Title: Gluon scattering in N=4 super-Yang-Mills theory from weak to strong coupling

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Abstract: TBA

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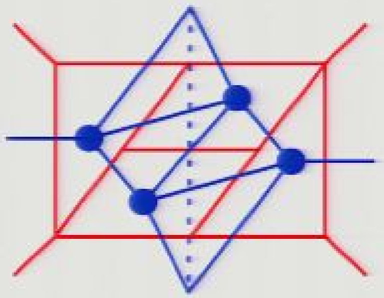

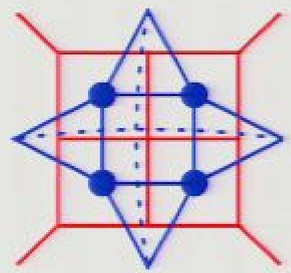

Lance Dixon (SLAC)
with Z. Bern, D. Kosower, R. Roiban,
V. Smirnov, M. Spradlin, C. Vergu, A. Volovich

Perimeter Institute, February 26, 2008

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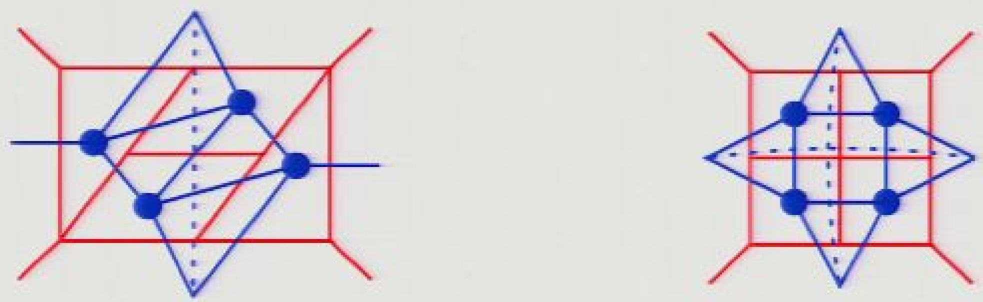

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# N=4 Super-Yang-Mills Theory

- N=4 SYM: most supersymmetric theory possible without gravity:

| | |
|---|---|
| *massless spin 1 gluon* | |
| *4 massless spin 1/2 gluinos* | |
| *6 massless spin 0 scalars* | |

all states in adjoint representation, all linked by N=4 supersymmetry

- Interactions uniquely specified by gauge group, say SU($N_c$), 1 coupling $g$
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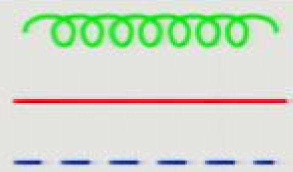

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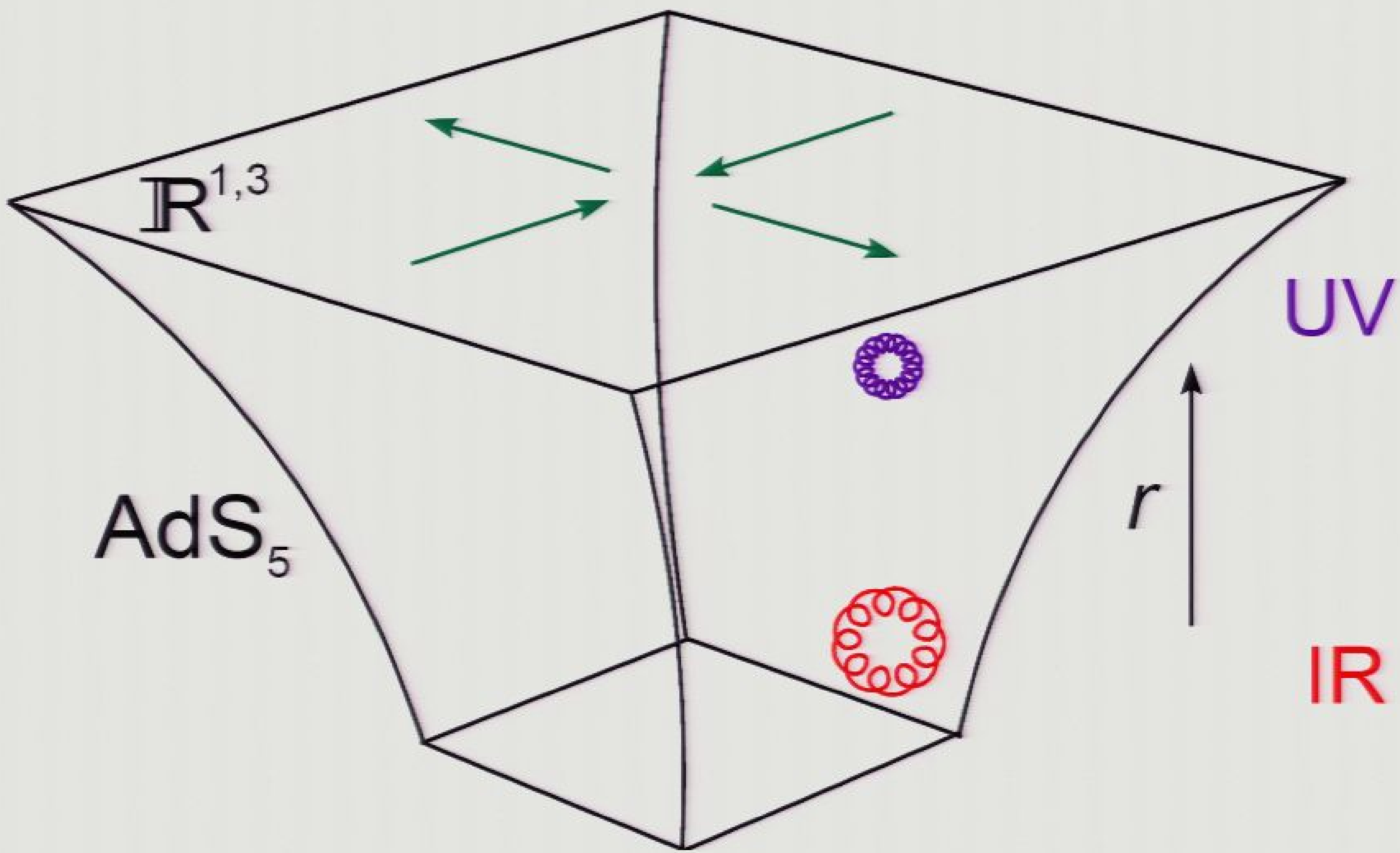
# Planar N=4 SYM and AdS/CFT

- Consider the 't Hooft limit,
  $N_c \to \infty$ , with $\lambda = g^2 N_c$ fixed,
  in which planar Feynman diagrams dominate
- AdS/CFT duality Maldacena; Gubser, Klebanov, Polyakov; Witten
  suggests that weak-coupling perturbation series in $\lambda$ for large-$N_c$ (planar) N=4 SYM should have special properties, because
  large $\lambda$ limit ←→ weakly-coupled gravity/string theory

# AdS/CFT in one picture



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- Recent strong-coupling confirmation for 2 → 2 scattering.
- **But:** problems for n gluons? Alday, Maldacena, 0705.0303[th], 0710.1060 [hep-th]; Drummond, Henn, Korchemsky, Sokatchev, 0712.4138[th]; Bartels, Lipatov, Sabio Vera, 0802.2065[th]

# Some questions you might have

- "What are gluons?"

  AdS/CFT most simply relates "glueballs"
  – color-singlet, gauge-invariant local operators
  – to modes of gravitational fields propagating in $AdS_5 \times S^5$. But gluons are colored states
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- What is the cusp anomalous dimension?
- What are the other functions of $\lambda$ entering the scattering amplitude?
- What is the evidence for this proposal, at weak and at strong coupling?

# Take a cue from perturbative QCD

- Gluons (in QCD, not N=4 SYM) are the objects colliding at the LHC (most of the time).

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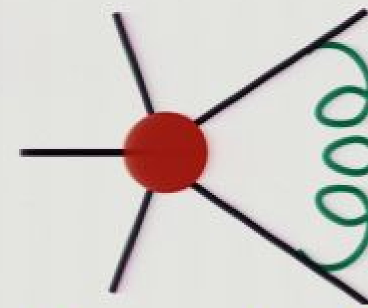
- Dim. reg. breaks conformal invariance. Recover it by Laurent expansion around $\varepsilon = 0$ through $O(\varepsilon^0)$.
- In string theory, gluons can be "discovered" by tying open string ends to a D-brane in the IR, and using kinematics (large $s$ and $t$) to force the string to stretch deep into the UV. But there is also a dim. reg. version of $AdS_5 \times S^5$ Alday, Maldacena

# Dimensional Regulation in the IR

**One-loop IR divergences are of two types:**



$D = 4 - 2\epsilon$

$\epsilon < 0$

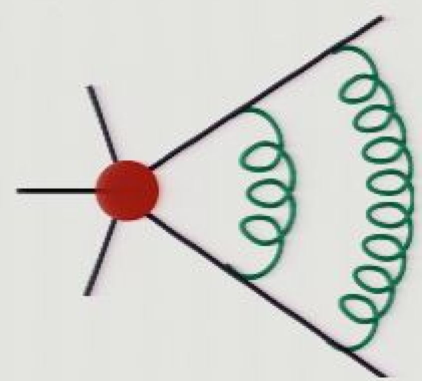
**Soft** $\int_0 \frac{d\omega}{\omega} \rightarrow \int_0 \frac{d\omega}{\omega^{1+\epsilon}} \propto \frac{1}{\epsilon}$

**Collinear** (with respect to massless emitting line)

$$\int_0 \frac{dk_T}{k_T} \rightarrow \int_0 \frac{dk_T}{k_T^{1+\epsilon}} \propto \frac{1}{\epsilon}$$

Overlapping soft + collinear divergences imply leading pole is $\frac{1}{\epsilon^2}$ at 1 loop

$\frac{1}{\epsilon^{2L}}$ at $L$ loops



# IR Structure in QCD and N=4 SYM

- Pole terms in $\varepsilon$ are predictable due to soft/collinear factorization and exponentiation – long-studied in QCD, straightforwardly applicable to N=4 SYM

Akhoury (1979); Mueller (1979); Collins (1980); Sen (1981); Sterman (1987); Botts, Sterman (1989); Catani, Trentadue (1989); Korchemsky (1989) **Magnea, Sterman (1990);** Korchemsky, Marchesini, hep-ph/9210281 Catani, hep-ph/9802439; Sterman, Tejeda-Yeomans, hep-ph/0210130

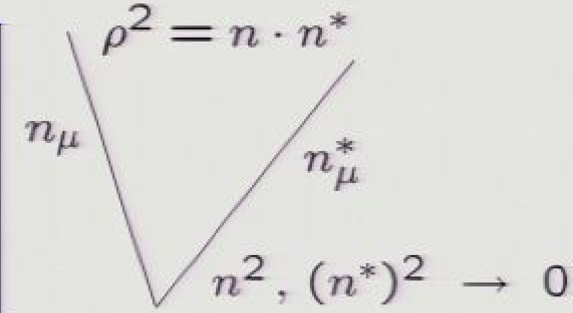
In the planar limit, for both QCD and N=4 SYM, pole terms are given in terms of:

- the beta function $\beta(\lambda)$ [ = 0 in N=4 SYM ]
- the cusp (or soft) anomalous dimension $\gamma_K(\lambda)$
- a "collinear" anomalous dimension $\mathcal{G}_0(\lambda)$

# Cusp anomalous dimension

VEV of Wilson line with kink or cusp in it obeys renormalization group equation:

$$\left(\rho\frac{\partial}{\partial\rho} + \beta(g)\frac{\partial}{\partial g}\right) \ln W(\rho, g) = -2\,\gamma_K(g)\, \ln \rho^2 + \mathcal{O}(\rho^0)$$

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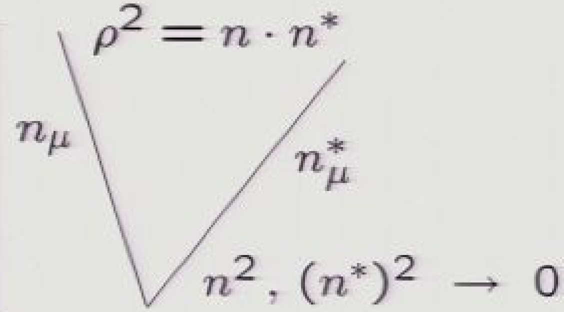


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Cusp (soft) anomalous dimension $\gamma_K(g)$ also controls large-spin limit of anomalous dimensions $\gamma_j$ of leading-twist operators with spin *j*: $\bar{q}(\gamma^+ D_+)^j q$

$$\gamma_j = \frac{1}{2}\gamma_K(g)\, \ln j \ +\ \mathcal{O}(j^0)$$

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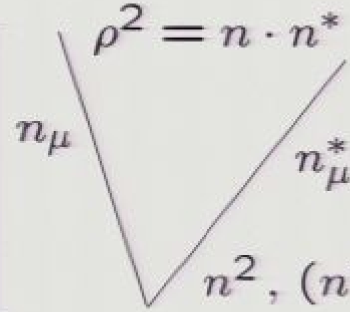


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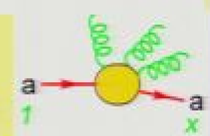
Related by Mellin transform to $x \longrightarrow 1$ limit of DGLAP kernel for evolving parton distribution functions $f(x, \mu_F)$:

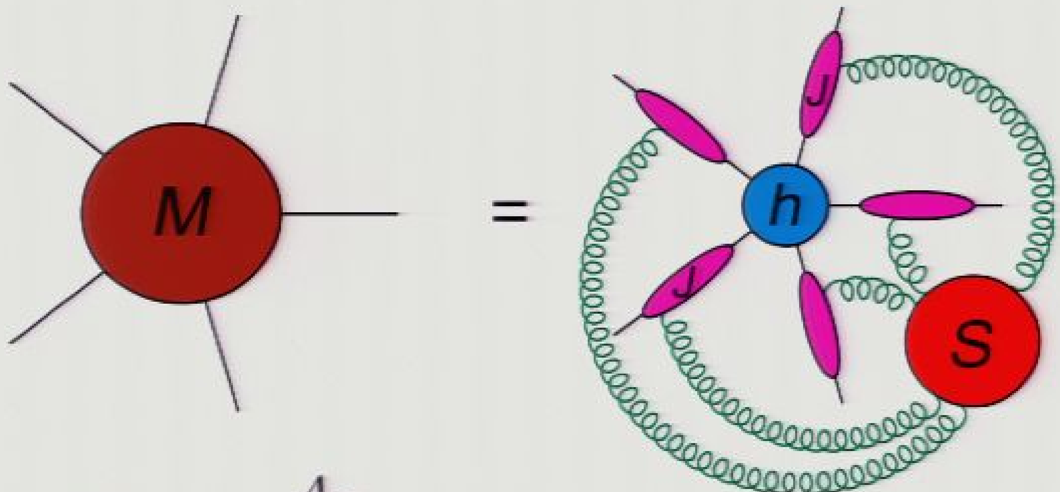
$$\gamma_j \equiv -\int_0^1 dx\, x^{j-1}\, P_{aa}(x)$$

$$P_{aa}(x) = \frac{1}{2}\frac{\gamma_K(g)}{(1-x)_+} + B(g)\,\delta(1-x) + \cdots$$

→ important for soft gluon resummations



# Soft/Collinear Factorization




Magnea, Sterman (1990);
Sterman, Tejeda-Yeomans,
hep-ph/0210130

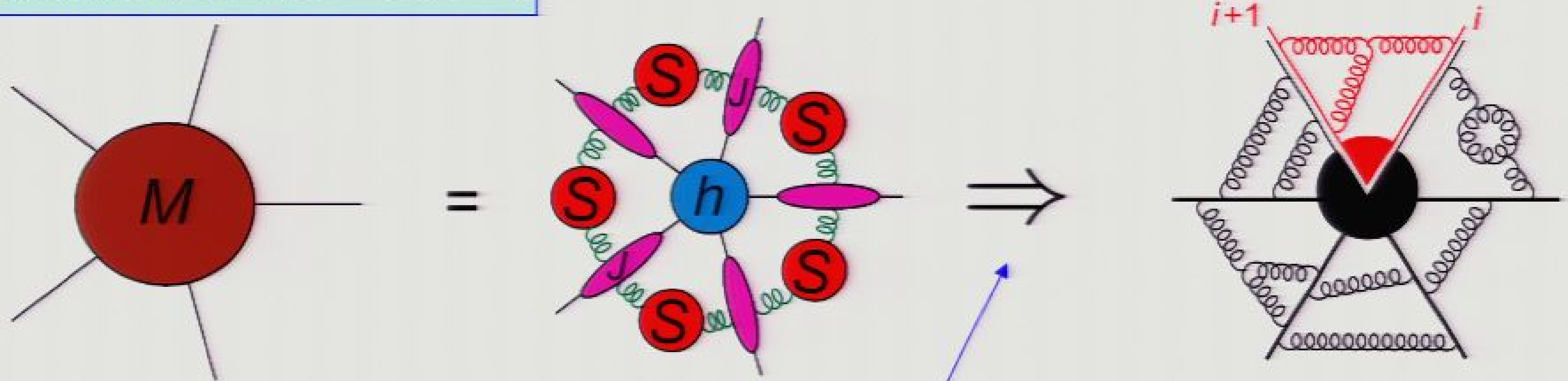
$$\mathcal{M}_n \equiv \frac{\mathcal{A}_n}{\mathcal{A}_n^{\text{tree}}}$$

$$\mathcal{M}_n = S(k_i, \mu, \alpha_s(\mu), \epsilon) \times \left[\prod_{i=1}^{n} J_i(\mu, \alpha_s(\mu), \epsilon)\right] \times h_n(k_i, \mu, \alpha_s(\mu), \epsilon)$$

- *S = soft function (only depends on color of $i^{th}$ particle)*
- *J = jet function (color-diagonal; depends on $i^{th}$ spin)*
- *$h_n$ = hard remainder function (finite as $\epsilon \to 0$ )*

# Simplification at Large $N_c$ (Planar Case)

coefficient of $\mathrm{Tr}[T^{a_1} \ldots T^{a_n}]$




- *Soft function* only defined up to a multiple of the identity matrix in color space
- Planar limit is color-trivial; can absorb $S$ into $J_i$
- If all $n$ particles are identical, say gluons, then each "wedge" is the square root of the "$gg \rightarrow 1$" process (Sudakov form factor):

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[ \mathcal{M}^{[gg\to 1]} \left( \frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n(k_i, \mu, \alpha_s, \epsilon)$$

# Sudakov form factor

- Factorization → differential equation for form factor

Mueller (1979); Collins (1980); Sen (1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{\partial}{\partial \ln Q^2} \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$
$$= \frac{1}{2}\Big[K(\epsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s(\mu), \epsilon)\Big] \times \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$

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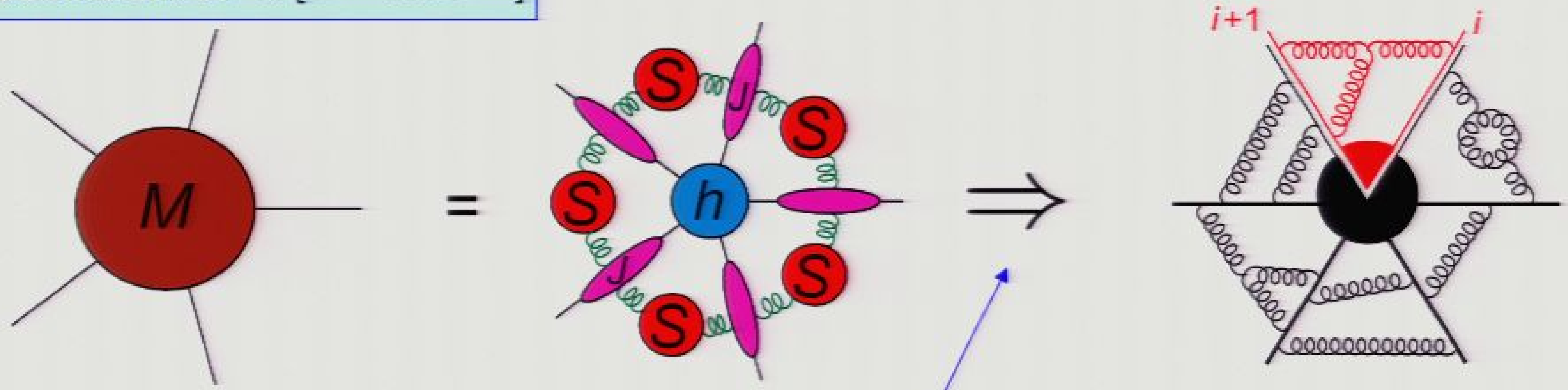


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like $\beta(\varepsilon,\alpha_s)$, single poles in $\varepsilon$ determine $K$ completely

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*K*, *G* also obey differential equations (ren. group):

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right)(K+G)=0 \qquad \left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right)K=-\gamma_K(\alpha_s)$$

cusp anomalous dimension

# General amplitude in planar N=4 SYM

- Solve differential equations for $K, G$. **Easy** because coupling doesn't run.
- Insert result for Sudakov form factor into *n*-point amplitude

$$\Rightarrow \quad \mathcal{M}_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp\Big[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{i,i+1}}\Big)^{l\epsilon}\Big] \times h_n$$

loop expansion parameter:

$$a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon} = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$$

$\hat{\gamma}_K^{(l)}, \hat{\mathcal{G}}_0^{(l)}$ are $l$-loop coefficients of $\gamma_K(a), \mathcal{G}_0(a)$

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looks like the one-loop amplitude,
but with $\varepsilon$ shifted to ($l\ \varepsilon$), up to finite terms

$\hat{\gamma}_K^{(l)}$, $\hat{\mathcal{G}}_0^{(l)}$ are $l$-loop coefficients of $\gamma_K(a)$, $\mathcal{G}_0(a)$

# General amplitude in planar N=4 SYM

- Solve differential equations for $K$, $G$. **Easy** because coupling doesn't run.
- Insert result for Sudakov form factor into $n$-point amplitude

$$\Rightarrow \quad \mathcal{M}_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp\Big[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{i,i+1}}\Big)^{l\epsilon}\Big] \times h_n$$

loop expansion parameter:

$$a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^\epsilon = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$$

looks like the one-loop amplitude, but with $\varepsilon$ shifted to $(l\,\varepsilon)$, up to finite terms

$\hat{\gamma}_K^{(l)}$, $\hat{\mathcal{G}}_0^{(l)}$ are $l$-loop coefficients of $\gamma_K(a)$, $\mathcal{G}_0(a)$

Rewrite as

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1})\Big)\Big]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

collects **3 series of constants**:

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)} \qquad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)} \qquad f_2^{(l)} = (???)$$

# Exponentiation in planar N=4 SYM

- For planar N=4 SYM, propose that the finite terms also exponentiate. That is, the hard remainder function $h_n^{(l)}$ defined by

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \Big)\Big]$$

is also a series of constants, $C^{(l)}$ [for **MHV** amplitudes]:

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon) \Big)\Big]$$

$$\Rightarrow \quad \mathcal{M}_4|_{\text{finite}} = \exp\Big[\frac{1}{8}\gamma_K(a) \ln^2\Big(\frac{s}{t}\Big) + \text{const.}\Big]$$

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Anastasiou, Bern, LD, Kosower, hep-th/0309040; Cachazo, Spradlin, Volovich, hep-th/0602228; Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

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**Evidence** based on two loops (*n=4,5*, plus collinear limits)
and three loops (for *n=4*) — Bern, LD, Smirnov, hep-th/0505205
and now strong coupling (*n=4,5 only?*) — Alday, Maldacena, 0705.0303 [hep-th]

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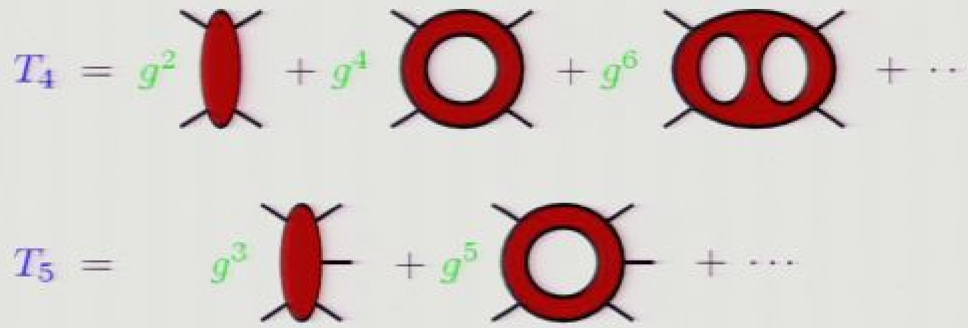
In contrast, for QCD, and non-planar N=4 SYM, two-loop amplitudes have been computed, and hard remainders are a mess of polylogarithms in *t/s*

# Evidence: from amplitudes computed via perturbative unitarity

Expand scattering matrix $T$ in coupling $g$



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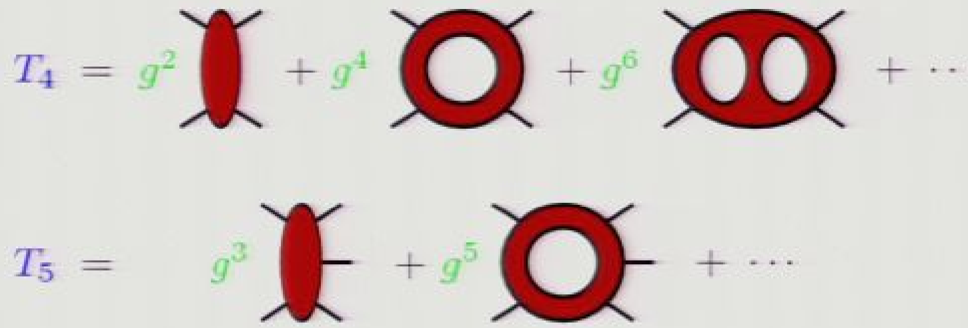


Insert expansion into unitarity relation

$$2\,\mathrm{Im}\,T = T^{\dagger}T$$

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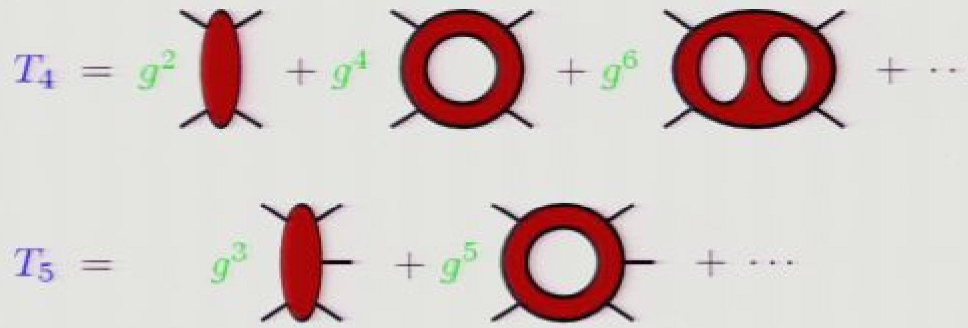


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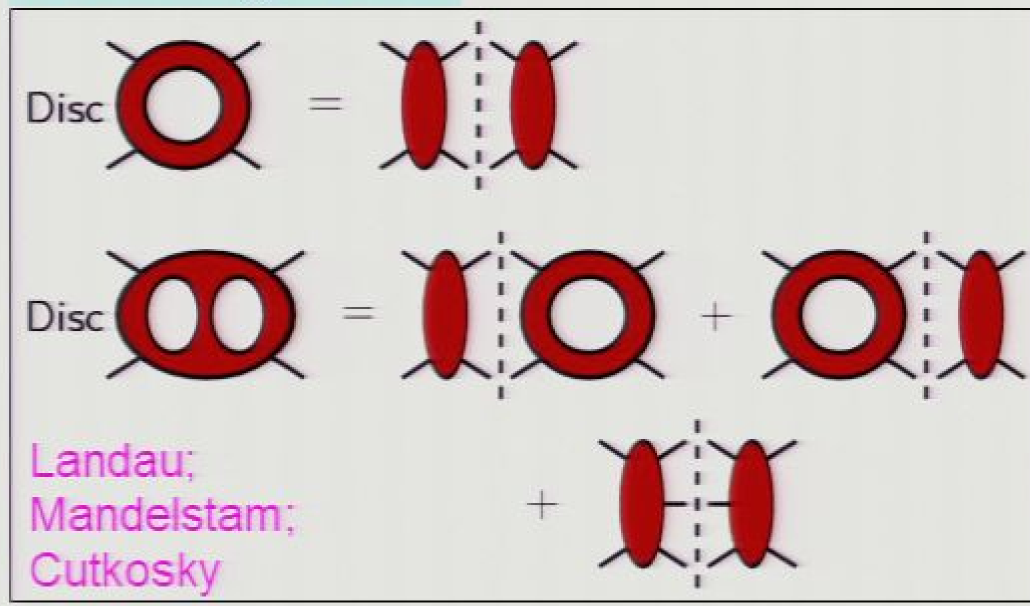

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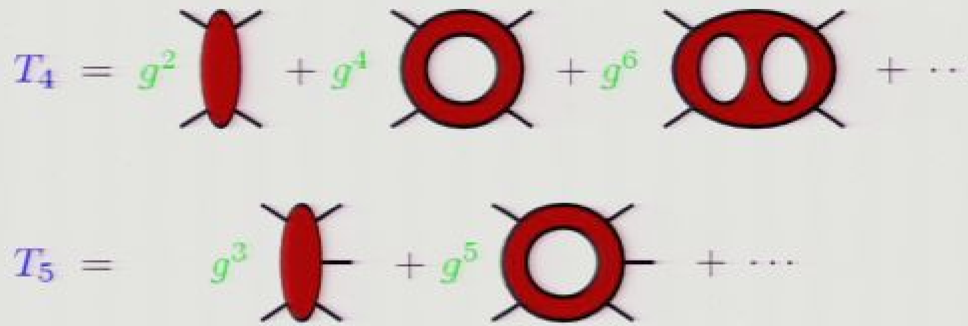


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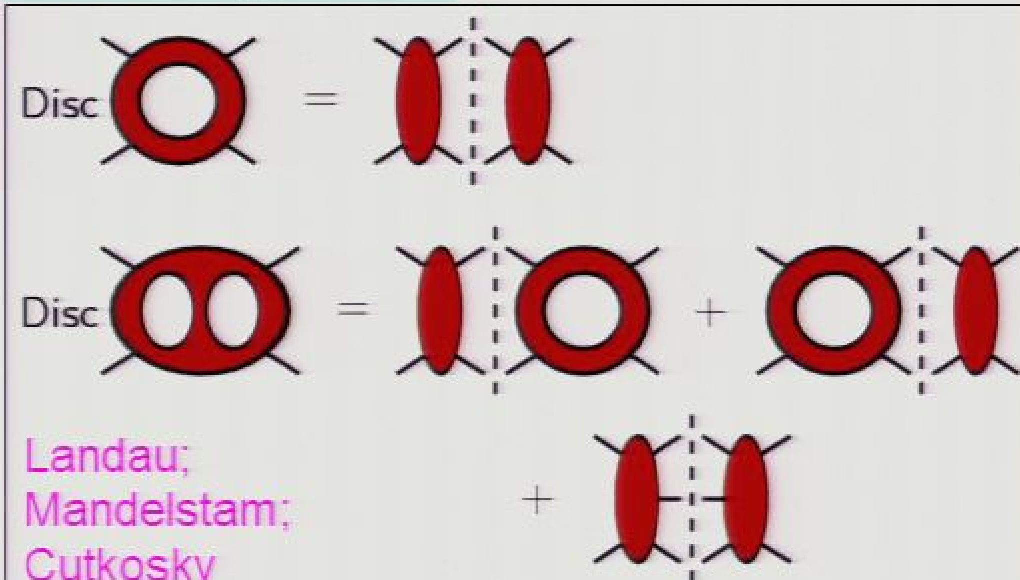


Landau;
Mandelstam;
Cutkosky

Find representations of amplitudes in terms of different loop integrals, matching all the cuts

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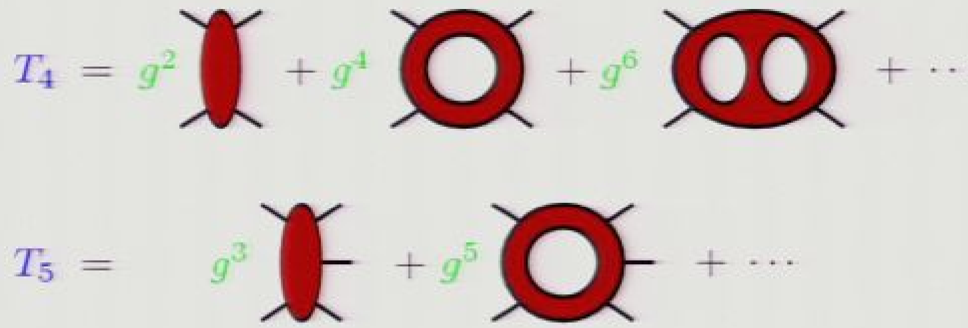


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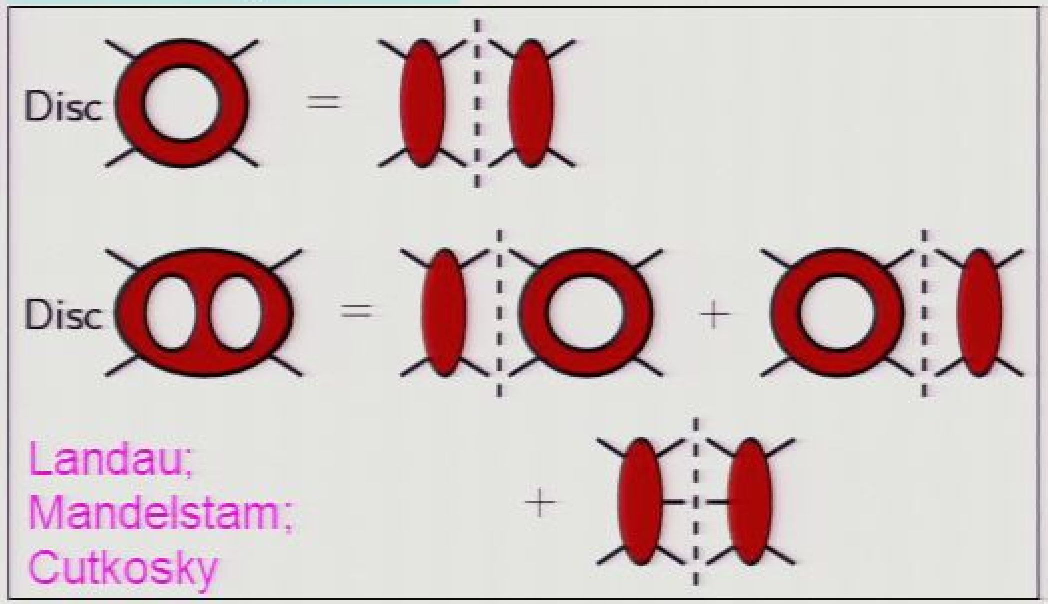



Find representations of amplitudes in terms of different loop integrals, matching all the cuts

Very efficient – especially for N=4 SYM – due to simple structure of tree helicity amplitudes, plus manifest N=4 SUSY

Bern, LD, Dunbar, Kosower (1994)

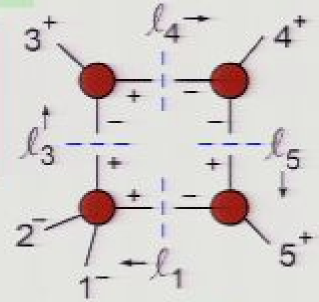
# Generalized unitarity

If one cut is good, surely more must be better


Multiple cut conditions connected with leading singularities

Eden, Landshoff, Olive, Polkinghorne (1966)

At one loop, efficiently extract coefficients of triangle integrals & especially box integrals from products of trees



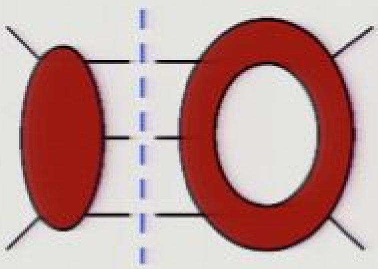

Bern, LD, Kosower (1997); Britto, Cachazo, Feng (2004);…

# Generalized unitarity at multi-loop level

Bern, LD, Kosower (2000); BCDKS (2006); BCJK (2007)

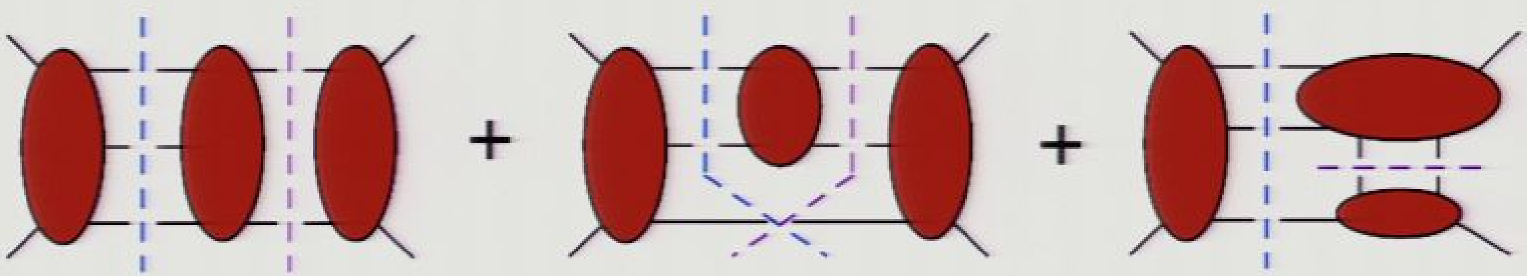
In matching loop-integral representations of amplitudes with the cuts, it is convenient to work with tree amplitudes only.

For example, at 3 loops, one encounters the product of a 5-point tree and a 5-point one-loop amplitude:

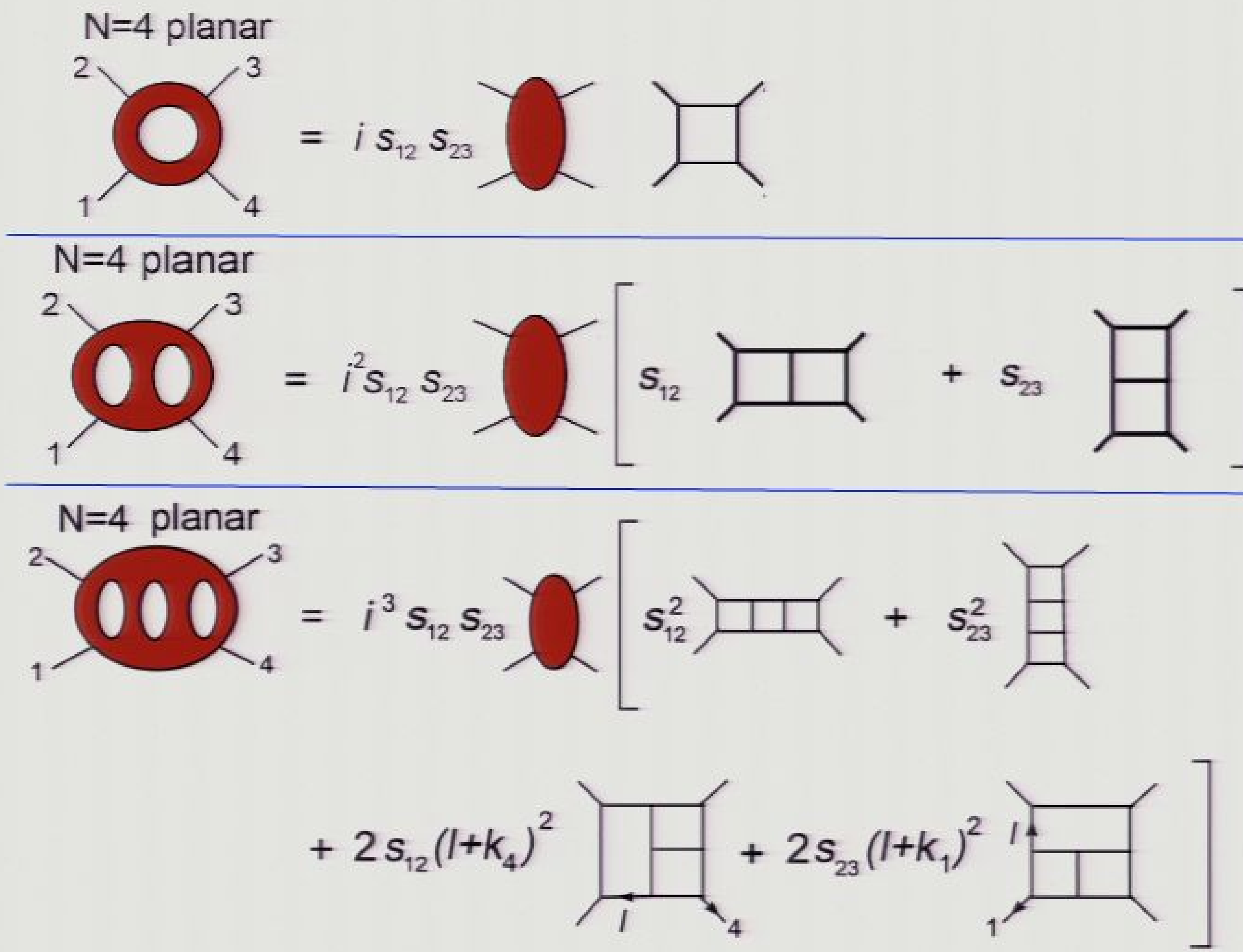


$\Leftrightarrow$

Cut 5-point loop amplitude further, into (4-point tree) x (5-point tree), in all inequivalent ways:



# Planar N=4 amplitudes from 1 to 3 loops

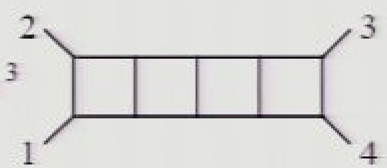



Green, Schwarz, Brink (1982)

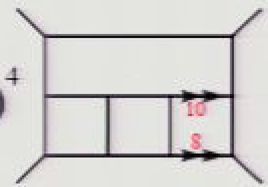
Bern, Rozowsky, Yan (1997)

# Integrals for planar amplitude at 4 loops

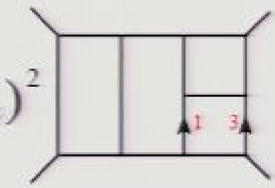
Bern, Czakon, LD, Kosower, Smirnov, hep-th/0610248

$s^3$


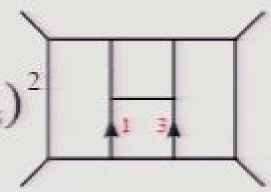
(a)

$t\,(l_8 + l_{10})^4$


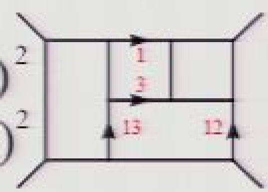
(b)

$s^2(l_1 + l_3)^2$


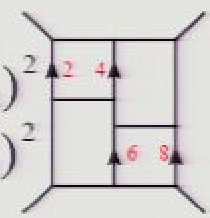
(c)

$s^2(l_1 + l_3)^2$


(d)

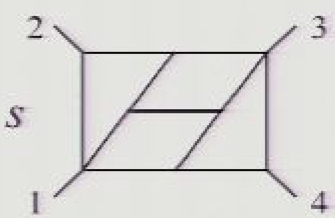
$s\,(l_1 + l_3)^2$
$\times\,(l_{12} + l_{13})^2$


(e)

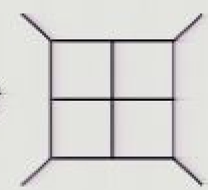
$s\,(l_2 + l_4)^2$
$\times\,(l_6 + l_8)^2$


(f)

diagrams with
no 2-particle cuts

$s$


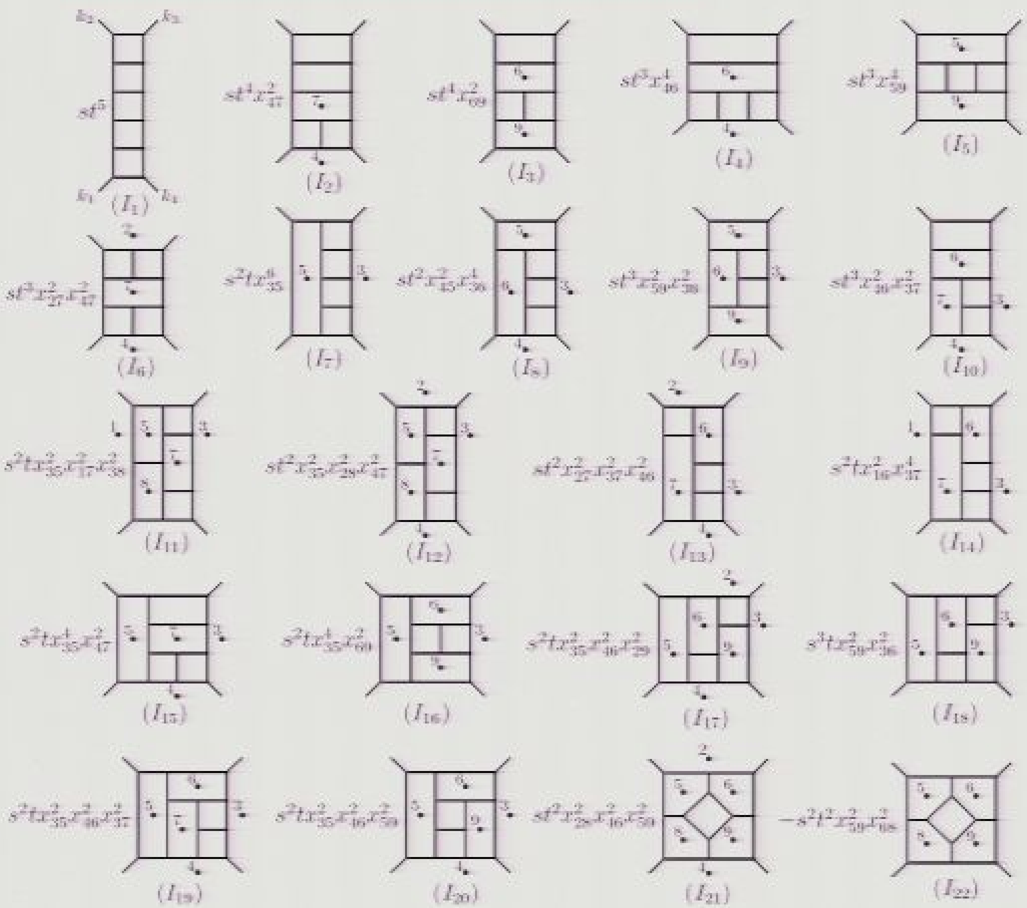
($d_2$)

$s\,t$

($f_2$)

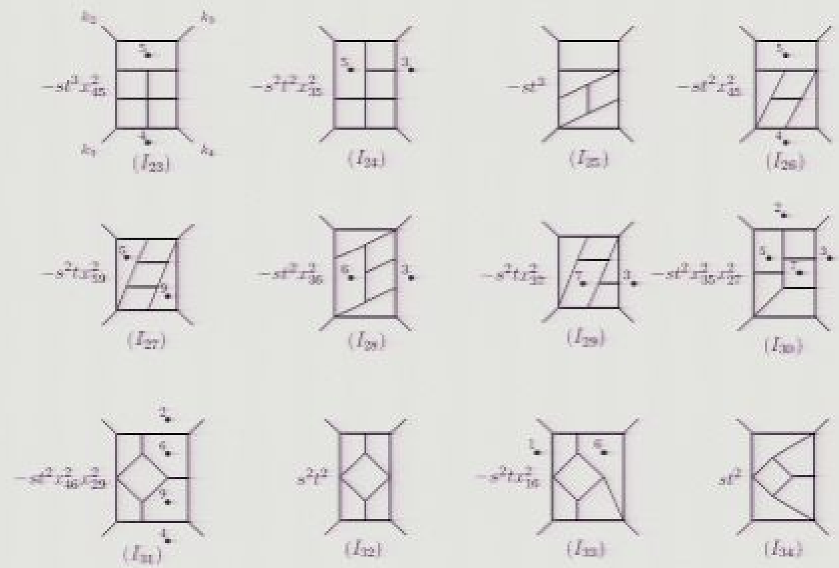
# Integrals for planar amplitude at 5 loops

Bern, Carrasco, Johansson, Kosower, 0705.1864[th]

only cubic vertices (22)

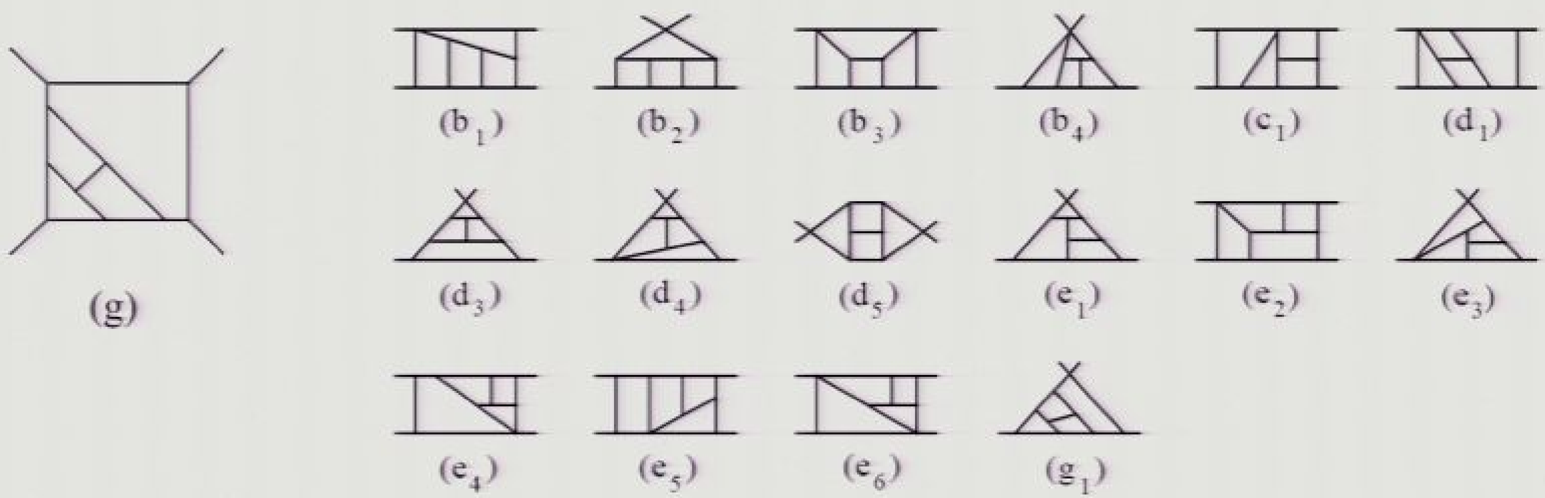


also quartic vertices (12)



# Patterns in the planar case

• At four loops, if we assume there are no triangle sub-diagrams, then besides the 8 contributing rung-rule & non-rung-rule diagrams, there are over a dozen additional possible integral topologies:




• Why do none of these topologies appear?
• What distinguishes them from the ones that do appear?

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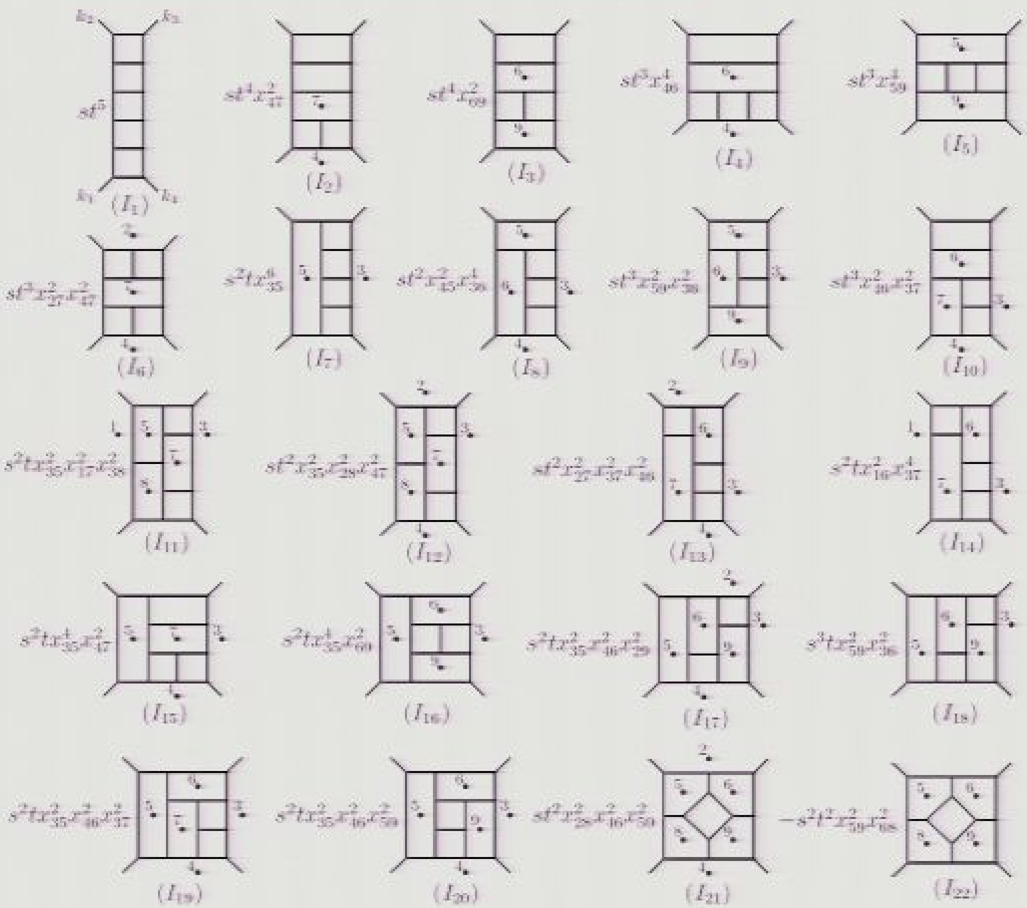

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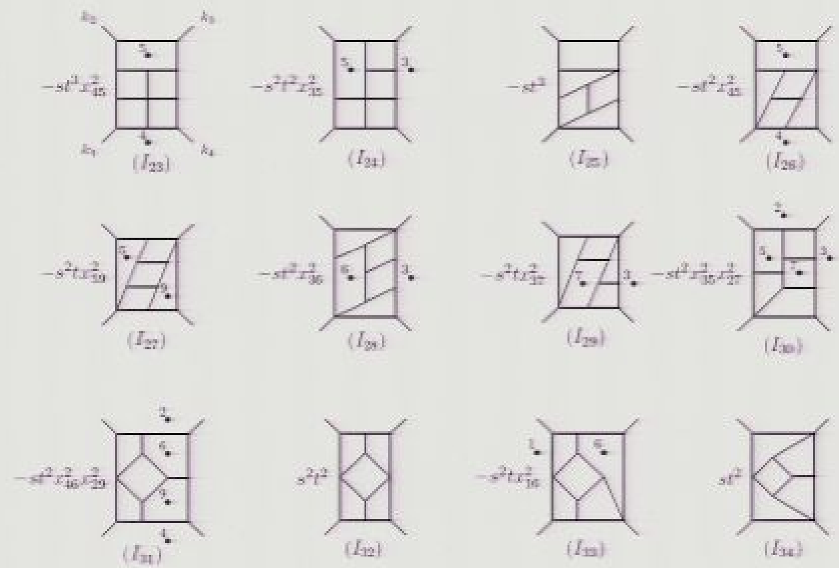

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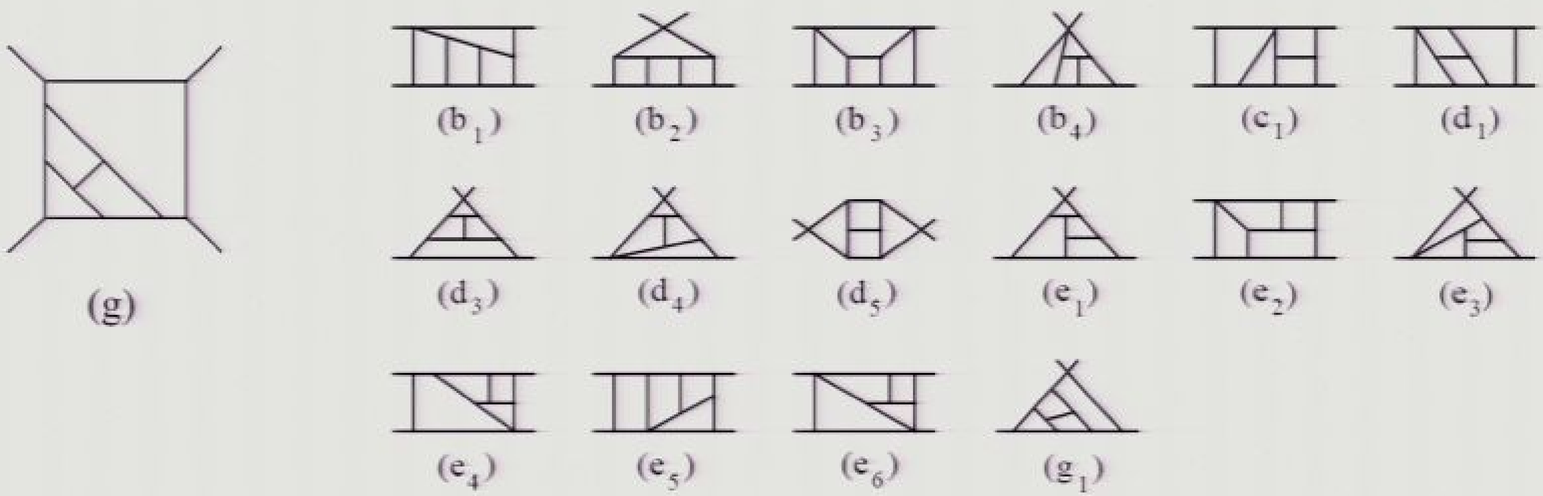


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- What distinguishes them from the ones that do appear?

# Surviving diagrams all have "dual conformal invariance"

- Although amplitude is evaluated in $D=4-2\varepsilon$, all non-contributing no-triangle diagrams can be eliminated by requiring $D=4$ "dual conformal invariance" and finiteness.
- Take $k_i^2 \neq 0$ to regulate integrals in $D=4$.
- Require inversion symmetry on dual variables $x_i^\mu$:

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}$$

Lipatov (2d) (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

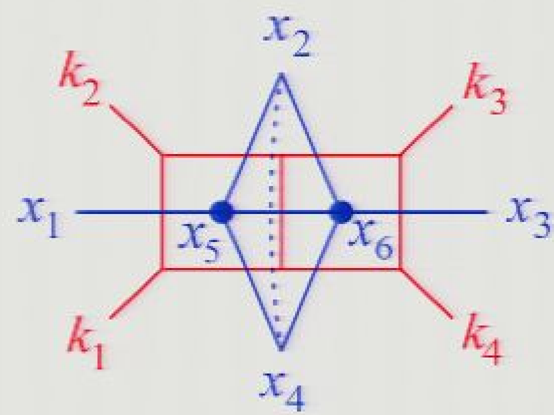
- No explicit $x_{i-1,i}^2 = k_i^2$ allowed (so $k_i^2 \rightarrow 0$ OK)

$$x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad d^4 x_i \rightarrow \frac{d^4 x_i}{x_i^8}$$

Requires 4 (net) lines out of every internal dual vertex, 1 (net) line out of every external one.
Dotted lines = numerator factors

Two-loop example

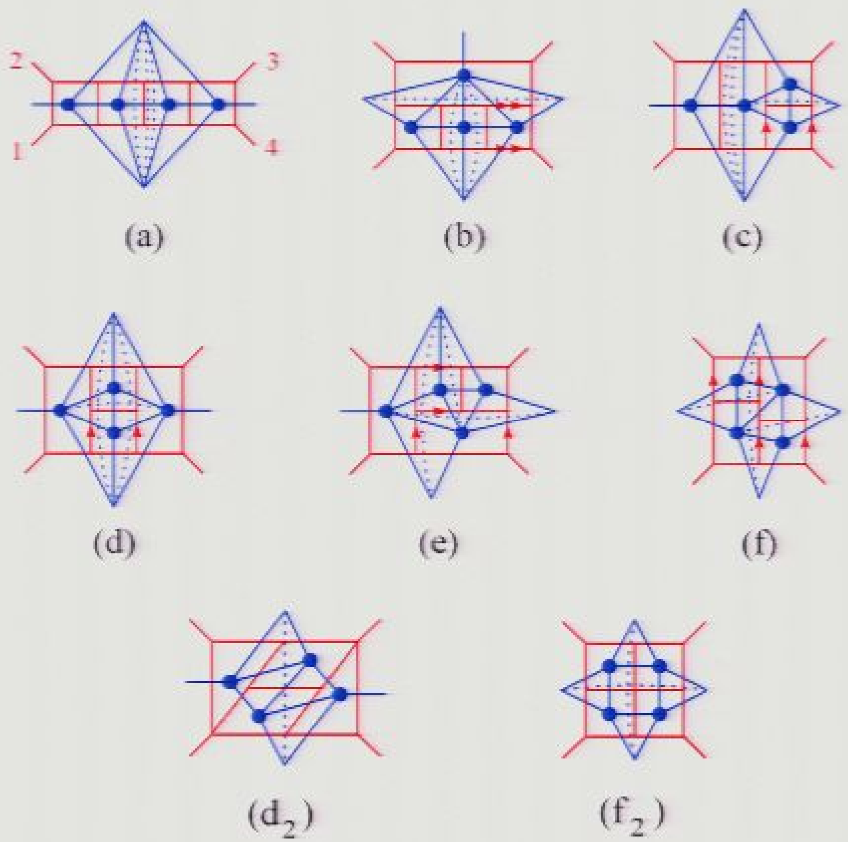
$$k_1 = x_{41}, \quad k_2 = x_{12}, \quad k_3 = x_{23}, \quad k_4 = x_{34}, \quad p = x_{45}, \quad q = x_{65}$$




numerator: $x_{42}^2 = (k_1 + k_2)^2 = s$

# Dual diagrams at four loops

Present in the amplitude




(a) (b) (c)

(d) (e) (f)

($d_2$) ($f_2$)

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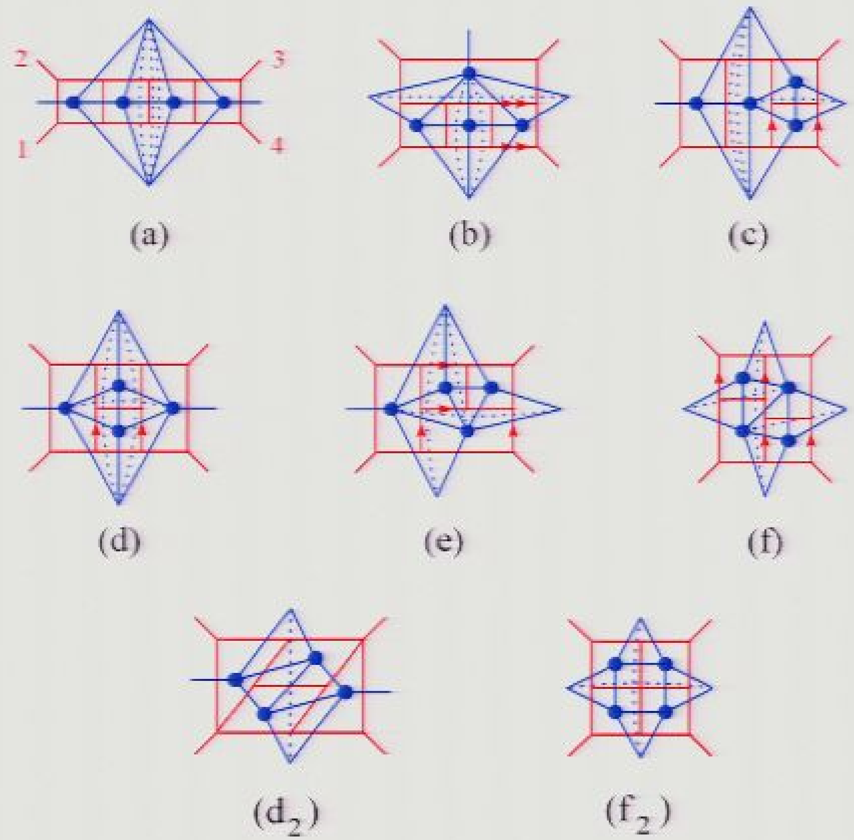
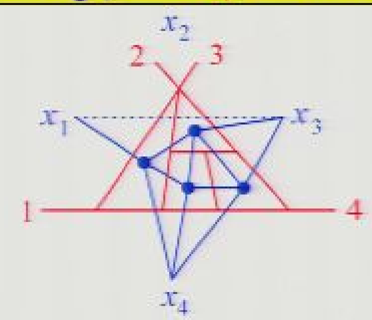



**Not present:**
Requires $x_{34}^2 = k_4^2 = 0$ *on shell*



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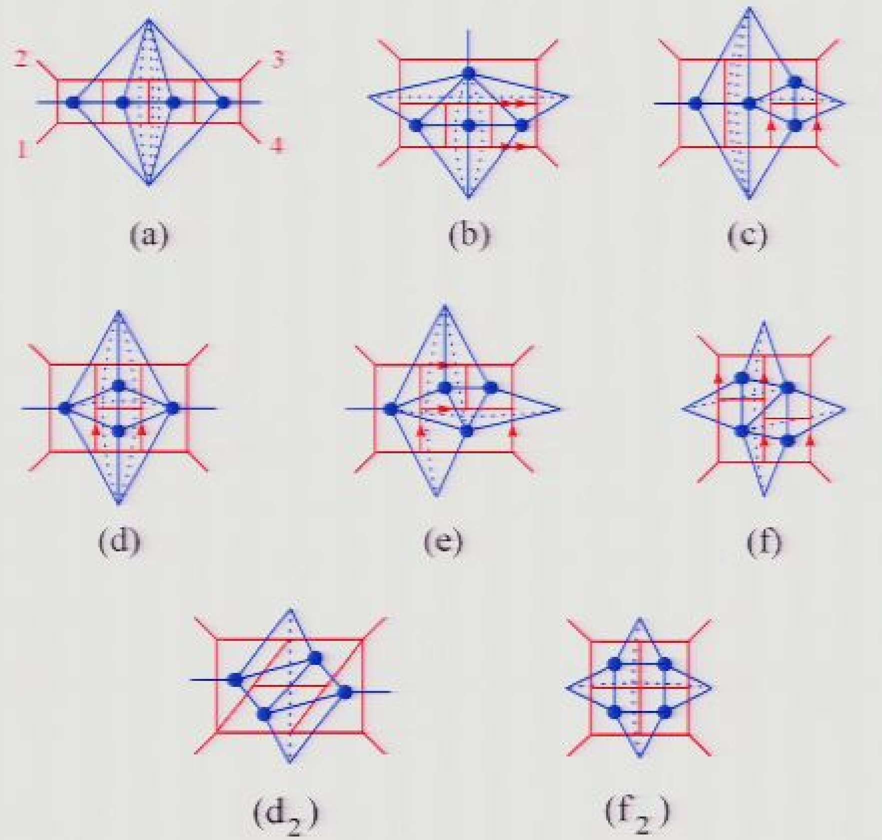


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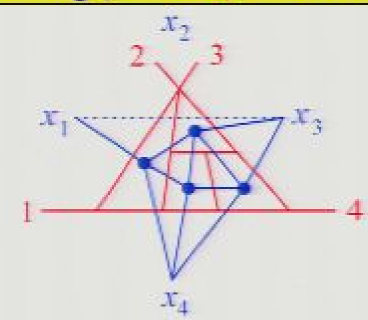
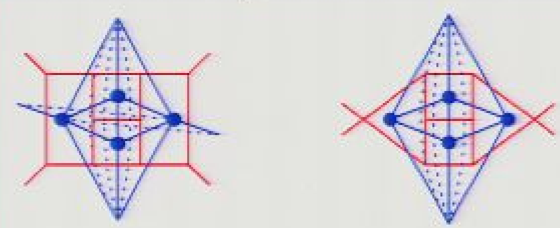



- 2 diagrams possess dual conformal invariance and a smooth $k_i^2 \to 0$ limit, yet are **not present** in the amplitude.
- But they are not finite in *D=4*



Drummond, Korchemsky, Sokatchev, 0707.0243[th]

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- All such integrals do so with weight $\pm 1$.

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- All such integrals do so with weight $\pm 1$.

It's a pity, but there does not (yet) seem to be a good notion of dual conformal invariance for nonplanar integrals…

# Back to exponentiation: the 3 loop case

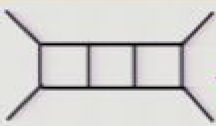
• *L*-loop formula:

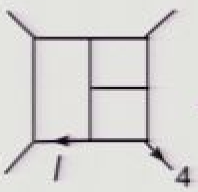
$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l\Big(f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

implies at 3 loops:

$$M_n^{(3)}(\epsilon) = -\frac{1}{3}\Big[M_n^{(1)}(\epsilon)\Big]^3 + M_n^{(1)}(\epsilon)M_n^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_n^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

• To check exponentiation at $\mathcal{O}(\epsilon^0)$ for $n$=4, need to evaluate just 4 integrals:

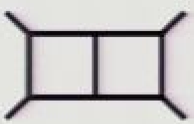




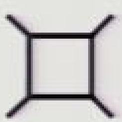

$\frac{1}{\epsilon^6}, \frac{1}{\epsilon^5}, \frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0$

Smirnov, hep-ph/0305142

Use Mellin-Barnes integration method



$\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2$



$\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4$ elementary

# Exponentiation at 3 loops (cont.)

• Inserting the values of the integrals (including those with $s \leftrightarrow t$) into

$$M_4^{(3)}(\epsilon) = -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_4^{(1)}(3\epsilon) + C^{(3)} + E_4^{(3)}(\epsilon)$$

using weight 6 harmonic polylogarithm identities, etc., relation was **verified**, and **3 of 4 constants** extracted:

BDS, hep-th/0505205

Agrees with Moch, Vermaseren, Vogt, hep-ph/0508055

$$f_0^{(3)} = \frac{11}{5}(\zeta_2)^2 \qquad f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3 \qquad f_2^{(3)} = c_1\zeta_6 + c_2\zeta_3^2$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_2$$

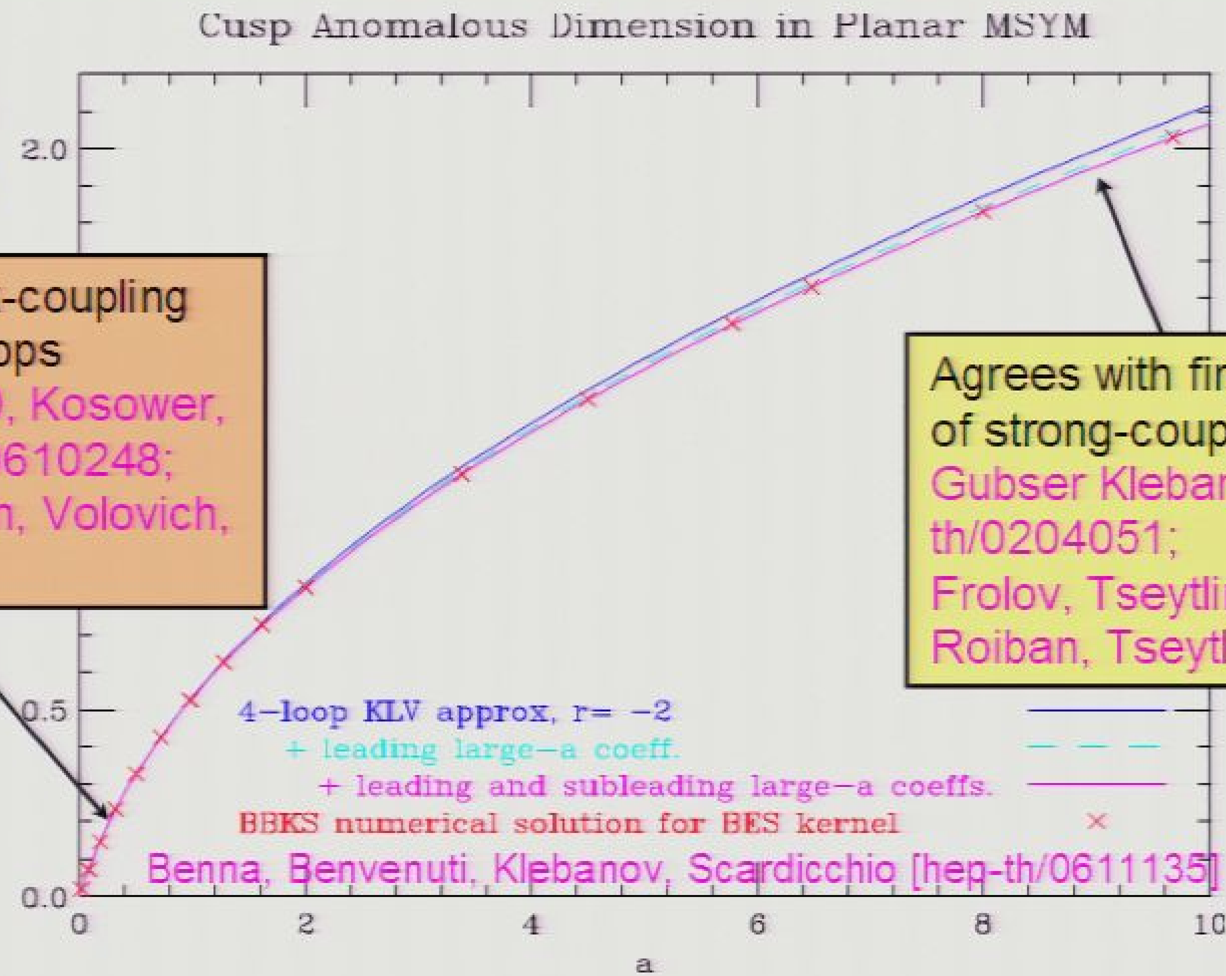
n-point information still required to separate

Confirmed result for 3-loop cusp anomalous dimension from maximum transcendentality
Kotikov, Lipatov, Onishchenko, Velizhanin, hep-th/0404092

# $\gamma_K(\lambda)$ to all orders

Beisert, Eden, Staudacher [hep-th/0610251] proposal based on **integrability**

$f_0(a)$
$\sim \gamma_K(\lambda)$




Benna, Benvenuti, Klebanov, Scardicchio [hep-th/0611135]

Agrees with weak-coupling data through 4 loops
Bern, Czakon, LD, Kosower, Smirnov, hep-th/0610248;
Cachazo, Spradlin, Volovich, hep-th/0612309

Agrees with first 3 terms of strong-coupling expansion
Gubser Klebanov, Polyakov, th/0204051;
Frolov, Tseytlin, th/0204226;
Roiban, Tseytlin, 0709.0681[th]

Full strong-coupling expansion
Basso,Korchemsky, Kotański, 0708.3933[th]

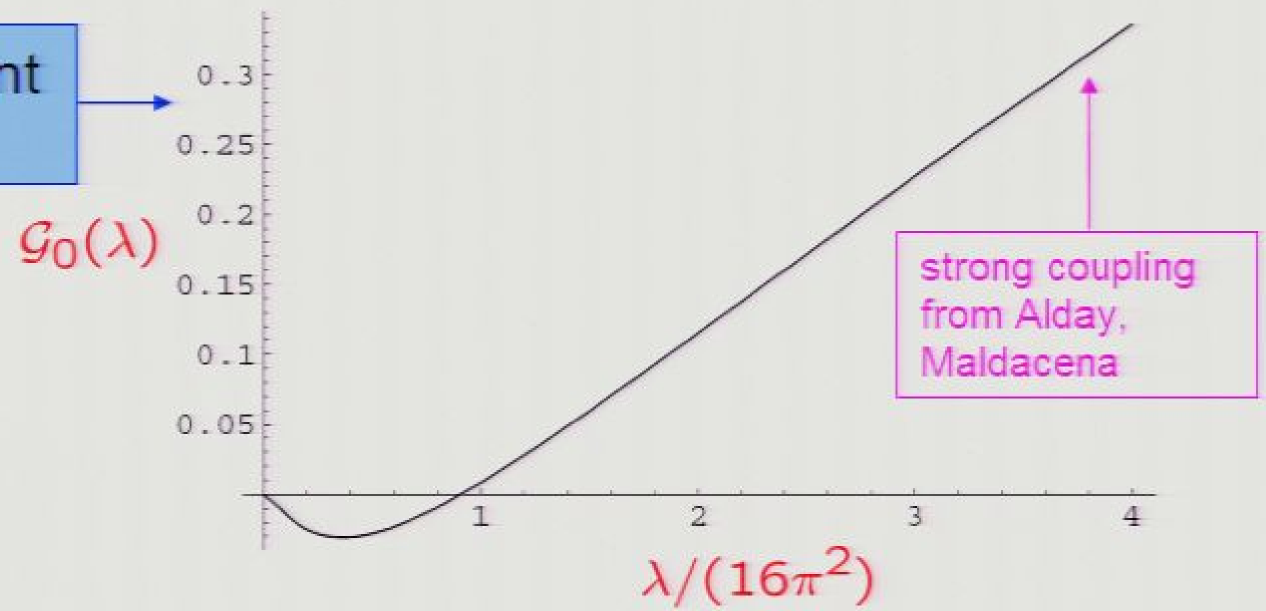
# Pinning down $\mathcal{G}_0(\lambda)$

Cachazo, Spradlin, Volovich, 0707.1903 [hep-th]

- CSV computed four-loop coefficient numerically by expanding same integrals needed for $\gamma_K^{(4)}(\lambda)$ to one higher power in $\varepsilon$

$$\mathcal{G}_0(\lambda) = -\zeta_3 \left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{2}{3}(6\zeta_5 + 5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3 - (77.69 \pm 0.06)\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots$$

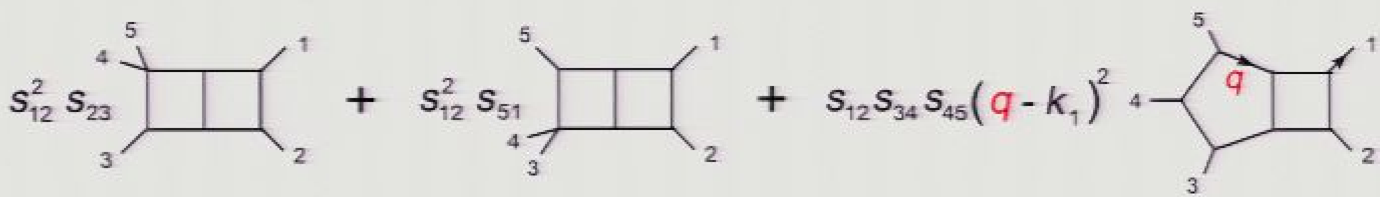
[3/2] Padé approximant incorporating all data




So far, no proposal for an exact solution for this quantity

# Two-loop directly for *n=5*

Using unitarity, first in *D=4*, later in *D=4-2ε*, the two-loop *n=5* amplitude was found to be:




+ cyclic

Bern, Rozowsky, Yan, hep-ph/9706392

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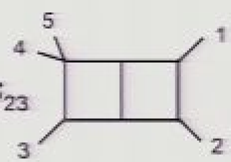
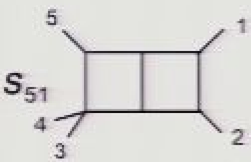
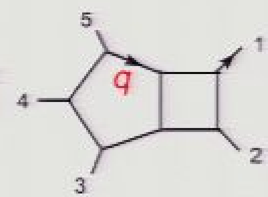
$s_{12}^2 s_{23}$ 

$+\ s_{12}^2 s_{51}$ 

$+\ s_{12} s_{34} s_{45} (q - k_1)^2$ 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Bern, Rozowsky, Yan, hep-ph/9706392

Cachazo, Spradlin, Volovich, hep-th/0602228

Even terms checked numerically with aid of Czakon, hep-ph/0511200

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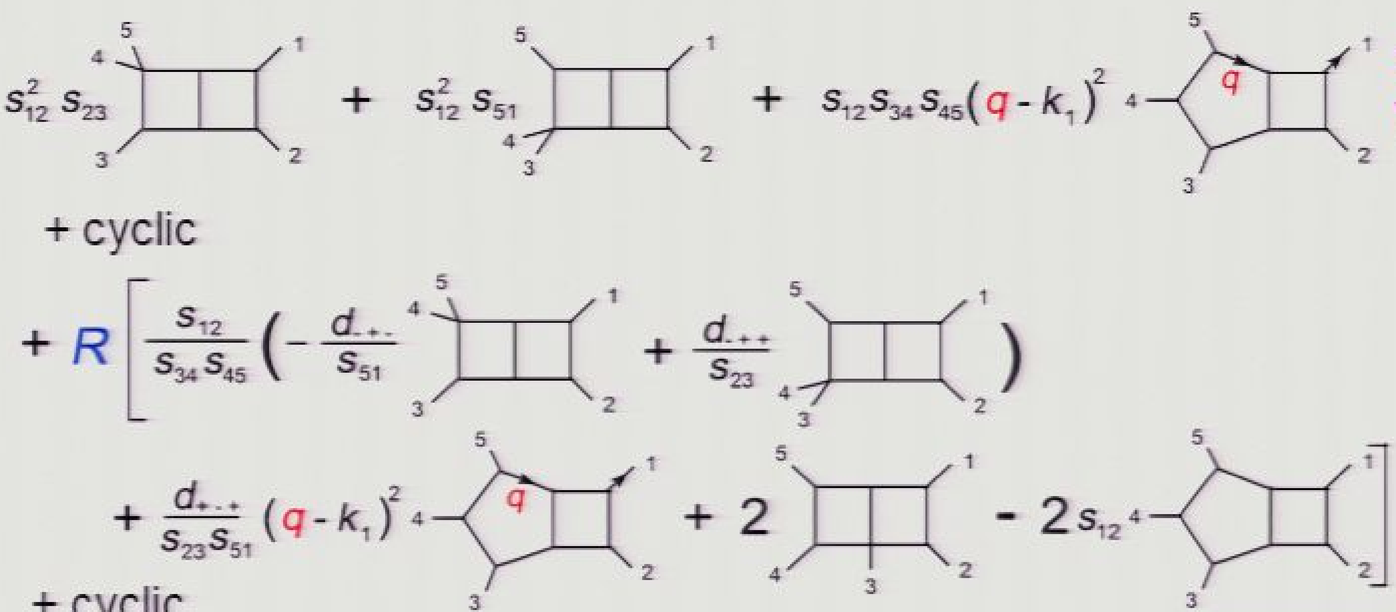



Bern, Rozowsky, Yan, hep-ph/9706392

Cachazo, Spradlin, Volovich, hep-th/0602228

$$R = \varepsilon(k_1,k_2,k_3,k_4) \times s_{12}s_{23}s_{34}s_{45}s_{51}/\det(s_{ij})|_{i,j=1,2,3,4}$$

Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

Even and odd terms checked numerically with aid of Czakon, hep-ph/0511200

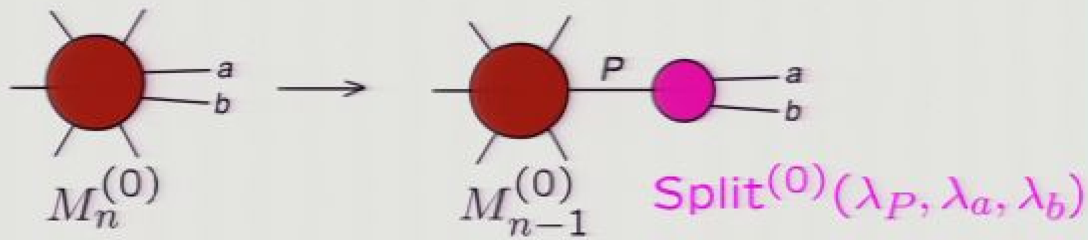
# Collinear limits

Bern, LD, Dunbar, Kosower (1994)

- Evidence for $n>4$: Use limits as 2 momenta become **collinear:**

$$k_a \to z k_P$$
$$k_b \to (1-z) k_P$$

- Tree amplitude behavior:



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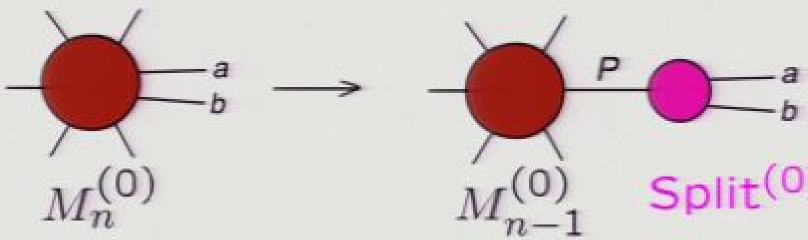
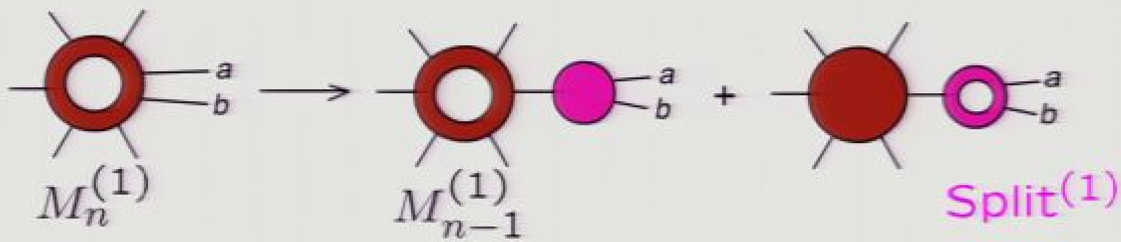



- One-loop behavior:



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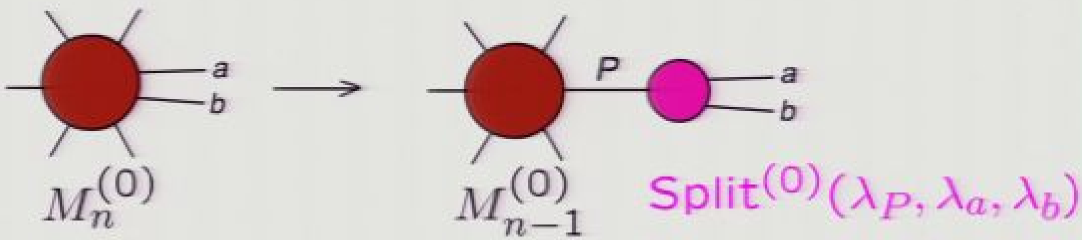

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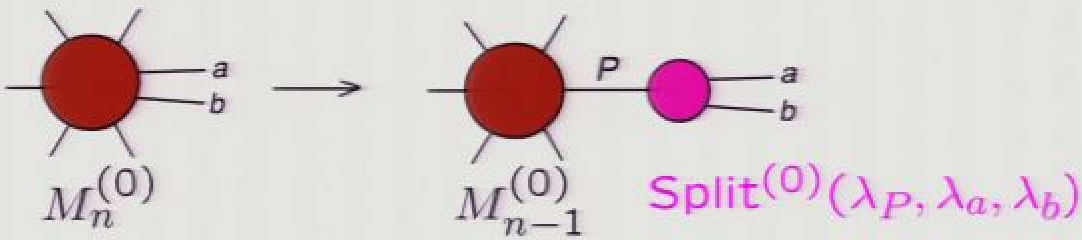


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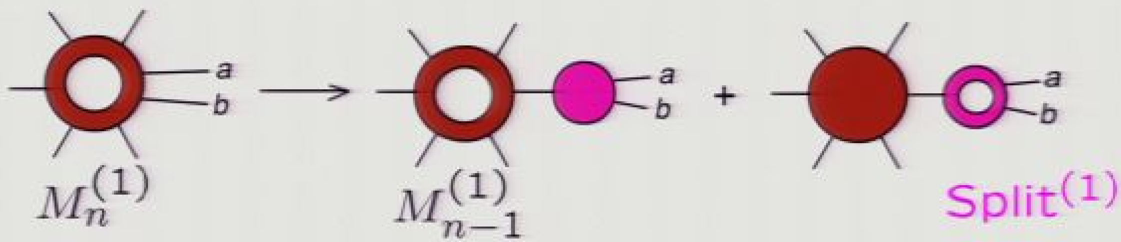

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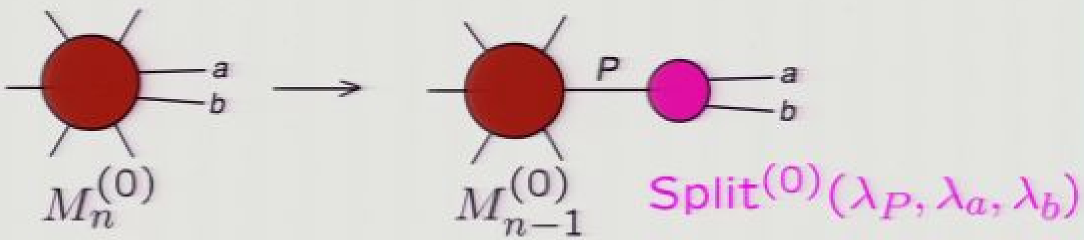


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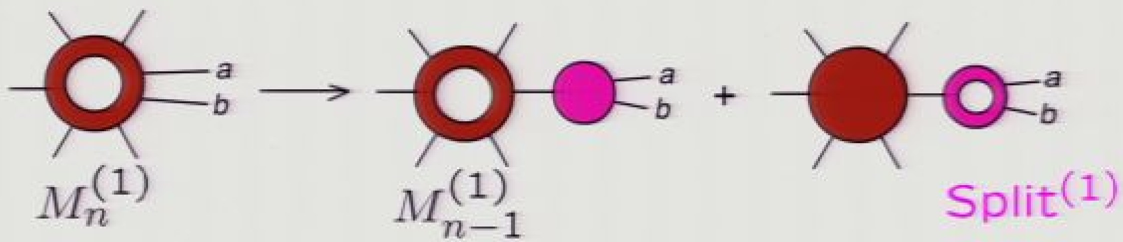


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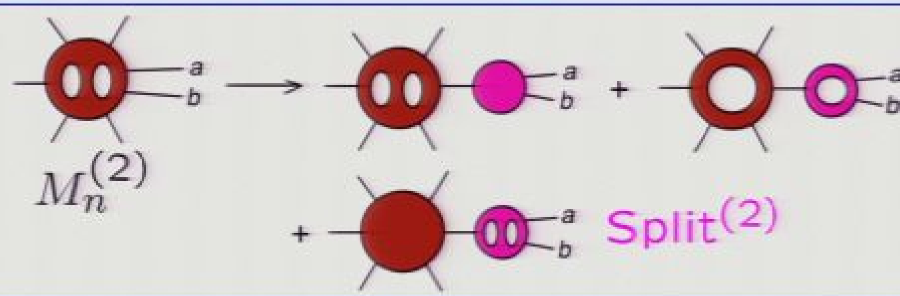

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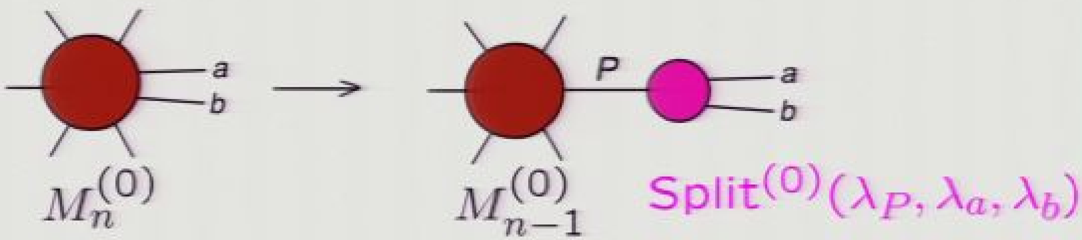


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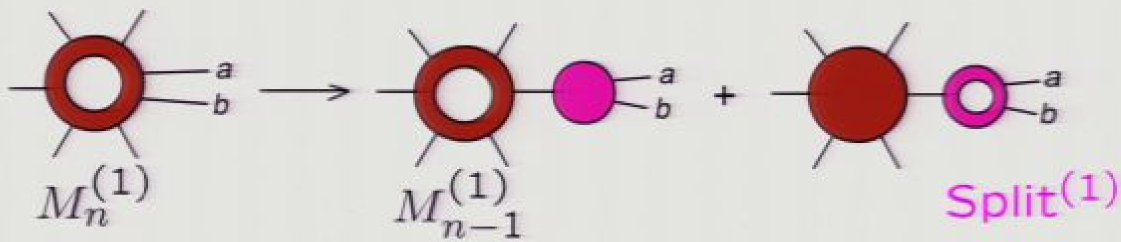


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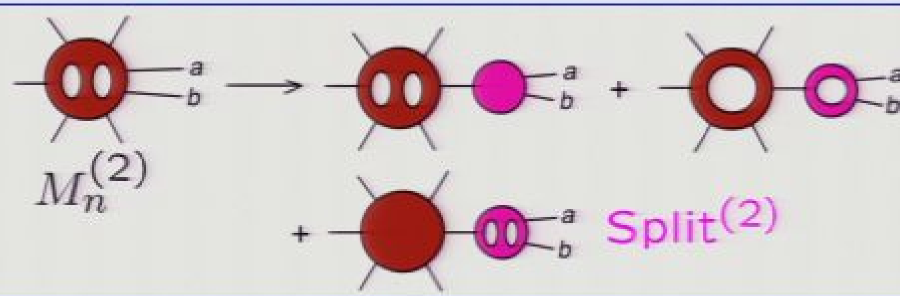


strong-coupling:
Komargodski,
0801.3274 [th]

# Collinear limits consistent at 2 loops

- In N=4 SYM, all MHV helicity configurations are equivalent, can write

$$\text{Split}^{(l)}(\lambda_P, \lambda_a, \lambda_b) = r_S^{(l)}(z, s_{ab}, \epsilon) \times \text{Split}^{(0)}(\lambda_P, \lambda_a, \lambda_b)$$

Anastasiou, Bern, LD, Kosower, hep-th/0309040

- The two-loop splitting amplitude obeys:

$$r_S^{(2)}(\epsilon) = \frac{1}{2}\left[r_S^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

which is consistent with the *n*-point amplitude ansatz

$$\mathcal{M}_n^{(2)}(\epsilon) = \frac{1}{2}\left[M_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + E_n^{(2)}(\epsilon)$$

and fixes $f_0^{(2)} = -\zeta_2$ $\quad f_1^{(2)} = -\zeta_3$ $\quad f_2^{(2)} = -\zeta_4$ $\quad C^{(2)} = -\frac{(\zeta_2)^2}{2}$

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n-point information required to separate these two

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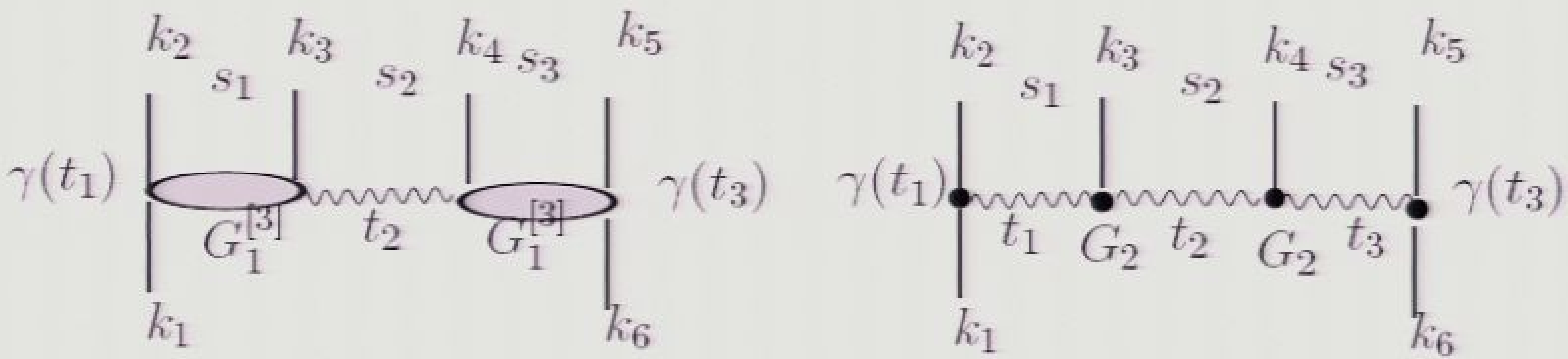
n-point information required to separate these two

Note: by definition $f_0^{(1)} = 1,\; f_1^{(1)} = f_2^{(1)} = C^{(1)} = E_n^{(1)}(\epsilon) = 0$

# Regge / high-energy behavior

Naculich, Schnitzer, 0708.3069 [hep-th]
Brower, Nastase, Schnitzer,Tan, 0801.3891 [hep-th];
Bartels, Lipatov, Sabio Vera, 0802.2065[th]



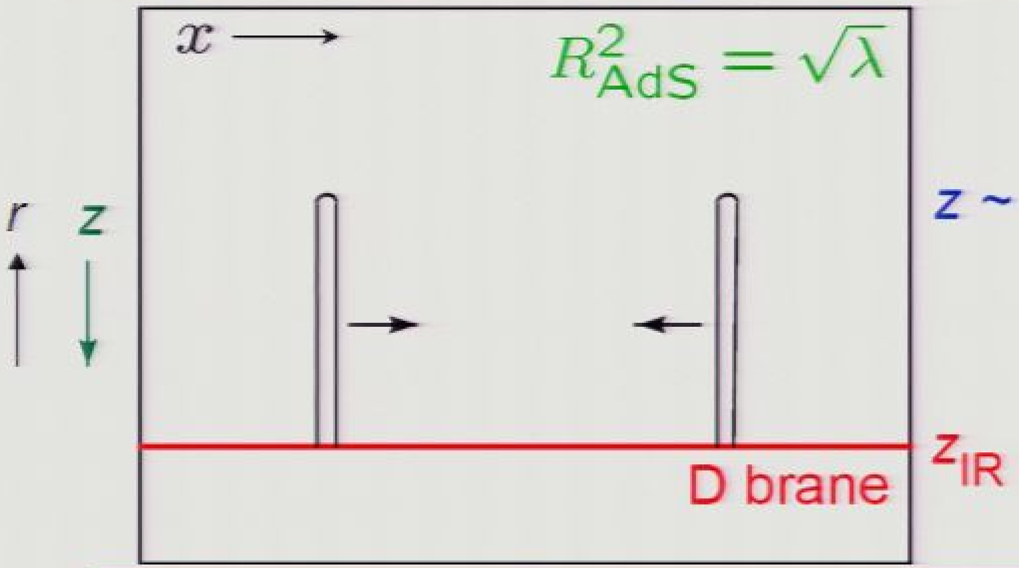

- Study limits with large rapidity separations between final-state gluons
- Everything consistent with Regge/BFKL factorization for *n=4,5*.
- BNST find consistency for *n>5,* but BLS (looking closer) do not, at *n=6*

# Scattering at strong coupling

Alday, Maldacena, 0705.0303 [hep-th]

- Use AdS/CFT to compute an appropriate scattering amplitude
- High energy scattering in string theory is semi-classical

Gross, Mende (1987,1988)




$z \sim s^{-1/2},\ t^{-1/2}$

$$z \equiv \frac{R^2_{\mathrm{AdS}}}{r}$$

Evaluated on the classical solution, action is imaginary → exponentially suppressed tunnelling configuration

$$A_4 \ \sim\ \exp[iS_{\mathrm{cl}}] \ \sim\ \exp[-(-iS_{\mathrm{cl}})] \ \sim\ \exp[-\sqrt{\lambda}\ln^2(z/z_{\mathrm{IR}})]$$

Better to use dimensional regularization instead of $z_{\mathrm{IR}}$

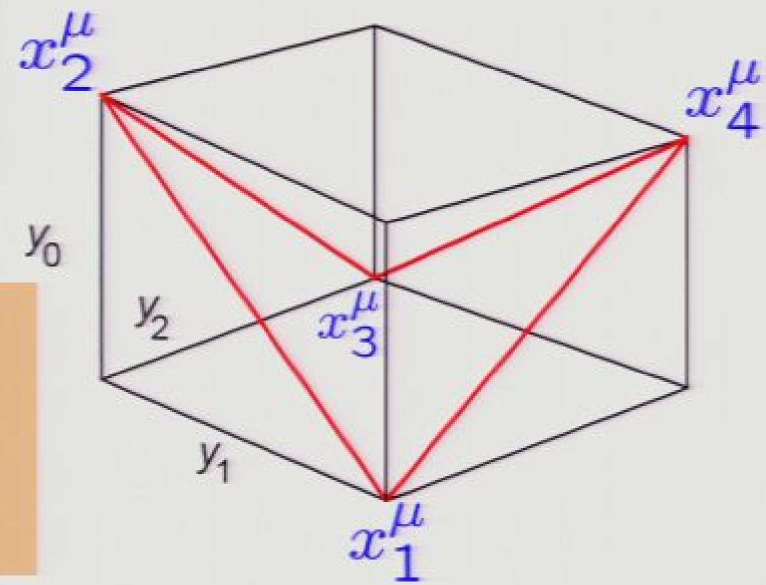
# Dual variables and strong coupling

• T-dual momentum variables $y^\mu$ introduced by Alday, Maldacena

• Boundary values for world-sheet are light-like segments in $y^\mu$:

$\Delta y^\mu = 2\pi k^\mu$ for gluon with momentum $k^\mu$

• For example,
for *gg* → *gg* 90-degree scattering,
*s = t = -u/2*, the boundary looks like:



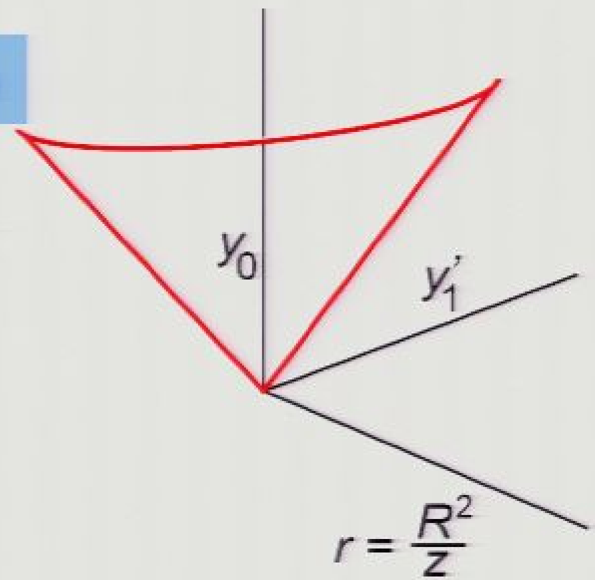

Corners (cusps) are located at $x_i^\mu$
– same dual momentum variables introduced above for discussing dual conformal invariance of integrals!!

# Cusps in the solution

- Near each corner, solution has a cusp

Kruczenski, hep-th/0210115

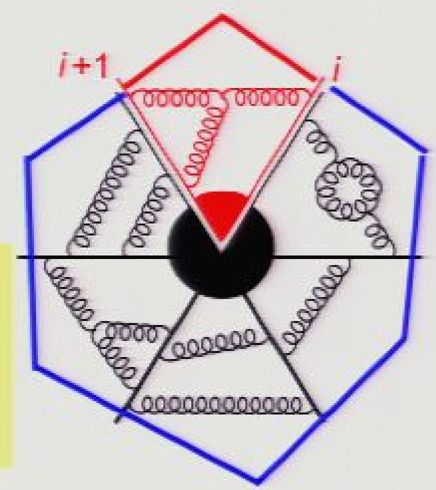
$$r = \sqrt{(2+\epsilon)(y_0^2 - y_1'^2)} \equiv \sqrt{(2+\epsilon)y^+y^-}$$




- Classical action divergence is regulated by $\epsilon$

$$iS = -S_E = -\frac{R^2}{4\pi}\int d\sigma d\tau$$

$$\to \; -R^2 \int_0 \frac{dy^+dy^-}{(y^+y^-)^{1+\epsilon/2}} \sim -\frac{\sqrt{\lambda}}{\epsilon^2} \sim -\frac{\gamma_K(\lambda)}{\epsilon^2}$$

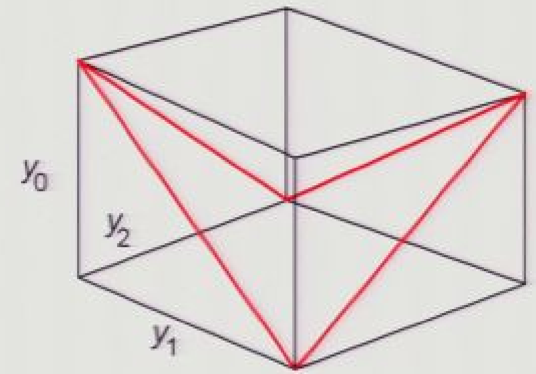



- Cusp in $(y,r)$ is the strong-coupling limit of the red wedge; i.e. the Sudakov form factor.
- See also Buchbinder, 0706.2015 [hep-th]

# The full solution

- Divergences only come from corners; can set *D=4* in interior.
- Evaluating the action as $\varepsilon \to 0$ gives:

Alday, Maldacena, 0705.0303 [hep-th]




$$A_4 = \exp(-S_E)$$

$$-S_E = \left(-\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}(1-\ln 2)\right)\left[\left(\frac{\mu^2}{-s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right] + \frac{\sqrt{\lambda}}{4\pi}\left[\ln^2\frac{s}{t} + \tilde{C}\right]$$

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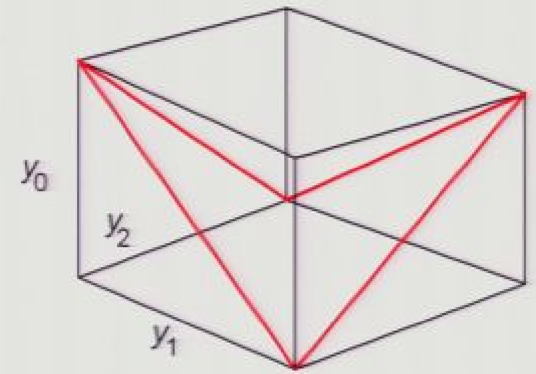
Alday, Maldacena, 0705.0303 [hep-th]

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$\gamma_K(\lambda)$ $\mathcal{G}_0(\lambda)$

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$\gamma_K(\lambda) \times M_4^{(1)}(s,t)$

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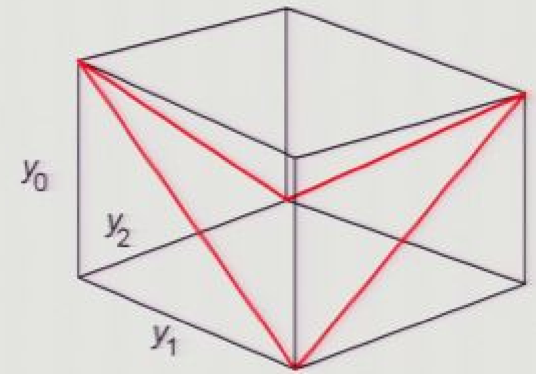


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$$+\frac{\sqrt{\lambda}}{4\pi}\left[\ln^2\frac{s}{t} + \tilde{C}\right]$$

combination of $f_2(\lambda) \oplus C(\lambda)$

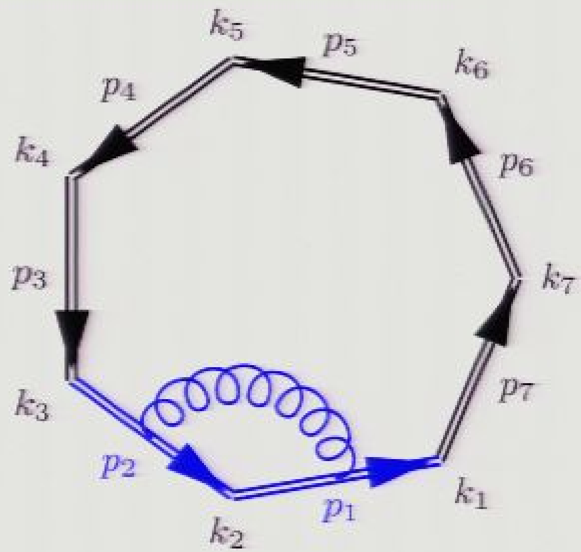
$\gamma_K(\lambda) \times M_4^{(1)}(s,t)$

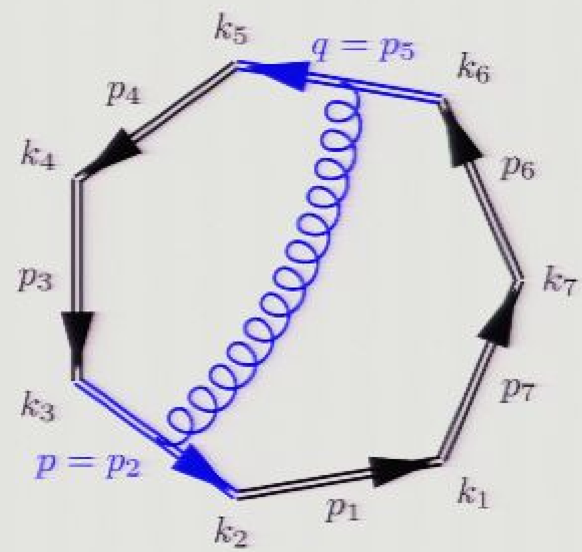
# Dual variables and Wilson lines at weak coupling

• Inspired by Alday, Maldacena, there has been a sequence of recent computations of Wilson-line configurations with same "dual momentum" boundary conditions:

• One loop, *n=4* — Drummond, Korchemsky, Sokatchev, 0707.0243[th]

• One loop, any *n* — Brandhuber, Heslop, Travaglini, 0707.1153[th]

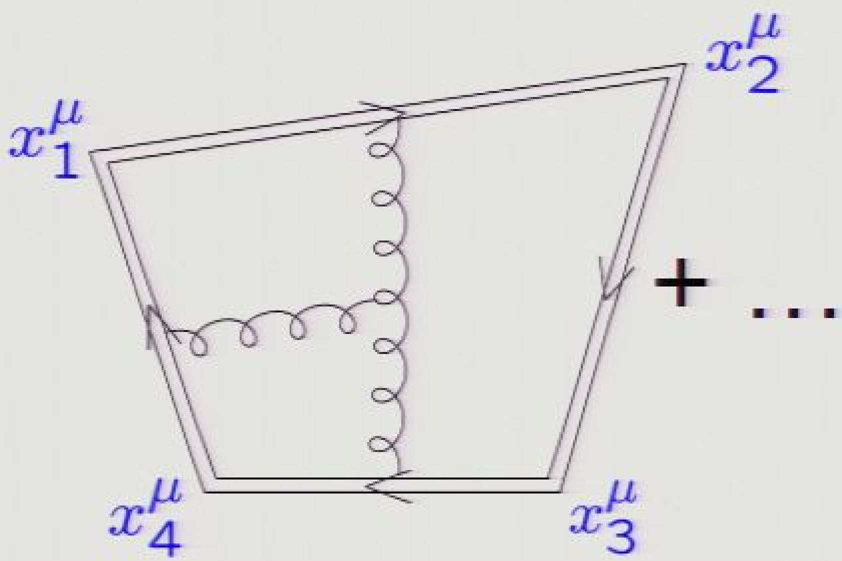


# Dual variables and Wilson lines at weak coupling (cont.)

• Two loops, *n=4,5*

Drummond, Henn, Korchemsky, Sokatchev, 0709.2368[th], 0712.1223[th]



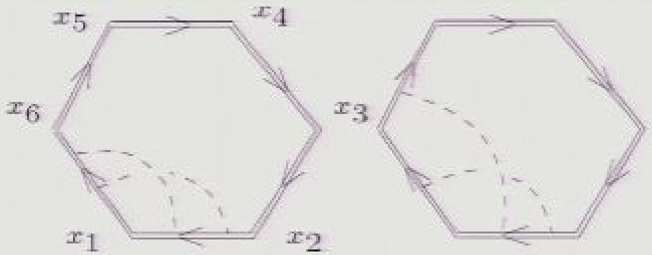

• In all such cases, Wilson-line results match the full scattering amplitude [the MHV case for *n>5*] (!) – up to an additive constants.

Wilson lines obey an "anomalous" (due to IR divergences) dual conformal Ward identity – totally fixes their structure for *n=4,5*.
DHKS, 0712.1223[th]

# Dual variables and Wilson lines at weak coupling (cont.)

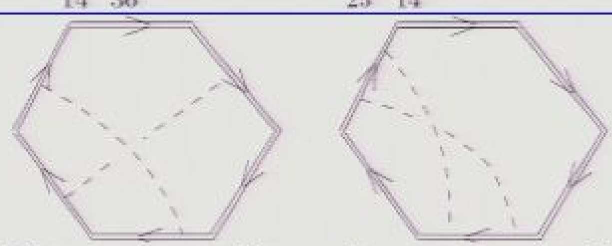
Assuming dual conformal invariance, first possible nontrivial "remainder" function from ABDK/BDS, for MHV amplitudes or for Wilson lines, is at $n=6$, where "cross-ratios" appear. [Not $n=4$ because $x^2_{i,i+1} = 0$.]

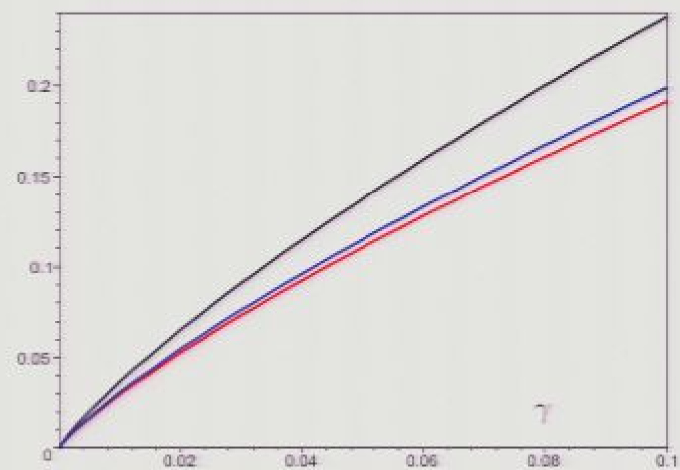
Drummond, Henn, Korchemsky, Sokatchev, 0712.4138 [th] computed the two-loop Wilson line for $n=6$, and found a **discrepancy**




+ ...

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



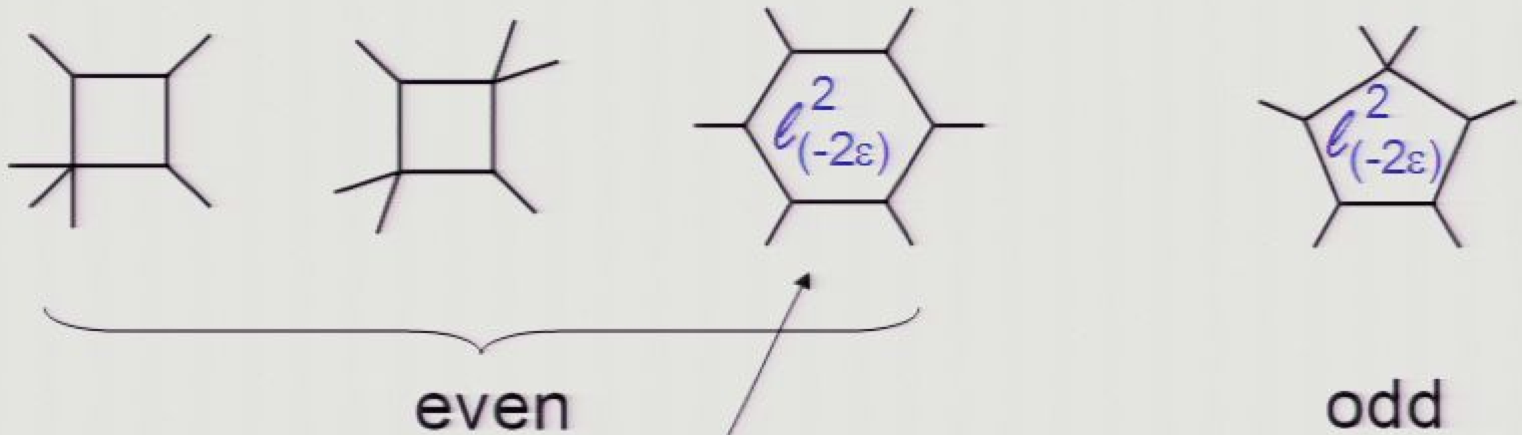



- What does this mean for amplitudes?

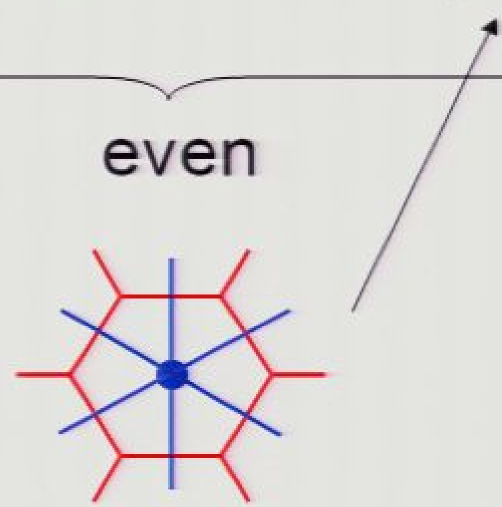
# Two-loop 6-point amplitude

Bern, LD, Kosower, R. Roiban, M. Spradlin, C. Vergu, A. Volovich, in progress

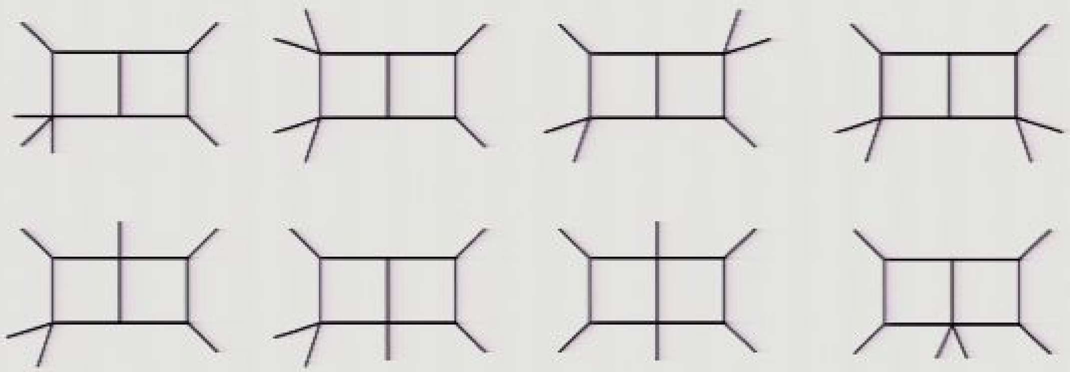
One loop n=6 integrals:



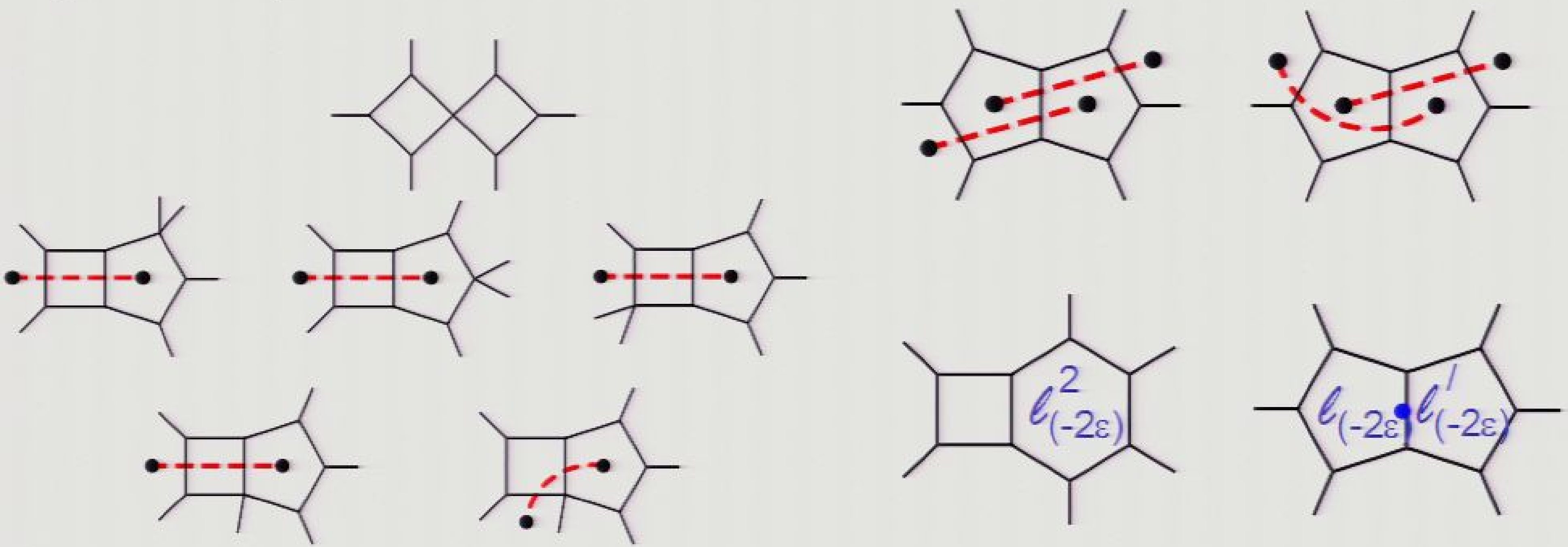

dual conformal invariant in *D=6*



# Two loop n=6 “even” integrals



all with dual conformal invariant integrands (including prefactors)



# Two loop n=6 status

- Expression on previous slide passes many consistency checks (though not all cuts have been evaluated in $D=4-2\varepsilon$)
- $1/\varepsilon^4$ , $1/\varepsilon^3$ , $1/\varepsilon^2$ , $1/\varepsilon$ poles all OK
- $O(\varepsilon^0)$ numerical evaluation confirms that ABDK/BDS ansatz for scattering amplitudes definitely needs correction.
- We also have decent direct numerical evidence that it is dual conformal invariant.

# Two loop n=6 status (cont.)

- We compared the "remainder function" for the amplitude with the corresponding one for the Wilson line

Drummond, Henn, Korchemsky, Sokatchev, 0712.4138 [th]

- They agree!

| kinematic point | $(u_1, u_2, u3)$ | $R_A - R_A^{(0)}$ | $R_W - R_W^{(0)}$ |
|---|---|---|---|
| $K^{(1)}$ | $(1/4, 1/4, 1/4)$ | $-0.0181 \pm 0.017$ | $< 10^{-5}$ |
| $K^{(2)}$ | $(0.547253,\ 0.203822,\ 0.88127)$ | $-2.753 \pm 0.012$ | $-2.7553$ |
| $K^{(3)}$ | $(28/17, 16/5, 112/85)$ | $-4.74445 \pm 0.00653$ | $-4.7446$ |
| $K^{(4)}$ | $(1/9, 1/9, 1/9)$ | $4.1161 \pm 0.10$ | $4.0914$ |
| $K^{(5)}$ | $(4/8, 4/81, 4/81)$ | $9.9963 \pm 0.50$ | $9.7255$ |

# Conclusions & Open Questions

- Remarkably, finite terms in planar gg → gg amplitudes in N=4 SYM exponentiate in a very similar way to the IR divergences. Full amplitude seems to depend on just 4 functions of $\lambda$ alone (one already "known" to all orders, so n=4 problem (also n=5) may be at least "1/4" solved!

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# Extra Slides

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