Title: Advanced General Relativity - Lecture 5A

Date: Feb 06, 2008 10:30 AM

URL: http://pirsa.org/08020016

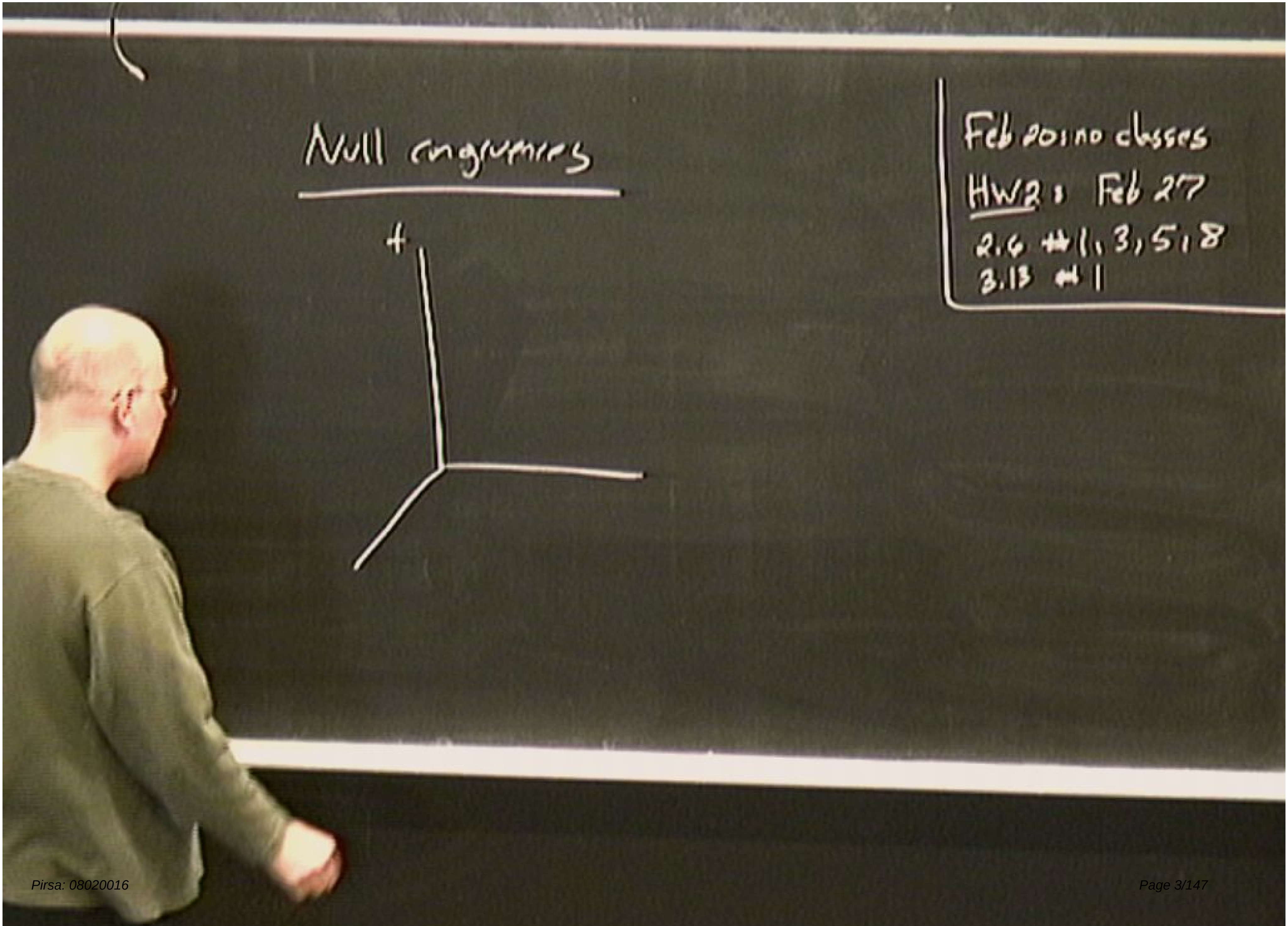
Abstract: Advanced General Relativity

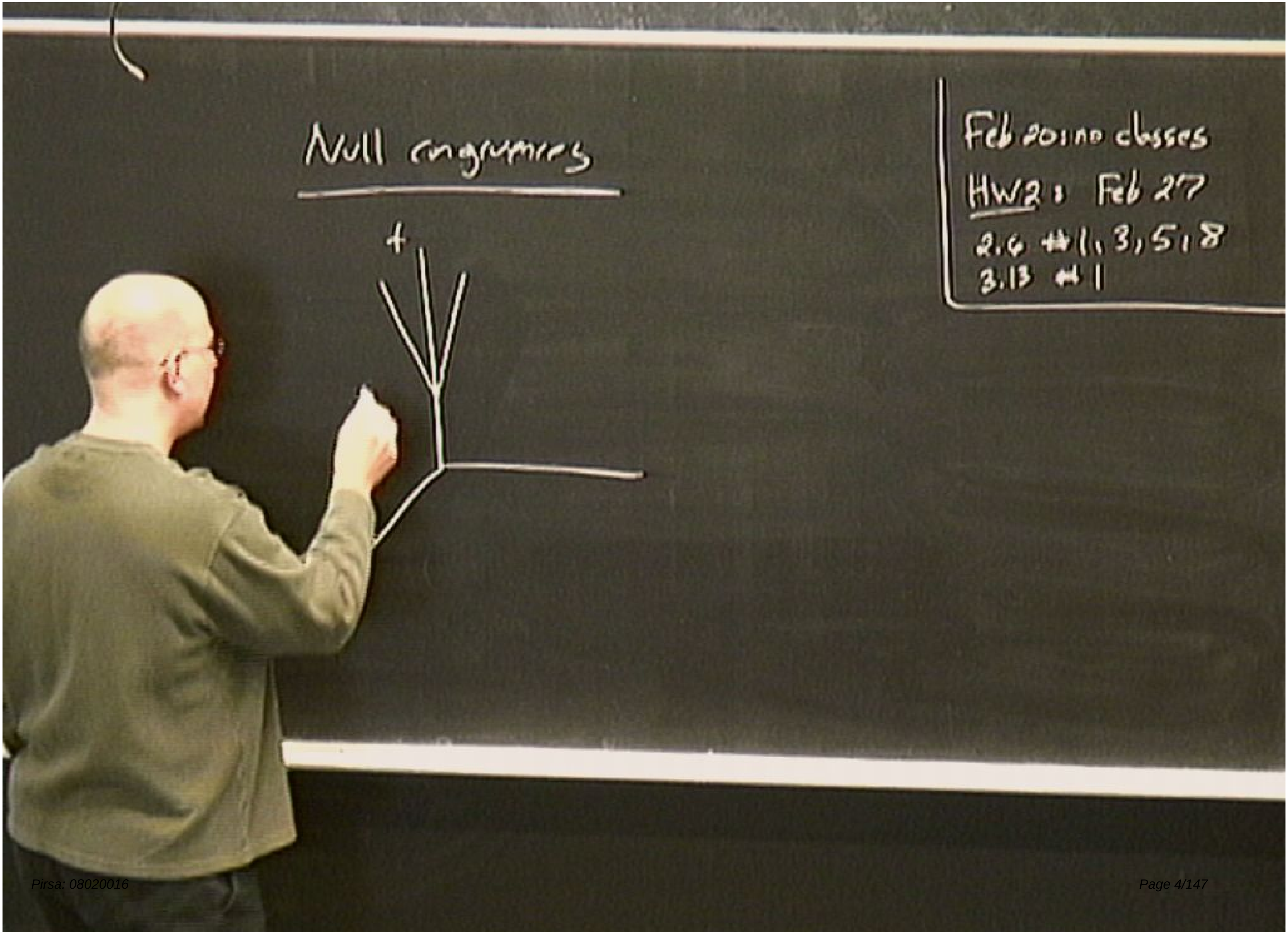
Feb 20: no classes

HW2: Feb 27

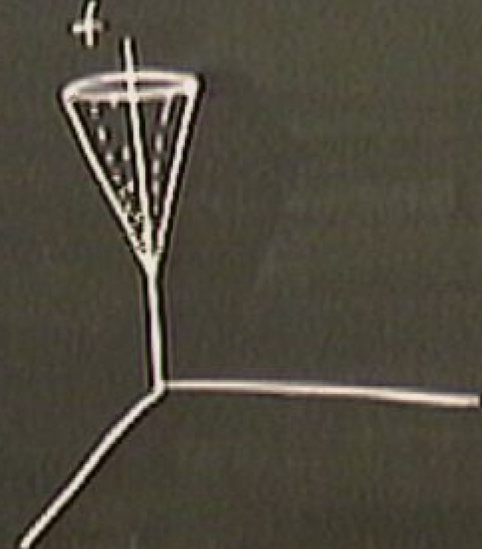
2.6 #1, 3, 5, 8

3.13 #1


Null congruences
t
Feb 20: no classes
HW2: Feb 27
2.6 #1, 3, 5, 8
3.13 #1


Null congruences
t
Feb 20: no classes
HW2: Feb 27
2.6 #1, 3, 5, 8
3.13 #1

Null congruences

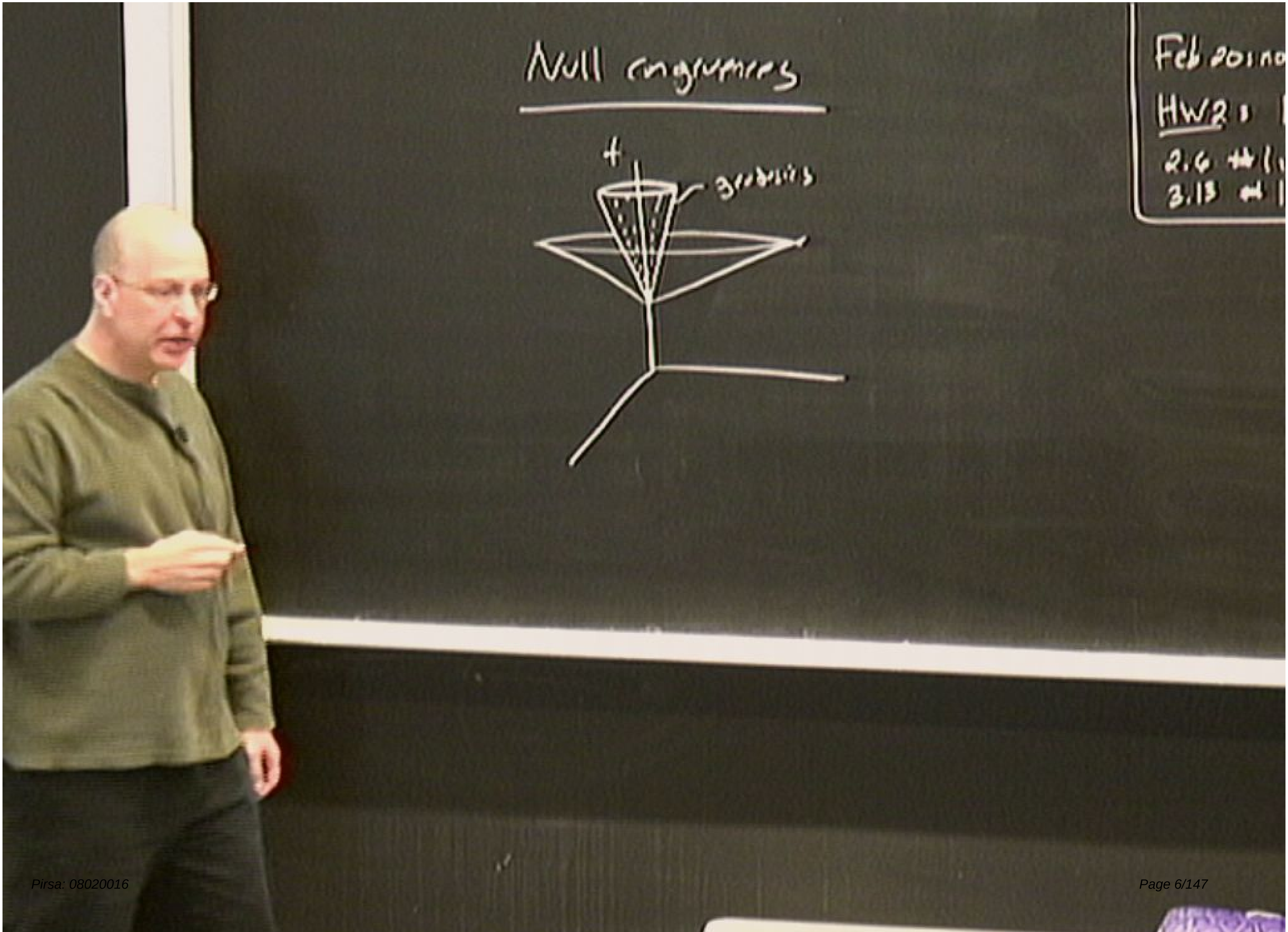


Feb 20: no classes

HW2: Feb 27

2.6 #1, 3, 5, 8

3.13 #1


Null congruences
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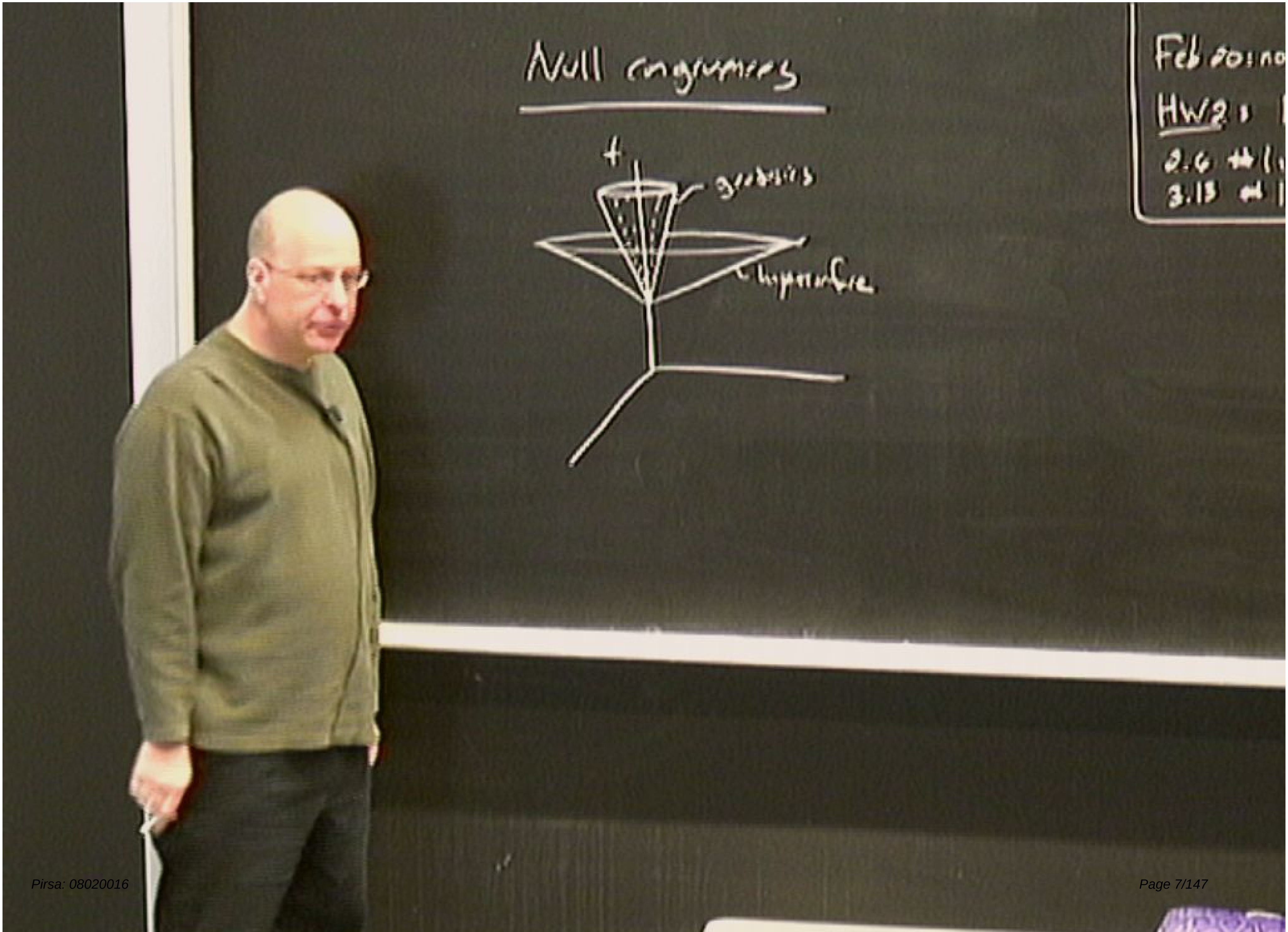
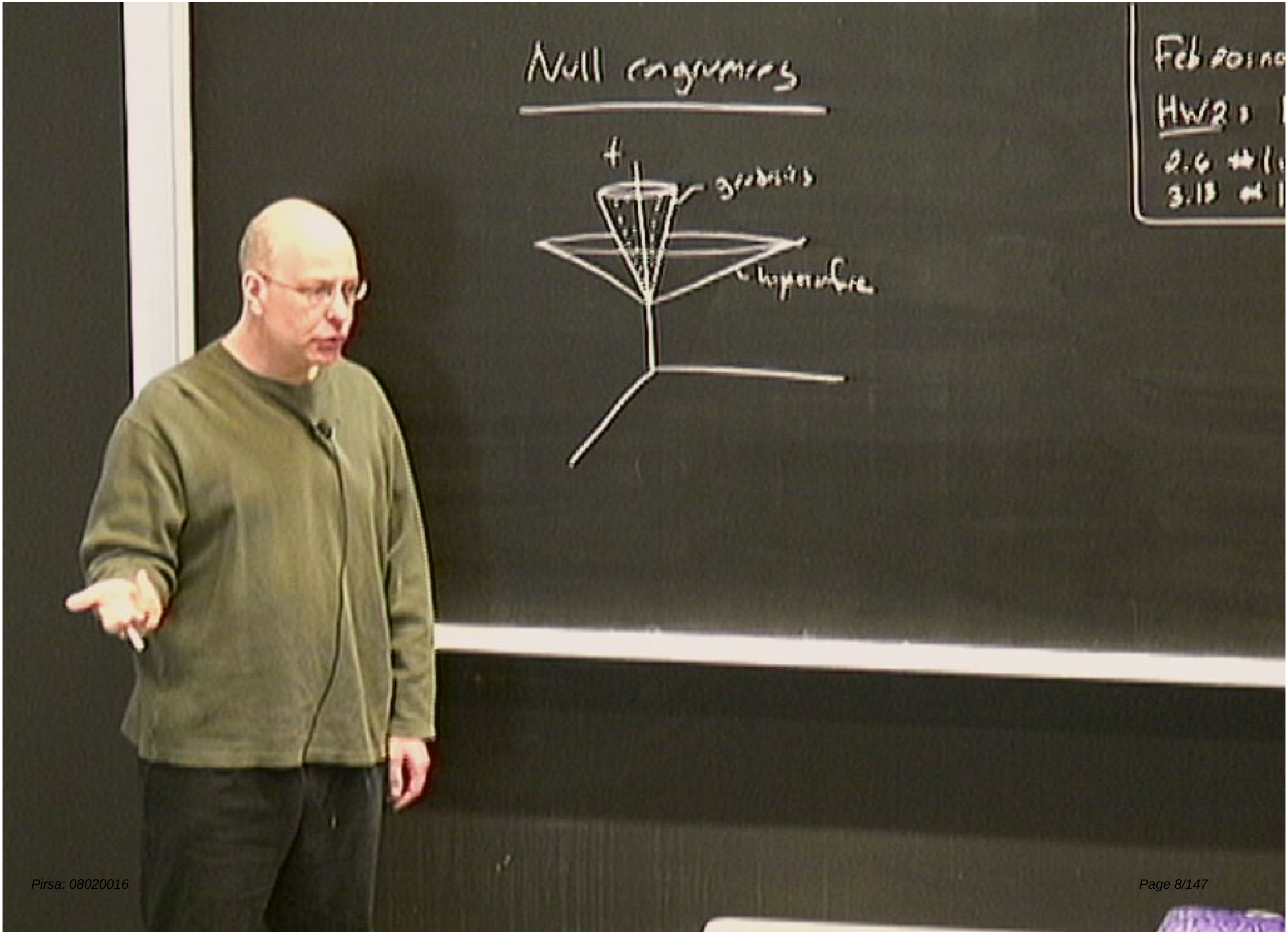
geodesics
HW2:
2.6
3.13


Null congruences
t
geodesics
hypersurface
Feb 20: no
HW2:
2.6
3.13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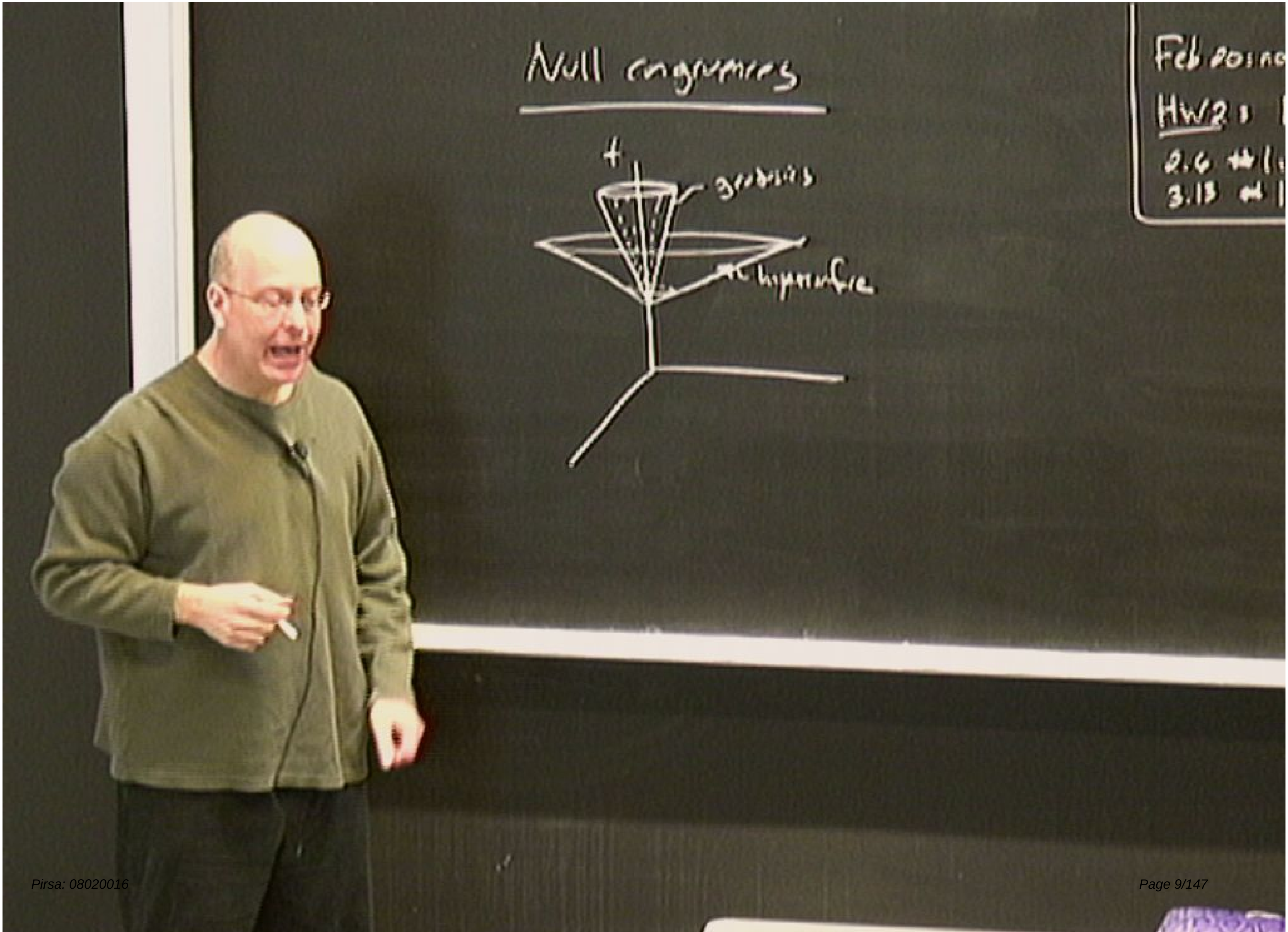
geodesics
HW2:


Null congruences
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HW2:
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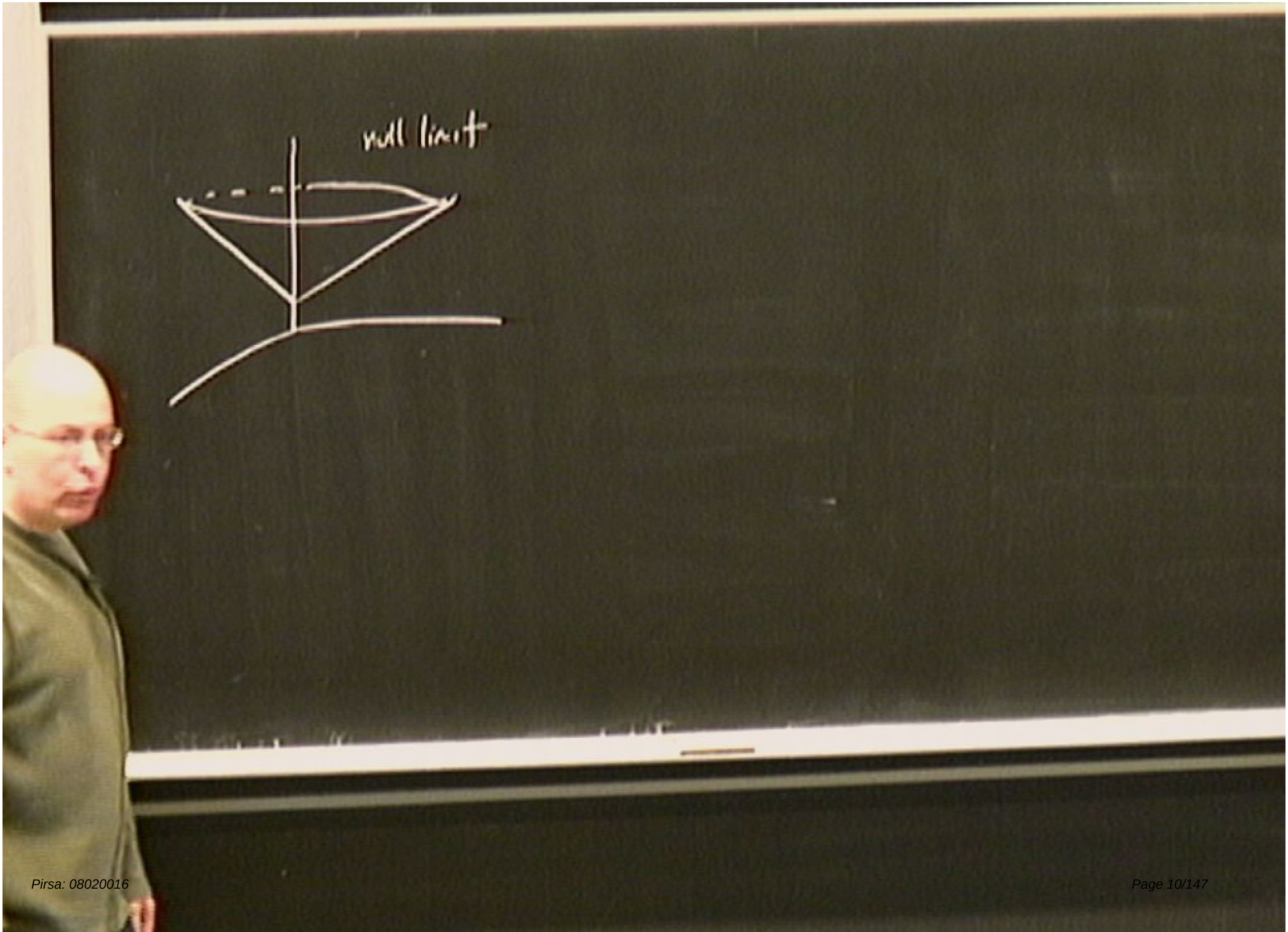
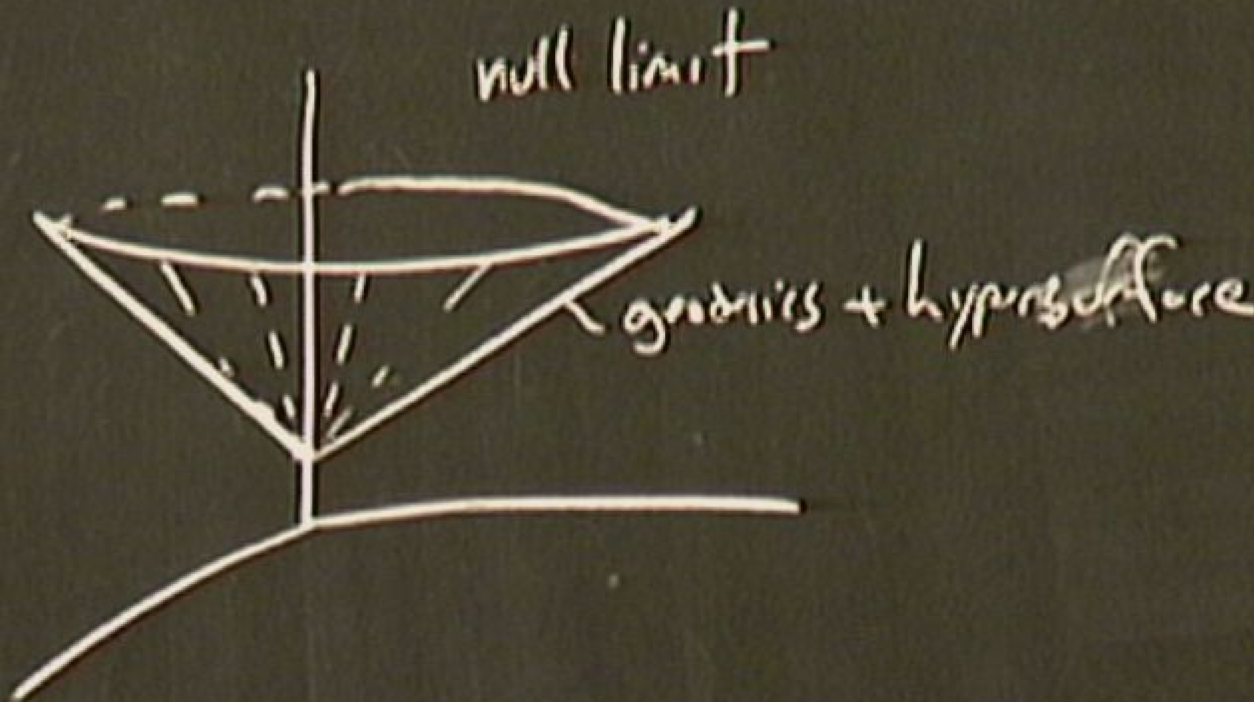
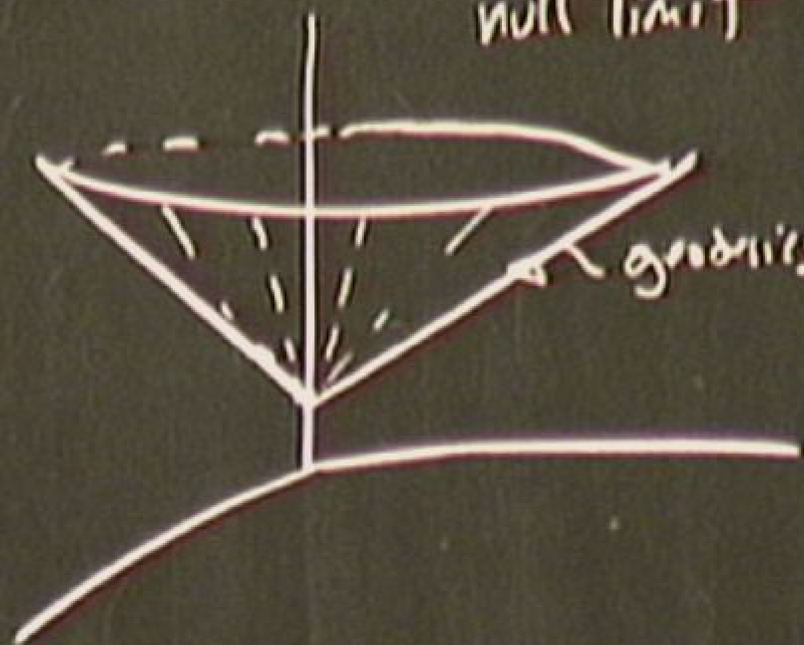
3.13

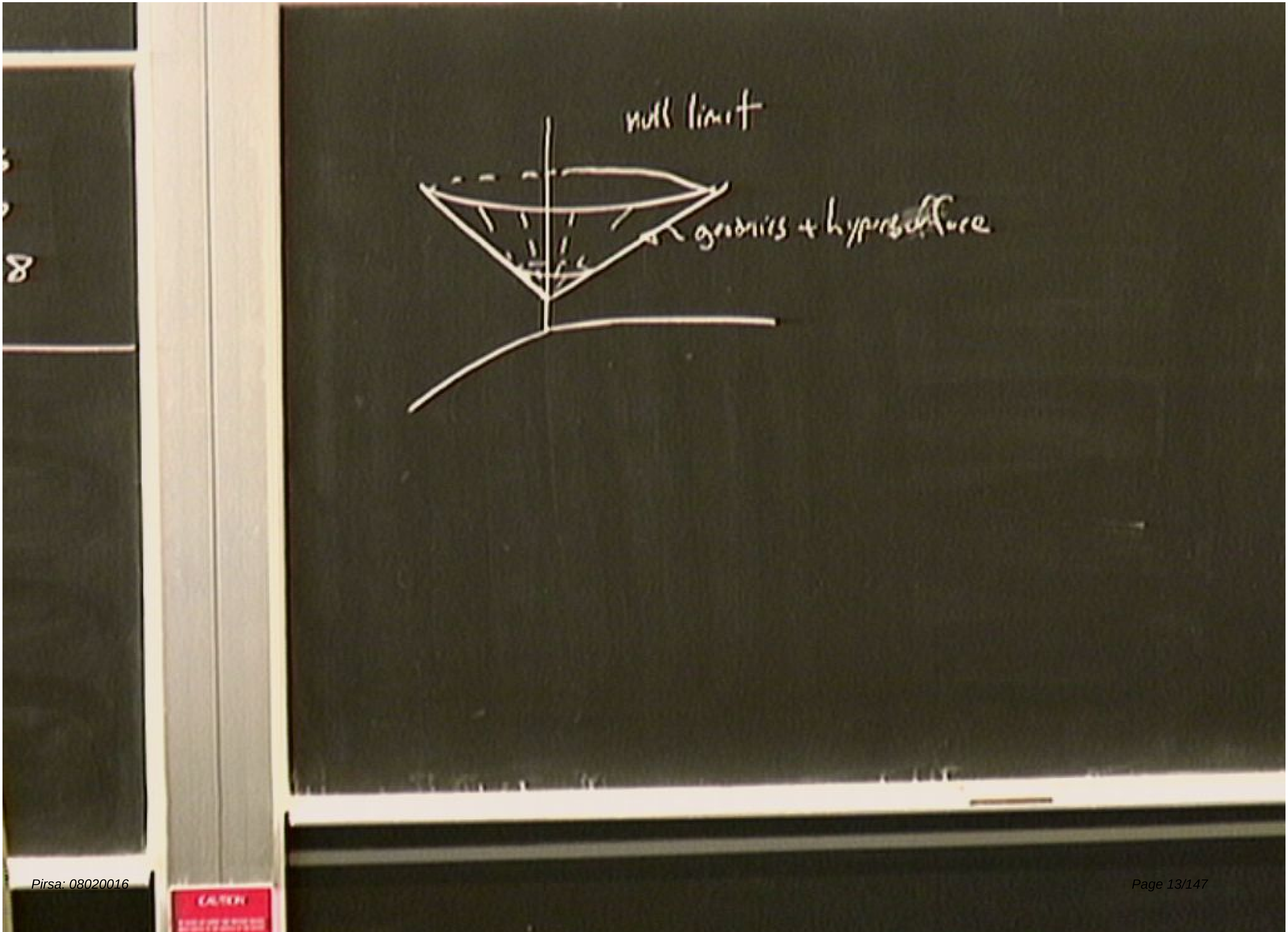

null limit


null limit
geodesics + hypersurface

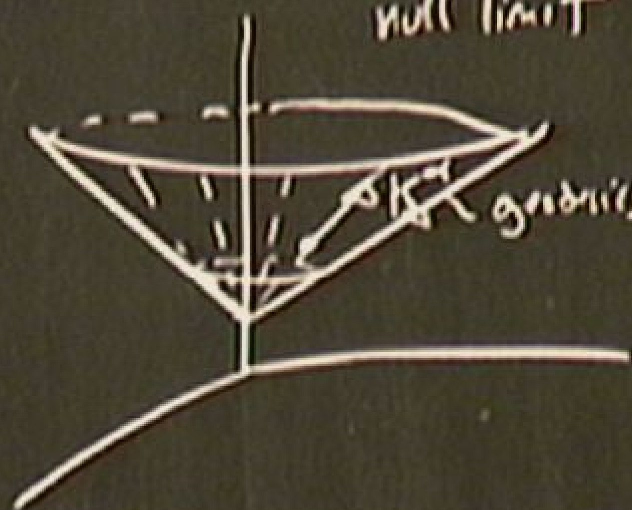
null limit



geodesics + hypersurface

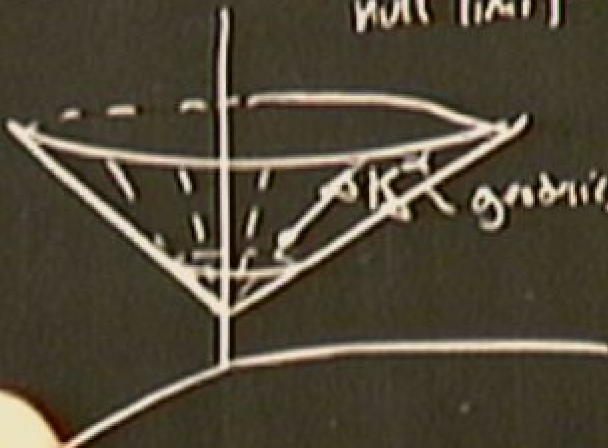

null limit

null limit



geodesics + hypersurface (metrically 2D)

null limit

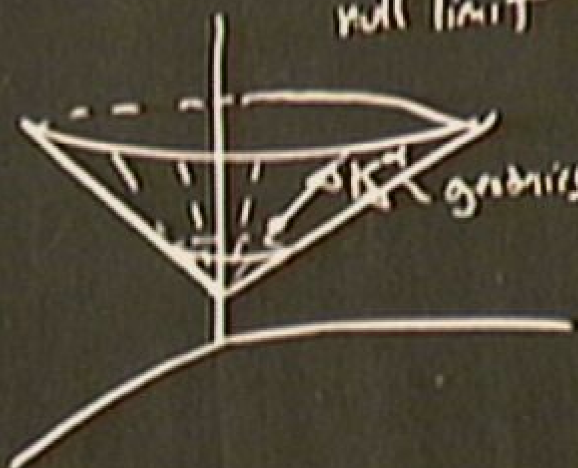


k^α geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics

: normal to hypersurface

null limit



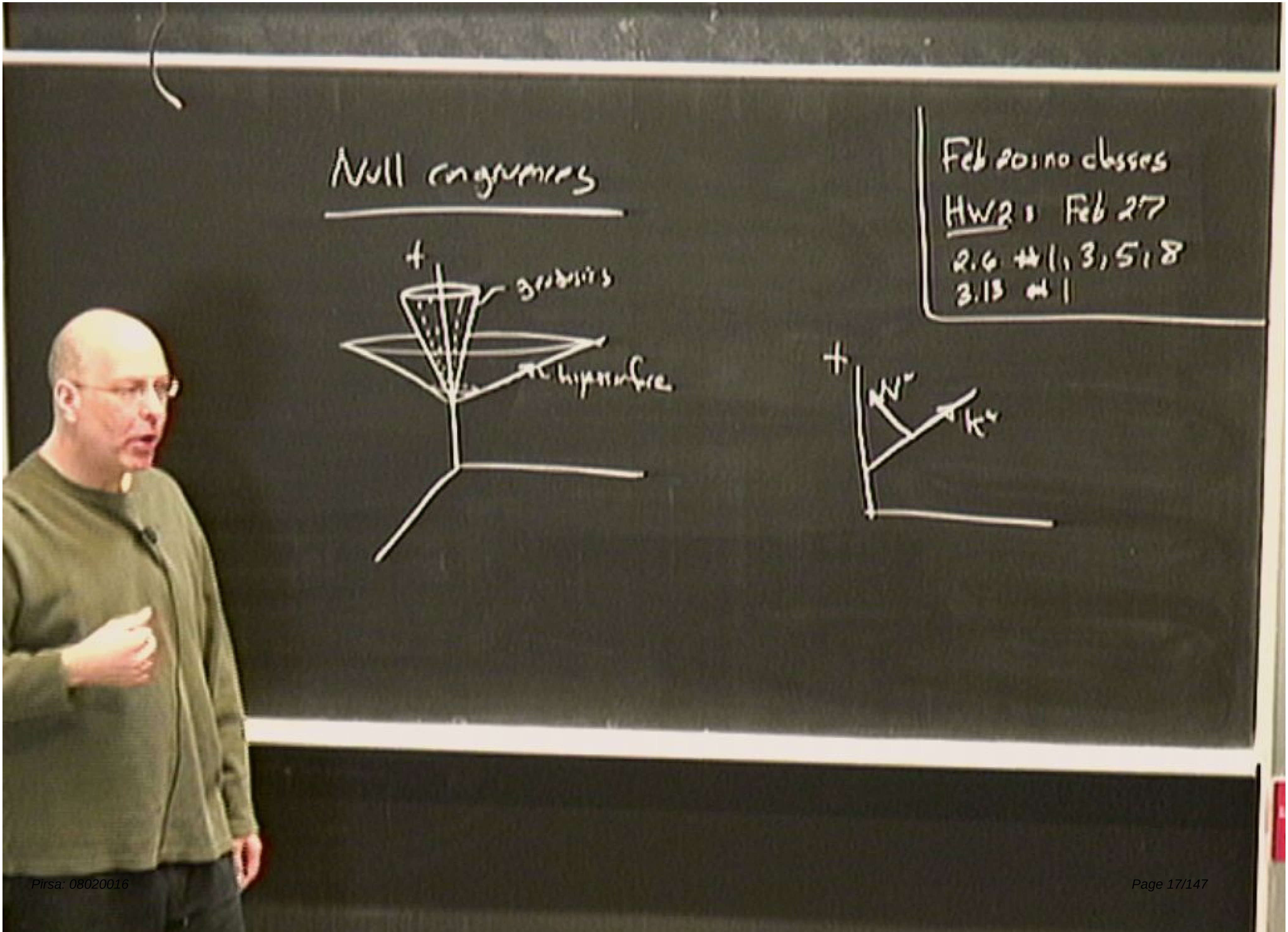
k^α geodesics + hypersurface (intrinsically 2D)

k^α: tangent to geodesics

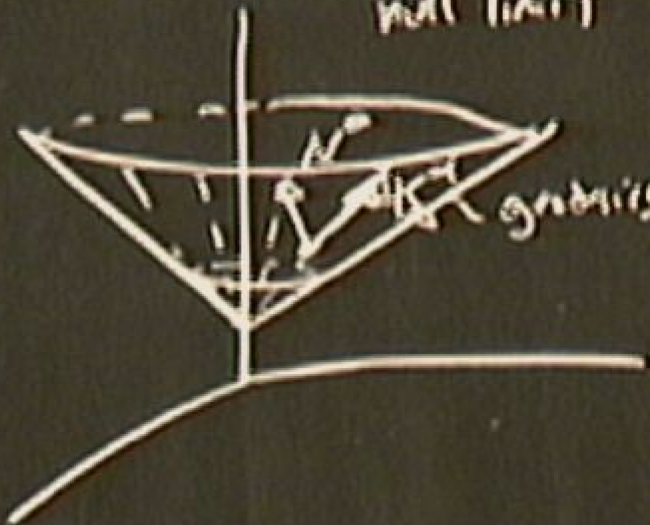
: normal to hypersurface

: tangent to hypersurface

$\left.\begin{array}{l} \text{normal to hypersurface} \\ \text{tangent to hypersurface} \end{array}\right\} k_\alpha k^\alpha = 0$


Null congruences
t
geodesics
hypersurface
Feb 20: no classes
HW2: Feb 27
2.6 #1, 3, 5, 8
3.13 #1
t
N^μ
k^μ

null limit



geodesics + hypersurface (intrinsically 2D)

k^α: tangent to geodesics

: normal to hypersurface

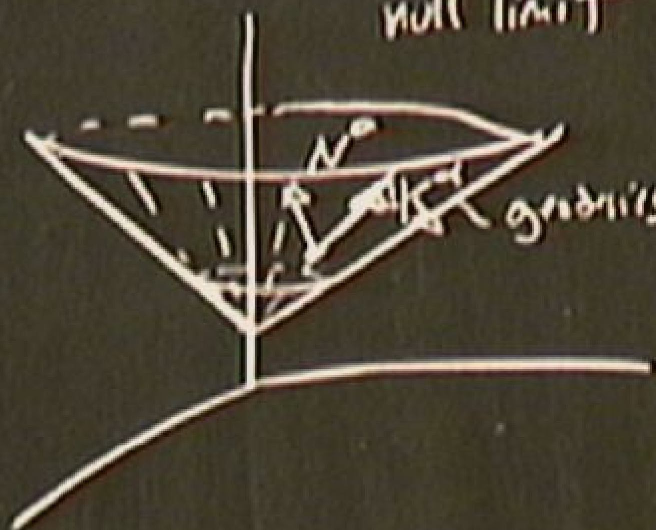
: tangent to hypersurface

$\Big\}\ k_\alpha k^\alpha =$

N^α: auxiliary null direction

$$N_\alpha N^\alpha = 0$$

null limit



geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics

: normal to hypersurface

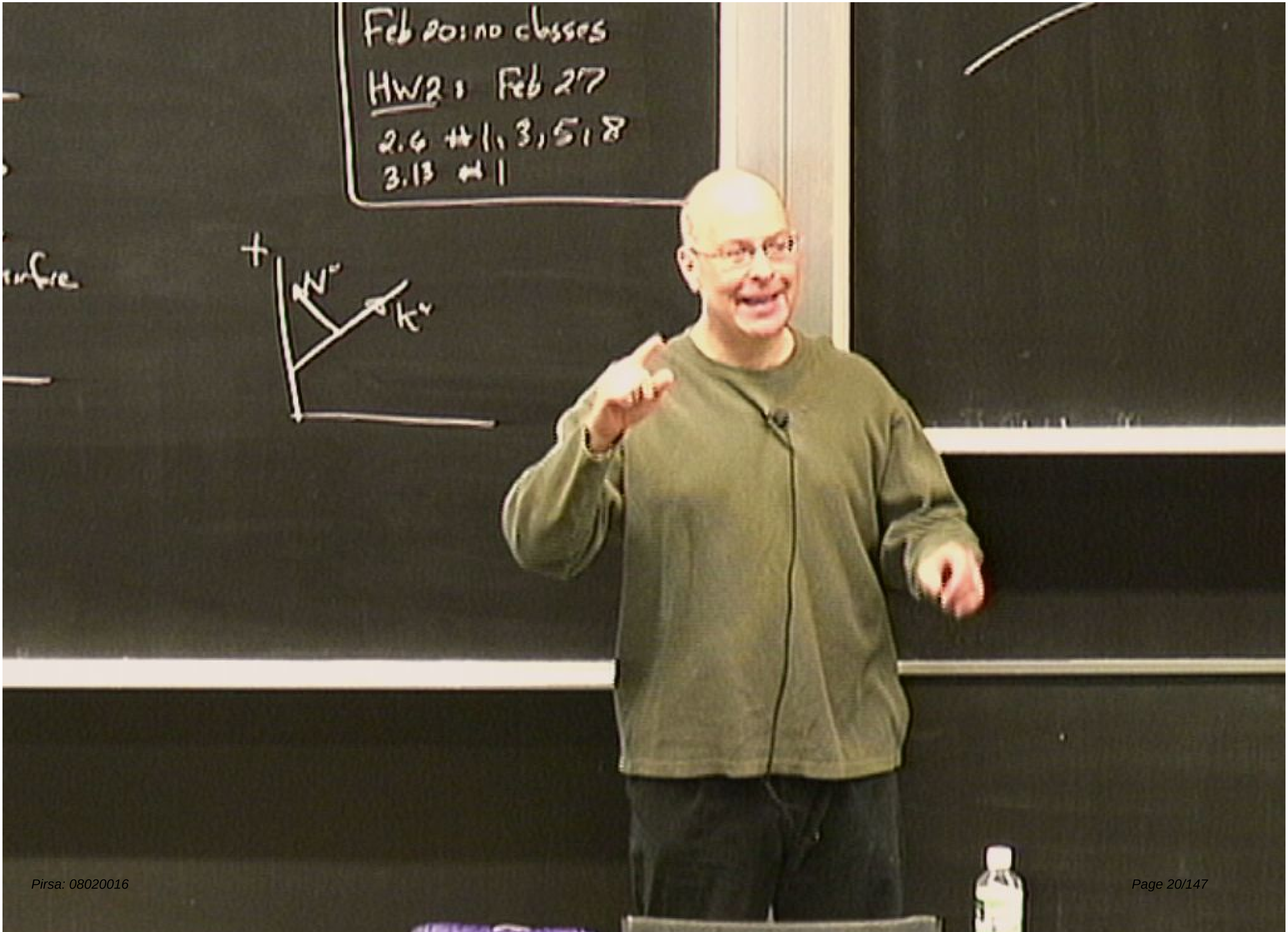
: tangent to hypersurface

$\Big\}\ k_\alpha k^\alpha = 0$

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha k^\alpha = -1$$


Feb 20: no classes
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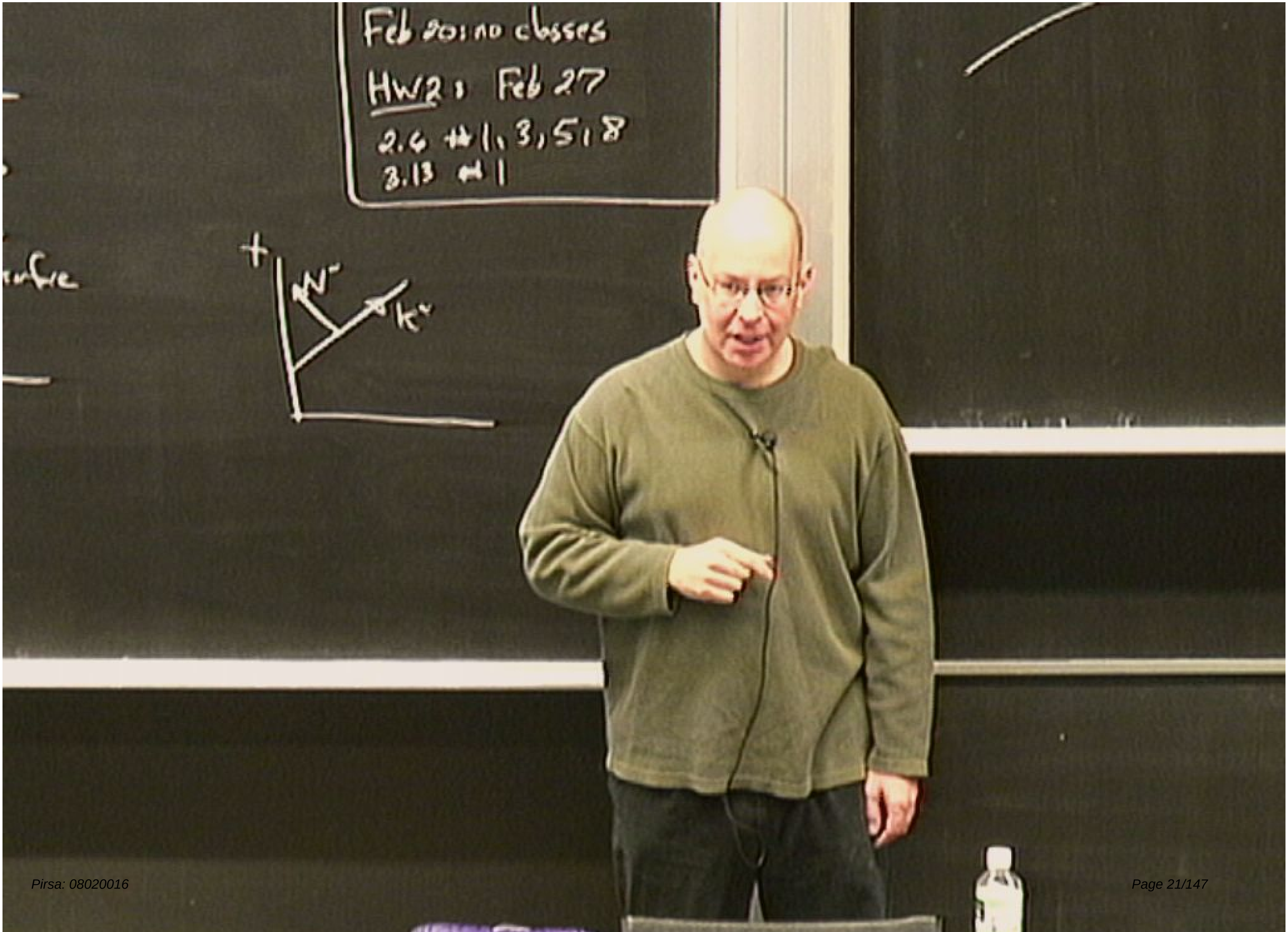
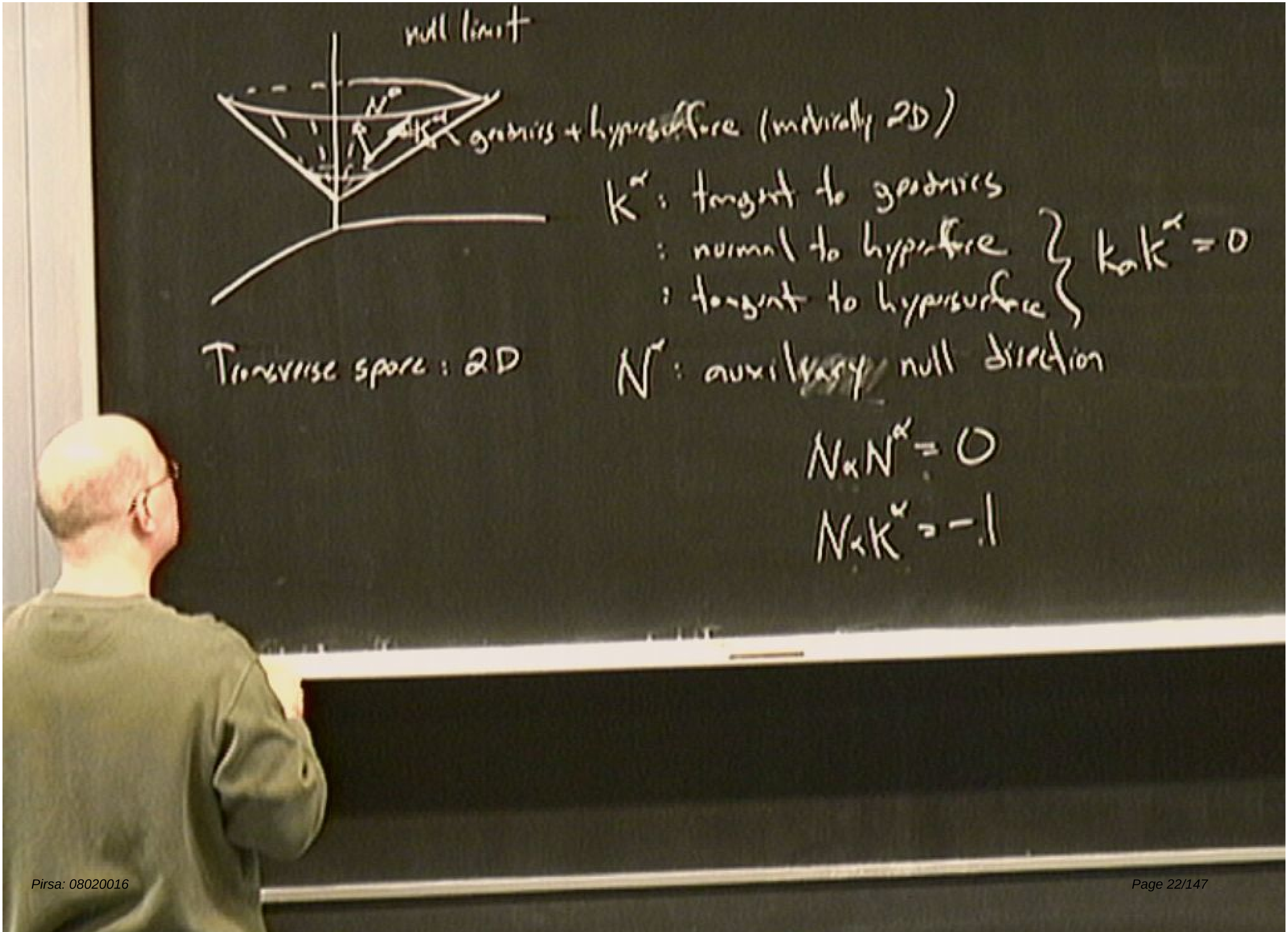
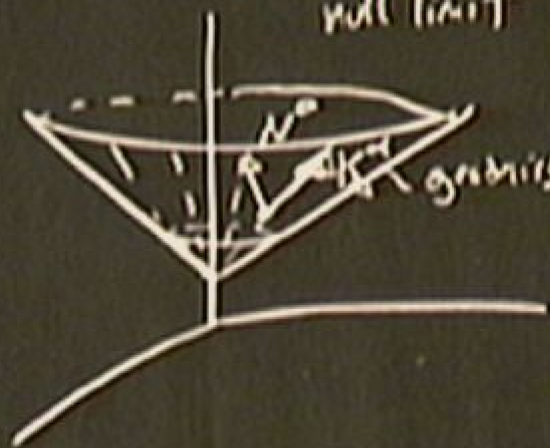

Feb 20: no classes
HW2: Feb 27
2.4 #1, 3, 5, 8
3.13 #1


null limit
N^α
K^α geodesics + hypersurface (individually 2D)
K^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface
$K_\alpha K^\alpha = 0$
Transverse space : 2D
N^α : auxiliary null direction
$N_\alpha N^\alpha = 0$
$N_\alpha K^\alpha = -1$

null limit



geodesics + hypersurface (intrinsically 2D)

K^α : tangent to geodesics

$\left.\begin{array}{l}: \text{normal to hypersurface} \\ : \text{tangent to hypersurface}\end{array}\right\} \; k_\alpha k^\alpha = 0$

Transverse space : 2D

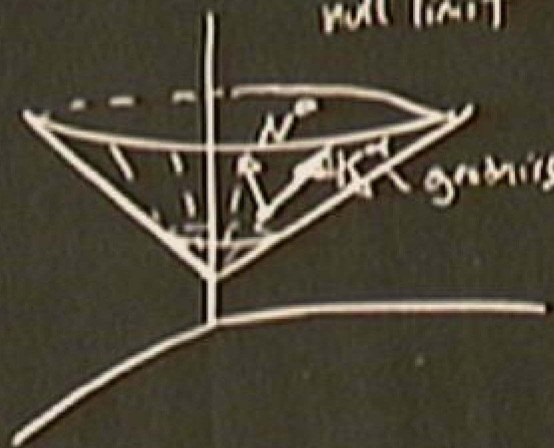
- orthogonal to both N^α and K^α

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$

null limit



geodesics + hypersurface (individually 2D)

k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface

$\Bigg\}\ k_\alpha k^\alpha = 0$

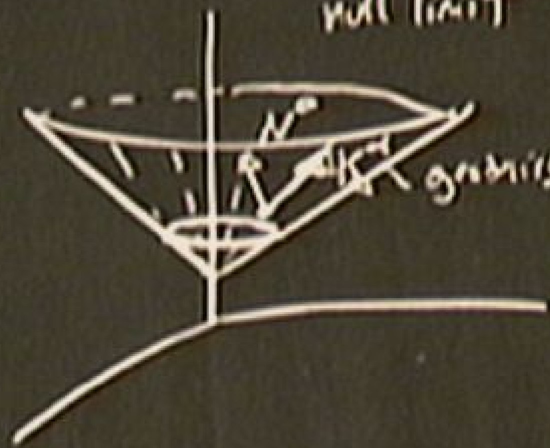
N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha k^\alpha = -1$$

Transverse space: 2D
- orthogonal to both $\underline{N}$ and $\underline{k}$

null limit



geodesics + hypersurface (individually 2D)

k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface

$$\left.\begin{array}{l}\text{tangent to geodesics}\\ \text{normal to hypersurface}\\ \text{tangent to hypersurface}\end{array}\right\} k_\alpha k^\alpha = 0$$

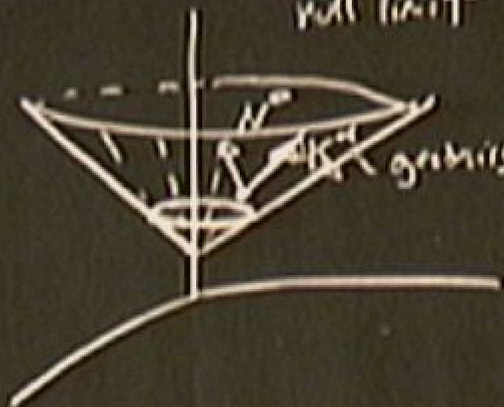
N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha k^\alpha = -1$$

Transverse space : 2D
- orthogonal to both $\underline{N}$ and $\underline{k}$

null limit



geodesics + hypersurface (intrinsically 2D)

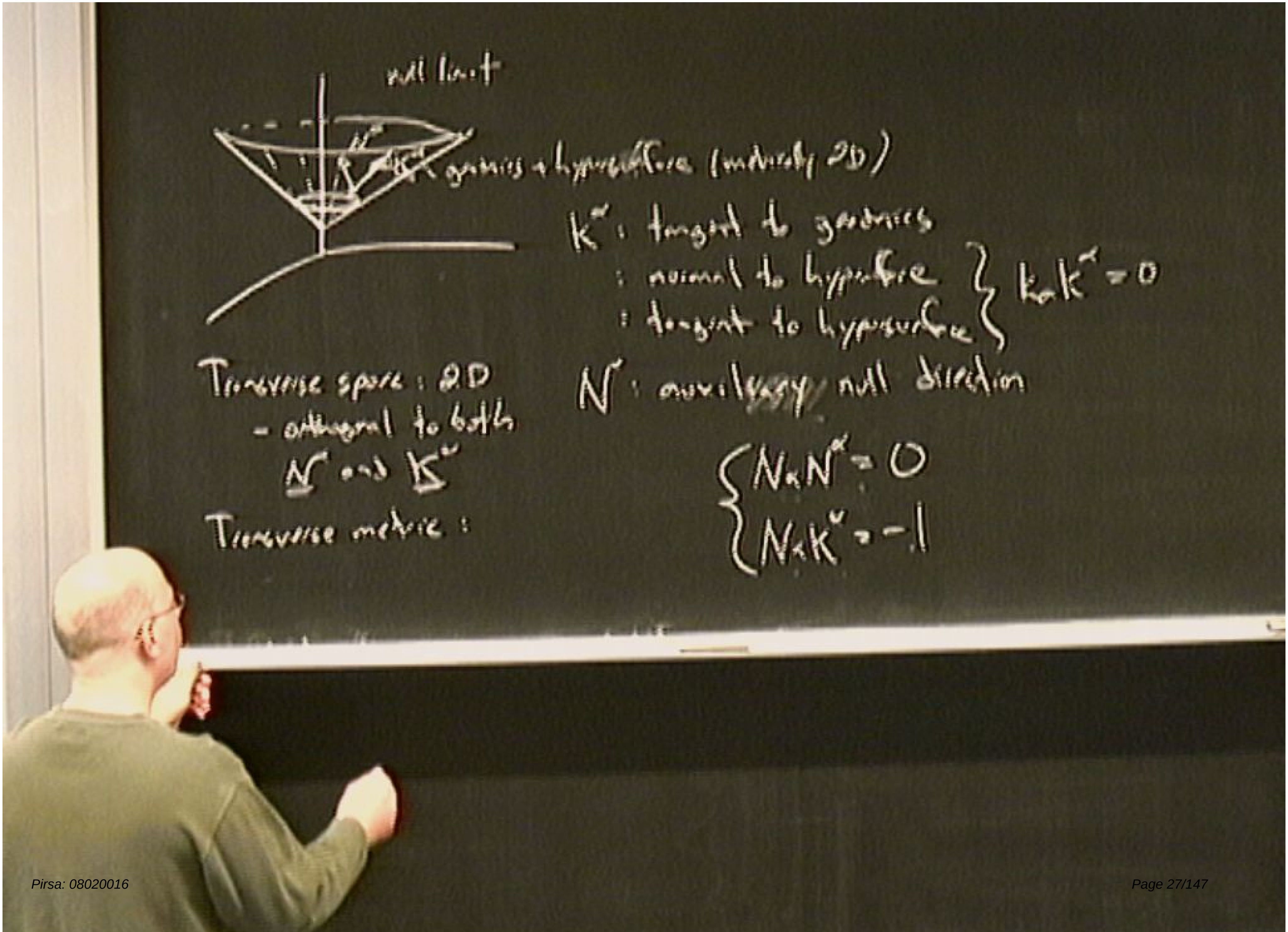
k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface

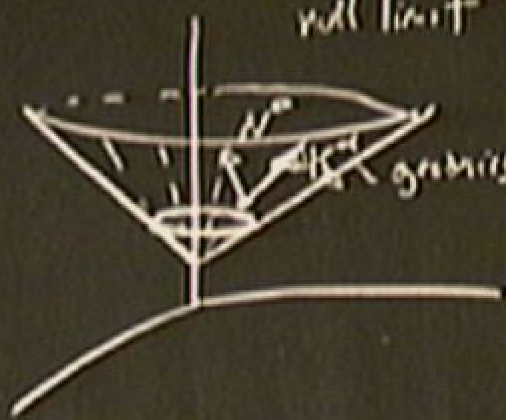
$\left.\begin{array}{l} \\ \\ \end{array}\right\} k_\alpha k^\alpha = 0$

Transverse space : 2D
- orthogonal to both
$\underline{N}^\alpha$ and $\underline{k}^\alpha$

N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha k^\alpha = -1 \end{cases}$$


null limit
(intrinsically 2D)
k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface
$k_\alpha k^\alpha = 0$
Transverse space : 2D
- orthogonal to both
N and k
Transverse metric :
N^α : auxiliary null direction
$N_\alpha N^\alpha = 0$
$N_\alpha k^\alpha = -1$



geodesics + hypersurface (inherently 2D)

K^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface

$\Big\}\ K_\alpha K^\alpha = 0$

N^α : auxiliary null direction

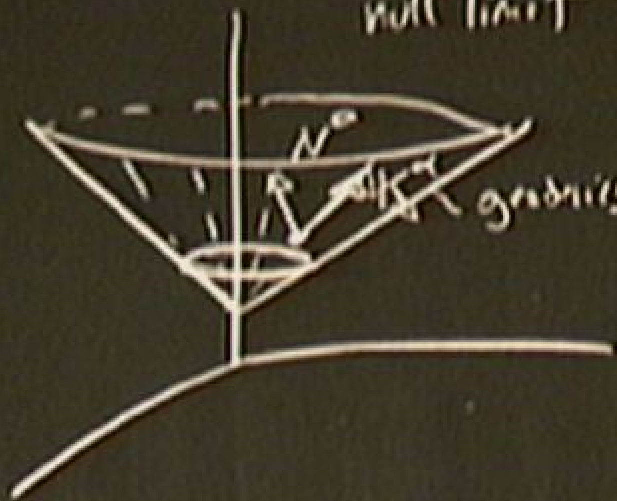
$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{cases}$$

Transverse space : 2D
- orthogonal to both $\underline{N}$ and $\underline{K}$

Transverse metric :

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

null limit



geodesics + hypersurface (intrinsically 2D)

$$k^\alpha : \left.\begin{array}{l}\text{tangent to geodesics} \\ \text{: normal to hypersurface} \\ \text{: tangent to hypersurface}\end{array}\right\} \; k_\alpha k^\alpha = 0$$

N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha k^\alpha = -1 \end{cases}$$

Transverse space : 2D

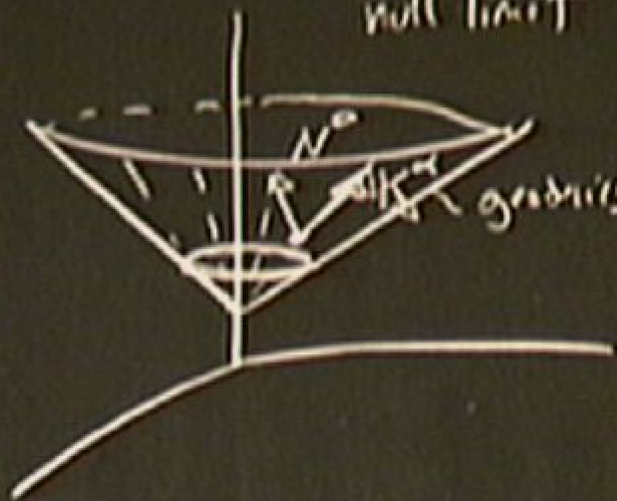
- orthogonal to both $\underline{N}^\alpha$ and $\underline{k}^\alpha$

Transverse metric :

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$h^\alpha{}_\alpha = 2$$

null limit



geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface

$$\left.\right\} k_\alpha k^\alpha = 0$$

N^α : auxiliary null direction

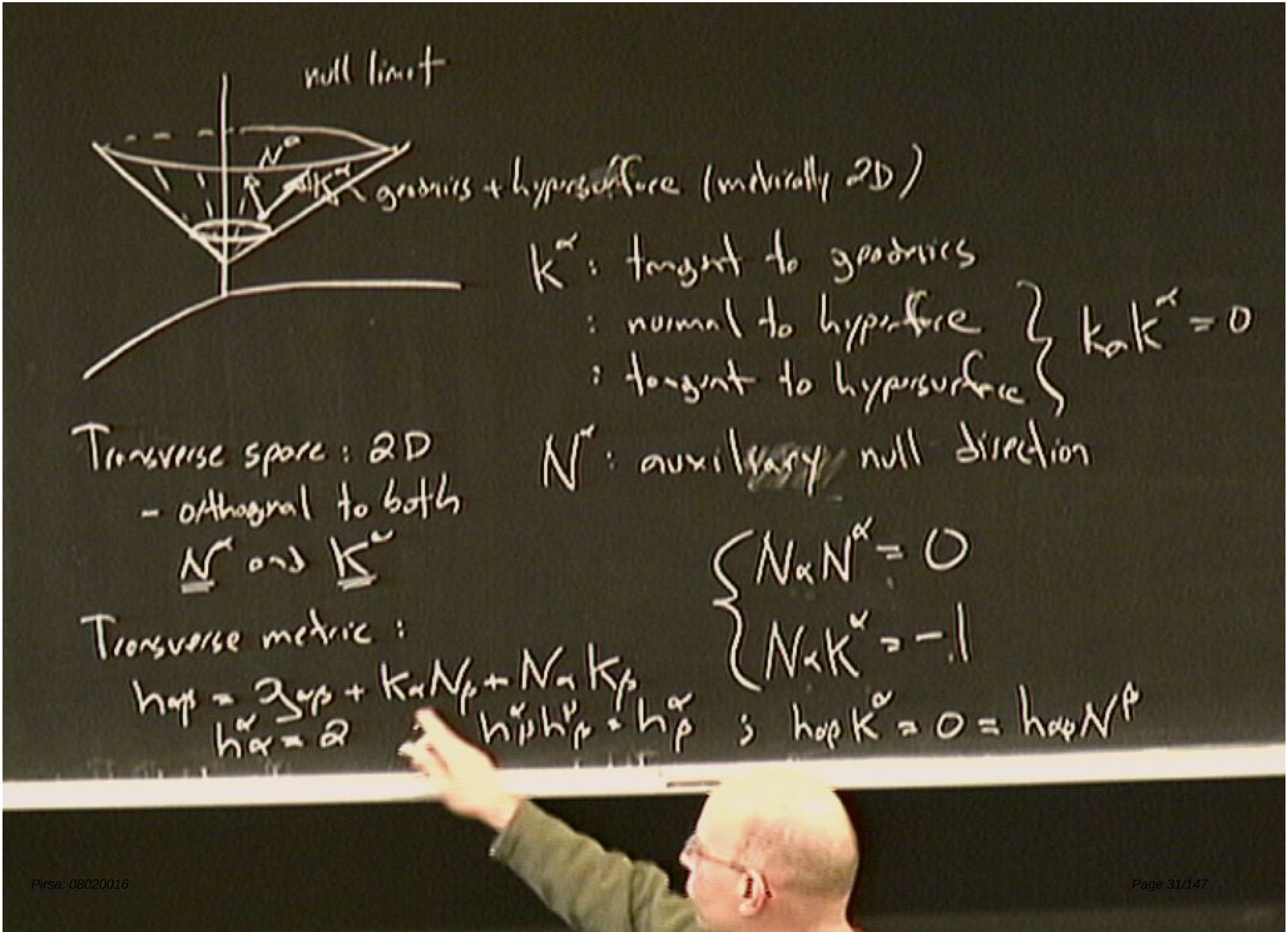
$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha k^\alpha = -1 \end{cases}$$

Transverse space : 2D
- orthogonal to both $\underline{N}^\alpha$ and $\underline{k}^\alpha$

Transverse metric :

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

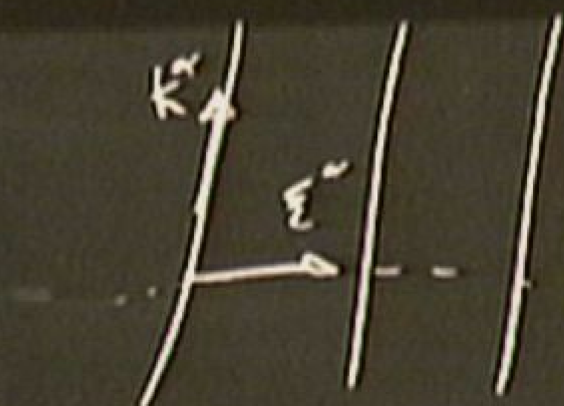
$$h^\alpha{}_\alpha = 2 \quad ; \quad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$$


null limit
N^α
k^α geodesics + hypersurface (mentally 2D)
k^α : tangent to geodesics
: normal to hypersurface
: tangent to hypersurface
$k_\alpha k^\alpha = 0$
Transverse space : 2D
- orthogonal to both
N^α and k^α
N^α : auxiliary null direction
$N_\alpha N^\alpha = 0$
$N_\alpha k^\alpha = -1$
Transverse metric :
$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$h^\alpha{}_\alpha = 2$
$h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$; $h_{\alpha\beta} k^\alpha = 0 = h_{\alpha\beta} N^\beta$

Transverse metric:

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta \qquad \{ N_\alpha k^\alpha = -1$$

$$h^\alpha{}_\alpha = 2 \;;\; h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta \;;\; h_{\alpha\beta} k^\alpha = 0 = h_{\alpha\beta} N^\beta$$



: tangent to hypersurface S

Transverse space : 2D
- orthogonal to both $\underline{N}^\alpha$ and $\underline{K}^\alpha$

N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{cases}$$

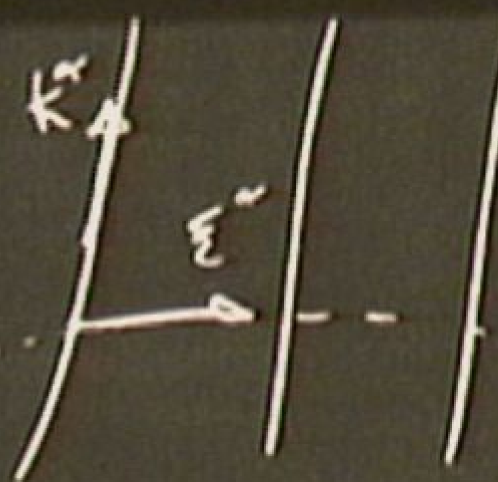
Transverse metric :

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$h^\alpha{}_\alpha = 2 \quad ; \quad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta \quad ; \quad h_{\alpha\beta} K^\alpha = 0 = h_{\alpha\beta} N^\beta$$

$$\xi^\alpha{}_{;\beta} K^\beta = K^\alpha{}_{;\beta}\, \xi^\beta$$

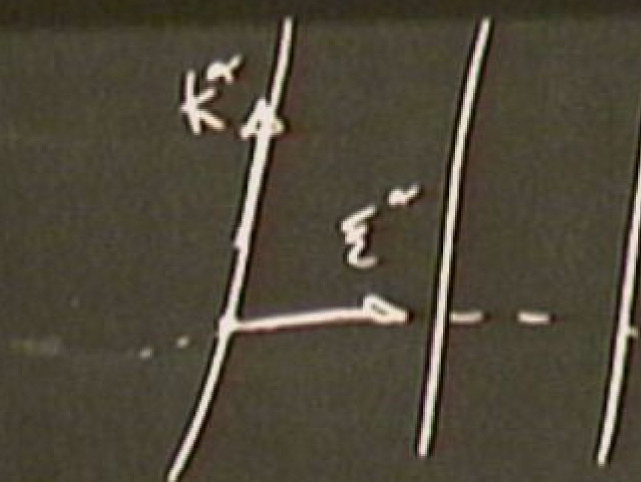
$h^{\alpha}_{\ \alpha} = 2 \ ; \ h^{\alpha}_{\ \beta} h^{\beta}_{\ \gamma} = h^{\alpha}_{\ \gamma} \ ; \ h_{\alpha\beta} k^{\beta} = 0 = h_{\alpha\beta} N^{\beta}$



$$\xi^{\alpha}_{\ ;\beta} k^{\beta} = k^{\alpha}_{\ ;\beta} \xi^{\beta}$$

$$\xi_{\alpha} k^{\alpha} = 0$$

$h_\alpha = a$; $h_\mu h^\mu{}_\rho = h_\rho$; $h_{\alpha\beta} K = 0 = h_{\alpha\beta} N$

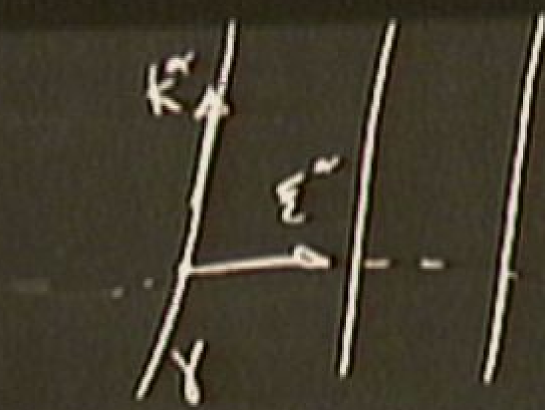


$$\xi^\alpha{}_{;\beta} K^\beta = K^\alpha{}_{;\beta}\, \xi^\beta$$

$$\xi_\alpha K^\alpha = 0$$

$$K^\alpha{}_{;\beta} K^\beta = 0$$

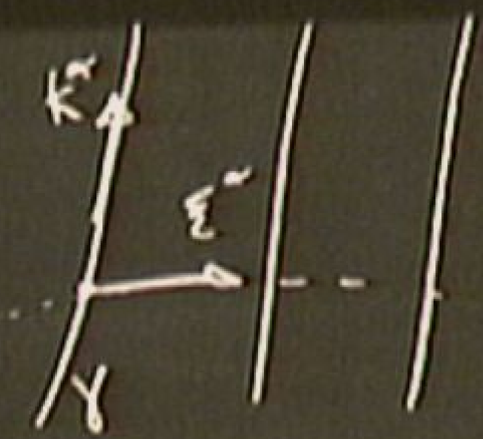
$$h^{\alpha}{}_{\alpha} = 2 \; ; \; h^{\alpha}{}_{\mu} h^{\mu}{}_{\beta} = h^{\alpha}{}_{\beta} \; ; \; h_{\alpha\beta} k^{\beta} = 0 = h_{\alpha\beta} N^{\beta}$$



$$\xi^{\alpha}{}_{;\beta} k^{\beta} = k^{\alpha}{}_{;\beta} \xi^{\beta}$$

$$\xi_{\alpha} k^{\alpha} = 0$$

$$k^{\alpha}{}_{;\beta} k^{\beta} = 0$$



$$\xi^\alpha{}_{;\beta}\, k^\beta = k^\alpha{}_{;\beta}\, \xi^\beta$$

$\xi_\alpha k^\alpha = 0$: does not rule out a component of ξ_α along k_α

$$k^\alpha{}_{;\beta}\, k^\beta = 0$$

– orthogonal to both

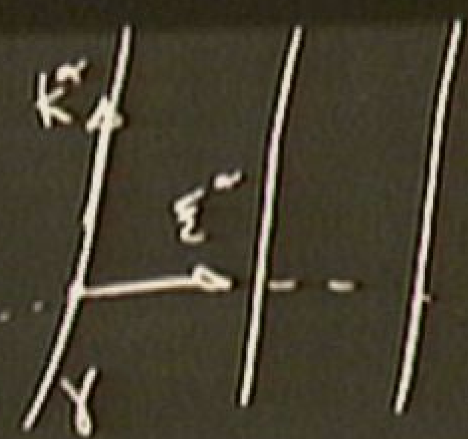
$\underline{N}^\alpha$ and $\underline{K}^\alpha$

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{cases}$$

Transverse metric:

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

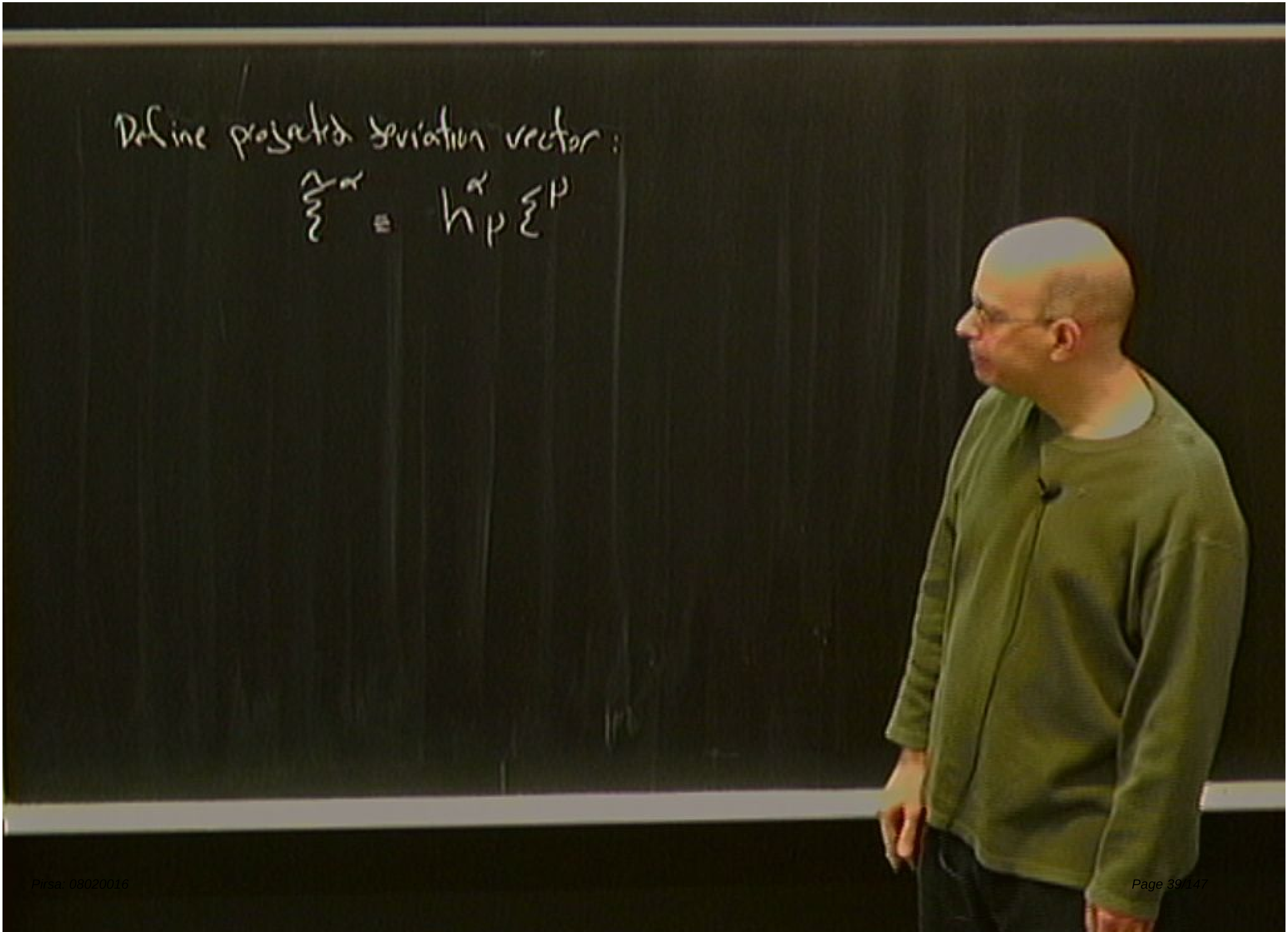
$$h^\alpha{}_\alpha = 2 \quad ; \quad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta \quad ; \quad h_{\alpha\beta} K^\alpha = 0 = h_{\alpha\beta} N^\beta$$



$$\xi^\alpha{}_{;\beta} K^\beta = K^\alpha{}_{;\beta}\, \xi^\beta$$

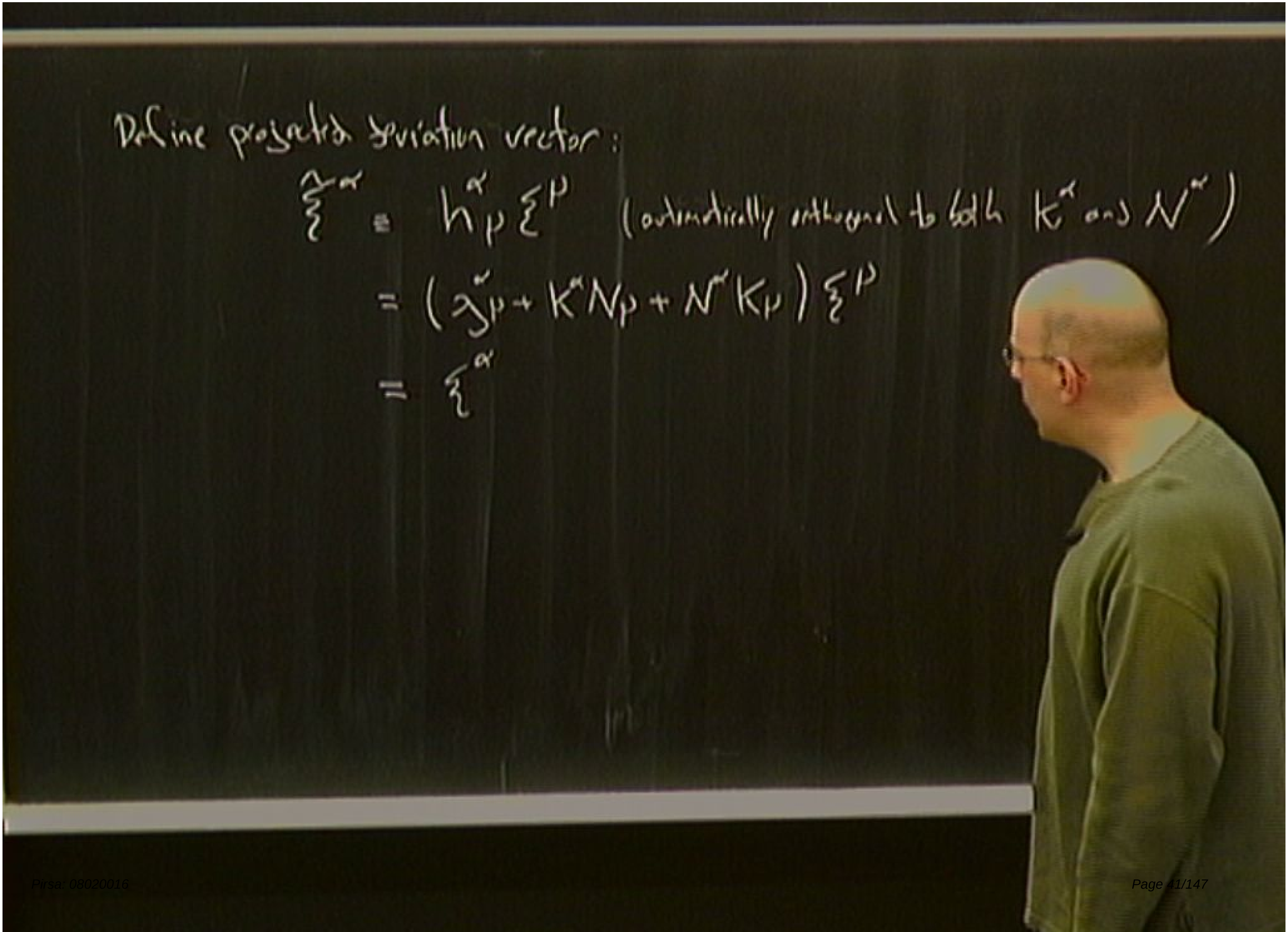
$\xi_\alpha K^\alpha = 0$: does not rule out a component of ξ^α along K^α

$$K^\alpha{}_{;\beta} K^\beta = 0$$


Define projection deviation vector:
$\tilde{\xi}^\alpha \equiv h^\alpha{}_\beta \xi^\beta$

Define projected deviation vector:

$$\tilde{\xi}^{\alpha} \equiv h^{\alpha}{}_{\mu}\, \xi^{\mu} \quad \text{(automatically orthogonal to both } K^{\alpha} \text{ and } N^{\alpha}\text{)}$$


Define projected deviation vector:
$\tilde{\xi}^\alpha \equiv h^\alpha{}_\beta\, \xi^\beta$ (automatically orthogonal to both K^α and N^α)
$= (g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta)\, \xi^\beta$
$= \xi^\alpha$

Define projected deviation vector:

$$\tilde{\xi}^\alpha \equiv h^\alpha{}_\beta \, \xi^\beta \quad \text{(automatically orthogonal to both } K^\alpha \text{ and } N^\alpha\text{)}$$

$$= \left(g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta \right) \xi^\beta$$

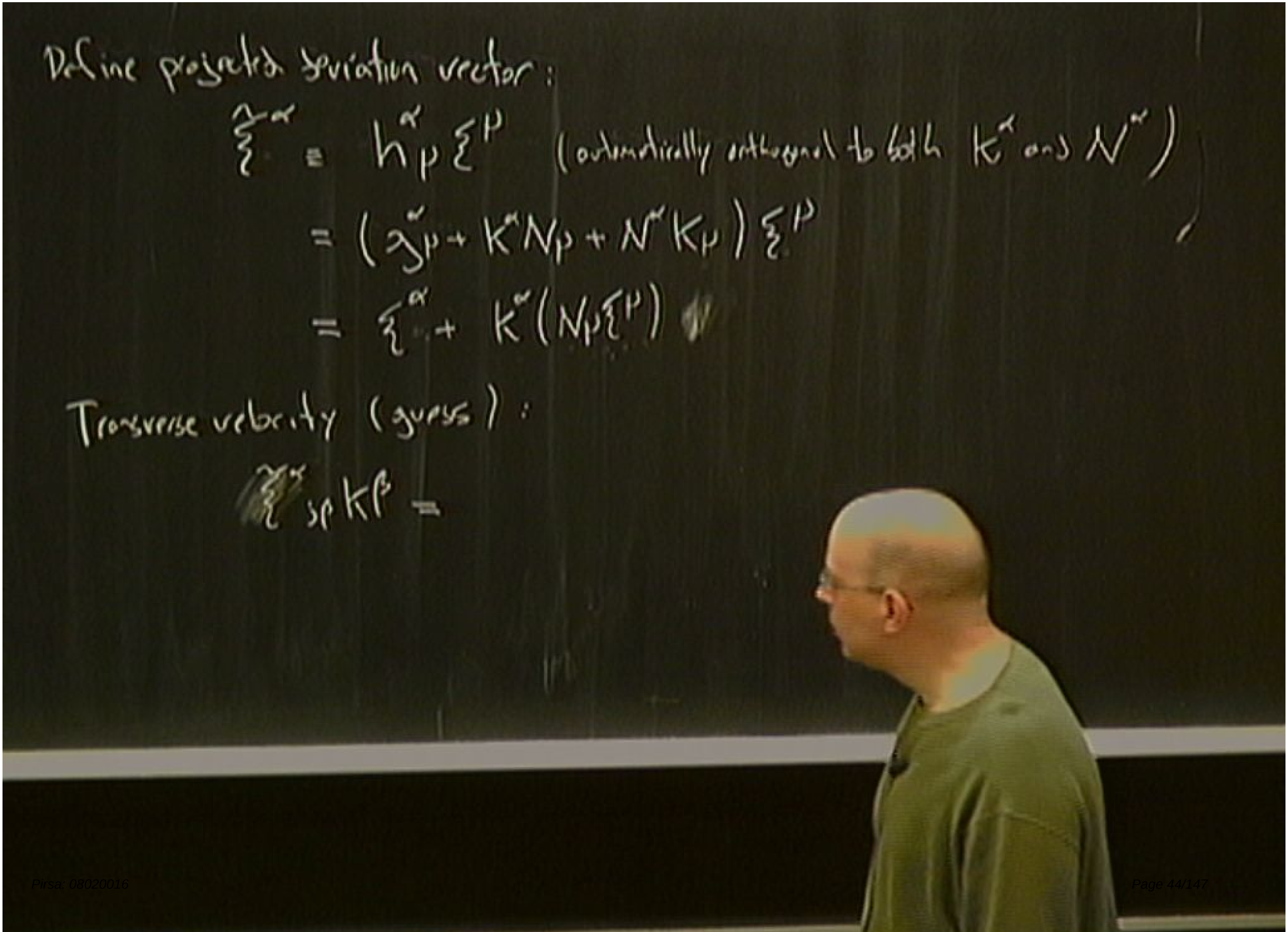
$$= \xi^\alpha + K^\alpha \left(N_\beta \xi^\beta \right)$$

Define projected deviation vector:

$$\tilde{\xi}^{\alpha} = h^{\alpha}{}_{\beta}\, \xi^{\beta} \quad \text{(automatically orthogonal to both } K^{\alpha} \text{ and } N^{\alpha}\text{)}$$

$$= \left(g^{\alpha}{}_{\beta} + K^{\alpha} N_{\beta} + N^{\alpha} K_{\beta} \right) \xi^{\beta}$$

$$= \xi^{\alpha} + K^{\alpha} \left(N_{\beta} \xi^{\beta} \right)$$


Define projected deviation vector:
$\tilde{\xi}^\alpha = h^\alpha{}_\mu \xi^\mu$ (automatically orthogonal to both K^α and N^α)
$= (g^\alpha{}_\mu + K^\alpha N_\mu + N^\alpha K_\mu)\xi^\mu$
$= \xi^\alpha + K^\alpha(N_\mu \xi^\mu)$
Transverse velocity (guess):
$\tilde{\xi}^\alpha{}_{;\beta} K^\beta =$

Define projected deviation vector:

$$\tilde{\xi}^\alpha = h^\alpha{}_\mu \xi^\mu \quad \text{(automatically orthogonal to both } K^\alpha \text{ and } N^\alpha\text{)}$$

$$= \left(g^\alpha{}_\mu + K^\alpha N_\mu + N^\alpha K_\mu \right) \xi^\mu$$

$$= \xi^\alpha + K^\alpha \left(N_\mu \xi^\mu \right)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = \left(h^\alpha{}_\mu \xi^\mu \right)_{;\beta} K^\beta$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha = h^\alpha{}_\mu \xi^\mu \quad \text{(automatically orthogonal to both } K^\alpha \text{ and } N^\alpha\text{)}$$

$$= (\delta^\alpha{}_\mu + K^\alpha N_\mu + N^\alpha K_\mu)\,\xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = (h^\alpha{}_\mu \xi^\mu)_{;\beta} K^\beta$$

$$= h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta$$

$$\tilde{\xi}^\alpha = h^\alpha{}_\mu \xi^\mu \quad \text{(automatically orthogonal to both } k^\alpha \text{ and } N^\alpha\text{)}$$

$$= \left(g^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu \right) \xi^\mu$$

$$= \xi^\alpha + k^\alpha \left(N_\mu \xi^\mu \right)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha{}_{;\beta} k^\beta = \left(h^\alpha{}_\mu \xi^\mu \right)_{;\beta} k^\beta$$

$$= h^\alpha{}_\mu \xi^\mu{}_{;\beta} k^\beta + h^\alpha{}_{\mu;\beta} \xi^\mu k^\beta$$

$$= \left(\delta^\alpha{}_\mu + K^\alpha N_\mu + N^\alpha K_\mu \right) \xi^\mu$$

$$= \xi^\alpha + K^\alpha \left(N_\mu \xi^\mu \right)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = \left(h^\alpha{}_\mu \xi^\mu \right)_{;\beta} K^\beta$$

$$= h^\alpha{}_\mu \, \xi^\mu{}_{;\beta} K^\beta + \underbrace{h^\alpha{}_{\mu;\beta}}_{K^\alpha{}_{;\beta} N_\mu + K^\alpha N_{\mu;\beta}} \xi^\mu K^\beta$$

$$= (\delta^\alpha{}_\mu + K^\alpha N_\mu + N^\alpha K_\mu)\,\xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\xi^\alpha{}_{;\beta} K^\beta = (h^\alpha{}_\mu \xi^\mu)_{;\beta} K^\beta$$

$$= h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta + \underbrace{h^\alpha{}_{\mu;\beta}}_{K^\alpha{}_{;\beta} N_\mu + K^\alpha N_{\mu;\beta} + N^\alpha{}_{;\beta} K_\mu + N^\alpha K_{\mu;\beta}} \xi^\mu K^\beta$$

$$= \left(g^\alpha{}_\beta + K^\alpha N_\beta + N^\alpha K_\beta\right)\xi^\beta$$

$$= \xi^\alpha + K^\alpha\left(N_\beta \xi^\beta\right)$$

Transverse velocity (guess):

$$\xi^\alpha{}_{;\beta} K^\beta = \left(h^\alpha{}_\beta \xi^\beta\right)_{;\beta} K^\beta$$

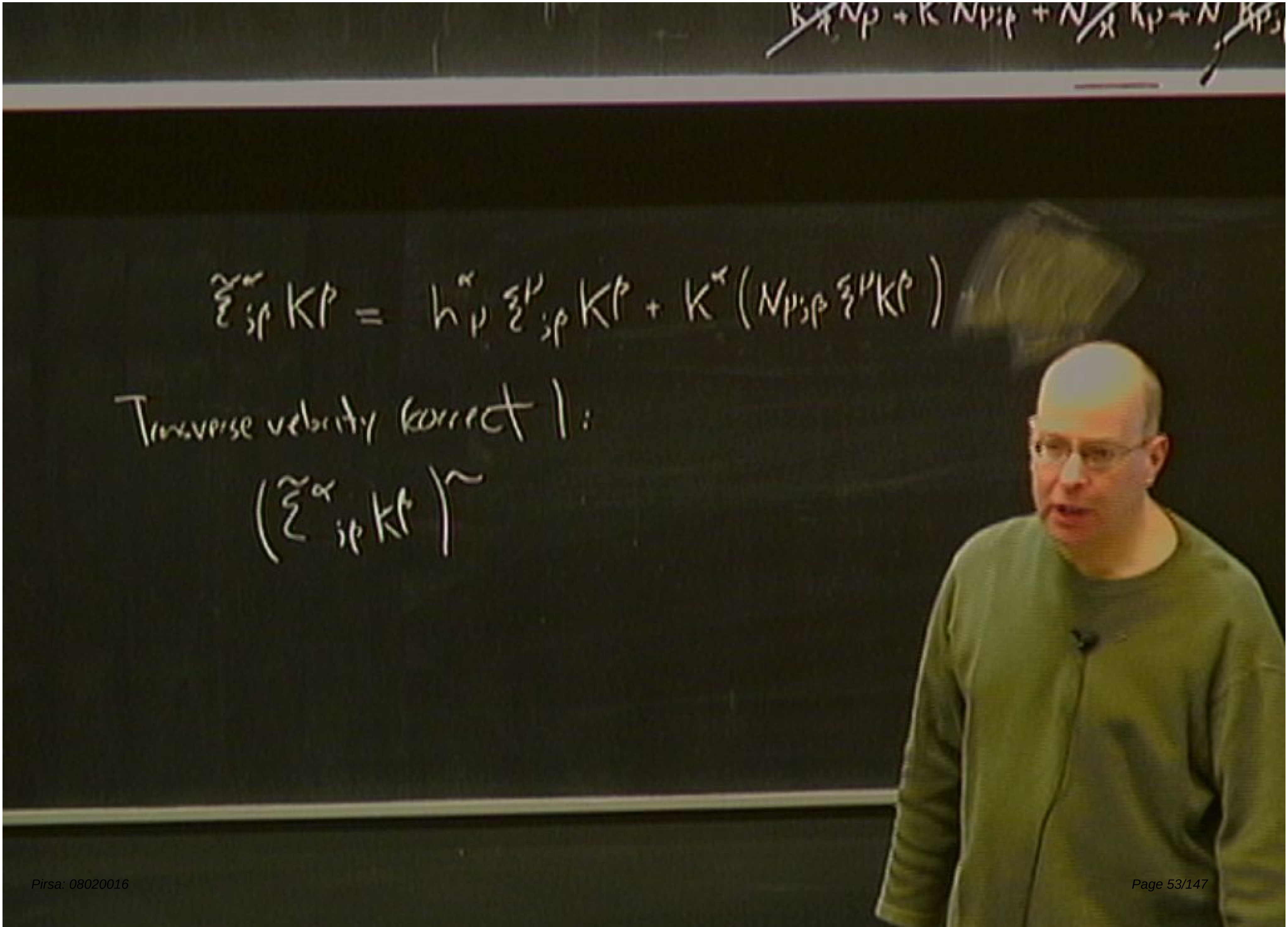
$$= h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta + \underbrace{h^\alpha{}_{\mu;\beta}}_{\cancel{K^\alpha{}_{;\beta}} N_\mu + K^\alpha N_{\mu;\beta} + N^\alpha{}_{;\beta} K_\mu + N^\alpha \cancel{K_{\mu;\beta}}} \xi^\mu K^\beta$$

$$= h^{\alpha}{}_{\mu}\, \xi^{\mu}{}_{;\rho} K^{\rho} + \underbrace{h^{\alpha}{}_{\mu;\rho}}_{\cancel{K^{\alpha}{}_{;\rho} N_{\mu}} + K^{\alpha} N_{\mu;\rho} + N^{\alpha}{}_{;\rho} K_{\mu} + \cancel{N^{\alpha} K_{\mu;\rho}}} \xi^{\mu} K^{\rho}$$

$$\tilde{\xi}^{\alpha}{}_{;\rho} K^{\rho} = h^{\alpha}{}_{\mu}\, \xi^{\mu}{}_{;\rho} K^{\rho}$$

$$= h^\alpha{}_\mu \, \xi^\mu{}_{;\rho} K^\rho + \underbrace{h^\alpha{}_{\mu;\rho}}_{\cancel{K^\alpha{}_{;\rho}} N_\mu + K^\alpha N_{\mu;\rho} + N^\alpha{}_{;\rho} K_\mu + N^\alpha \cancel{K_{\mu;\rho}}} \xi^\mu K^\rho$$

$$\tilde{\xi}^\alpha{}_{;\rho} K^\rho = h^\alpha{}_\mu \, \xi^\mu{}_{;\rho} K^\rho + K^\alpha \left(N_{\mu;\rho} \, \xi^\mu K^\rho \right)$$


$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta + K^\alpha \left(N_{\mu;\beta} \xi^\mu K^\beta \right)$
Transverse velocity correct!:
$\left(\tilde{\xi}^\alpha{}_{;\beta} K^\beta \right)^\sim$

$$\cancel{K_{;\alpha}} N_\beta + K N_{\beta;\beta} + \cancel{N_{;\alpha}} K_\beta + N \cancel{K_{\beta;}}$$

$$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta + K^\alpha \left(N_{\mu;\beta} \xi^\mu K^\beta \right)$$

Transverse velocity (correct!):

$$\left(\tilde{\xi}^\alpha{}_{;\beta} K^\beta \right)^\sim \equiv h^\alpha{}_\mu \tilde{\xi}^\mu{}_{;\beta} K^\beta$$

$$\tilde{\xi}^{\alpha}{}_{;\beta} K^{\beta} = h^{\alpha}{}_{\mu}\, \xi^{\mu}{}_{;\beta} K^{\beta} + K^{\alpha} \left(N_{\mu;\beta}\, \xi^{\mu} K^{\beta} \right)$$

Transverse velocity (correct):

$$\left(\tilde{\xi}^{\alpha}{}_{;\beta} K^{\beta} \right)^{\sim} \equiv h^{\alpha}{}_{\mu}\, \tilde{\xi}^{\mu}{}_{;\beta} K^{\beta}$$

$$= h^{\alpha}{}_{\mu}\, h^{\mu}{}_{\nu}\, \xi^{\nu}{}_{;\beta} K^{\beta}$$

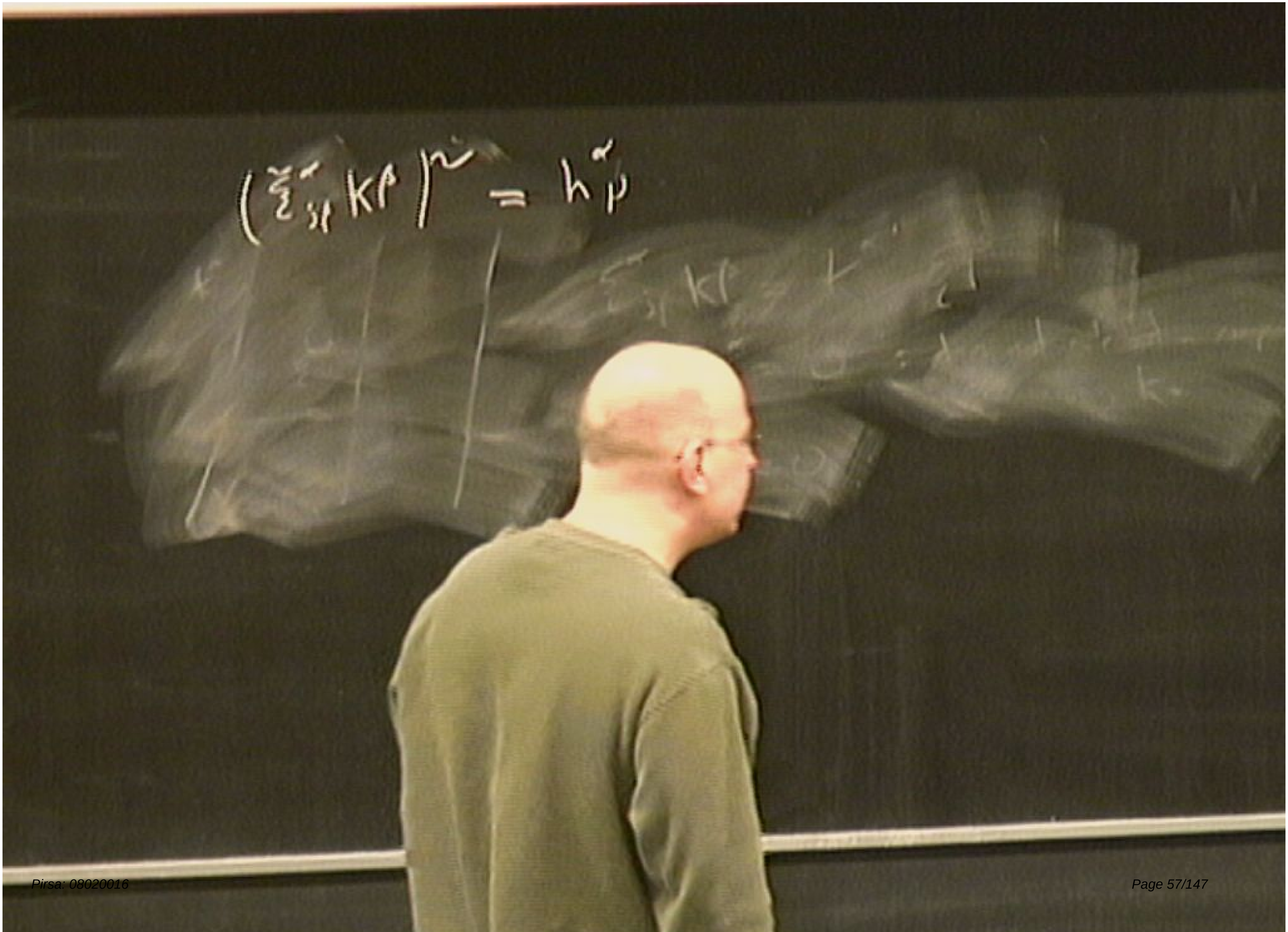
$$\tilde{\xi}^\alpha{}_{;\beta} K^\beta = h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta + K^\alpha \left(N_{\mu;\beta} \xi^\mu K^\beta \right)$$

Transverse velocity (correct):

$$\left(\tilde{\xi}^\alpha{}_{;\beta} K^\beta \right)^\sim = h^\alpha{}_\mu \tilde{\xi}^\mu{}_{;\beta} K^\beta$$

$$= h^\alpha{}_\mu h^\mu{}_\nu \xi^\nu{}_{;\beta} K^\beta$$

$$= h^\alpha{}_\mu \xi^\mu{}_{;\beta} K^\beta$$



$$\left(\tilde{\xi}^{\alpha}{}_{;\mu}\, k^{\mu}\right)^{\sim} = h^{\alpha}{}_{\mu}\, k^{\mu}{}_{;\beta}\, \xi^{\beta}$$

$$B_{\alpha\beta} = k_{\alpha;\beta}$$

$$(\xi^{\alpha}{}_{;\beta}\, k^{\beta})^{\sim} = h^{\alpha}{}_{\mu}\, k^{\mu}{}_{;\beta}\, \xi^{\beta}$$

$$= h^{\alpha}{}_{\mu}\, B^{\mu}{}_{\beta}\, \xi^{\beta}$$

$$B_{\alpha\beta} = k_{\alpha;\beta}$$

$$\left(\tilde{\xi}^\alpha{}_{;\beta}\, k^\beta\right)^\sim = h^\alpha{}_\mu\, k^\mu{}_{;\beta}\, \xi^\beta \qquad B_{\alpha\beta} = k_{\alpha;\beta}$$

$$= h^\alpha{}_\mu\, B^\mu{}_\beta\, \xi^\beta$$

$$= h^\alpha{}_\mu\, B^\mu{}_\beta \left(\tilde{\xi}^\beta \right.$$

$$B_{\alpha\beta} = K_{\alpha;\beta}$$

$$\left(\tilde{\xi}^\alpha{}_{;\rho} K^\rho\right)^\sim = h^\alpha{}_\mu K^\mu{}_{;\rho}\, \xi^\rho$$

$$= h^\alpha{}_\mu B^\mu{}_\rho\, \xi^\rho$$

$$= h^\alpha{}_\mu B^\mu{}_\rho \left(\tilde{\xi}^\rho - K^\rho N_\nu \xi^\nu\right)$$

$$(\tilde{\xi}^\alpha{}_{;\beta}\, k^\beta)^\sim = h^\alpha{}_\mu\, k^\mu{}_{;\rho}\, \xi^\rho$$

$$= h^\alpha{}_\mu\, B^\mu{}_\rho\, \xi^\rho$$

$$= h^\alpha{}_\mu\, B^\mu{}_\rho \left(\tilde{\xi}^\rho - k^\rho \cancel{k_\nu \xi^\nu} \right)$$

$$B_{\alpha\beta} = k_{\alpha;\beta}$$

$$B_{\alpha\beta}\, k^\beta = 0$$

$$B_{\alpha\beta} = k_{\alpha;\beta}$$

$$B_{\alpha\beta} k^\beta = 0$$

$$(\tilde{\xi}^\alpha{}_{;\rho} k^\rho)^\sim = h^\alpha{}_\mu k^\mu{}_{;\rho} \xi^\rho$$

$$= h^\alpha{}_\mu B^\mu{}_\rho \xi^\rho$$

$$= h^\alpha{}_\mu B^\mu{}_\rho \left(\tilde{\xi}^\rho - \cancel{k^\rho v_\nu \xi^\nu} \right)$$

$$= h^\alpha{}_\mu B^\mu{}_\rho \tilde{\xi}^\rho$$

$$= h^\alpha{}_\mu B^\mu{}_\rho h^\rho{}_\nu \tilde{\xi}^\nu$$

$$\left(\tilde{\xi}^{\alpha}{}_{;\rho} K^{\rho}\right)^{\sim} = h^{\alpha}{}_{\mu} K^{\mu}{}_{;\rho}\, \xi^{\rho}$$

$$= h^{\alpha}{}_{\mu} B^{\mu}{}_{\rho}\, \xi^{\rho}$$

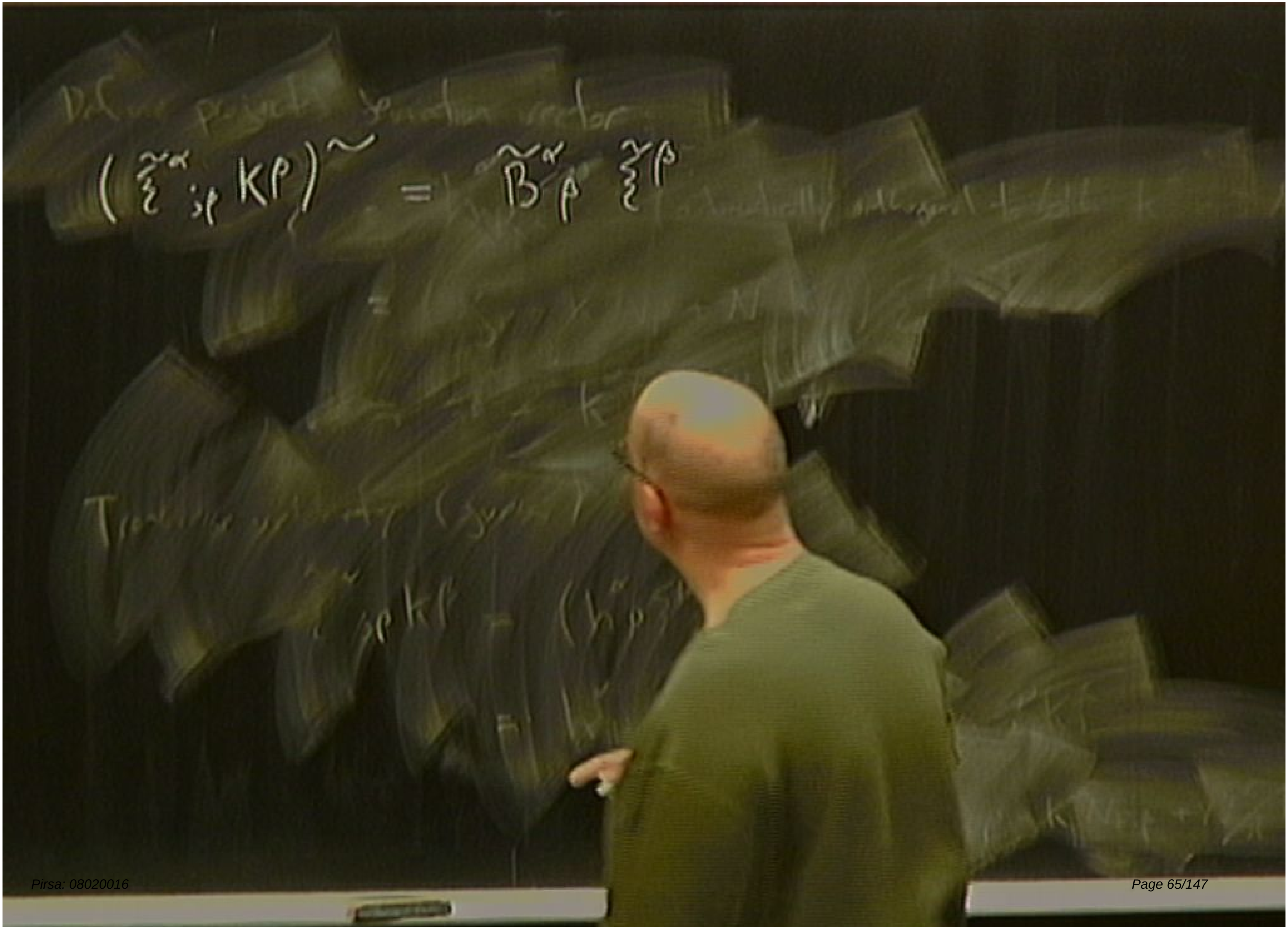
$$= h^{\alpha}{}_{\mu} B^{\mu}{}_{\rho} \left(\tilde{\xi}^{\rho} - K^{\rho} \cancel{K_{\nu} \xi^{\nu}} \right)$$

$$= h^{\alpha}{}_{\mu} B^{\mu}{}_{\rho}\, \tilde{\xi}^{\rho}$$

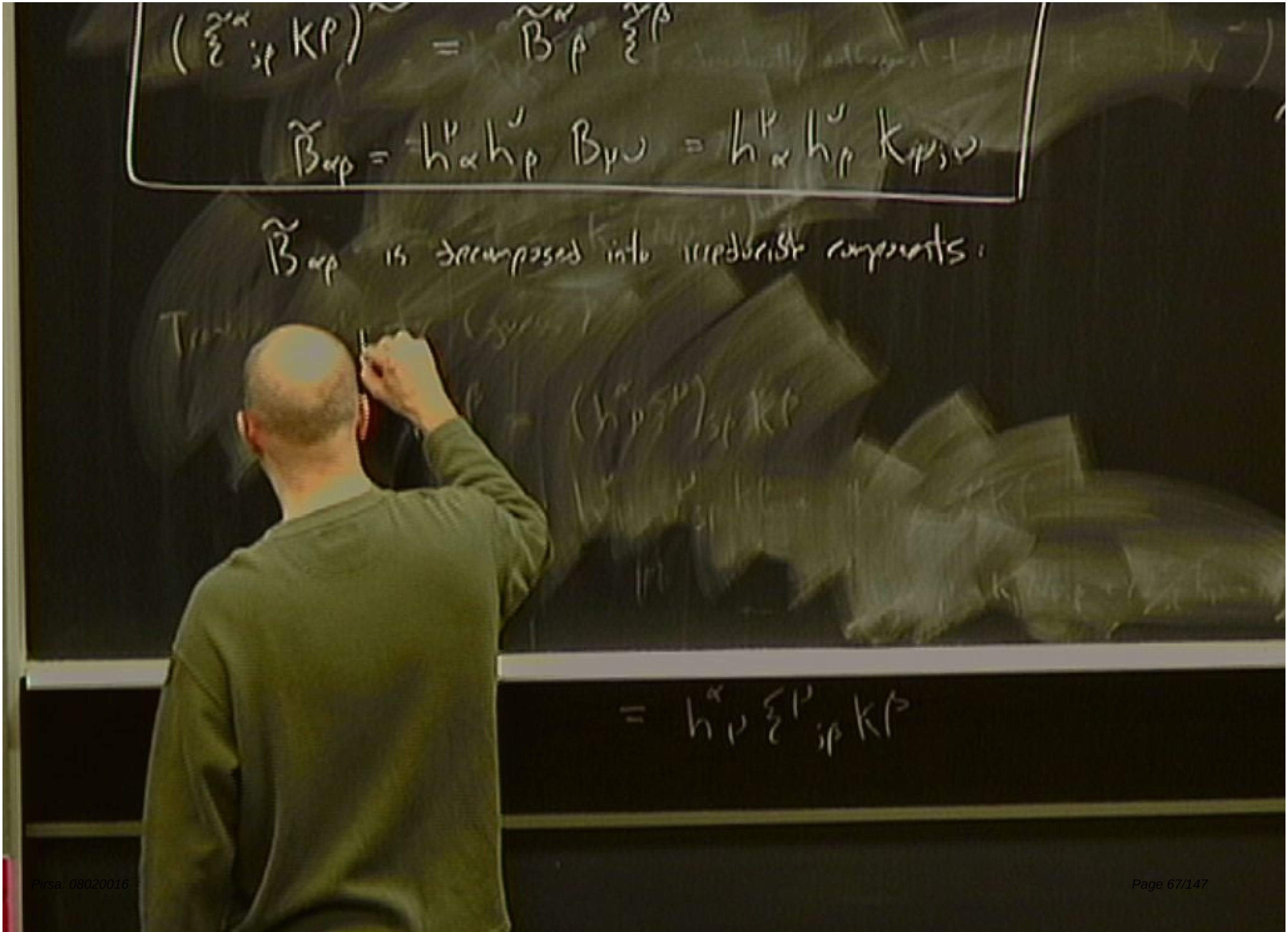
$$= \underbrace{h^{\alpha}{}_{\mu} B^{\mu}{}_{\rho} h^{\rho}{}_{\nu}}\, \tilde{\xi}^{\nu}$$

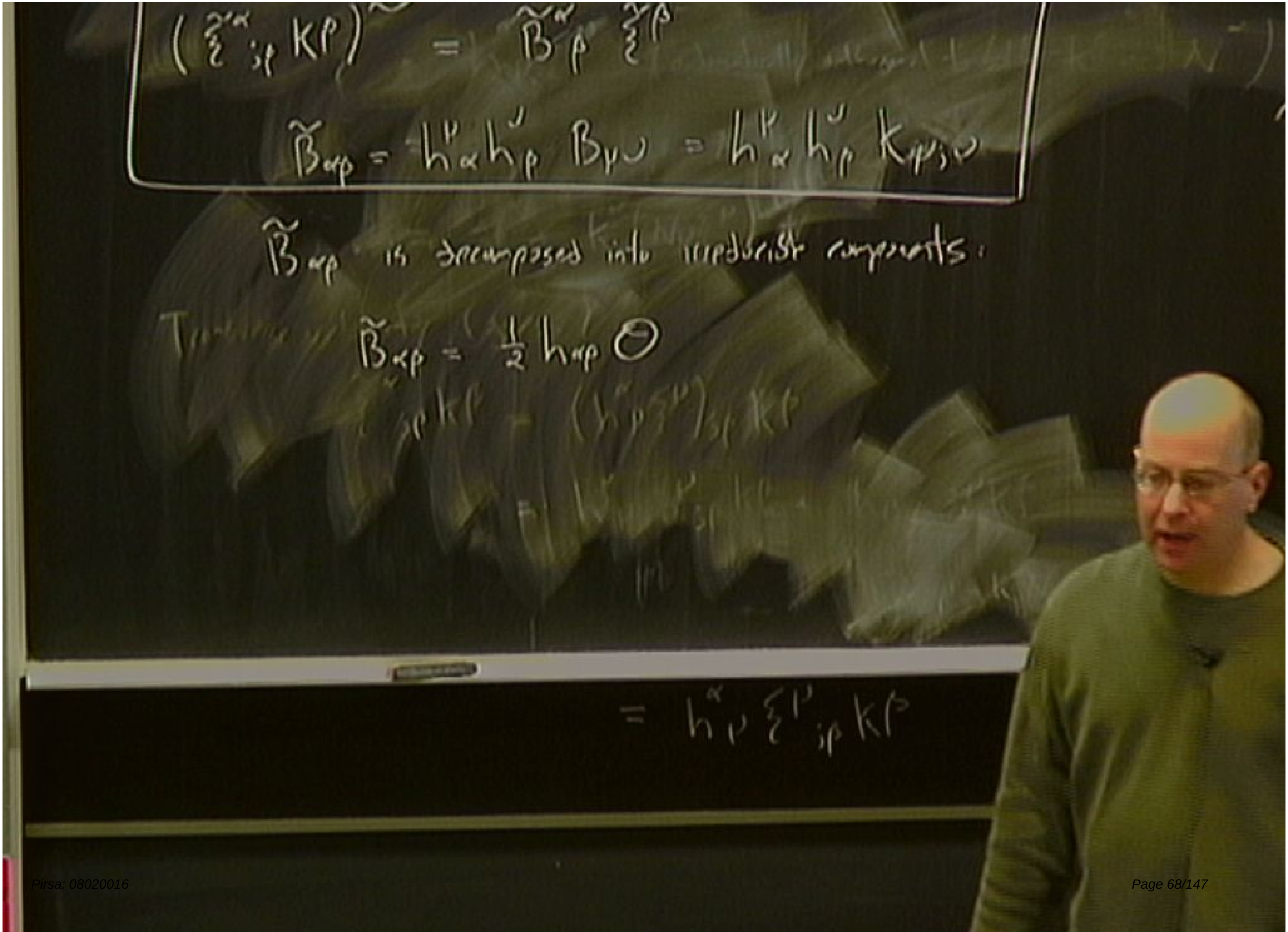
$$B_{\alpha\beta} = K_{\alpha;\beta}$$

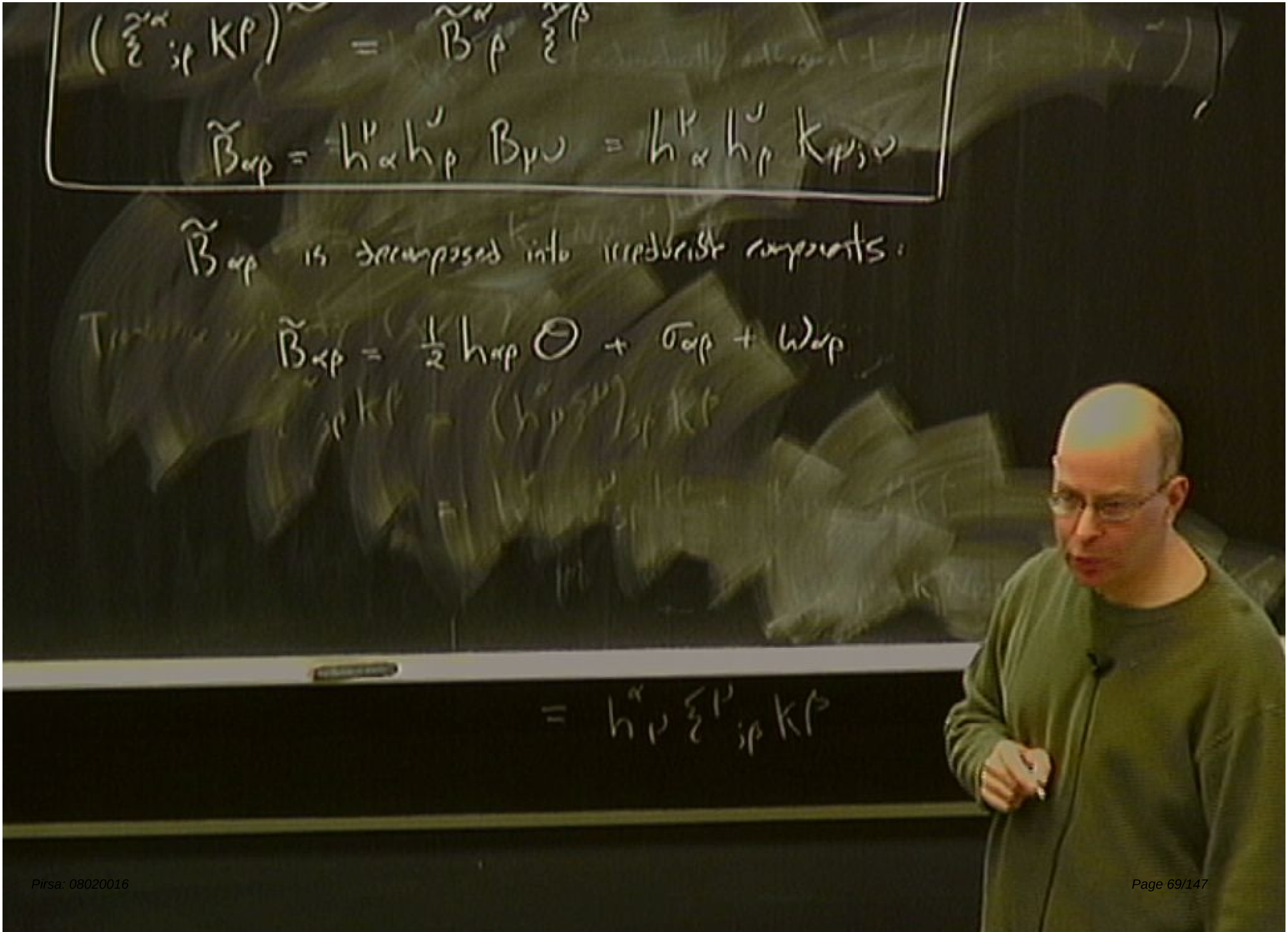
$$B_{\alpha\beta} K^{\beta} = 0$$

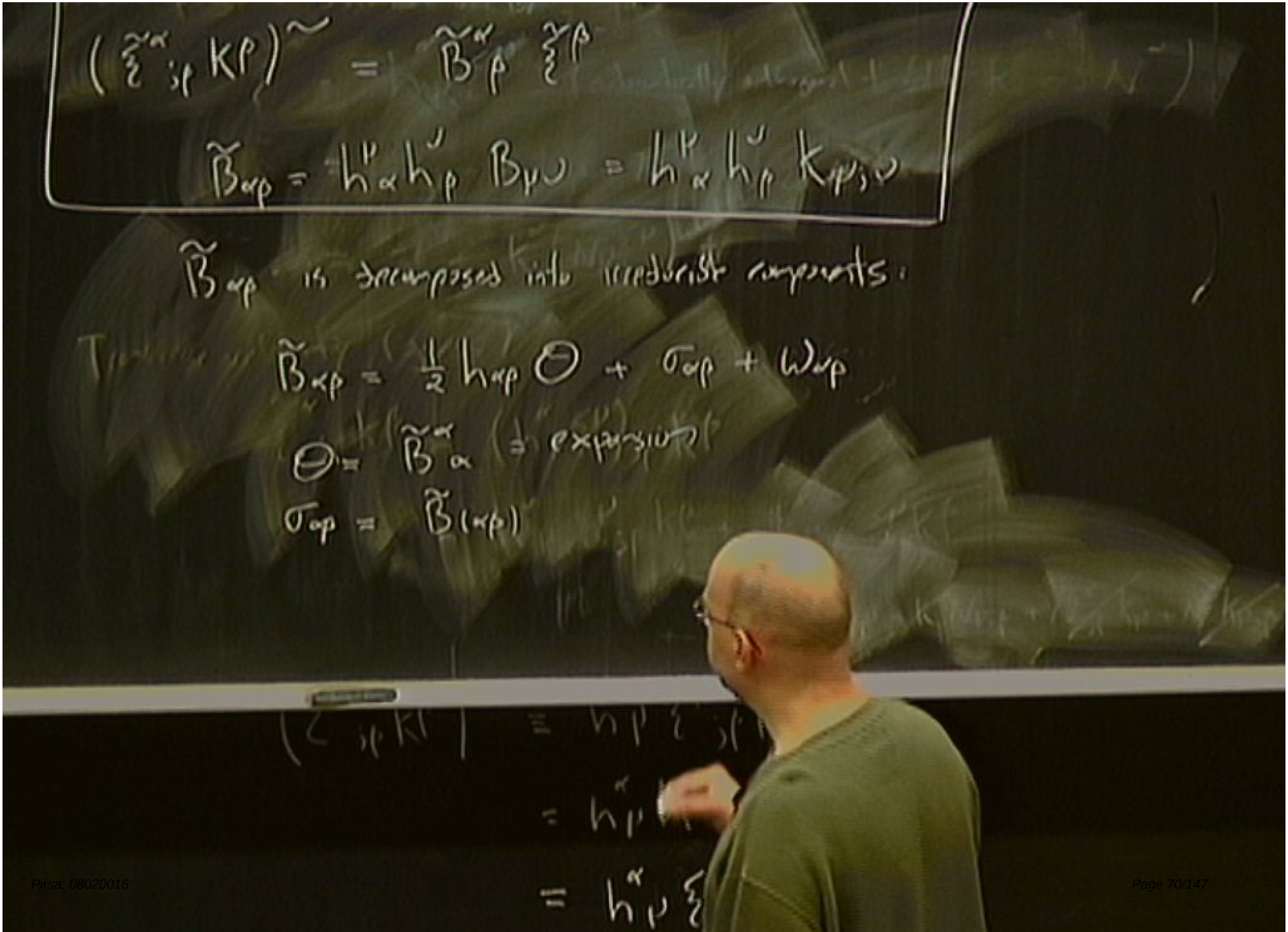

$(\tilde{\xi}^{\alpha}{}_{;\rho} K^{\rho})^{\sim} = \tilde{B}^{\alpha}{}_{\rho} \tilde{\xi}^{\rho}$

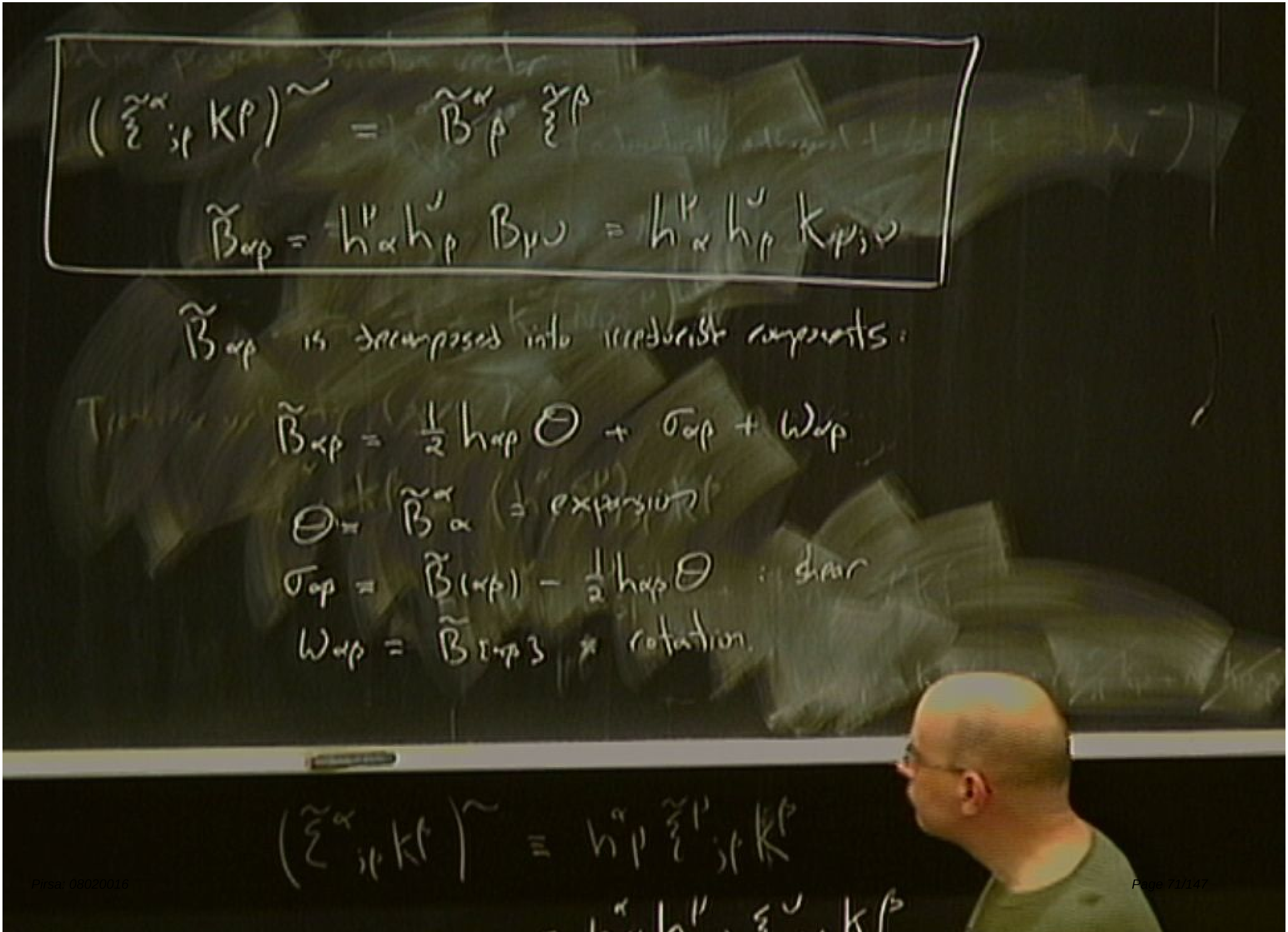
$$\boxed{\begin{aligned}(\tilde{\xi}^{\alpha}{}_{;\rho}\, k^{\rho})^{\sim} &= \tilde{B}^{\alpha}{}_{\beta}\, \tilde{\xi}^{\beta} \\ \tilde{B}_{\alpha\beta} = h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}\, B_{\mu\nu} &= h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}\, k_{\mu;\nu}\end{aligned}}$$


$(\tilde{\xi}^\alpha{}_{;\rho} K^\rho)^\sim = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta K_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components.
$= h^\alpha{}_\rho \xi^\rho{}_{;\rho} K^\rho$


$(\tilde{\xi}^\alpha{}_{;\rho} k^\rho)^\sim = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta$
$= h^\alpha{}_\rho \xi^\rho{}_{;\rho} k^\rho$


$(\tilde{\xi}^{\alpha}{}_{;\rho} k^{\rho})^{\sim} = \tilde{B}^{\alpha}{}_{\beta} \tilde{\xi}^{\beta}$
$\tilde{B}_{\alpha\beta} = h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} B_{\mu\nu} = h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$= h^{\alpha}{}_{\mu} \xi^{\mu}{}_{;\rho} k^{\rho}$


$(\tilde{\xi}^\alpha{}_{;\beta} k^\beta)^\sim = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$\Theta = \tilde{B}^\alpha{}_\alpha$ = expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)}$



$(\tilde{\xi}^\alpha{}_{;\beta} k^\beta)^\sim = \tilde{B}^\alpha{}_\beta \, \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$\Theta = \tilde{B}^\alpha{}_\alpha$: expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta$: shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$: rotation
$(\xi^\alpha{}_{;\beta} k^\beta)^\sim = h^\alpha{}_\beta \, \xi^\beta{}_{;\mu} k^\mu$

$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$\Theta = \tilde{B}^{\alpha}{}_{\alpha}$: expansion

$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\Theta$: shear

$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$: rotation.

$$\Theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} K_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\,\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\Theta = \tilde{B}^{\alpha}{}_{\alpha} \quad : \text{ expansion}$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\,\Theta \quad : \text{ shear}$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad : \text{ rotation}$$

$$\Theta = g^{\alpha\beta}\,\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\, h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$= h^{\mu\beta} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\,\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\Theta = \tilde{B}^{\alpha}{}_{\alpha} \quad : \text{expansion}$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\,\Theta \quad : \text{shear}$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad : \text{rotation}$$

$$\Theta = g^{\alpha\beta}\,\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\, h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

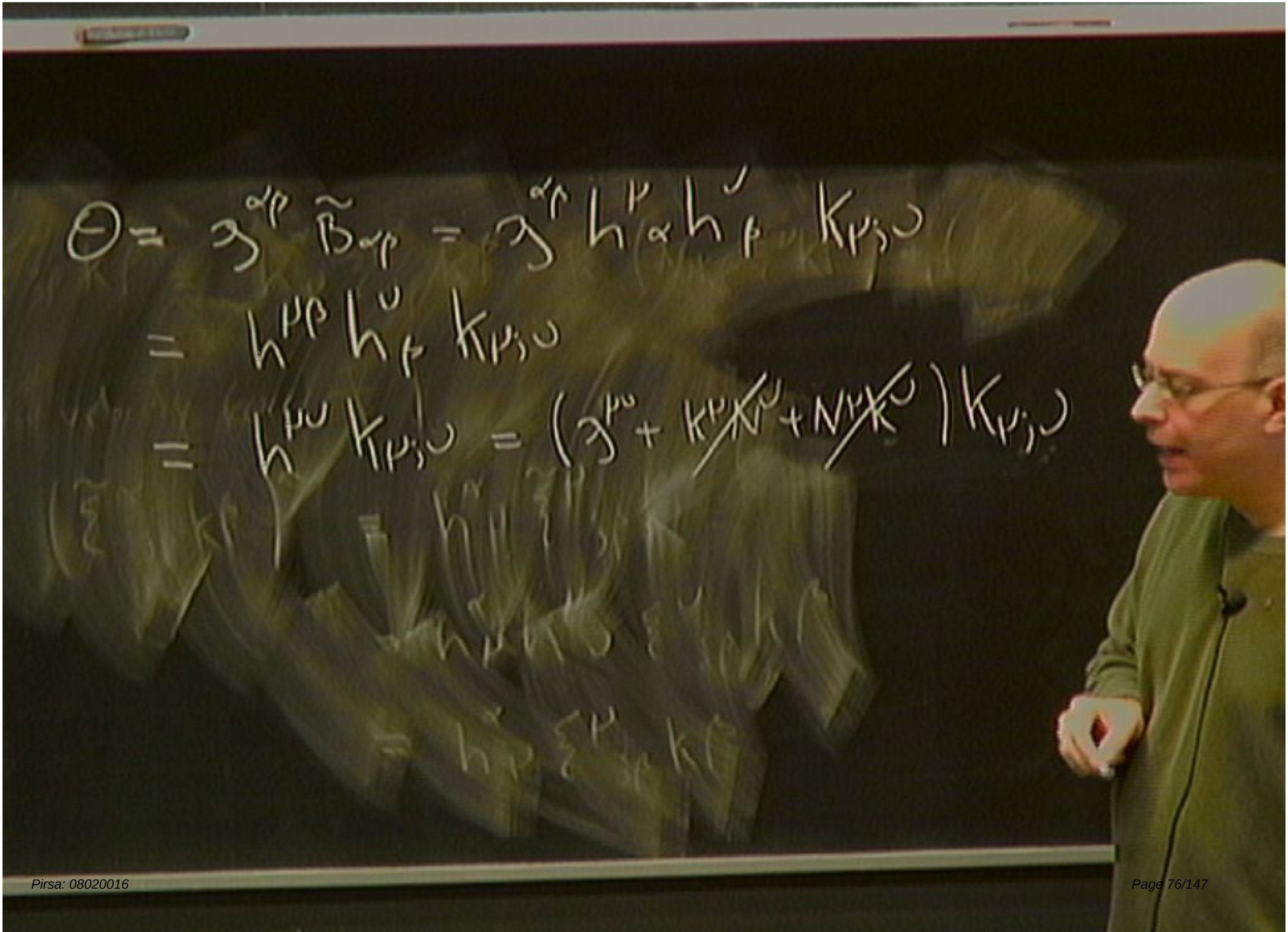
$$= h^{\mu\beta}\, h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$= h^{\mu\nu}\, k_{\mu;\nu}$$

$$\Theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$= h^{\mu\beta} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$= h^{\mu\nu} k_{\mu;\nu} = \left(g^{\mu\nu} + k^{\mu}N^{\nu} + N^{\mu}k^{\nu}\right) k_{\mu;\nu}$$


$\Theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} K_{\mu;\nu}$
$= h^{\mu\beta} h^{\nu}{}_{\beta} K_{\mu;\nu}$
$= h^{\mu\nu} K_{\mu;\nu} = (g^{\mu\nu} + k^{\mu}\cancel{k^{\nu}} + N^{\mu}\cancel{k^{\nu}}) K_{\mu;\nu}$

$$\Theta = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\beta} h^{\nu}{}_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\nu} K_{\mu;\nu} = (g^{\mu\nu} + \cancel{K^{\mu}K^{\nu}} + N^{\mu}\cancel{K^{\nu}}) K$$

$$\boxed{\Theta = K^{\alpha}{}_{;\alpha}}$$

$$\Theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\beta} h^{\nu}{}_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\nu} K_{\mu;\nu} = (g^{\mu\nu} + \cancel{K^{\mu}N^{\nu}} + \cancel{N^{\mu}K^{\nu}}) K_{\mu;\nu}$$

$$\boxed{\Theta = K^{\alpha}{}_{;\alpha}} \qquad K^{\mu} K_{\mu;\nu} = \tfrac{1}{2}(K^{\mu}K_{\mu})_{;\nu}$$

$$\Theta = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} \, k_{\mu;\nu}$$

$$= h^{\mu\beta} h^{\nu}{}_{\beta} \, k_{\mu;\nu}$$

$$= h^{\mu\nu} k_{\mu;\nu} = \left(g^{\mu\nu} + \cancel{k^{\mu} N^{\nu}} + \cancel{N^{\mu} k^{\nu}} \right) k_{\mu;\nu}$$

$$\boxed{\Theta = k^{\alpha}{}_{;\alpha}}$$

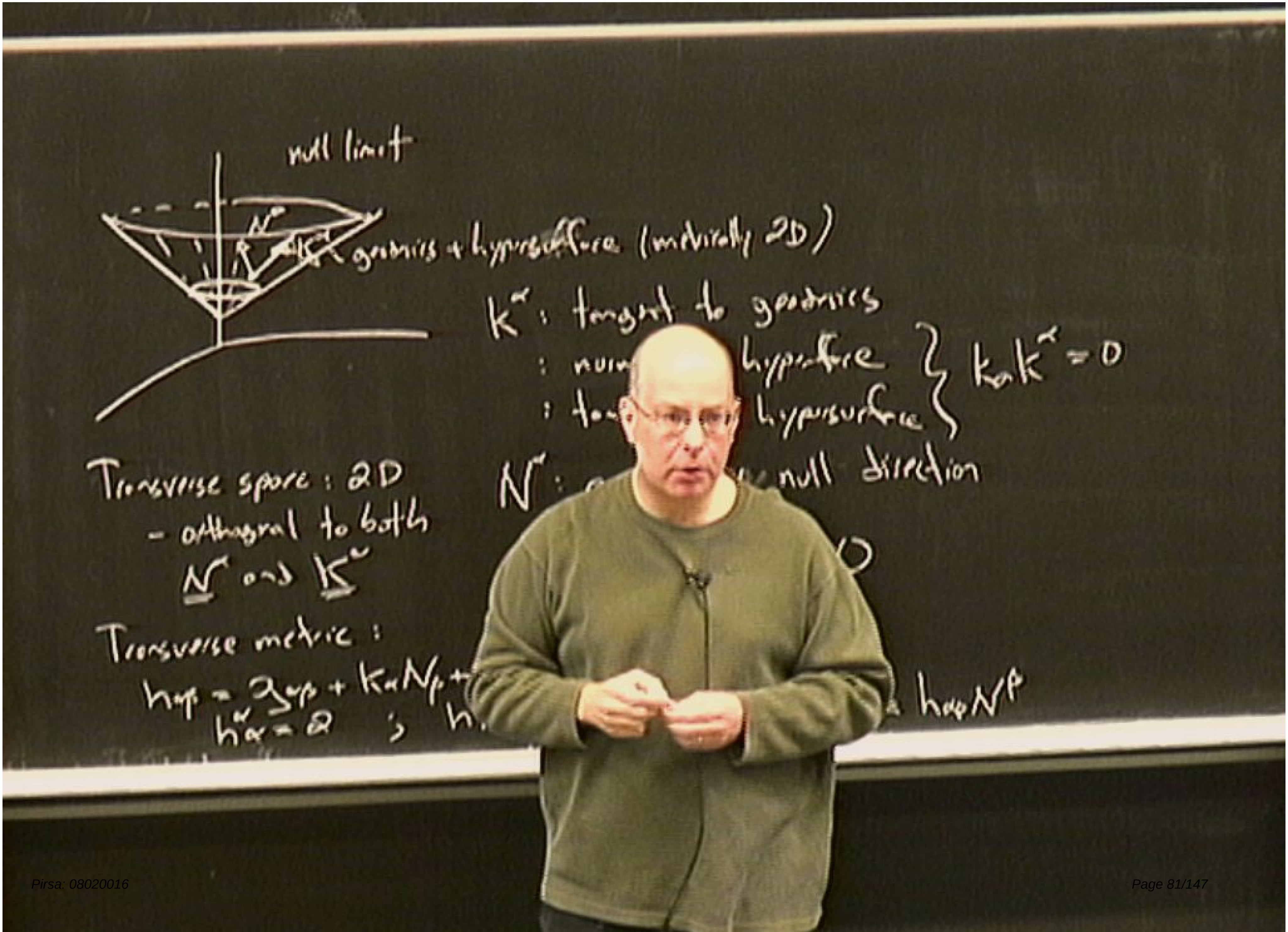
$$k^{\mu} k_{\mu;\nu} = \tfrac{1}{2}(k^{\mu} k_{\mu})_{;\nu} =$$

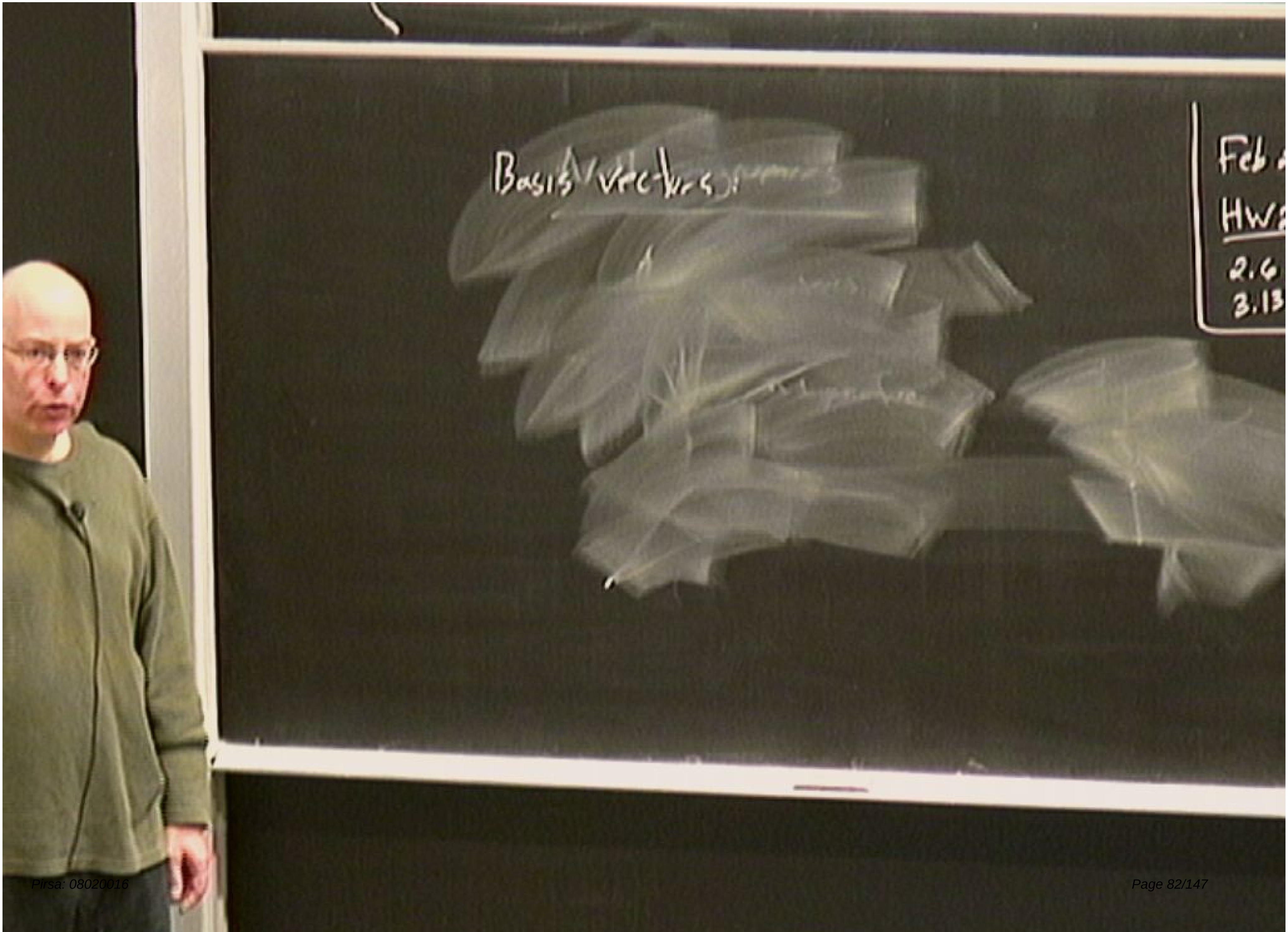
$$\Theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

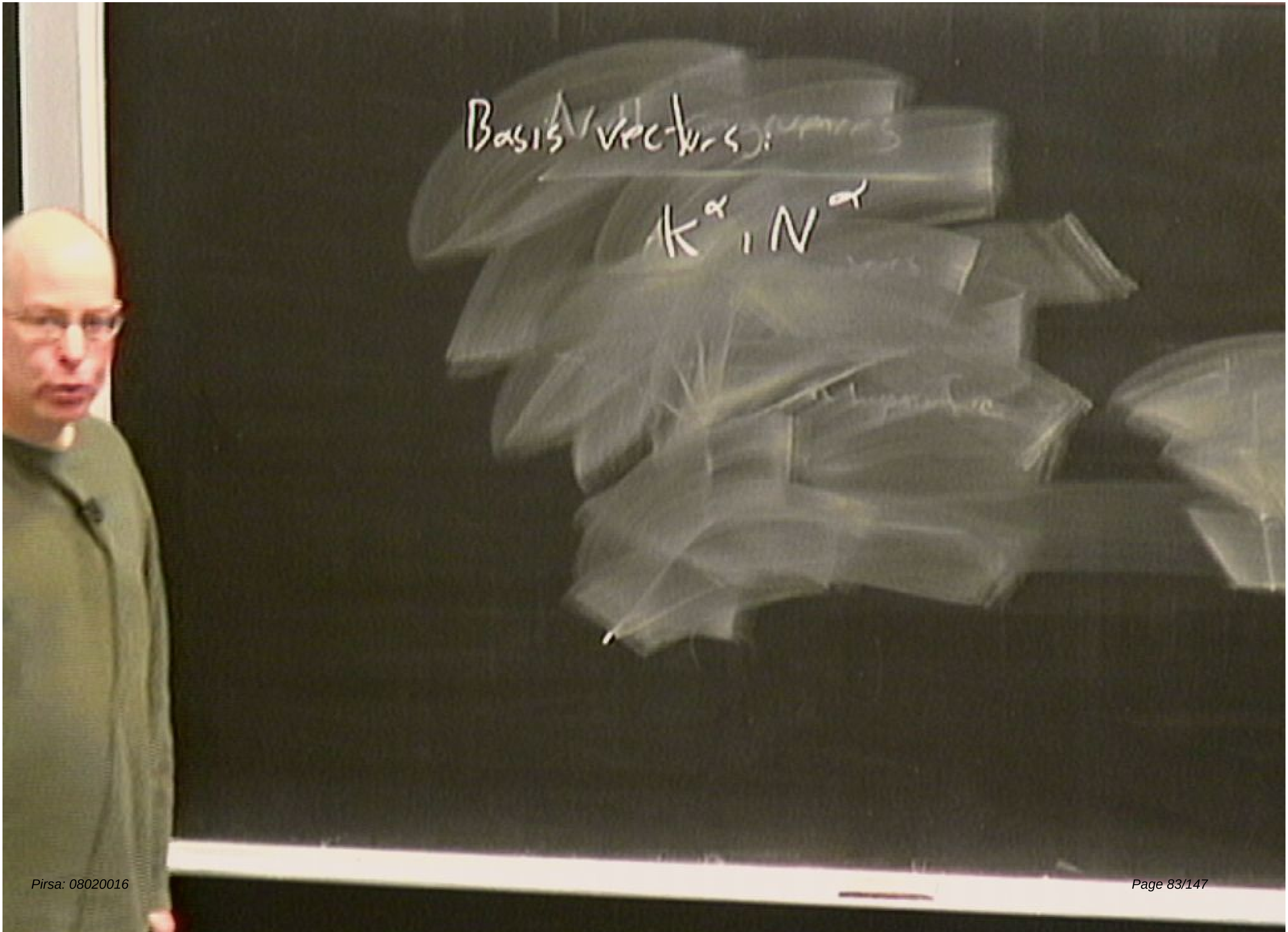
$$= h^{\mu\beta} h^{\nu}{}_{\beta}\, k_{\mu;\nu}$$

$$= h^{\mu\nu} k_{\mu;\nu} = (g^{\mu\nu} + k^{\mu}\cancel{N^{\nu}} + N^{\mu}\cancel{k^{\nu}})\, k_{\mu;\nu}$$

$$\boxed{\Theta = k^{\alpha}{}_{;\alpha}} \qquad k^{\mu} k_{\mu;\nu} = \tfrac{1}{2}(k^{\mu}k_{\mu})_{;\nu} = 0$$


null limit
geodesics + hypersurface (intrinsically 2D)
K^α : tangent to geodesics
hypersurface
hypersurface
$k_\alpha k^\alpha = 0$
null direction
Transverse space : 2D
- orthogonal to both
N^α and K^α
Transverse metric :
$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta +$
$h_{\alpha\beta} N^\beta$
$h^\alpha{}_\alpha = 2$;


Basis vectors:
Feb
HW
2.6
3.13


Basis vectors:
k^α, N^α

Basis vectors:

$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$

Feb 20: no

HW2: Fe

2.6 #1,

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha k^\alpha = -1$$

Feb 20: no

HW2: F

2.6 #1,

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$$

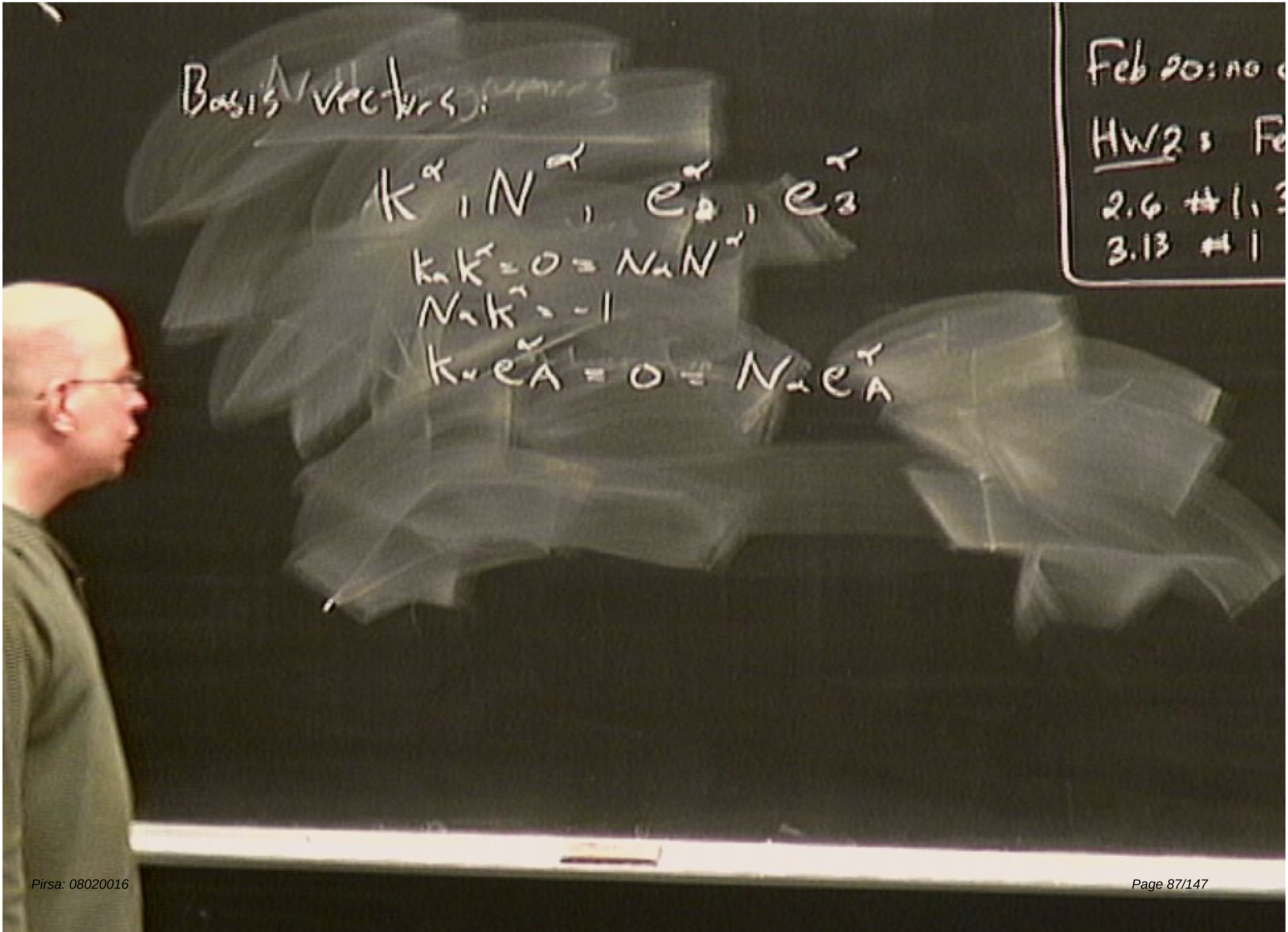
$$N_\alpha k^\alpha = -1$$

$$k_\alpha e_A^\alpha$$

HW2: R

2.6 #1,

3.13 #1


Basis vectors:
$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$
$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$
$N_\alpha k^\alpha = -1$
$k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha$
HW2:
2.6 #1,
3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha k^\alpha = -1$$

$$k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha$$

$$g_{\alpha\beta} e_A^\alpha e^\beta_B = \delta_{AB}$$

Feb 20: no c

HW2: Fe

2.6 #1, 3

3.13 #1

Basis vectors:

$$k^\alpha,\ N^\alpha,\ e^\alpha_2,\ e^\alpha_3$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta}\, e^\alpha_A e^\beta_B = \delta_{AB} \end{cases}$$

Feb 20: no c

HW2: Fe

2.6 #1, 3

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ g_{\alpha\beta}\, e_A^\alpha e^\beta_B = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e_A^\alpha$$

Basis vectors:

$$k^\alpha, N^\alpha, e^\alpha_2, e^\alpha_3$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e^\alpha_A$$

$$k_\alpha \xi^\alpha = -b$$

Feb 20: ...

HW2:

2.6 #1...

3.13 #...

Feb 20: n

HW2:

2.6 #

3.13 #

Basis vectors:

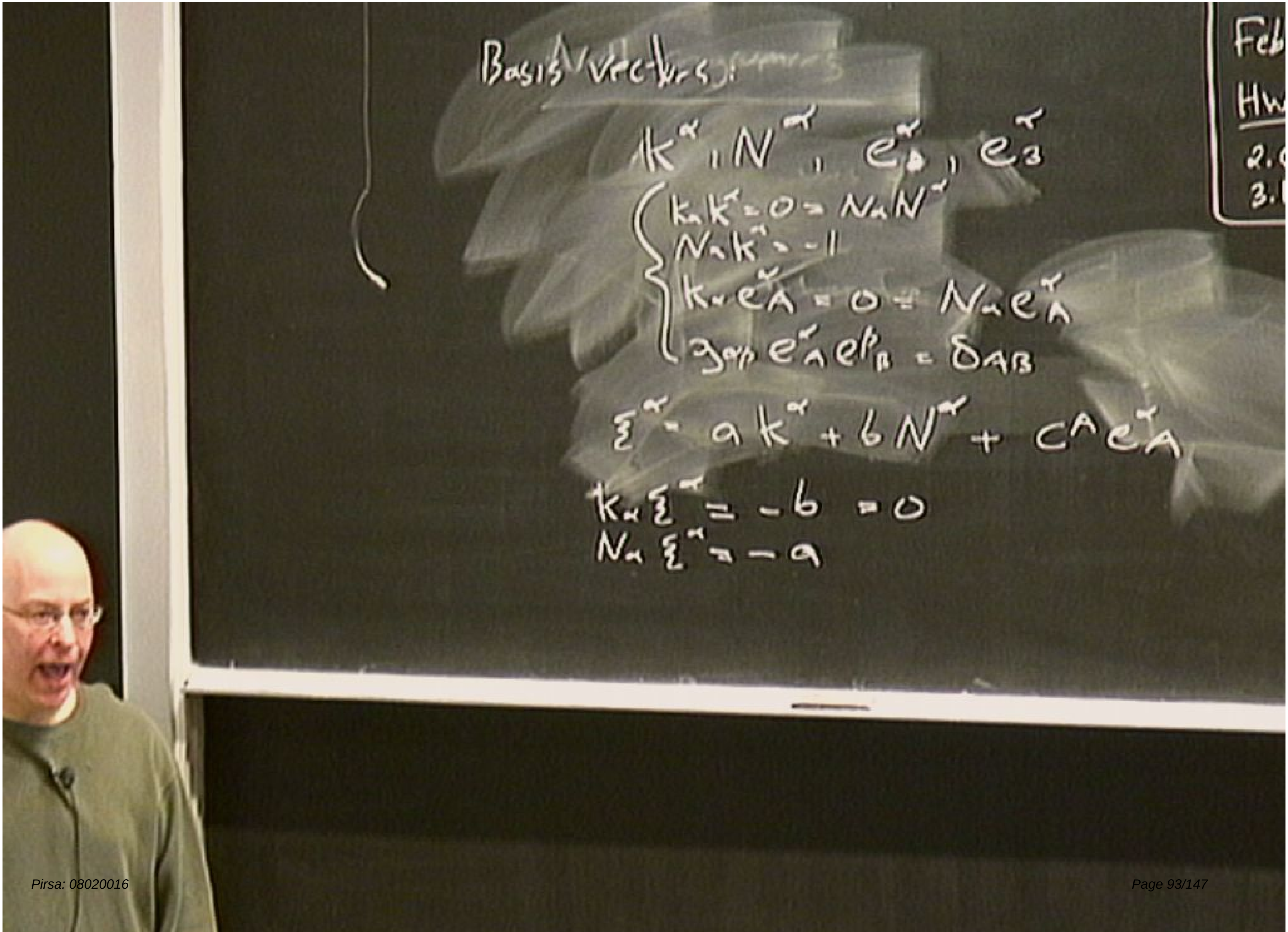
$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ g_{\alpha\beta} e_A^\alpha e_B^\beta = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e_A^\alpha$$

$$k_\alpha \xi^\alpha = -b$$

$$N_\alpha \xi^\alpha = -a$$


Basis vectors:
$k^\alpha, N^\alpha, e^\alpha_2, e^\alpha_3$
$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$
$N_\alpha k^\alpha = -1$
$k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A$
$g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB}$
$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e^\alpha_A$
$k_\alpha \xi^\alpha = -b = 0$
$N_\alpha \xi^\alpha = -a$

Basis vectors:

$$k^\alpha, N^\alpha, e^\alpha_2, e^\alpha_3$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta}\, e^\alpha_A e^\beta_B = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e^\alpha_A$$

$$k_\alpha \xi^\alpha = -b = 0$$

$$N_\alpha \xi^\alpha = -a$$

Basis vectors:

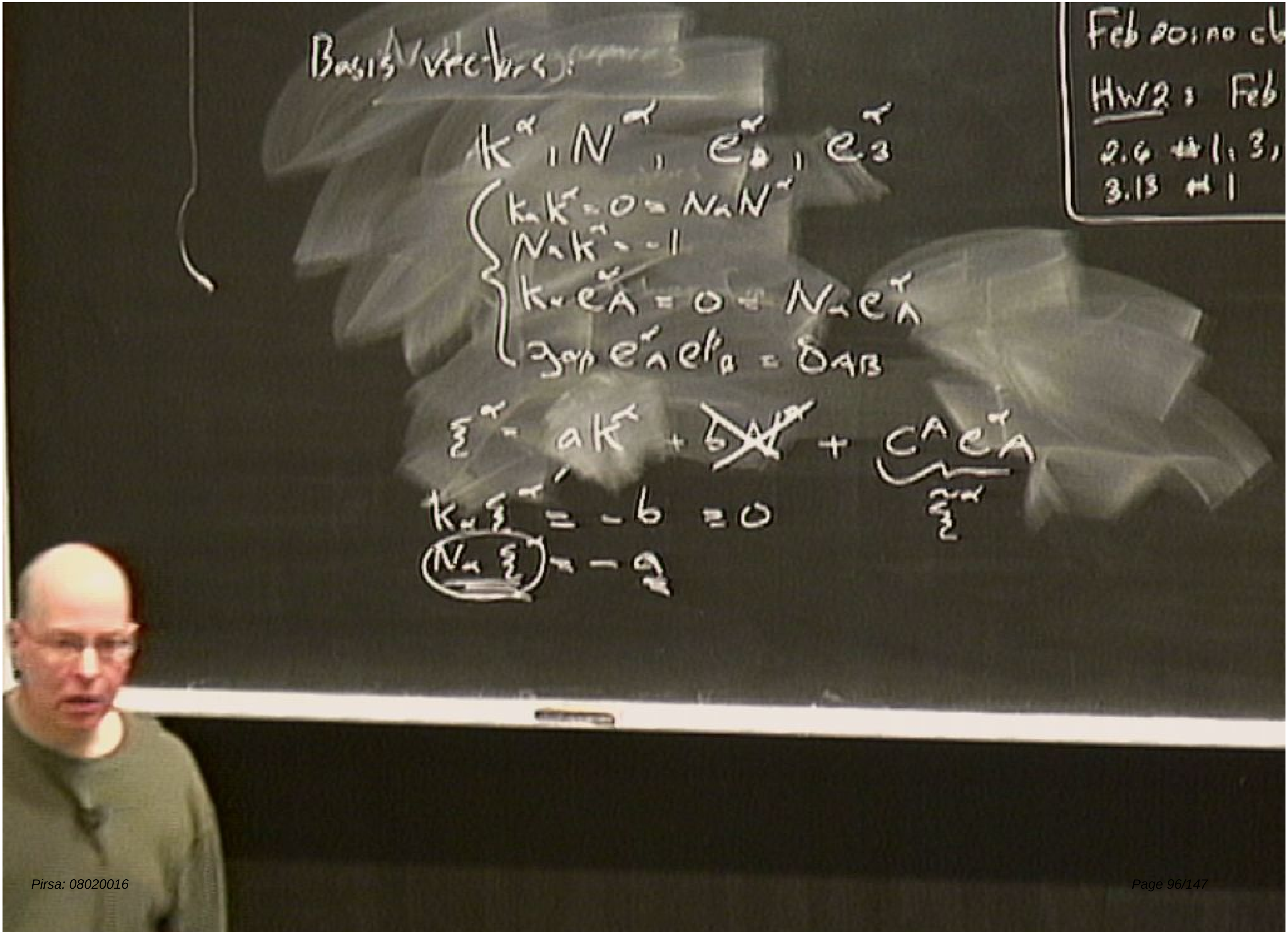
$$k^\alpha, N^\alpha, e^\alpha_2, e^\alpha_3$$

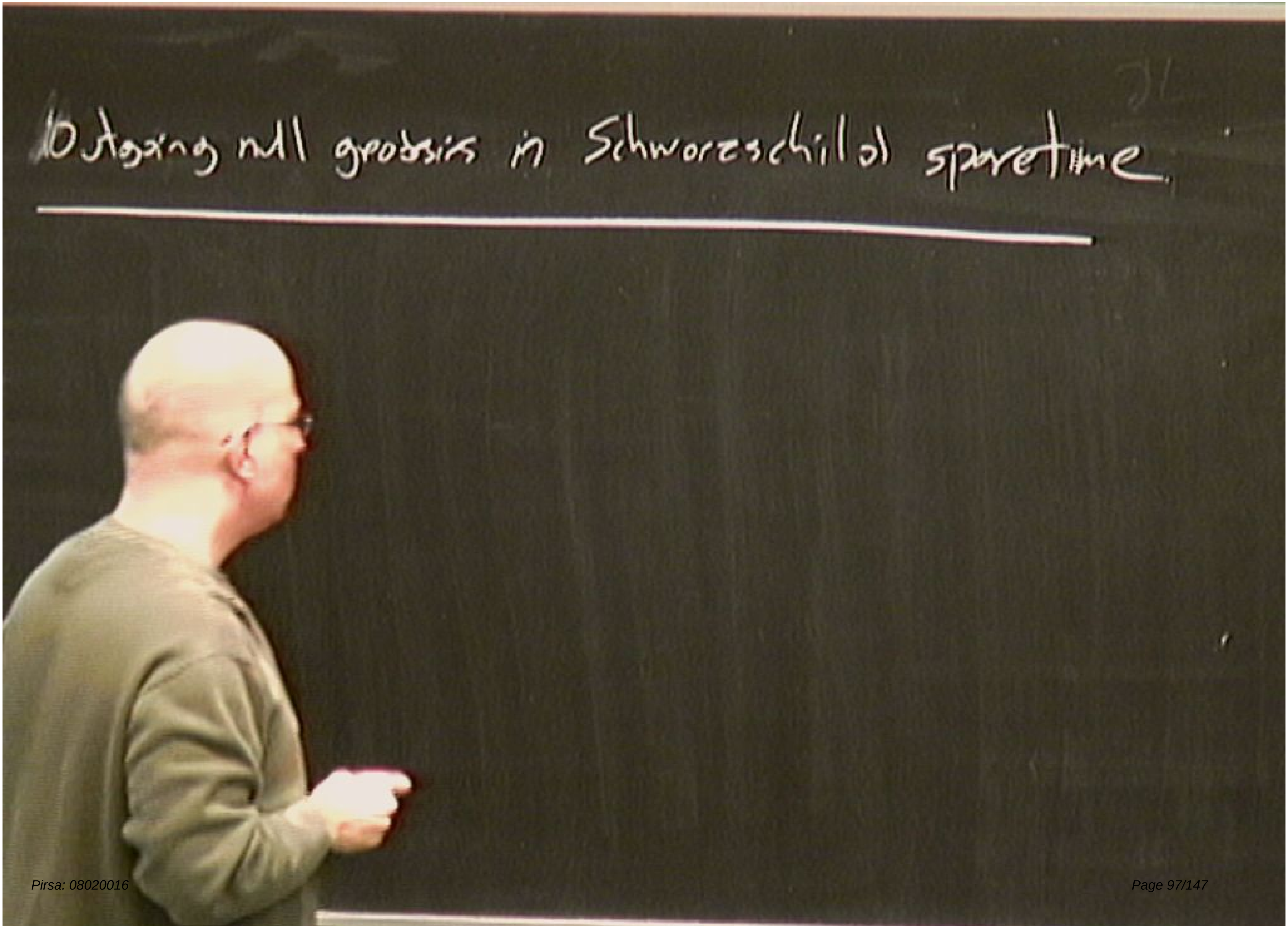
$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + \cancel{b N^\alpha} + c^A e^\alpha_A$$

$$k_\alpha \xi^\alpha = -b = 0$$

$$N_\alpha \xi^\alpha = -a$$


Basis vectors:
$k^\alpha, N^\alpha, e^\alpha_2, e^\alpha_3$
$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$
$N_\alpha k^\alpha = -1$
$k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A$
$g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB}$
$\xi^\alpha = a k^\alpha + \cancel{b N^\alpha} + \underbrace{C^A e^\alpha_A}_{\tilde{\xi}^\alpha}$
$k_\alpha \xi^\alpha = -b = 0$
$N_\alpha \xi^\alpha = -a$
Feb 20: no cl
HW2: Feb
2.6 #1, 3,
3.13 #1


10 Outgoing null geodesics in Schwarzschild spacetime

Outgoing null geodesics in Schwarzschild spacetime

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

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$$= -f(dt - f^{-1}dr)(dt + f^{-1}dr) + r^2 d\Omega^2$$

Outgoing null geodesics in Schwarzschild spacetime

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$$= -f\left(dt^2 - f^{-2}\,dr^2\right) + r^2\,d\Omega^2$$

$$= -f\left(dt - f^{-1}\,dr\right)\left(dt + f^{-1}\,dr\right) + r^2\,d\Omega^2$$

Outgoing null geodesics in Schwarzschild spacetime

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$$= -f\left(dt^2 - f^{-2}\,dr^2\right) + r^2\,d\Omega^2$$

$$= -f\,(\underbrace{dt - f^{-1}\,dr}_{dU})(\underbrace{dt + f^{-1}\,dr}_{dV}) + r^2\,d\Omega^2$$

Outgoing null geodesics in Schwarzschild spacetime

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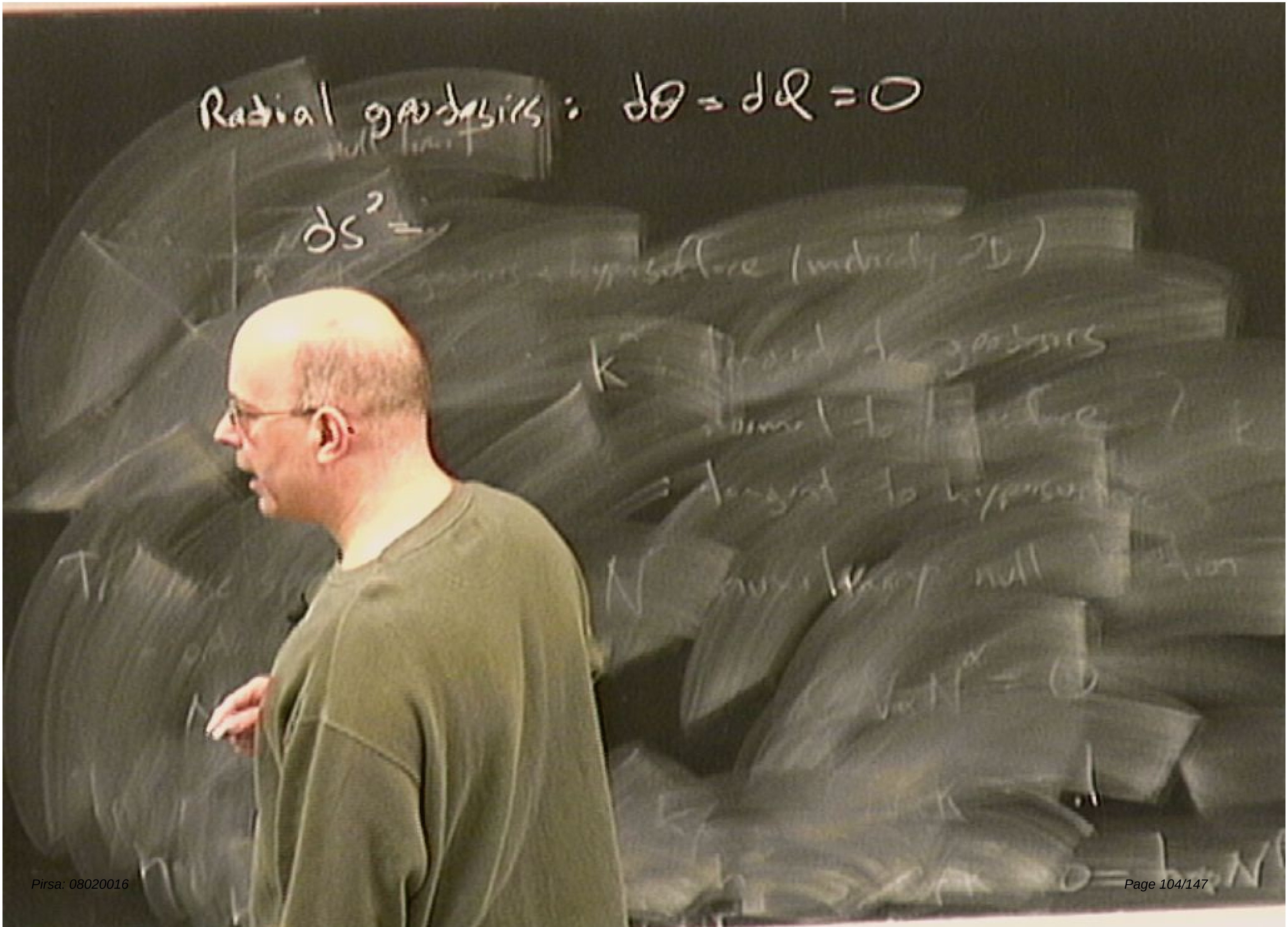
$$= -f\left(dt^2 - f^{-2}dr^2\right) + r^2 d\Omega^2$$

$$= -f\,(\underbrace{dt - f^{-1}dr}_{du})(\underbrace{dt + f^{-1}dr}_{dv}) + r^2 d\Omega^2$$

$$= -f\,du\,dv + r^2 d\Omega^2$$

$$u = t - \int \frac{dr}{f} \qquad v = t + \int \frac{dr}{f}$$


Radial geodesics : $d\theta = d\varphi = 0$
$ds^2 =$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du = 0$$

Radial geodesics : $d\theta = d\phi = 0$

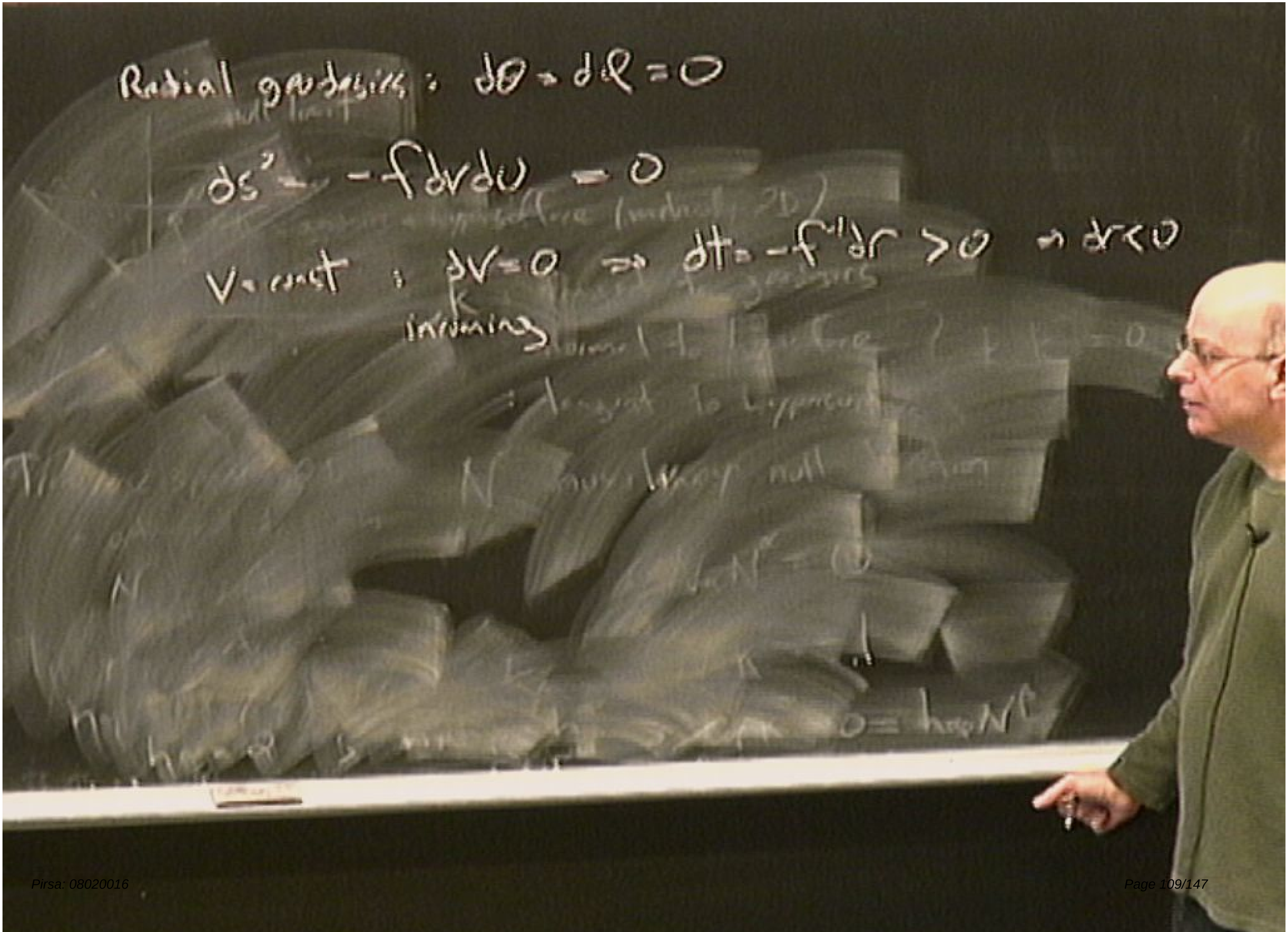
$$ds^2 = -f\,dv\,du = 0$$

$V = \text{const}$: $dV = 0 \Rightarrow dt = -f^{-1} dr$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du = 0$$

$v = \text{const}$: $dv = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$


Radial geodesics : $d\theta = d\varphi = 0$
$ds^2 = -f\,dv\,du = 0$
$V = \text{const} : dV = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$
k
incoming

Radial geodesics: $d\theta = d\varphi = 0$

$ds^2 = -f\,dv\,du = 0$

$V = \text{const}$: $dV = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$

incoming

$U = \text{const}$: $dU = 0 \Rightarrow dt = f^{-1}dr > 0$

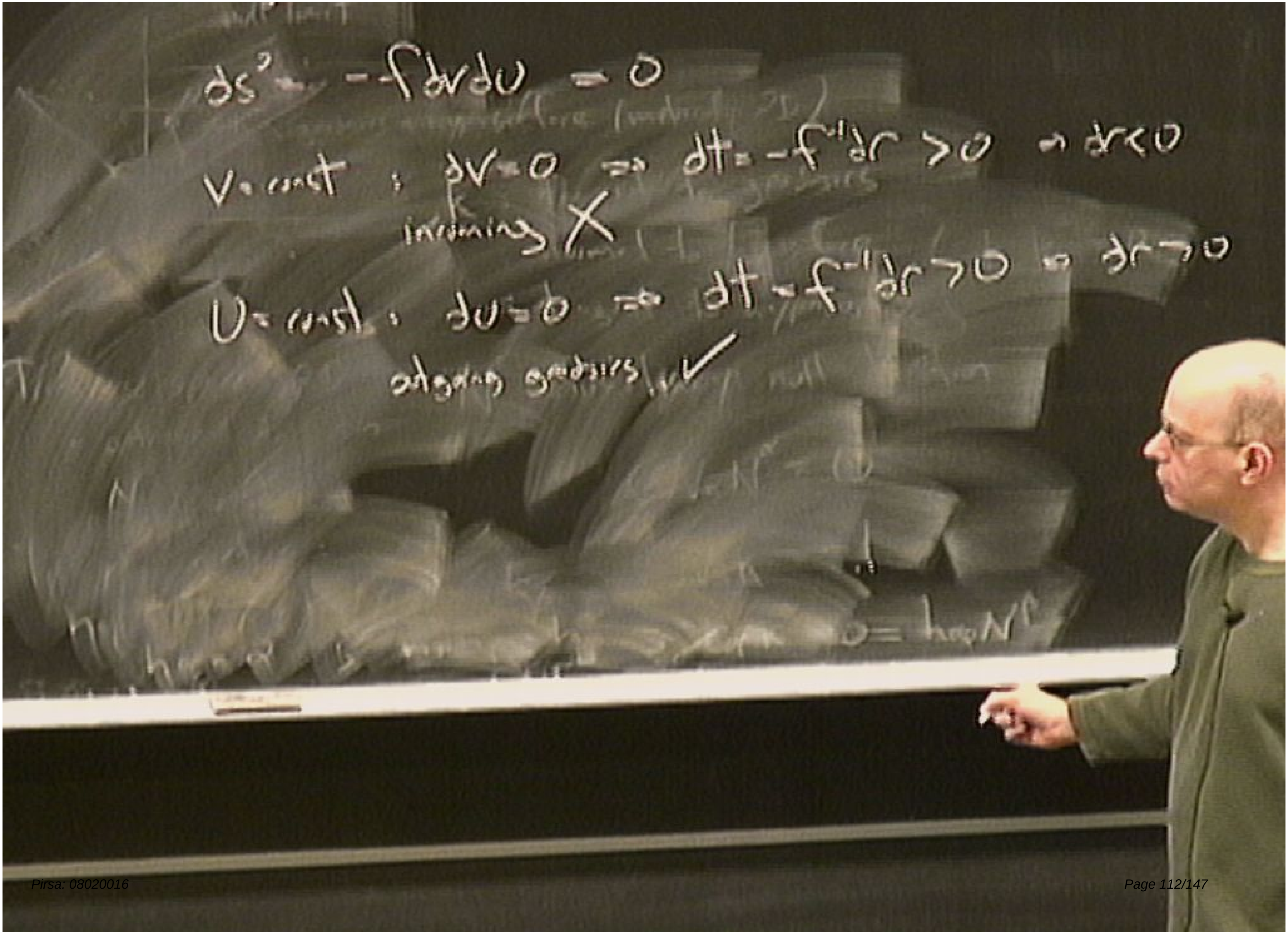
Radial geodesics : $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du = 0$$

$V = $ const : $dV = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$

incoming

$U = $ const : $dU = 0 \Rightarrow dt = f^{-1}dr > 0 \Rightarrow dr > 0$



$ds^2 = -f\,dv\,du = 0$
$v = \text{const}: \; dv = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$
incoming ✗
$u = \text{const}: \; du = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$
outgoing geodesics ✓

Radial geodesics: $d\theta = d\varphi = 0$

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incoming X

$U = \text{const}$: $du = 0 \Rightarrow dt = f^{-1}dr > 0 \Rightarrow dr > 0$

outgoing geodesics ✓

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0)$$

$= h^\alpha{}_\rho B^\rho{}_\mu \xi$

$= h^\alpha{}_\rho B^\rho{}_\mu h^\mu{}_\nu \xi^\nu$

Radial geodesics : $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du = 0$$

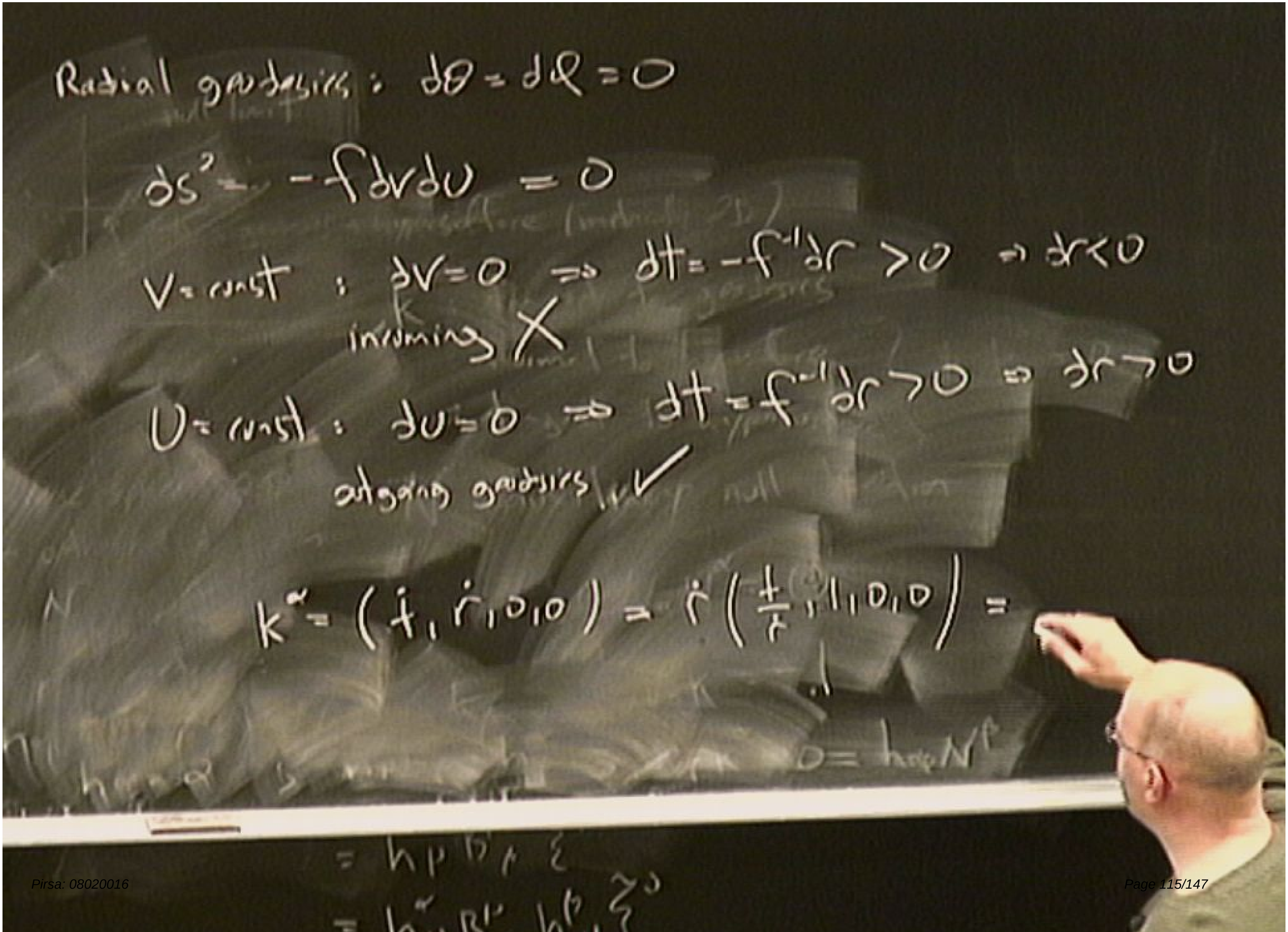
$v = \text{const}$: $dv = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$

incoming $\times$

$u = \text{const}$: $du = 0 \Rightarrow dt = f^{-1}dr > 0 \Rightarrow dr > 0$

outgoing geodesics $\checkmark$

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r}\left(\frac{\dot{t}}{\dot{r}}, 1, 0, 0\right)$$


Radial geodesics : $d\theta = d\varphi = 0$
$ds^2 = -f\,dv\,du = 0$
$V =$ const : $dV = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$
incoming ✗
$U =$ const : $dU = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$
outgoing geodesics ✓
$k^\alpha = (\dot t, \dot r, 0, 0) = \dot r\left(\frac{\dot t}{\dot r}, 1, 0, 0\right) =$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f\,dv\,du = 0$$

$v = \text{const}$: $dv = 0 \Rightarrow dt = -f^{-1}dr > 0 \Rightarrow dr < 0$

incoming ✗

$u = \text{const}$: $du = 0 \Rightarrow dt = f^{-1}dr > 0 \Rightarrow dr > 0$

outgoing geodesics ✓

$$k^\alpha = (\dot t, \dot r, 0, 0) = \dot r\left(\frac{\dot t}{\dot r}, 1, 0, 0\right) = \left(\frac{dt}{dr}, 1, 0, 0\right)$$

$V = \text{const} : \partial V = 0 \Rightarrow \ldots$

incoming ✗

$U = \text{const} : \partial U = 0 \Rightarrow \partial t = f^{-1} \partial r > 0 \Rightarrow \partial r > 0$

outgoing geodesics ✓

$$k^\mu = (\dot t, \dot r, 0, 0) = \dot r \left(\frac{\dot t}{\dot r}, 1, 0, 0 \right) = \left(\frac{dt}{dr}, 1, 0, 0 \right)$$

choose $\lambda = r$

$0 = h_{\mu\rho} N^\rho$

$B_{\nu\rho} k^\rho = 0$

$$= h^\mu{}_\nu B^\nu{}_\rho \xi^\rho$$

$$= h^\mu{}_\nu B^\nu{}_\rho \left(\tilde\xi^\rho - \cancel{k^\rho N_\sigma \xi^\sigma} \right)$$

$$= h^\mu{}_\nu B^\nu{}_\rho \tilde\xi^\rho$$

$$= h^\mu{}_\nu B^\nu{}_\sigma h^\sigma{}_\rho \tilde\xi^\rho$$

const : $\partial v = 0 \Rightarrow \partial t = -f^{-1}\,\partial r > 0 \Rightarrow \partial r < 0$

incoming ✗

$=$ const : $\partial u = 0 \Rightarrow \partial t = f^{-1}\,\partial r > 0 \Rightarrow \partial r > 0$

outgoing geodesics ✓

$$k^{\alpha} = (\dot{t}, \dot{r}, 0, 0) = \dot{r}\left(\frac{\dot{t}}{\dot{r}}, 1, 0, 0\right) = \left(\frac{dt}{dr}, 1, 0, 0\right)$$

choose $\lambda \equiv r \equiv$ affine parameter.

const : $\partial v = 0 \Rightarrow \partial t = -f^{-1}\, \partial r > 0 \Rightarrow \partial r < 0$

incoming ✗

const : $\partial u = 0 \Rightarrow \partial t = f^{-1}\, \partial r > 0 \Rightarrow \partial r > 0$

outgoing geodesics ✓

$$k^\alpha = (\dot t, \dot r, 0, 0) = \dot r \left(\frac{\dot t}{\dot r}, 1, 0, 0 \right) = \left(\frac{dt}{dr}, 1, 0, 0 \right)$$

choose $\lambda \equiv r \equiv$ affine parameter.

$U = \text{const}: \quad dU = 0 \Rightarrow dt = f^{-1} dr$

outgoing geodesics ✓

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r}\left(\frac{\dot{t}}{\dot{r}}, 1, 0, 0\right) = \left(\underbrace{\frac{dt}{dr}}_{\frac{1}{f}}, 1, 0, 0\right)$$

choose $\lambda \equiv r$ = affine parameter.

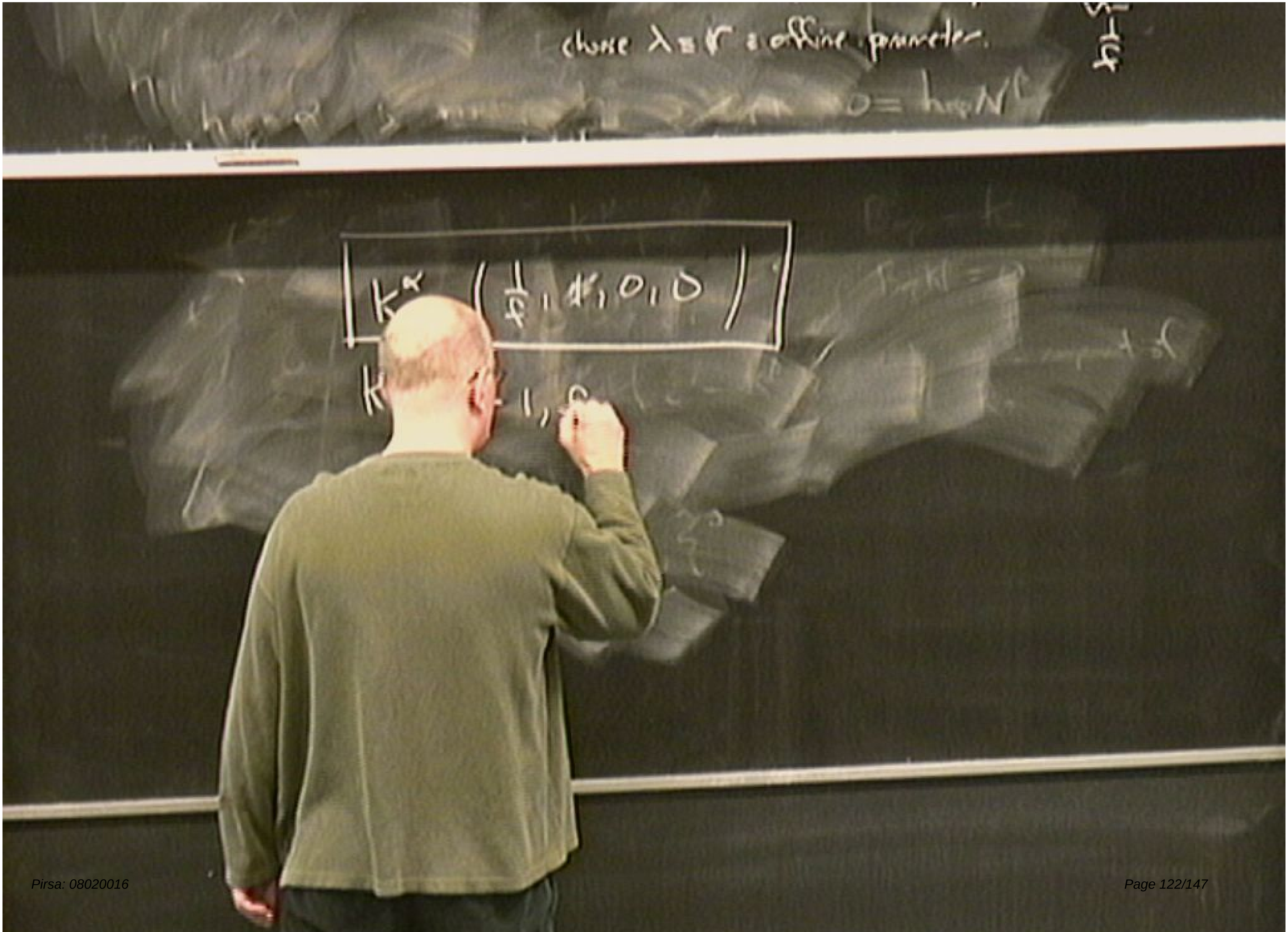
$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

choose $\lambda \equiv r$ = affine parameter.

$0 = h_{\alpha\beta} N^\beta$

$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

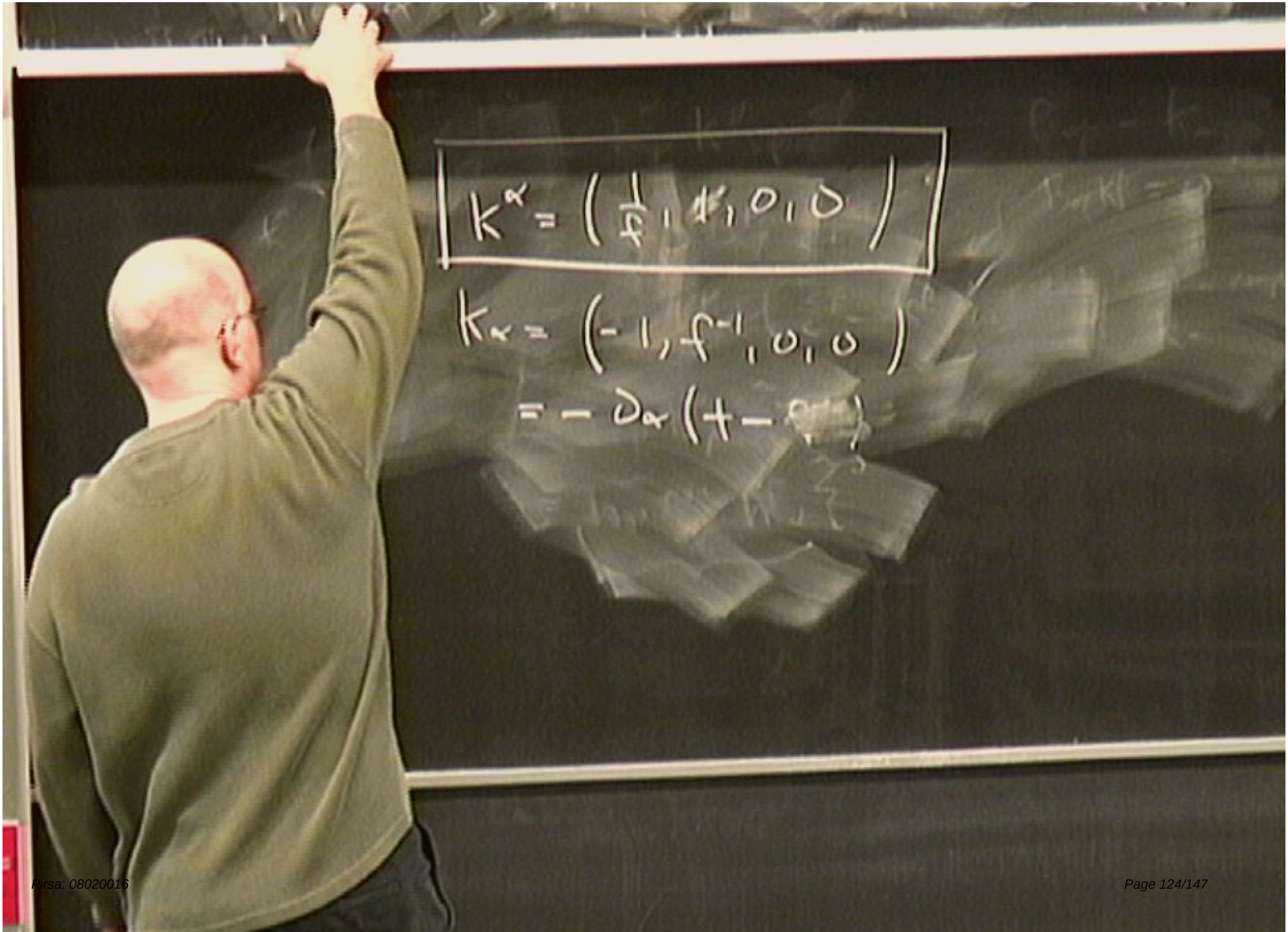
$k_\alpha = (-1,$

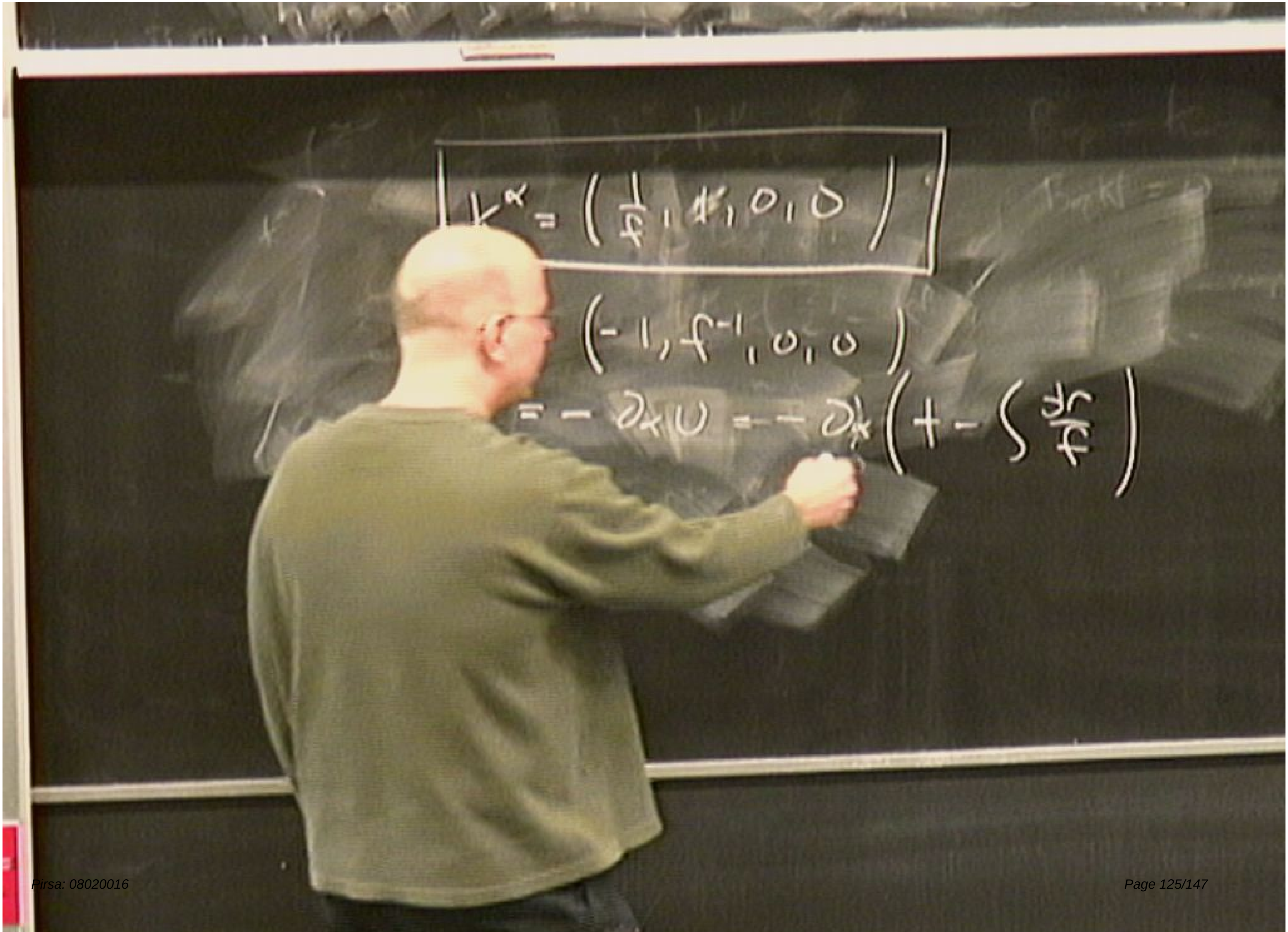

choose λ ≡ r : affine parameter.

choose $\lambda \equiv r$ = affine parameter.

$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)}$$

$$k_\alpha = \left(-1, f^{-1}, 0, 0 \right)$$


Kα = (1/f , 1, 0, 0)
Kα = (-1, f-1, 0, 0)
= - ∂α (t -



k^α = (1/f , 1 , 0 , 0)
(-1 , f^-1 , 0 , 0)
= - ∂_α U = - ∂_α (t - ∫ dr/f)

$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

$$k_\alpha = \left(-1, f^{-1}, 0, 0\right)$$

$$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$$

$$\boxed{K^\alpha = \left(\frac{1}{f}, \cancel{1}, 0, 0\right)}$$

$$K_\alpha = \left(-1, f^{-1}, 0, 0\right)$$

$$= -\partial_\alpha u = -\partial_\alpha \left(t - \int \frac{dr}{f}\right)$$

$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

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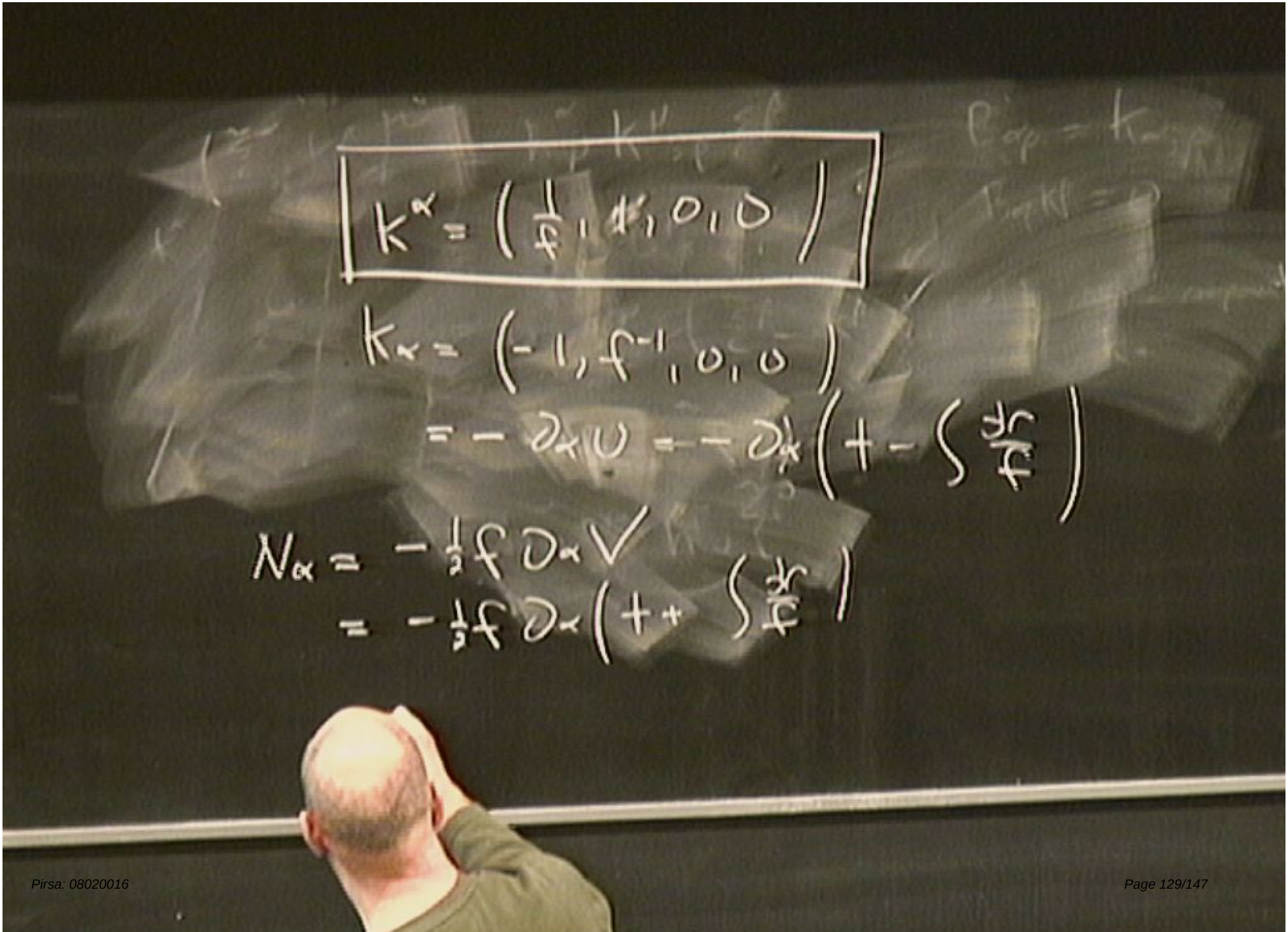
$$= -\partial_\alpha u = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$$

$$N_\alpha =$$


$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$
$k_\alpha = \left(-1, f^{-1}, 0, 0\right)$
$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$
$N_\alpha = -\frac{1}{2} f \partial_\alpha V$
$= -\frac{1}{2} f \partial_\alpha\left(t + \int \frac{dr}{f}\right)$

$$\boxed{K^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

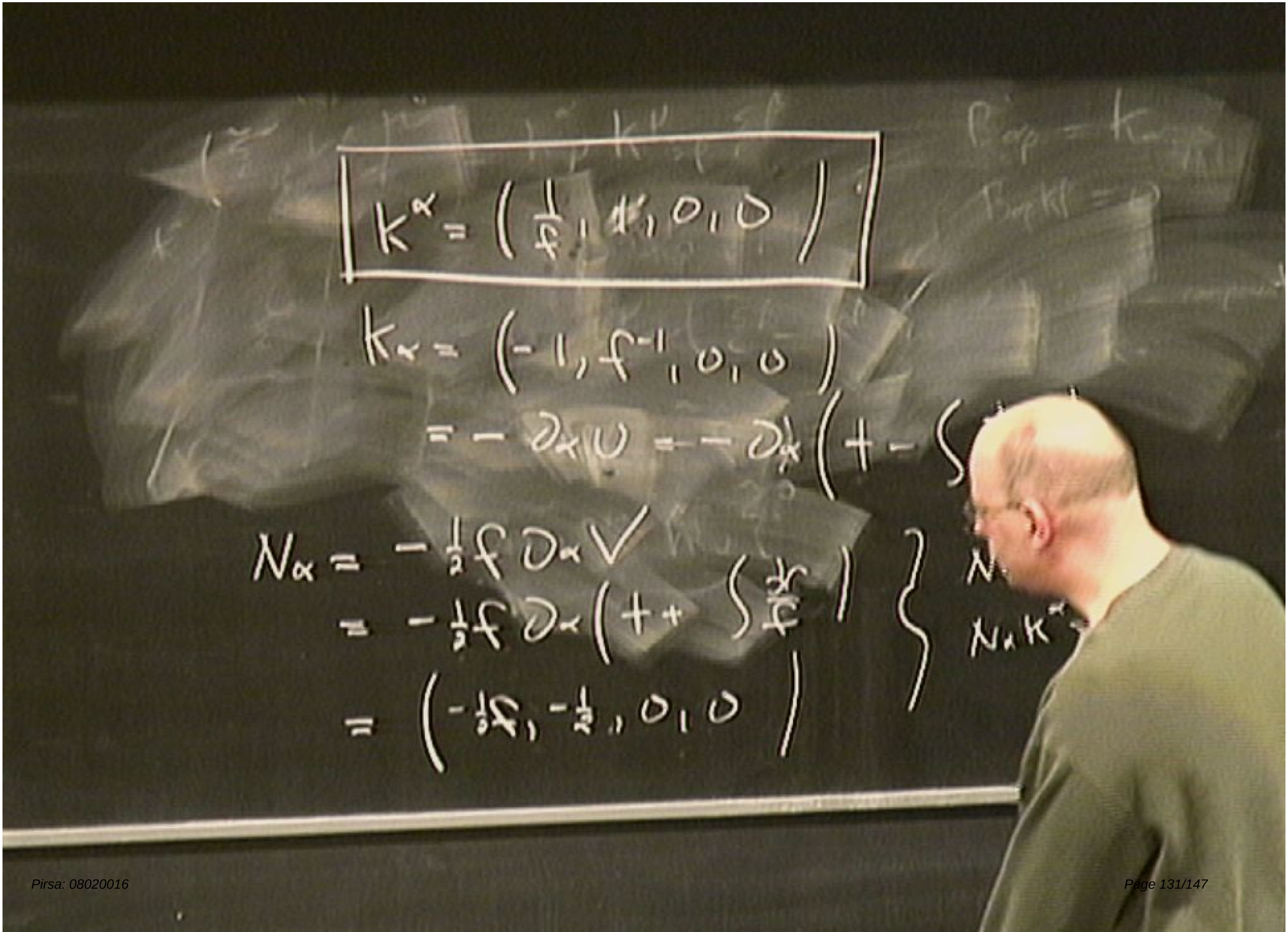
$$K_\alpha = \left(-1, f^{-1}, 0, 0\right)$$

$$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$$

$$N_\alpha = -\frac{1}{2} f \,\partial_\alpha V$$

$$= -\frac{1}{2} f \,\partial_\alpha\left(t + \int \frac{dr}{f}\right)$$

$$= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0\right)$$

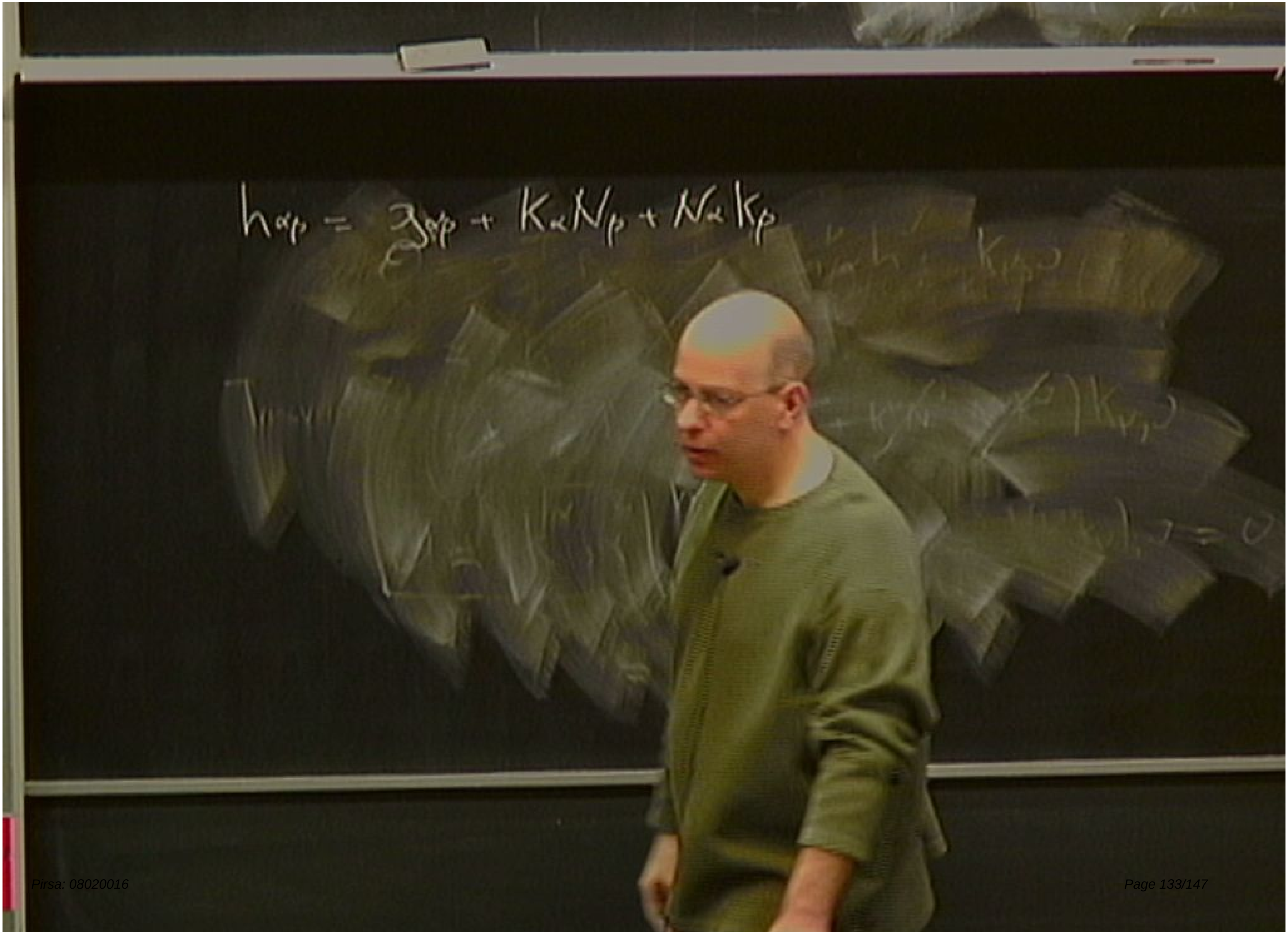

$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$
$K_\alpha = \left(-1, f^{-1}, 0, 0 \right)$
$= -\partial_\alpha U = -\partial_\alpha \left(t - \int \right.$
$N_\alpha = -\frac{1}{2} f \partial_\alpha V$
$= -\frac{1}{2} f \partial_\alpha \left(t + \int \frac{dr}{f} \right)$
$= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right)$
$N_\alpha K^\alpha$

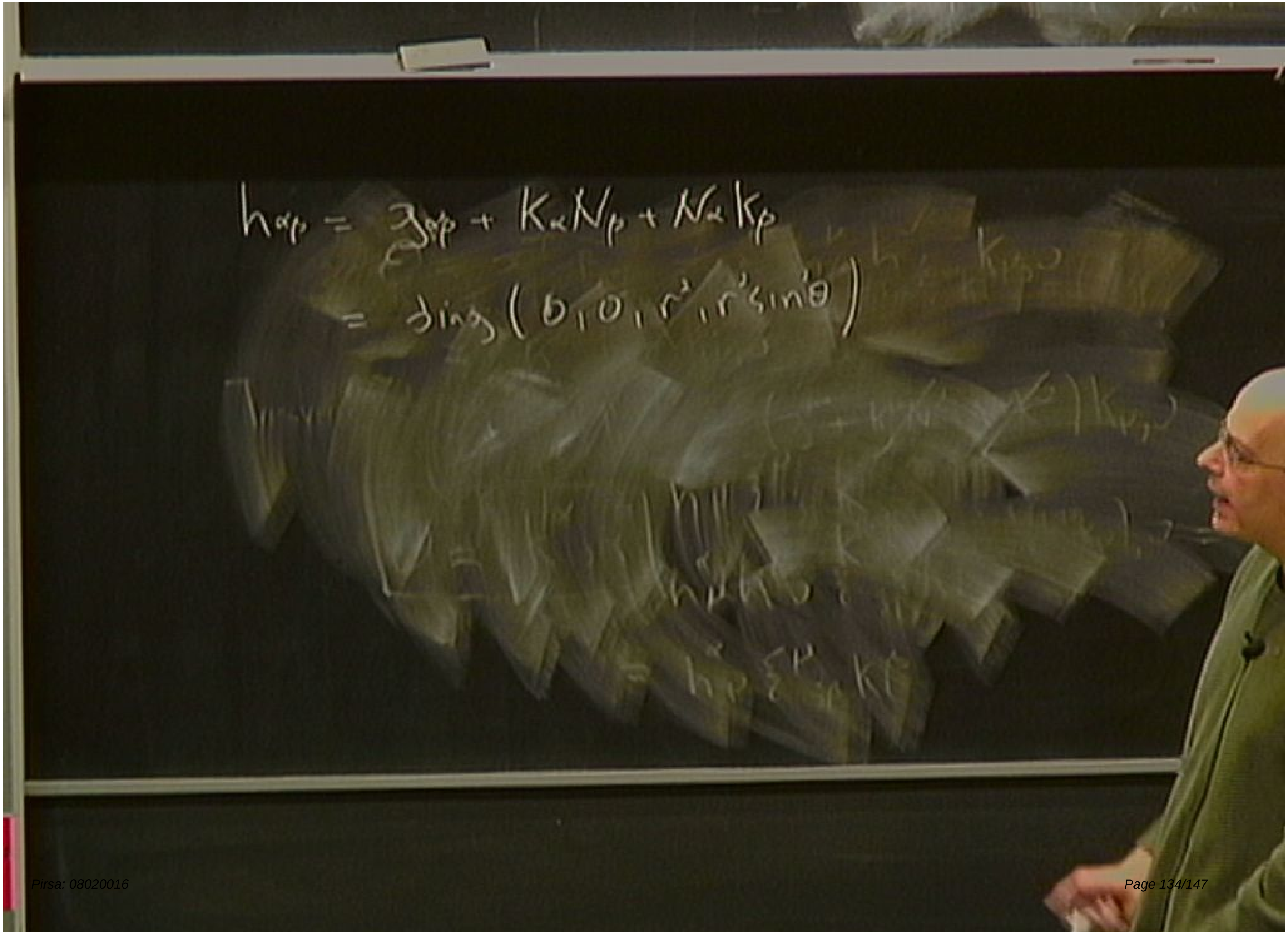
$$\boxed{k^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)}$$

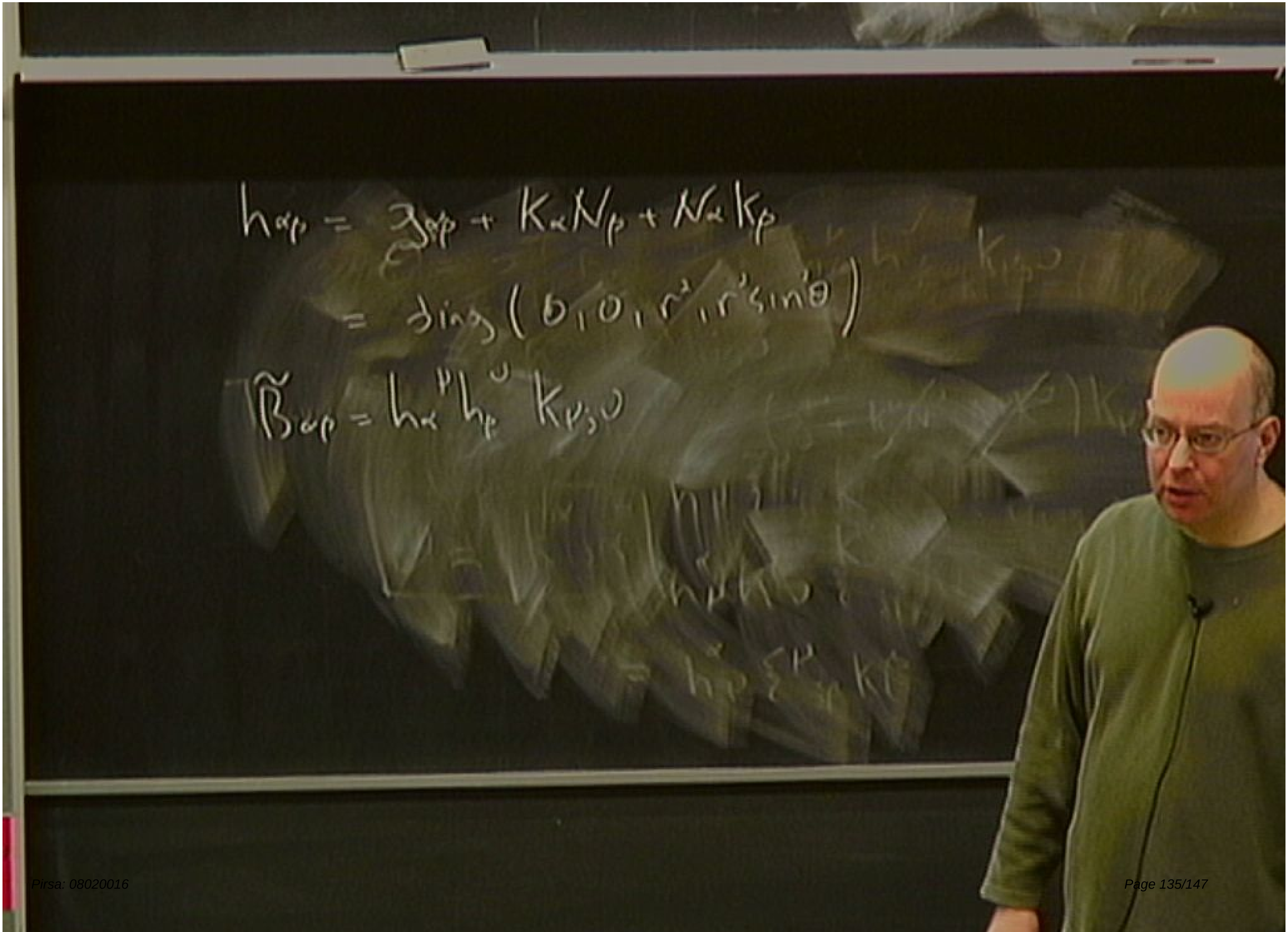
$$k_\alpha = \left(-1, f^{-1}, 0, 0\right)$$

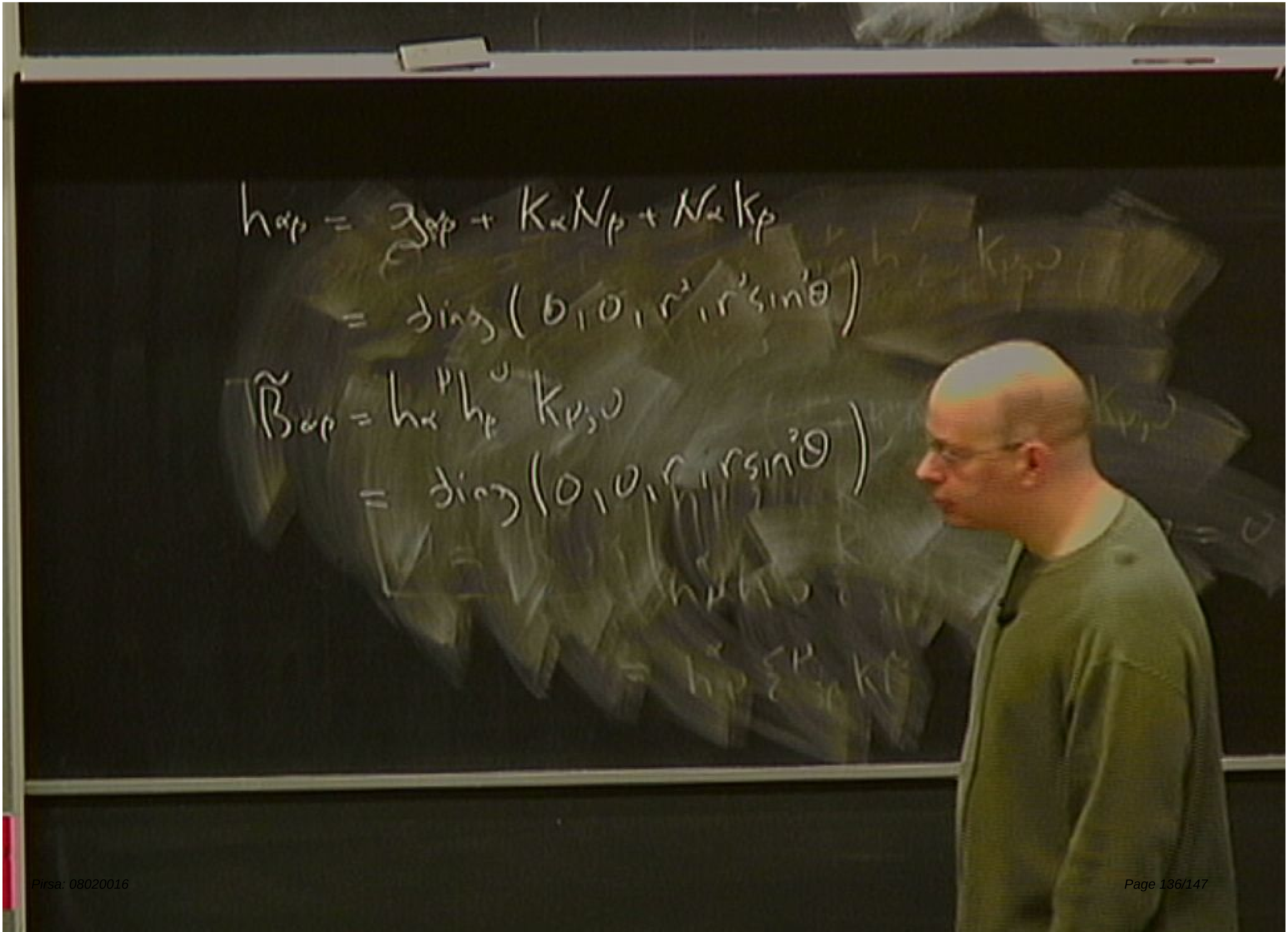
$$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$$

$$\left.\begin{aligned} N_\alpha &= -\tfrac{1}{2} f\, \partial_\alpha V \\ &= -\tfrac{1}{2} f\, \partial_\alpha\left(t + \int \frac{dr}{f}\right) \\ &= \left(-\tfrac{1}{2} f, -\tfrac{1}{2}, 0, 0\right) \end{aligned}\right\} \quad \begin{aligned} N_\alpha N^\alpha &= 0 \\ N_\alpha k^\alpha &= -1 \end{aligned}$$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$
$= \mathrm{diag}(0, 0, r^2, r^2\sin^2\theta)$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$
$= \mathrm{diag}(0, 0, r^2, r^2\sin^2\theta)$
$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$
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$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$
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$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$= \mathrm{diag}(0, 0, r^2, r^2\sin^2\theta)$$

$$\tilde{B}_{\alpha\beta} = h_\alpha^{\ \mu} h_\beta^{\ \nu} K_{\mu;\nu}$$

$$= \mathrm{diag}(0, 0, r, r\sin^2\theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$= \text{diag}(0, 0, r^2, r^2\sin^2\theta)$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$$

$$= \text{diag}(0, 0, r, r\sin^2\theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$= \operatorname{diag}(0, 0, r^2, r^2 \sin^2\theta)$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$$

$$= \operatorname{diag}(0, 0, r, r \sin^2\theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$= \mathrm{diag}(0, 0, r^2, r^2 \sin^2\theta)$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$$

$$= \mathrm{diag}(0, 0, r, r \sin^2\theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$= \frac{1}{4\pi r^2} \frac{\partial}{\partial\lambda} (4\pi r^2)$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$= \mathrm{diag}\left(0, 0, r^2, r^2\sin^2\theta\right)$$

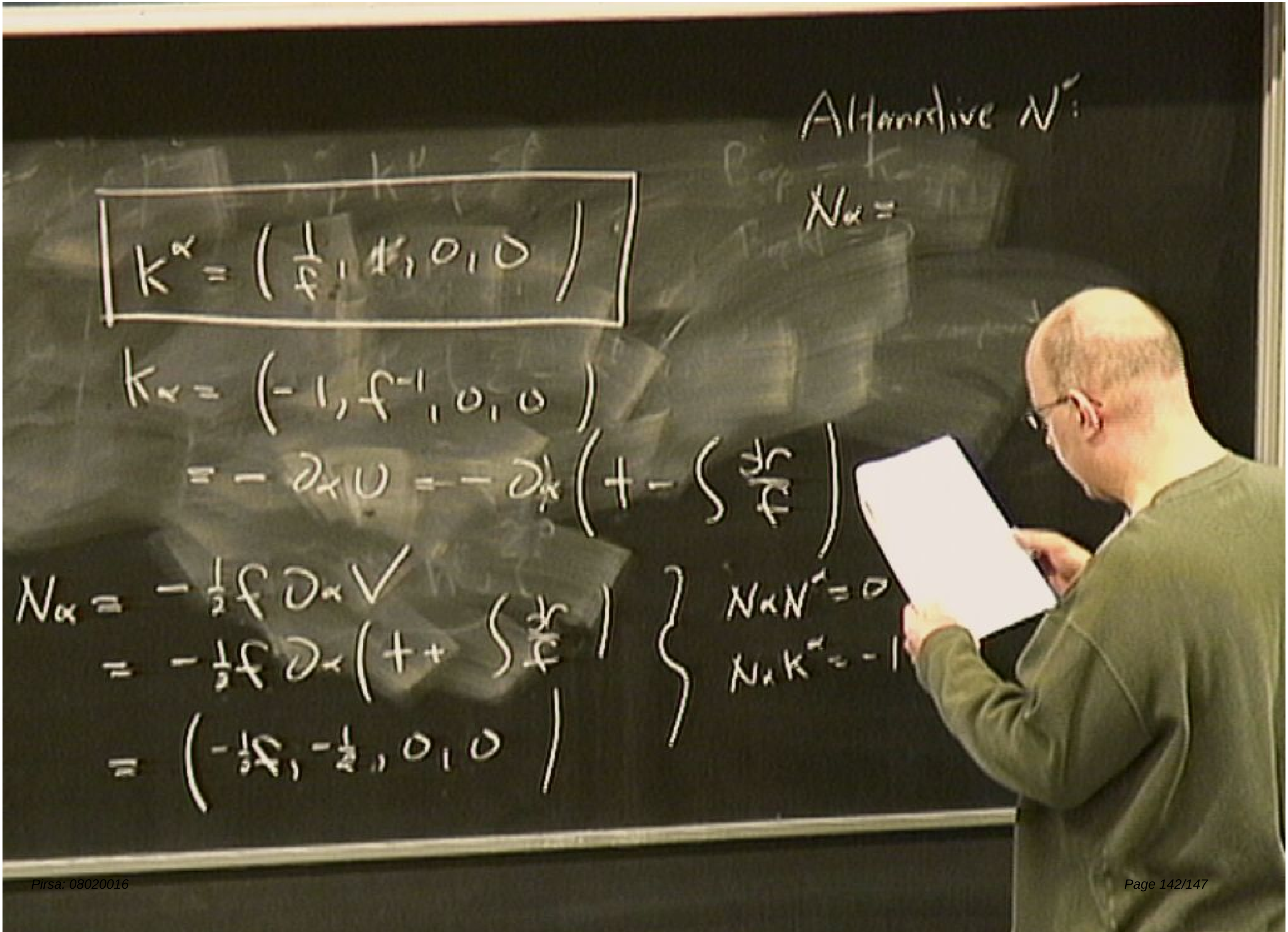
$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$$

$$= \mathrm{diag}\left(0, 0, r, r\sin^2\theta\right)$$

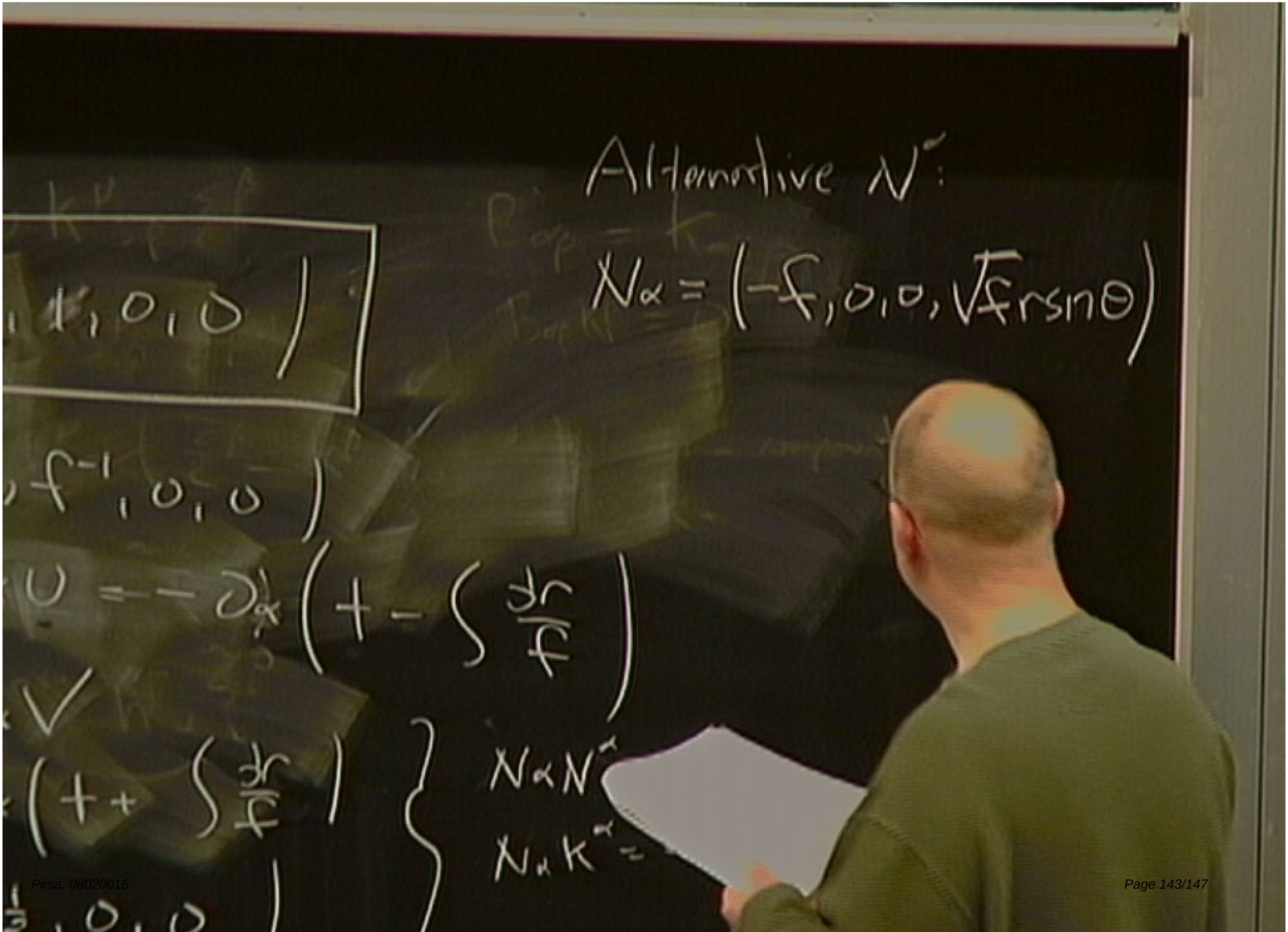
$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$= \frac{1}{4\pi r^2}\frac{\partial}{\partial\lambda}\left(4\pi r^2\right)$$



Alternative N^α:
$N_\alpha =$
$K^\alpha = \left(\frac{1}{f}, 1, 0, 0\right)$
$K_\alpha = \left(-1, f^{-1}, 0, 0\right)$
$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$
$N_\alpha = -\frac{1}{2} f\, \partial_\alpha V$
$= -\frac{1}{2} f\, \partial_\alpha\left(t + \int \frac{dr}{f}\right)$
$= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0\right)$
$N_\alpha N^\alpha = 0$
$N_\alpha K^\alpha = -1$


Alternative N^α:
$N_\alpha = (-f, 0, 0, \sqrt{f} r \sin\theta)$
$N_\alpha N^\alpha$
$N_\alpha K^\alpha =$

Alternative N^α:

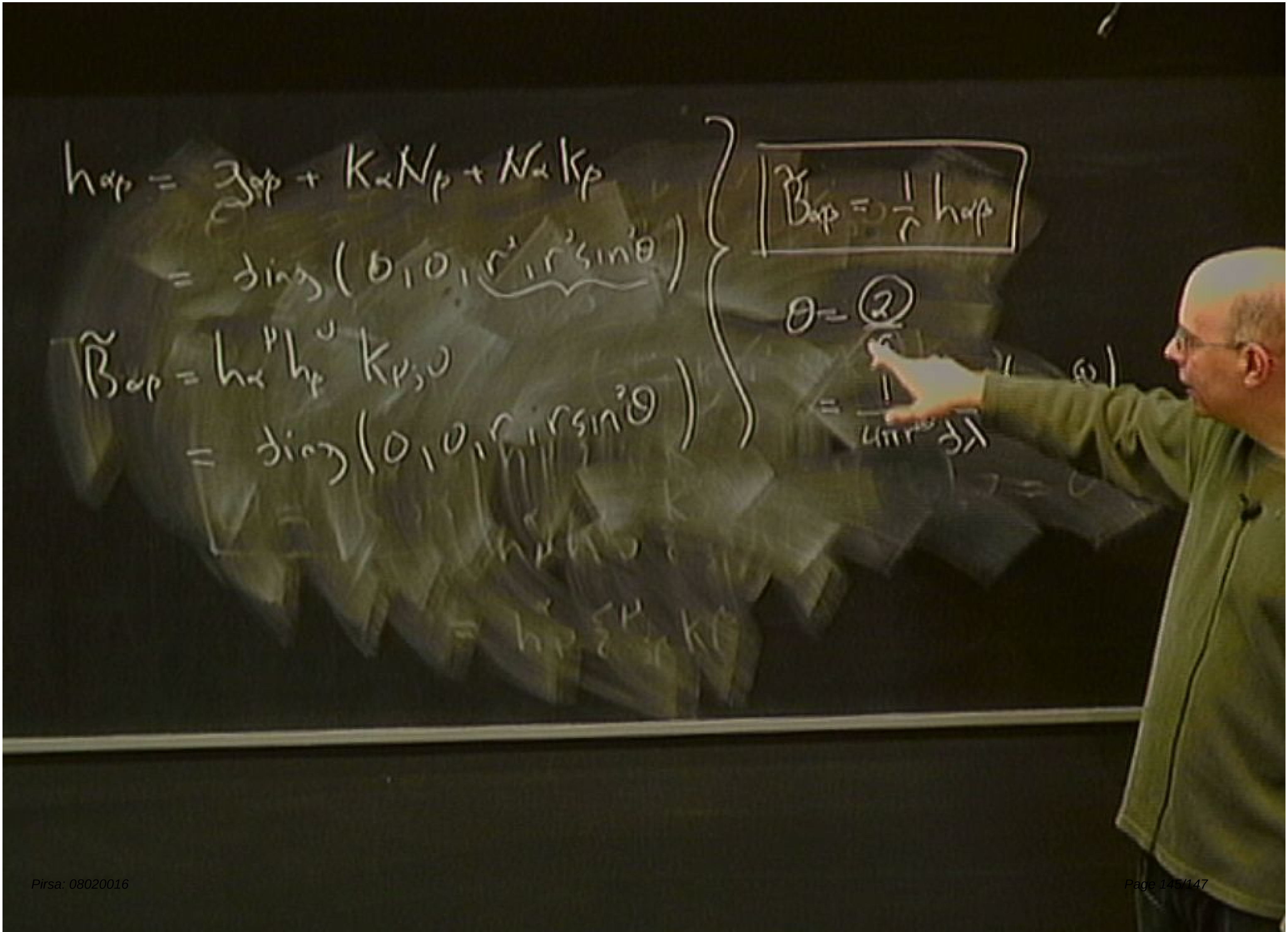
$$N_\alpha = \left(-f, 0, 0, \sqrt{f}\, r \sin\theta\right)$$

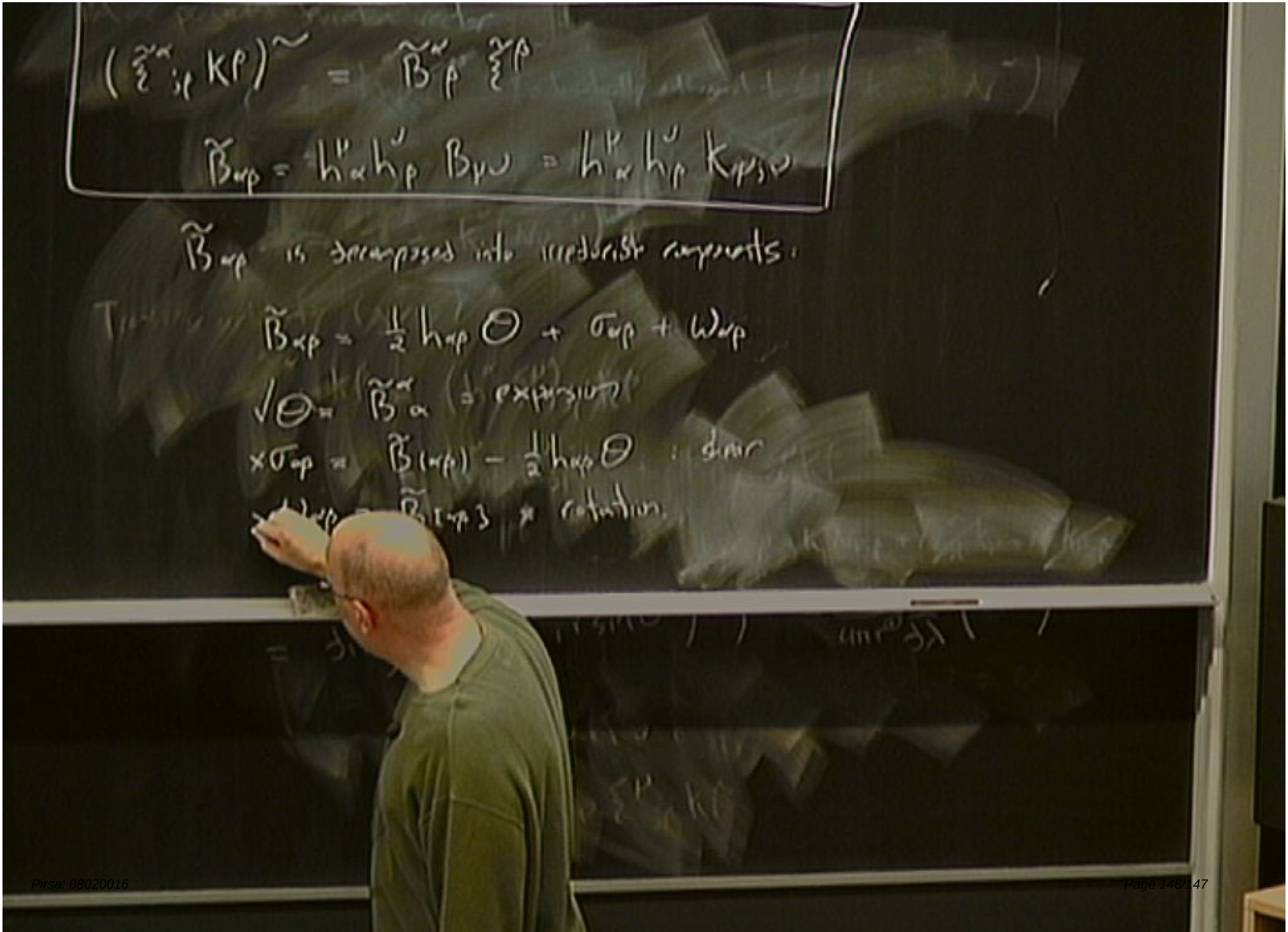
$$\boxed{K^\alpha = \left(\frac{1}{f}, f, 0, 0\right)}$$

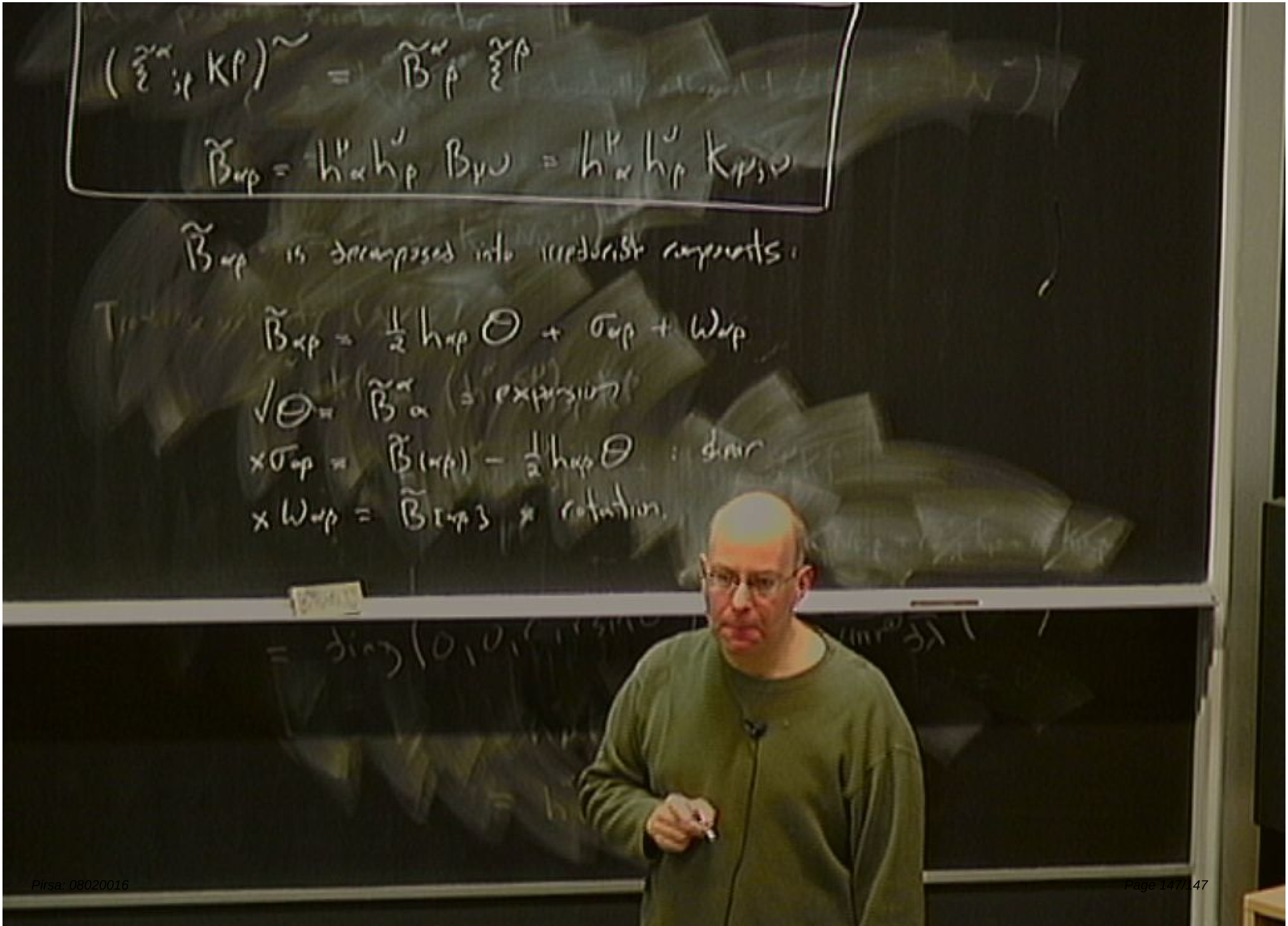
$$K_\alpha = \left(-1, f^{-1}, 0, 0\right)$$

$$= -\partial_\alpha U = -\partial_\alpha\left(t - \int \frac{dr}{f}\right)$$

$$\left.\begin{aligned} N_\alpha &= -\tfrac{1}{2} f\, \partial_\alpha V \\ &= -\tfrac{1}{2} f\, \partial_\alpha\left(t + \int \frac{dr}{f}\right) \\ &= \left(-\tfrac{1}{2}f, -\tfrac{1}{2}, 0, 0\right) \end{aligned}\right\} \begin{aligned} N_\alpha N^\alpha &= 0 \\ N_\alpha K^\alpha &= -1 \end{aligned}$$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$
$= \mathrm{diag}(0, 0, r^2, r^2\sin^2\theta)$
$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu K_{\mu;\nu}$
$= \mathrm{diag}(0, 0, r, r\sin^2\theta)$
$\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}$
$\theta = \frac{2}{r}$


$(\tilde{\xi}^\alpha{}_{;\beta} k^\beta)^\sim = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$\Theta = \tilde{B}^\alpha{}_\alpha$ = expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta$: shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$ = rotation


$(\tilde{\xi}^{\alpha}{}_{;\beta} K^{\beta})^{\sim} = \tilde{B}^{\alpha}{}_{\beta}\, \tilde{\xi}^{\beta}$
$\tilde{B}_{\alpha\beta} = h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} B_{\mu\nu} = h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} K_{\mu;\nu}$
$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$\Theta = \tilde{B}^{\alpha}{}_{\alpha}$: expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta$: shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$: rotation