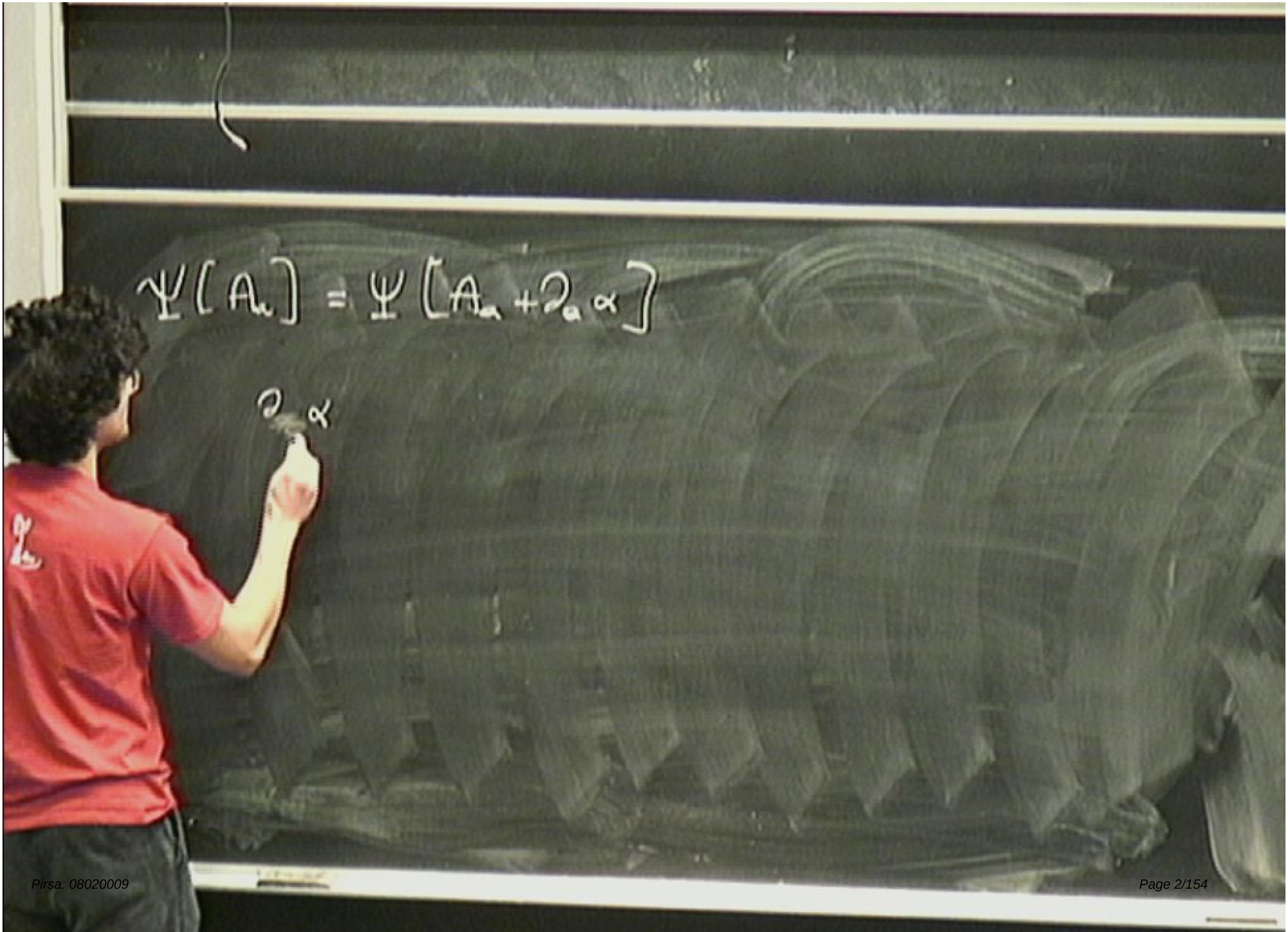
Title: Loop Quantum gravity #3

Date: Feb 12, 2008 06:30 PM

URL: http://pirsa.org/08020009

Abstract: Constrained systems, meaning of diffeomorphism invariance, loops and spin networks.


$\Psi[A_a] = \Psi[A_a + \partial_a \alpha]$
$\partial_a \alpha$

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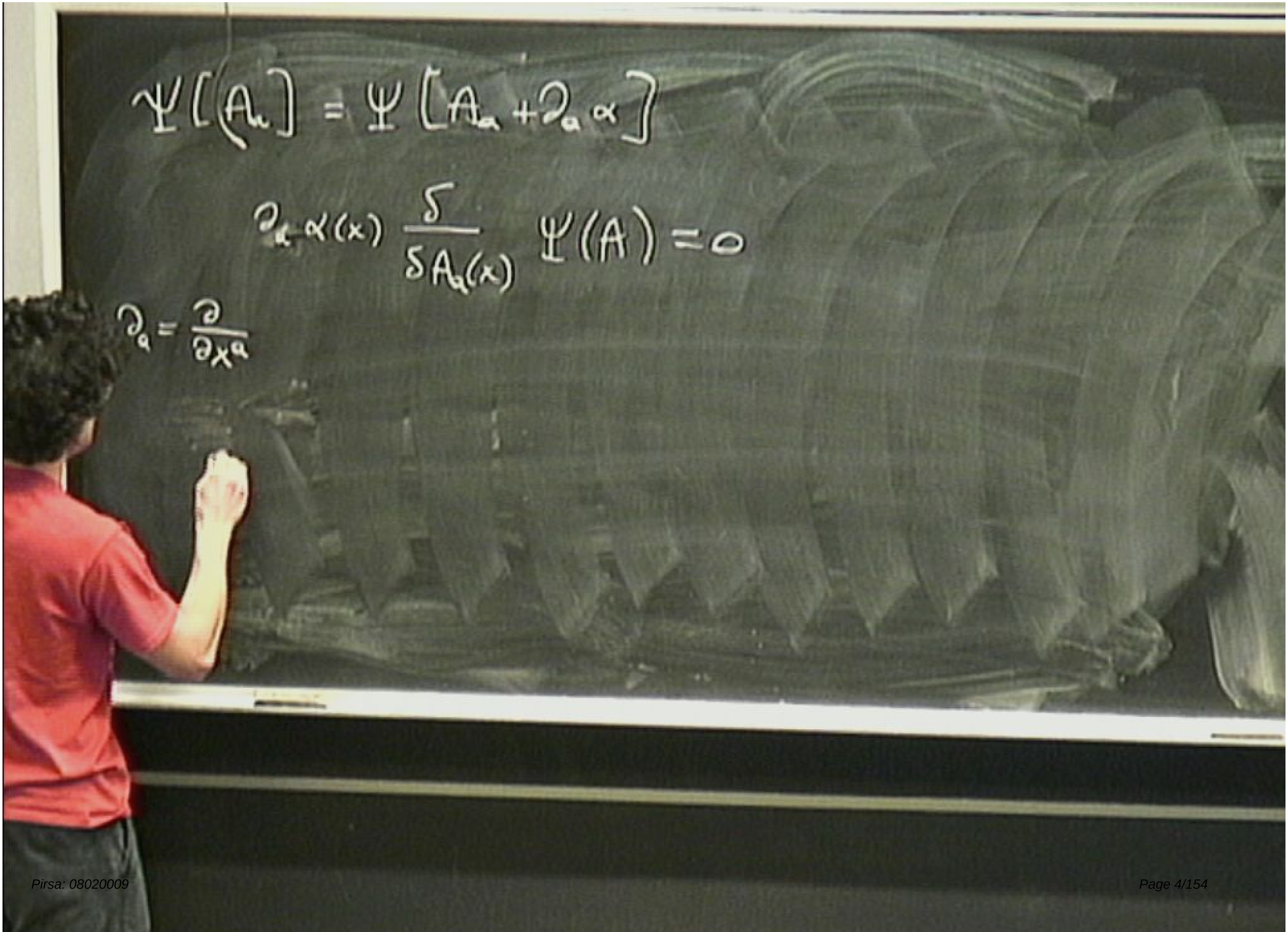
$$\partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0$$

$$\partial_a = \frac{\partial}{\partial x^a}$$



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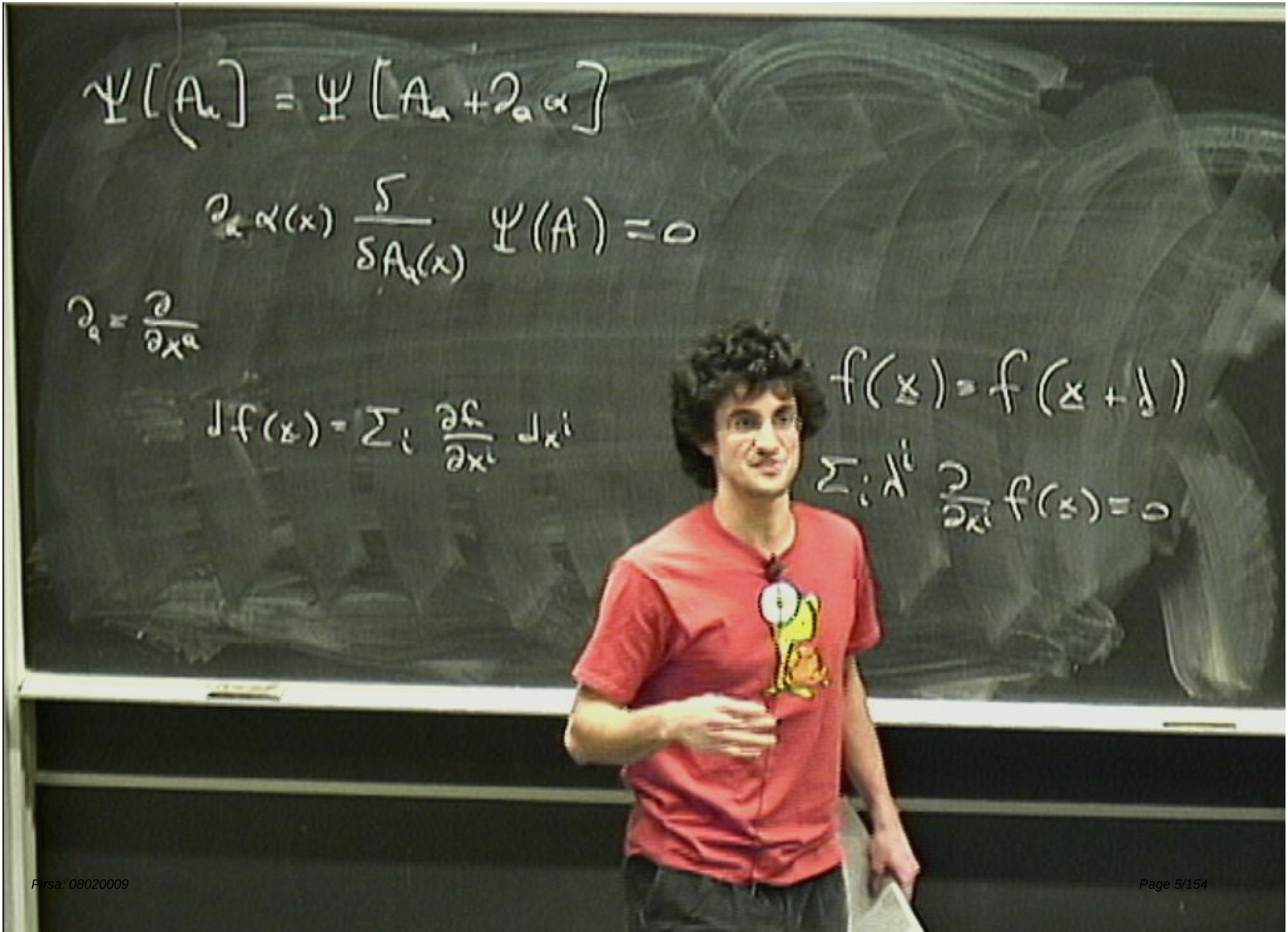
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$\partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0$
$\partial_a = \frac{\partial}{\partial x^a}$
$df(x) = \sum_i \frac{\partial f}{\partial x^i} dx^i$
$f(\underline{x}) = f(\underline{x} + \underline{\lambda})$
$\sum_i \lambda^i \frac{\partial}{\partial x^i} f(\underline{x}) = 0$

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$$\sum_i \lambda^i \frac{\partial}{\partial x^i} f(x) = 0$$

$$\int dx \, \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$$

$$\Psi[A_a] = \Psi[A_a + \partial_a \alpha]$$

$$\int dx\, \partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0$$

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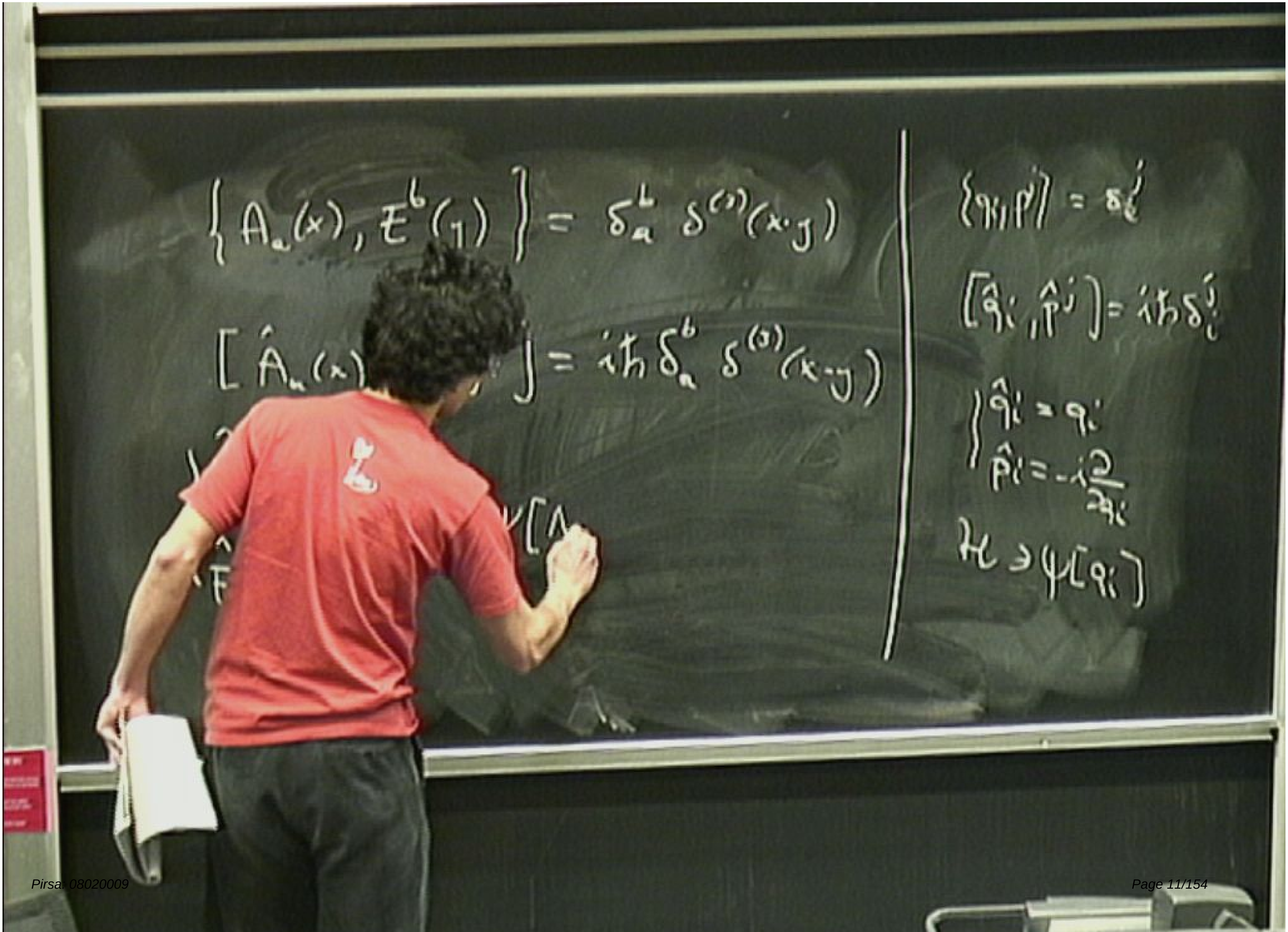
$$\int dx \, \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$$

$$\{ A_a(x), E^b(y) \} = \delta_a^b \, \delta^{(3)}(x-y)$$


$\{ A_a(x), E^b(y) \} = \delta_a^b \, \delta^{(3)}(x-y)$
$[\hat{A}_a(x)$
$] = i\hbar \, \delta_a^b \, \delta^{(3)}(x-y)$
$\{q_i, p^j\} = \delta_i^j$
$[\hat{q}_i, \hat{p}^j] = i\hbar \, \delta_i^j$
$\hat{q}_i = q_i$
$\hat{p}_i = -i \frac{\partial}{\partial q_i}$
$\mathcal{H} \ni \psi[q_i]$

$$\{ A_a(x), E^b(y) \} = \delta_a^b \, \delta^{(3)}(x-y)$$

$$[\hat{A}_a(x), \hat{E}^b(y)] = i\hbar \, \delta_a^b \, \delta^{(3)}(x-y)$$

$$\begin{cases} \hat{A}_a = A_a \\ \hat{E}^a = \dfrac{\delta}{\delta A_a} \end{cases} \qquad \Psi[A_a]$$

$$\{ q_i, p^j \} = \delta_i^j$$

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$$\Psi[A_a(x)]$$

$$\{q_i, p^j\} = \delta_i^j$$

$$[\hat{q}_i, \hat{p}^j] = i\hbar \delta$$

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$$\{ A_a(x), E^b(y) \} = \delta_a^b \, \delta^{(3)}(x-y)$$

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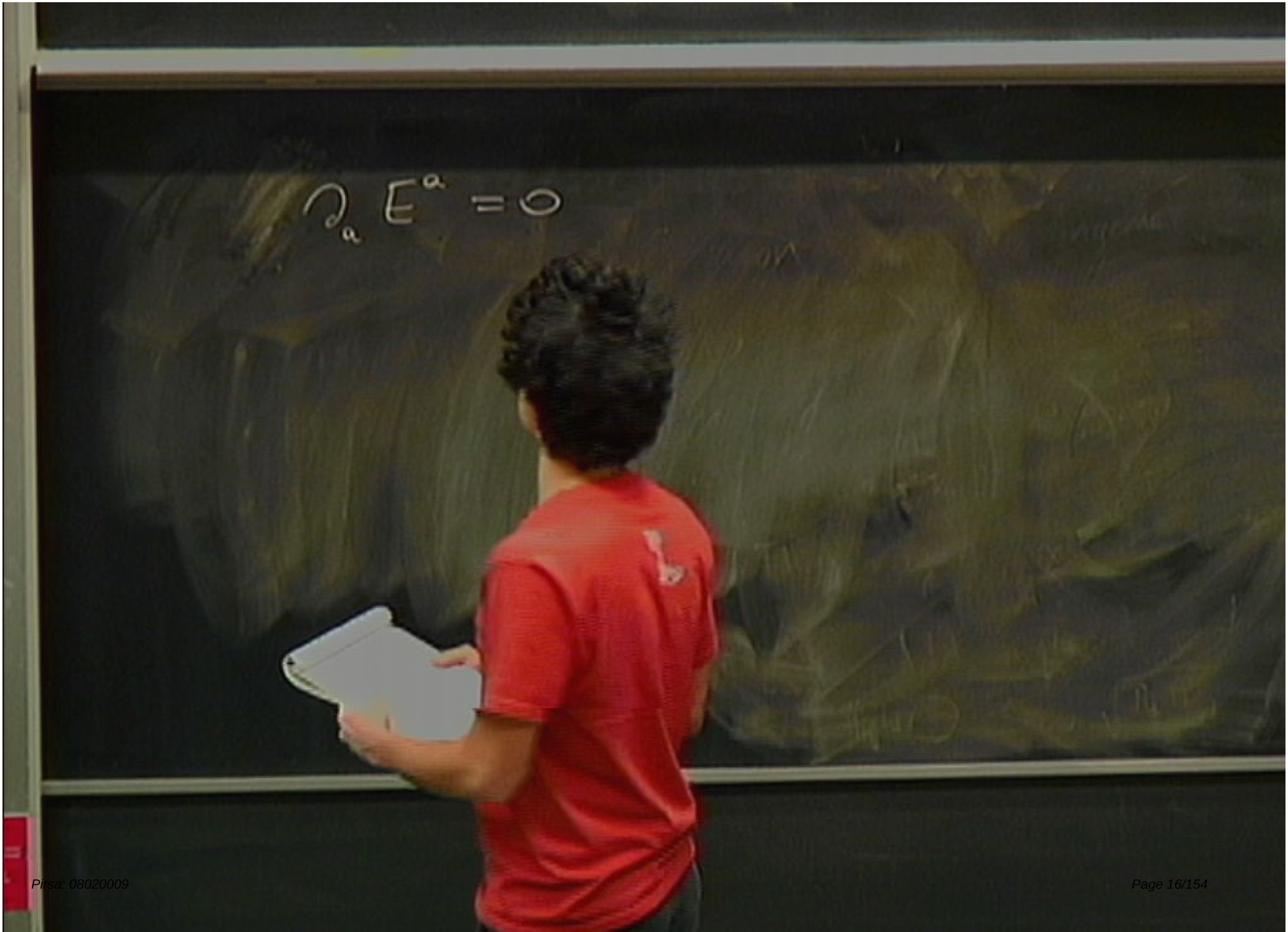

$\partial_a E^a = 0$

$$\partial_a E^a = 0$$

a, b, t	H
a, b	$H(a,b) = 0$

$$\hat{H}\psi = 0$$

$$\Psi[A_a] = \Psi[A_a + \partial_a \alpha]$$

$$\int dx\, \partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0 =$$

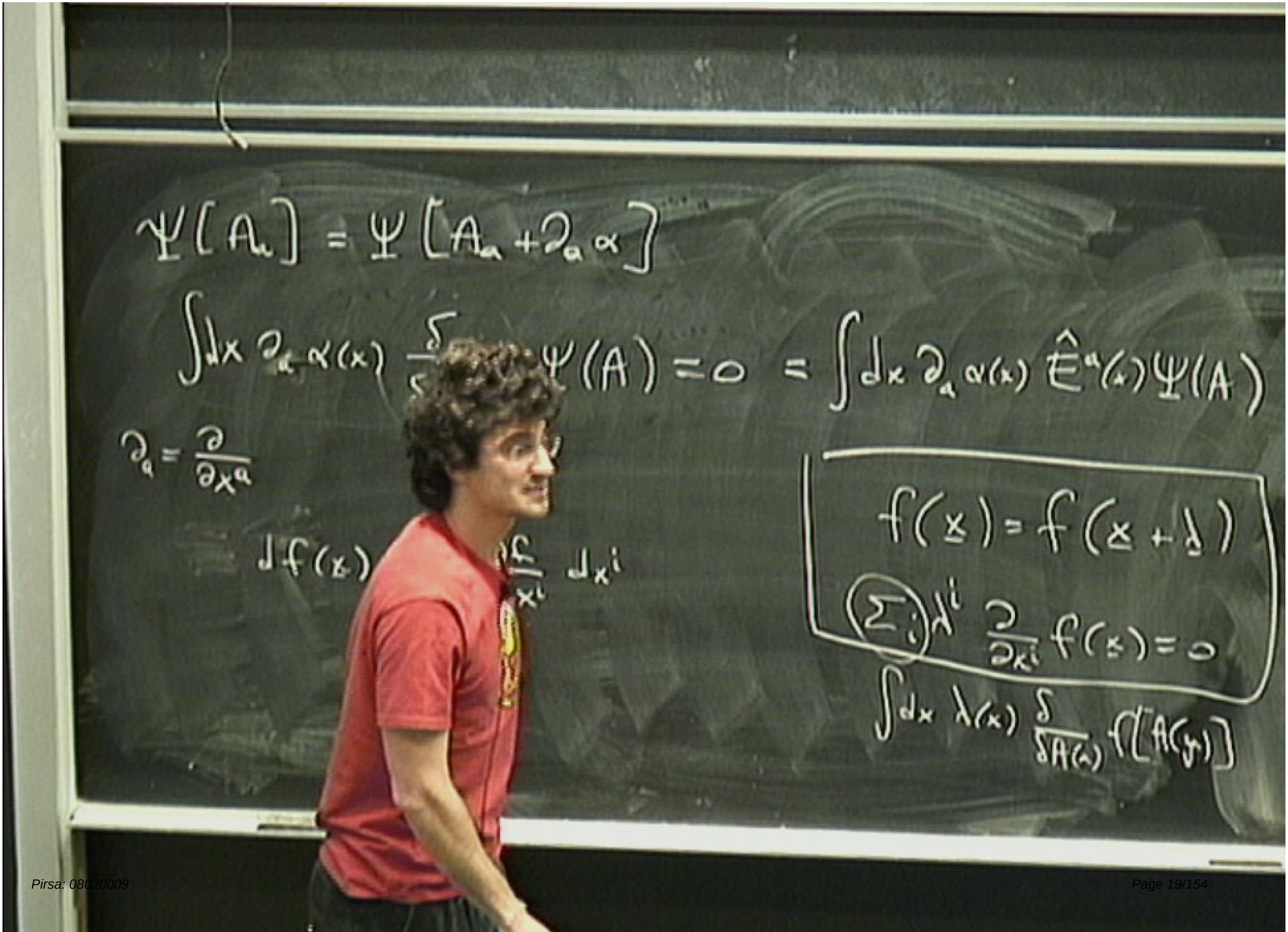
$$\partial_a = \frac{\partial}{\partial x^a}$$

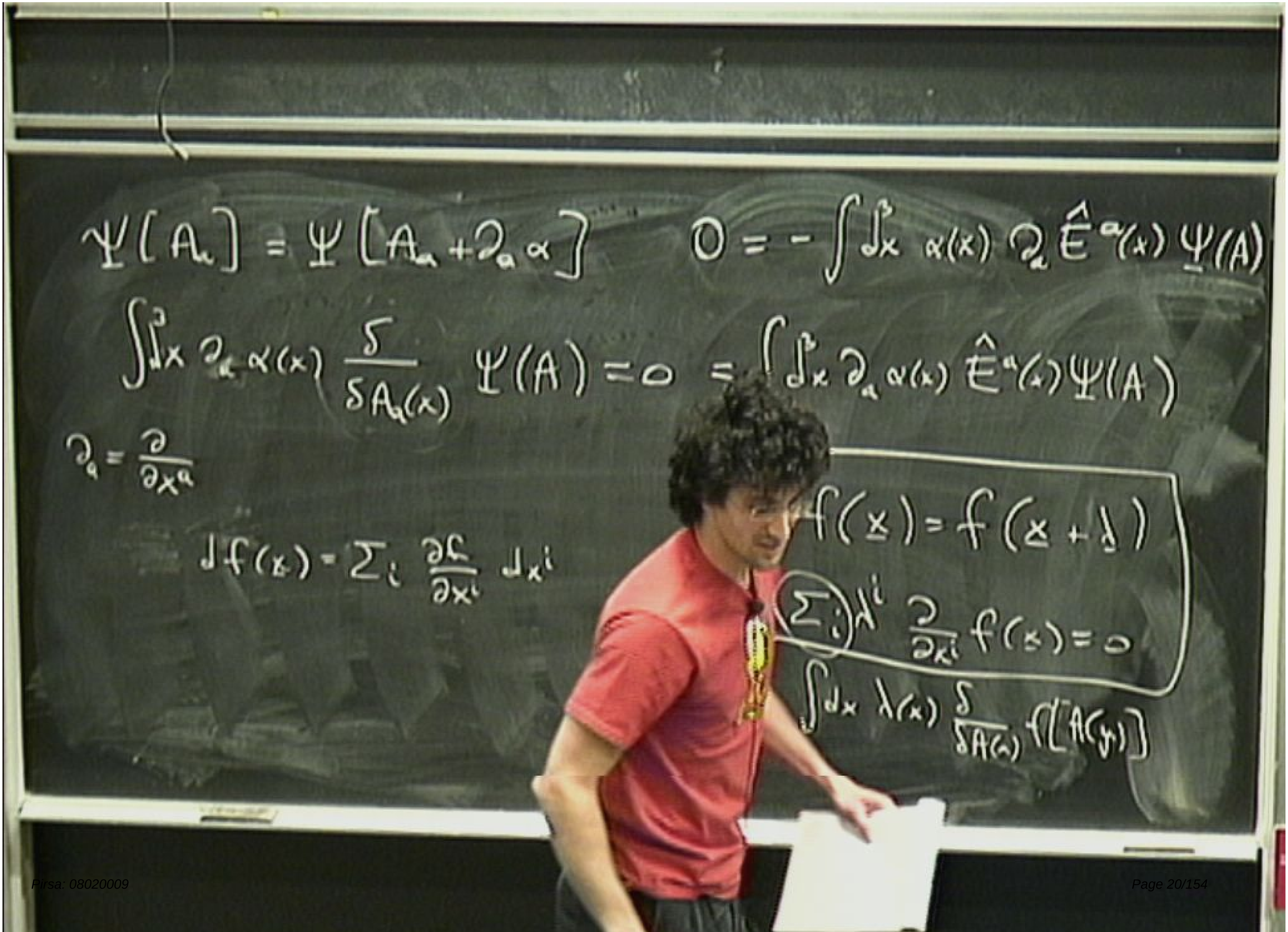
$$df(x) = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$$f(\underline{x}) = f(\underline{x} + \underline{\lambda})$$

$$\sum_i \lambda^i \frac{\partial}{\partial x^i} f(x)$$

$$\int dx\, \lambda(x) \frac{\delta}{\delta A(x)} \Psi(y$$


$\Psi[A_a] = \Psi[A_a + \partial_a \alpha]$
$\partial_a = \frac{\partial}{\partial x^a}$
$\int dx\, \partial_a \alpha(x)\, \hat{E}^a(x) \Psi(A)$
$f(\underline{x}) = f(\underline{x} + \underline{\lambda})$
$\Sigma_i \lambda^i \frac{\partial}{\partial x^i} f(\underline{x}) = 0$
$\int dx\, \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$


$\Psi[A_a] = \Psi[A_a + \partial_a \alpha]$
$0 = -\int d^3x\, \alpha(x)\, \partial_a \hat{E}^a(x)\, \Psi(A)$
$\int d^3x\, \partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0 = \int d^3x\, \partial_a \alpha(x)\, \hat{E}^a(x) \Psi(A)$
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$df(x) = \sum_i \frac{\partial f}{\partial x^i} dx^i$
$f(\underline{x}) = f(\underline{x} + \underline{\lambda})$
$\sum_i \lambda^i \frac{\partial}{\partial x^i} f(\underline{x}) = 0$
$\int dx\, \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$

$$\Psi[A_a] = \Psi[A_a + \partial_a \alpha] \qquad 0 = -\int d^3x \; \alpha(x)\, \partial_a \hat{E}^a(x)\, \Psi(A)$$

$$\int d^3x \; \partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0 = \int d^3x \; \partial_a \alpha(x)\, \hat{E}^a(x) \Psi(A)$$

$$\partial_a = \frac{\partial}{\partial x^a}$$

$$df(x) = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$$\boxed{\begin{aligned} f(\underline{x}) &= f(\underline{x} + \underline{\lambda}) \\ \sum_i \lambda^i \frac{\partial}{\partial x^i} f(\underline{x}) &= 0 \end{aligned}}$$

$$\int dx \; \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$$

$$\Psi[A_a] = \Psi[A_a + \partial_a \alpha] \qquad 0 = -\int d^3x \; \alpha(x)\, \partial_a \hat{E}^a(x)\, \Psi(A)$$

$$\int d^3x \; \partial_a \alpha(x) \frac{\delta}{\delta A_a(x)} \Psi(A) = 0 = \int d^3x \; \partial_a \alpha(x)\, \hat{E}^a(x) \Psi(A)$$

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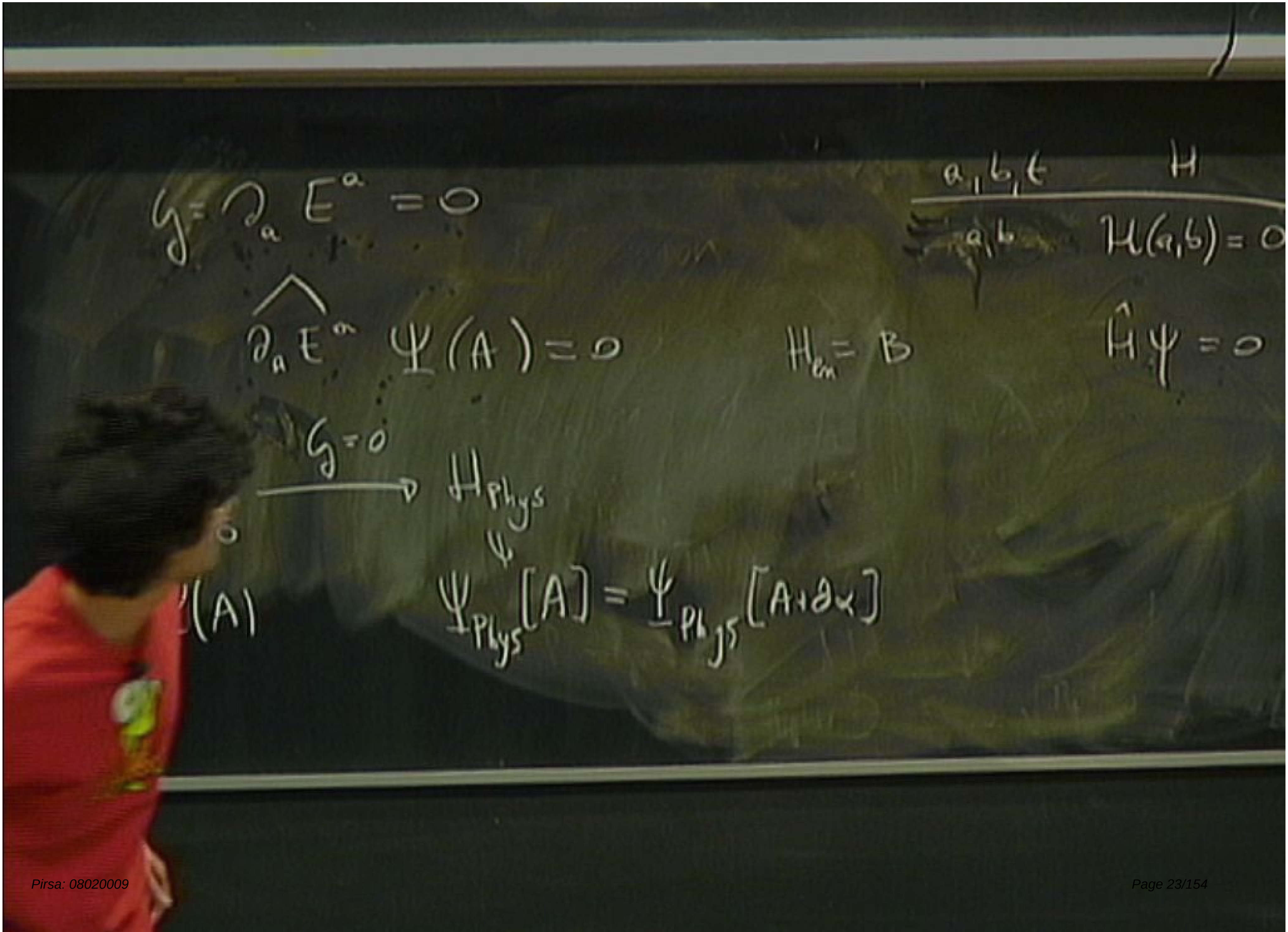
$$\boxed{\begin{gathered} f(\underline{x}) = f(\underline{x} + \underline{\lambda}) \\ \sum_i \lambda^i \frac{\partial}{\partial x^i} f(\underline{x}) = 0 \end{gathered}}$$

$$\int dx \; \lambda(x) \frac{\delta}{\delta A(x)} f[A(y)]$$



$\mathcal{G} = \partial_a E^a = 0$
$\widehat{\partial_a E^a}\, \Psi(A) = 0$
$H_{em} = B$
a, b, t
H
a, b
$\mathcal{H}(a,b) = 0$
$\hat{H}\psi = 0$
$\mathcal{G} = 0$
$\longrightarrow \mathcal{H}_{phys}$
$\Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$
(A)

$$\mathcal{G} = \partial_a E^a = 0$$

$$\frac{a, b, t \qquad H}{a, b \qquad H(a,b) = 0}$$

$$\widehat{\partial_a E^a}\, \Psi(A) = 0 \qquad H_{em} = B^2 + E^2 \qquad \hat{H}\psi = 0$$

$$\mathcal{H}_0 \xrightarrow{\mathcal{G}=0} \mathcal{H}_{phys}$$

$$\Psi(A) \in \mathcal{H}_0 \qquad \Psi_{phys} \in \mathcal{H}_{phys}$$

$$\Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$$

$$\mathcal{G} = \partial_a E^a = 0$$

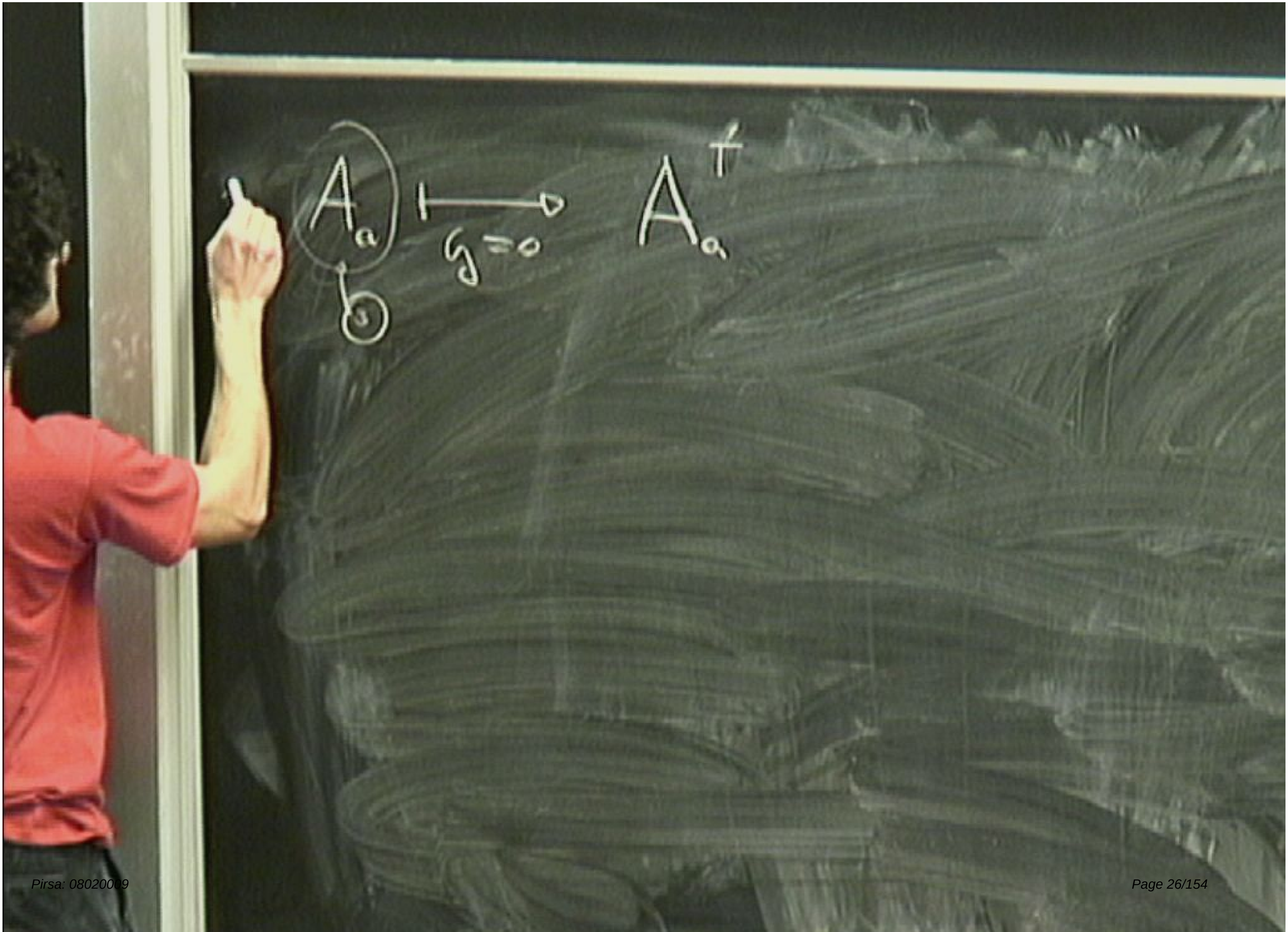
$$\widehat{\partial_a E^a}\, \Psi(A) = 0$$

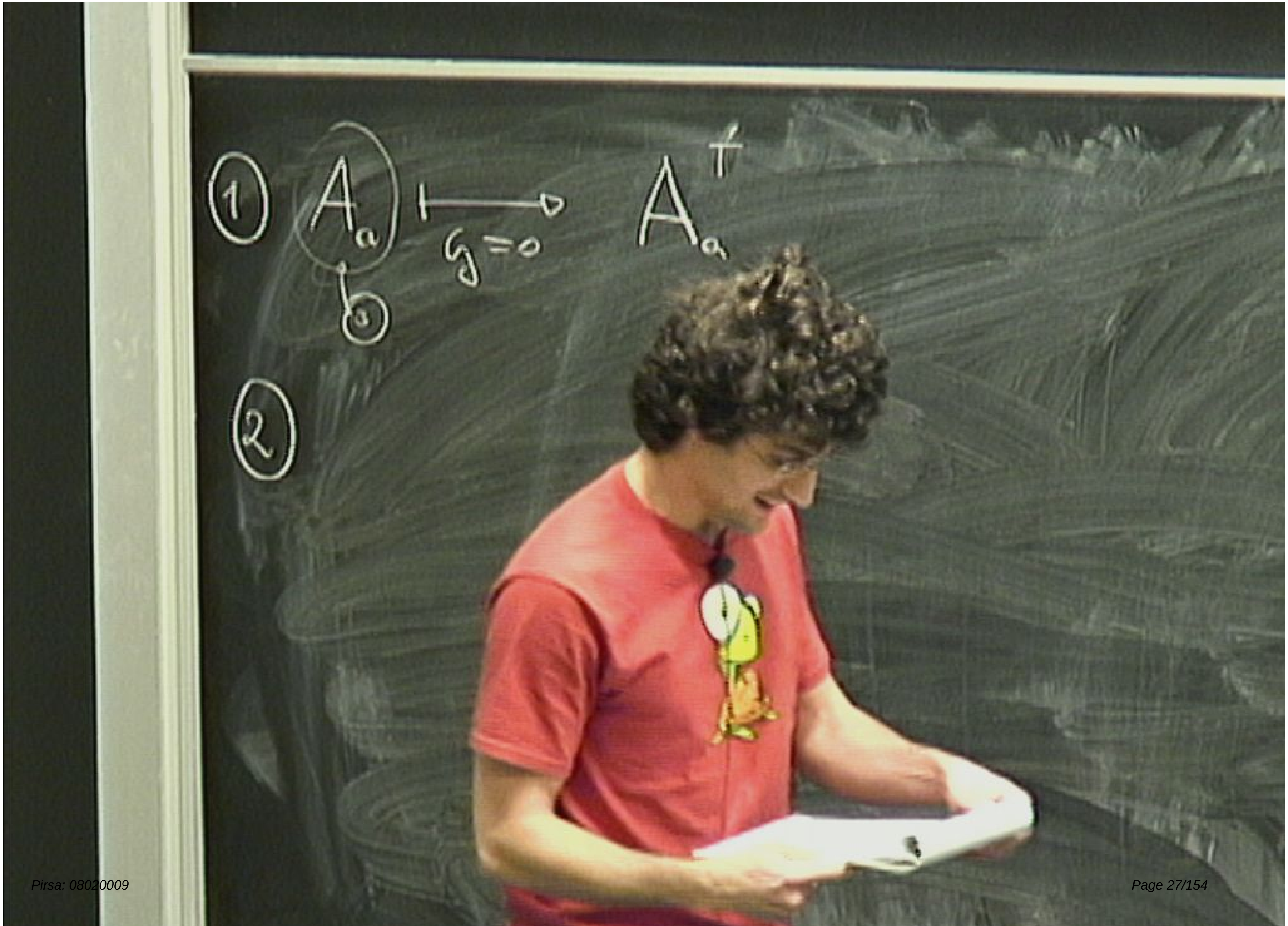
$$\mathcal{H}_0 \xrightarrow{\mathcal{G}=0} \mathcal{H}_{phys}$$

$$\Psi(A) \qquad \Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$$

a, b, t	H
a, b	$\mathcal{H}(a,b) = 0$

$$H_{em} = B^2 + E^2 \qquad \hat{H}\Psi = 0$$


$A_a \xmapsto[G=0]{} A_a^f$



1 $A_a \longmapsto A_a^f$
$g=0$
0
2

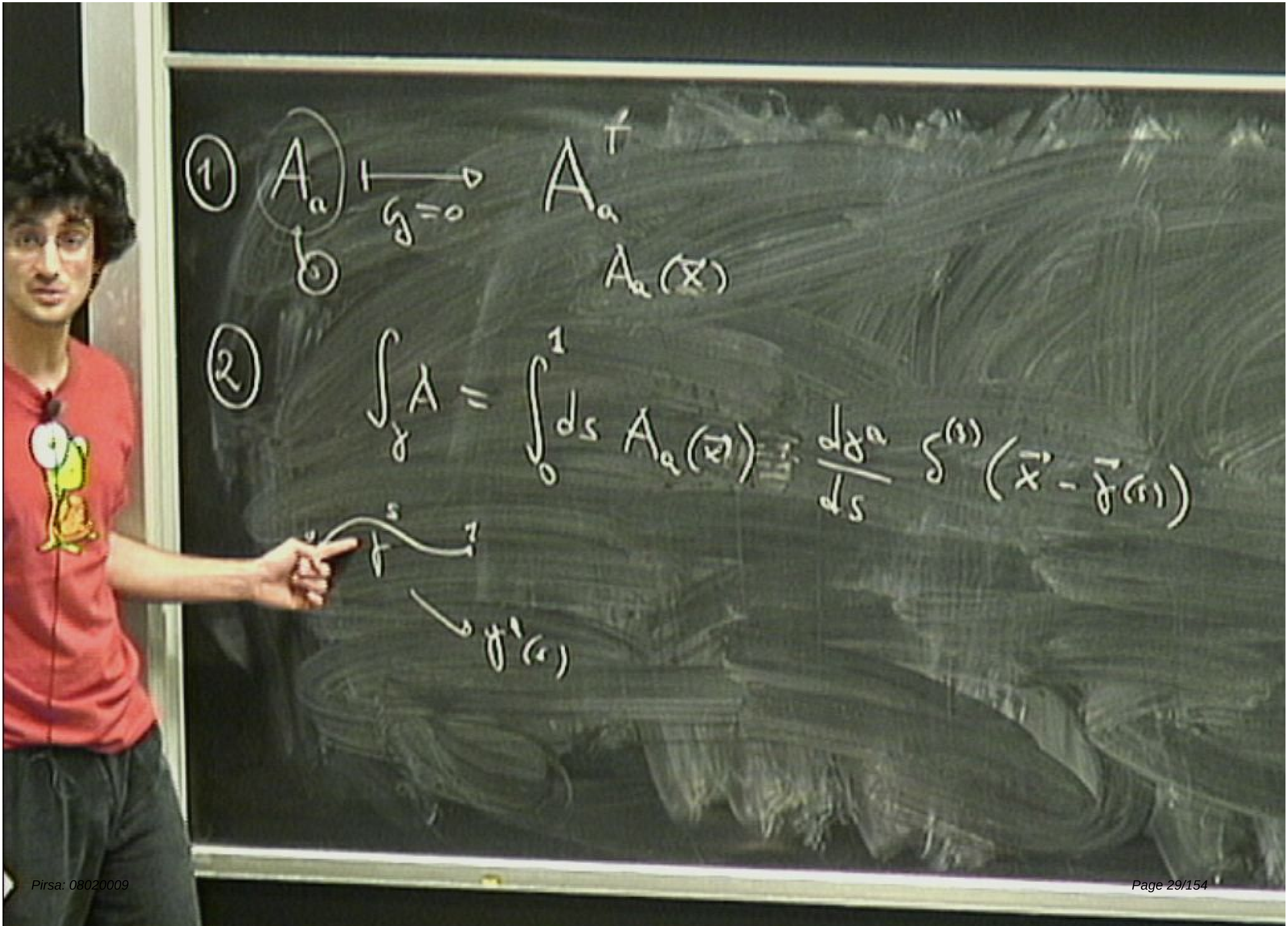
(1) $A_a \xmapsto[g=0]{} A_a^{\dagger}$

(3)

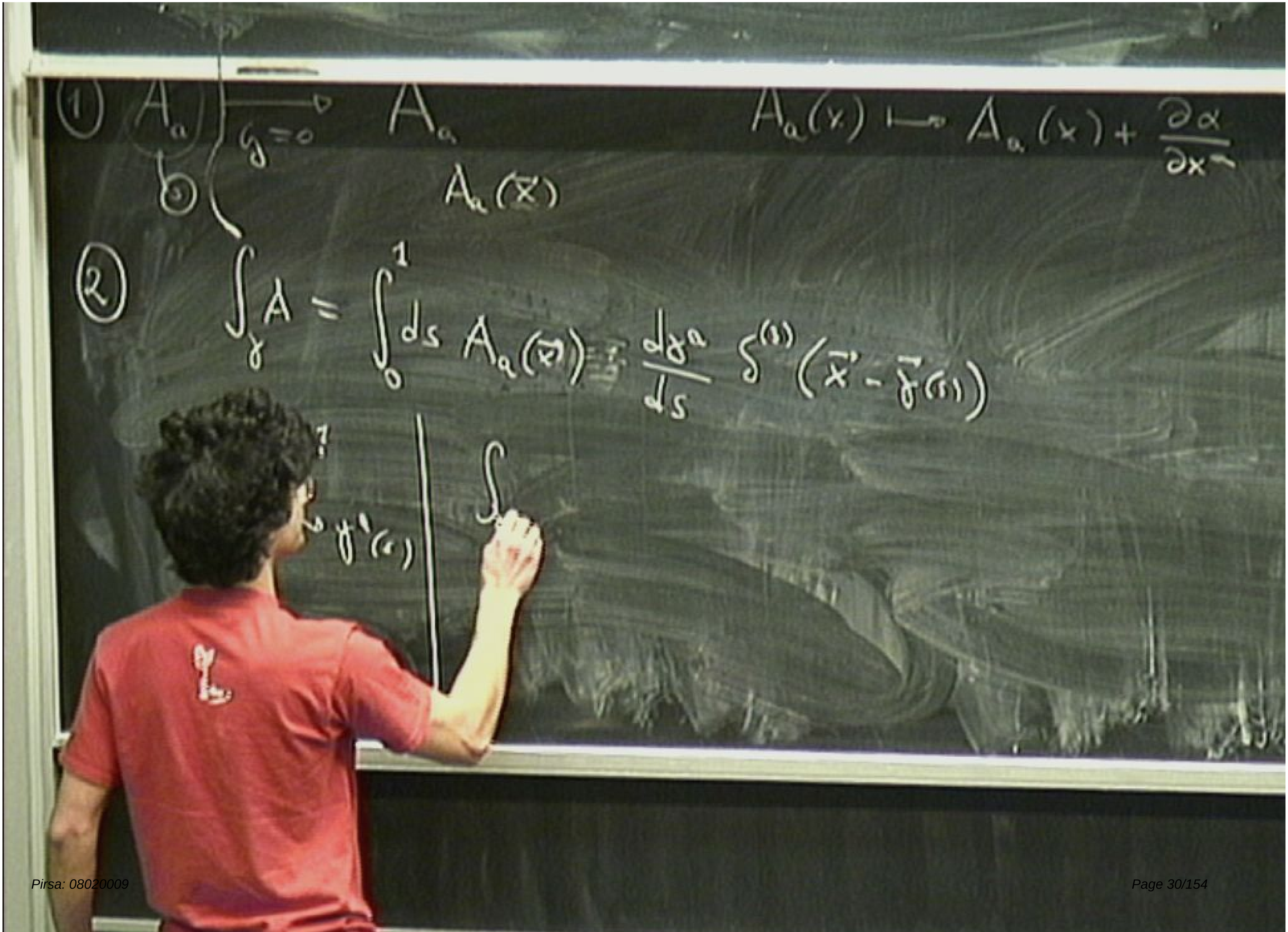
(2) $$\int_\gamma A = \int_0^1 ds\, A_a(x(s))\, \frac{d\gamma^a}{ds}$$

$0 \quad s \quad 1$

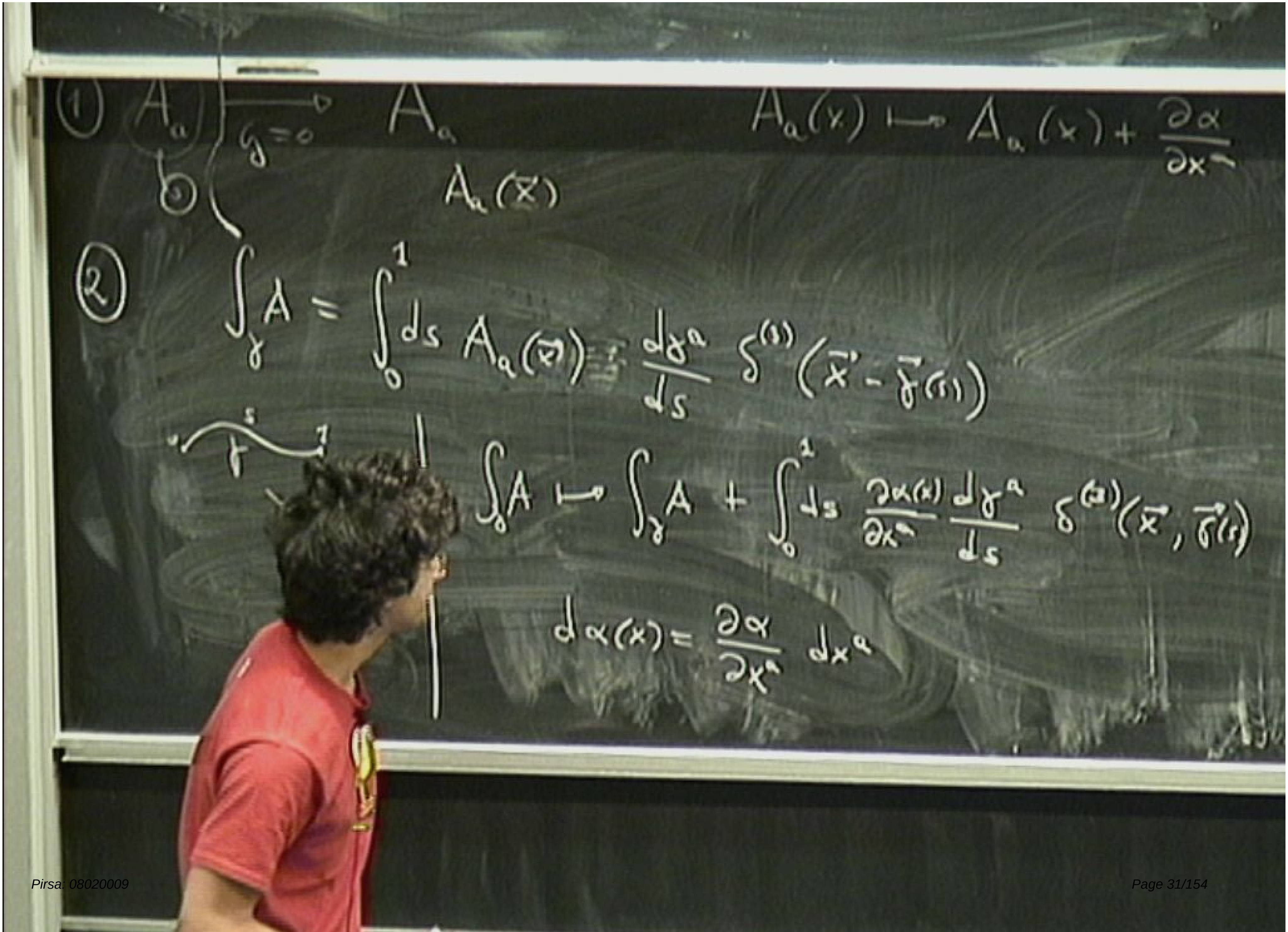
$\gamma \longrightarrow \gamma'(s)$



① $A_a \xmapsto{g=0} A_a^T$
$A_a(\vec{x})$
② $\int_\gamma A = \int_0^1 ds\, A_a(\vec{x}) \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x}-\vec{\gamma}(s))$
0 s γ 1
$\gamma^a(s)$



(1) $A_a \xrightarrow{\gamma = 0} A_a$
$A_a(x) \mapsto A_a(x) + \frac{\partial \alpha}{\partial x^a}$
(b)
$A_a(\vec{x})$
(2) $\int_\gamma A = \int_0^1 ds\, A_a(\vec{x}) \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$
$\gamma^a(s)$


① $A_a \xmapsto{\gamma=0} A_a$
$A_a(x) \mapsto A_a(x) + \frac{\partial\alpha}{\partial x^a}$
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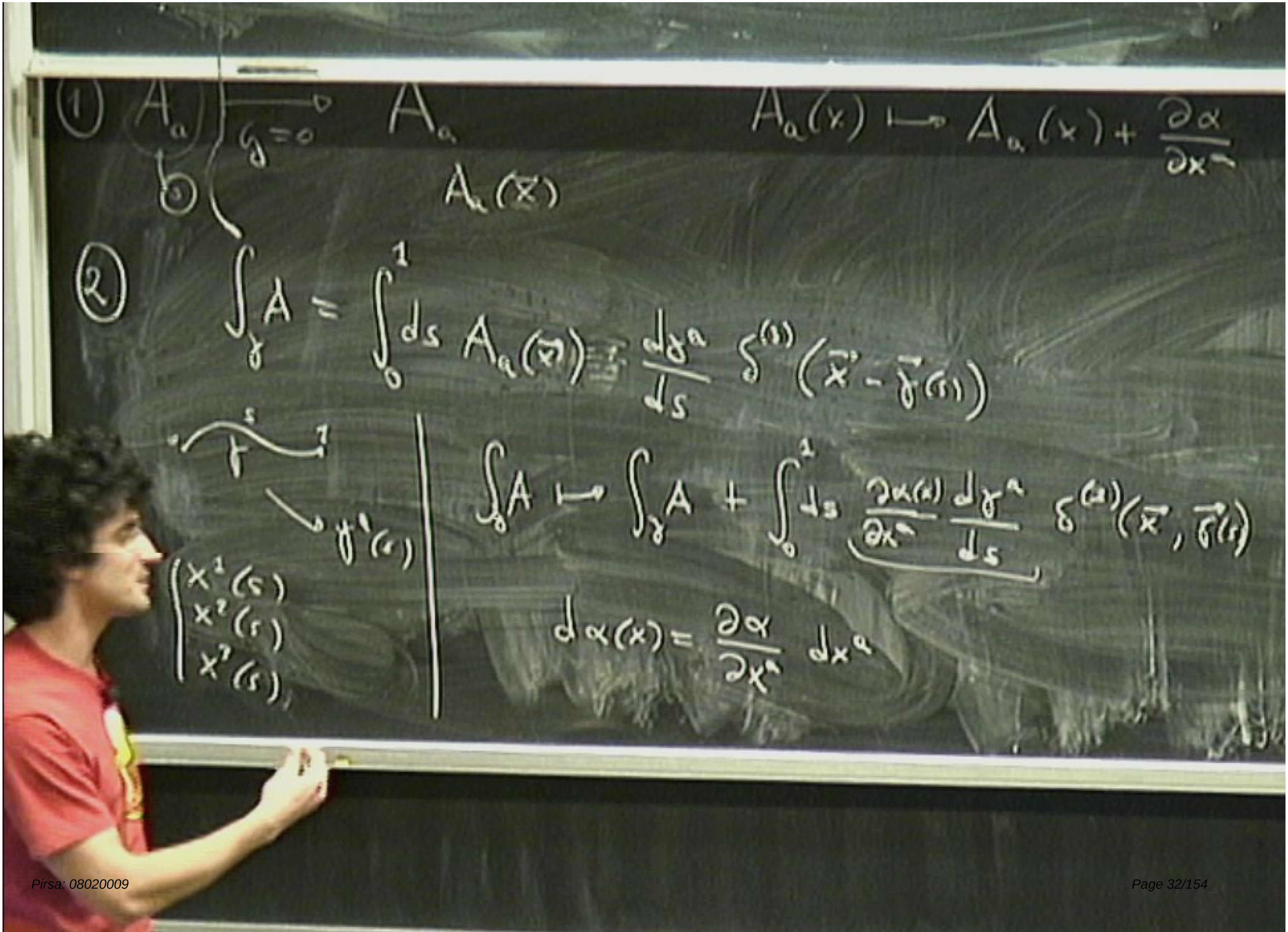
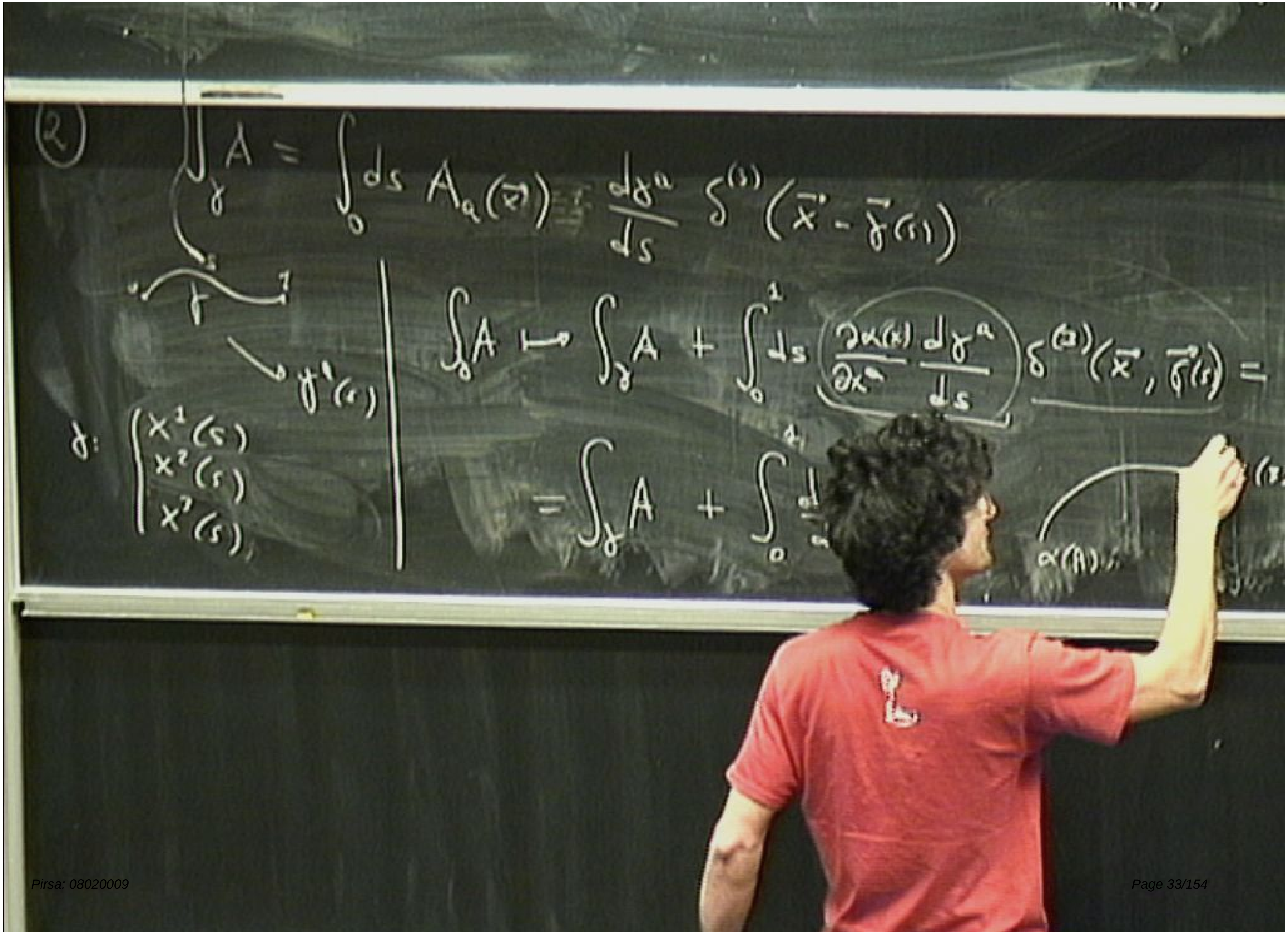
② $\int_\gamma A = \int_0^1 ds\, A_a(\vec{x}) \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$
$\int_\gamma A \mapsto \int_\gamma A + \int_0^1 ds\, \frac{\partial\alpha(x)}{\partial x^a} \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x}, \vec{\gamma}(s))$
$d\alpha(x) = \frac{\partial\alpha}{\partial x^a} dx^a$



① $A_a \xrightarrow{g=0} A_a$
$A_a(x) \mapsto A_a(x) + \frac{\partial \alpha}{\partial x^a}$
$A_a(\vec{x})$
②
$\int_\gamma A = \int_0^1 ds\, A_a(\vec{x}) \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$
$\gamma^a(s)$
$x^1(s)$
$x^2(s)$
$x^3(s)$
$\int_\gamma A \mapsto \int_\gamma A + \int_0^1 ds \frac{\partial \alpha(x)}{\partial x^a} \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x}, \vec{\gamma}(s))$
$d\alpha(x) = \frac{\partial \alpha}{\partial x^a} dx^a$



$\oint_\gamma A = \int_0^1 ds\, A_a(\vec{x}) \frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$
$\gamma: \begin{cases} x^1(s) \\ x^2(s) \\ x^3(s) \end{cases}$
$\gamma'(s)$
$\oint_\gamma A \mapsto \int_\gamma A + \int_0^1 ds \frac{\partial\alpha(x)}{\partial x^a}\frac{d\gamma^a}{ds} \delta^{(3)}(\vec{x}, \vec{\gamma}(s)) =$
$= \int_\gamma A + \int_0$
$\alpha(A)$

(2)

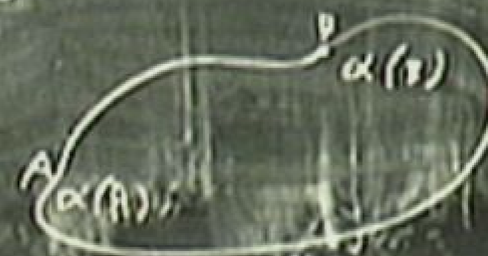
$$\int_\gamma A = \int_0^1 ds\, A_a(\vec{x})\, \frac{d\gamma^a}{ds}\, \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$$

$\gamma'(s)$

$$\gamma: \begin{cases} x^1(s) \\ x^2(s) \\ x^3(s) \end{cases}$$

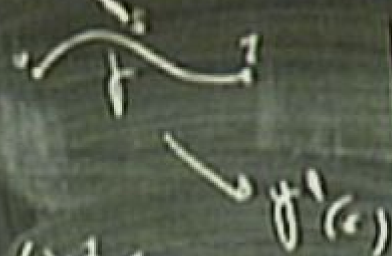
$$\int_\gamma A \mapsto \int_\gamma A + \int_0^1 ds\, \frac{\partial \alpha(x)}{\partial x^a} \frac{d\gamma^a}{ds}\, \delta^{(3)}(\vec{x}, \vec{\gamma}(s)) =$$

$$= \int_\gamma A + \int_0^1 \frac{d\alpha}{ds}\, ds$$



(2)

$$\int_\gamma A = \int_0 ds\, A_a(\vec{x})\, \frac{d\gamma^a}{ds}\, \delta^{(3)}(\vec{x} - \vec{\gamma}(s))$$

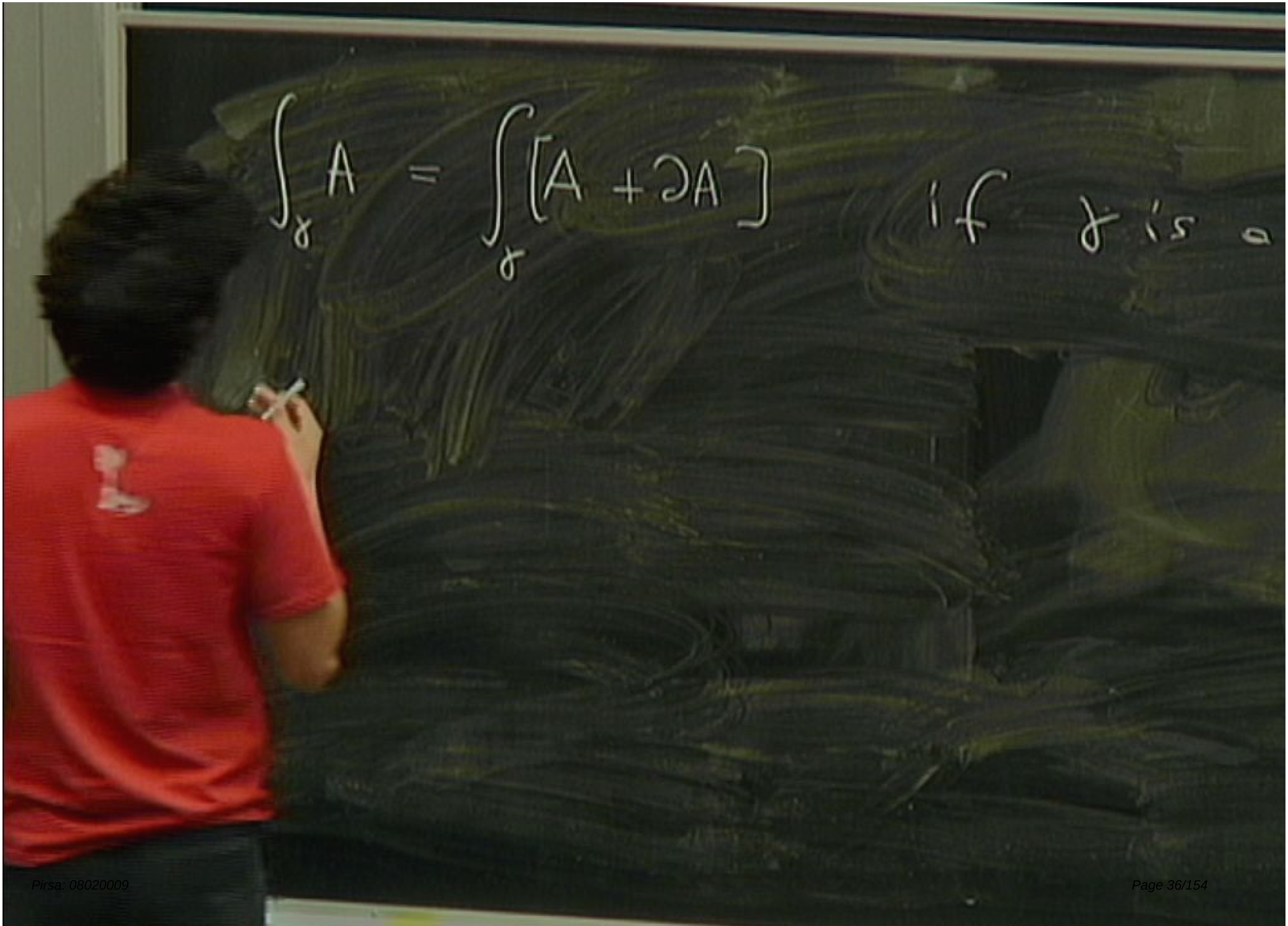


$$\gamma : \begin{cases} x^1(s) \\ x^2(s) \\ x^3(s) \end{cases}$$

$$\int_\gamma A \mapsto \int_\gamma A + \int_0^1 ds\, \frac{\partial \alpha(x)}{\partial x^a} \frac{d\gamma^a}{ds}\, \delta^{(3)}(\vec{x}, \vec{\gamma}(s)) =$$

$$= \int_\gamma A + \int_0^1 \frac{d\alpha}{ds}\, ds = 0$$

$\alpha(B)$ $\alpha(A)$


$\int_\gamma A = \int_\gamma [A + \partial A]$ if γ is a

$$\oint_\gamma A = \oint_\gamma [A + \partial A] \quad \text{if } \gamma \text{ is a loop.}$$

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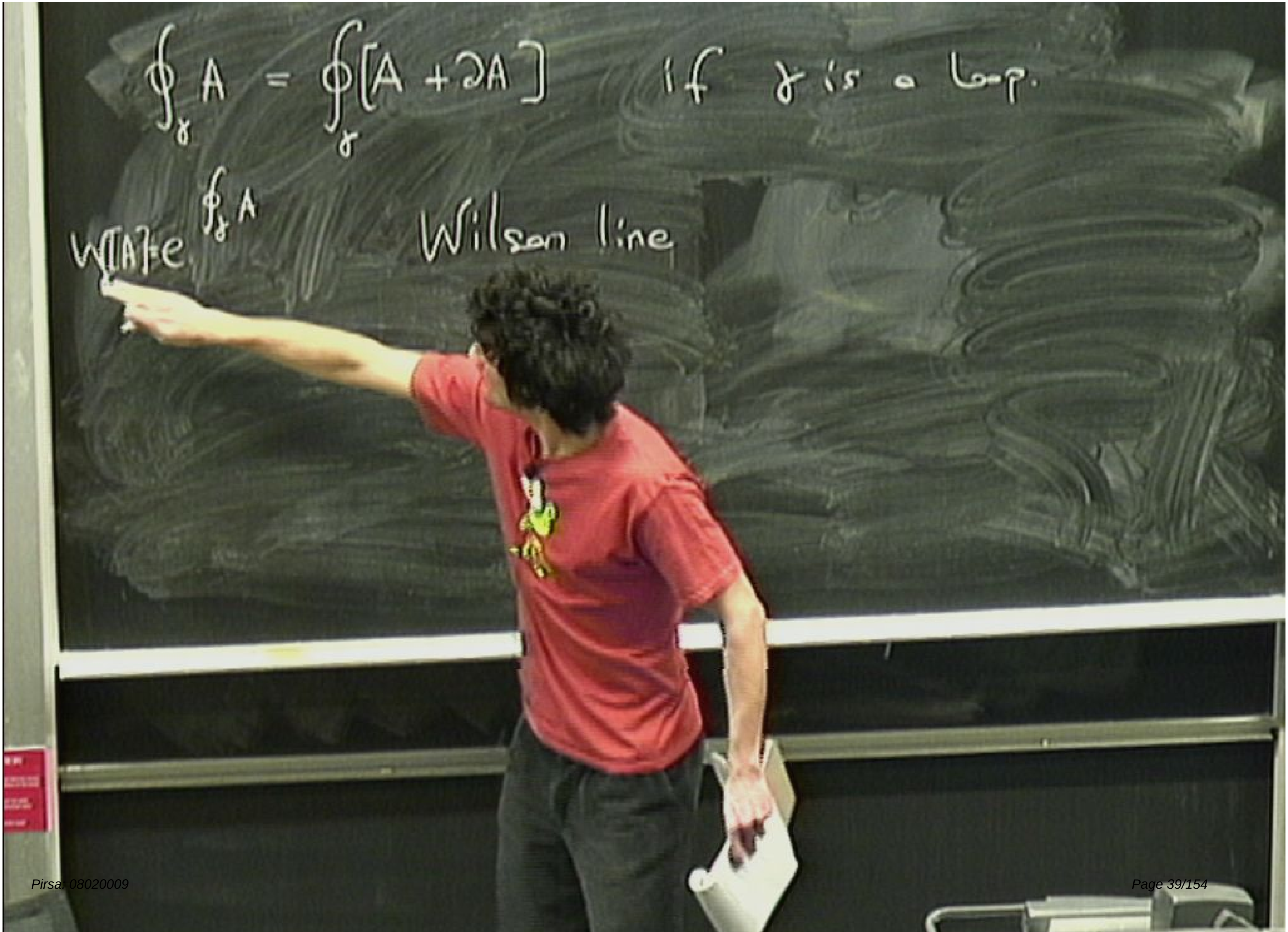
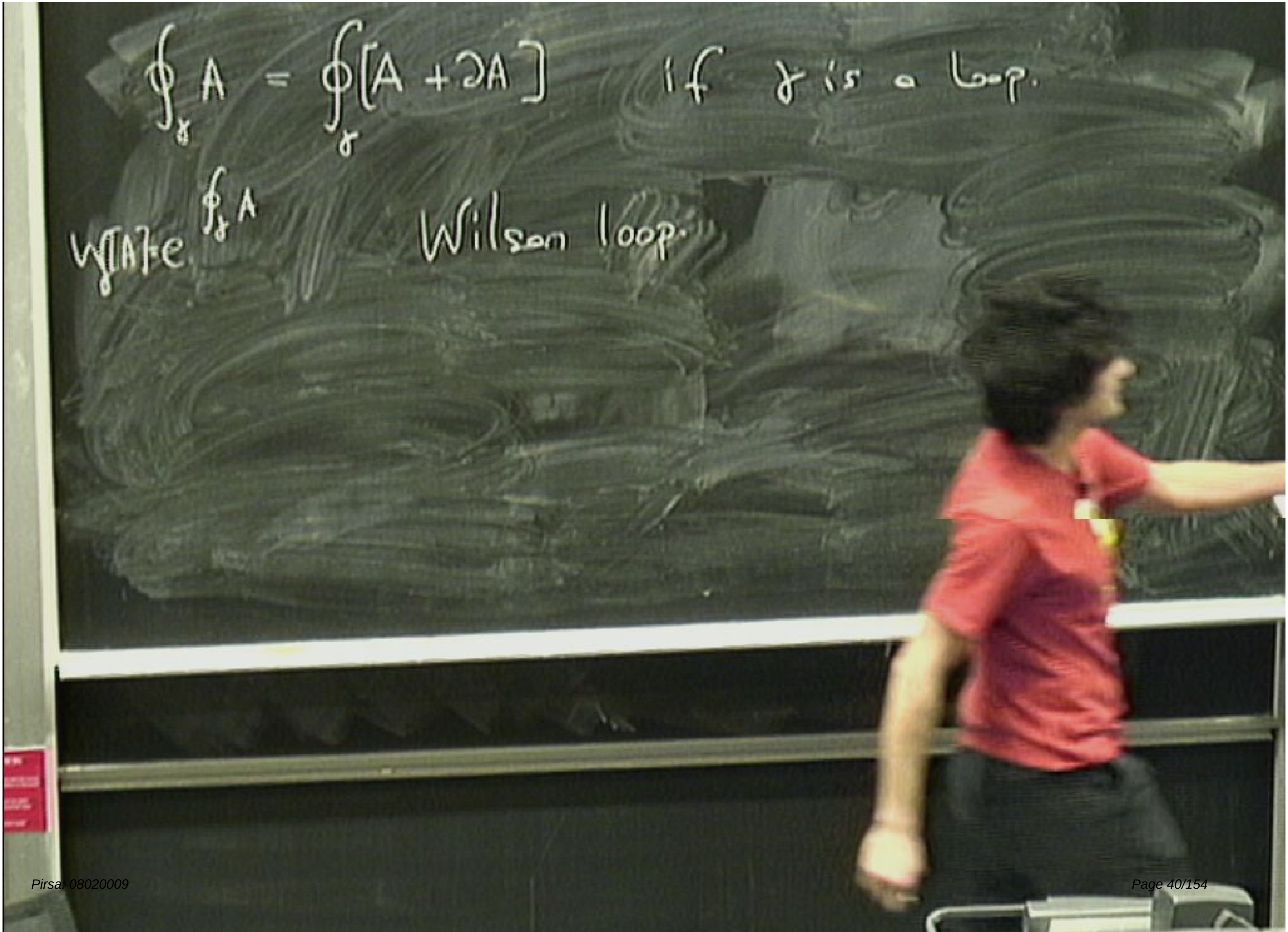
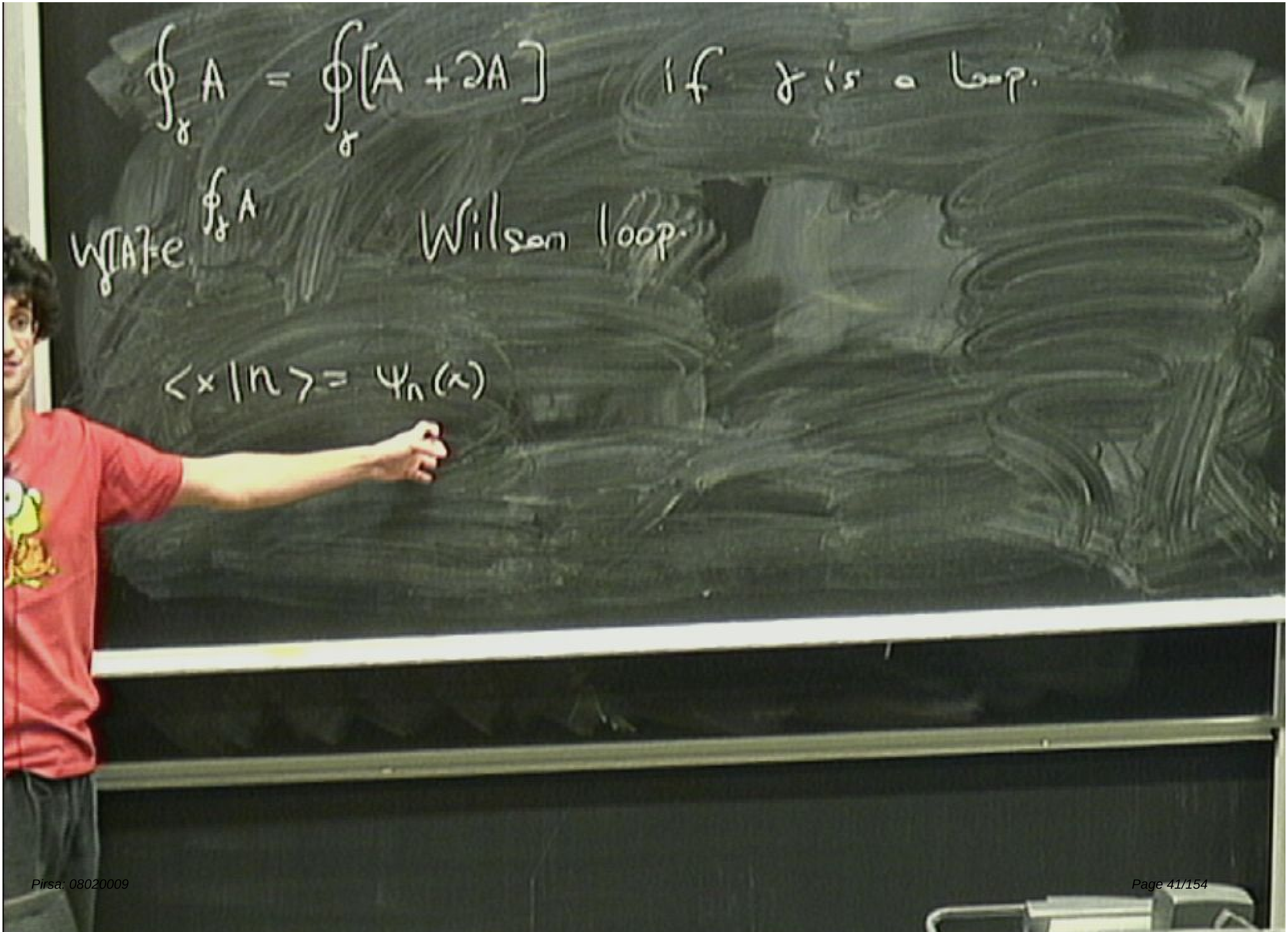
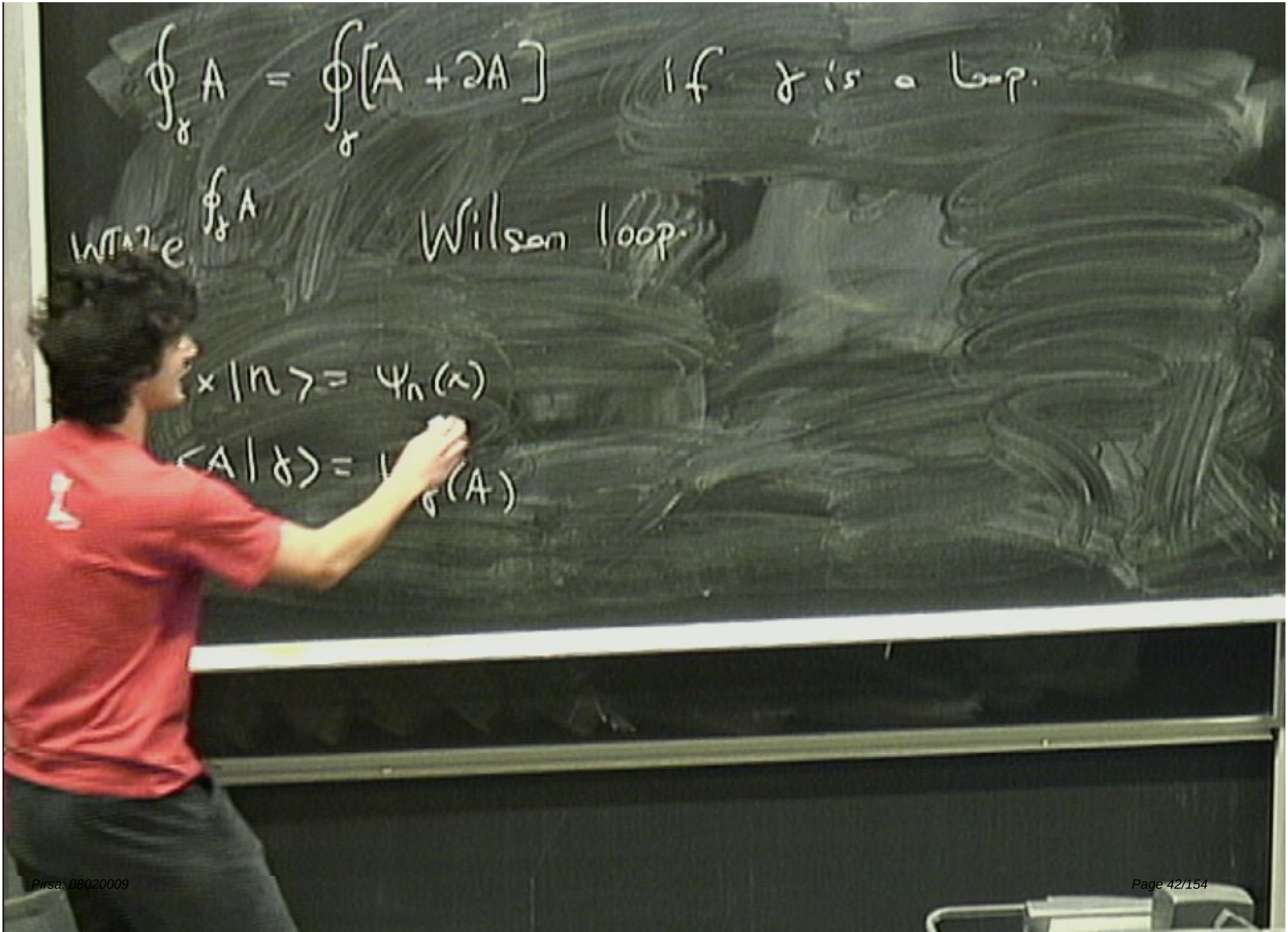
$$W[A] = e^{\oint_\gamma A}$$


$\oint_\gamma A = \oint_\gamma [A + \partial A]$ if γ is a loop.
$W[A] = e^{\oint_\gamma A}$
Wilson line


$\oint_\gamma A = \oint_\gamma [A + \partial A]$ if γ is a loop.
$W_\gamma[A] = e^{\oint_\gamma A}$ Wilson loop.


$\oint_\gamma A = \oint_\gamma [A + \partial A]$ if γ is a loop.
$W_\gamma[A] = e^{\oint_\gamma A}$ Wilson loop.
$\langle x | n \rangle = \psi_n(x)$

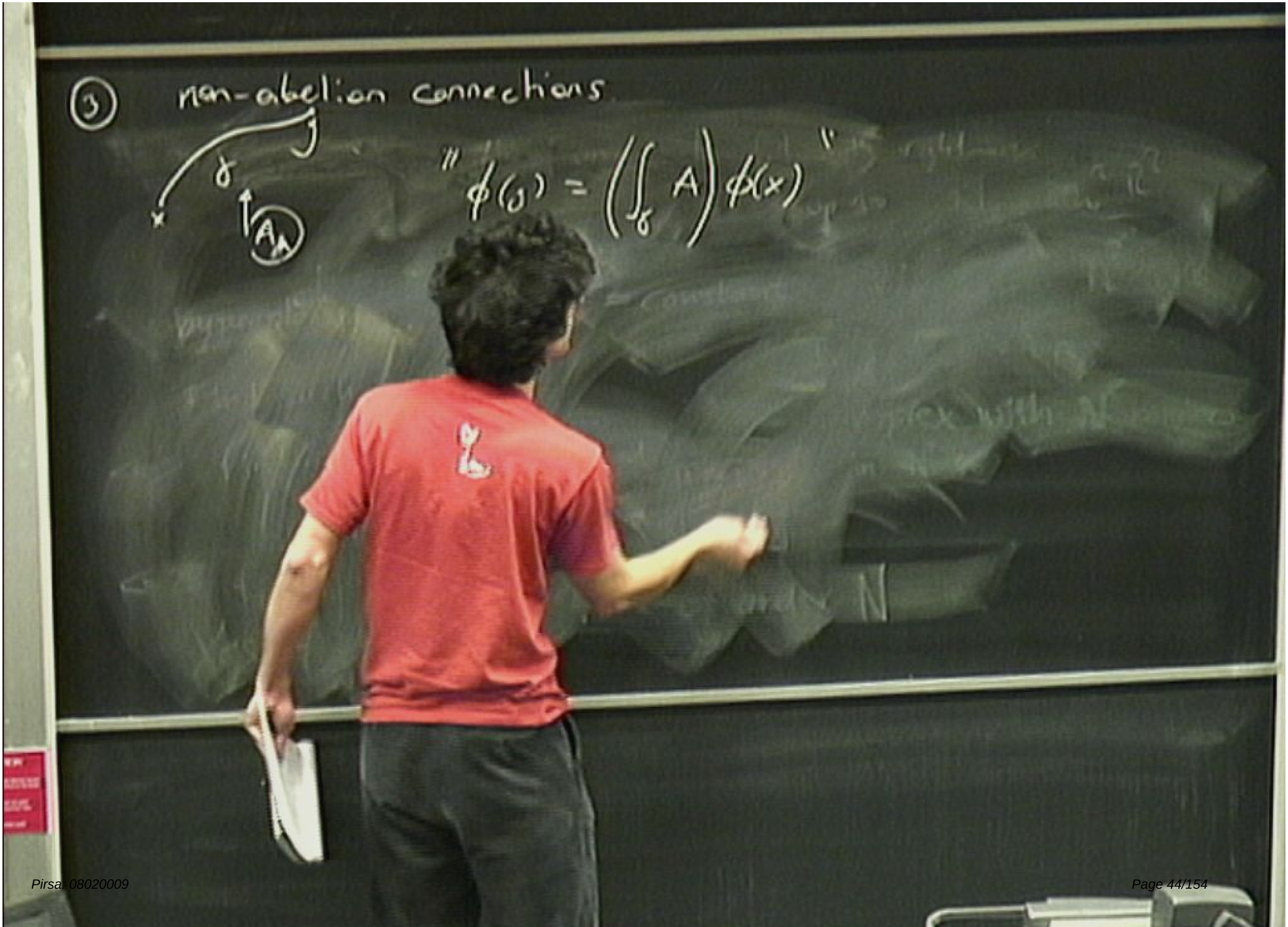

$\oint_\gamma A = \oint_\gamma [A + \partial A]$ if γ is a loop.
where $e^{\oint_\gamma A}$ Wilson loop.
$\langle x | n \rangle = \psi_n(x)$
$\langle A | \gamma \rangle = \psi_\gamma(A)$

$\oint_\gamma A = \oint_\gamma [A + \partial A]$ if γ is a loop.

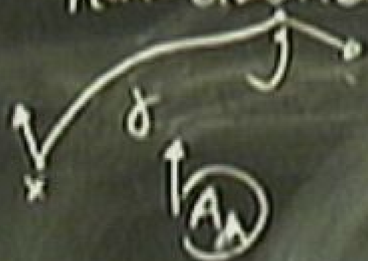
$W_\gamma[A] = e^{\oint_\gamma A}$ Wilson loop.

$\langle x | n \rangle = \psi_n(x)$ h.o.

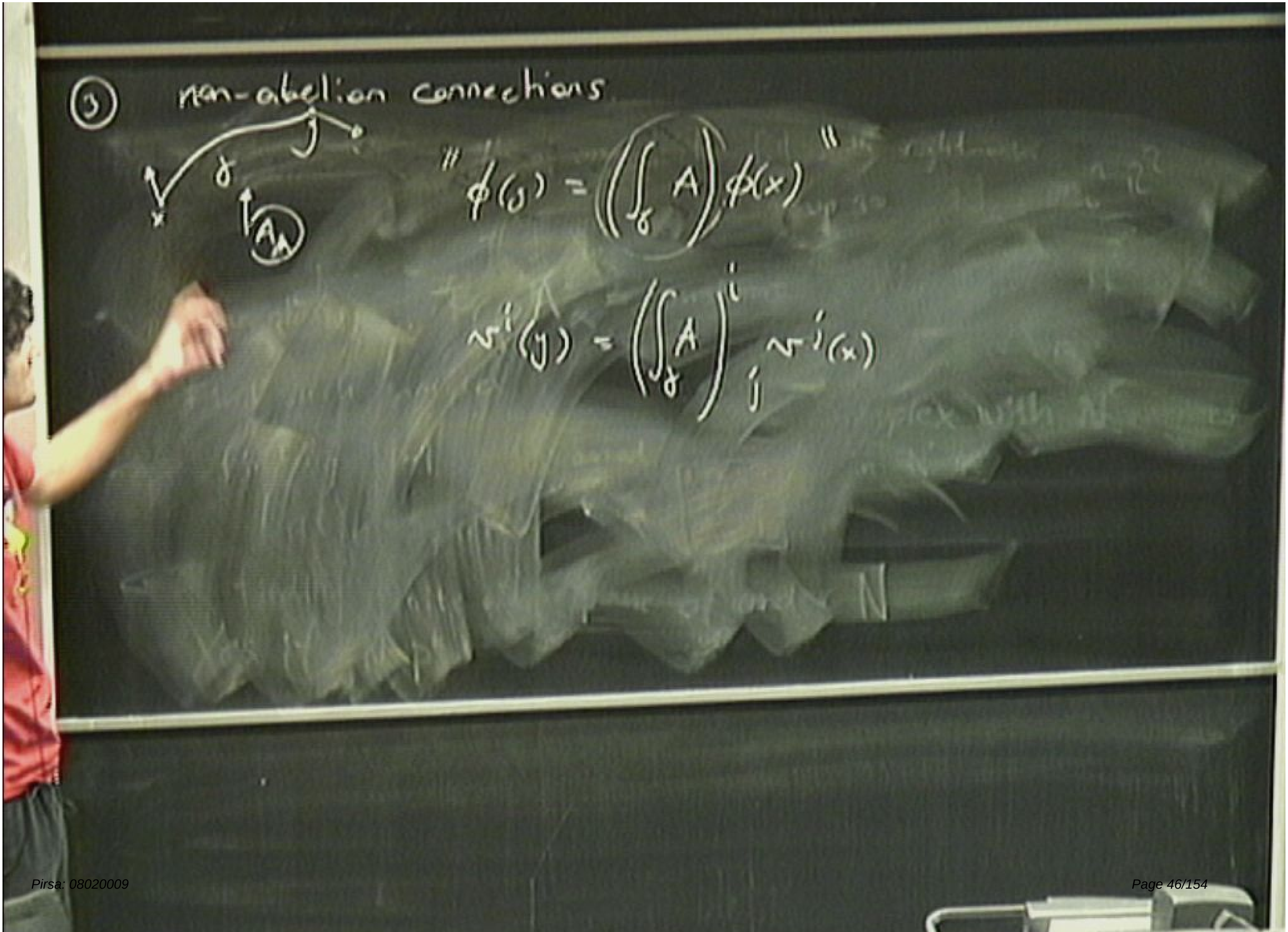
$\langle A | \gamma \rangle = W_\gamma(A)$ em.


③ non-abelian connections
"φ(y) = (∫γ A) φ(x)"
Aμ

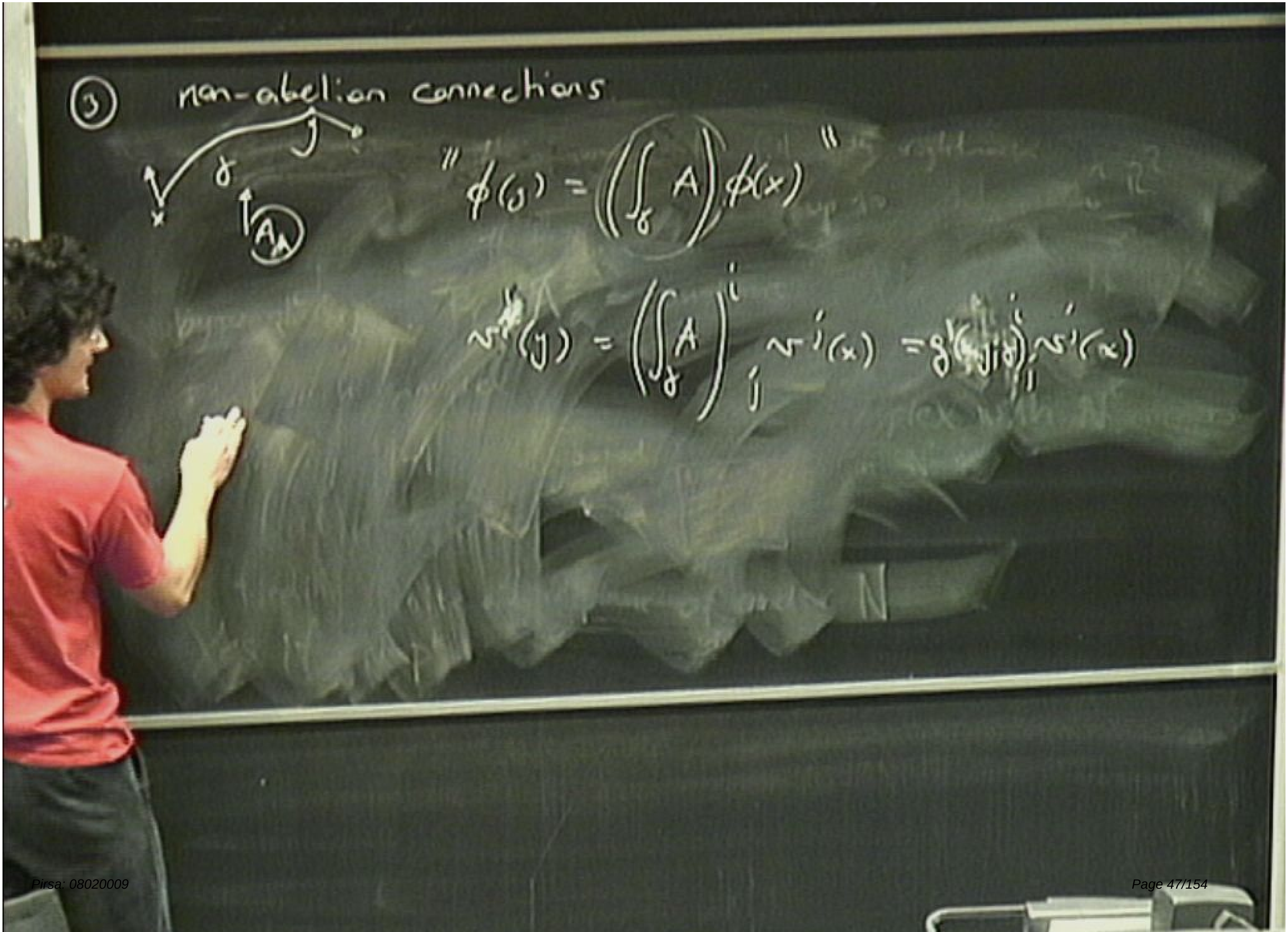
③ non-abelian connections



$$\text{"}\ \phi(\gamma) = \left(\int_\gamma A\right) \phi(x)\ \text{"}$$



③ non-abelian connections
y
x
γ
A_μ
"$\phi(y) = \left(\int_\gamma A\right)\phi(x)$"
$v^i(y) = \left(\int_\gamma A\right)^i_{\ j} v^j(x)$



3 non-abelian connections
"φ(y) = (∫_γ A) φ(x)"
v^i(y) = (∫_γ A)^i_j v^j(x) = g(y|γ)^i_j v^j(x)

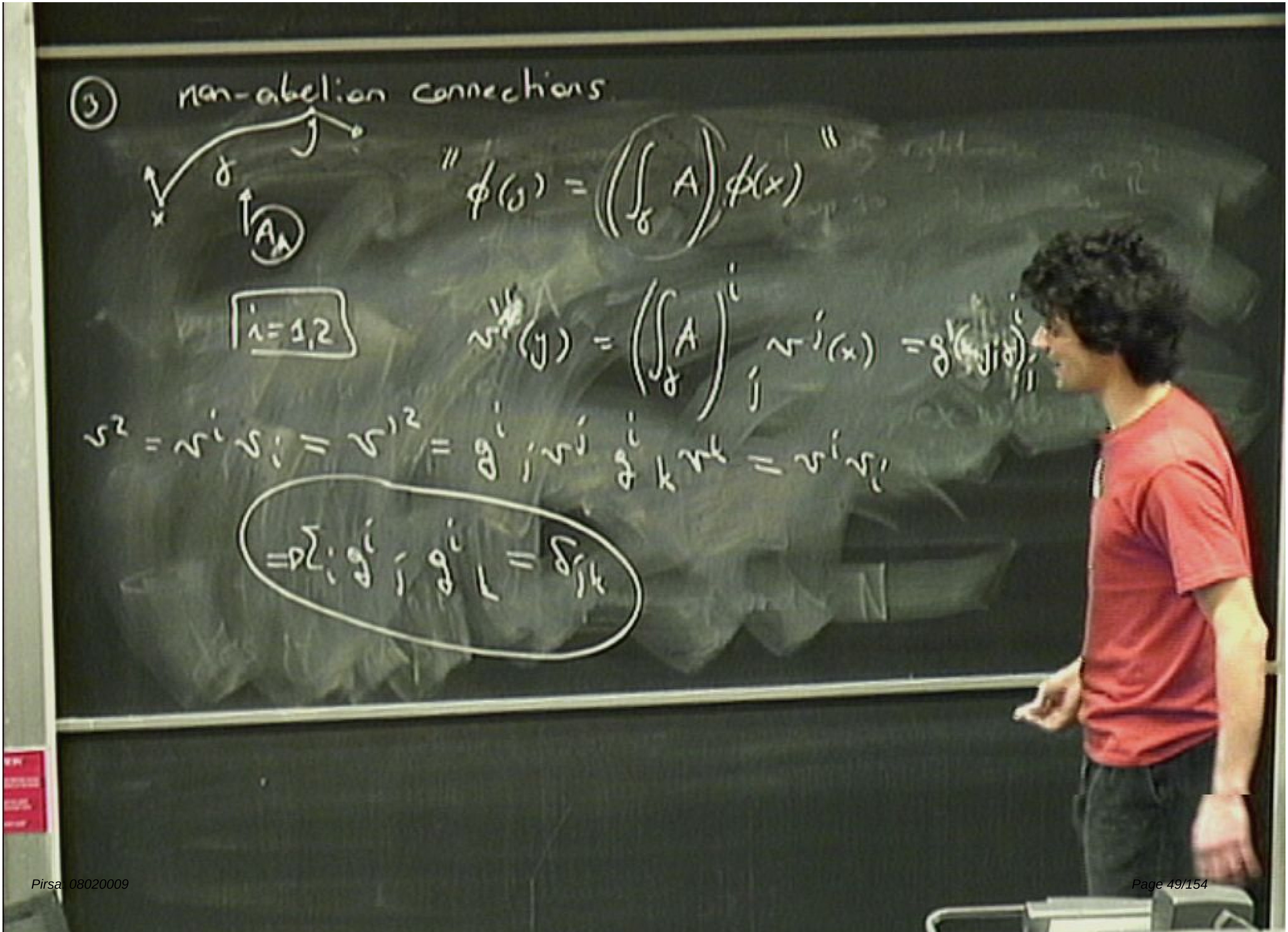
③ non-abelian connections

$$\text{“}\,\phi(y) = \left(\int_\gamma A\right)\phi(x)\,\text{”}$$

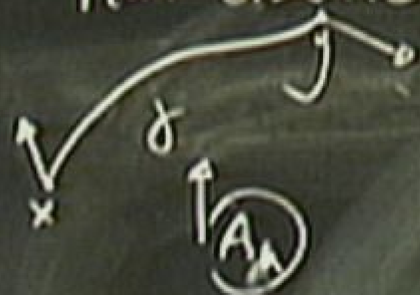
$$v^i(y) = \left(\int_\gamma A\right)^i{}_j v^j(x) = g(y,x,\gamma)^i{}_j\, v^j(x)$$

$$v^2 = v^i v_i = v'^2 = g^i{}_j v^j\, g^i{}_k v^k = v^i v_i$$

$$\Rightarrow \sum_i g^i{}_j\, g^i{}_k = \delta_{jk}$$


3 non-abelian connections
γ
A
i = 1,2
" φ(y) = (∫_γ A) φ(x) "
v^i(y) = (∫_γ A)^i_j v^j(x)
v^2 = v^i v_i = v'^2 = g^i_j v^j g^i_k v^k = v^i v_i
⇒ Σ_i g^i_j g^i_k = δ_jk

③ non-abelian connections



$$\text{"}\ \phi(y) = \left(\int_\gamma A\right) \phi(x)\ \text{"}$$

$\boxed{i = 1,2}$

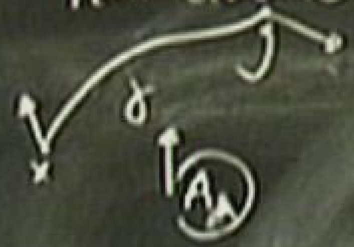
$$v^i(y) = \left(\int_\gamma A\right)^i{}_j\, v^j(x) = g^i{}_j(y,\gamma,x)\, v^j(x)$$

$$v^2 = v^i v_i = v'^2 = g^i{}_j v^j\, g^i{}_k v^k = v^i v_i$$

$$\Rightarrow \sum_i g^i{}_j\, g^i{}_k = \delta_{jk}$$

SU(2)

③ non-abelian connections



"$\phi(y) = \left(\int_\gamma A\right)\phi(x)$"

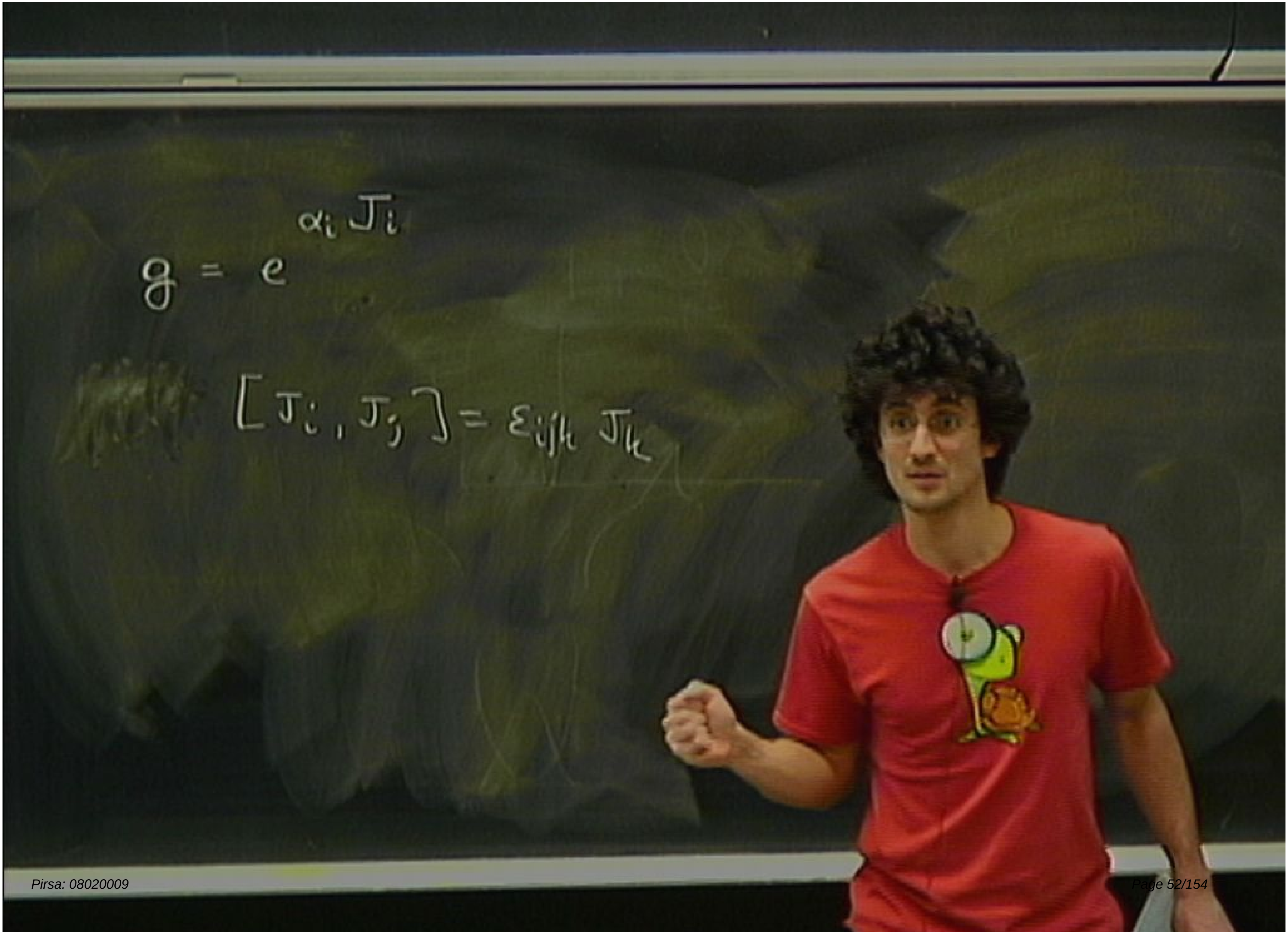
$\boxed{i=1,2}$

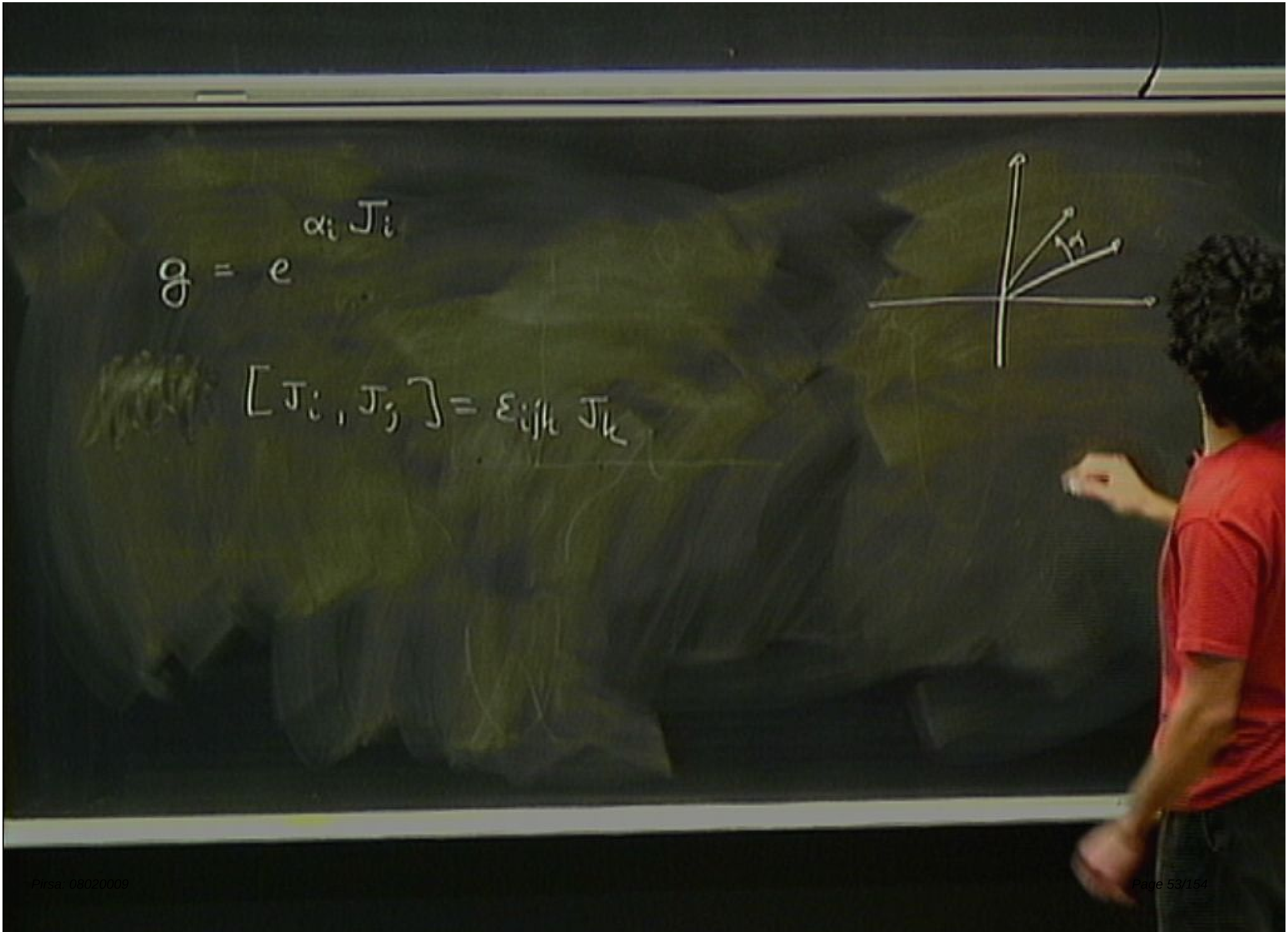
$v^i(y) = \left(\int_\gamma A\right)^i{}_j v^j(x) = g^i{}_j(\gamma)\, v^j(x)$

$v^2 = v^i v_i = v'^2 = g^i{}_j v^j g^i{}_k v^k = v^i v_i$

$\Rightarrow \sum_i g^i{}_j\, g^i{}_k = \delta_{jk}$

SU(2)


$g = e^{\alpha_i J_i}$
$[J_i, J_j] = \varepsilon_{ijk} J_k$


$g = e^{\alpha_i J_i}$
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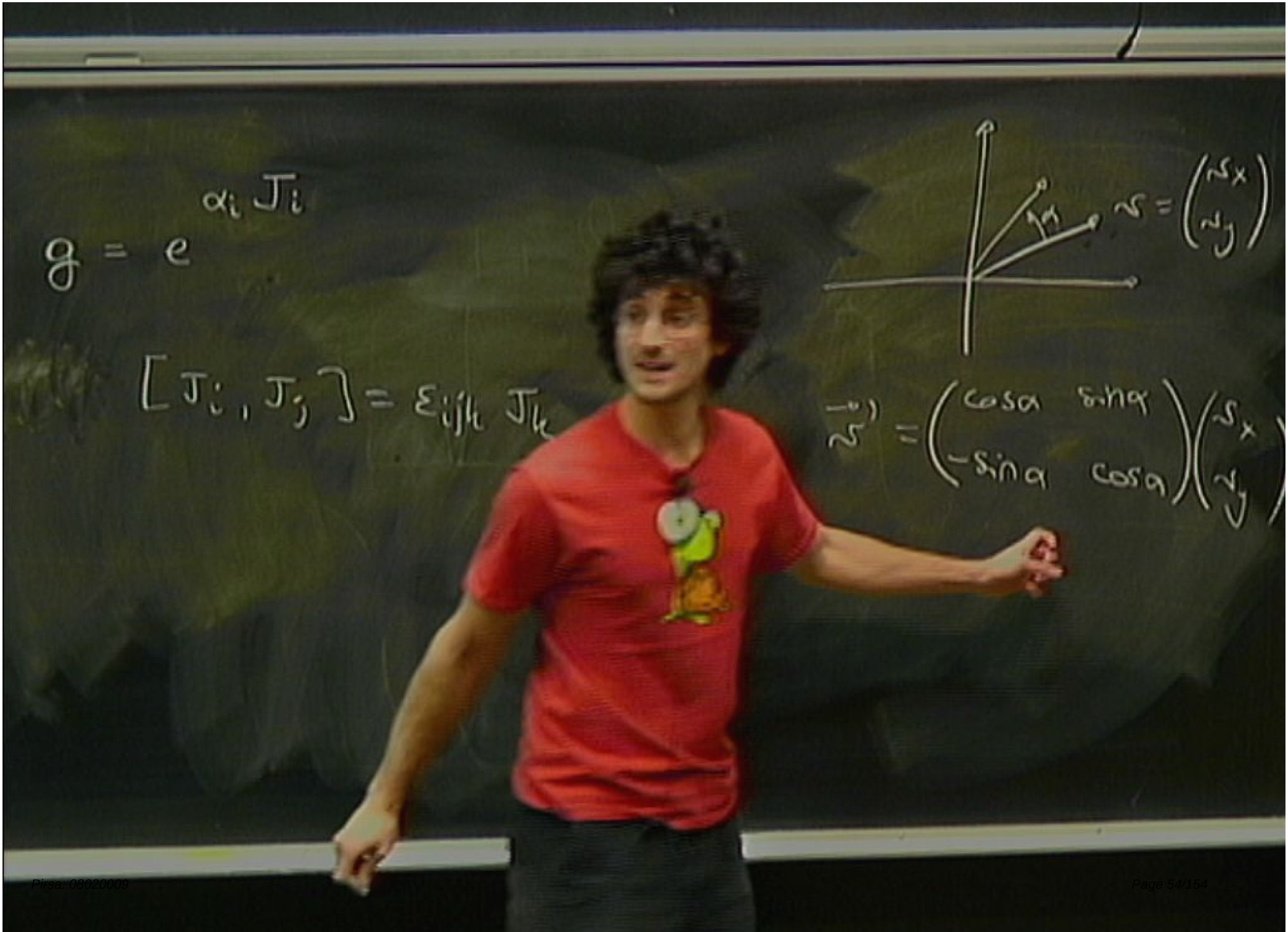
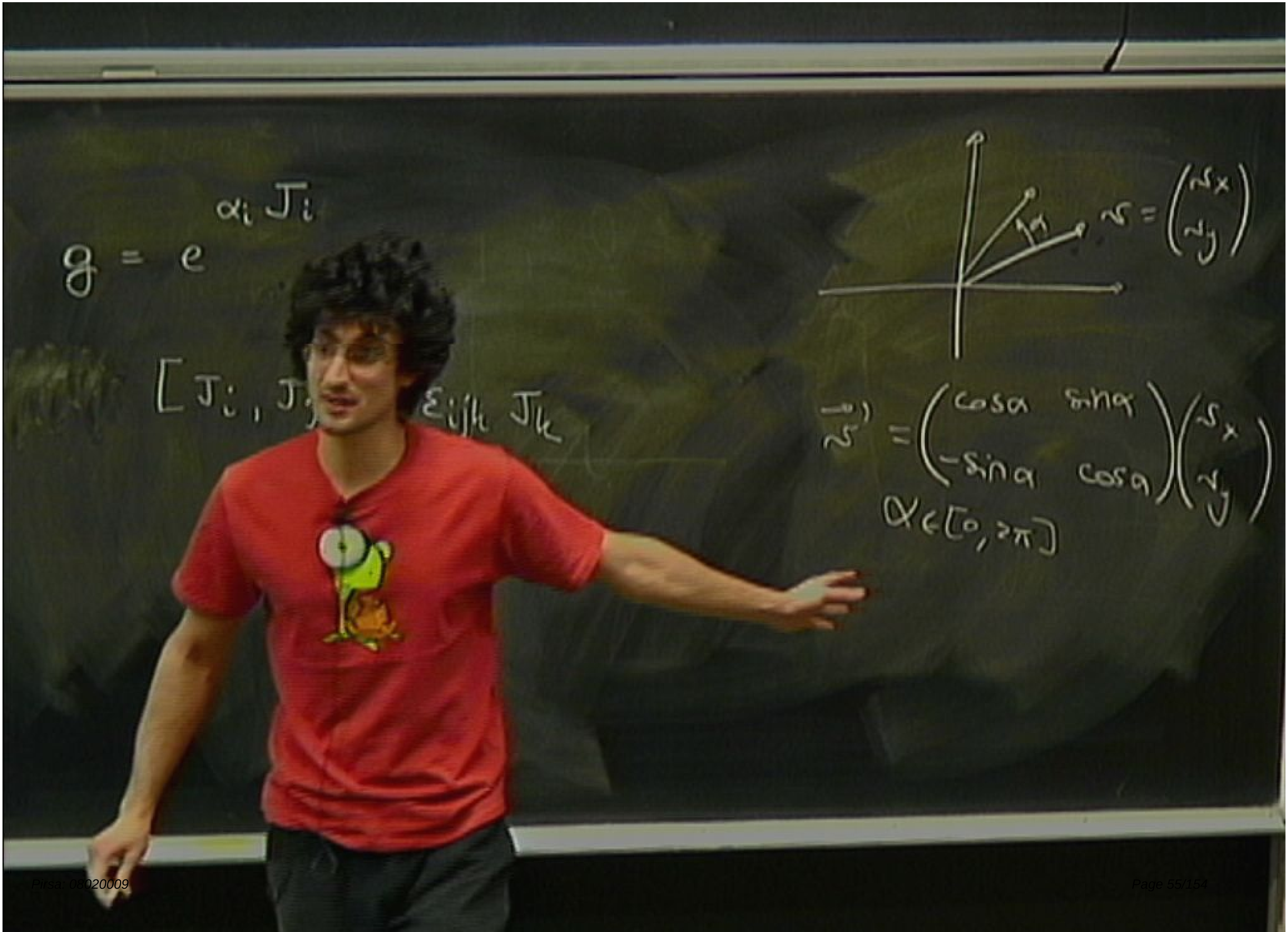
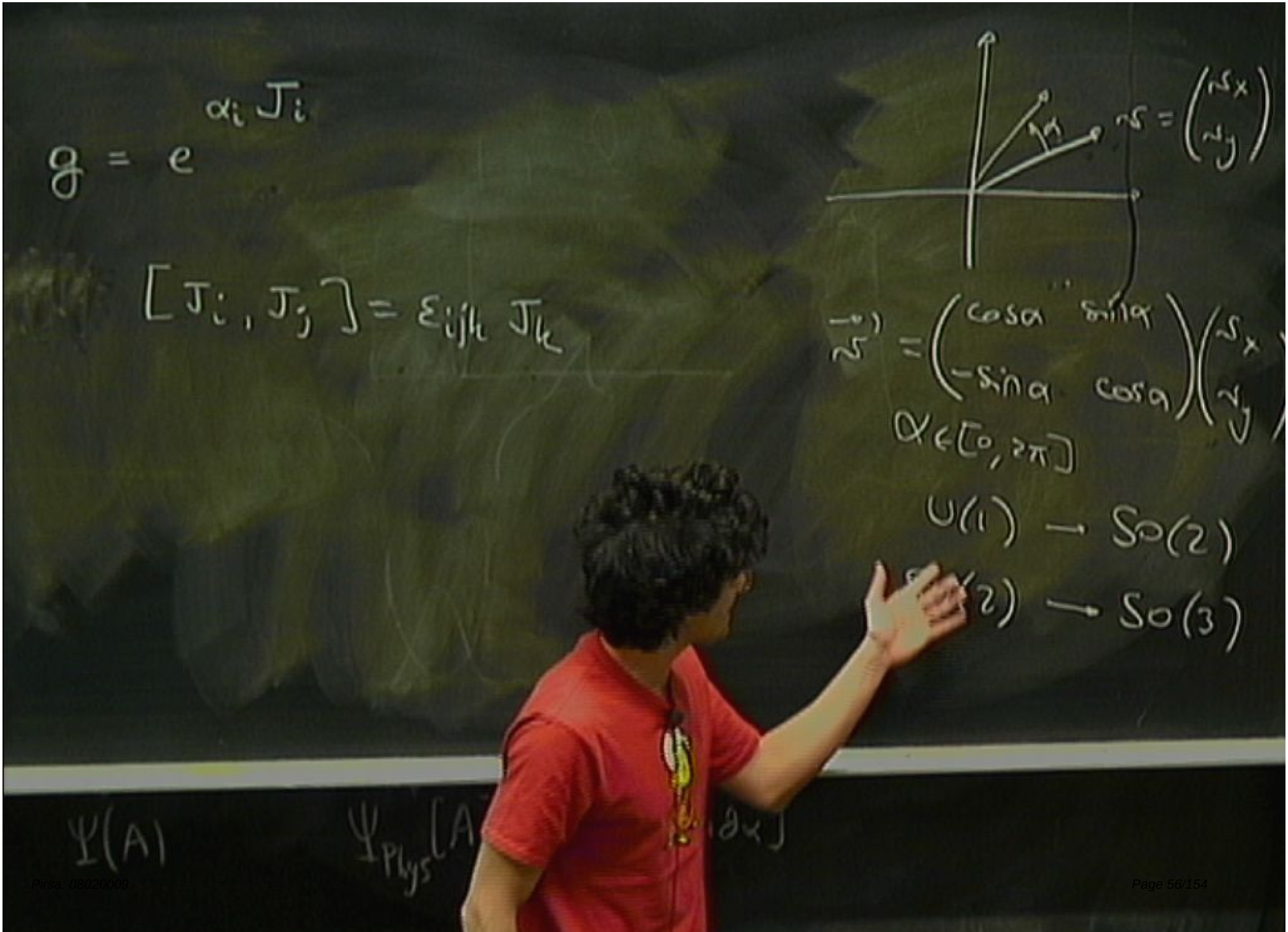
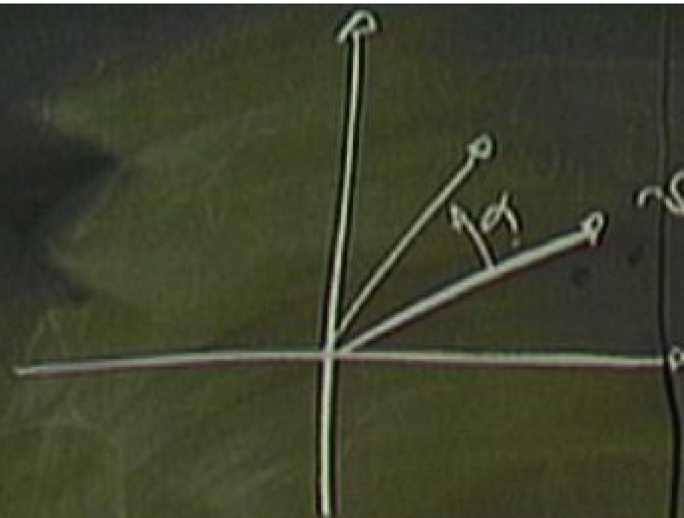


$g = e^{\alpha_i J_i}$
$[J_i, J_j] = \varepsilon_{ijk} J_k$
$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$


$g = e^{\alpha_i J_i}$
$[J_i , J_j$
$\epsilon_{ijk} J_k$
$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
α
$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\alpha \in [0, 2\pi]$


$g = e^{\alpha_i J_i}$
$[J_i, J_j] = \varepsilon_{ijk} J_k$
$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\alpha \in [0, 2\pi]$
$U(1) \to SO(2)$
$(2) \to SO(3)$
$\Psi(A)$



$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\alpha \in [0, 2\pi]$$

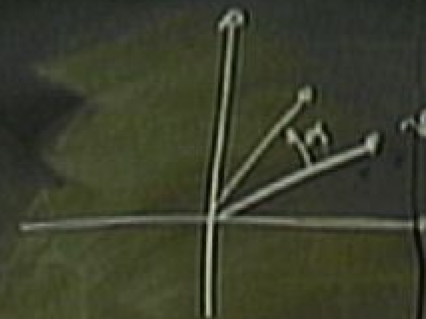
$$U(1) \longrightarrow SO(2)$$

$$SU(2) \longrightarrow SO(3)$$

$$g = e^{\alpha_i J_i}$$

$$[J_i, J_j] = \varepsilon_{ijk} J_k$$

$$SU(2) \ni g^i_j = e^{\int_\gamma A^i_j}$$



$$\vec{s} = \begin{pmatrix} s_x \\ s_y \end{pmatrix}$$

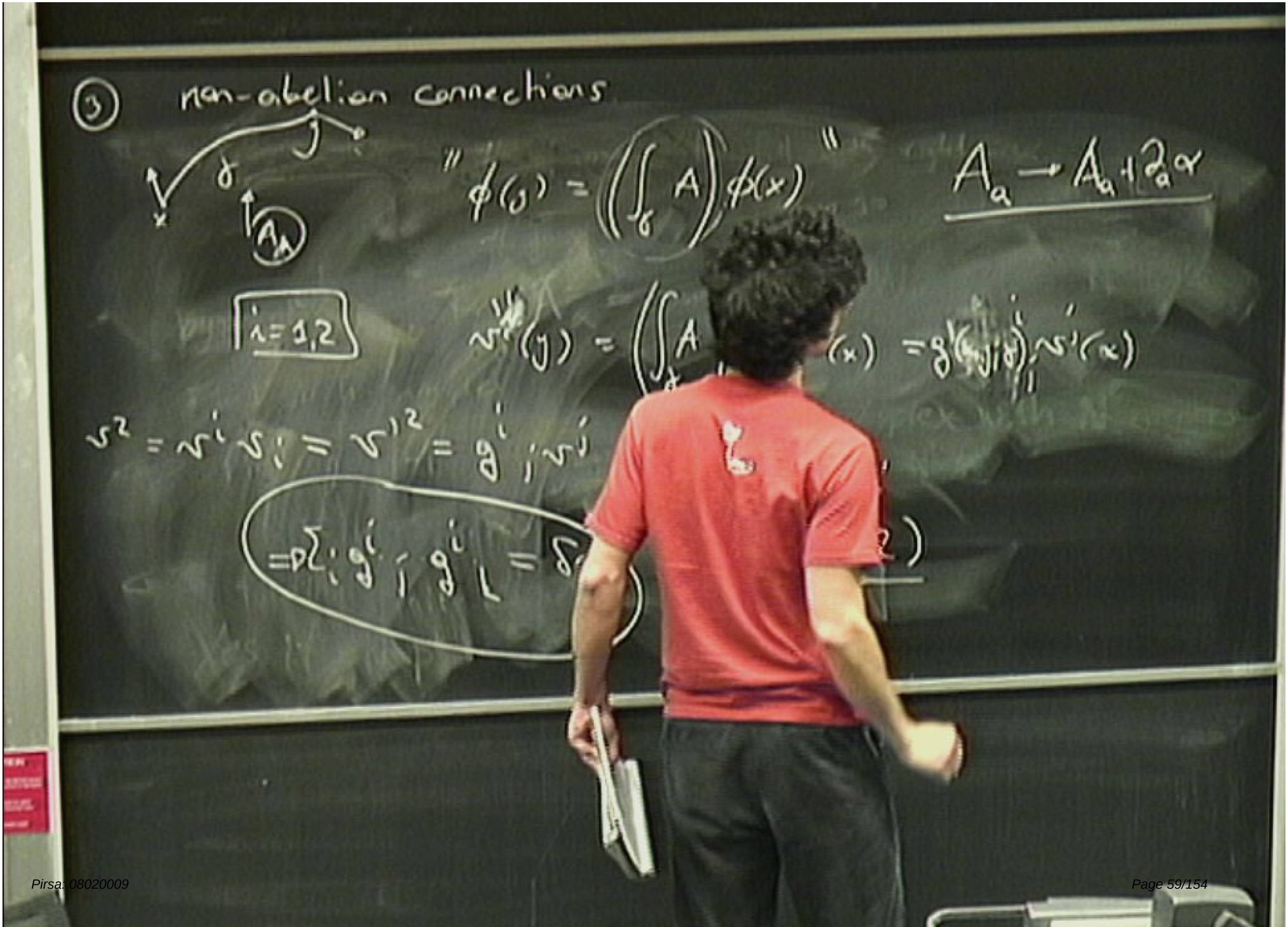
$$\vec{s}\,' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} s_x \\ s_y \end{pmatrix}$$

$\alpha \in [0, 2\pi]$

$U(1) \to SO(2)$

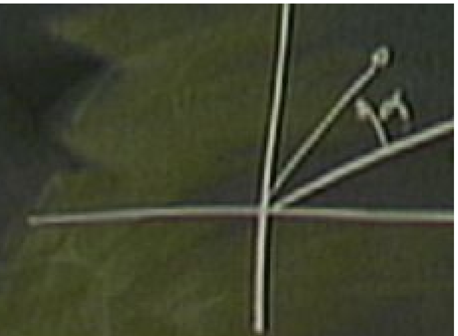
$SU(2) \to SO(3)$

$$\Psi(A) \qquad \Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$$



③ non-abelian connections
"φ(y) = (∫γ A) φ(x)"
Aa → Aa + ∂a α
i = 1,2
v²= v^i v_i = v'² = g^i_j v^j

$$g = e^{\alpha_i J_i}$$



$$[J_i , J_j] = \varepsilon_{ijk} J_k$$

$$\vec{v}' = \begin{pmatrix} \cos\alpha & \\ -\sin\alpha & \end{pmatrix}$$

$$\alpha \in [0, 2\pi]$$

$$SU(2) \ni g^i_j = e^{\int_\gamma A^i_j}$$

$U(1) \longrightarrow$

$A($

$SU(2) \longrightarrow$

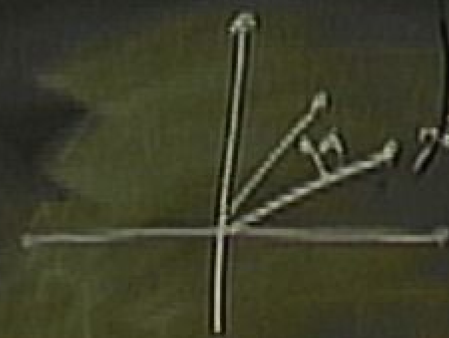
$$\Psi(A) \qquad \Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$$

$g = e^{\alpha_i J_i}$ $\boxed{i=1,2,3}$

$[J_i, J_j] = \varepsilon_{ijk} J_k$

$SU(2) \ni g_\Gamma = e^{\int_\gamma A^a J_a}$

$A($



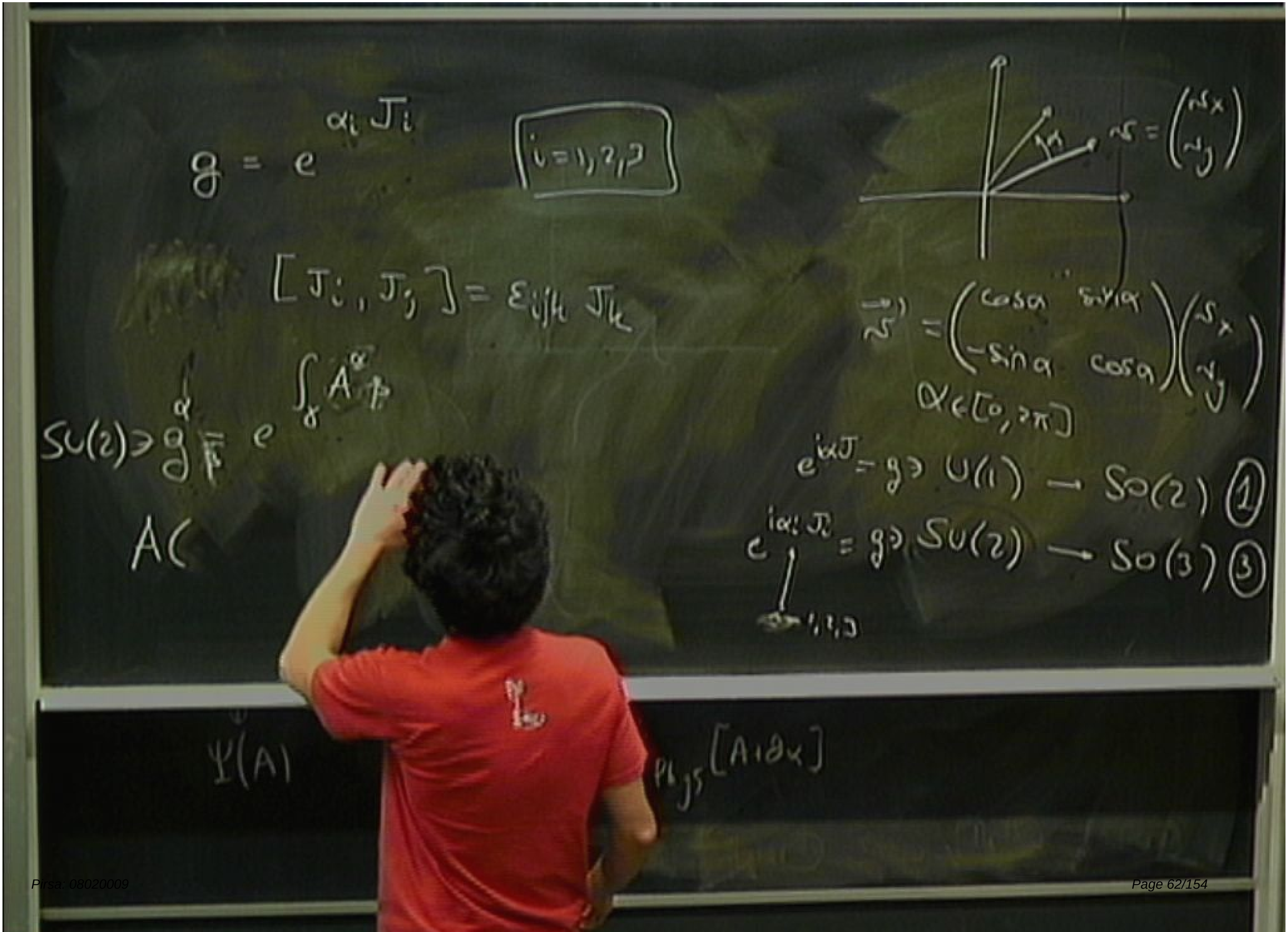
$\vec{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$

$\vec{n}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}($

$\alpha \in [0, 2\pi]$

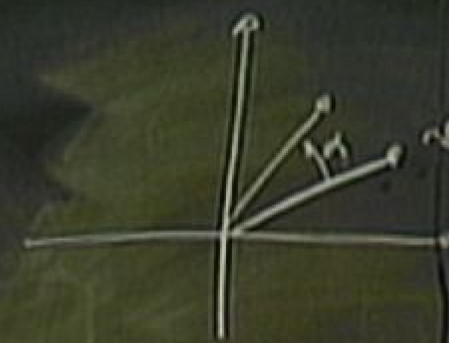
$U(1) \to SO($

$SU(2) \to$



$g = e^{\alpha_i J_i}$
$i = 1, 2, 3$
$[J_i, J_j] = \varepsilon_{ijk} J_k$
$SU(2) \ni g^{\alpha}_{\beta} = e^{\int_\gamma A^\alpha_\beta}$
A(
$\Psi(A)$
$\Psi_{phys}[A + \partial\alpha]$
$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\alpha \in [0, 2\pi]$
$e^{i\alpha J} = g \ni U(1) \to SO(2)$ 1
$e^{i\alpha_i J_i} = g \ni SU(2) \to SO(3)$ 3
1,2,3

$g = e^{\alpha_i J_i}$ $\boxed{i=1,2,3}$



$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

$[J_i, J_j] = \varepsilon_{ijk} J_k$

$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

$\alpha \in [0, 2\pi]$

$SU(2) \ni g^{\alpha}_{\beta} = e^{\int_\gamma A^{\alpha}_{\beta}}$

$e^{i\alpha J} = g \ni U(1) \to SO(2)$ 1

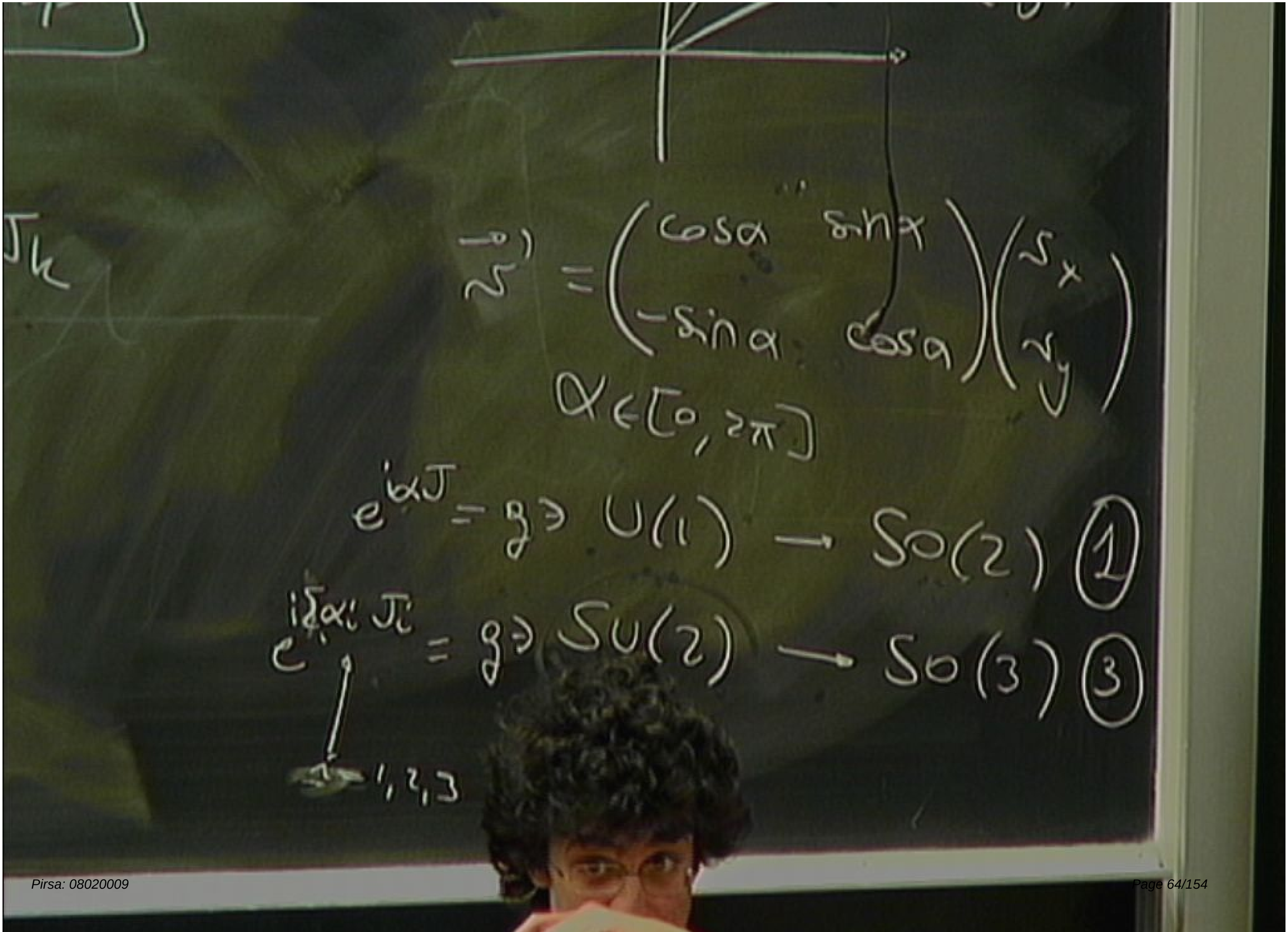
$e^{i\alpha_i J_i} = g \ni SU(2) \to SO(3)$ 3

A(

$1,2,3$

$\Psi(A)$

$\Psi_{Phys}[A] = \Psi_{Phys}$


$\vec{v}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
$\alpha \in [0, 2\pi]$
$e^{i\alpha J} = g \ni U(1) \to SO(2)$ 1
$e^{i\sum \alpha_i J_i} = g \ni SU(2) \to SO(3)$ 3
$i = 1,2,3$

$$\mathcal{G} = \partial_a E^a = 0$$

$$\widehat{\partial_a E^a} \, \Psi(A) = 0$$

$$\mathcal{H}_0 \xrightarrow{\mathcal{G}=0} \mathcal{H}_{phys}$$

$$\Psi(A) \in \mathcal{H}_0 \qquad \Psi \in \mathcal{H}_{phys}$$

$$\Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$$

a, b, t	H
a, b	$H(a,b) = 0$

$$H_{em} = B^2 + E^2$$

$$\hat{H}\Psi = 0$$

A group has a mult. rule, an identity element, and an inverse element.

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$\mathbb{R}, \oplus$

A group has a mult. rule, an identity element, and an inverse element.

$$\mathbb{R}, \odot \longmapsto \quad x \cdot y = z \in \mathbb{R}, \quad \begin{cases} x \cdot 1 = x \\ x \cdot \frac{1}{x} = 1 \end{cases}$$

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$$g_1, g_2 \in SU(2)$$

$$g_1 \cdot g_2 = g_3 \in SU(2)$$

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$$g_1, g_2 \in SU(2)$$

$$\begin{cases} g_1 \cdot g_2 = g_3 \in SU(2) \\ g_1 \cdot 1 = g_1 \\ g_1 \cdot g_1^{-1} - 1 \end{cases}$$

$$\boxed{g = e^{\sum_i \alpha_i J_i}}$$

$$\alpha_i \in I \subseteq \mathbb{R}$$

A group has a mult. rule, an identity element, and an inverse element.

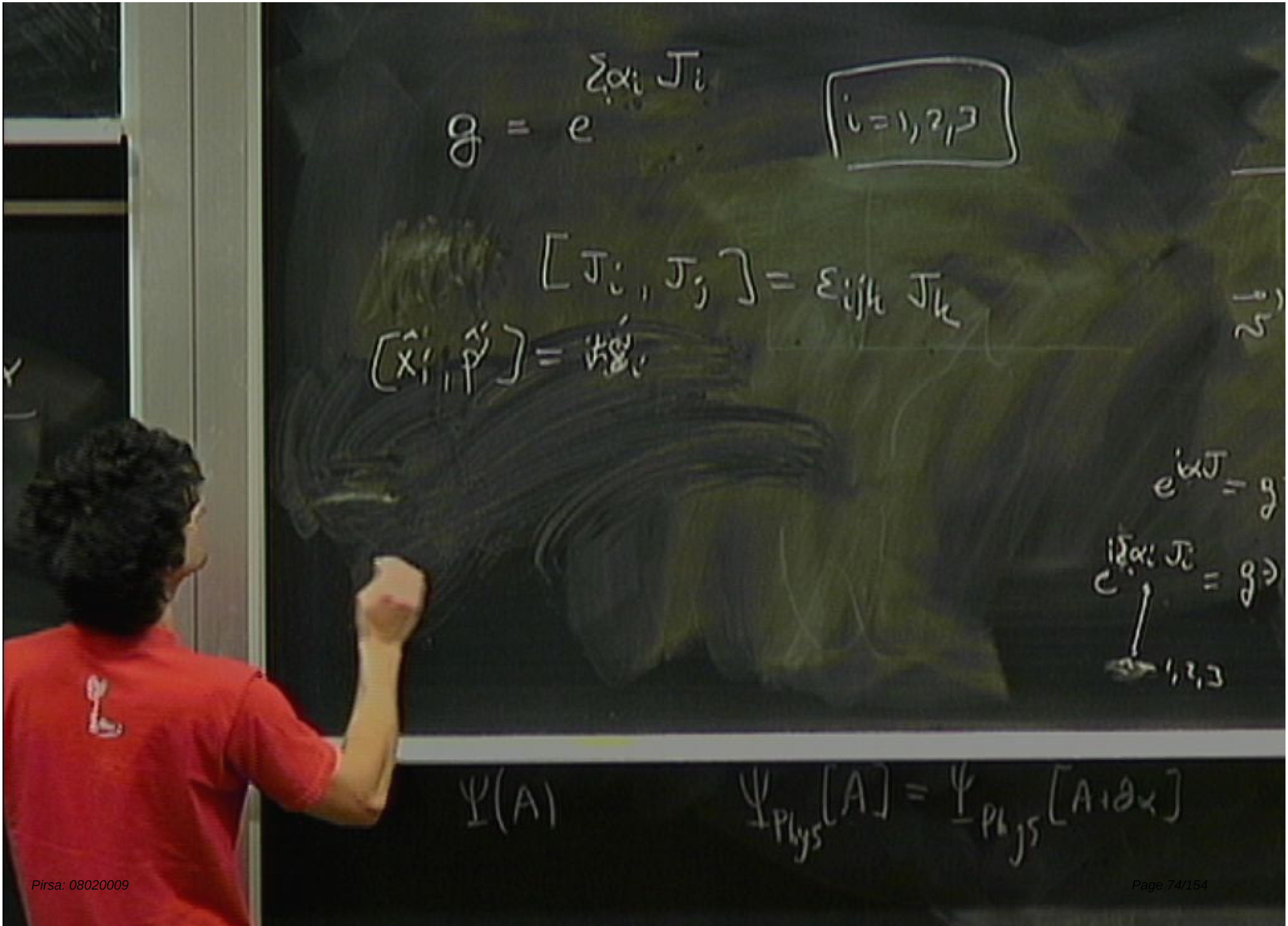
$\mathbb{R}, \odot \longmapsto x \cdot y = z \in \mathbb{R}, \quad \begin{cases} x \cdot 1 = x \\ x \cdot \frac{1}{x} = 1 \end{cases}$

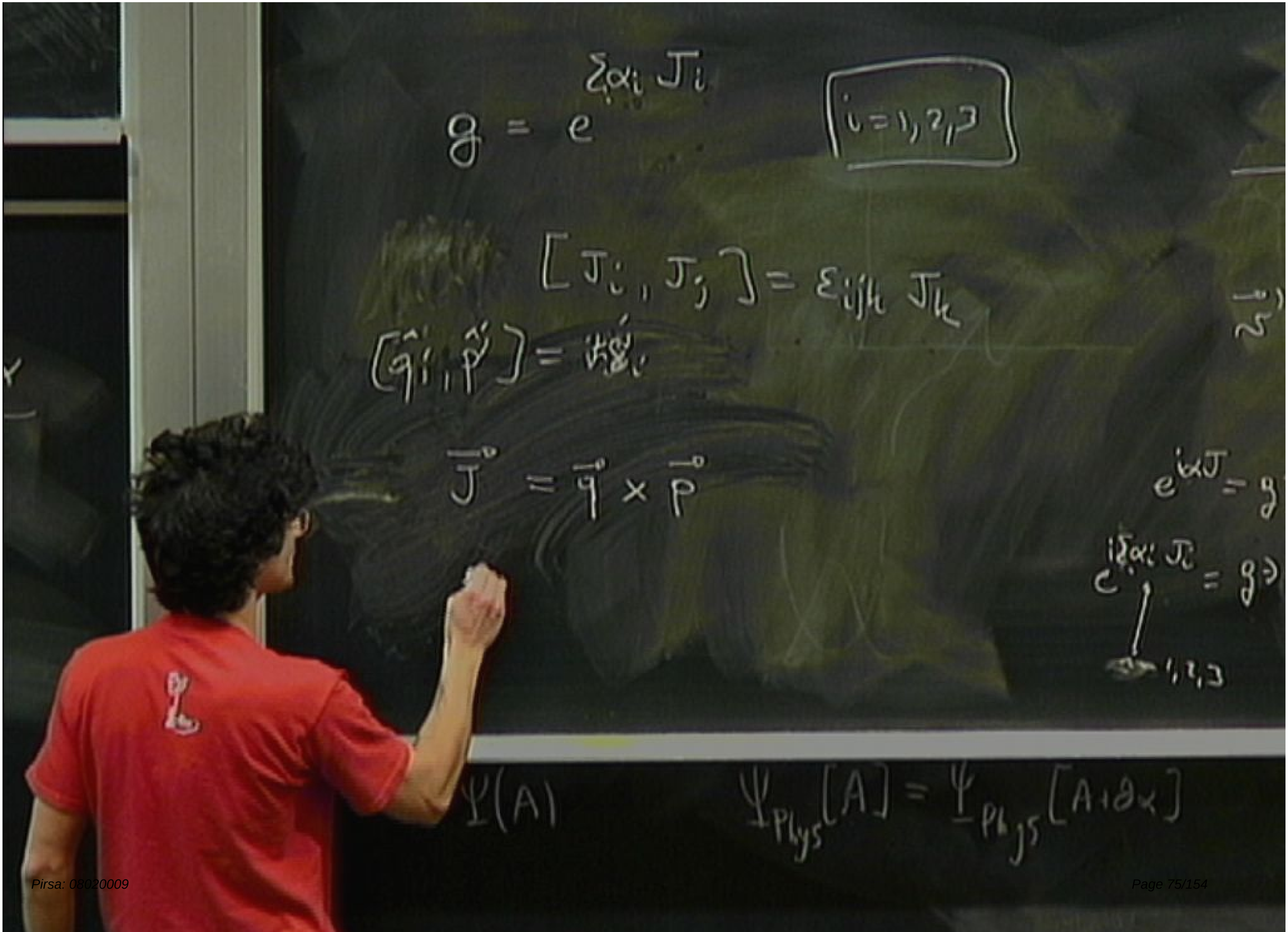
$g_1, g_2 \in SU(2)$

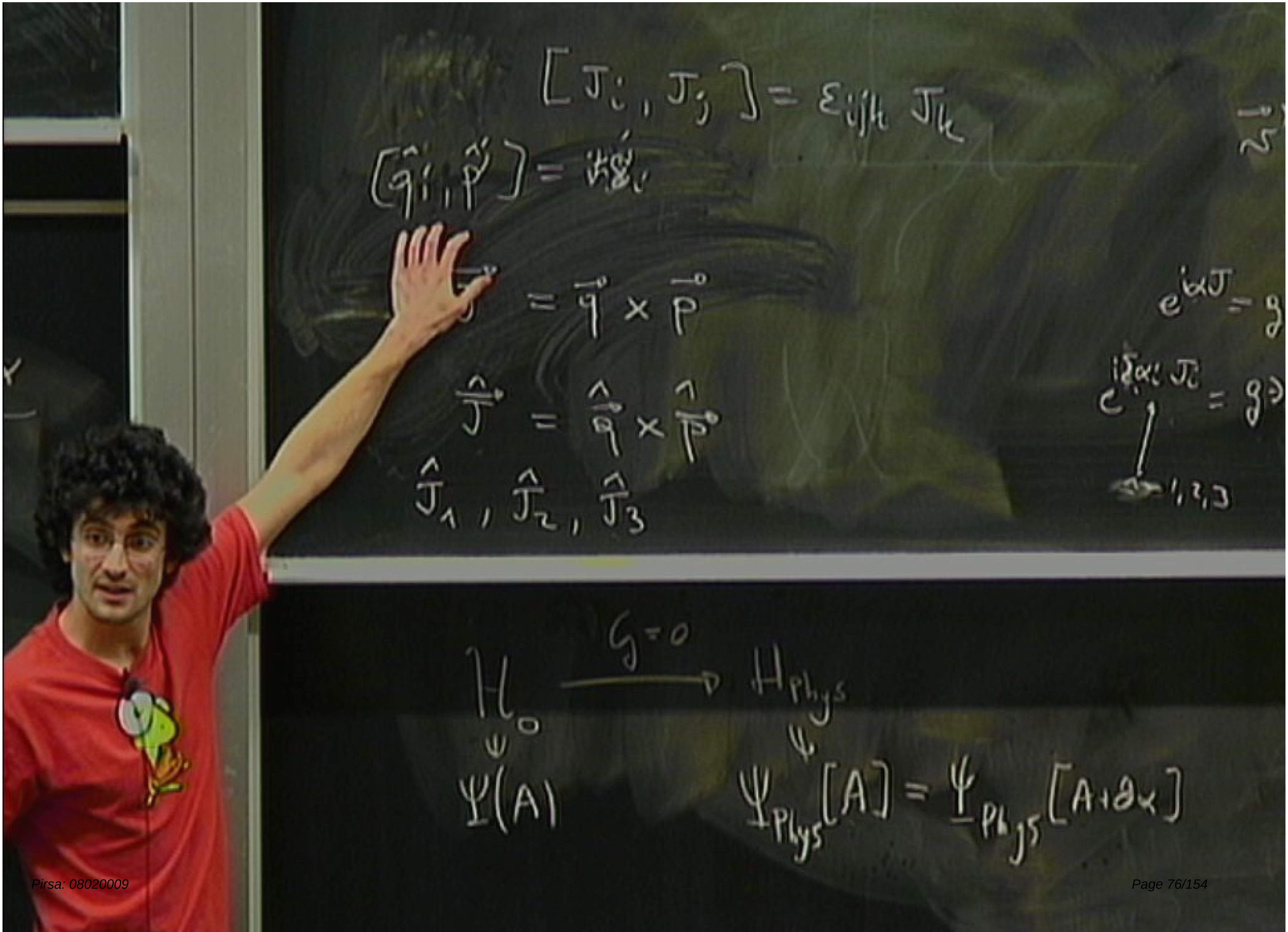
$$\begin{cases} g_1 \cdot g_2 = g_3 \in SU(2) \\ g_1 \cdot 1 = g_1 \\ g_1 \cdot g_1^{-1} = 1 \end{cases}$$

$$\boxed{g = e^{\sum_i \alpha_i T_i}}$$

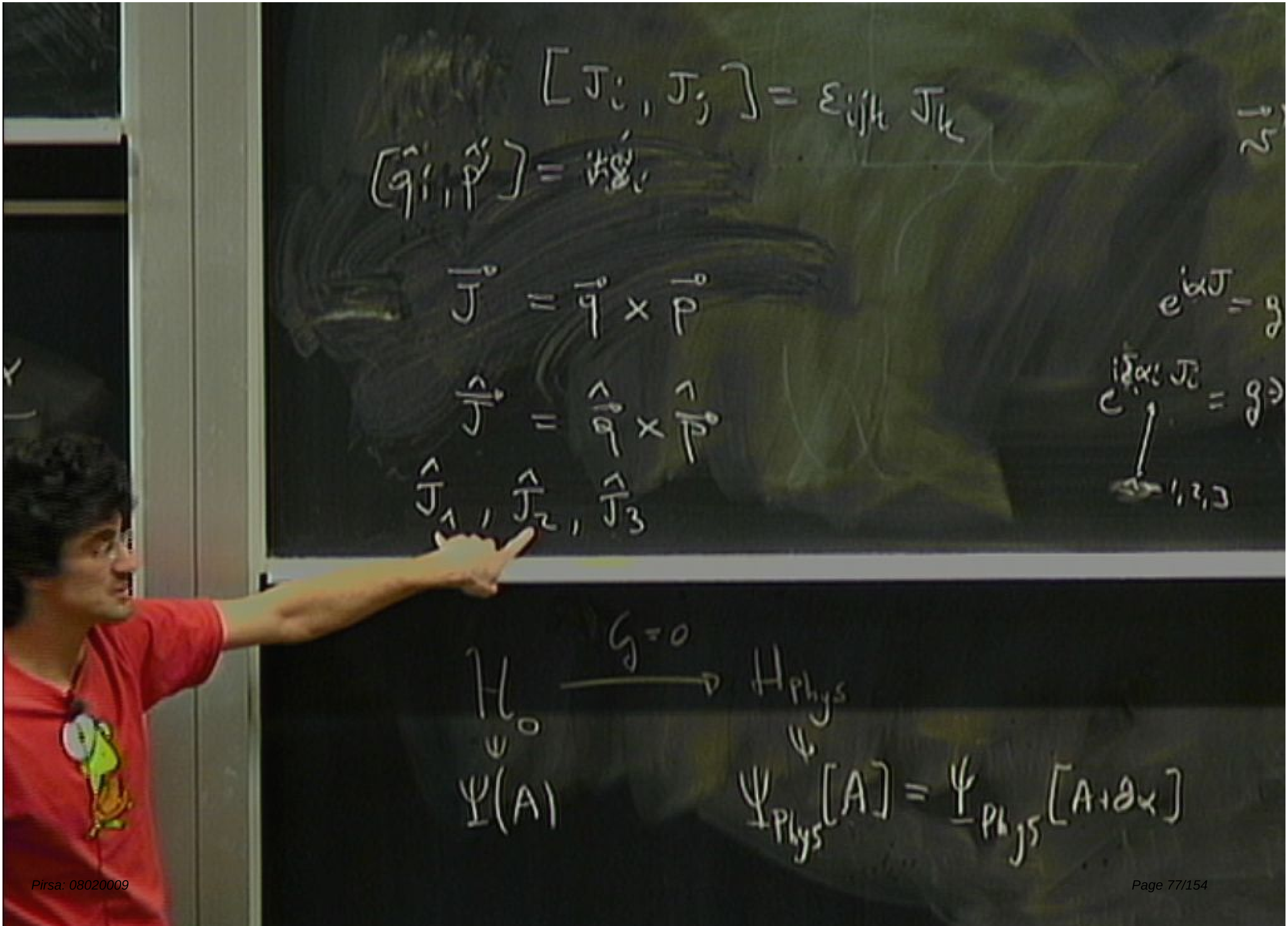
$\alpha_i \in I \subseteq \mathbb{R}$


$g = e^{\sum_i \alpha_i J_i}$
$i = 1, 2, 3$
$[J_i, J_j] = \varepsilon_{ijk} J_k$
$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$
$e^{i\alpha J} = g$
$e^{i\sum \alpha_i J_i} = g$
$1, 2, 3$
$\Psi(A)$
$\Psi_{Phys}[A] = \Psi_{Phys}[A + \partial\alpha]$


g = e^{Σ α_i J_i}
i = 1, 2, 3
[J_i , J_j] = ε_ijk J_k
[q̂^i , p̂^j] = iħδ^ij
J = q × p
e^{iαJ} = g
e^{iΣ α_i J_i} = g
1, 2, 3
Ψ(A)
Ψ_phys[A] = Ψ_phys[A + ∂α]



[J_i , J_j] = ε_ijk J_k
[q̂^i , p̂^j] = iħδ^j_i
J = q × p
Ĵ = q̂ × p̂
Ĵ_1 , Ĵ_2 , Ĵ_3
e^{iα J} = g
e^{i Σ α_i J_i} = g
1,2,3
G=0
H_0 → H_phys
Ψ(A)
Ψ_phys[A] = Ψ_phys[A+∂α]



$[J_i, J_j] = \varepsilon_{ijk} J_k$
$[\hat{q}^i, \hat{p}^j] = i\hbar\delta^j_i$
$\vec{J} = \vec{q} \times \vec{p}$
$\hat{\vec{J}} = \hat{\vec{q}} \times \hat{\vec{p}}$
$\hat{J}_1, \hat{J}_2, \hat{J}_3$
$e^{i\alpha J} = g$
$e^{i\sum \alpha_i J_i} = g$
$1,2,3$
$H_0 \xrightarrow{G=0} H_{phys}$
$\Psi(A)$
$\Psi_{phys}[A] = \Psi_{phys}[A + \partial\alpha]$

$$g = e^{\sum \alpha_i J_i} \qquad \boxed{i = 1,2,3}$$

$$\boxed{[J_i, J_j] = \varepsilon_{ijk} J_k}$$

$$\varepsilon_{123} = 1 \qquad \varepsilon_{132} = -1 \qquad \varepsilon_{112} = 0$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \Rightarrow$$

$$[J_1, J_2] = J_3$$

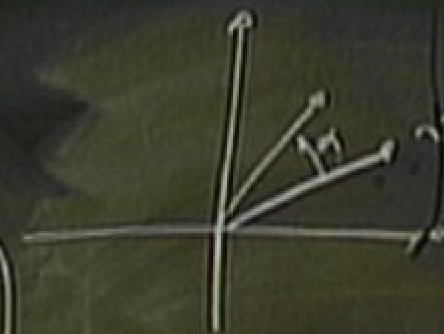
$$[J_1, J_1] = 0$$

$$[J_1, J_3] = -J_2$$

$$\vec{J} = \vec{q} \times \vec{p}$$

$$\hat{\vec{J}} = \hat{\vec{q}} \times \hat{\vec{p}}$$

$$\hat{J}_1, \hat{J}_2, \hat{J}_3$$



$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

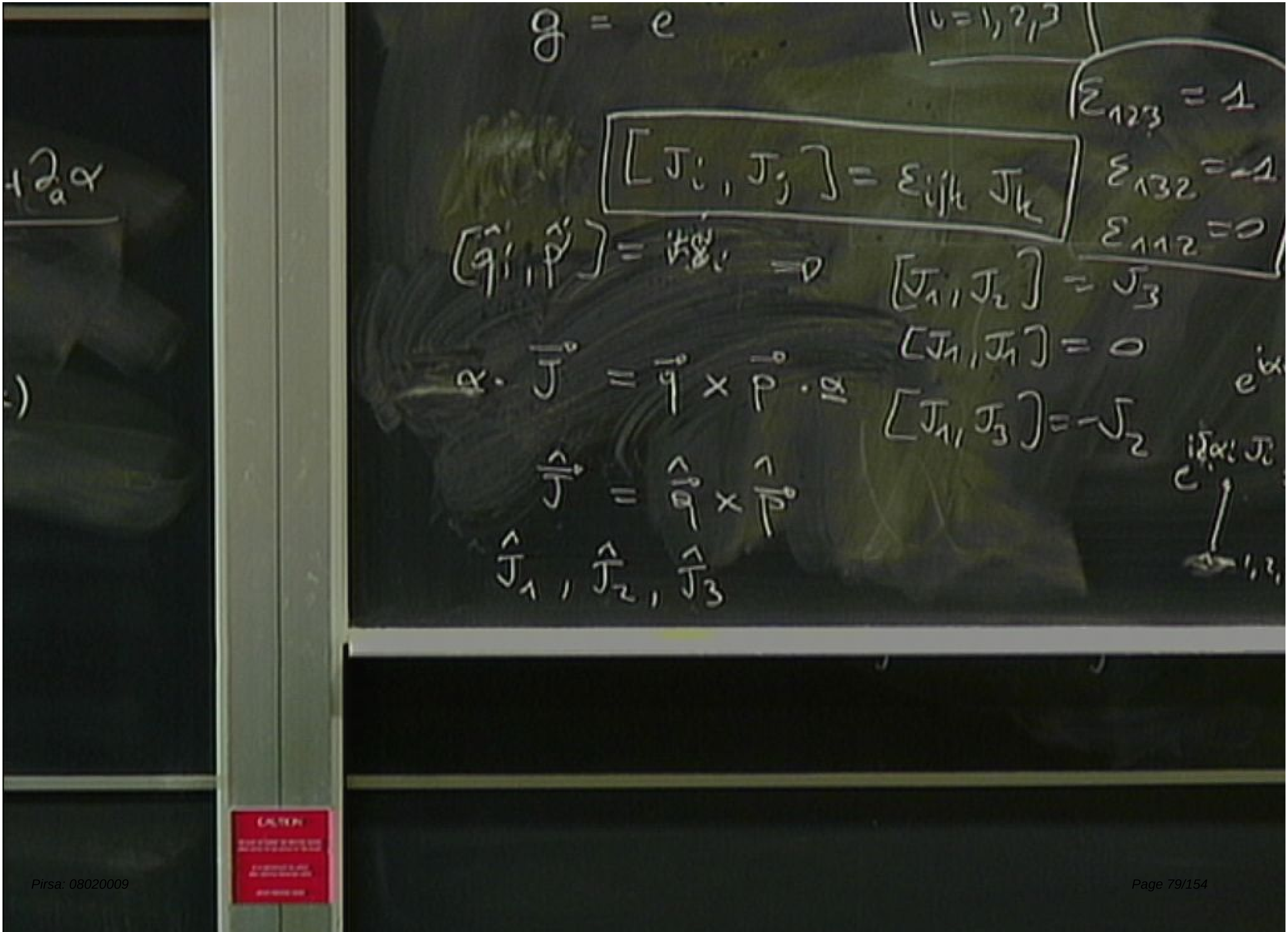
$$\vec{v}\,' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\alpha \in [0, 2\pi]$$

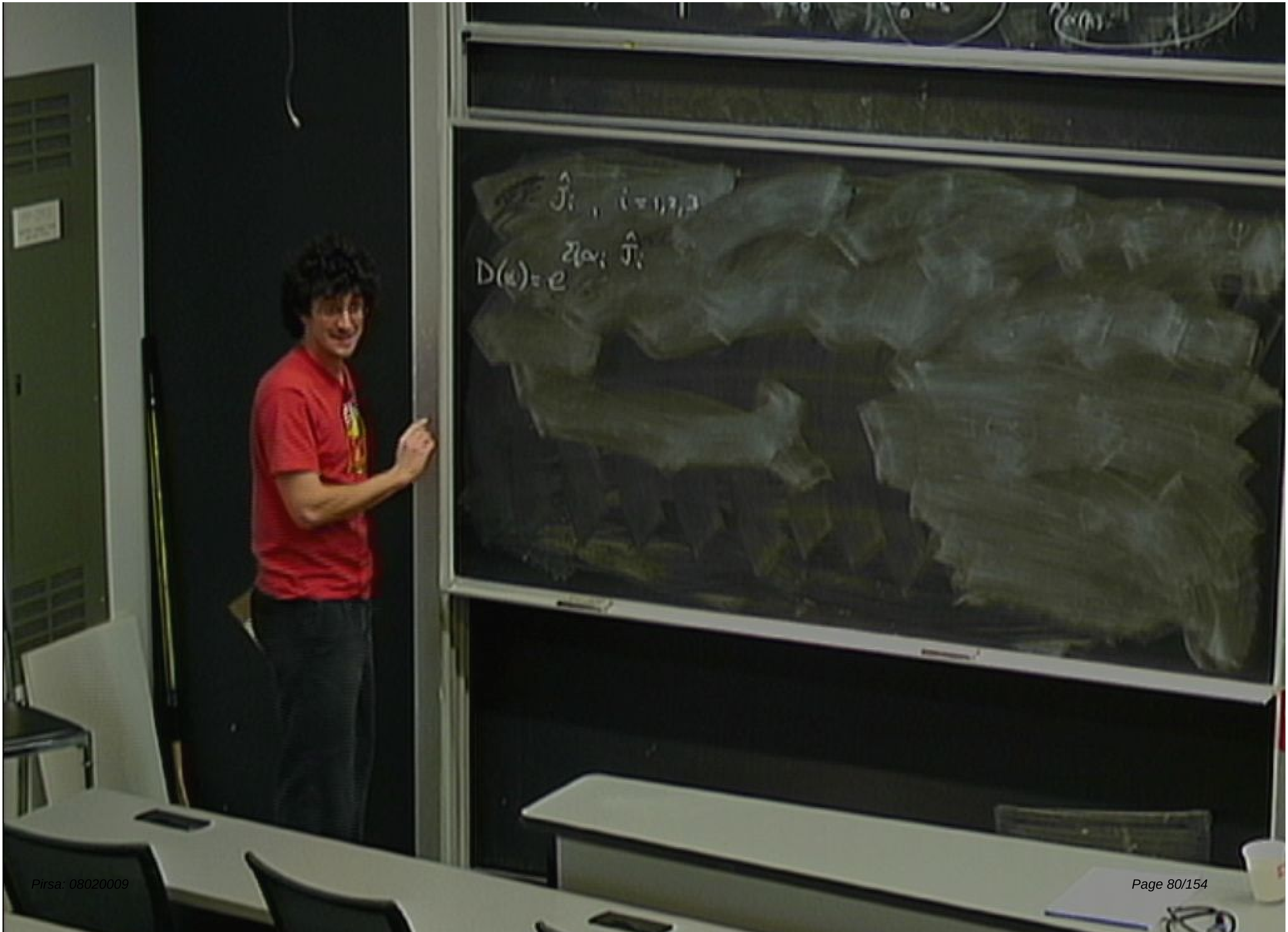
$$e^{i\alpha J} = g \ni U(1) \to SO(2) \quad 1$$

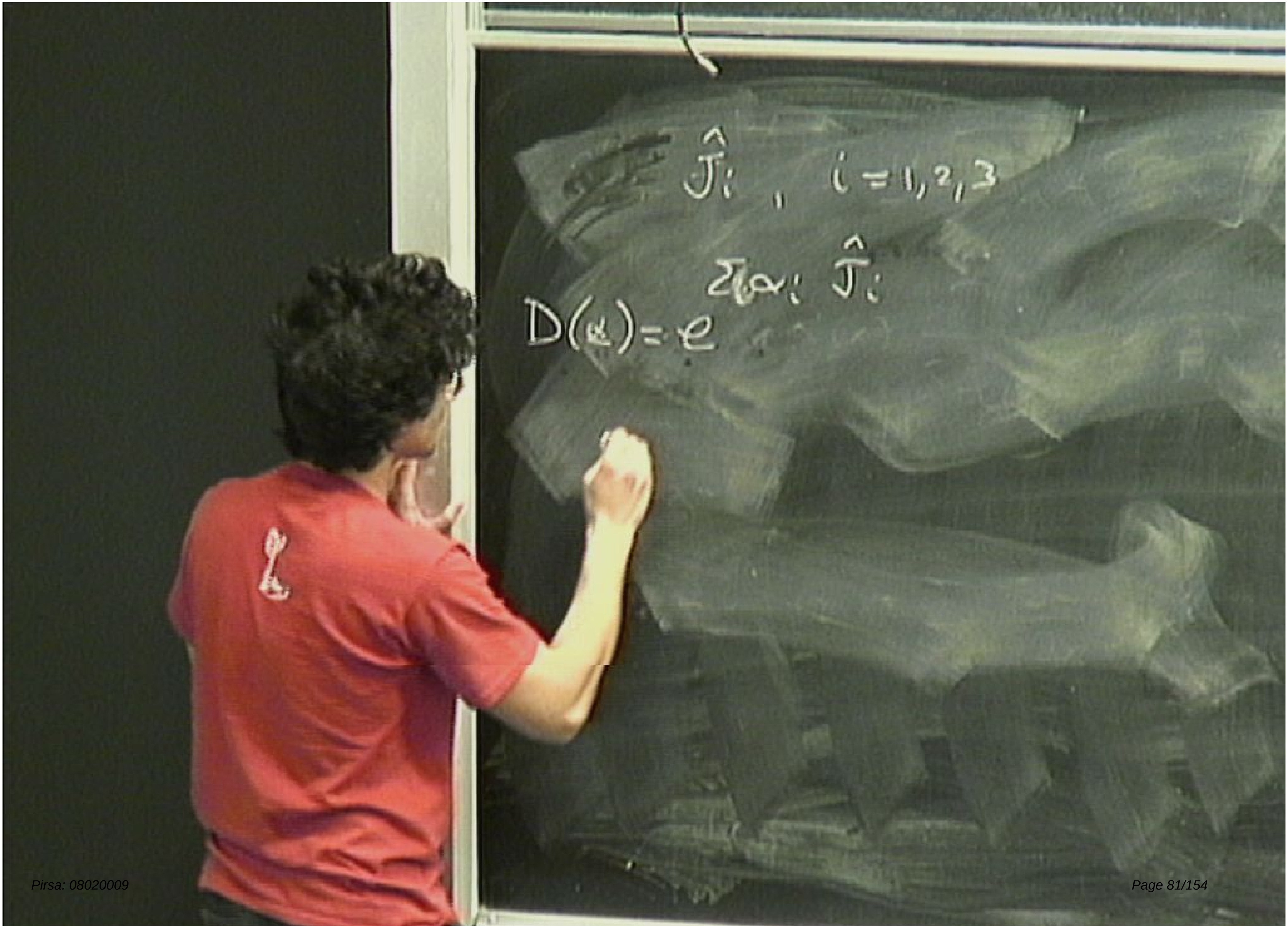
$$e^{i\sum \alpha_i J_i} = g \ni SU(2) \to SO(3) \quad 3$$

$$i = 1,2,3$$



g = e
i = 1, 2, 3
ε_123 = 1
ε_132 = -1
ε_112 = 0
[J_i , J_j] = ε_ijk J_k
[q̂^i , p̂^j] = iħδ_ij
[J_1 , J_2] = J_3
[J_1 , J_1] = 0
[J_1 , J_3] = -J_2
α · J = q × p · α
Ĵ = q̂ × p̂
Ĵ_1 , Ĵ_2 , Ĵ_3
CAUTION


$\hat{J}_i$, $i=1,2,3$
$D(\alpha)=e^{i\alpha_i \hat{J}_i}$


$\hat{J}_i \,,\quad i = 1,2,3$
$D(\alpha) = e^{\sum \alpha_i \hat{J}_i}$

$\hat{J}_i \; , \quad i = 1, 2, 3$

$$D(\alpha) = e^{\sum \alpha_i \hat{J}_i}$$

$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles

$\hat{J}_i \quad , \quad i = 1, 2, 3$

$$D(\underline{\alpha}) = e^{\sum_i \alpha_i \hat{J}_i}$$

$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles.

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$

$\hat{J}_i \,,\quad i=1,2,3$

$$D(\alpha) = e^{\sum \alpha_i \hat{J}_i}$$

(3x3) acting on vectors in 3d space.

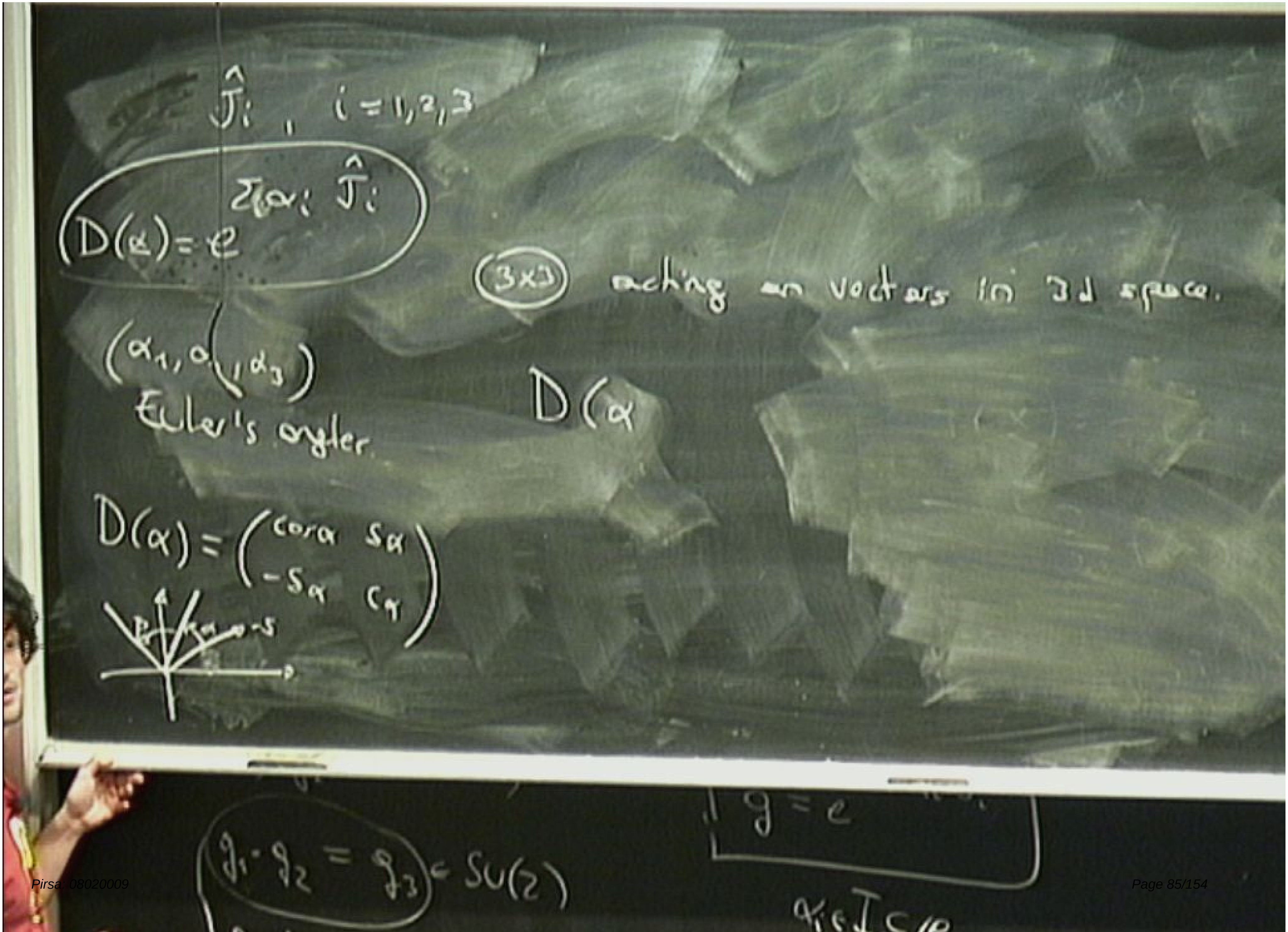
$(\alpha_1, \alpha_2, \alpha_3)$
Euler's angles

$D(\alpha$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$

$g = e$

$g_1 \cdot g_2 = g_3 \in SU(2)$


Ĵi , i = 1,2,3
D(α) = e^(Σ αi Ĵi)
(3x3) acting on vectors in 3d space.
(α1, α2, α3)
Euler's angles
D(α
D(α) = (cos α sα / −sα cα)
g = e
g1 · g2 = g3 ∈ SU(2)

$\hat{J}_i \, , \quad i = 1,2,3$

$$D(\alpha) = e^{\sum \alpha_i \hat{J}_i}$$

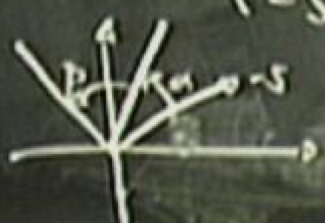
(3x3) acting on vectors in 3d space.

$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles

$D(\alpha$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$



$g = e$

$g_1 \cdot g_2 = g_3 \in SU(2)$

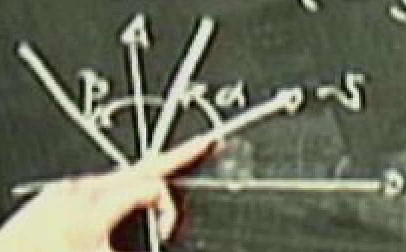
$\alpha_i \in \mathbb{R}$

$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles.

$D(\alpha$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$



$$D(\alpha)\cdot D(\beta)$$

$g_1, g_2 \in SU(2)$

$$\boxed{g = e^{\sum_i i\alpha_i J_i}}$$

$\alpha_i \in I \subseteq \mathbb{R}$

$$g_1 \cdot g_2 = g_3 \in SU(2)$$

$$g_1 \cdot 1 = g_1$$

$\hat{J}_i \, , \quad i = 1,2,3$

$$D(\alpha) = e^{\sum_i \alpha_i \hat{J}_i}$$

(3x3) acting on vectors in 3d space.

$(\alpha_1, \alpha_2, \alpha_3)$
Euler's angles

$D(\alpha$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$

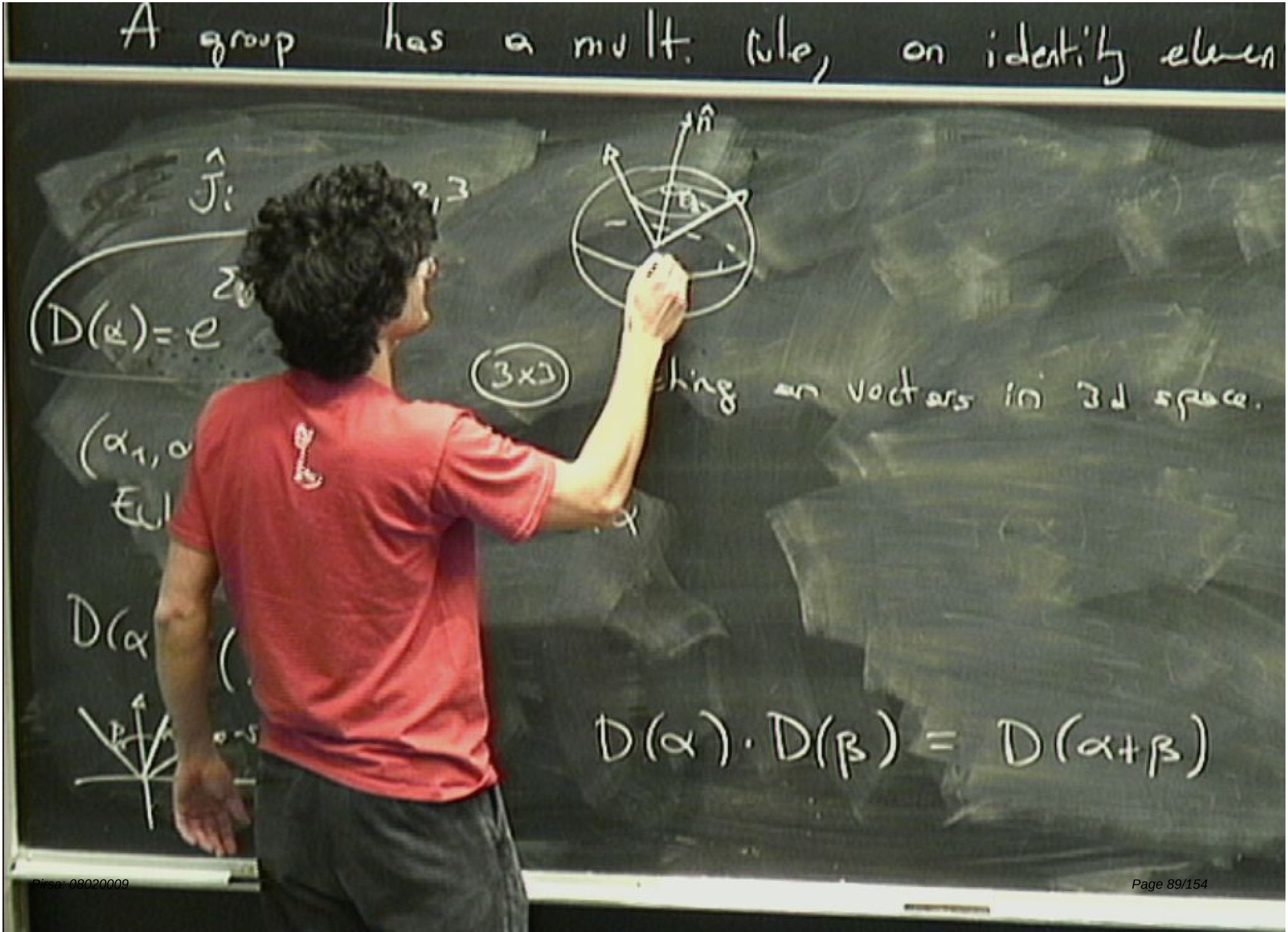
$$D(\alpha) \cdot D(\beta) = D(\alpha + \beta)$$

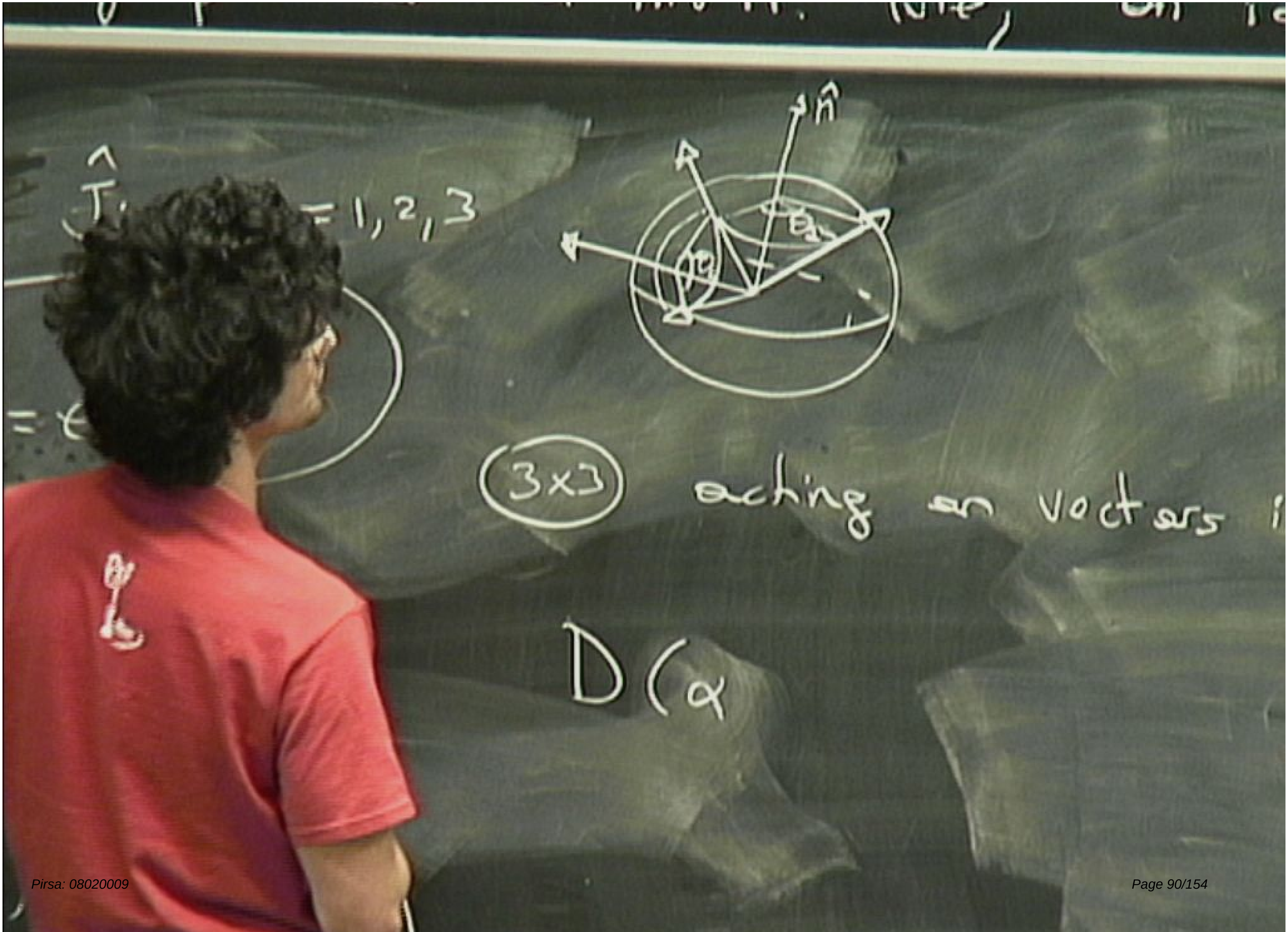
$g_1 , g_2 \in SU(2)$

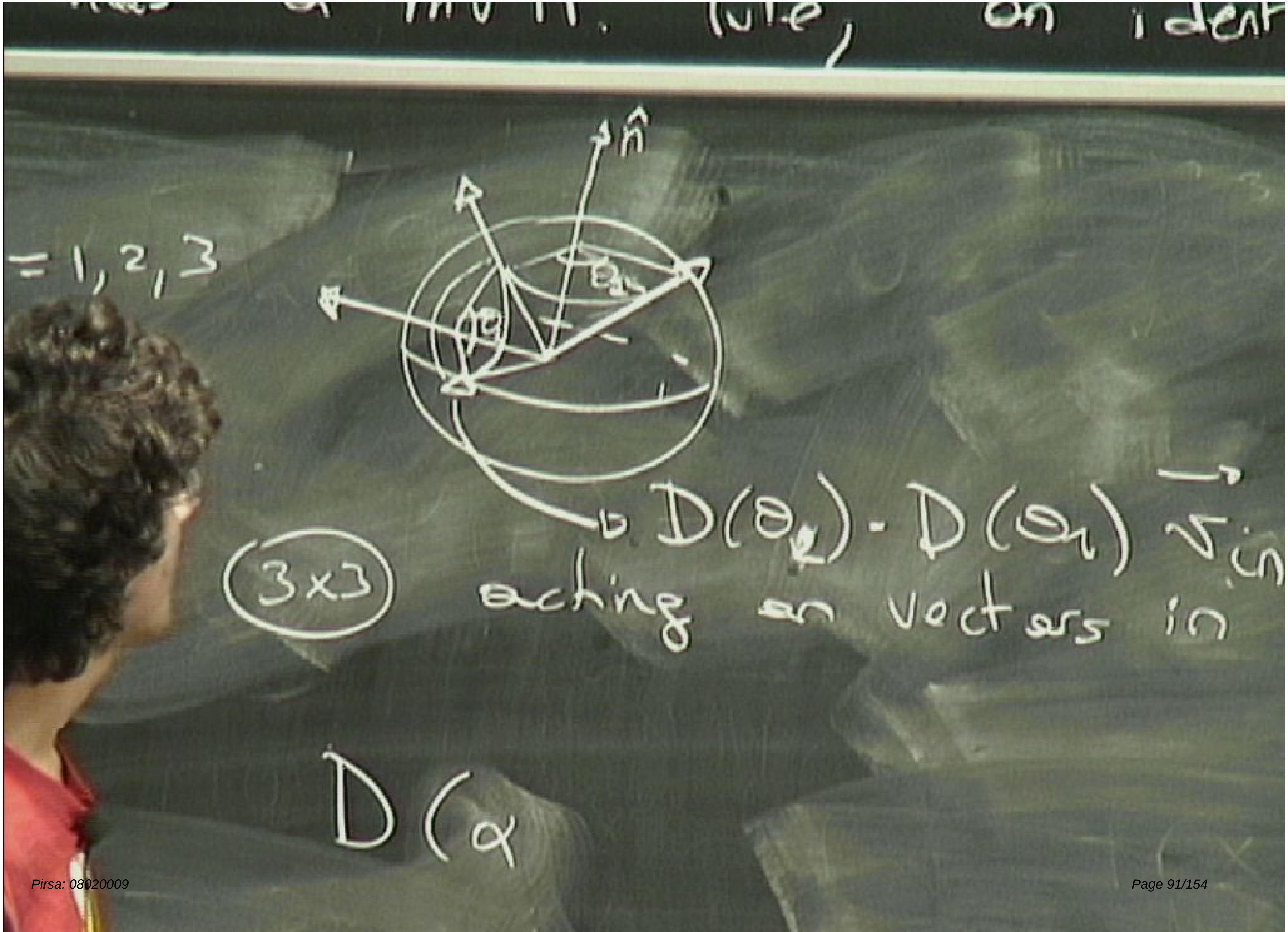
$$\boxed{g = e^{\sum_i \alpha_i J_i}}$$

$g_1 \cdot g_2 = g_3 \in SU(2)$

$\alpha_i \in I \subseteq \mathbb{R}$


A group has a mult. rule, an identity
3x3
vectors in 3d space.
D(α)·D(β) = D(α+β)


n̂
=1,2,3
3x3
acting on vectors
D(α


= 1, 2, 3
n̂
D(θ2) · D(θ1) v_in
3x3
acting on vectors in
D(α

$D(\alpha$

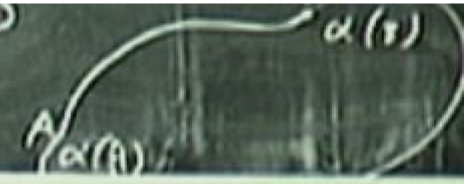
$$D(\beta)\cdot D(\alpha) = D(\beta+\alpha) =$$

$$D(\alpha)\cdot D(\beta) = D(\alpha+\beta) =$$

$x^2(r)$

$x^3(s)$

$$= \int_\gamma A + \int_0 \frac{d\alpha}{ds}\, ds = 0$$



$\hat{J}_i \,, \quad i = 1,2,3$

$$D(\alpha) = e^{i\alpha_i \hat{J}_i}$$

$$D(\theta_2)\cdot D(\theta_1)\,\vec{v}_{in} \neq D(\theta_1)\,D(\theta_2)\,\vec{v}_{in}$$

(3×3) acting on vectors in 3d space.

$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles

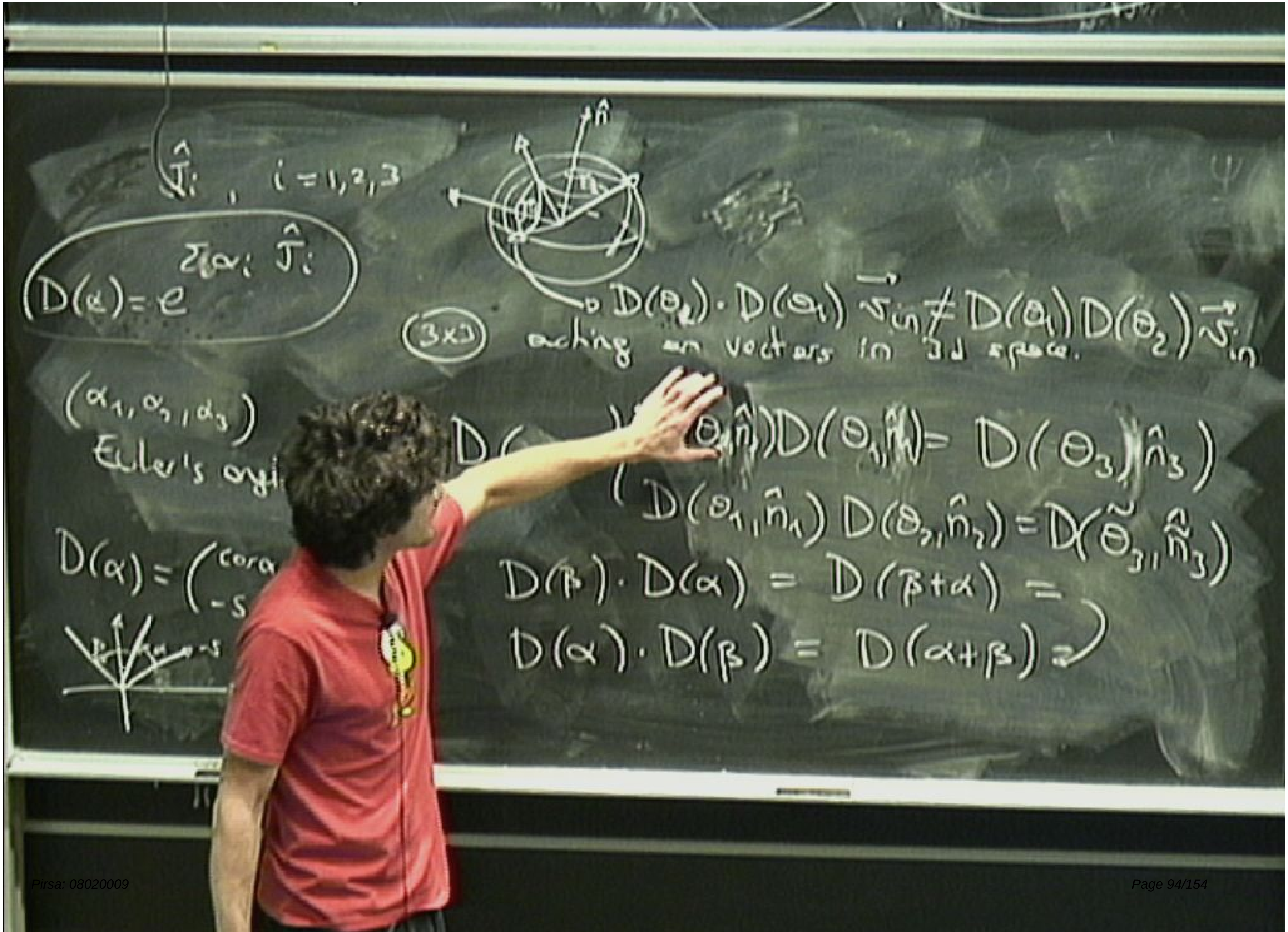
$D(\alpha$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$

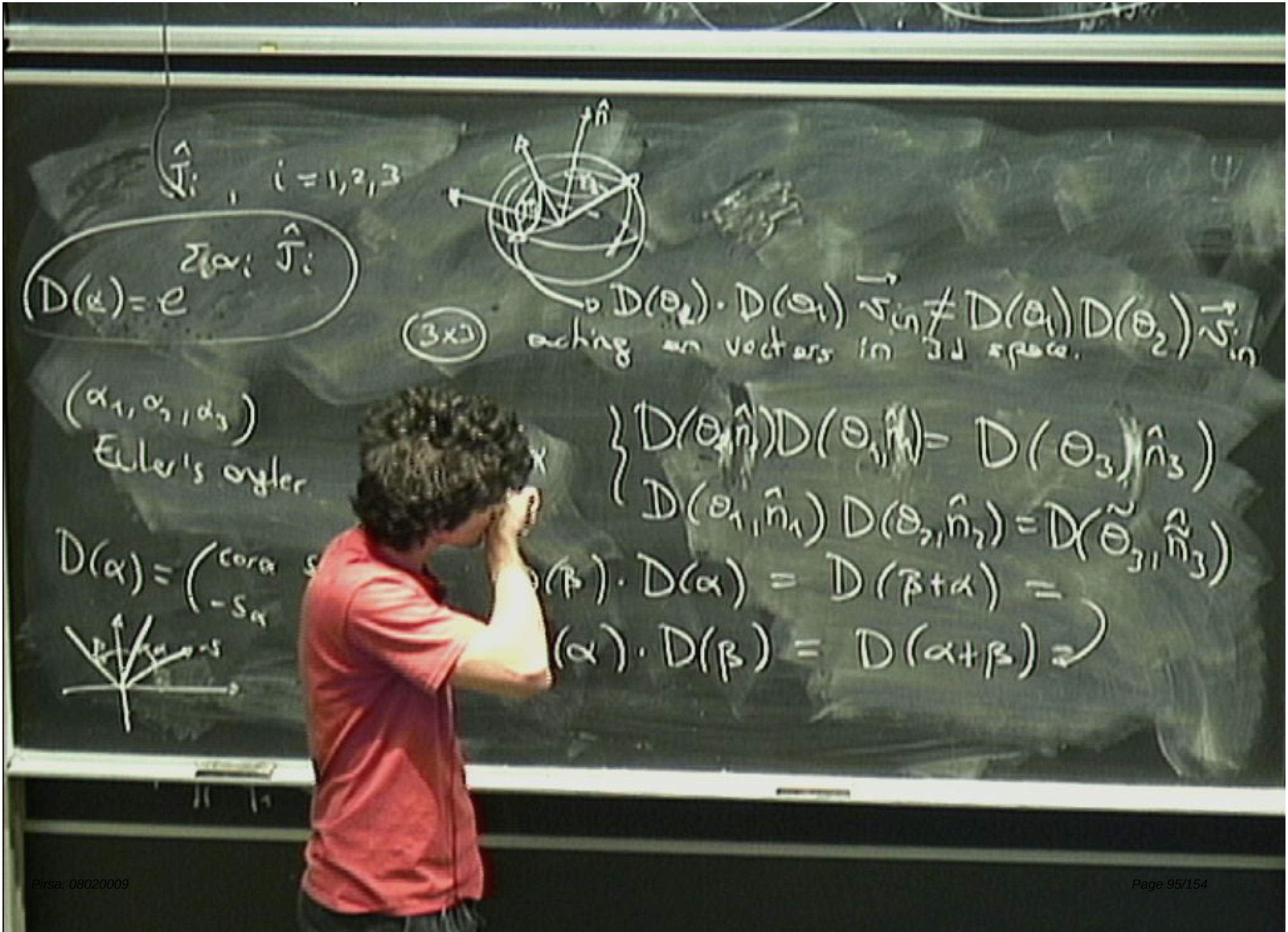
$$D(\beta)\cdot D(\alpha) = D(\beta+\alpha) =$$

$$D(\alpha)\cdot D(\beta) = D(\alpha+\beta)$$

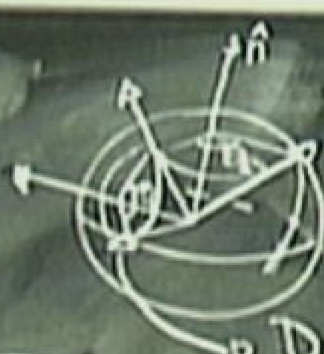
$q_1 \cdot q_1^{-1} - 1$



$\hat{J}_i$, $i = 1,2,3$
$D(\alpha) = e^{\Sigma \alpha_i \hat{J}_i}$
$D(\theta_2)\cdot D(\theta_1)\vec{v}_{in} \neq D(\theta_1)D(\theta_2)\vec{v}_{in}$
(3x3) acting on vectors in 3d space.
$(\alpha_1, \alpha_2, \alpha_3)$
Euler's angl
$D(\theta_2,\hat{n}_2)D(\theta_1,\hat{n}_1) = D(\theta_3,\hat{n}_3)$
$D(\theta_1,\hat{n}_1)\,D(\theta_2,\hat{n}_2) = D(\tilde{\theta}_3,\hat{\tilde{n}}_3)$
$D(\beta)\cdot D(\alpha) = D(\beta+\alpha) =$
$D(\alpha)\cdot D(\beta) = D(\alpha+\beta) =$


Ĵ_i , i = 1,2,3
D(α) = e^{Σ α_i Ĵ_i}
D(θ_2)·D(θ_1) s_in ≠ D(θ_1) D(θ_2) s_in
(3x3) acting on vectors in 3d space.
(α_1, α_2, α_3)
Euler's angles
D(θ_1, n̂_1) D(θ_2, n̂_2) = D(θ̃_3, n̂̃_3)
D(α) = (cos α ; −s_α)
D(β)·D(α) = D(β+α) =
D(α)·D(β) = D(α+β)

$\hat{J}_i \,, \quad i=1,2,3$



$$D(\alpha) = e^{\sum \alpha_i \hat{J}_i}$$

$D(\theta_2)\cdot D(\theta_1)\vec{s}_{in} \neq D(\theta_1)D(\theta_2)\vec{s}_{in}$

(3×3) acting on vectors in 3d space.

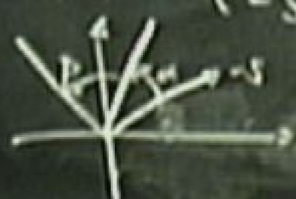
$(\alpha_1, \alpha_2, \alpha_3)$

Euler's angles

$D(\alpha$

$$\begin{cases} D(\theta_2,\hat{n}_2)D(\theta_1,\hat{n}_1) = D(\theta_3,\hat{n}_3) \\ D(\theta_1,\hat{n}_1)\,D(\theta_2,\hat{n}_2) = D(\tilde{\theta}_3,\hat{\tilde{n}}_3) \end{cases}$$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}$$



$$D(\beta)\cdot D(\alpha) = D(\beta+\alpha) =$$

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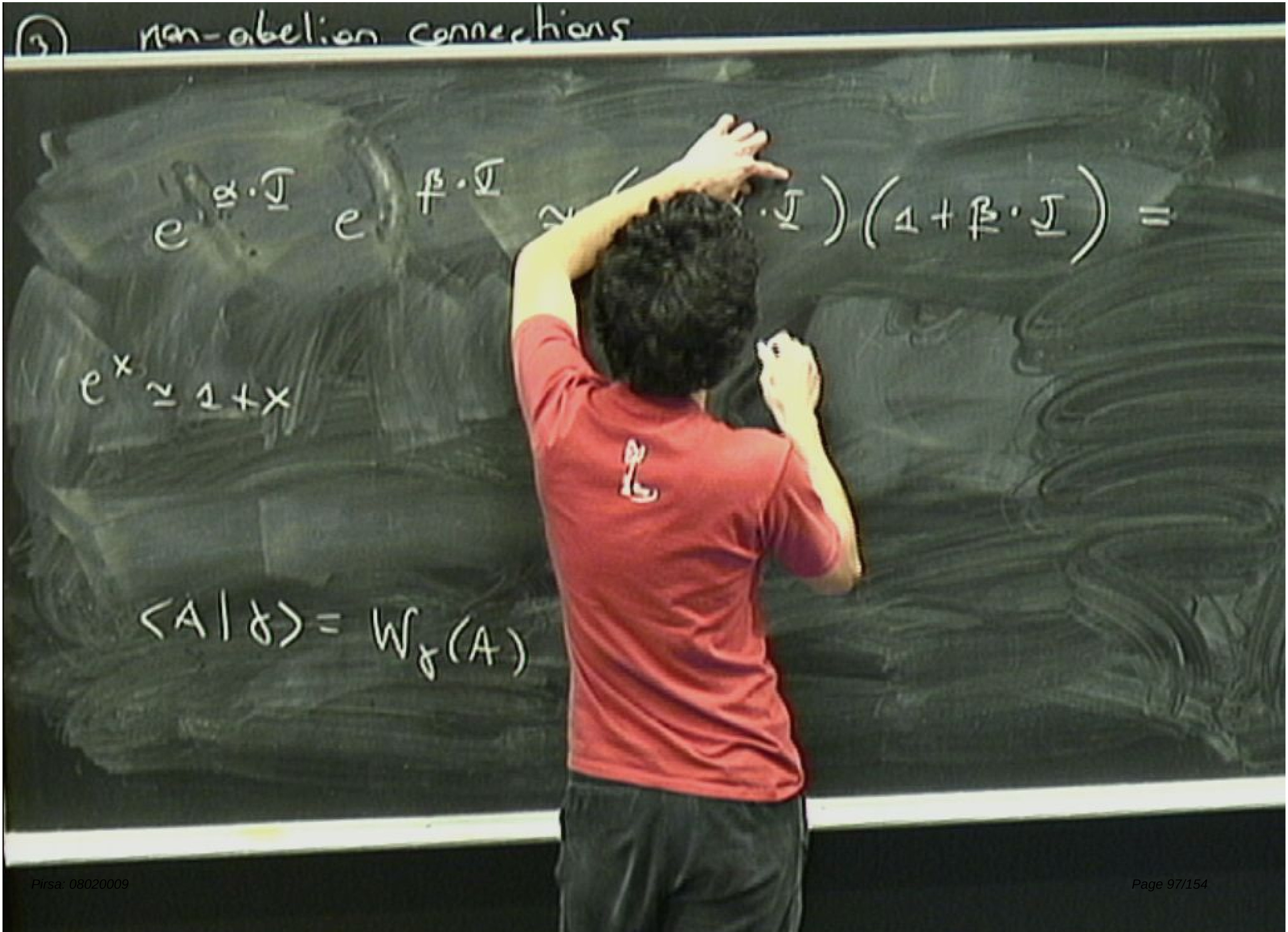
③ non-abelian connections
$e^{\underline{\alpha}\cdot\underline{J}}\, e^{\underline{\beta}\cdot\underline{J}} \simeq (\ \cdot\underline{J})(1+\underline{\beta}\cdot\underline{J}) =$
$e^{x} \simeq 1+x$
$\langle A|\gamma\rangle = W_\gamma(A)$

③ non-abelian connections

$$e^{\underline{\alpha}\cdot\underline{J}}\cdot e^{\underline{\beta}\cdot\underline{J}} \simeq \left(1+\underline{\alpha}\cdot\underline{J}\right)\left(1+\underline{\beta}\cdot\underline{J}+\frac{1}{2}\beta_i\beta_k J_i J_k\right) =$$

$$= 1 + \alpha_i J_i + \beta_k J_k + \alpha_i \beta_j J_i J_j$$

$$e^x \simeq 1 + x$$

$$e^{\beta\cdot}$$

$$\gamma\rangle = W_\gamma(A) \quad \text{em}$$

③ non-abelian connections

$$e^{\underline{\alpha}\cdot\underline{J}}\cdot e^{\underline{\beta}\cdot\underline{J}} \simeq (1+\underline{\alpha}\cdot\underline{J})(1+\underline{\beta}\cdot\underline{J}+\tfrac{1}{2}\beta_i\beta_k J_i J_k) =$$

$$e^x \simeq 1+x$$

$$= 1+\alpha_i J_i + \beta_k J_k + \alpha_i \beta_j J_i J_j$$

$$e^{\underline{\beta}\cdot\underline{J}}\cdot e^{\underline{\alpha}\cdot\underline{J}}$$

$$\langle A|\gamma\rangle = W_\gamma(A) \quad \text{em}$$

③ non-abelian connections

$$e^{\underline{\alpha}\cdot\underline{J}}\cdot e^{\underline{\beta}\cdot\underline{J}} \simeq \left(1+\underline{\alpha}\cdot\underline{J}\right)\left(1+\underline{\beta}\cdot\underline{J}+\frac{1}{2}\beta_i\beta_k J_i J_k\right) =$$

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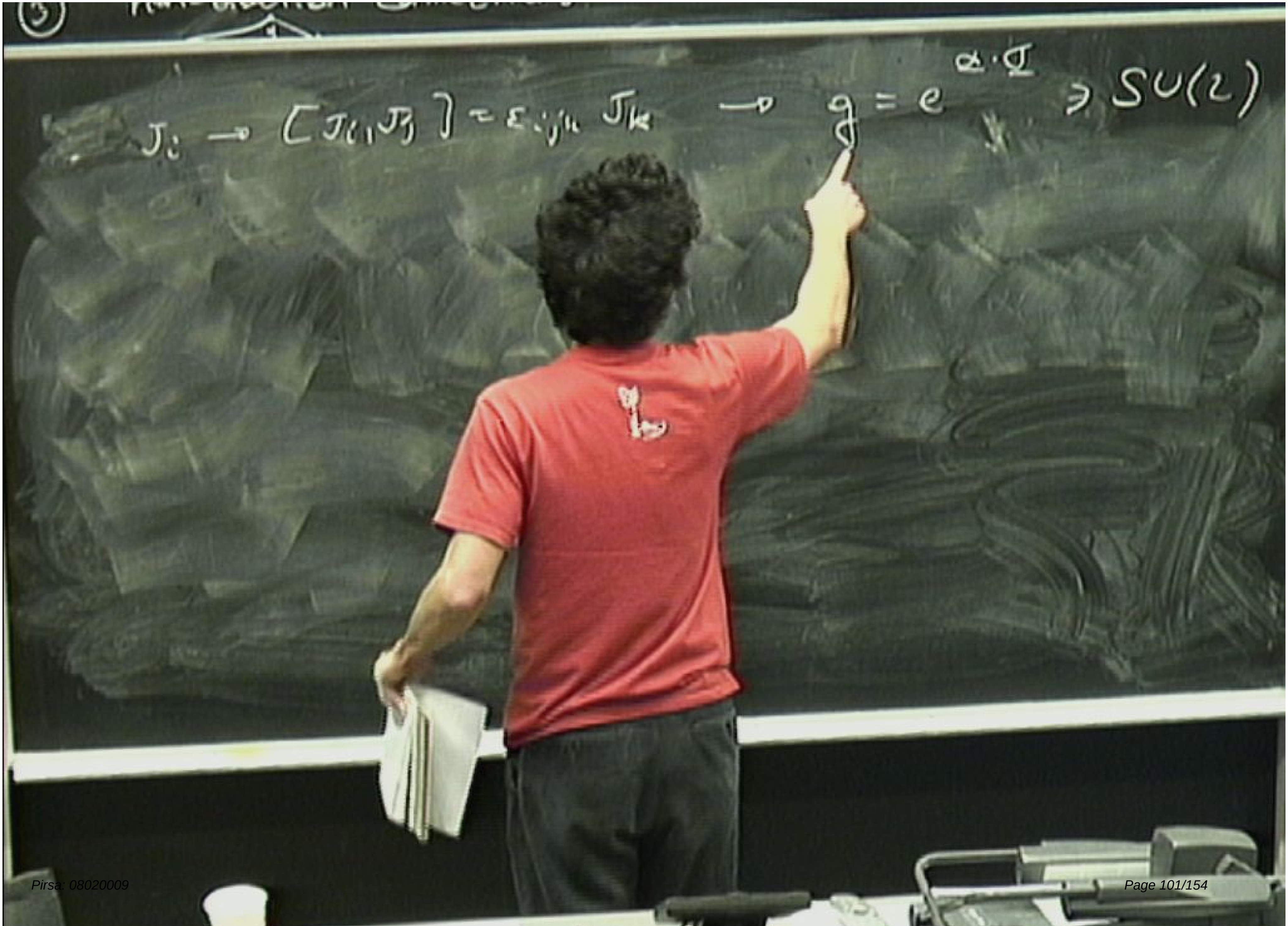
$$e^{x} \simeq 1 + x$$

$$e^{\underline{\beta}\cdot\underline{J}}\cdot e^{\underline{\alpha}\cdot\underline{J}}$$

$$\langle A|\gamma\rangle = W_\gamma(A) \qquad \text{em.}$$


$J_i \to [J_i, J_j] = \varepsilon_{ijk} J_k \to g = e^{\alpha \cdot J} \ni SU(2)$

$J_i \to [J_i, J_j] = \varepsilon_{ijk} J_k \quad \to \quad g = e^{\alpha \cdot J} \ni SU(2)$

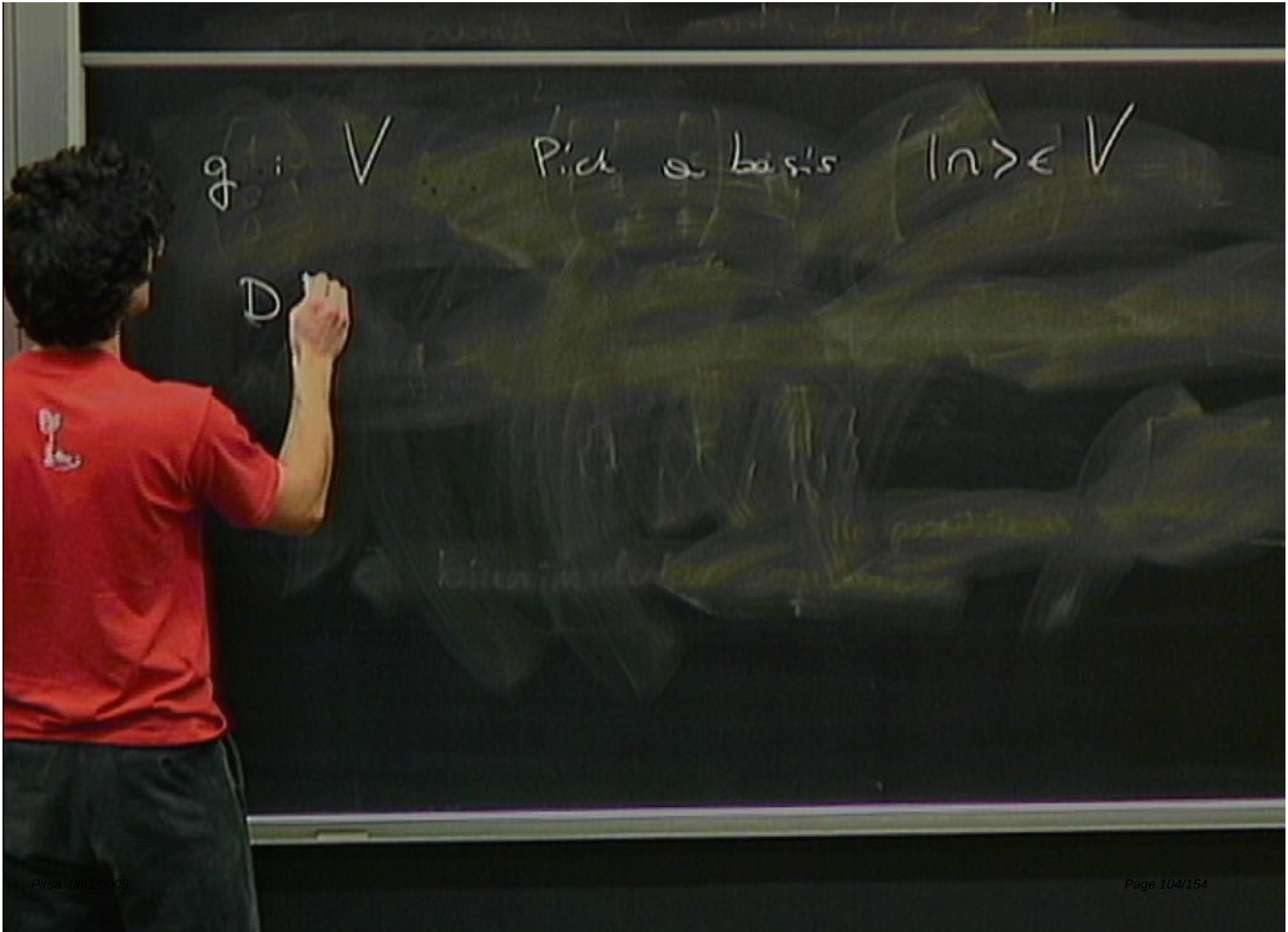
Any vector space V over which the group acts as endomorphism

(given $v \in V$, $gv \in V$) is called a <u>representation</u>

$J_i \rightarrow [J_i, J_j] = \varepsilon_{ijk} J_k \quad \rightarrow \quad g = e^{\alpha \cdot J} \ni SU(2)$

Any vector space V over which the group acts as endomorphism
($\text{given } v \in V, \ gv \in V$) is called a <u>representation</u>

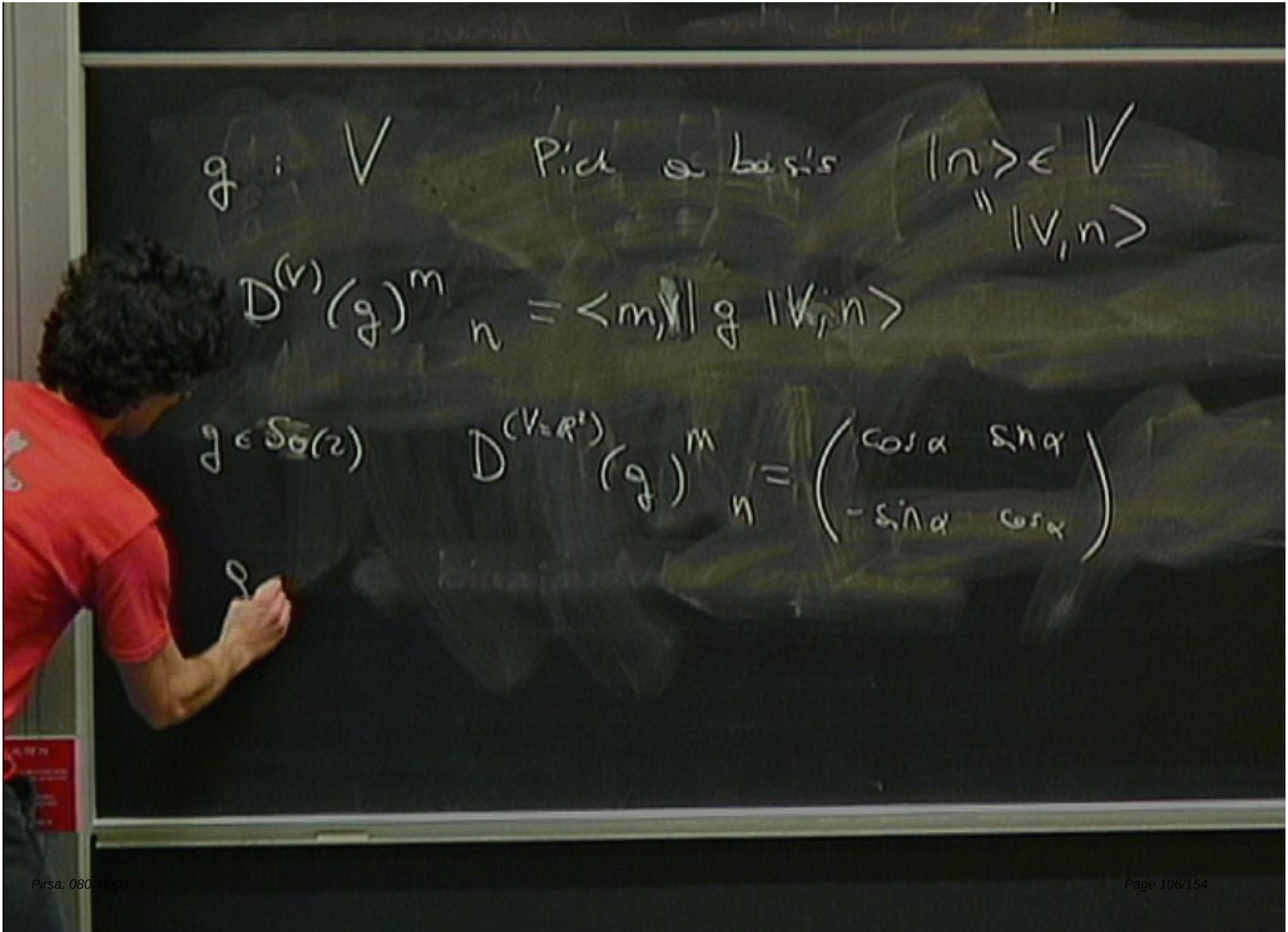
The V with the smallest dimension is called <u>fundamental</u>


g : V
Pick a basis |n> ∈ V
D

$g : V$ Pick a basis $|n\rangle \in V$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \; |V,n\rangle$

$$D^{(V)}(g)^m{}_n = \langle m,V|\, g \,|V,n\rangle$$


g : V
Pick a basis |n> ∈ V
= |V,n>
D^(V)(g)^m_n = <m,V| g |V,n>
g ∈ SO(2)
D^(V=R^2)(g)^m_n = (cos α sin α ; -sin α cos α)
g

$g: V$ Pick a basis $|n\rangle \in V$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \| \; |V, n\rangle$

$$D^{(V)}(g)^m{}_n = \langle m, V | g | V, n\rangle$$

$g \in SO(2)$ $\quad D^{(V=\mathbb{R}^2)}(g)^m{}_n = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$g \in SO(3)$
$\quad\;\; \wr$
$\quad SU(2)$

$g : V$ Pick a basis $|n\rangle \in V$
$= |V, n\rangle$

$$D^{(V)}(g)^m{}_n = \langle m, V| g |V, n\rangle$$

$g \in SO(2)$ $D^{(V=\mathbb{R}^2)}(g)^m{}_n = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$g \in SO(3) \simeq SU(2)$ $D^{(V=\mathbb{R}^3)}(g)^m{}_n = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}_{3\times 3}$ $(\alpha_1, \alpha_2, \alpha_3)$

③ non-abelian connections

$$\text{If} \quad D^{(\nu)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \quad \forall g \in G$$

③ non-abelian connections

If $$D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \quad \forall g \in G$$

the V is said reducible.

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the V is said reducible.

In the case of SU(2), the irreps are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}$

j	n
0	1
1/2	2
1	3

$\Rightarrow \sum_i g^i{}_j \, g^i{}_k = \delta_{jk}$

SU(2)

If $D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \quad \forall g \in G$

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j	n
0	1
1/2	2
1	3
3/2	4

If $D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \; \forall g \in G$

the V is said reducible.

In the case of SU(2), the irreps are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$|m\rangle$; $m = -j, \ldots 0, \ldots j$

j	n	$\|m\rangle$
0	1	0
1/2	2	
1	3	
3/2	4	

$\Rightarrow \sum_i g^i{}_j \, g^i{}_k = \delta_{jk}$

SU(2)

If $D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \; \forall g \in G$

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In the case of $SU(2)$, the irreps are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$|m\rangle$; $m = -j, \ldots, 0, \ldots, j$

j	n	$\vert m\rangle$
0	1	0
$1/2$	2	$\vert -\frac{1}{2}\rangle, \vert +\frac{1}{2}\rangle$
1	3	$\vert -1\rangle, \vert 0\rangle, \vert 1\rangle$
$3/2$	4	$\vert -3/2\rangle, \vert -1/2\rangle, \vert 1/2\rangle, \vert 3/2\rangle$

If $D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \forall g \in G$

the V is said reducible.

In the case of SU(2), the irreps are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$|m\rangle$; $m = -j, \ldots 0, \ldots j$

j	n	$\|m\rangle$
0	1	0
1/2	2	$\|-\frac{1}{2}\rangle, \|+\frac{1}{2}\rangle$
1	3	$\|-1\rangle, \|0\rangle, \|1\rangle$
3/2	4	$\|-3/2\rangle, \|-1/2\rangle, \|1/2\rangle, \|3/2\rangle$

If $D^{(V)}(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \; \forall g \in G$

the V is said reducible.

In the case of $SU(2)$, the <u>irreps</u> are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$|m\rangle \; ; \; m = -j, \ldots, 0, \ldots, j$

j	n	$\lvert m\rangle$
0	1	0
1/2	2	$\lvert -\frac{1}{2}\rangle, \lvert +\frac{1}{2}\rangle$
1	3	$\lvert -1\rangle, \lvert 0\rangle, \lvert 1\rangle$
3/2	4	$\lvert -\frac{3}{2}\rangle, \lvert -\frac{1}{2}\rangle, \lvert \frac{1}{2}\rangle, \lvert \frac{3}{2}\rangle$

③ non-abelian connections

$f(\theta, \varphi)$

In the case of $SU(2)$, the irreps are the vector spaces

integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2} \dots$

$|m\rangle$; $m = -j, \dots 0, \dots j$

j	n	$\lvert m\rangle$
0	1	0
1/2	2	$\lvert -\frac{1}{2}\rangle, \lvert +\frac{1}{2}\rangle$
1	3	$\lvert -1\rangle, \lvert 0\rangle, \lvert 1\rangle$
3/2	4	$\lvert -3/2\rangle, \lvert -1/2\rangle, \lvert 1/2\rangle, \lvert 3/2\rangle$

A group has a mult. rule, an identity element, and an inverse element.

$$f(\theta) = \sum_n \hat{f}_n \, e^{in\theta}$$

$$\begin{cases} g_1 \cdot g_2 = g_3 \in SU(2) \\ g_1 \cdot 1 = g_1 \\ g_1 \cdot g_1^{-1} = 1 \end{cases}$$

③ non-abelian connections

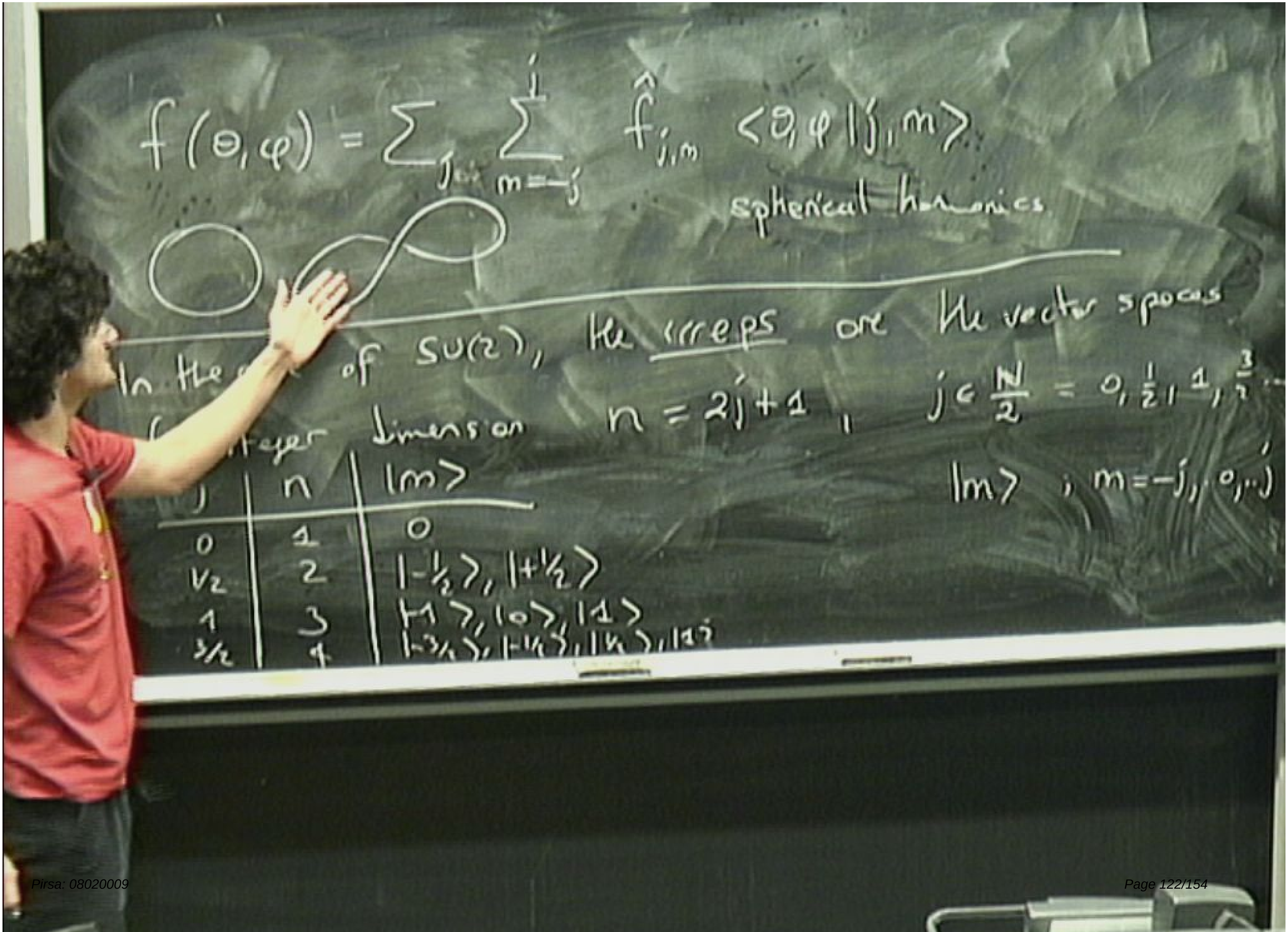
$$f(\theta,\varphi) = \sum_{j=0} \sum_{m=-j}^{j} \hat{f}_{j,m} \langle \theta,\varphi | j,m \rangle$$

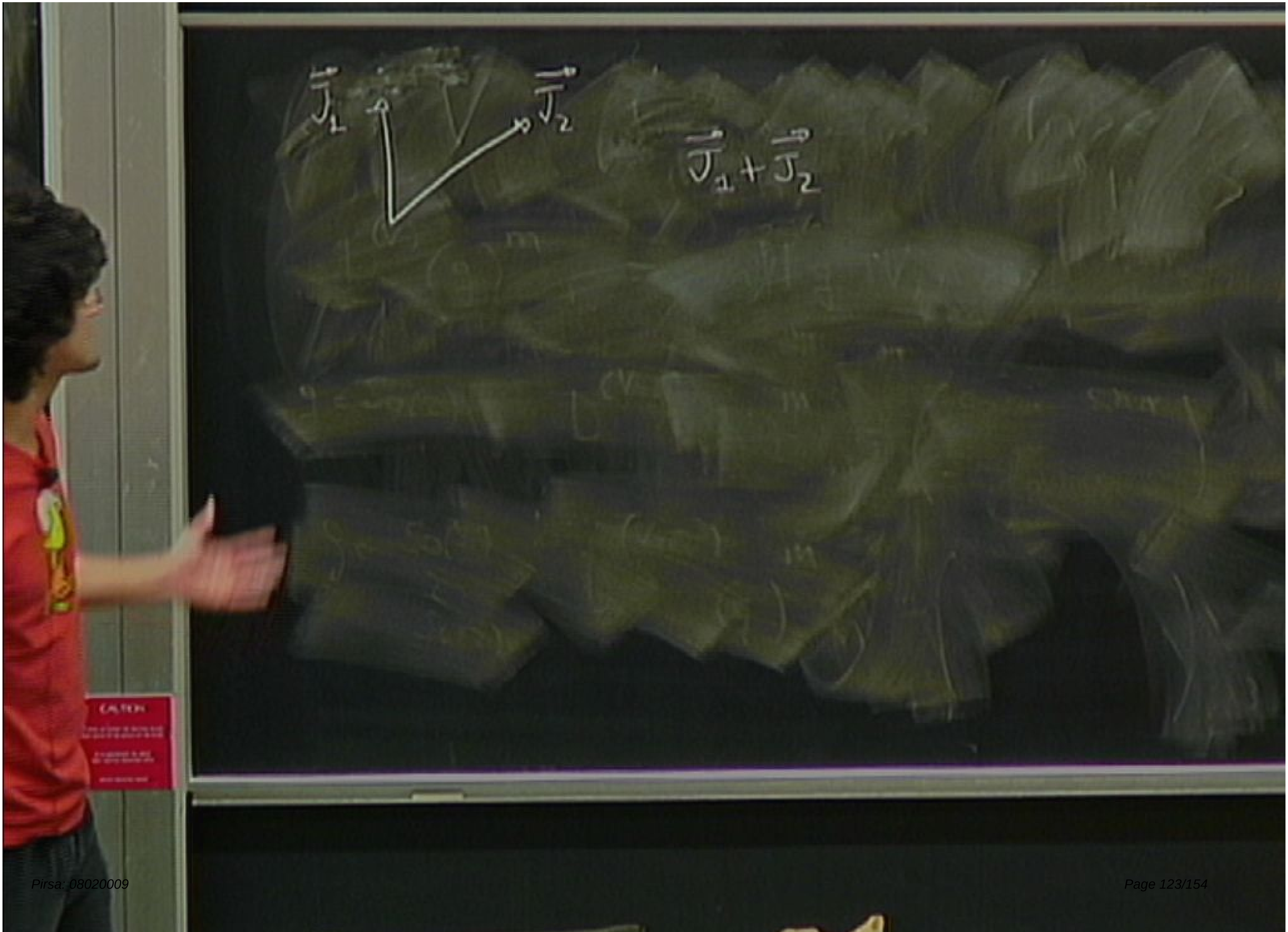
spherical harmonics

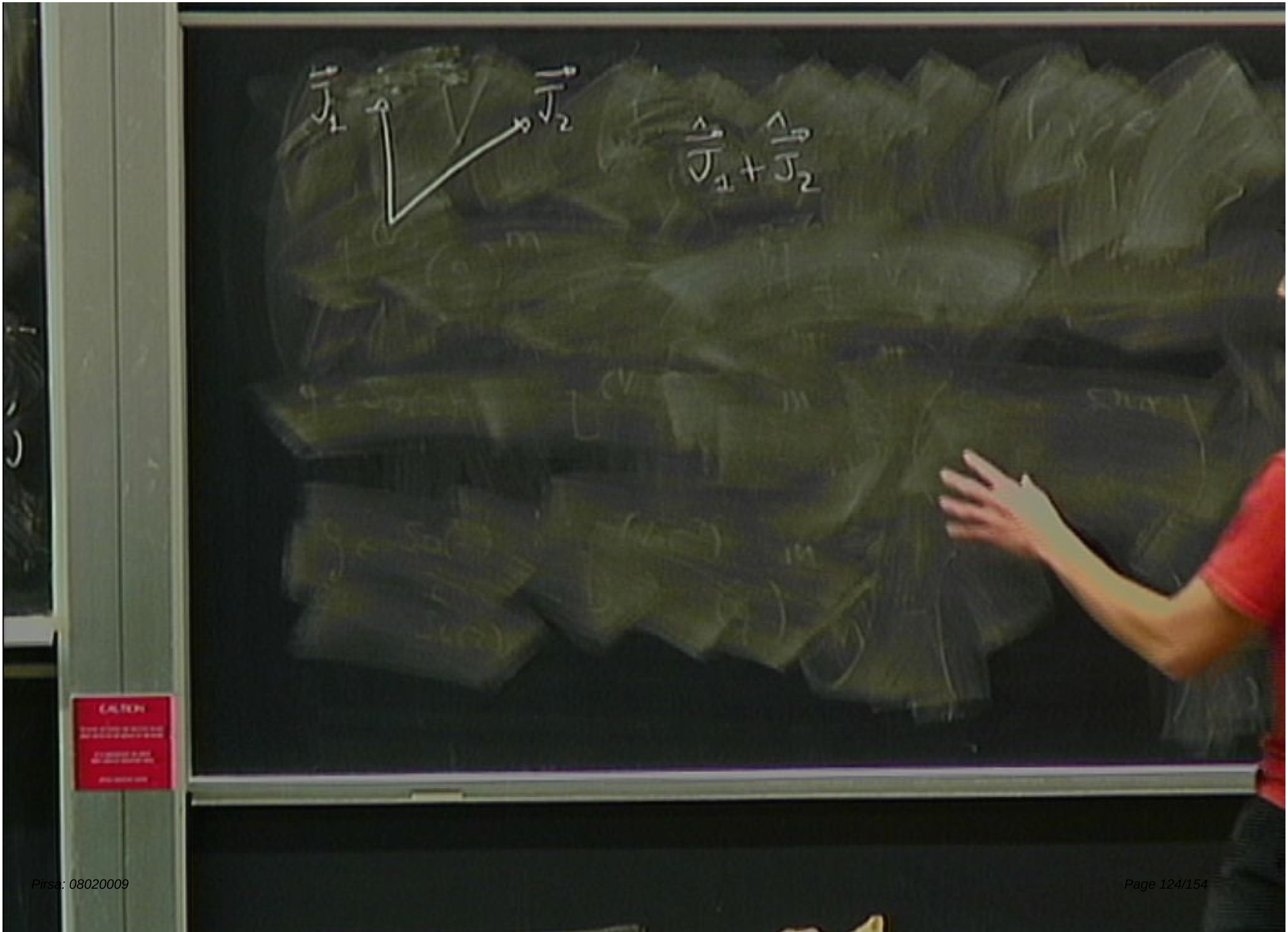
In the case of SU(2), the irreps are the vector spaces of integer dimension $n = 2j+1$, $j \in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$|m\rangle$; $m = -j, \ldots, 0, \ldots, j$

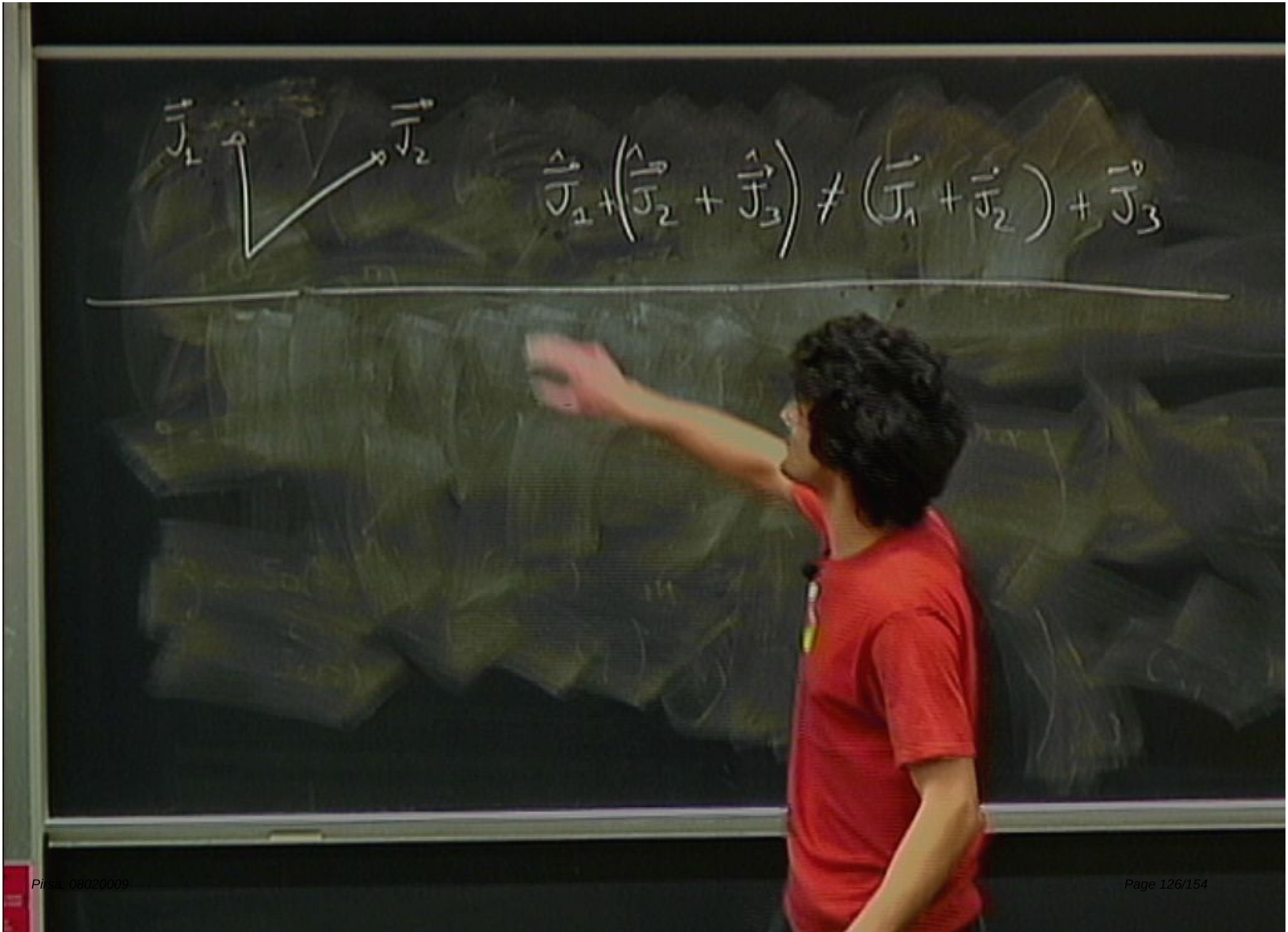
j	n	$\vert m\rangle$
0	1	0
1/2	2	$\vert -\frac{1}{2}\rangle, \vert +\frac{1}{2}\rangle$
1	3	$\vert -1\rangle, \vert 0\rangle, \vert 1\rangle$
3/2	4	$\vert -3/2\rangle, \vert -1/2\rangle, \vert 1/2\rangle, \vert 3/2\rangle$

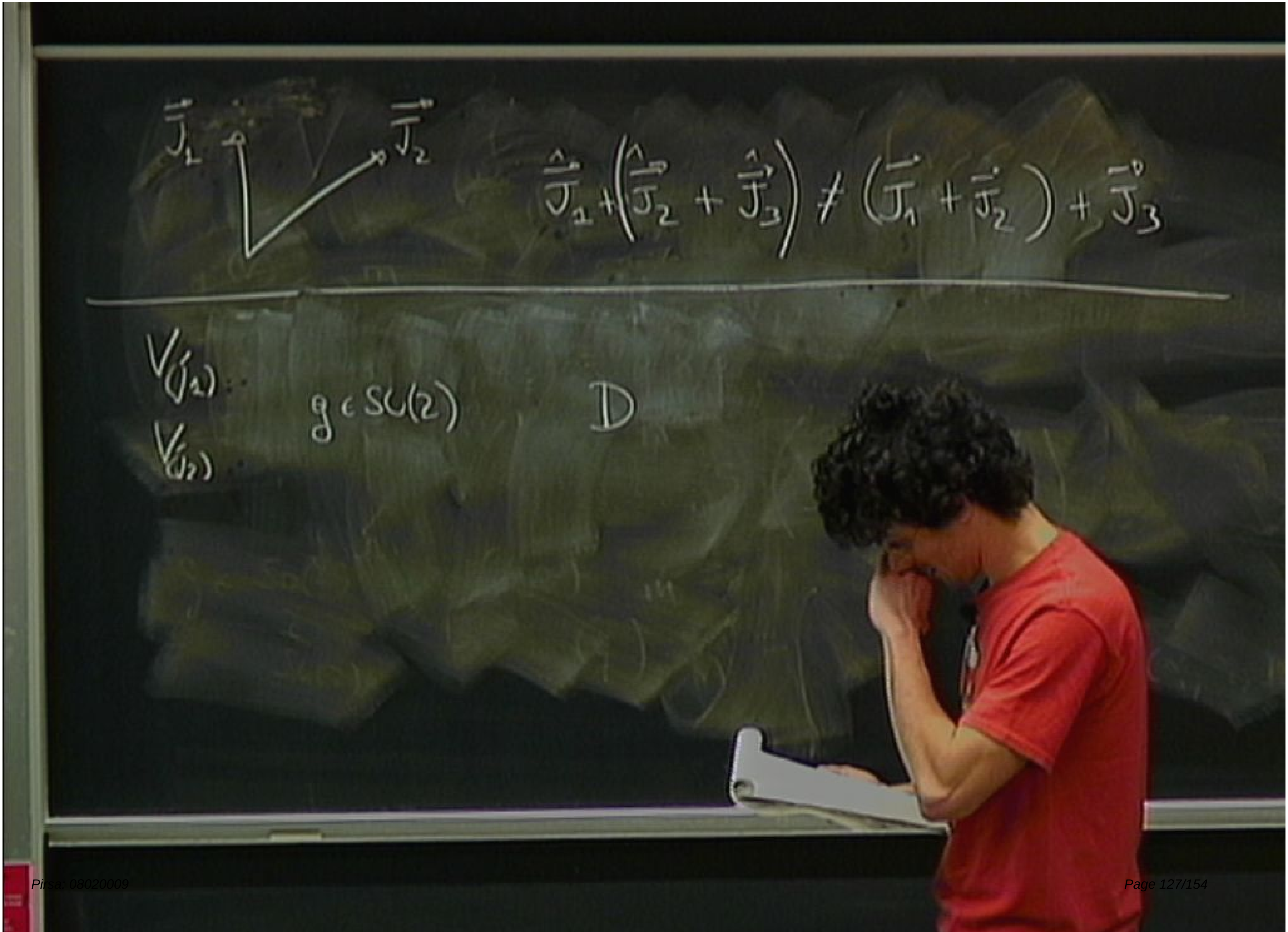

$f(\theta,\varphi) = \sum_{j} \sum_{m=-j}^{j} \hat{f}_{j,m} \langle \theta,\varphi | j,m \rangle$
spherical harmonics
In the ... of SU(2), the irreps are the vector spaces
$n = 2j+1$, $j \in \frac{N}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
integer dimension
$|m\rangle$; $m = -j, \ldots, 0, \ldots, j$
j | n | $|m\rangle$
0 | 1 | 0
1/2 | 2 | $|-\frac{1}{2}\rangle, |+\frac{1}{2}\rangle$
1 | 3 | $|-1\rangle, |0\rangle, |1\rangle$
3/2 | 4 | $|-3/2\rangle, |-1/2\rangle, |1/2\rangle, |3/2\rangle$

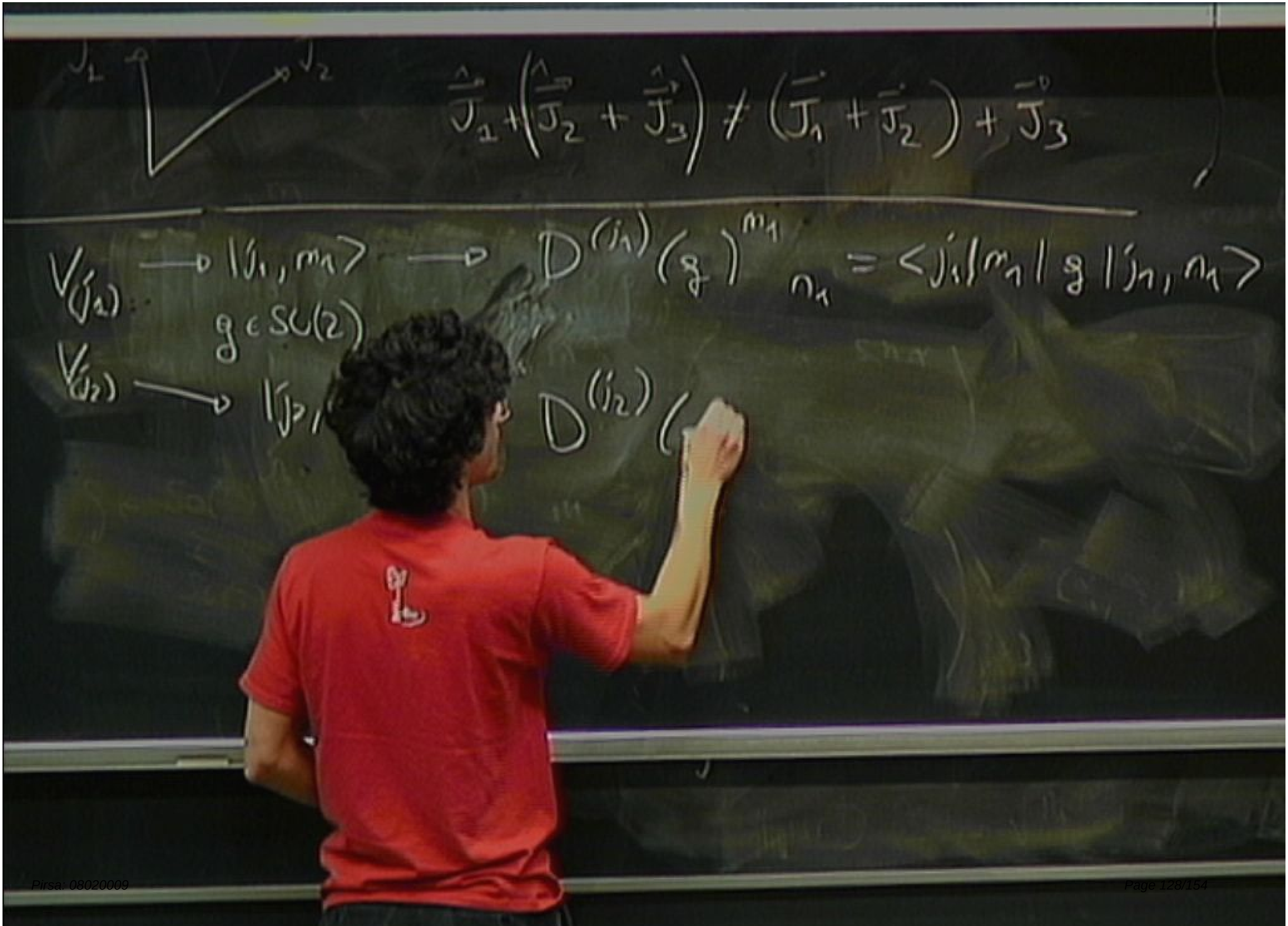

$\vec{J}_1$
$\vec{J}_2$
$\vec{J}_1 + \vec{J}_2$
CAUTION

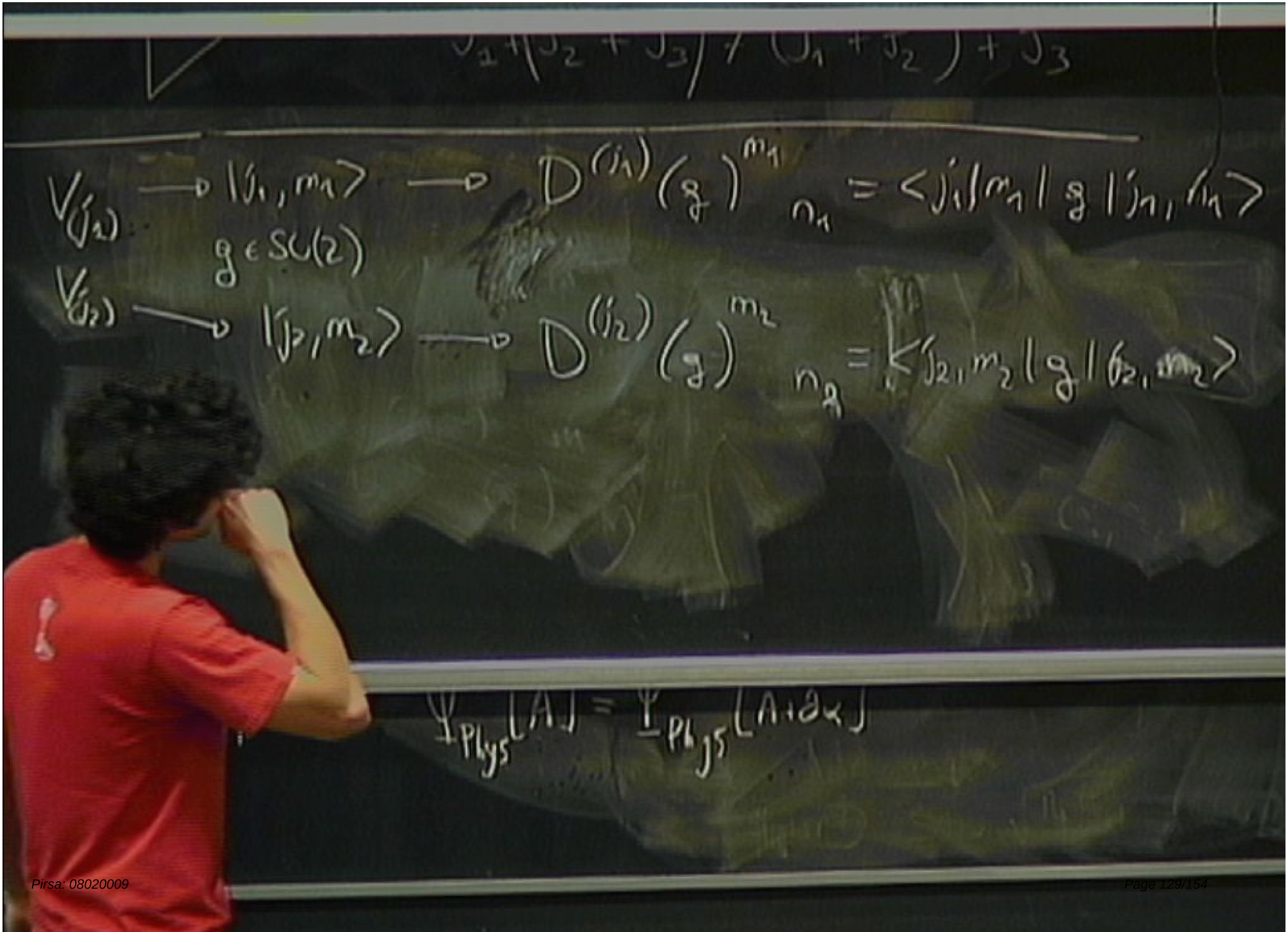

$\vec{J}_1$
$\vec{J}_2$
$\hat{\vec{J}}_1 + \hat{\vec{J}}_2$

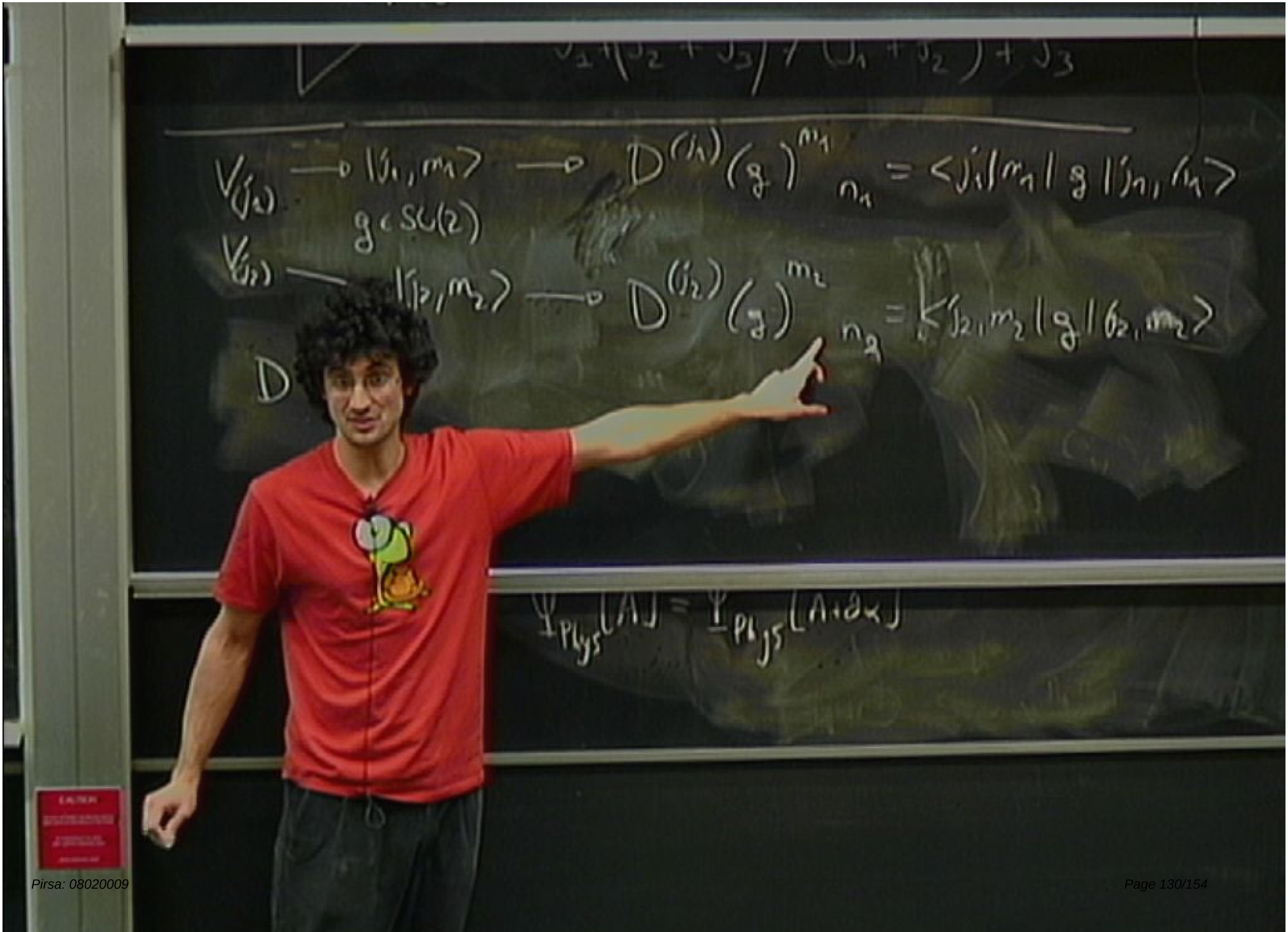
$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$


$\vec{J}_1$
$\vec{J}_2$
$\hat{\vec{J}}_1 + (\hat{\vec{J}}_2 + \hat{\vec{J}}_3) \neq (\vec{J}_1 + \vec{J}_2) + \vec{J}_3$

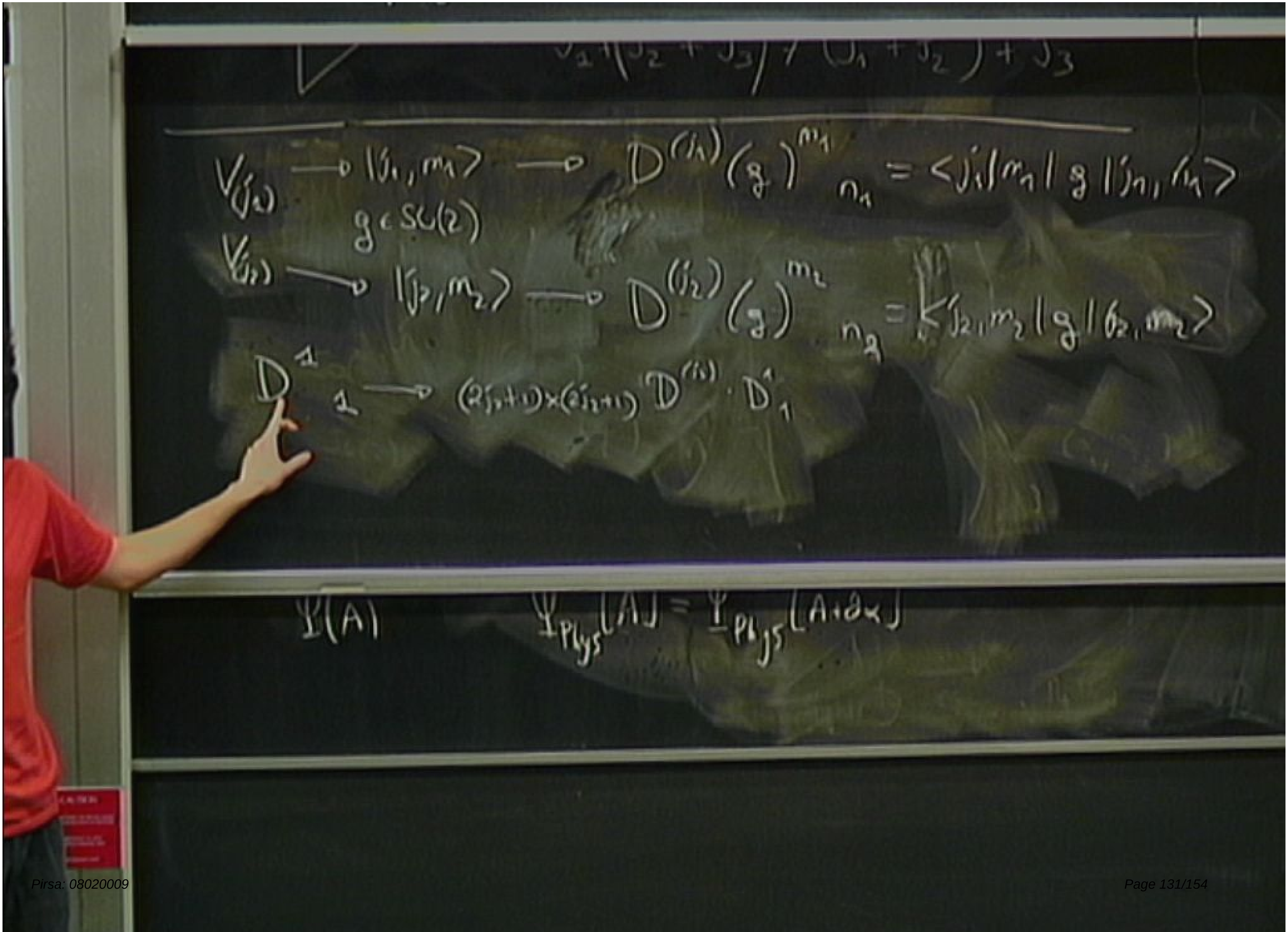

$\vec{J}_1$
$\vec{J}_2$
$\hat{\vec{J}}_1 + (\hat{\vec{J}}_2 + \hat{\vec{J}}_3) \neq (\vec{J}_1 + \vec{J}_2) + \vec{J}_3$
$V_{(j_1)}$
$V_{(j_2)}$
$g \in SU(2)$
D


j1 j2
J1 + (J2 + J3) ≠ (J1 + J2) + J3
V(j1) → |j1, m1> → D^(j1)(g)^m1 n1 = <j1, m1 | g | j1, n1>
g ∈ SU(2)
V(j2) → |j2, D^(j2)(

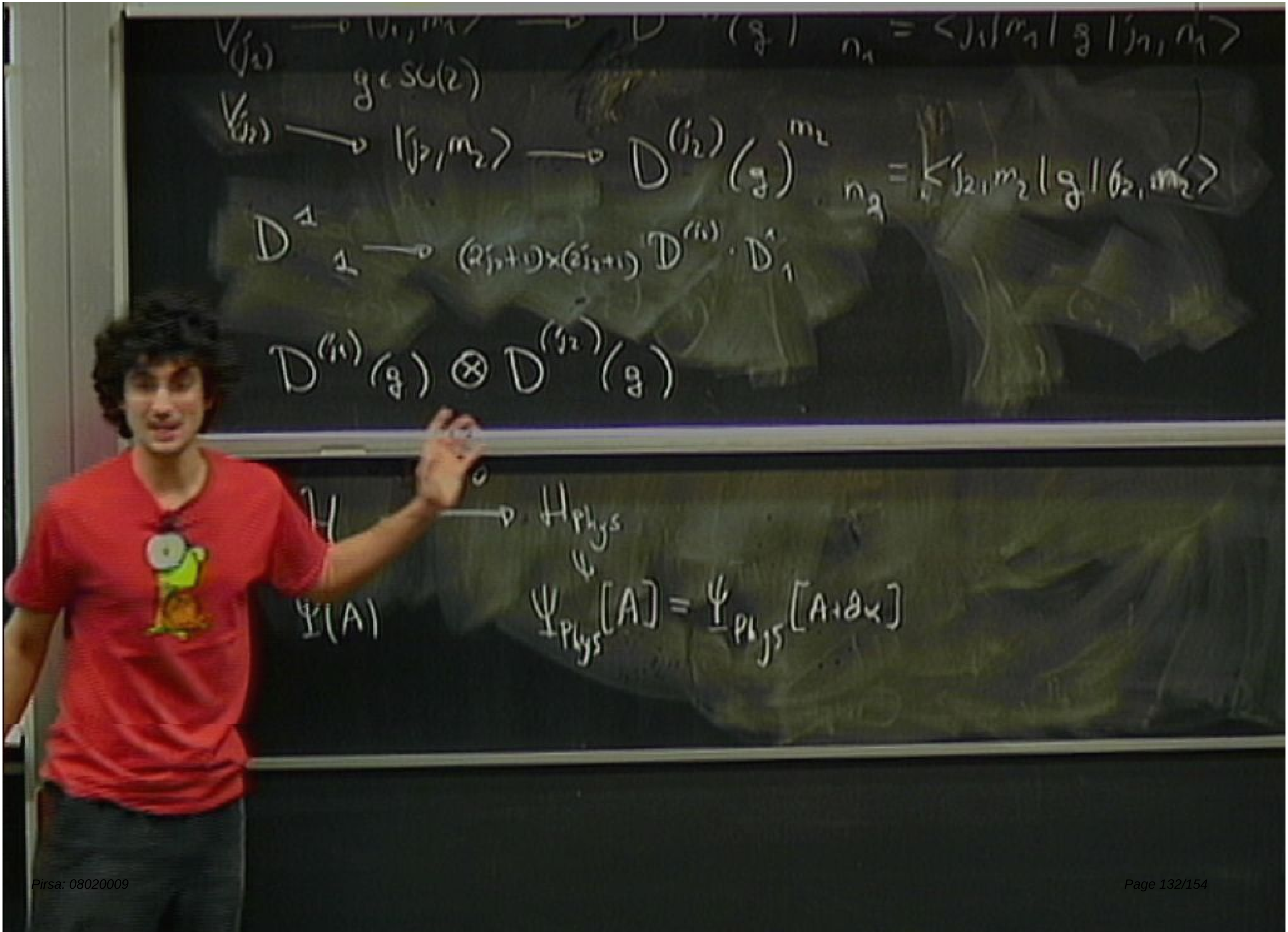

V(j1) → |j1, m1⟩ → D^(j1)(g)^m1_n1 = ⟨j1, m1 | g | j1, n1⟩
g ∈ SU(2)
V(j2) → |j2, m2⟩ → D^(j2)(g)^m2_n2 = ⟨j2, m2 | g | j2, n2⟩
Ψ_Phys[A] = Ψ_Phys[A + ∂α]

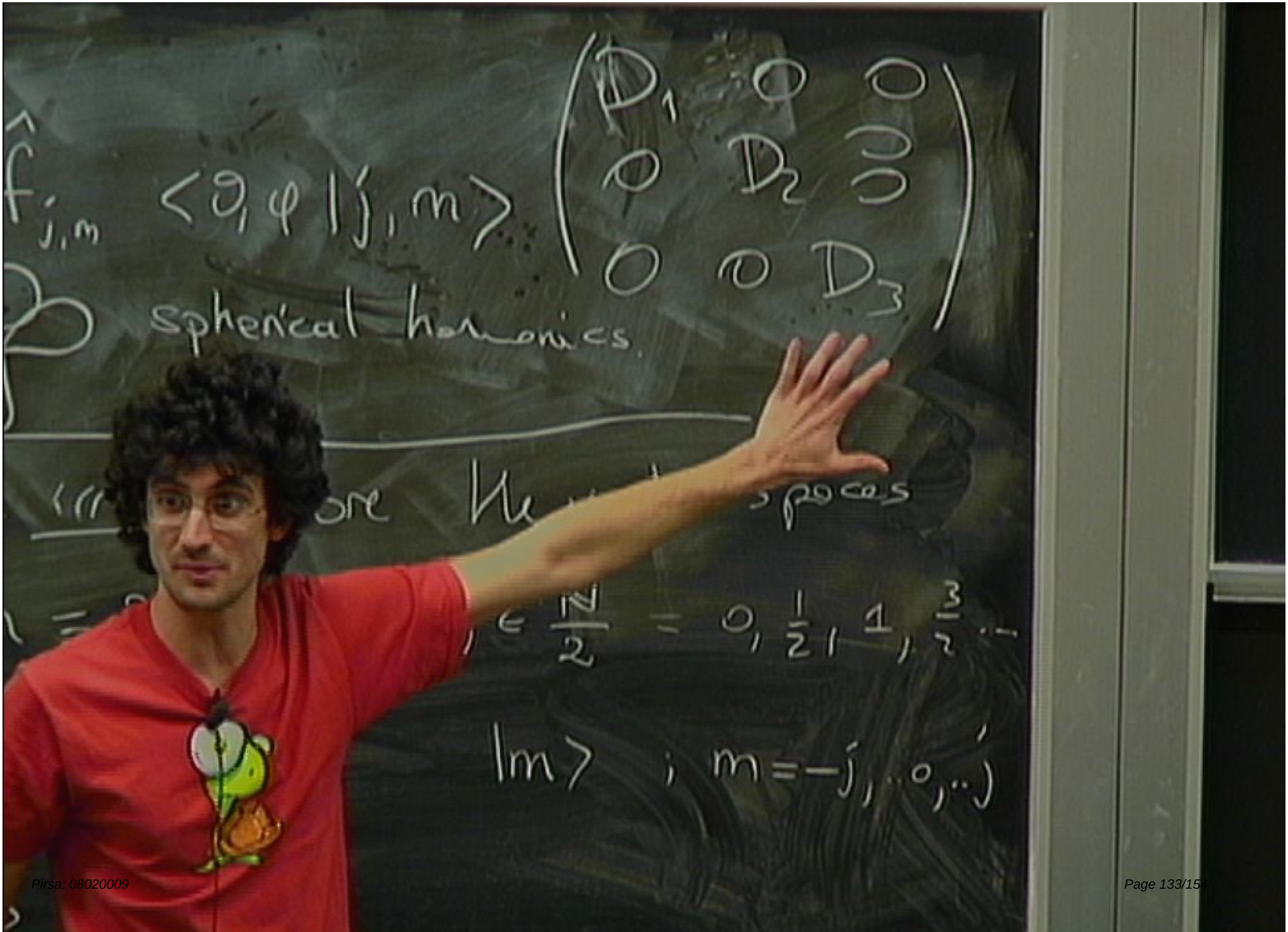


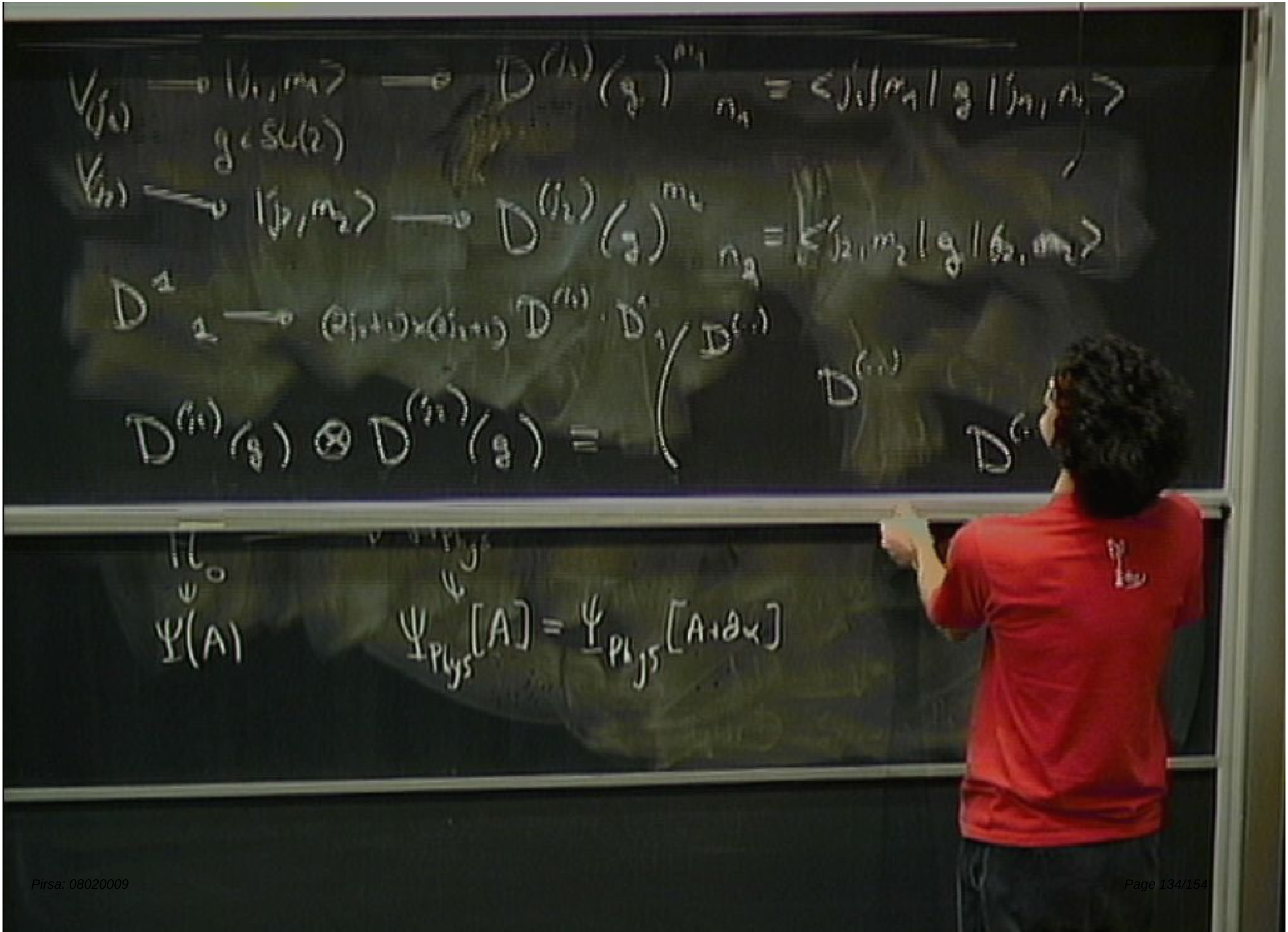
V(j1) → |j1, m1> → D^(j1)(g)^m1_n1 = <j1, m1 | g | j1, n1>
g ∈ SU(2)
V(j2) → |j2, m2> → D^(j2)(g)^m2_n2 = <j2, m2 | g | j2, n2>
D
Ψ_phys[A] = Ψ_phys[A+∂α]



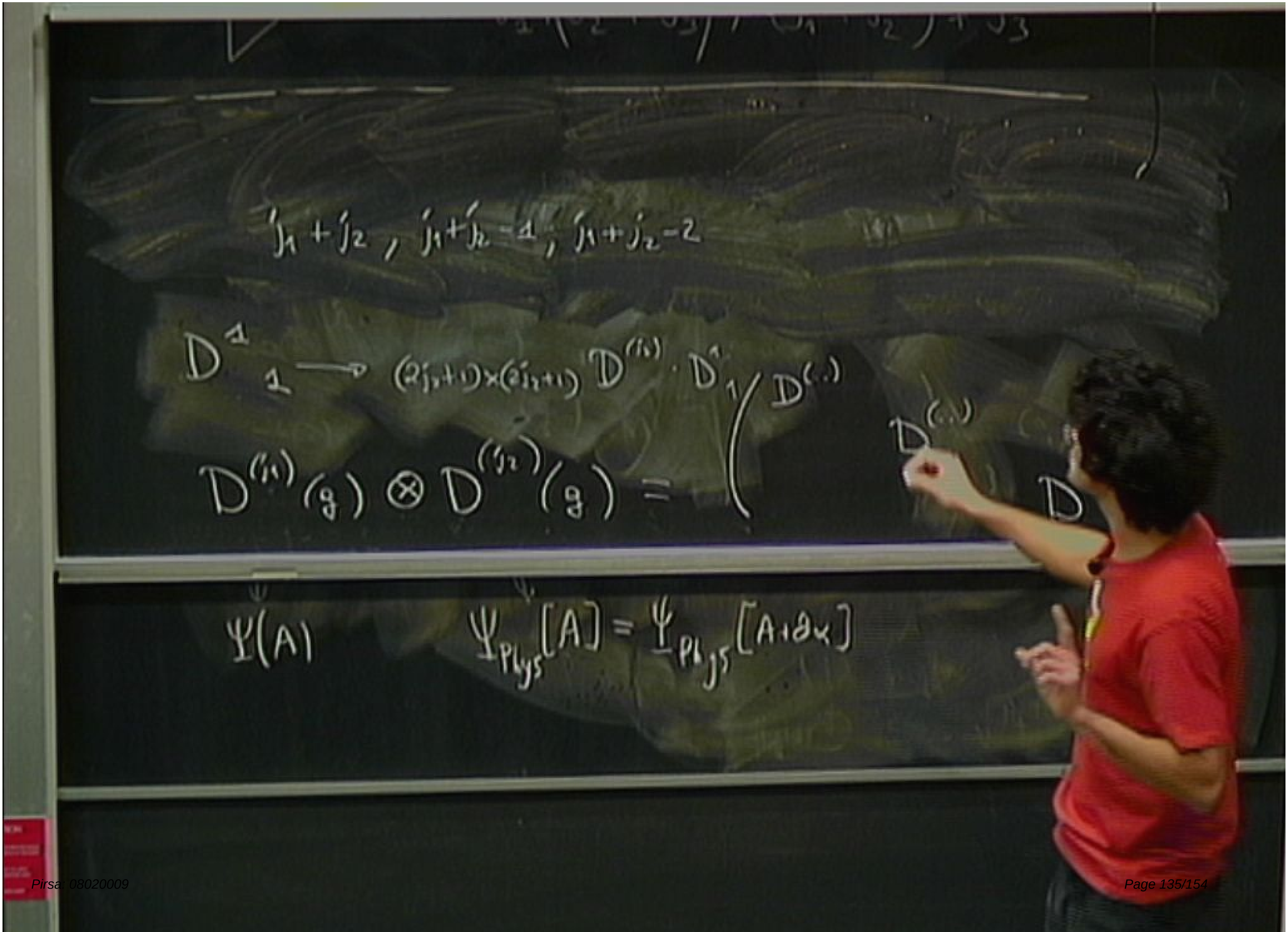
$j_1 + (j_2 + j_3) \neq (j_1 + j_2) + j_3$
$V_{(j_1)} \longrightarrow |j_1, m_1\rangle \longrightarrow D^{(j_1)}(g)^{m_1}{}_{n_1} = \langle j_1, m_1 | g | j_1, n_1 \rangle$
$g \in SU(2)$
$V_{(j_2)} \longrightarrow |j_2, m_2\rangle \longrightarrow D^{(j_2)}(g)^{m_2}{}_{n_2} = \langle j_2, m_2 | g | j_2, n_2 \rangle$
$D^{1}{}_{1} \longrightarrow (2j_1+1)\times(2j_2+1)\ D^{(j_1)} \cdot D^{1}{}_{1}$
$\Psi(A)$
$\Psi_{Phys}[A] = \Psi_{Phys}[A + \partial\alpha]$

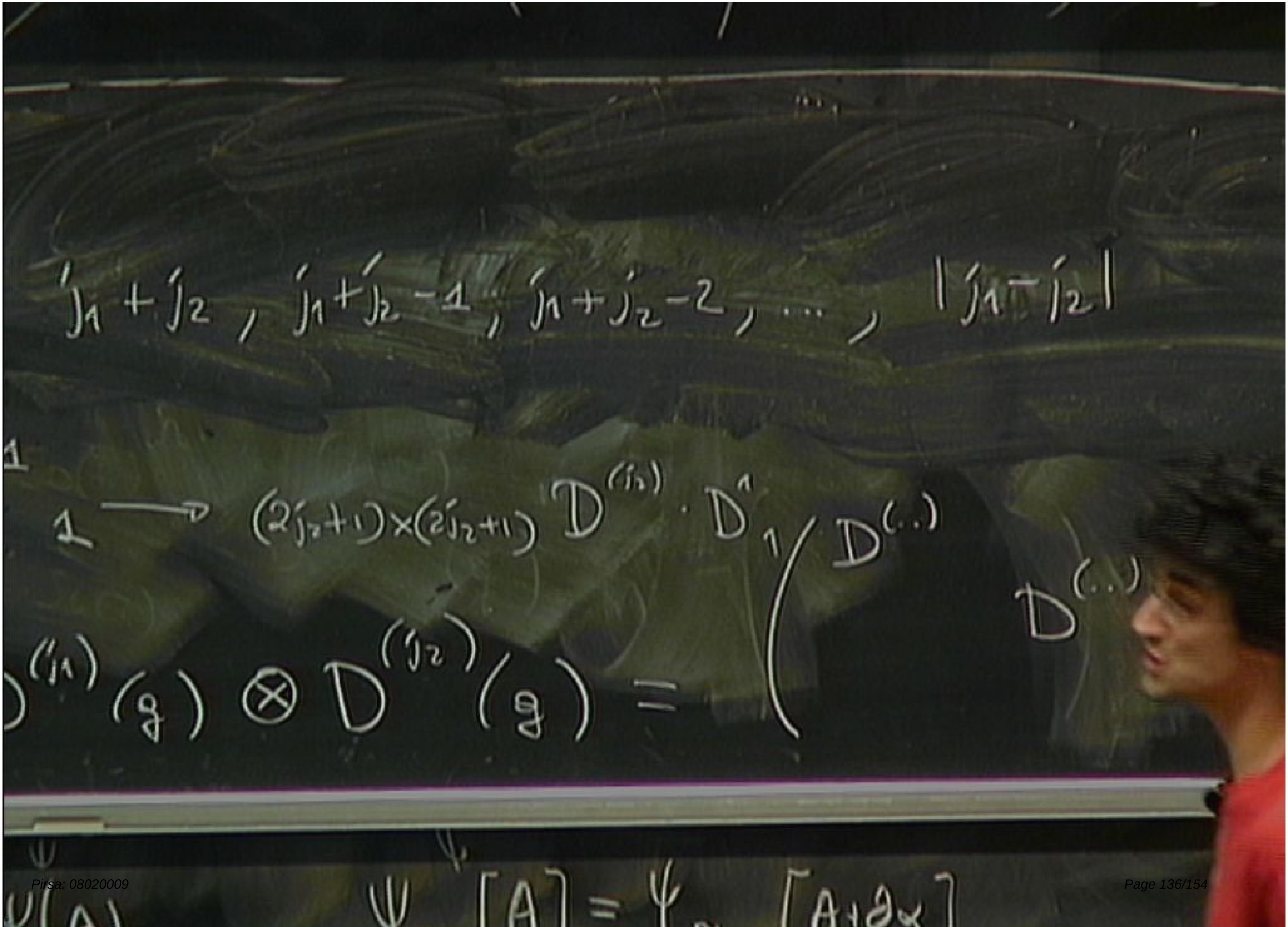

V(j1) → |j1, m1> → D(j1)(g) n1 = <j1, m1 | g | j1, n1>
g ∈ SU(2)
V(j2) → |j2, m2> → D(j2)(g)m2 n2 = <j2, m2 | g | j2, n2>
D1 1 → (2j1+1)×(2j2+1) D(j1) · D1 1
D(j1)(g) ⊗ D(j2)(g)
H → Hphys
Ψ
Ψ(A)   Ψphys[A] = Ψphys[A+∂α]

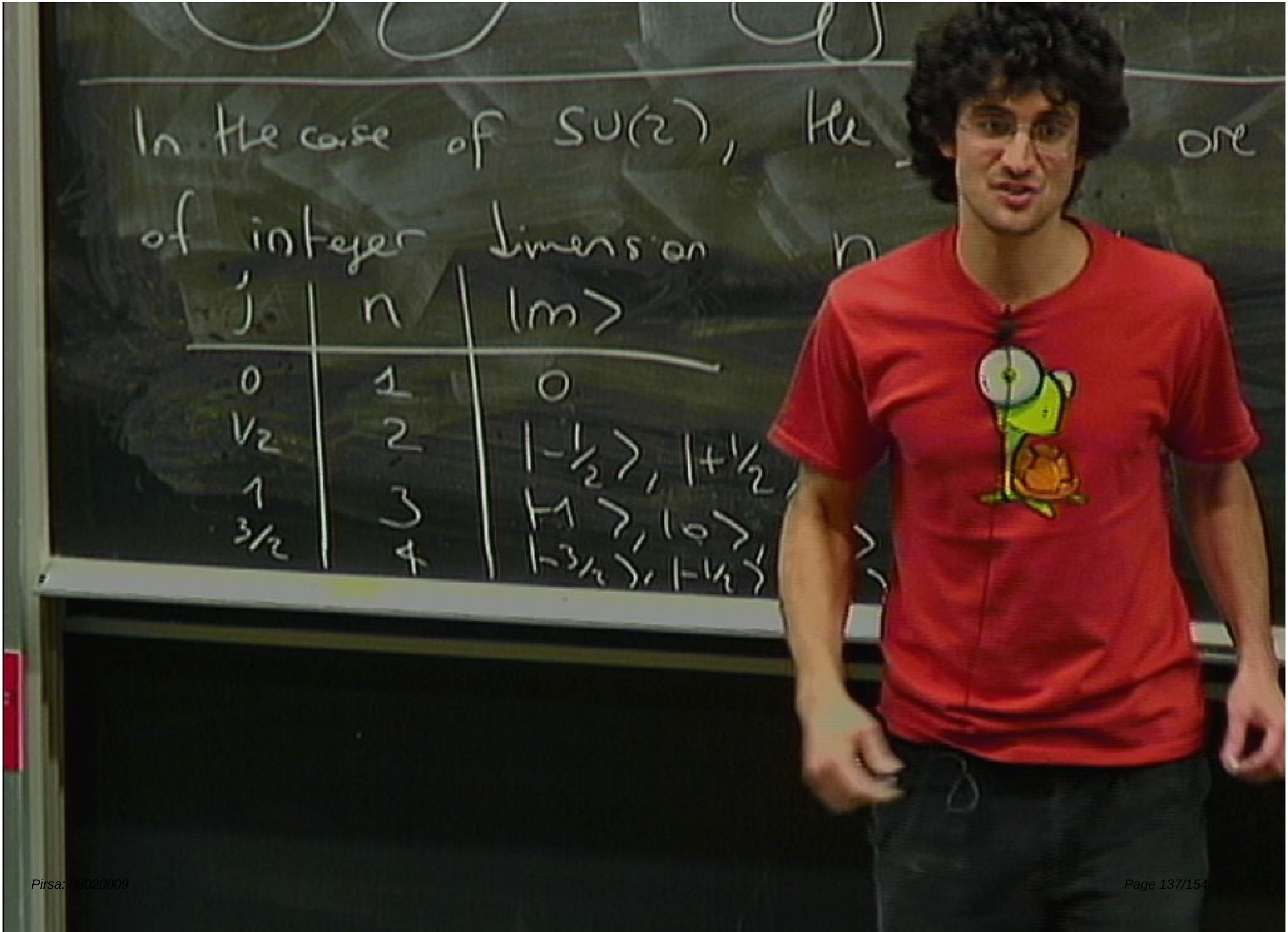

$\hat{f}_{j,m}$ $\langle \vartheta, \varphi | j, m \rangle$
$\begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$
spherical harmonics
spaces
$\in \frac{\mathbb{N}}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
$|m\rangle$; $m = -j, .. 0, .. j$

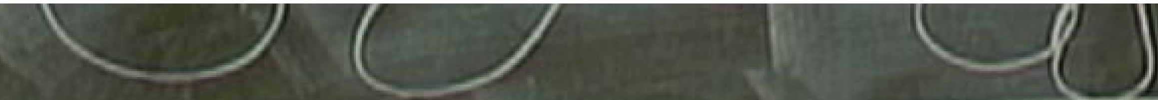


$V_{(j_1)} \longrightarrow |j_1, m_1\rangle \longrightarrow D^{(j_1)}(g)^{m_1}{}_{n_1} = \langle j_1, m_1| g |j_1, n_1\rangle$
$g \in SU(2)$
$V_{(j_2)} \longrightarrow |j_2, m_2\rangle \longrightarrow D^{(j_2)}(g)^{m_2}{}_{n_2} = \langle j_2, m_2| g |j_2, n_2\rangle$
$D^1{}_1 \longrightarrow (2j_1+1)\times(2j_2+1)$
$D^{(j_1)}(g) \otimes D^{(j_2)}(g) =$
$\Psi(A)$
$\Psi_{Phys}[A] = \Psi_{Phys}[A+\partial\alpha]$


j1 + j2 , j1 + j2 - 1 , j1 + j2 - 2
D^1_1 → (2j1+1)×(2j2+1) D^(j1) · D^1
D^(j1)(g) ⊗ D^(j2)(g) =
Ψ(A)
Ψ_Phys[A] = Ψ_Phys[A+∂α]


$j_1+j_2,\ j_1+j_2-1,\ j_1+j_2-2,\ \dots,\ |j_1-j_2|$
$1 \longrightarrow (2j_1+1)\times(2j_2+1)$
$D^{(j_1)} \cdot D^{1}$
$D^{(..)}$
$D^{(..)}$
$D^{(j_1)}(g) \otimes D^{(j_2)}(g) =$
$\Psi[A] = \Psi[A+\partial\alpha]$


In the case of SU(2), the
are
of integer dimension
n
j n |m>
0 1 0
1/2 2 |-1/2>, |+1/2
1 3 |-1>, |0>,
3/2 4 |-3/2>, |-1/2>,



In the case of $SU(2)$, the irreps are of integer dimension $n = 2j+1$

j	n	$\|m\rangle$
0	1	0
$1/2$	2	$\|-1/2\rangle, \|+1/2\rangle$
1	3	$\|-1\rangle, \|0\rangle, \|1\rangle$
$3/2$	4	$\|-3/2\rangle, \|-1/2\rangle, \|1/2\rangle,$ [illegible]

In the case of SU(2), the irreps on

of integer dimension $n = 2j+1$,

j	n	$\|m\rangle$
0	1	0
$1/2$	2	$\|-\frac{1}{2}\rangle, \|+\frac{1}{2}\rangle$
1	3	$\|-1\rangle, \|0\rangle, \|1\rangle$
$3/2$	4	$\|-3/2\rangle, \|-1/2\rangle, \|1/2\rangle, \|3$

$$D^{1}{}_{1} \longrightarrow (2j_1+1)\times(2j_2+1) \quad D^{(j_1)} \cdot D^{1}{}_{1}$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & & 0 & 0 \\ & & \ddots & D^{(|j_1-j_2|)} & \end{pmatrix}$$

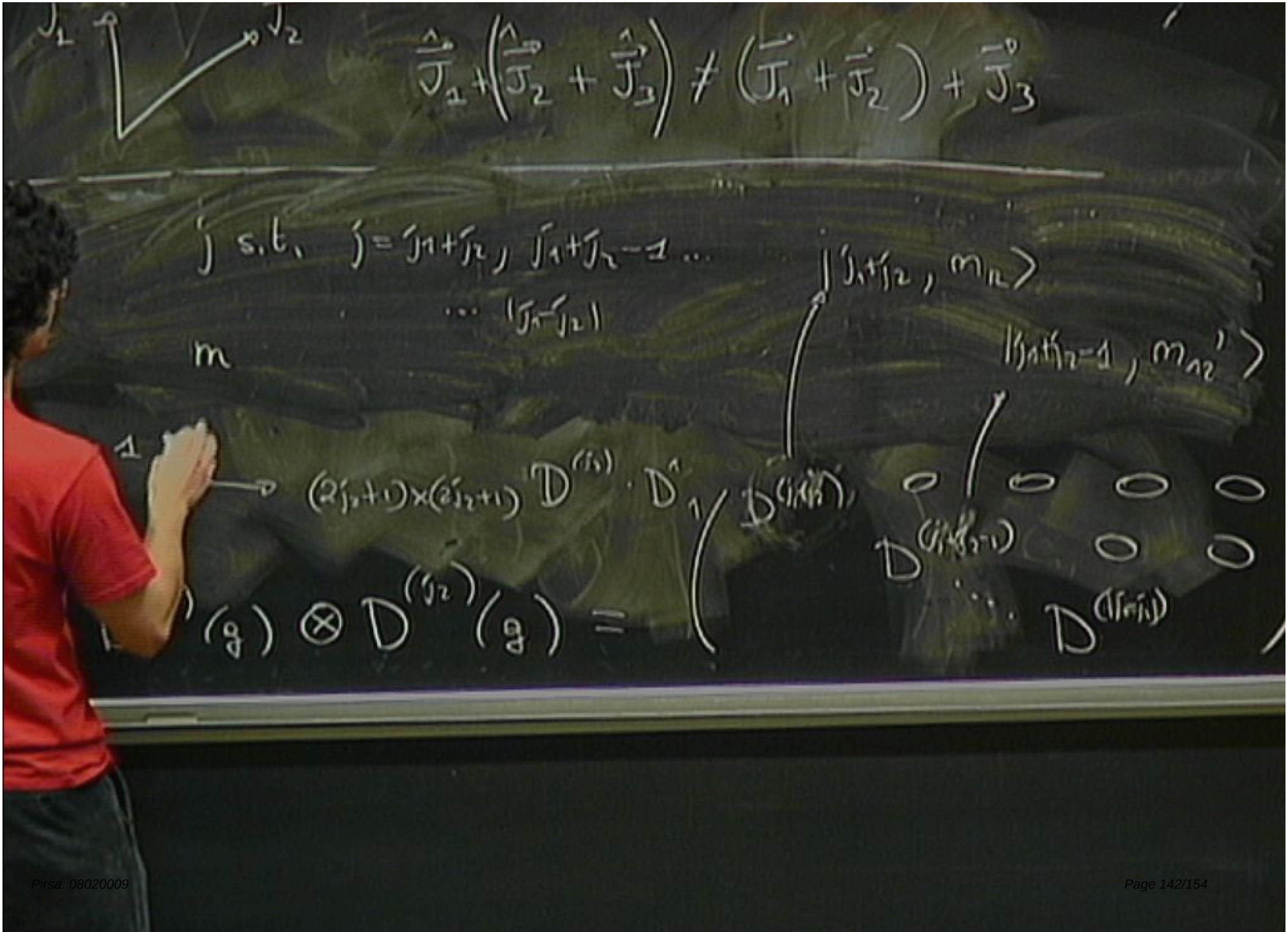
$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$

$$|j_1+j_2\,,\ m_{12}\rangle$$

$$|j_1+j_2-1\,,\ m_{12}'\rangle$$

$$D^1{}_1 \longrightarrow (2j_1+1)\times(2j_2+1)\ D^{(j_1)}\cdot D^{(j_2)}$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & & 0 & 0 \\ & & \ddots & & \\ & & & D^{(|j_1-j_2|)} & \end{pmatrix}$$


J1 J2
J1 + (J2 + J3) ≠ (J1 + J2) + J3
j s.t. j = j1+j2, j1+j2−1 ...
... |j1−j2|
|j1+j2, m12>
m
|j1+j2−1, m12'>
(2j1+1)×(2j2+1)
(g) ⊗ D(j2)(g) =

$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$

$$\begin{cases} j \text{ s.t. } j = j_1 + j_2,\ j_1 + j_2 - 1 \ldots \\ \qquad\qquad \ldots\ |j_1 - j_2| \\ m = -j, \ldots, j \end{cases}$$

$|j_1 + j_2, m_{12}\rangle$

$|j_1 + j_2 - 1, m_{12}'\rangle$

$$D^1{}_1 \longrightarrow (2j_1+1)\times(2j_2+1) \quad D^{(j_1)} \cdot D^{j}{}_1$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & & 0 & 0 \\ & & \ddots & & D^{(|j_1-j_2|)} \end{pmatrix}$$

$\vec{J}_1 \quad \vec{J}_2$

$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$

$|j, m\rangle$

$$\begin{cases} j \text{ s.t. } j = j_1 + j_2,\ j_1 + j_2 - 1 \ldots \\ \qquad \ldots |j_1 - j_2| \\ m = -j, \ldots, j \end{cases}$$

$|j_1 + j_2, m_{12}\rangle$

$|j_1 + j_2 - 1, m_{12}'\rangle$

$$D^{1}{}_{1} \longrightarrow (2j_1+1)\times(2j_2+1) \quad D^{(j_1)} \cdot D^{1}{}_{1}$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & & 0 & 0 \\ & & \ddots & D^{(|j_1-j_2|)} & \end{pmatrix}$$

$\vec{J}_1$ $\vec{J}_2$

$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$

$$\begin{cases} j \text{ s.t. } j = j_1 + j_2,\ j_1 + j_2 - 1 \ldots \\ \qquad\qquad \ldots\ |j_1 - j_2| \\ m = -j, \ldots, j \end{cases}$$

$|j, m\rangle$

$|j_1 + j_2, m_{12}\rangle$

$|j_1 + j_2 - 1, m_{12}'\rangle$

$$D^1{}_1 \longrightarrow (2j_1+1)\times(2j_2+1) \quad D^{(j_1)} \cdot D^1{}_1$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & & 0 & 0 \\ & & \ddots & D^{(|j_1-j_2|)} & \end{pmatrix}$$

$$\hat{\vec{J}}_1 + \left(\hat{\vec{J}}_2 + \hat{\vec{J}}_3\right) \neq \left(\vec{J}_1 + \vec{J}_2\right) + \vec{J}_3$$

$$\begin{cases} j \text{ s.t. } j = j_1 + j_2,\ j_1 + j_2 - 1 \ldots \\ \qquad \ldots\ |j_1 - j_2| \\ m = -j, \ldots, j \end{cases}$$

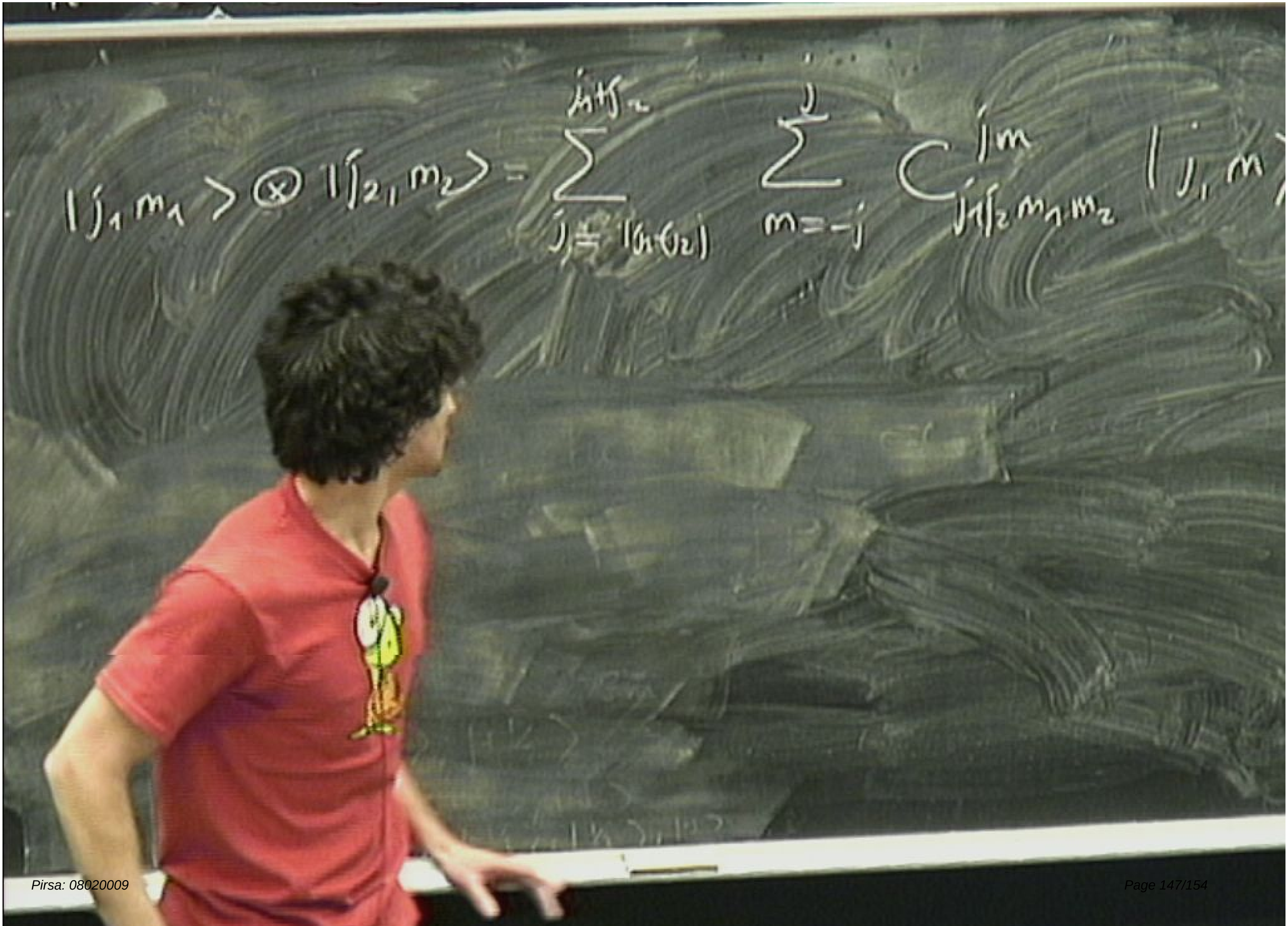
$|j, m\rangle$

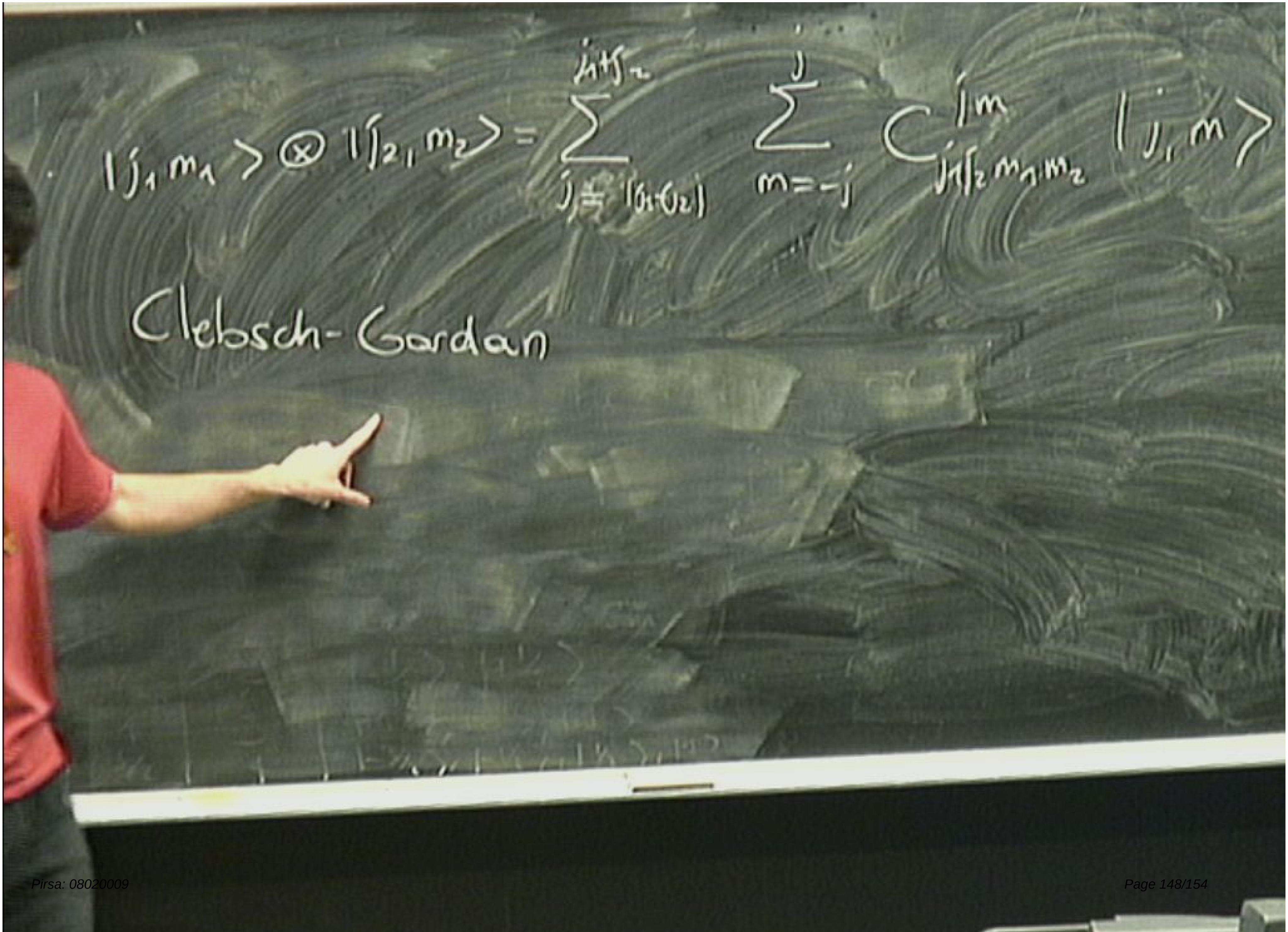
$|j_1 + j_2, m_{12}\rangle$

$|j_1 + j_2 - 1, m_{12}'\rangle$

$$D^1{}_1 \longrightarrow (2j_1+1)\times(2j_2+1) \quad D^{(j_1)} \cdot D^1{}_1$$

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \begin{pmatrix} D^{(j_1+j_2)} & 0 & 0 & 0 \\ & D^{(j_1+j_2-1)} & 0 & 0 \\ & & \ddots & \\ & & & D^{(|j_1-j_2|)} \end{pmatrix}$$


$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$



$|j_1 m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$
Clebsch-Gordan

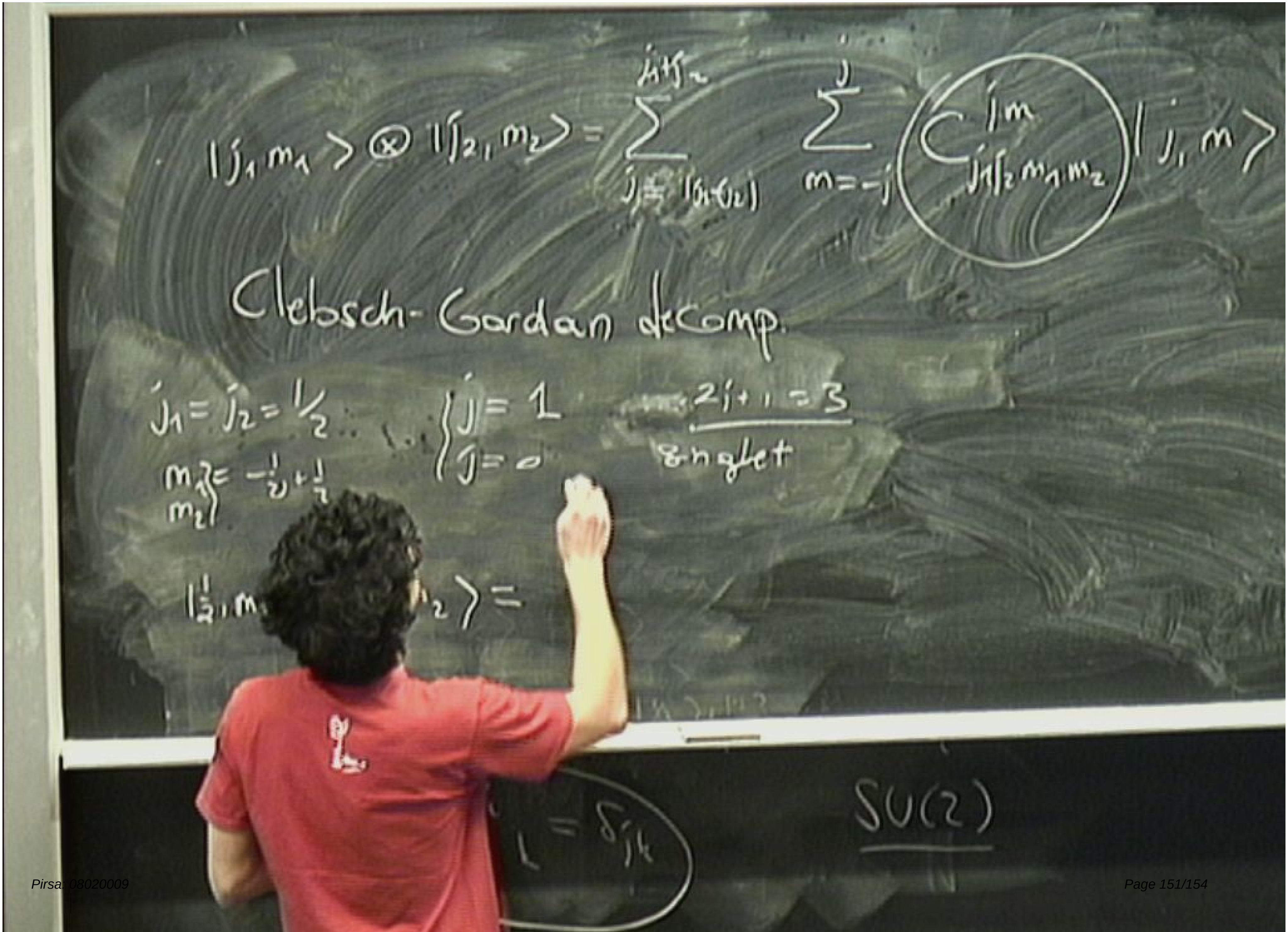
$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$$

Clebsch-Gordan decomp.

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$$

Clebsch-Gordan decomp.

$j_1 = j_2 = 1/2$



$|j_1 m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$
Clebsch-Gordan decomp.
$j_1 = j_2 = 1/2$
$j = 1$
$2j+1 = 3$
$j = 0$
singlet
SU(2)

$$|j_1 m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$$

Clebsch-Gordan decomp.

$j_1 = j_2 = 1/2$ $\quad \begin{cases} j = 1 & 2j+1 = 3 \\ j = 0 & \text{singlet} \end{cases}$

$\left.\begin{matrix} m_1 \\ m_2 \end{matrix}\right\} = -\frac{1}{2}, +\frac{1}{2}$

$$|\tfrac{1}{2}, m_1\rangle \otimes |\tfrac{1}{2}, m_2\rangle = C^{00}_{\frac{1}{2}\frac{1}{2}m_1 m_2} |0,0\rangle + \sum_{m=-1}^{1} C^{1m}_{\frac{1}{2}\frac{1}{2}m_1 m_2} |1, m\rangle$$

$= \sum_i g^i{}_j \, g^i{}_k = \delta_{jk}$

SU(2)

$$|j_1 m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$$

$$\begin{array}{c|cc} & + & - \\ \hline + & & \\ - & & \end{array}\quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$m=1$

$$|\tfrac{1}{2}, m_1\rangle \otimes |\tfrac{1}{2}, m_2\rangle = C^{00}_{\frac{1}{2}\frac{1}{2} m_1 m_2} |0,0\rangle + \ldots C^{1m} \ldots |1, m\rangle$$

$$= \sum_i g^i{}_j \, g^i{}_k = \delta_{jk}$$

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} C^{jm}_{j_1 j_2 m_1 m_2} |j, m\rangle$$

$$\begin{array}{c} \\ + \\ - \end{array}\frac{1}{\sqrt{2}}\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} m=1 \\ \begin{array}{cc} + & - \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} + \frac{1}{\sqrt{2}} \begin{array}{c} m=0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} m=-1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\| \qquad\qquad \|$$

$$|\tfrac{1}{2}, m_1\rangle \otimes |\tfrac{1}{2}, m_2\rangle = C^{00}_{\frac{1}{2}\frac{1}{2} m_1 m_2} |0,0\rangle + \sum_{m=-1}^{1} C^{1m}_{\frac{1}{2}\frac{1}{2} m_1 m_2} |1, m\rangle$$

$$\Rightarrow \sum_i g^i{}_j \, g^i{}_k = \delta_{jk}$$

SU(2)