

Title: Advanced General Relativity - Lecture 5A

Date: Feb 06, 2008 10:30 AM

URL: <http://pirsa.org/08020016>

Abstract: Advanced General Relativity

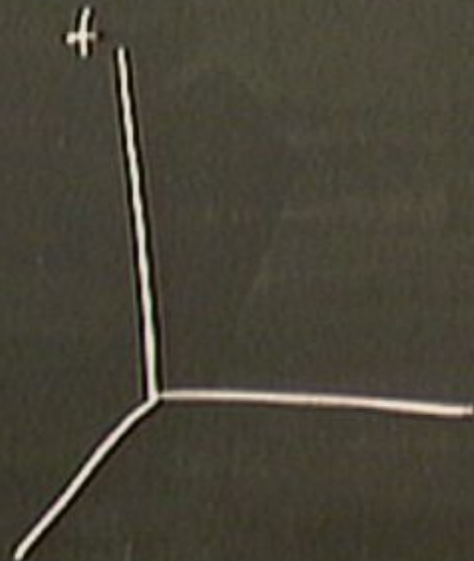
Feb 20: no classes

HW2: Feb 27

2.6 #1, 3, 5, 8

3.13 #1

Null congruences



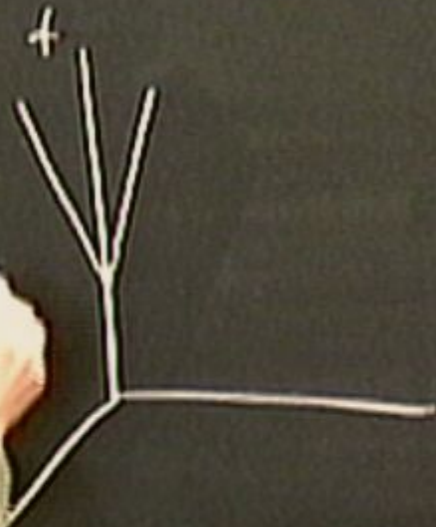
Feb 20, no classes

HW2: Feb 27

2.6 $\rightarrow$ 1, 3, 5, 8

3.13 $\rightarrow$ 1

Null congruences



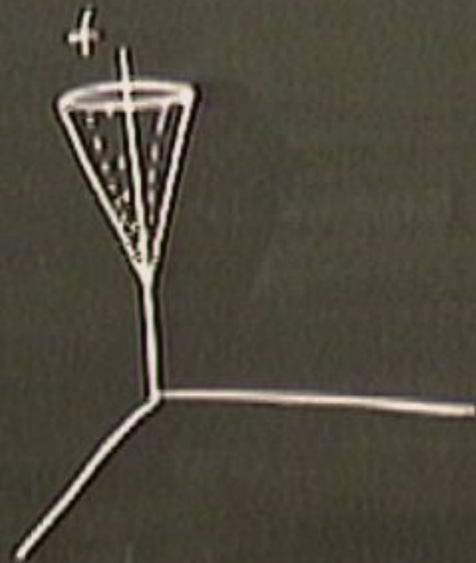
Feb 2010 no classes

HW2: Feb 27

2.6 $\rightarrow$ 1, 3, 5, 8

3.13 $\rightarrow$ 1

Null congruences



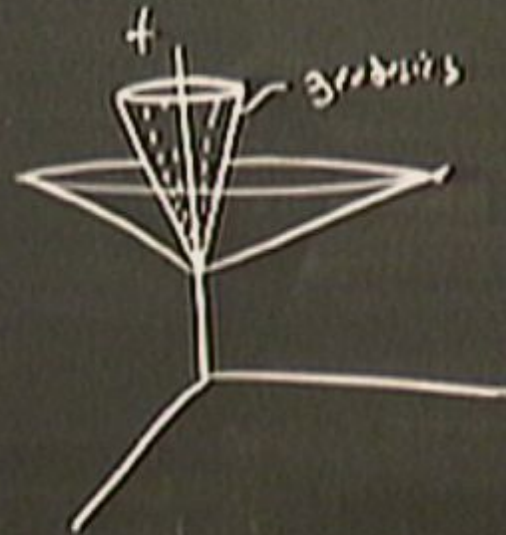
Feb 20: no classes

HW2: Feb 27

2.4 $\rightarrow$ 1, 3, 5, 8

3.13 $\rightarrow$ 1

Null engineering



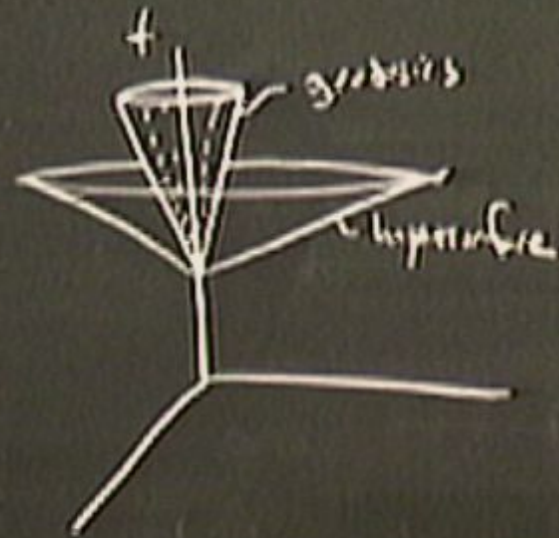
Feb 2011 no

HW2

2.6 11/1

3.13 11/1

Null engineering



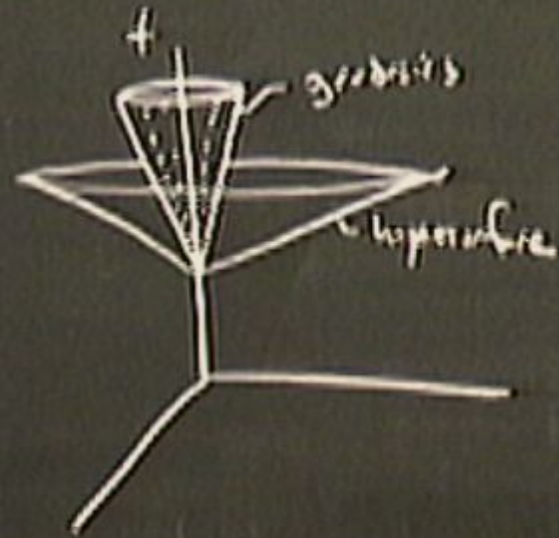
Feb 20: no

HW2:

2.6 + 1

3.13 + 1

Null congruences



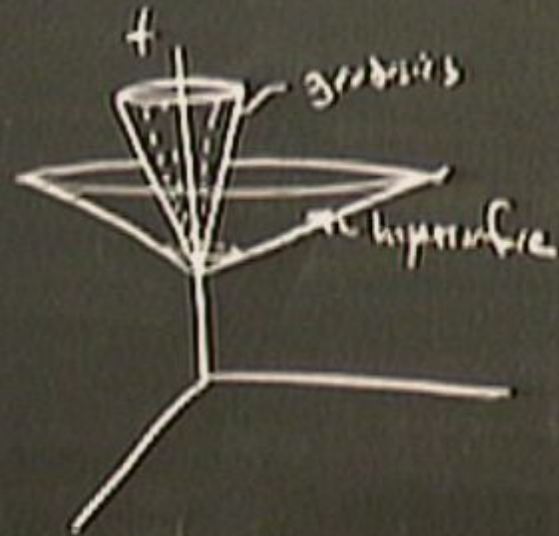
Feb 20, 1985

HW2, 1

2.6 $\rightarrow$ (1)

3.13 $\rightarrow$ (1)

Null engineering



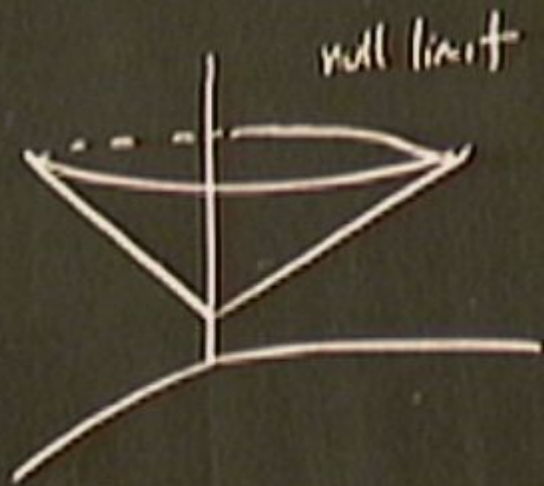
Feb 2016

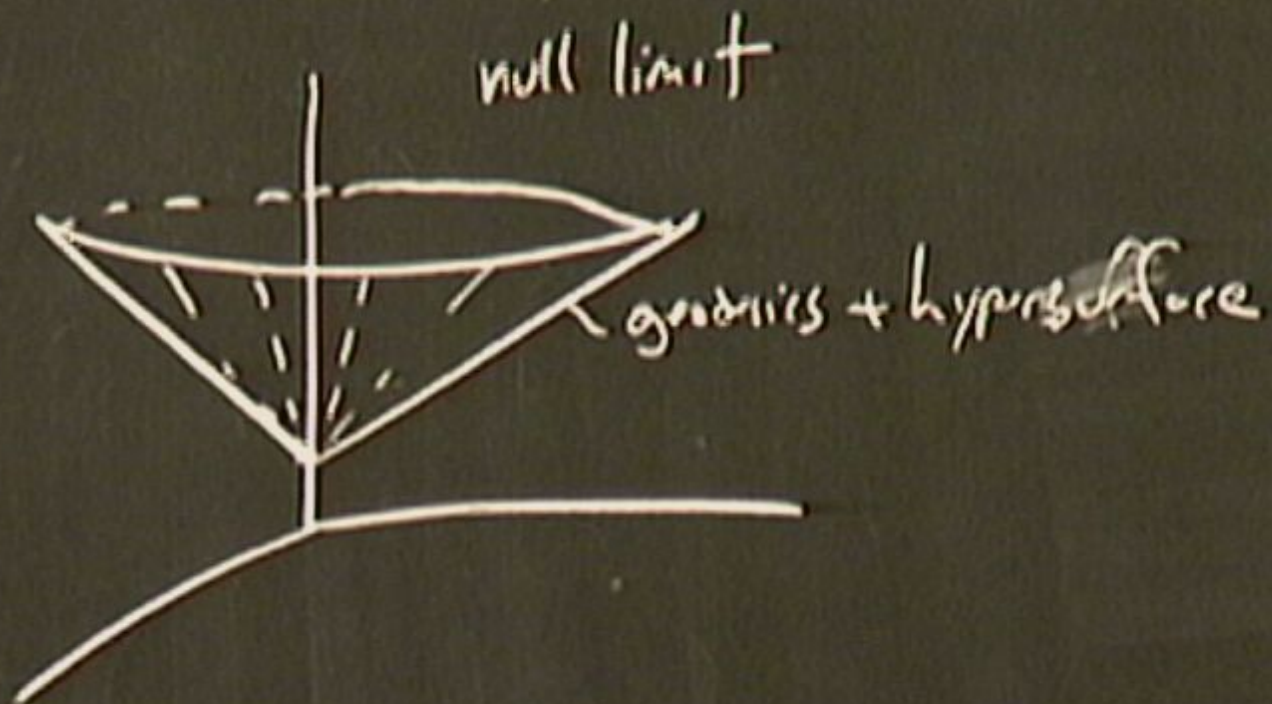
HW2

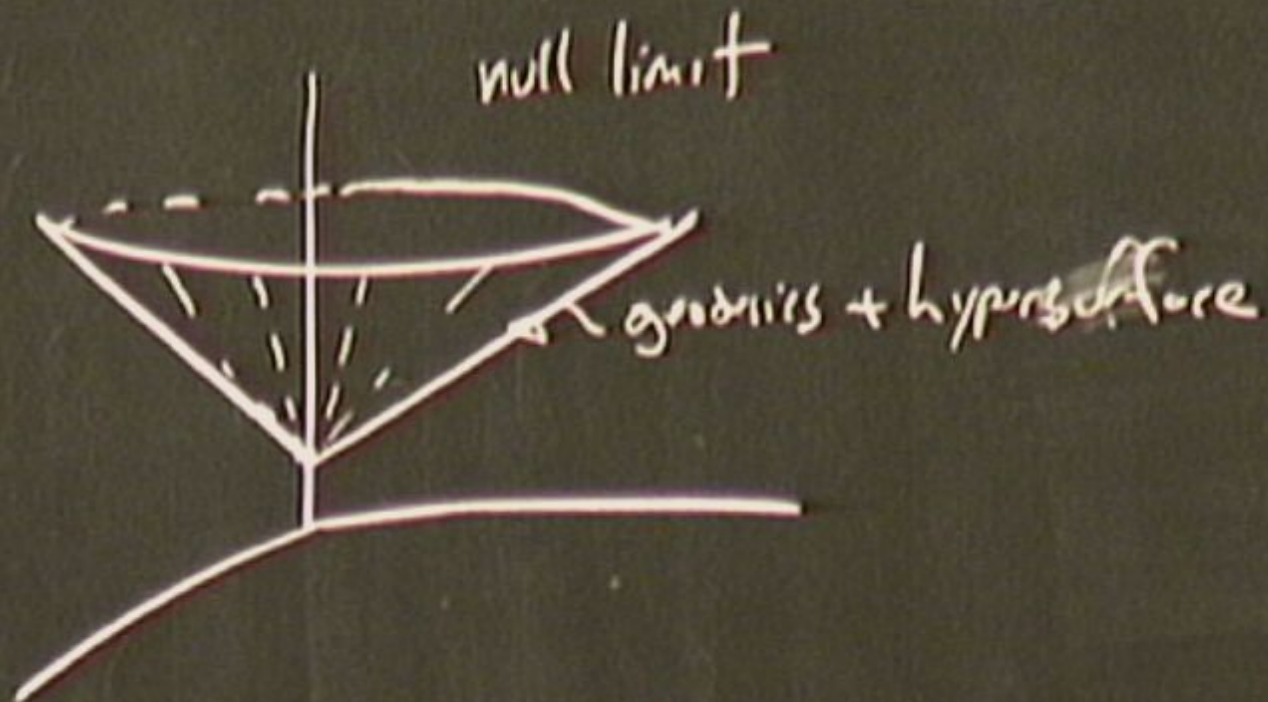
2.6 11/1

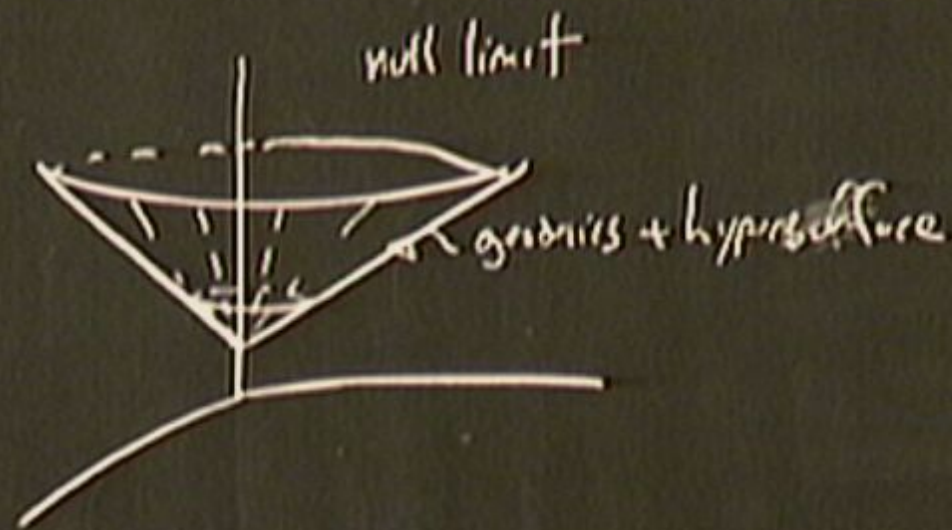
3.13 11/1

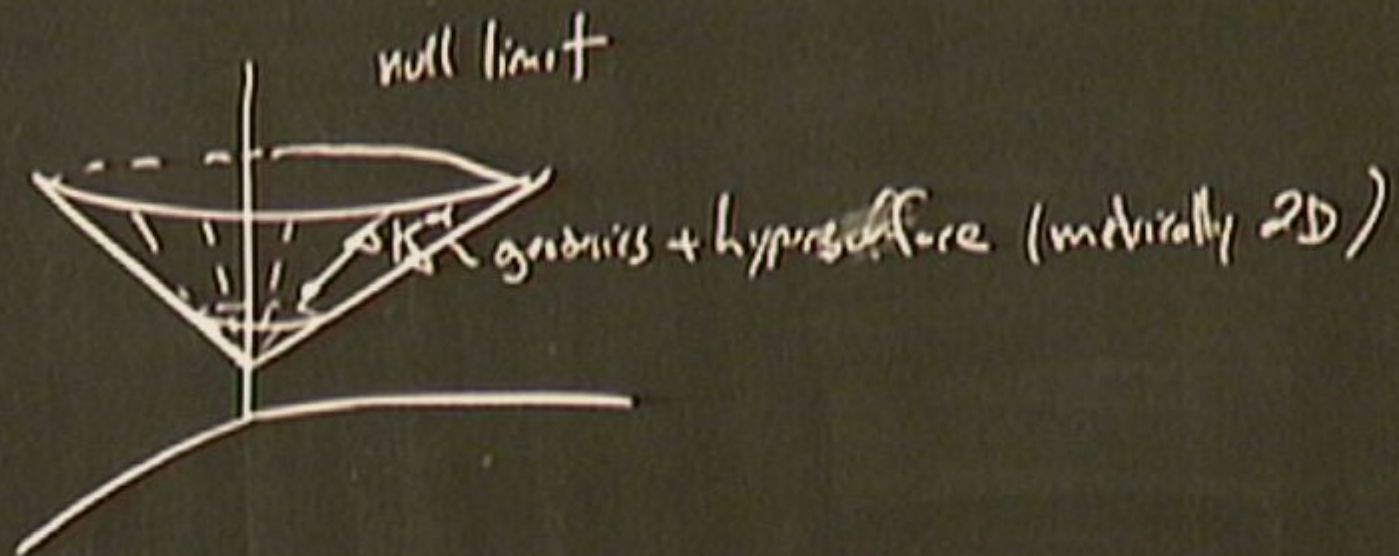


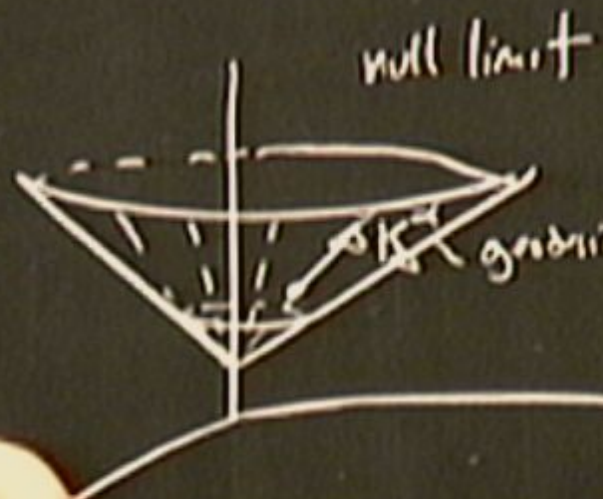








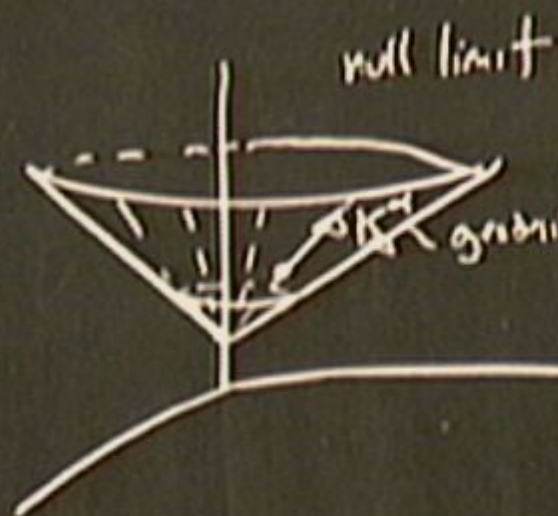




null limit

geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics
: normal to hypersurface



null limit

geodesics + hypersurface (metrically 2D)

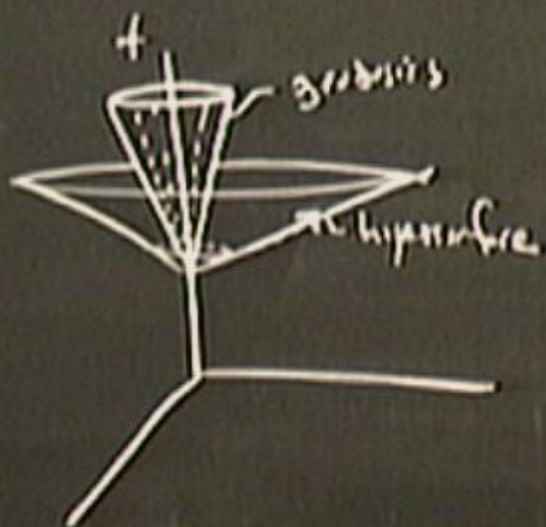
k^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} \text{tangent to geodesics} \\ \text{normal to hypersurface} \\ \text{tangent to hypersurface} \end{array} \right\} k_\alpha k^\alpha = 0$$

Null congruences

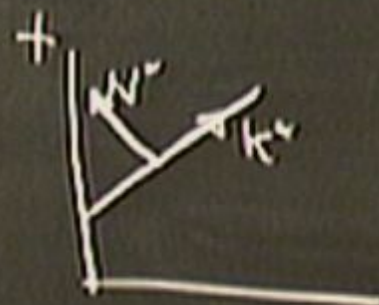


Feb 20, no classes

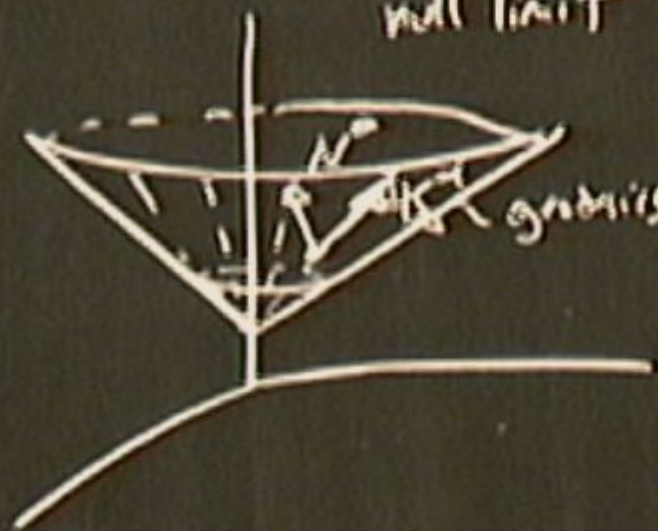
HW2: Feb 27

2.6 + 1, 3, 5, 8

3.13 + 1



null limit



geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics

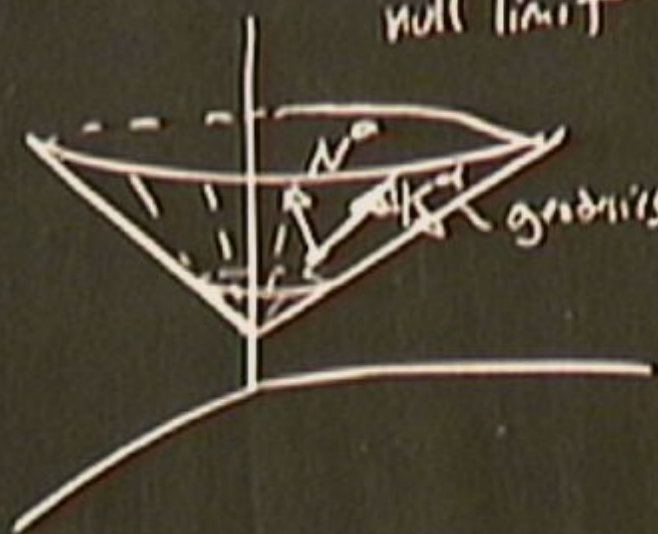
N^α : normal to hypersurface

k^α : tangent to hypersurface

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

null limit



geodesics + hypersurface (metrically 2D)

k^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} k^\alpha \\ : \text{normal to hypersurface} \\ : \text{tangent to hypersurface} \end{array} \right\} k_\alpha k^\alpha = 0$$

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

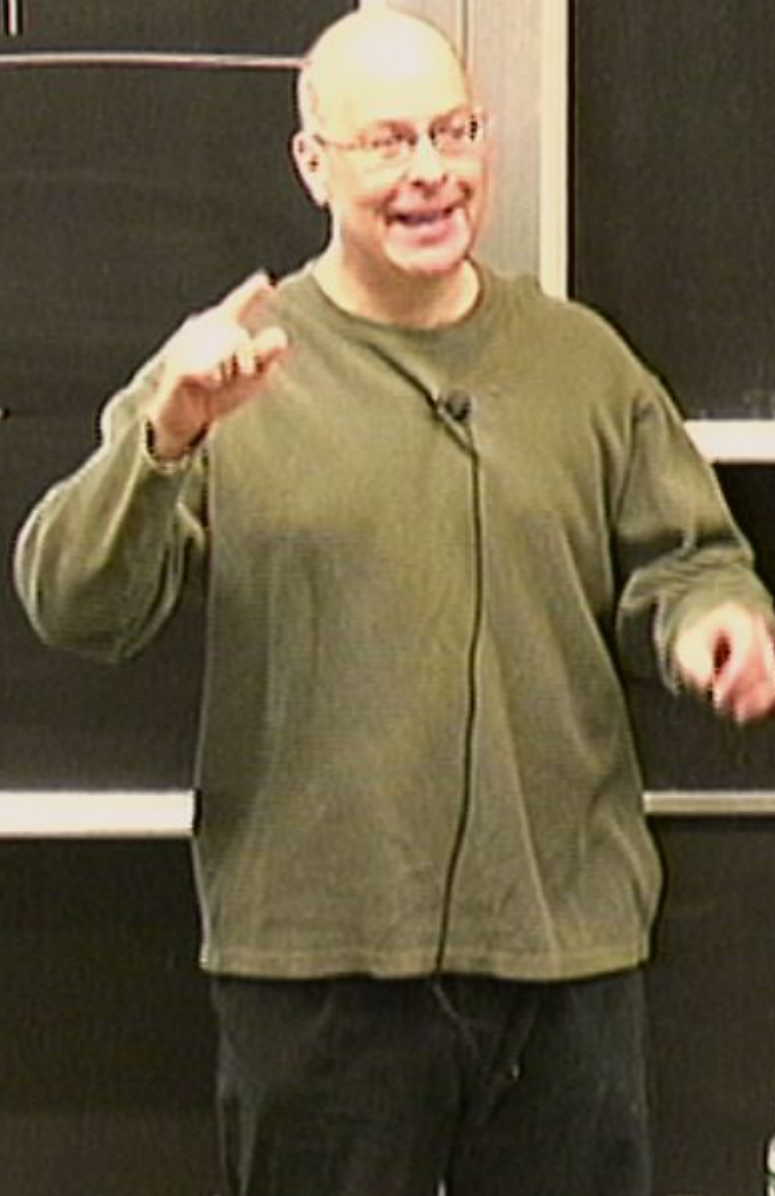
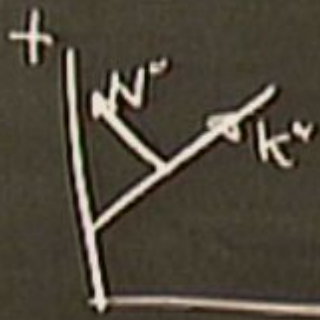
$$N_\alpha k^\alpha = -1$$

Feb 20: no classes

HW2: Feb 27

2.6 #1, 3, 5, 8

3.13 #1

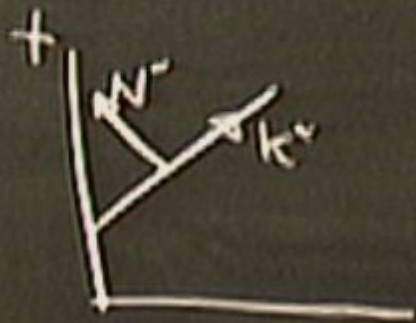


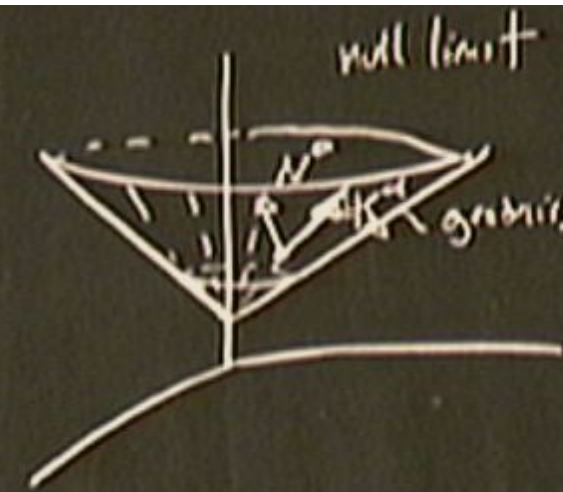
Feb 20: no classes

HW2: Feb 27

2.4 #1, 3, 5, 8

3.13 #1





null limit

geonics + hypersurface (mutually 2D)

k^α : tangent to geonics

: normal to hypersurface

: tangent to hypersurface

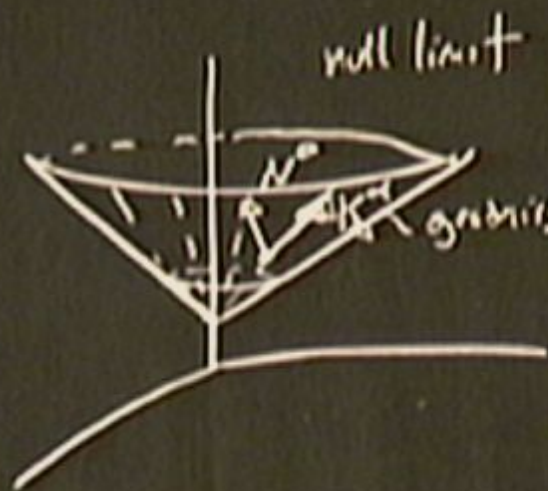
$$\left. \begin{array}{l} k^\alpha \\ : \\ : \end{array} \right\} k_\alpha k^\alpha = 0$$

N^α : auxiliary null direction

Transverse space: 2D

$$N_\alpha N^\alpha = 0$$

$$N_\alpha k^\alpha = -1$$



null limit

geonics + hypersurface (intrinsically 2D)

K^α : tangent to geonics

: normal to hypersurface

: tangent to hypersurface

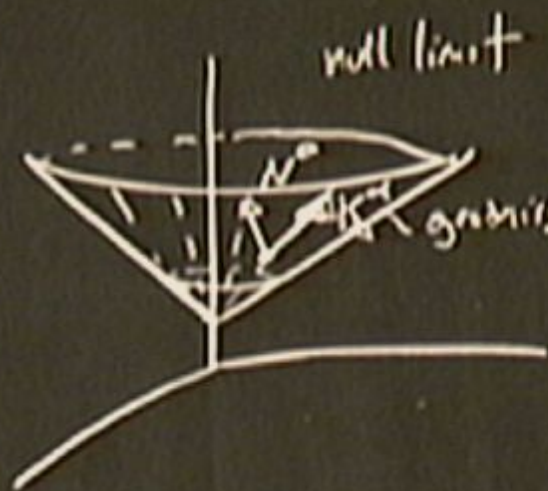
$$K_\alpha K^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 N^α and K^α

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$



null limit

geonics + hypersurface (mutually 2D)

K^α : tangent to geonics

: normal to hypersurface

: tangent to hypersurface

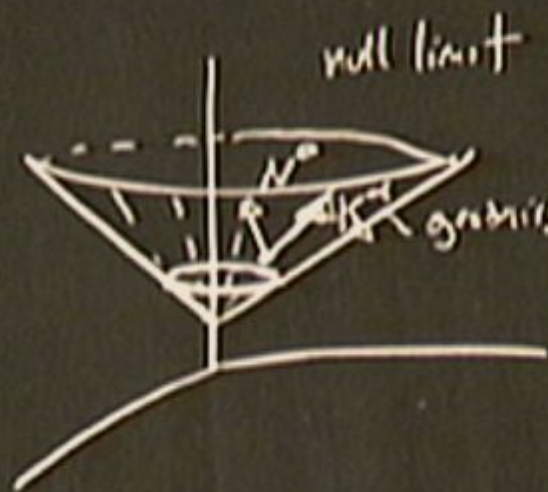
$$K_\alpha K^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 N^α and K^α

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$



null limit

geodesics + hypersurface (individually 2D)

K^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

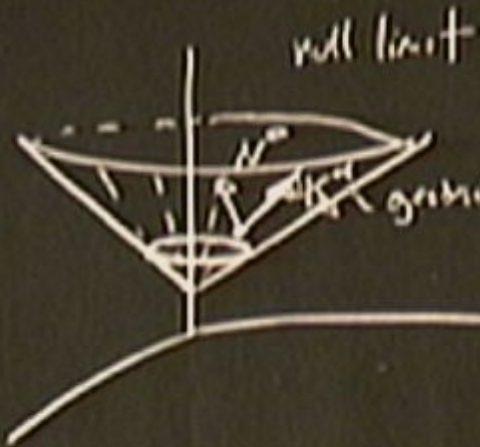
$$\left. \begin{array}{l} \text{normal to hypersurface} \\ \text{tangent to hypersurface} \end{array} \right\} K_\alpha K^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 N^α and K^α

N^α : auxiliary null direction

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$



null limit

geodesics + hypersurface (intrinsically 2D)

k^α : tangent to geodesics

: normal to hypersurface

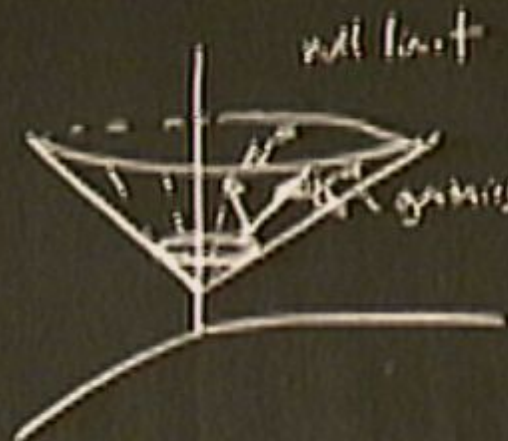
: tangent to hypersurface

$$\left. \begin{array}{l} k^\alpha \\ : \text{normal to hypersurface} \\ : \text{tangent to hypersurface} \end{array} \right\} k_\alpha k^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 N^α and k^α

N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha k^\alpha = -1 \end{cases}$$



null line

generators + hypersurface (intrinsically 2D)

k^μ : tangent to generators
 $\quad$: normal to hypersurface
 $\quad$: tangent to hypersurface

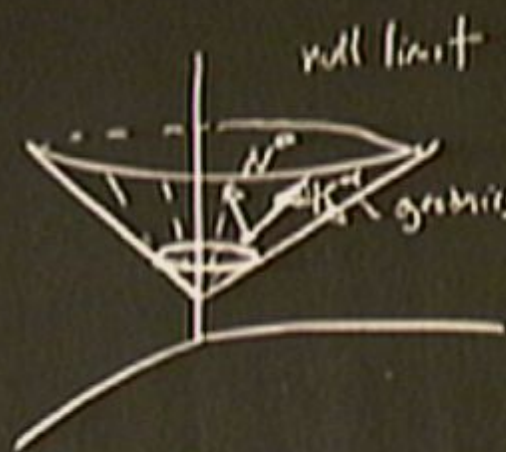
$$\left. \begin{array}{l} \text{normal to hypersurface} \\ \text{tangent to hypersurface} \end{array} \right\} k_\mu k^\mu = 0$$

Transverse space: 2D
 - orthogonal to both
 N^μ and k^μ

N^μ : auxiliary null direction

$$\begin{cases} N_\mu N^\mu = 0 \\ N_\mu k^\mu = -1 \end{cases}$$

Transverse metric:



null limit

geodesics + hypersurface (intrinsically 2D)

k^μ : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} \text{normal to hypersurface} \\ \text{tangent to hypersurface} \end{array} \right\} k_\mu k^\mu = 0$$

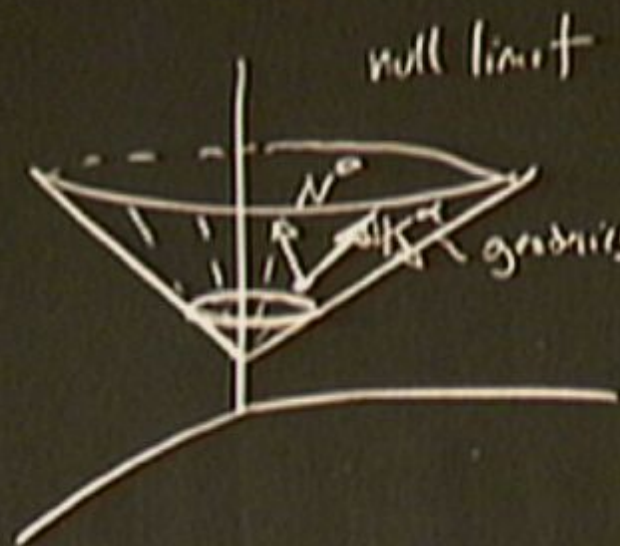
Transverse space: 2D
- orthogonal to both
 $\underline{N}$ and $\underline{K}$

N^μ : auxiliary null direction

Transverse metric:

$$h_{\mu\nu} = g_{\mu\nu} + k_\mu N_\nu + N_\mu k_\nu$$

$$\begin{cases} N_\mu N^\mu = 0 \\ N_\mu k^\mu = -1 \end{cases}$$



null limit

geodesics + hypersurface (metrically 2D)

k^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} k^\alpha \\ : \text{normal to hypersurface} \\ : \text{tangent to hypersurface} \end{array} \right\} k_\alpha k^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 $\underline{N}$ and $\underline{K}$

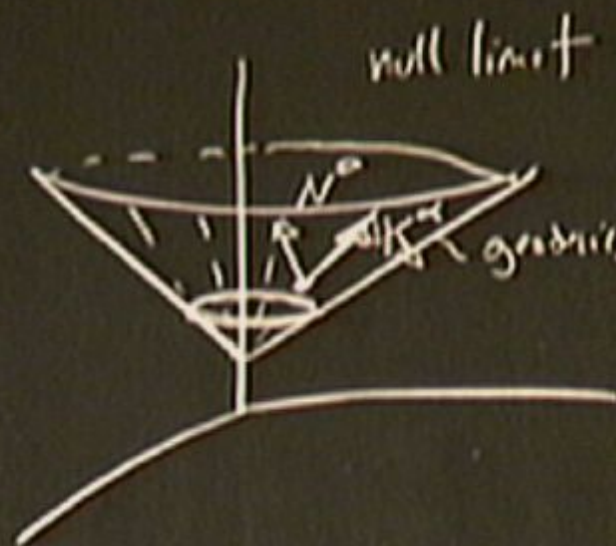
N^α : auxiliary null direction

Transverse metric:

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$h^\alpha_\alpha = 2$$

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha k^\alpha = -1 \end{cases}$$



null limit

geodesics + hypersurface (metrically 2D)

K^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} K^\alpha \text{ : tangent to geodesics} \\ \text{ : normal to hypersurface} \\ \text{ : tangent to hypersurface} \end{array} \right\} K_\alpha K^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 $\underline{N}$ and $\underline{K}$

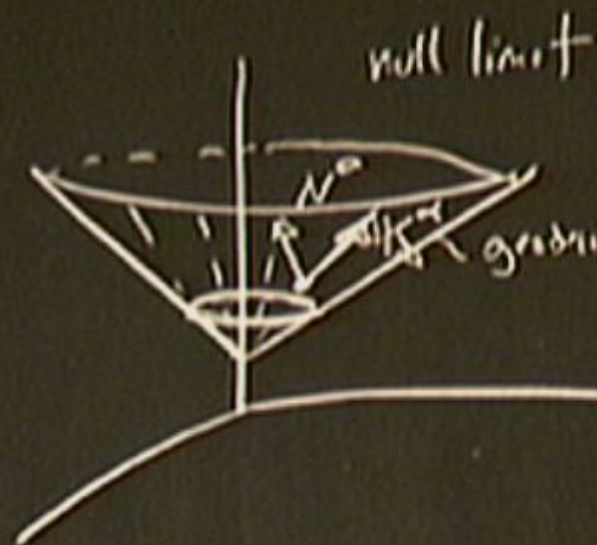
N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{cases}$$

Transverse metric:

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$h^\alpha_\alpha = 2 \quad ; \quad h^\mu_\nu h^\nu_\rho = h^\mu_\rho$$



null limit

geodesics + hypersurface (metrically 2D)

K^α : tangent to geodesics

: normal to hypersurface

: tangent to hypersurface

$$\left. \begin{array}{l} K^\alpha \text{ : tangent to geodesics} \\ \text{ : normal to hypersurface} \\ \text{ : tangent to hypersurface} \end{array} \right\} K_\alpha K^\alpha = 0$$

Transverse space: 2D
- orthogonal to both
 $\underline{N}$ and $\underline{K}$

N^α : auxiliary null direction

$$\begin{cases} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{cases}$$

Transverse metric:

$$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$$

$$h^\alpha_\alpha = 2$$

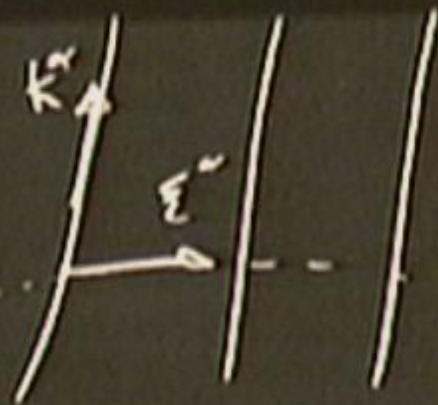
$$h^\mu_\nu h^\nu_\rho = h^\mu_\rho$$

$$h_{\alpha\beta} K^\alpha = 0 = h_{\alpha\beta} N^\alpha$$

Transverse metric :

$$h_{\alpha\beta} = \gamma_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$\left. \begin{aligned} h^{\alpha}{}_{\alpha} = 3 \quad ; \quad h^{\mu}{}_{\mu} h^{\nu}{}_{\nu} = h^{\alpha}{}_{\alpha} \quad ; \quad h_{\alpha\beta} K^{\alpha} = 0 = h_{\alpha\beta} N^{\alpha} \end{aligned} \right\} N_{\alpha} K^{\alpha} = -1$$



Transverse space: 2D
 - orthogonal to both
 $\underline{N}$ and $\underline{K}$

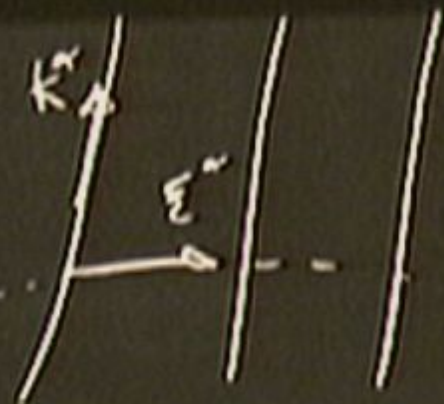
Transverse metric:

$$h_{\mu\nu} = \gamma_{\mu\nu} + K_{\mu} N_{\nu} + N_{\mu} K_{\nu}$$

$$h^{\alpha}_{\alpha} = 2 \quad ; \quad h^{\mu\nu} h^{\rho}_{\nu} = h^{\mu\rho} \quad ; \quad h_{\mu\rho} K^{\rho} = 0 = h_{\mu\rho} N^{\rho}$$

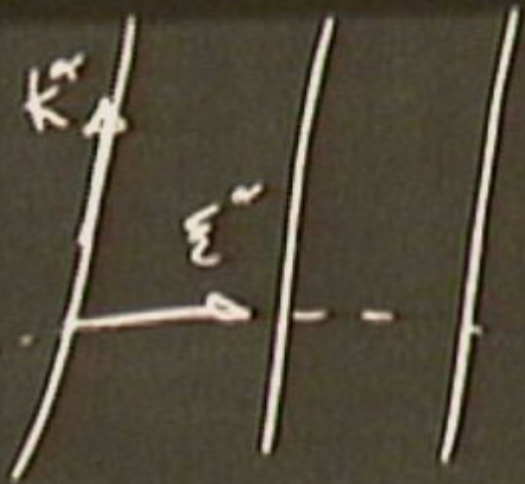
: tangent to hypersurface
 $\underline{N}$: auxiliary null direction

$$\begin{cases} N_{\alpha} N^{\alpha} = 0 \\ N_{\alpha} K^{\alpha} = -1 \end{cases}$$



$$\underline{\epsilon}^{\mu}_{;\rho} K^{\rho} = K^{\mu}_{;\rho} \underline{\epsilon}^{\rho}$$

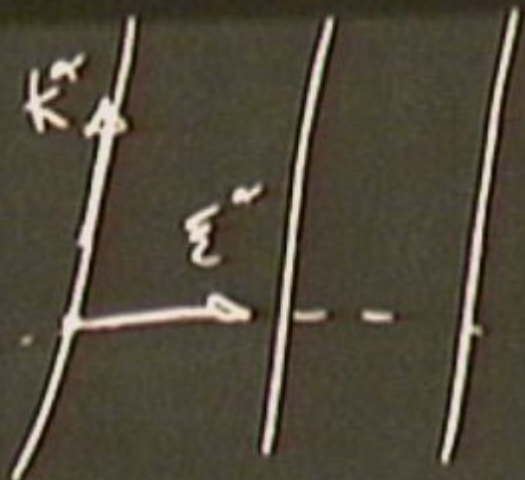
$$h^\alpha = 2 \quad ; \quad h^\mu h^\mu = h^\mu \quad ; \quad h^\mu K = 0 = h^\mu N$$



$$\xi^\alpha K^\beta = K^\alpha \xi^\beta$$

$$\xi_\alpha K^\alpha = 0$$

$$h^\alpha = \partial \quad ; \quad h^\mu h^\nu = h^\nu \quad ; \quad h^\alpha K = 0 = h^\alpha N$$

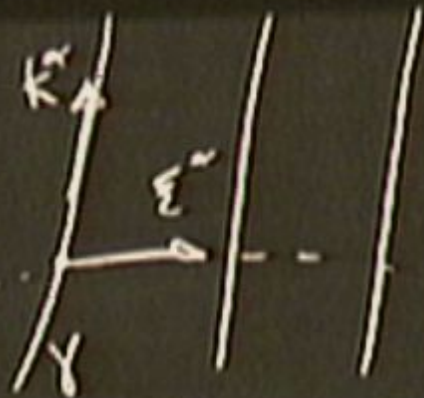


$$\xi^\alpha_{; \beta} K^\beta = K^\alpha_{; \beta} \xi^\beta$$

$$\xi_\alpha K^\alpha = 0$$

$$K^\alpha_{; \beta} K^\beta = 0$$

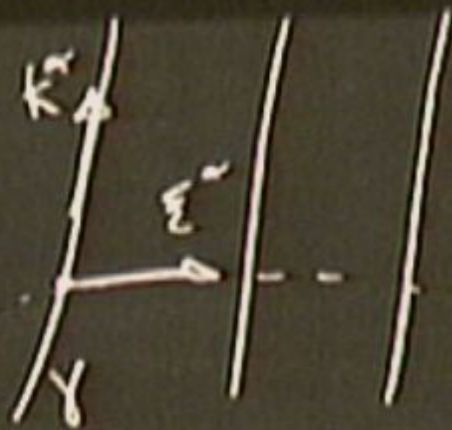
$$h^\mu_\alpha = \delta^\mu_\alpha \quad ; \quad h^\mu_\mu h^\nu_\nu = h^\mu_\mu \quad ; \quad h^\mu_\mu K^\nu = 0 = h^\mu_\mu N^\nu$$



$$\xi^\alpha_{;\beta} K^\beta = K^\alpha_{;\beta} \xi^\beta$$

$$\xi_\alpha K^\alpha = 0$$

$$K^\alpha_{;\beta} K^\beta = 0$$



$$\epsilon^\alpha_{\beta\gamma} k^\beta = k^\alpha_{\beta\gamma} \epsilon^\beta$$

$$\epsilon^\alpha_{\beta\gamma} k^\beta = 0 \quad : \text{ does not take out a component of } \epsilon^\alpha \text{ along } k^\alpha$$

$$k^\alpha_{\beta\gamma} k^\beta = 0$$

- orthogonal to both

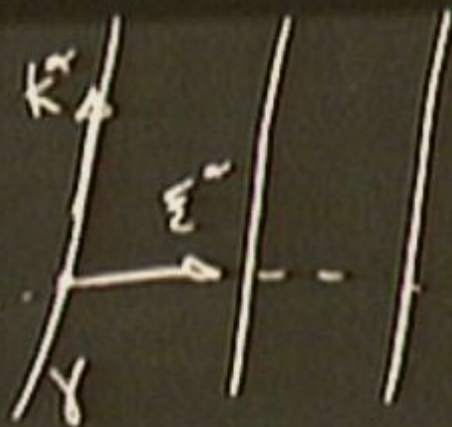
$\underline{N}$ and $\underline{K}$

Transverse metric:

$$h_{\alpha\beta} = \gamma_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$h^{\alpha}{}_{\alpha} = 2 \quad ; \quad h^{\mu}{}_{\mu} h^{\nu}{}_{\nu} = h^{\alpha}{}_{\alpha} \quad ; \quad h_{\alpha\beta} K^{\alpha} = 0 = h_{\alpha\beta} N^{\alpha}$$

$$\begin{cases} N_{\alpha} N^{\alpha} = 0 \\ N_{\alpha} K^{\alpha} = -1 \end{cases}$$



$$\xi^{\alpha}{}_{;\beta} K^{\beta} = K^{\alpha}{}_{;\beta} \xi^{\beta}$$

$$\xi_{\alpha} K^{\alpha} = 0 \quad ; \quad \text{does not rule out a component of } \xi_{\alpha} \text{ along } K_{\alpha}$$

$$K^{\alpha}{}_{;\beta} K^{\beta} = 0$$

Define projected deviation vector:

$$\hat{\xi}^\alpha = h^\alpha_\mu \xi^\mu$$



Define projected deviation vector:

$$\hat{\xi}^\alpha \equiv h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha = h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\gamma^\mu_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha \equiv h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\gamma^\mu_\nu + K^\mu N_\nu + N^\mu K_\nu) \xi^\nu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha = h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\gamma^\mu_\nu + K^\mu N_\nu + N^\mu K_\nu) \xi^\nu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$



Define projected deviation vector:

$$\tilde{\xi}^\alpha \equiv h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\gamma^\alpha_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha \propto K^\alpha =$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha \equiv h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\gamma^\alpha_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha_{sp} K^\beta = (h^\alpha_\mu \xi^\mu)_{sp} K^\beta$$

Define projected deviation vector:

$$\tilde{\xi}^\alpha \equiv h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha)$$

$$= (\tilde{\gamma}_{\mu\nu} + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha_{; \mu} K^\mu = (h^\alpha_\mu \xi^\mu)_{; \mu} K^\mu$$

$$= h^\alpha_\mu \xi^\mu_{; \mu} K^\mu$$

$$\begin{aligned}
 \tilde{\xi}^\alpha &= h^\alpha_\mu \xi^\mu \quad (\text{automatically orthogonal to both } K^\alpha \text{ and } N^\alpha) \\
 &= (\tilde{\gamma}^\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu \\
 &= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)
 \end{aligned}$$

Transverse velocity (guess):

$$\begin{aligned}
 \tilde{\xi}^\alpha_{; \beta} K^\beta &= (h^\alpha_\mu \xi^\mu)_{; \beta} K^\beta \\
 &= h^\alpha_\mu \xi^\mu_{; \beta} K^\beta + h^\alpha_{\mu; \beta} \xi^\mu K^\beta
 \end{aligned}$$

$$= (\tilde{\gamma}_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha_{;\mu} K^\mu = (h^\alpha_\mu \xi^\mu)_{;\mu} K^\mu$$

$$= h^\alpha_\mu \xi^\mu_{;\mu} K^\mu + \underbrace{h^\alpha_{\mu;\mu}}_{K^\alpha_\mu N_\mu + K^\mu N_{\mu;\mu}} \xi^\mu K^\mu$$

$$= (\tilde{\gamma}_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha_{;\rho} K^\rho = (h^\alpha_\rho \xi^\mu)_{;\rho} K^\rho$$

$$= h^\alpha_\mu \xi^\mu_{;\rho} K^\rho + \underbrace{h^\alpha_{\mu;\rho} \xi^\mu K^\rho}_{K^\alpha N_\rho + K^\alpha N_{\rho;\alpha} + N^\alpha_{;\alpha} K_\rho + N^\alpha K_{\rho;\alpha}}$$

$$= (\tilde{\gamma}_\mu + K^\alpha N_\mu + N^\alpha K_\mu) \xi^\mu$$

$$= \xi^\alpha + K^\alpha (N_\mu \xi^\mu)$$

Transverse velocity (guess):

$$\tilde{\xi}^\alpha_{;\mu} K^\mu = (h^\alpha_\mu \xi^\mu)_{;\mu} K^\mu$$

$$= h^\alpha_\mu \xi^\mu_{;\mu} K^\mu + \underbrace{h^\alpha_{\mu;\mu}}_{\text{cancel}} \xi^\mu K^\mu$$

$$\cancel{K^\alpha N_\mu} + K^\alpha N_{\mu;\mu} + N^\alpha_{;\mu} K^\mu + \cancel{N^\alpha K_{\mu;\mu}}$$

$$= h^{\alpha}_{\mu} \xi^{\mu}_{;\rho} K^{\rho} + \underbrace{h^{\alpha}_{\rho;\mu} \xi^{\mu} K^{\rho}}$$

$$\cancel{K^{\alpha}_{;\mu} N^{\mu}} + K^{\alpha} N^{\mu}_{;\rho} + N^{\alpha}_{;\mu} K^{\mu} + \cancel{N^{\alpha}_{;\rho} K^{\rho}}$$

$$\tilde{\xi}^{\alpha}_{;\rho} K^{\rho} = h^{\alpha}_{\mu} \xi^{\mu}_{;\rho} K^{\rho}$$

$$= h^\alpha{}_\mu \xi^\mu{}_{;\rho} K^\rho + \underbrace{h^\alpha{}_{\rho;\mu} \xi^\mu K^\rho}$$

$$\cancel{K^\alpha N_\rho} + \cancel{K^\alpha N_{\rho;\mu}} + N^\alpha{}_\mu K^\mu + \cancel{N^\alpha{}_\mu K_{\rho;\mu}}$$

$$\tilde{\xi}^\alpha{}_{;\rho} K^\rho = h^\alpha{}_\mu \xi^\mu{}_{;\rho} K^\rho + K^\alpha (N_{\rho;\mu} \xi^\mu K^\rho)$$

$$\cancel{K^\alpha N_\rho} + K^\alpha N_{\rho;\beta} + \cancel{N^\alpha K_\rho} + \cancel{N^\alpha K_{\rho;\beta}}$$

$$\tilde{\xi}^\alpha_{;\rho} K^\rho = h^\alpha_\mu \tilde{\xi}^\mu_{;\rho} K^\rho + K^\alpha (N_{\rho;\beta} \tilde{\xi}^\rho K^\beta)$$

Transverse velocity correct 1:

$$(\tilde{\xi}^\alpha_{;\rho} K^\rho)^\sim$$



$$\tilde{\xi}^{\alpha}_{; \rho} K^{\rho} = h^{\alpha}_{\mu} \tilde{\xi}^{\mu}_{; \rho} K^{\rho} + K^{\alpha} (N_{\rho; \beta} \tilde{\xi}^{\rho} K^{\beta})$$

Transverse velocity correct |:

$$(\tilde{\xi}^{\alpha}_{; \rho} K^{\rho})^{\sim} \equiv h^{\alpha}_{\mu} \tilde{\xi}^{\mu}_{; \rho} K^{\rho}$$

$$\tilde{\xi}^{\alpha}_{; \rho} K^{\rho} = h^{\alpha}_{\mu} \tilde{\xi}^{\mu}_{; \rho} K^{\rho} + K^{\alpha} (N_{\mu; \rho} \xi^{\mu} K^{\rho})$$

Transverse velocity correct |:

$$\begin{aligned} (\tilde{\xi}^{\alpha}_{; \rho} K^{\rho})^{\sim} &\equiv h^{\alpha}_{\mu} \tilde{\xi}^{\mu}_{; \rho} K^{\rho} \\ &= h^{\alpha}_{\mu} h^{\mu}_{\nu} \xi^{\nu}_{; \rho} K^{\rho} \end{aligned}$$

$$\tilde{\xi}^{\alpha}_{; \rho} K^{\rho} = h^{\alpha}_{\mu} \xi^{\mu}_{; \rho} K^{\rho} + K^{\alpha} (N_{\rho; \beta} \xi^{\rho} K^{\beta})$$

Transverse velocity correct |:

$$\begin{aligned} (\tilde{\xi}^{\alpha}_{; \rho} K^{\rho})^{\sim} &= h^{\alpha}_{\mu} \tilde{\xi}^{\mu}_{; \rho} K^{\rho} \\ &= h^{\alpha}_{\mu} h^{\mu}_{\nu} \xi^{\nu}_{; \rho} K^{\rho} \\ &= h^{\alpha}_{\mu} \xi^{\mu}_{; \rho} K^{\rho} \end{aligned}$$

$$(\sum_{\alpha} k_{\alpha}^{\alpha})^2 = h^2 p^2$$



$$(\tilde{\xi}_{\alpha}^{\mu} k^{\rho})^{\sim} = \hbar^{\mu}{}_{\rho} k^{\nu}{}_{;\rho} \xi^{\rho}$$

$$B_{\alpha\rho} = k_{\alpha;\rho}$$

$$(\tilde{\xi}_{\sigma}^{\mu} k^{\rho})^{\sim} = h^{\mu}_{\rho} k^{\nu}_{\sigma} \xi^{\rho}$$

$$B_{\sigma\rho} = K_{\sigma\rho}$$

$$= h^{\mu}_{\rho} B^{\nu}_{\sigma} \xi^{\rho}$$

$$(\tilde{\xi}_{\sigma}^{\mu} k^{\rho})^{\sim} = h^{\mu}_{\nu} k^{\nu}_{\sigma\rho} \xi^{\rho}$$

$$B_{\sigma\rho} = K_{\sigma\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\sigma\rho} \xi^{\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\sigma\rho} (\tilde{\xi}^{\rho})$$

$$(\tilde{\xi}_{\alpha}^{\mu} k^{\rho})^{\sim} = h^{\mu}_{\nu} k^{\nu}_{\alpha\rho} \xi^{\rho}$$

$$B_{\alpha\rho} = k_{\alpha\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\alpha} \xi^{\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\alpha} (\tilde{\xi}^{\rho} - k^{\rho} \Lambda_{\alpha} \tilde{\xi}^{\nu})$$

$$(\tilde{\xi}_{\alpha}^{\mu} k^{\rho})^{\sim} = h^{\mu}_{\nu} k^{\nu}_{\alpha\rho} \xi^{\rho}$$

$$B_{\alpha\rho} = k_{\alpha\rho}$$

$$B_{\alpha\rho} k^{\rho} = 0$$

$$= h^{\mu}_{\nu} B^{\nu}_{\alpha} \xi^{\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\alpha} (\tilde{\xi}^{\rho} - k^{\rho} \tilde{\xi}^{\rho})$$

$$(\tilde{\xi}_{; \rho} K^{\rho})^{\sim} = h^{\mu}{}_{\rho} K^{\rho}{}_{; \rho} \tilde{\xi}^{\rho}$$

$$B_{\alpha\rho} = K_{\alpha; \rho}$$

$$B_{\alpha\rho} K^{\rho} = 0$$

$$= h^{\mu}{}_{\rho} B^{\rho}{}_{\rho} \tilde{\xi}^{\rho}$$

$$= h^{\mu}{}_{\rho} B^{\rho}{}_{\rho} (\tilde{\xi}^{\rho} - \cancel{K^{\rho}{}_{\nu} \tilde{\xi}^{\nu}})$$

$$= h^{\mu}{}_{\rho} B^{\rho}{}_{\rho} \tilde{\xi}^{\rho}$$

$$= h^{\mu}{}_{\rho} B^{\rho}{}_{\rho} h^{\rho}{}_{\nu} \tilde{\xi}^{\nu}$$

$$(\tilde{\xi}_{\sigma}^{\mu} k^{\rho})^{\sim} = h^{\mu}_{\nu} k^{\nu}_{\sigma\rho} \xi^{\rho}$$

$$B_{\sigma\rho} = k_{\sigma\rho}$$

$$B_{\sigma\rho} k^{\rho} = 0$$

$$= h^{\mu}_{\nu} B^{\nu}_{\sigma} \xi^{\rho}$$

$$= h^{\mu}_{\nu} B^{\nu}_{\sigma} (\tilde{\xi}^{\rho} - k^{\rho} \cancel{\nu_{\sigma}} \tilde{\xi}^{\nu})$$

$$= h^{\mu}_{\nu} B^{\nu}_{\sigma} \tilde{\xi}^{\rho}$$

$$= \underbrace{h^{\mu}_{\nu} B^{\nu}_{\sigma} h^{\rho}_{\nu}}_{\sim} \tilde{\xi}^{\rho}$$

Define projective vector

$$(\xi^\alpha; k_P)^\sim = \tilde{B}^\alpha_P \xi^P$$

(arbitrarily assigned to follow k)

$$= \sum_{P=1}^N x_P (h^\alpha_P + N^{\alpha P})$$

k

Transverse velocity (given)

$$v^\alpha; k_P = (h^\alpha_P + N^{\alpha P})$$

Value projector function vector

$$(\tilde{\xi}^\alpha; K_P)^\sim = \tilde{B}_P^\alpha \tilde{\xi}^P$$

$$\tilde{B}_{\alpha p} = h^\mu_\alpha h^\nu_p B_{\mu\nu} = h^\mu_\alpha h^\nu_p K_{p;\mu\nu}$$

$$(\xi^\alpha;_{ip} K^P)^\sim = \tilde{B}^\alpha_\beta \xi^\beta$$

$$\tilde{B}_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta B_{\mu\nu} = h^\mu_\alpha h^\nu_\beta K_{\mu\nu}$$

$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:

Traceless part (symmetric)

$$(\tilde{h}^\alpha_\beta \xi^\beta)_{ip} K^P$$

$$= h^\alpha_\nu \xi^\nu_{ip} K^P$$

$$(\xi^\alpha_{;p} K^p)^\sim = \tilde{B}^\alpha_p \xi^\beta$$

$$\tilde{B}_{\alpha p} = h^\nu_\alpha h^\nu_p B_{\nu\omega} = h^\nu_\alpha h^\nu_p K_{\nu;\omega}$$

$\tilde{B}_{\alpha p}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha p} = \frac{1}{2} h_{\alpha p} \Theta$$

$$\xi^\alpha_{;p} K^p = (h^\alpha_p \xi^\nu)_{;p} K^p$$

$$= h^\alpha_\nu \xi^\nu_{;p} K^p$$

$$(\xi^\alpha;_{ip} K^P)^\sim = \tilde{B}^\alpha_\beta \xi^\beta$$

$$\tilde{B}_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta B_{\mu\nu} = h^\mu_\alpha h^\nu_\beta K_{\mu\nu}$$

$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$_{ip} K^P = (h^\mu_{ip} \xi^\mu)_{ip} K^P$$

$$= h^\mu_\nu \xi^\mu_{ip} K^P$$

$$(\tilde{\xi}^\alpha; \kappa^\rho)^\sim = \tilde{B}^\alpha_\rho \tilde{\xi}^\rho$$

$$\tilde{B}_{\alpha\rho} = h^\mu_\alpha h^\nu_\rho B_{\mu\nu} = h^\mu_\alpha h^\nu_\rho \kappa_{\mu\nu}$$

$\tilde{B}_{\alpha\rho}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha\rho} = \frac{1}{2} h_{\alpha\rho} \Theta + \sigma_{\alpha\rho} + \omega_{\alpha\rho}$$

$$\Theta = \tilde{B}^\alpha_\alpha \quad (= \text{expansion})$$

$$\sigma_{\alpha\rho} = \tilde{B}_{(\alpha\rho)}$$

$$(\tilde{\xi}^\alpha; k^p)^\sim = \tilde{B}^\alpha_p \tilde{\xi}^p$$

$$\tilde{B}_{\alpha p} = h^\mu_\alpha h^\nu_p B_{\mu\nu} = h^\mu_\alpha h^\nu_p K_{\mu\nu}$$

$\tilde{B}_{\alpha p}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha p} = \frac{1}{2} h_{\alpha p} \Theta + \sigma_{\alpha p} + \omega_{\alpha p}$$

$$\Theta = \tilde{B}^\alpha_\alpha \quad (= \text{expansion})$$

$$\sigma_{\alpha p} = \tilde{B}_{(\alpha p)} - \frac{1}{2} h_{\alpha p} \Theta \quad : \text{shear}$$

$$\omega_{\alpha p} = \tilde{B}_{[\alpha p]} \quad : \text{rotation}$$

$$(\tilde{\xi}^\alpha; k^p)^\sim = h^\alpha_p \tilde{\xi}^p; k^p$$

$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\Theta = \tilde{B}^{\alpha}_{\alpha} \quad (= \text{expansion})$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta \quad : \text{shear}$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad : \text{rotation.}$$

$$\Theta = \gamma^{\alpha\beta} \tilde{B}_{\alpha\beta} = \gamma^{\alpha\beta} h^{\mu}_{\alpha} h^{\nu}_{\beta} K_{\mu\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\Theta = \tilde{B}^{\alpha}_{\alpha} \quad (= \text{expansion})$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta \quad : \text{shear}$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad : \text{rotation.}$$

$$\begin{aligned} \Theta &= \gamma^{\alpha\beta} \tilde{B}_{\alpha\beta} = \gamma^{\alpha\beta} h^{\mu}_{\alpha} h^{\nu}_{\beta} K_{\mu\nu} \\ &= h^{\mu\rho} h^{\nu}_{\rho} K_{\mu\nu} \end{aligned}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\Theta = \tilde{B}^{\alpha}_{\alpha} \quad (\text{expansion})$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta \quad \text{shear}$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad \text{rotation}$$

$$\begin{aligned} \Theta &= \gamma^{\alpha\beta} \tilde{B}_{\alpha\beta} = \gamma^{\alpha\beta} h^{\mu}_{\alpha} h^{\nu}_{\beta} K_{\mu\nu} \\ &= h^{\mu\beta} h^{\nu}_{\beta} K_{\mu\nu} \\ &= h^{\mu\nu} K_{\mu\nu} \end{aligned}$$

$$\Theta = \mathcal{J}^{\alpha\beta} \tilde{B}_{\alpha\beta} = \mathcal{J}^{\alpha\beta} h_{\alpha}^{\mu} h_{\beta}^{\nu} K_{\mu;\nu}$$

$$= h^{\mu\rho} h_{\rho}^{\nu} K_{\mu;\nu}$$

$$= h^{\mu\nu} K_{\mu;\nu} = (\mathcal{J}^{\mu\nu} + K^{\mu} N^{\nu} + N^{\mu} K^{\nu}) K_{\mu;\nu}$$

$$\Theta = \eta^{\alpha\beta} \tilde{B}_{\alpha\beta} = \eta^{\alpha\beta} h^\mu_{\alpha} h^\nu_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\beta} h^\nu_{\beta} K_{\mu;\nu}$$

$$= h^{\mu\nu} K_{\mu;\nu} = (\eta^{\mu\nu} + \cancel{K^\mu K^\nu} + \cancel{N^\mu N^\nu}) K_{\mu;\nu}$$

$$\Theta = \mathcal{J}^{\alpha\beta} \tilde{B}_{\alpha\beta} = \mathcal{J}^{\alpha\beta} h^\mu{}_\alpha h^\nu{}_\beta K_{\mu;\nu}$$

$$= h^{\mu\beta} h^\nu{}_\beta K_{\mu;\nu}$$

$$= h^{\mu\nu} K_{\mu;\nu} = (\mathcal{J}^{\mu\nu} + \cancel{K^\mu K^\nu} + \cancel{N^\mu X^\nu}) K_{\mu;\nu}$$

$$\boxed{\Theta = K^\alpha{}_{;\alpha}}$$

$$\Theta = \eta^{\alpha\beta} \tilde{B}_{\alpha\beta} = \eta^{\alpha\beta} h^\mu_\alpha h^\nu_\beta k_{\mu;\nu}$$

$$= h^{\mu\beta} h^\nu_\beta k_{\mu;\nu}$$

$$= h^{\mu\nu} k_{\mu;\nu} = (\eta^{\mu\nu} + \cancel{k^\mu N^\nu} + \cancel{N^\mu k^\nu}) k_{\mu;\nu}$$

$$\boxed{\Theta = k^\alpha_{;\alpha}}$$

$$k^\mu k_{\mu;\nu} = \frac{1}{2} (k^\mu k_\mu)_{;\nu}$$

$$\Theta = \eta^{\alpha\mu} \tilde{B}_{\alpha\mu} = \eta^{\alpha\mu} h^\mu_\alpha h^\nu_\mu k_{\nu;\nu}$$

$$= h^{\mu\mu} h^\nu_\mu k_{\nu;\nu}$$

$$= h^{\mu\mu} k_{\mu;\nu} = (\eta^{\mu\nu} + \cancel{k^\mu N^\nu} + \cancel{N^\mu k^\nu}) k_{\mu;\nu}$$

$$\boxed{\Theta = k^\alpha_{;\alpha}}$$

$$k^\mu k_{\mu;\nu} = \frac{1}{2} (k^\mu k_\mu)_{;\nu} =$$

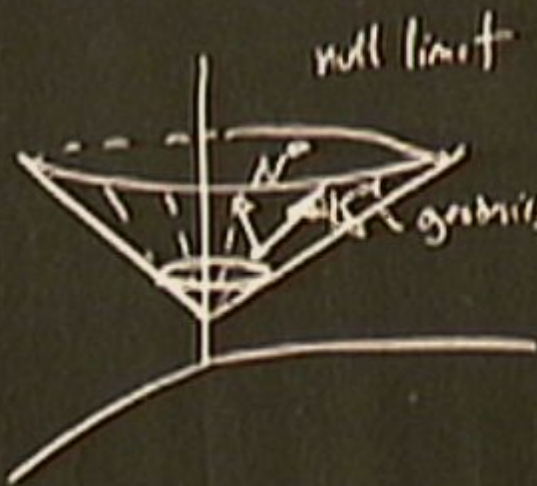
$$\Theta = \eta^{\alpha\mu} \tilde{B}_{\alpha\mu} = \eta^{\alpha\mu} h^\mu_\alpha h^\nu_\mu k_{\nu;\nu}$$

$$= h^{\mu\mu} h^\nu_\mu k_{\nu;\nu}$$

$$= h^{\mu\mu} k_{\mu;\nu} = (\eta^{\mu\nu} + \cancel{k^\mu k^\nu} + \cancel{N^\mu k^\nu}) k_{\mu;\nu}$$

$$\boxed{\Theta = k^\alpha_{;\alpha}}$$

$$k^\mu k_{\mu;\nu} = \frac{1}{2} (k^\mu k_\mu)_{;\nu} = 0$$



null limit

geodesics + hypersurface (intrinsically 2D)

K^μ : tangent to geodesics

N^μ : normal to hypersurface

K^μ : tangent to hypersurface

N^μ : null direction

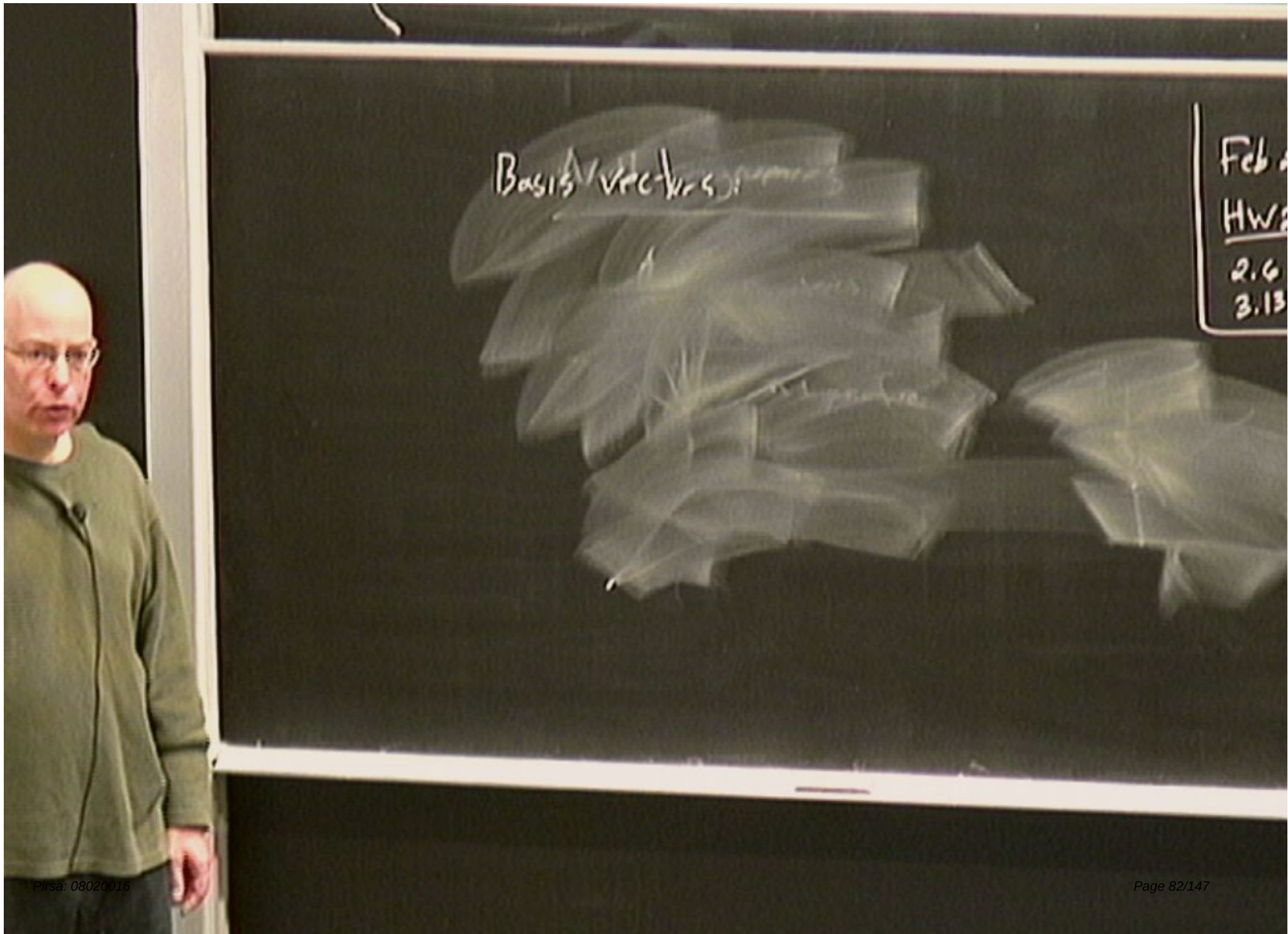
Transverse space: 2D
- orthogonal to both
 N^μ and K^μ

Transverse metric:

$$h_{\mu\nu} = g_{\mu\nu} + K_\mu N_\nu + N_\mu K_\nu$$

$$h^\mu{}_\mu = 2$$

$$h_{\mu\nu} N^\mu N^\nu = 0$$



Basis Vectors

Feb 4

HW

2.6

3.13

Basis vectors:

$$K^\alpha, N^\alpha$$

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

Feb 20: no

HW2 3 Feb

2.6 #1, 2

3.13 #1

Basis vectors:

$$K^\alpha, N^\alpha, e_1^\alpha, e_3^\alpha$$

$$K_\alpha K^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha K^\alpha = -1$$

Feb 20: no

HW2: F

2.6 #1,

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_2^\alpha, e_3^\alpha$$

$$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha k^\alpha = -1$$

$$k_\alpha e^\alpha_A$$

HW2: R

2.6 #1.

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_1^\alpha, e_2^\alpha$$

$$k_\alpha k^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha k^\alpha = -1$$

$$k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha$$

Feb 20: no class

HW2: Feb

2.6 #1, 3

3.13 #1

Basis vectors:

$$K^\alpha, N^\alpha, e^\alpha_1, e^\alpha_2$$

$$K_\alpha K^\alpha = 0 = N_\alpha N^\alpha$$

$$N_\alpha K^\alpha = -1$$

$$K_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A$$

$$\exists \alpha \beta e^\alpha_A e^\beta_B = \delta_{AB}$$

Feb 20: no class

HW2: Feb

2.6 #1, 2

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e^\alpha, e^\alpha_3$$

$$\left\{ \begin{array}{l} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \end{array} \right.$$

$$\left\{ \begin{array}{l} k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB} \end{array} \right.$$

Feb 20: no class

HW2: Feb

2.6 #1, 3

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_A^\alpha, e_3^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ g_{\alpha\beta} e_A^\alpha e_B^\beta = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e_A^\alpha$$

Feb

Hw

2.6

3.13

Basis vectors:

$$k^\alpha, N^\alpha, e^\alpha, e_3^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha = 0 = N_\alpha e^\alpha \\ \text{for } e^\alpha e^\beta = \delta_{\alpha\beta} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^\alpha e^\alpha$$

$$k_\alpha \xi^\alpha = -b$$

Feb 20: n

HW2:

2.6 #1

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e_A^\alpha, e_3^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ \text{for } e_A^\alpha e_B^\alpha = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e_A^\alpha$$

$$k_\alpha \xi^\alpha = -b$$

$$N_\alpha \xi^\alpha = -a$$

Feb 2011

HW2:

2.6 #1

3.13 #1

Basis vectors:

$$k^\alpha, N^\alpha, e^\alpha_1, e^\alpha_2$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A \\ g_{\alpha\beta} e^\alpha_A e^\beta_B = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e^\alpha_A$$

$$k_\alpha \xi^\alpha = -b = 0$$

$$N_\alpha \xi^\alpha = -a$$

Feb

Hw

2.0

3.0

Basis vectors:

$$k^\alpha, N^\alpha, e_A^\alpha, e_B^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ \text{for } e_A^\alpha e_B^\alpha = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + b N^\alpha + c^A e_A^\alpha$$

$$k_\alpha \xi^\alpha = -b = 0$$

$$(N_\alpha \xi^\alpha) = -a$$

Basis vectors:

$$k^\alpha, N^\alpha, e_A^\alpha, e_B^\alpha$$

$$\begin{cases} k_\alpha k^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha k^\alpha = -1 \\ k_\alpha e_A^\alpha = 0 = N_\alpha e_A^\alpha \\ \text{for } e_A^\alpha e_B^\alpha = \delta_{AB} \end{cases}$$

$$\xi^\alpha = a k^\alpha + \cancel{b N^\alpha} + c_A e_A^\alpha$$

$$k_\alpha \xi^\alpha = -b = 0$$

$$\boxed{N_\alpha \xi^\alpha} = -a$$

Basis vectors:

$$K^\alpha, N^\alpha, e^\alpha, e_3$$

$$\begin{cases} K_\alpha K^\alpha = 0 = N_\alpha N^\alpha \\ N_\alpha K^\alpha = -1 \\ K_\alpha e^\alpha = 0 = N_\alpha e^\alpha \\ \text{for } e^\alpha e^\beta = \delta^{\alpha\beta} \end{cases}$$

$$\xi^\alpha = a K^\alpha + \cancel{b N^\alpha} + \underbrace{c_\alpha e^\alpha}_{\xi^\alpha}$$

$$K_\alpha \xi^\alpha = -b = 0$$

$$(N_\alpha \xi^\alpha) = -a$$

Feb 20, no cl

HW2: Feb

2.6 + 1, 3,

3.13 + 1

36
Abgang null geodesics in Schwarzschild spacetime

Outgoing null geodesics in Schwarzschild spacetime

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

36
Outgoing null geodesics in Schwarzschild spacetime

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= -f (dt^2 - f^{-2} dr^2) + r^2 d\Omega^2 \end{aligned}$$

Obtaining null geodesics in Schwarzschild spacetime

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= -f (dt^2 - f^{-2} dr^2) + r^2 d\Omega^2 \\ &= -f (dt - f^{-1} dr) (dt + f^{-1} dr) + r^2 d\Omega^2 \end{aligned}$$

Obtaining null geodesics in Schwarzschild spacetime

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= -f (dt^2 - f^{-2} dr^2) + r^2 d\Omega^2 \\ &= -f (dt - f^{-1} dr) (dt + f^{-1} dr) + r^2 d\Omega^2 \end{aligned}$$

Obtaining null geodesics in Schwarzschild spacetime

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= -f (dt^2 - f^{-2} dr^2) + r^2 d\Omega^2 \\ &= -f \underbrace{(dt - f^{-1} dr)}_{du} \underbrace{(dt + f^{-1} dr)}_{dv} + r^2 d\Omega^2 \end{aligned}$$

Outgoing null geodesics in Schwarzschild spacetime

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= -f (dt^2 - f^{-2} dr^2) + r^2 d\Omega^2 \\ &= -f \underbrace{\left(dt - f^{-1} dr \right)}_{du} \underbrace{\left(dt + f^{-1} dr \right)}_{dv} + r^2 d\Omega^2 \end{aligned}$$

$$= -f du dv + r^2 d\Omega^2$$

$$u = t - \int \frac{dr}{f} \quad v = t + \int \frac{dr}{f}$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 =$$

k^μ is tangent to geodesics

normal to hypersurface

$\Rightarrow$ tangent to hypersurface

N^μ must be a null vector

$$g_{\mu\nu} N^\mu N^\nu = 0$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dr du$$

hypersurface (involving 2D)

k^μ tangent to geodesics

normal to hypersurface

tangent to hypersurface

N^μ null vector null condition

$$g_{\mu\nu} N^\mu N^\nu = 0$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dv du = 0$$

geodesics are hypersurface (independently 2D)

k^μ is tangent to geodesics

normal to hypersurface

tangent to hypersurface

N must keep null condition

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dv du = 0$$

$$v = \text{const} : \quad \frac{dv}{dr} = 0 \Rightarrow dt = -f^{-1} dr$$

normal to hypersurface $k \cdot k = 0$

tangent to hypersurface

N null vector

$$N \cdot N = 0$$

$$k \cdot k = 0$$

$$k \cdot N = 0 = k \cdot N^{\mu} \partial_{\mu}$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dr du = 0$$

$$v = \text{const} : \frac{dr}{du} = 0 \Rightarrow dt = -f' dr > 0 \Rightarrow dr < 0$$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f(r)dr^2 = 0$$

$$V = \text{const} : \partial V = 0 \Rightarrow dt = -f'(r)dr > 0 \Rightarrow dr < 0$$

increasing

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dr du = 0$$

$V = \text{const} : \dot{V} = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$
incoming

$U = \text{const} : \dot{U} = 0 \Rightarrow dt = f^{-1} dr > 0$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dr du = 0$$

$V = \text{const}$: $\partial V = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$
incoming

$U = \text{const}$: $\partial U = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$

$$ds^2 = -f dr du = 0$$

$$V = \text{const} : dv = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$$

incoming $\times$

$$U = \text{const} : du = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$$

outgoing geodesics $\checkmark$

Radial geodesics: $d\theta = d\varphi = 0$

$$ds^2 = -f dr du = 0$$

$V = \text{const} : dv = 0 \Rightarrow dt = -f^{-1} dr > 0 \Rightarrow dr < 0$
incoming $\times$

$U = \text{const} : du = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$
outgoing geodesics $\checkmark$

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0)$$

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$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{1}{f}, 1, 0, 0 \right)$$

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 outgoing geodesics $\checkmark$

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{1}{f}, 1, 0, 0 \right) = \left(\frac{\dot{t}}{\dot{r}}, 1, 0, 0 \right)$$

$V = \text{const} : pV = 0 \Rightarrow \text{incoming } X$

$U = \text{const} : dU = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$
 outgoing geodesics ✓

$$k^\mu = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{1}{f}, 1, 0, 0 \right) = \left(\frac{dt}{dr}, 1, 0, 0 \right)$$

choice $\lambda = r$

$$0 = h_{\mu\nu} N^\mu$$

$$B_{\mu\nu} k^\mu = 0$$

$$= h^\mu_\nu B^\nu_\mu \xi^\mu$$

$$= h^\mu_\nu B^\nu_\mu \left(\tilde{\xi}^\mu - k^\mu \cancel{\nu_0 \xi^\nu} \right)$$

$$= h^\mu_\nu B^\nu_\mu \tilde{\xi}^\mu$$

$$= h^\mu_\nu B^\nu_\mu h^\mu_\alpha \tilde{\xi}^\alpha$$

const : $\partial V = 0 \Rightarrow dt = -\frac{dr}{f}$ or > 0 and < 0
 incoming $\times$

const : $\partial U = 0 \Rightarrow dt = f^{-1} dr > 0 \Rightarrow dr > 0$
 outgoing geodesics $\checkmark$

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{1}{f}, 1, 0, 0 \right) = \left(\frac{\partial t}{\partial r}, 1, 0, 0 \right)$$

choose $\lambda \equiv r \equiv$ affine parameter.

const : $\partial V = 0 \Rightarrow \partial T = -\frac{1}{f}$ or > 0 or < 0

incoming $\times$

const : $\partial U = 0 \Rightarrow \partial t = f^{-1} \partial r > 0 \Rightarrow \partial r > 0$

outgoing geodesics ✓

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{1}{f}, 1, 0, 0 \right) = \left(\frac{\partial t}{\partial r}, 1, 0, 0 \right)$$

choose $\lambda \equiv r \equiv$ affine parameter.

$$U = \text{const} : dU = 0 \Rightarrow dt = f(r) dr > 0$$

along geodesics ✓

$$k^\alpha = (\dot{t}, \dot{r}, 0, 0) = \dot{r} \left(\frac{t}{r}, 1, 0, 0 \right) = \left(\frac{dt}{dr}, 1, 0, 0 \right)$$

choose $\lambda = r$: affine parameter.

$\frac{dt}{dr} = 14$

$$k^\alpha = \left(\frac{t}{r}, 1, 0, 0 \right)$$

choose $\lambda \approx 1$ as offline parameter.

5-14

$$K^x = \begin{pmatrix} \frac{1}{x} & 1 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} -1, \end{pmatrix}$$

choose $\lambda \in \mathbb{R}$ a affine parameter.

5-14

$$K^2 \left(\frac{1}{x}, 1, 0, 0 \right)$$

$$K^2 \left(\frac{1}{x}, 1, 0 \right)$$

choose $\lambda \approx \tau$: offline parameter.

5-14

$$K^x = \begin{pmatrix} \frac{1}{f} & 1 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} -1 & f^{-1} & 0 & 0 \end{pmatrix}$$

$$K^x = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_x = \left(-1, f^{-1}, 0, 0 \right)$$

$$= -\partial_x \left(t - \frac{1}{f} \right)$$

$$K^x = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$\left(-1, f^{-1}, 0, 0 \right)$$

$$= -\partial_x U = -\partial_x \left(1 - \left(\frac{y}{f} \right)^2 \right)$$

$$K^x = \left(\frac{1}{f}, 1, 0, 0 \right)$$

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$$= -\partial_x U = -\partial_x \left(1 - \left(\frac{y}{f} \right)^2 \right)$$

$$K^x = \left(\frac{1}{f}, f, 0, 0 \right)$$

$$K_x = \left(-1, f^{-1}, 0, 0 \right)$$

$$= -\partial_x U = -\partial_x \left(+ - \left(\frac{y}{f} \right) \right)$$

$$K^x = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_x = \left(-1, f^{-1}, 0, 0 \right)$$

$$= -\partial_x U = -\partial_x \left(1 - \left(\frac{y}{f} \right)^2 \right)$$

$$N_\alpha =$$

$$K^x = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_x = (-1, f^{-1}, 0, 0)$$

$$= -\partial_x U = -\partial_x \left(+ - \left(\frac{y}{f} \right) \right)$$

$$N_\alpha = -\frac{1}{2} f \partial_\alpha \sqrt{\quad}$$

$$= -\frac{1}{2} f \partial_\alpha \left(+ + \left(\frac{y}{f} \right) \right)$$

$$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_\alpha = \left(-1, f^{-1}, 0, 0 \right)$$

$$= -\partial_\alpha U = -\partial_\alpha \left(+ - \int \frac{dx}{f} \right)$$

$$\begin{aligned} N_\alpha &= -\frac{1}{2} f \partial_\alpha V \\ &= -\frac{1}{2} f \partial_\alpha \left(+ - \int \frac{dx}{f} \right) \\ &= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right) \end{aligned}$$

$$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_\alpha = (-1, f^{-1}, 0, 0)$$

$$= -\partial_\alpha U = -\partial_\alpha \left(+ - \left(\right) \right)$$

$$N_\alpha = -\frac{1}{2} f \partial_\alpha V$$

$$= -\frac{1}{2} f \partial_\alpha \left(+ + \left(\frac{1}{f} \right) \right)$$

$$= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right)$$

$N_\alpha K^\alpha$

$$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_\alpha = (-1, f^{-1}, 0, 0)$$

$$= -\partial_\alpha U = -\partial_\alpha \left(+ - \left(\frac{dr}{f} \right) \right)$$

$$\left. \begin{aligned} N_\alpha &= -\frac{1}{2} f \partial_\alpha V \\ &= -\frac{1}{2} f \partial_\alpha \left(+ + \left(\frac{dr}{f} \right) \right) \\ &= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right) \end{aligned} \right\} \begin{aligned} N_\alpha N^\alpha &= 0 \\ N_\alpha K^\alpha &= -1 \end{aligned}$$

$$h_{\alpha p} = z_{\alpha p} + K_{\alpha} N_p + N_{\alpha} K_p$$

$$h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$= \text{diag}(0, 0, r^2, r^2 \sin^2 \theta)$$

$$h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$= \text{diag}(0, 0, r^2, r^2 \sin^2 \theta)$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}^{\mu} h_{\beta}^{\nu} K_{\mu\nu}$$

$$h_{\alpha\beta} = \mathcal{Z}_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$= \text{diag}(0, 0, r^2, r'^2 \sin^2 \theta)$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}^{\mu} h_{\beta}^{\nu} K_{\mu\nu}$$

$$= \text{diag}(0, 0, r, r \sin^2 \theta)$$

$$\begin{aligned}
 h_{ap} &= \mathcal{J}_{ap} + K_{\alpha} N_p + N_{\alpha} K_p \\
 &= \text{diag}(0, 0, r^2, r'^2 \sin^2 \theta) \\
 \tilde{B}_{ap} &= h_{\alpha}^{\mu} h_{\mu}^{\nu} K_{\nu, \omega} \\
 &= \text{diag}(0, 0, r, r \sin^2 \theta)
 \end{aligned}
 \left\{ \begin{array}{l} \boxed{\tilde{B}_{ap} = \frac{1}{r} h_{ap}} \end{array} \right.$$

$$\begin{aligned}
 h_{ap} &= z_{ap} + K_{\alpha} N_p + N_{\alpha} K_p \\
 &= \text{diag}(0, 0, r^2, r'^2 \sin^2 \theta) \\
 \tilde{B}_{ap} &= h_{\alpha}^{\mu} h_{\mu}^{\nu} K_{\nu, \sigma} \\
 &= \text{diag}(0, 0, r, r \sin^2 \theta)
 \end{aligned}
 \left\{ \begin{array}{l} \boxed{\tilde{B}_{ap} = \frac{1}{r} h_{ap}} \\ \theta = \frac{2}{r} \end{array} \right.$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$= \text{diag}(0, 0, r^2, r'^2 \sin^2 \theta)$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}^{\mu} h_{\beta}^{\nu} K_{\mu\nu}$$

$$= \text{diag}(0, 0, r, r \sin^2 \theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + K_{\alpha} N_{\beta} + N_{\alpha} K_{\beta}$$

$$= \delta \ln g (0, 0, r^2, r'^2 \sin^2 \theta)$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}^{\mu} h_{\beta}^{\nu} K_{\mu\nu}$$

$$= \delta \ln g (0, 0, r, r \sin^2 \theta)$$

$$\boxed{\tilde{B}_{\alpha\beta} = \frac{1}{r} h_{\alpha\beta}}$$

$$\theta = \frac{2}{r}$$

$$= \frac{1}{4\pi r^2} \frac{\partial}{\partial \lambda} (4\pi r^2)$$

$$\begin{aligned}
 h_{\mu\nu} &= \gamma_{\mu\nu} + K_{\mu} N_{\nu} + N_{\mu} K_{\nu} \\
 &= \text{diag}(0, 0, r^2, r^2 \sin^2 \theta) \\
 \tilde{B}_{\mu\nu} &= h_{\mu}^{\rho} h_{\nu}^{\sigma} K_{\rho\sigma} \\
 &= \text{diag}(0, 0, r, r \sin^2 \theta)
 \end{aligned}
 \left\{
 \begin{array}{l}
 \boxed{\tilde{B}_{\mu\nu} = \frac{1}{r} h_{\mu\nu}} \\
 \theta = \frac{\pi}{2} \\
 = \frac{1}{4\pi r^2} \frac{d}{d\lambda} (4\pi r^2)
 \end{array}
 \right.$$

$$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_\alpha = (-1, f^{-1}, 0, 0)$$

$$= -\partial_\alpha U = -\partial_\alpha \left(+ - \left(\frac{dr}{f} \right) \right)$$

$$N_\alpha = -\frac{1}{2} f \partial_\alpha V$$

$$= -\frac{1}{2} f \partial_\alpha \left(+ + \left(\frac{dr}{f} \right) \right)$$

$$= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right)$$

$$N_\alpha N^\alpha = 0$$

$$N_\alpha K^\alpha = -1$$

Alternative N^α :

$$N_\alpha =$$

Alternative N' :

$$N_\alpha = (-f, 0, 0, \sqrt{f} \sin \theta)$$

$$\left(\frac{1}{f}, 0, 0 \right)$$

$$\left(f^{-1}, 0, 0 \right)$$

$$U = -\partial_\alpha \left(+ - \left(\frac{\partial}{\partial t} \right) \right)$$

$$\left(+ + \left(\frac{\partial}{\partial t} \right) \right)$$

$$N_\alpha N^\alpha$$

$$N_\alpha K^\alpha =$$

$$K^\alpha = \left(\frac{1}{f}, 1, 0, 0 \right)$$

$$K_\alpha = (-1, f^{-1}, 0, 0)$$

$$= -\partial_\alpha U = -\partial_\alpha \left(+ - \left(\frac{y}{f} \right) \right)$$

$$\begin{aligned} N_\alpha &= -\frac{1}{2} f \partial_\alpha \sqrt{\dots} \\ &= -\frac{1}{2} f \partial_\alpha \left(+ + \left(\frac{y}{f} \right) \right) \\ &= \left(-\frac{1}{2} f, -\frac{1}{2}, 0, 0 \right) \end{aligned} \quad \left. \begin{array}{l} N_\alpha N^\alpha = 0 \\ N_\alpha K^\alpha = -1 \end{array} \right\}$$

Alternative N^α :

$$N_\alpha = (-f, 0, 0, \sqrt{f} \sin \theta)$$

$$h_{ap} = \mathcal{Z}_{ap} + K_{\alpha} N_p + N_{\alpha} k_p$$

$$= \text{diag}(0, 0, \underbrace{r', r' \sin \theta})$$

$$\tilde{B}_{ap} = h_{\alpha}^{\mu} h_p^{\nu} K_{\mu\nu}$$

$$= \text{diag}(0, 0, r, r \sin^2 \theta)$$

$$\tilde{B}_{ap} = \frac{1}{r} h_{ap}$$

$$\theta = \frac{\pi}{2}$$

$$= \frac{1}{4\pi r^2} \frac{\partial \lambda}{\partial \lambda}$$

$$(\tilde{\xi}^\alpha; K_P)^\sim = \tilde{B}^\alpha_\beta \tilde{\xi}^\beta$$

$$\tilde{B}_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta B_{\mu\nu} = h^\mu_\alpha h^\nu_\beta K_{\mu\nu\lambda\rho}$$

$\tilde{B}_{\alpha\beta}$ is decomposed into irreducible components:

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\sqrt{\Theta} = \tilde{B}^\alpha_\alpha \quad (= \text{expansion})$$

$$\times \sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta \quad : \text{shear}$$

$$\times \omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} \quad : \text{rotation}$$

$$(\tilde{\xi}^\alpha; K_P)^\sim = \tilde{B}^\alpha_P \tilde{\xi}^P$$

$$\tilde{B}_{\mu\rho} = h^\mu_\alpha h^\rho_\beta B_{\mu\nu} = h^\mu_\alpha h^\rho_\beta K_{\mu\nu}$$

$\tilde{B}_{\mu\rho}$ is decomposed into irreducible components:

$$\tilde{B}_{\mu\rho} = \frac{1}{2} h_{\mu\rho} \Theta + \sigma_{\mu\rho} + \omega_{\mu\rho}$$

$$\sqrt{\Theta} = \tilde{B}^\alpha_\alpha \quad (\text{expansion})$$

$$\times \sigma_{\mu\rho} = \tilde{B}_{\mu\rho} - \frac{1}{2} h_{\mu\rho} \Theta \quad : \text{shear}$$

$$\times \omega_{\mu\rho} = \tilde{B}_{\mu\rho} \quad : \text{rotation}$$