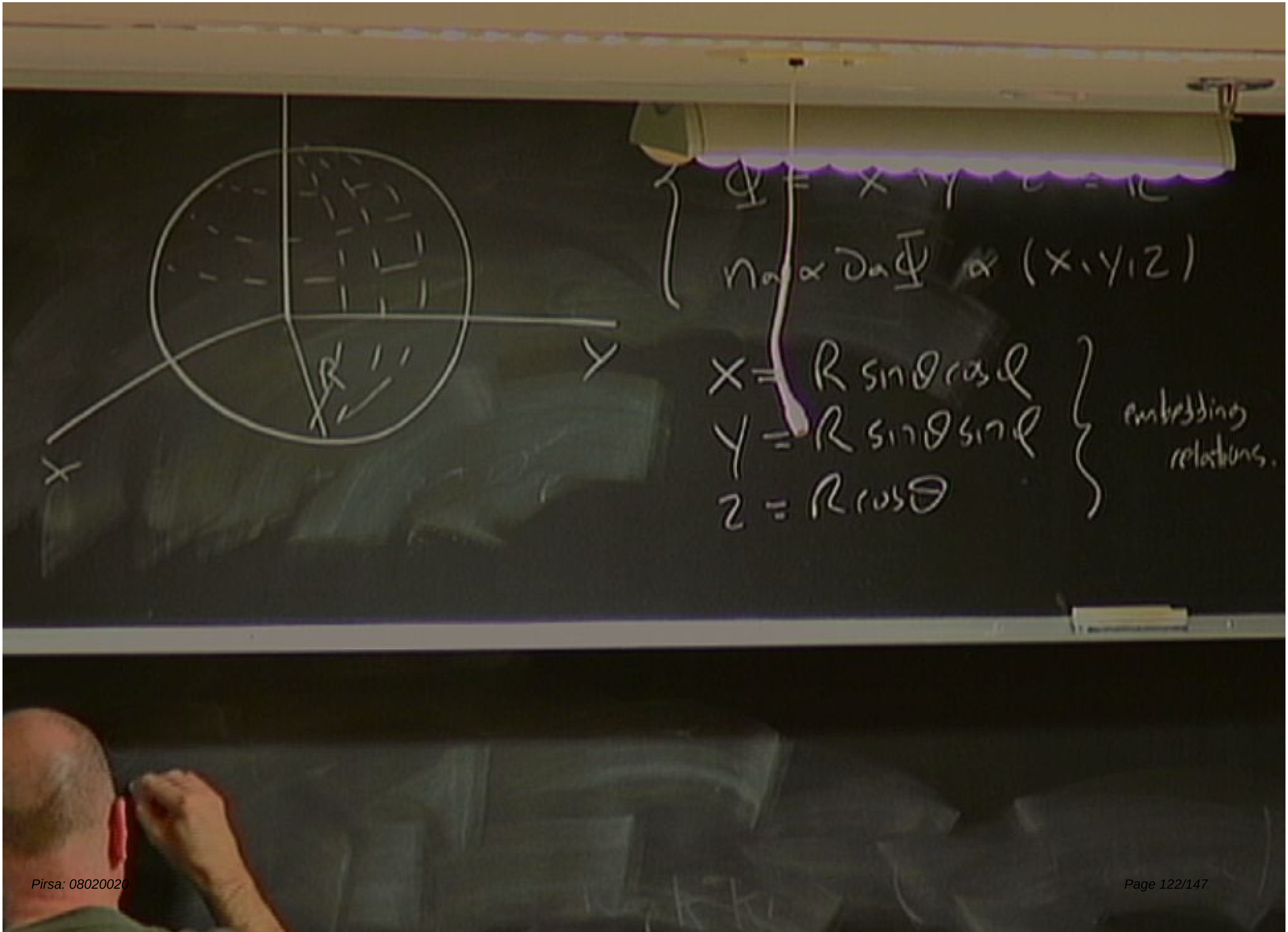

2D sphere in 3D flat space
z
x
y
R
Φ(x,y,z) = c
Φ = x² + y² + z² = R²
n_a ∝ ∂_a Φ ∝ (x,y,z)

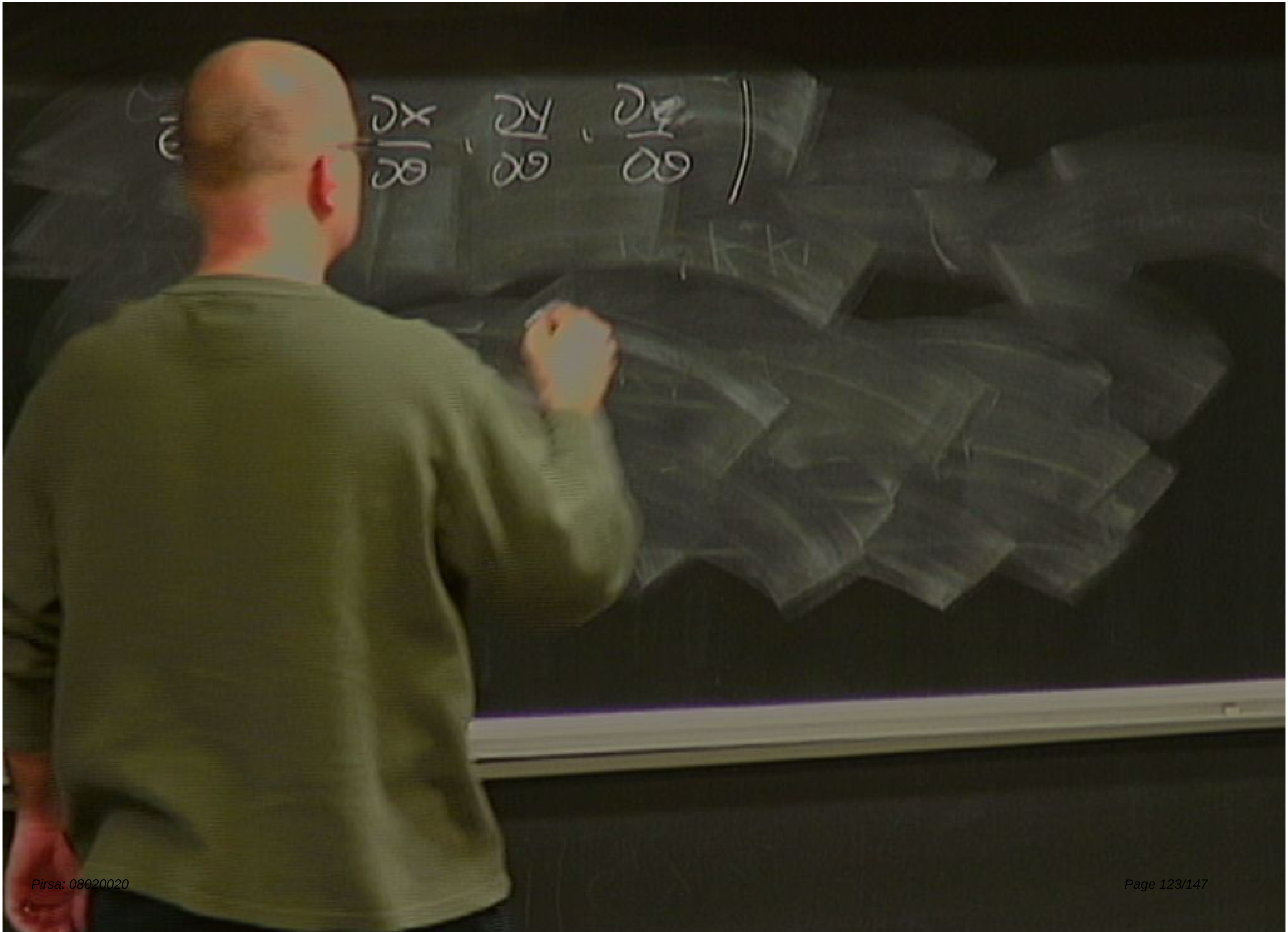
2D sphere in 3D flat space



$$\begin{cases} \Phi(x,y,z) = c \\ \Phi = x^2+y^2+z^2 = R^2 \\ n_a \propto \partial_a \Phi \propto (x,y,z) \end{cases}$$

$$x = R \sin\theta \cos\varphi$$
$$y = R \sin\theta \sin\varphi$$
$$z = R \cos\theta$$


n_a ∝ ∂_a Φ ∝ (x, y, z)
x = R sinθ cosφ
y = R sinθ sinφ
z = R cosθ
embedding relations.
X
Y
R


$\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}$

$$\vec{e}_\theta = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) = \left(R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta \right)$$

$$\vec{e}_\theta = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) = \left(R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta \right)$$

$$\vec{e}_\varphi = \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right) = \left(-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0 \right)$$

Induced metric on S^2:

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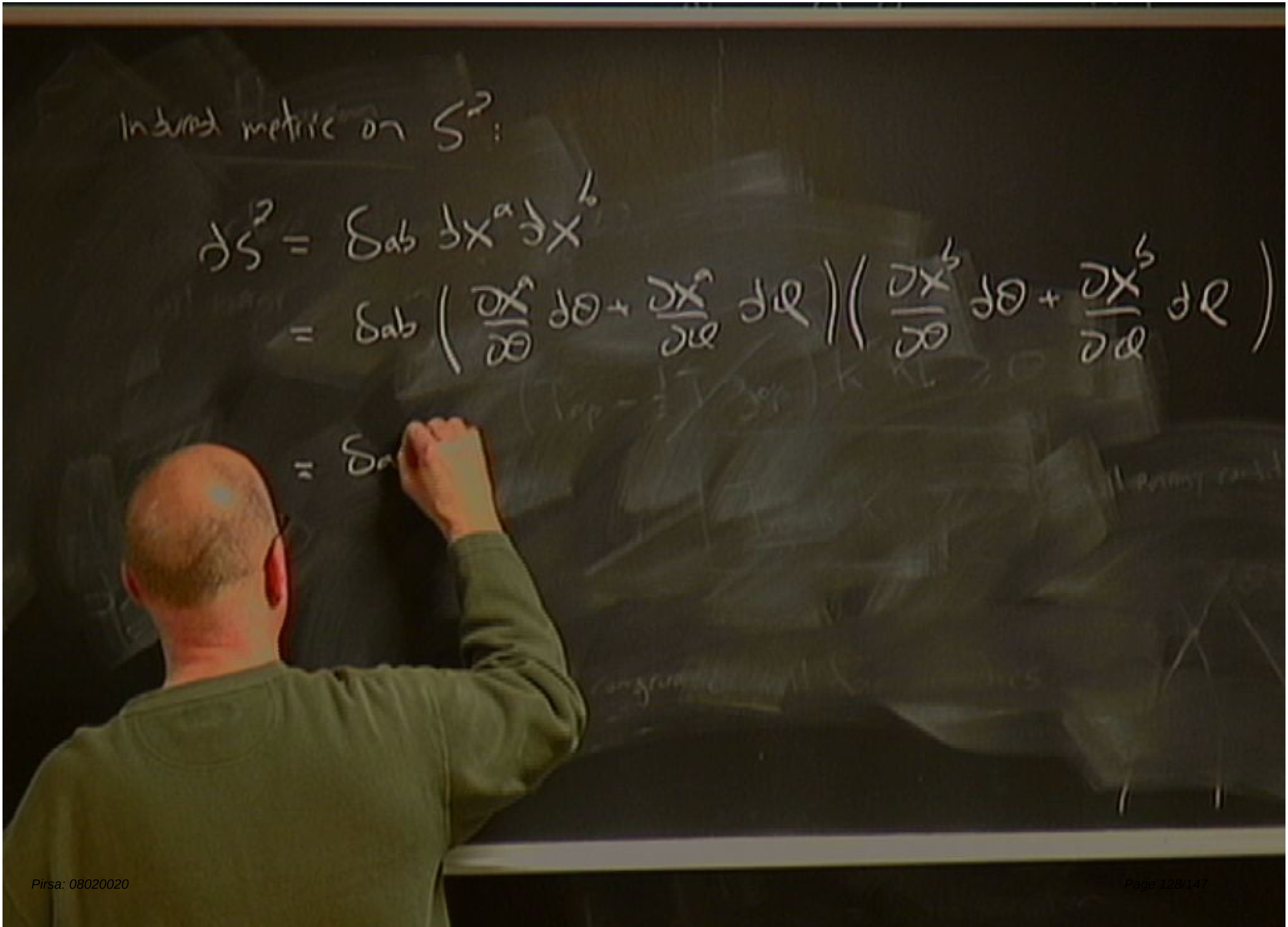
$$ds^2 = \delta_{ab}\, dx^a dx^b$$


Induced metric on S²:
ds² = δab dxᵃ dxᵇ
= δab (∂xᵃ/∂θ dθ + ∂xᵃ/∂φ dφ)(∂xᵇ/∂θ dθ + ∂xᵇ/∂φ dφ)
= δa

Induced metric on S^2:

$$ds^2 = \delta_{ab}\, dx^a dx^b$$

$$= \delta_{ab} \left(\frac{\partial x^a}{\partial \theta} d\theta + \frac{\partial x^a}{\partial \varphi} d\varphi \right)\left(\frac{\partial x^b}{\partial \theta} d\theta + \frac{\partial x^b}{\partial \varphi} d\varphi \right)$$

$$= \delta_{ab} \frac{\partial x^a}{\partial \theta} \frac{\partial x^b}{\partial \theta}\, d\theta^2 + \ldots$$

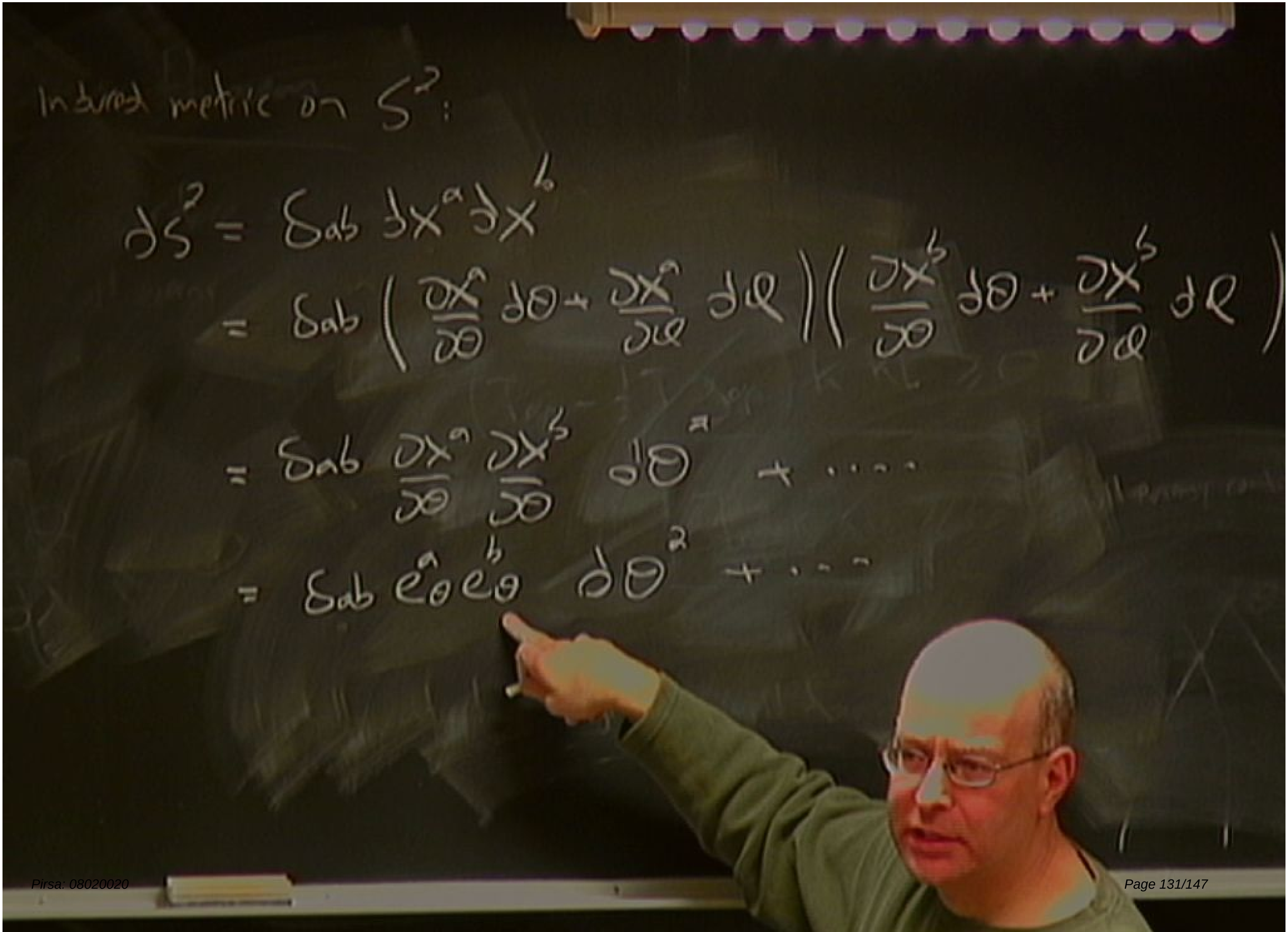
Induced metric on S^2:

$$ds^2 = \delta_{ab}\, dx^a dx^b$$

$$= \delta_{ab} \left(\frac{\partial x^a}{\partial \theta} d\theta + \frac{\partial x^a}{\partial \varphi} d\varphi \right) \left(\frac{\partial x^b}{\partial \theta} d\theta + \frac{\partial x^b}{\partial \varphi} d\varphi \right)$$

$$= \delta_{ab} \frac{\partial x^a}{\partial \theta} \frac{\partial x^b}{\partial \theta} d\theta^2 + \cdots$$

$$= \delta_{ab}\, e^a_\theta e^b_\theta$$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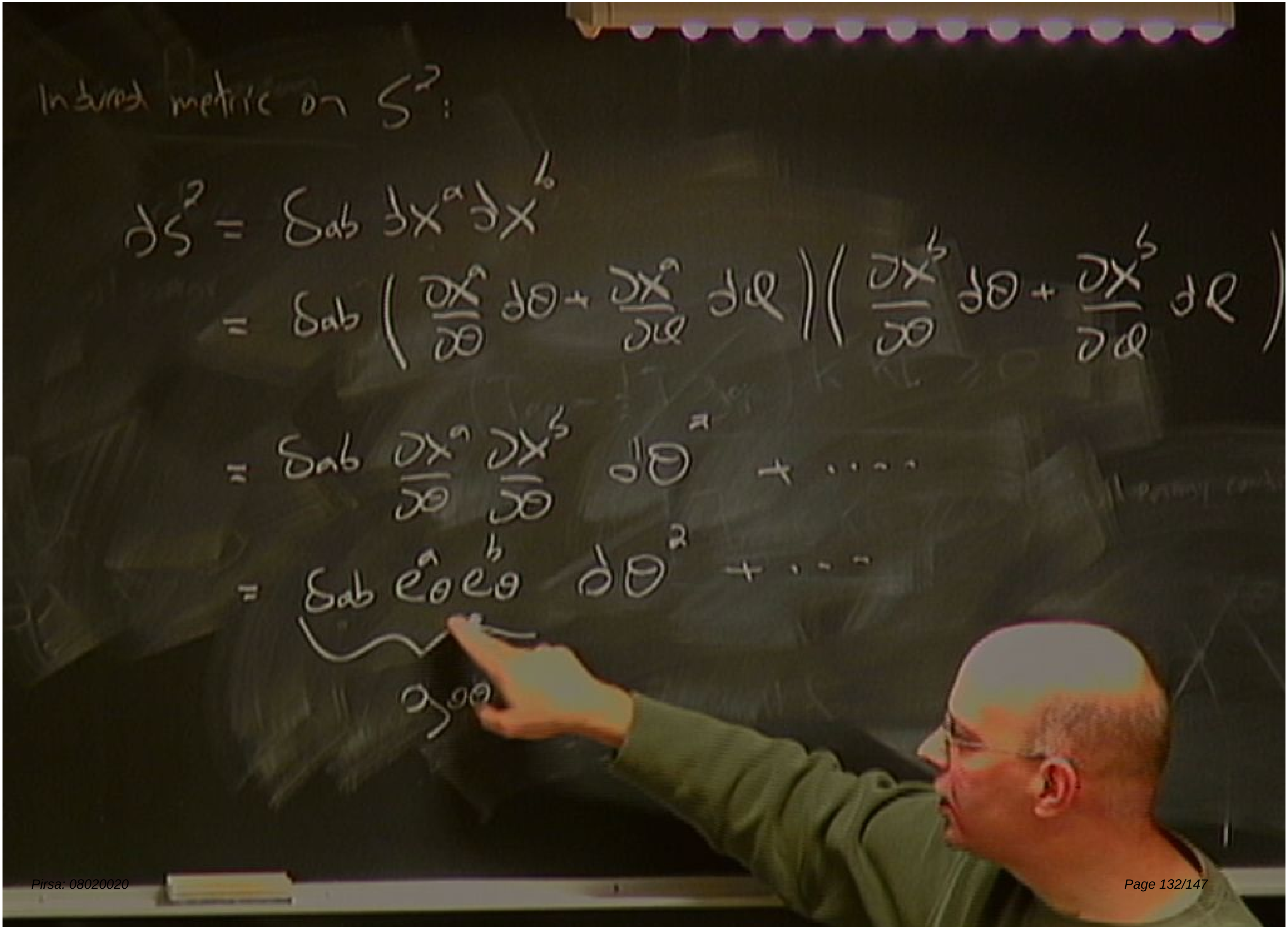
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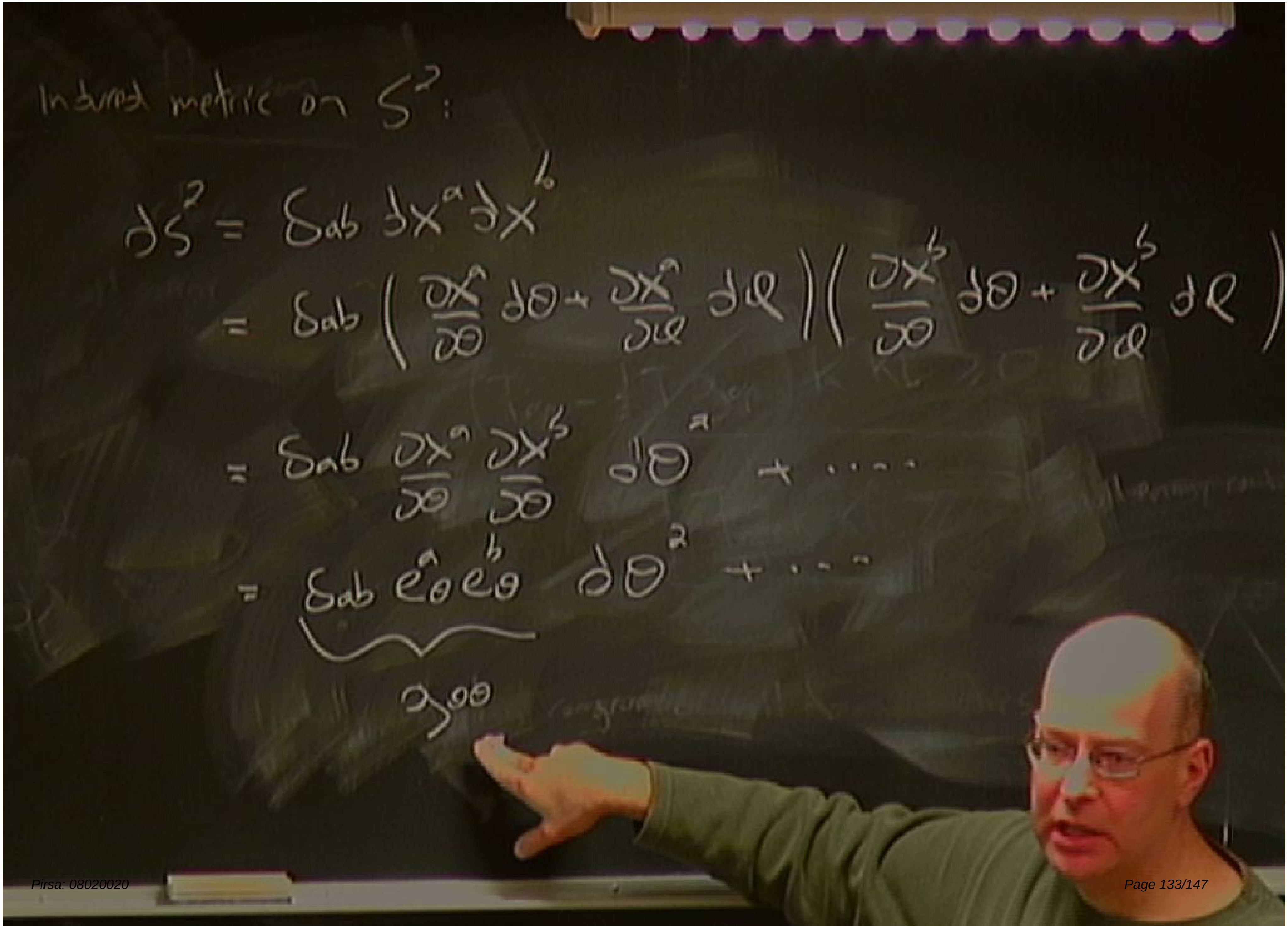
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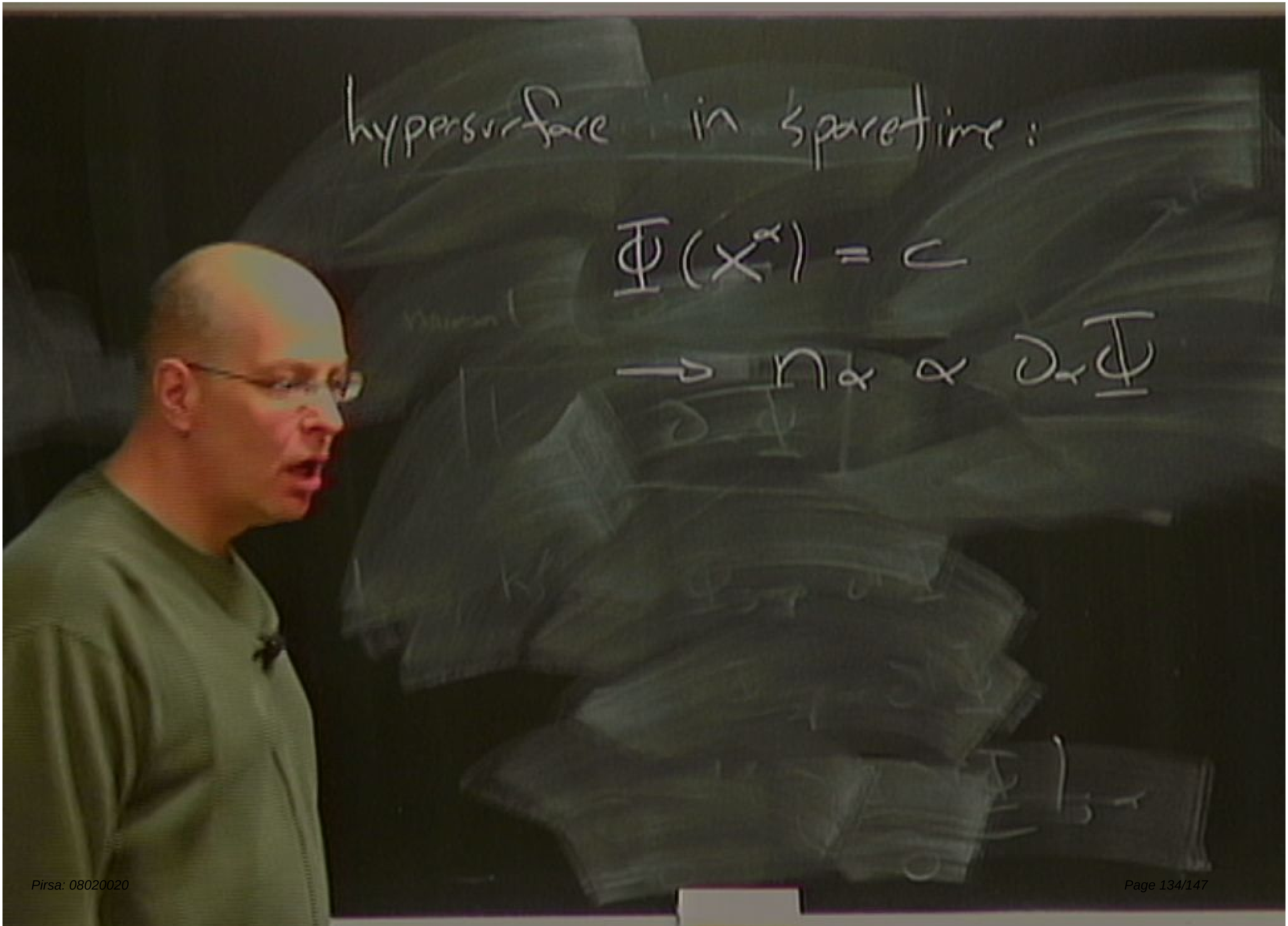
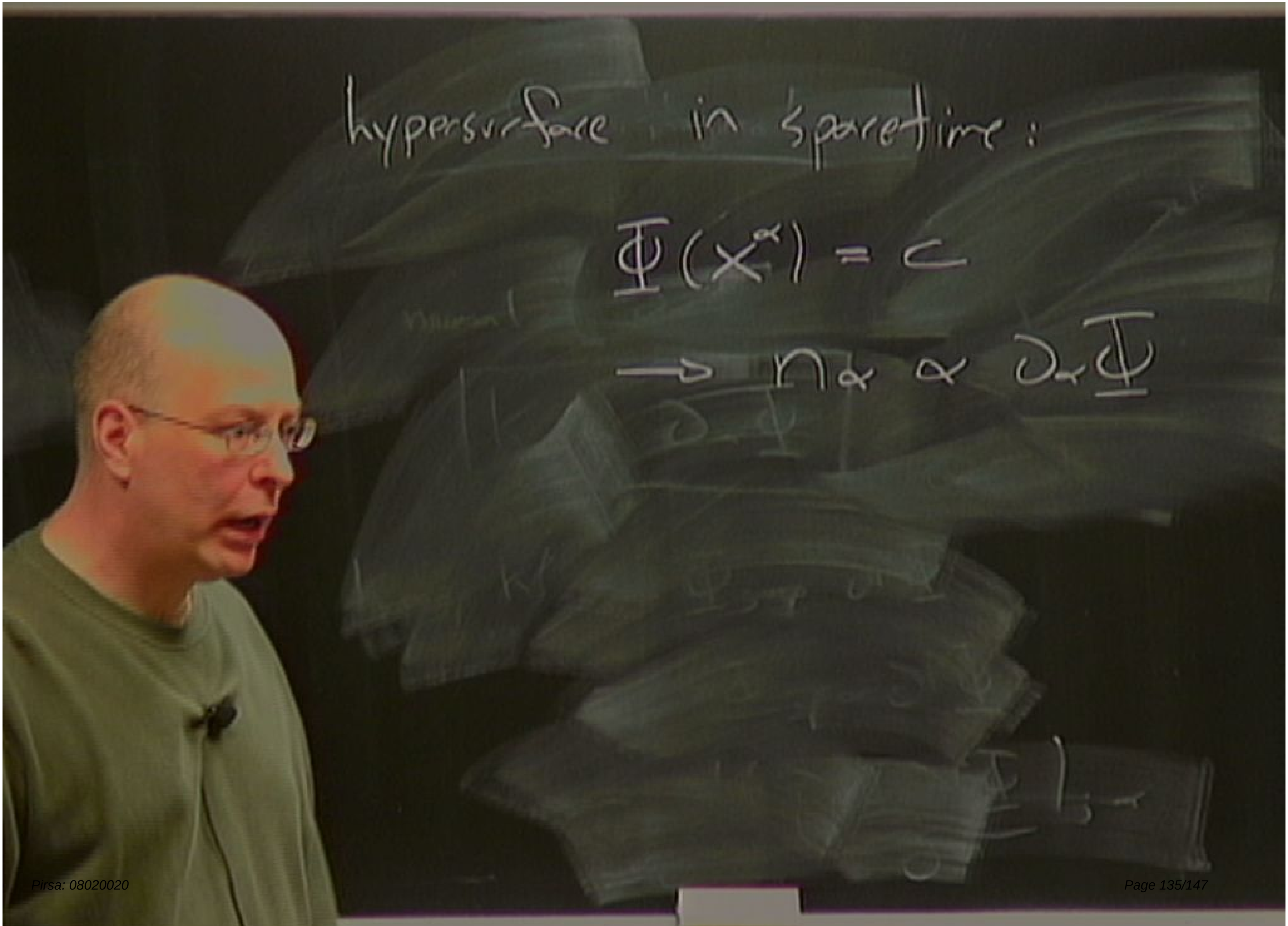

hypersurface in spacetime:
$\Phi(x^\alpha) = c$
$\rightarrow n_\alpha \propto \partial_\alpha \Phi$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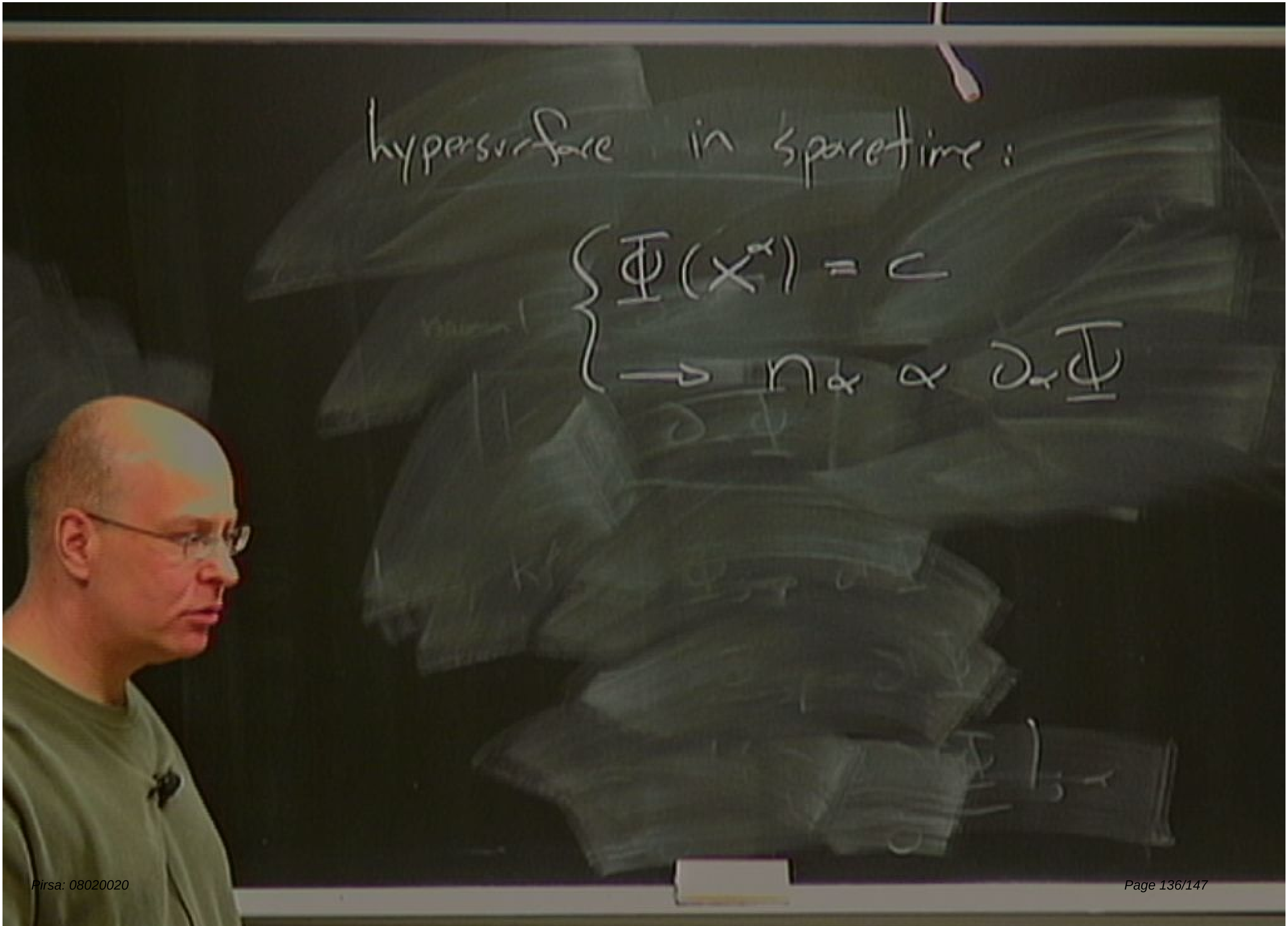
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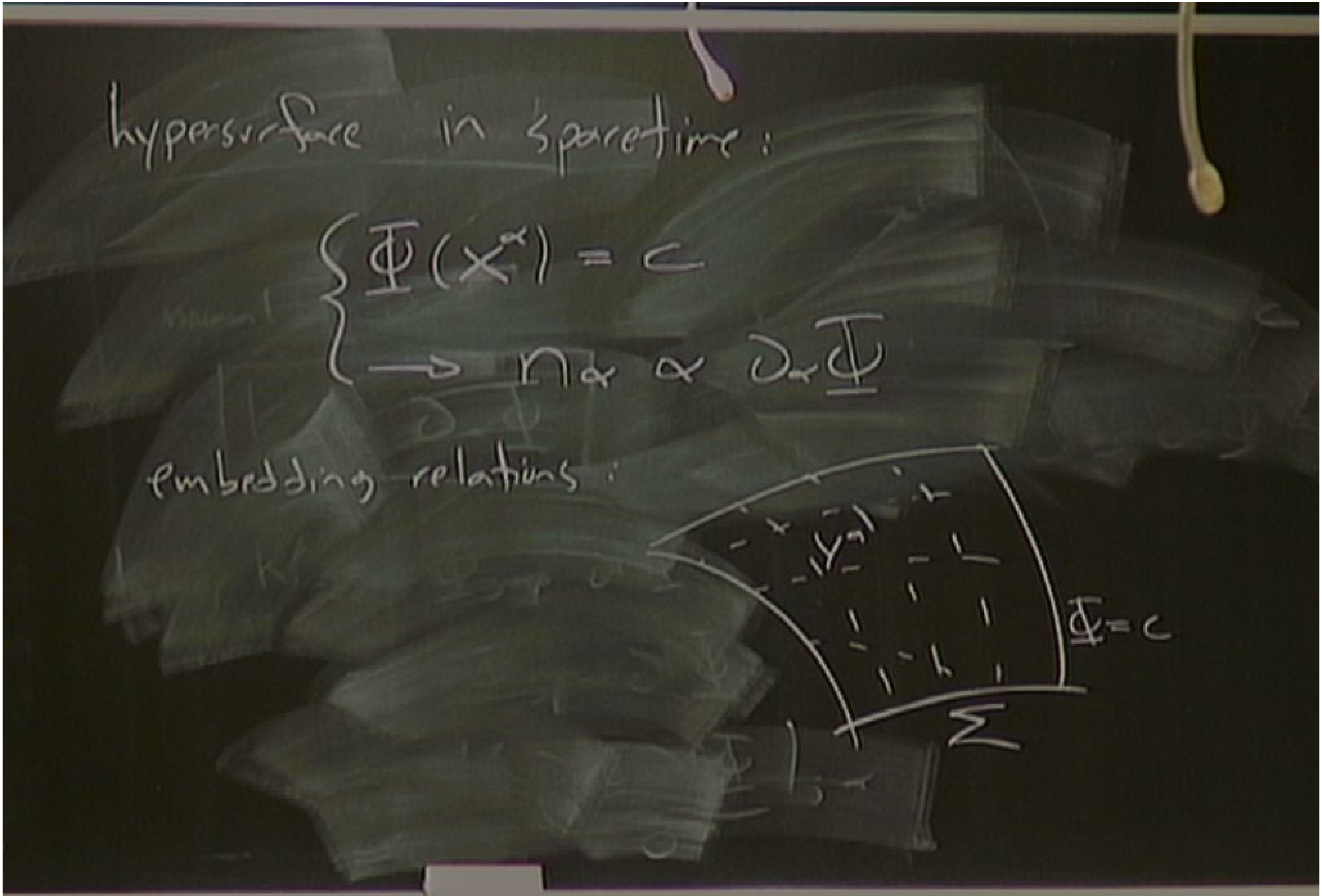
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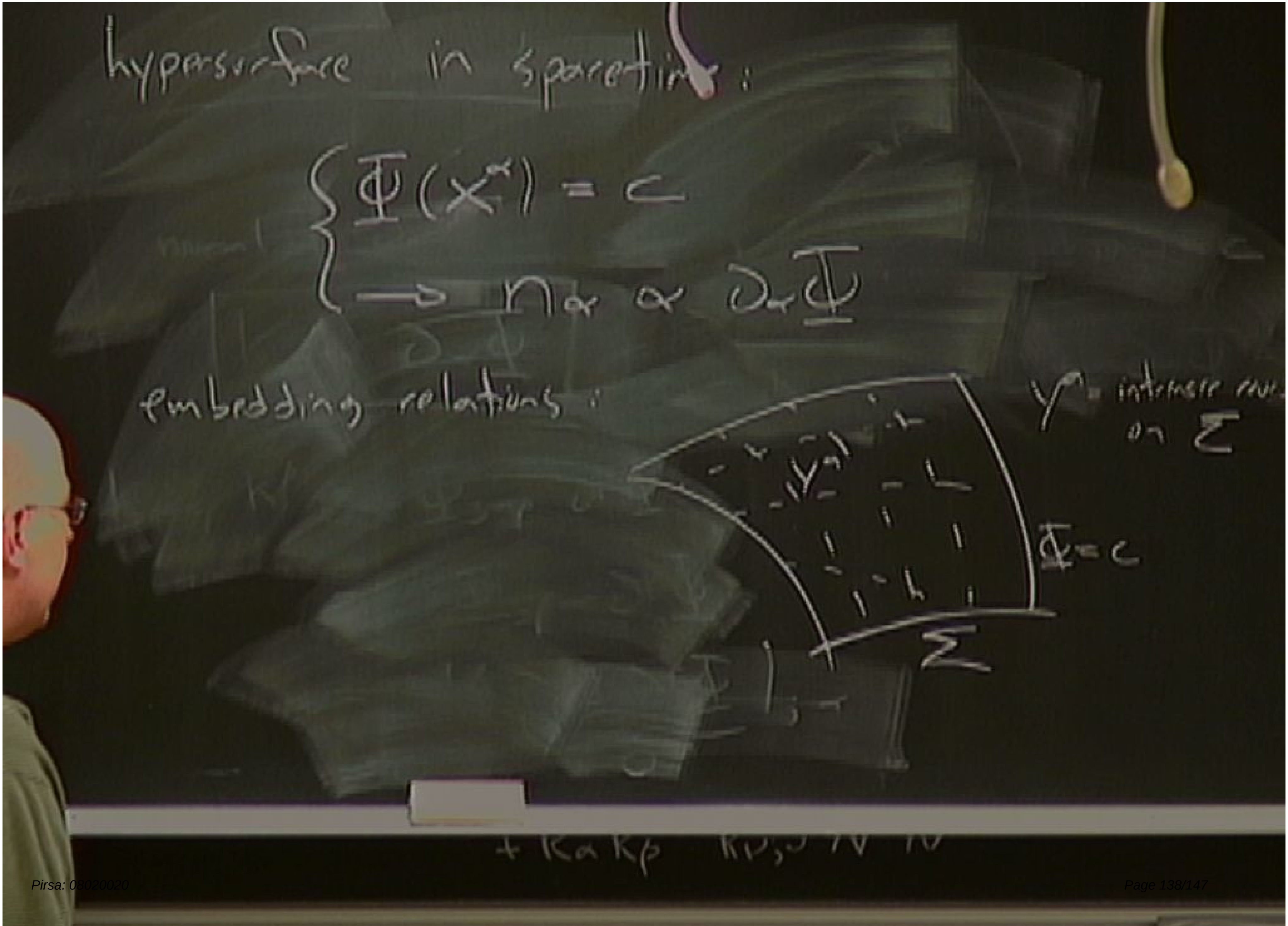
Φ = c
Σ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y^a intrinsic coo...
on Σ
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hypersurface in spacetime:

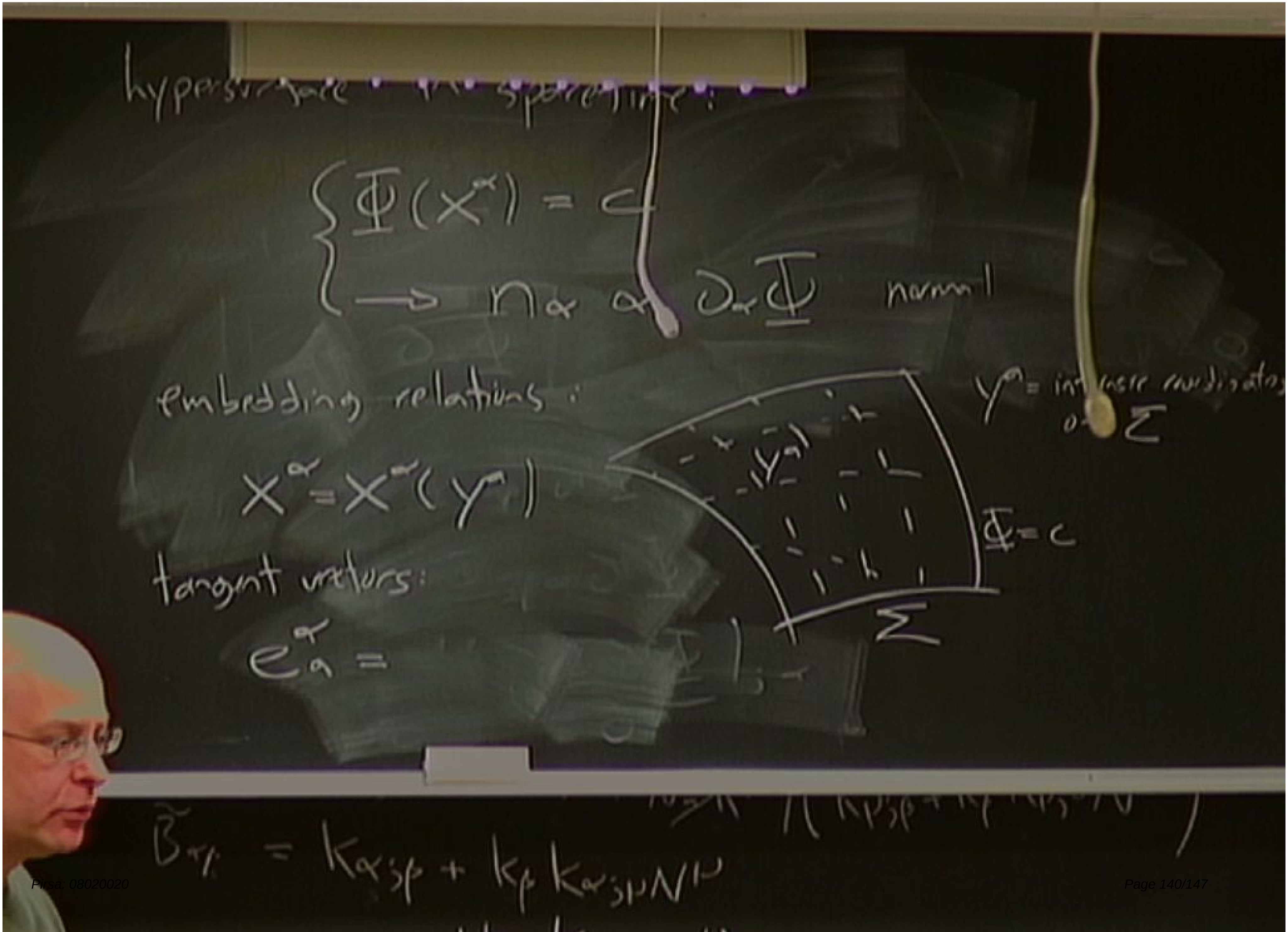
$$\begin{cases} \Phi(x^\alpha) = c \\ \rightarrow n_\alpha \propto \partial_\alpha \Phi \end{cases}$$

embedding relations:

$$x^\alpha = x^\alpha(y^a)$$

y^a = intrinsic coordinates on Σ

$$+ k_\alpha k_\beta \; K_{\rho\sigma} N^\rho N^\sigma$$


hypersurface in spacetime:
Φ(x^α) = c
→ n_α ∝ ∂_α Φ normal
embedding relations:
x^α = x^α(y^a)
y^a = intrinsic coordinates of Σ
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Σ
tangent vectors:
e^α_a =
B̃_αβ = K_α;β + k_β K_α;μ N^μ

hypersurface in spacetime:

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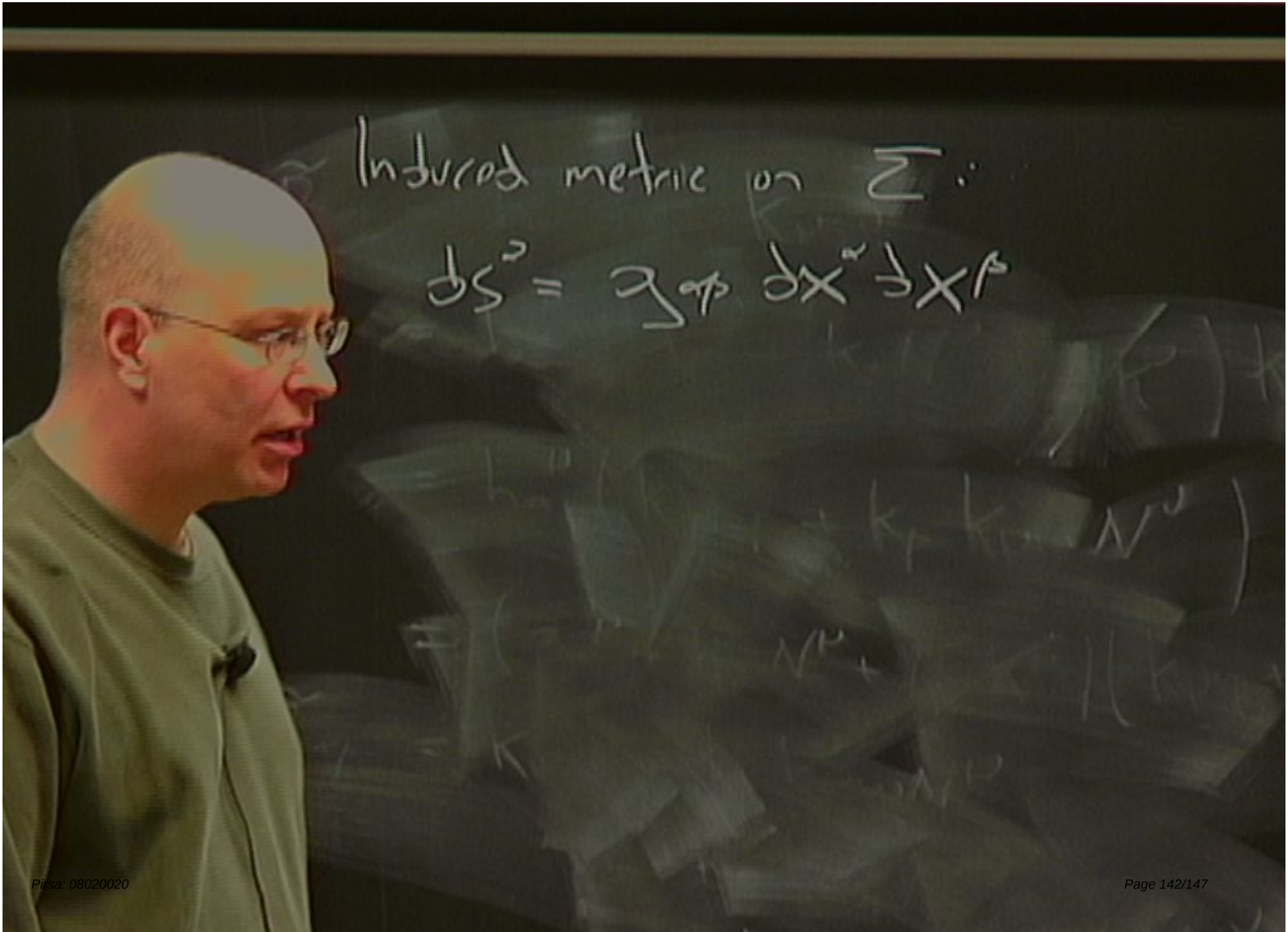
Induced metric on Σ:
ds² = g_αβ dx^α dx^β

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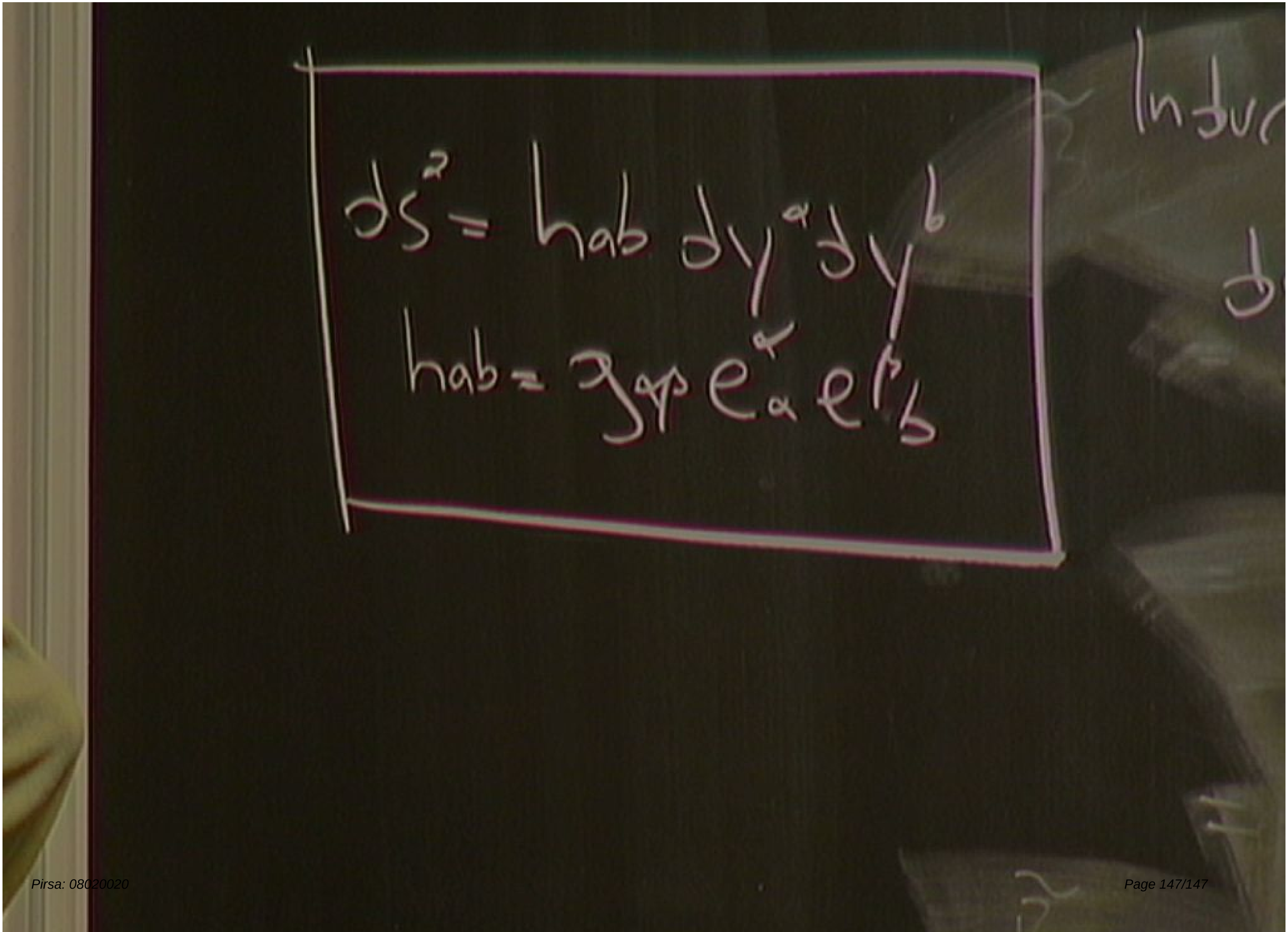
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