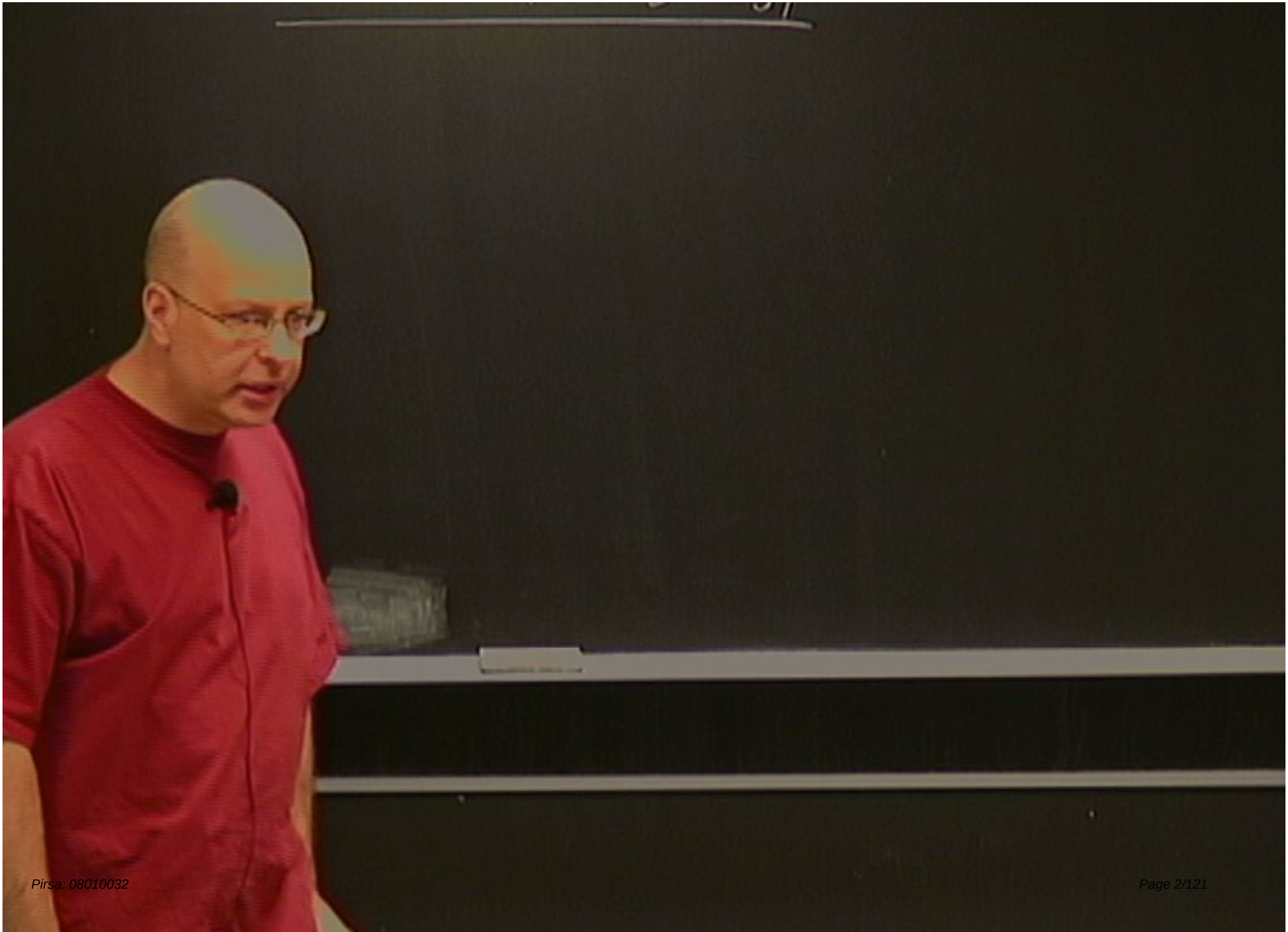
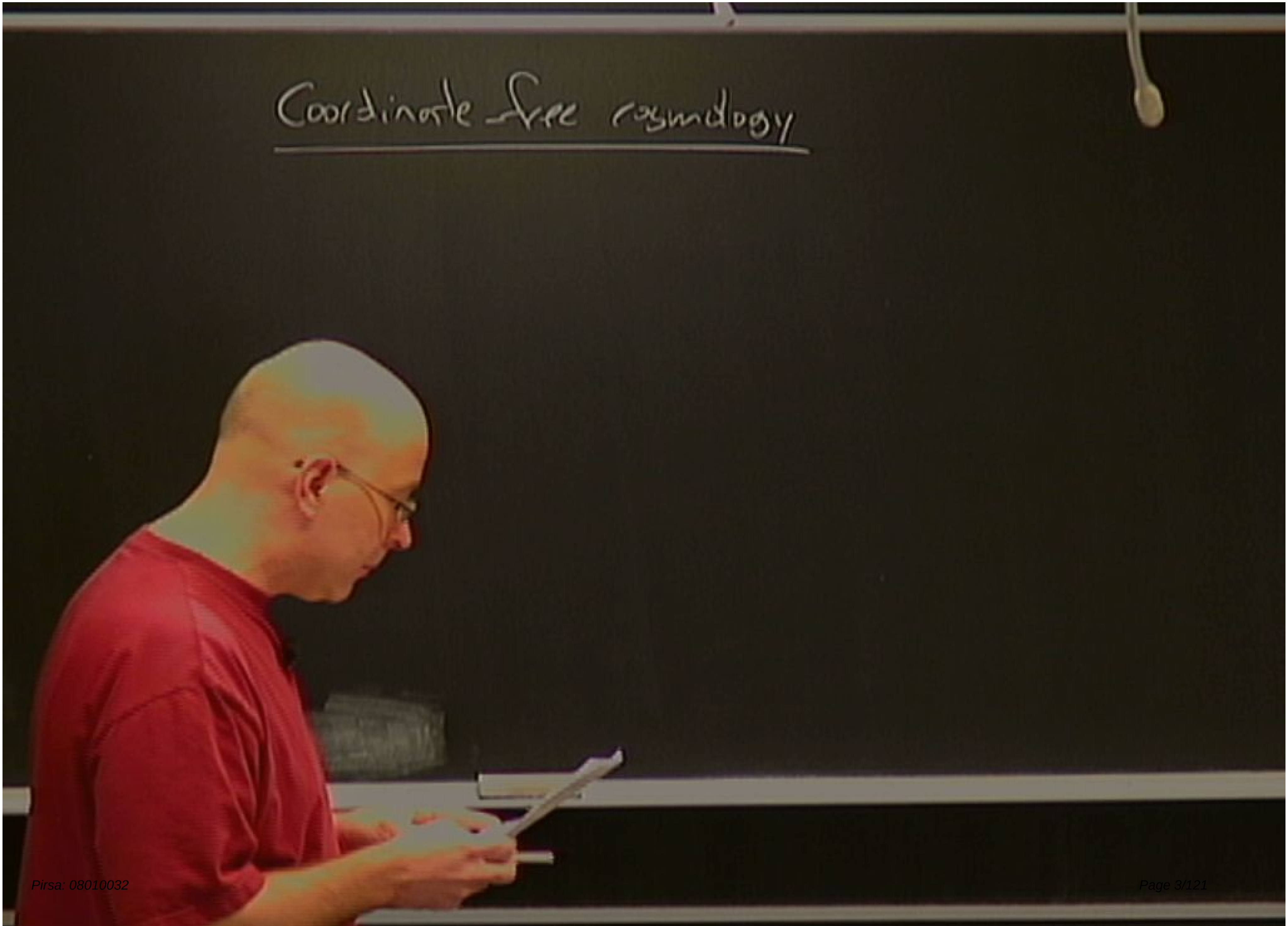
Title: Advanced General Relativity - Lecture 4B

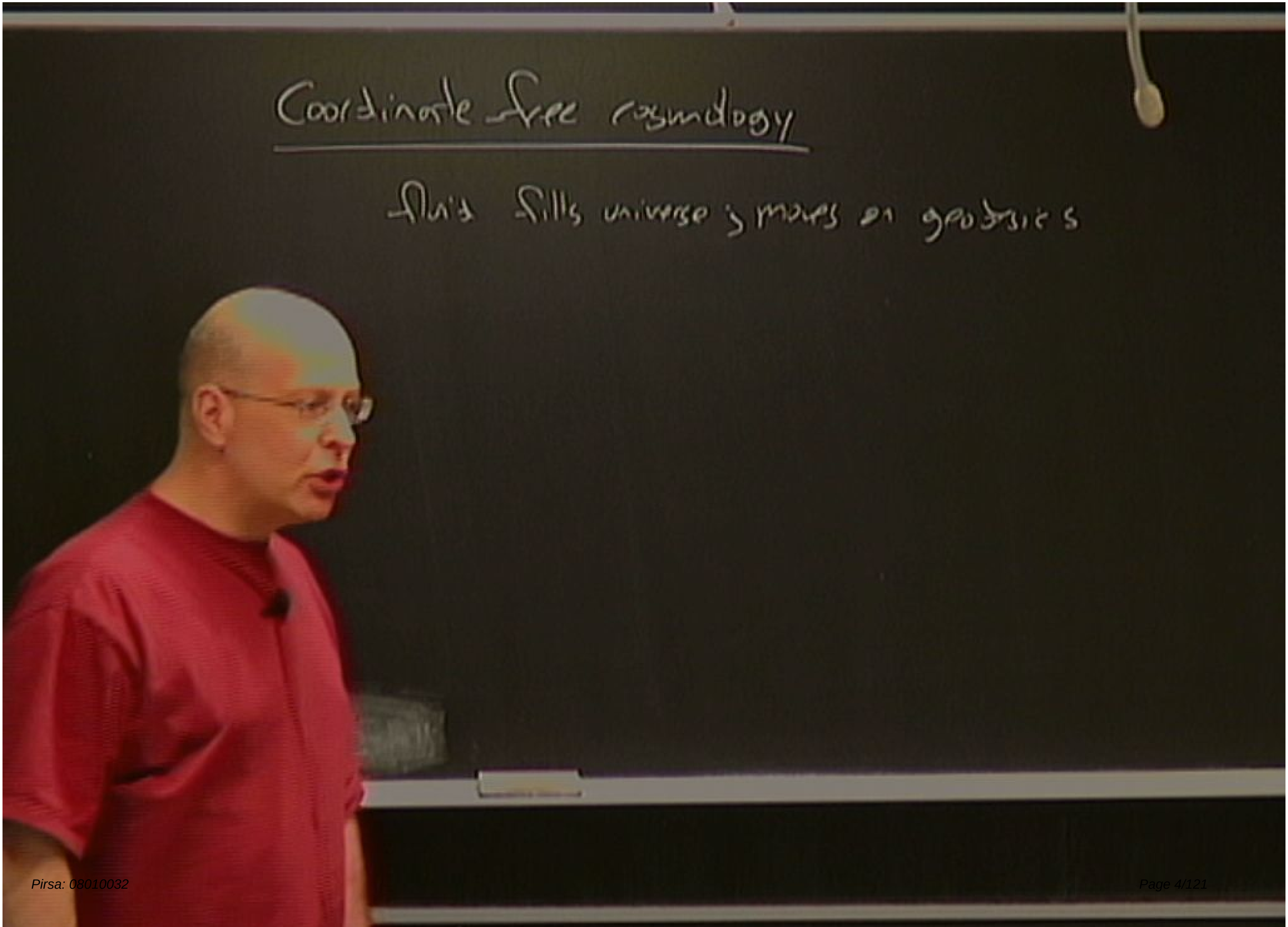
Date: Jan 30, 2008 04:00 PM

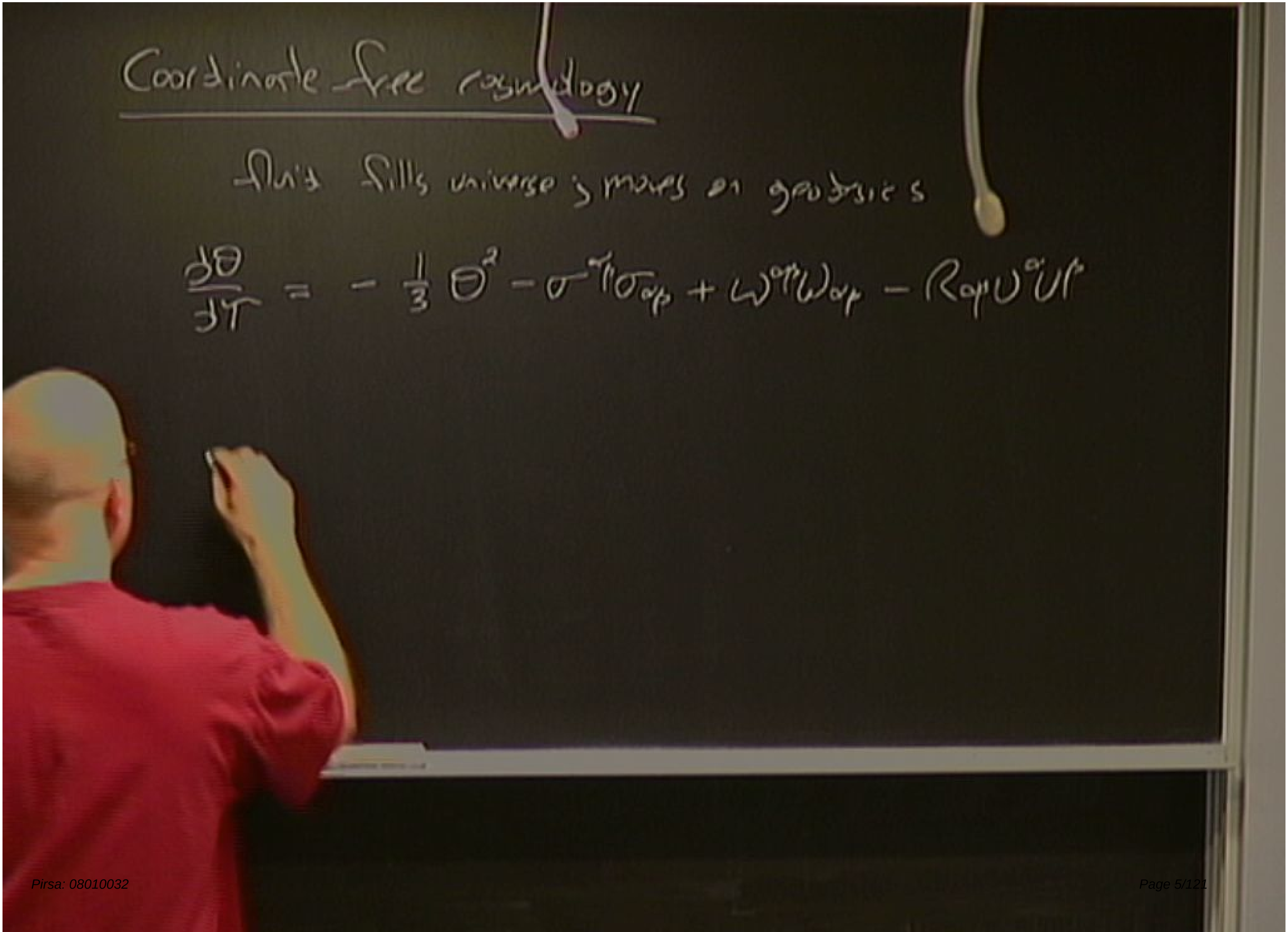
URL: http://pirsa.org/08010032

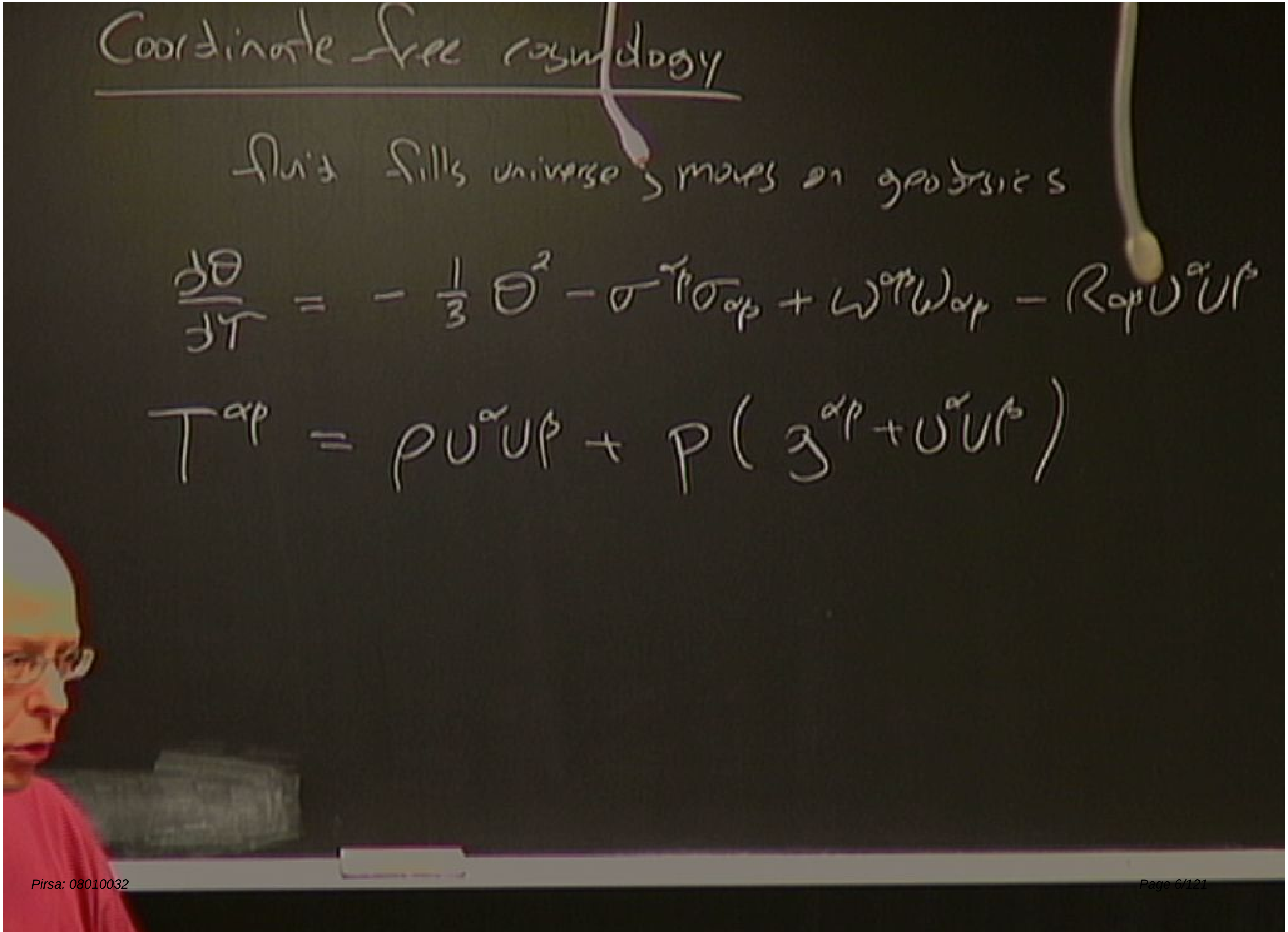
Abstract: Advanced General Relativity




Coordinate-free cosmology


Coordinate-free cosmology
fluid fills universe & moves on geodesics


Coordinate-free cosmology
Fluid fills universe; moves on geodesics
$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$


Coordinate-free cosmology
fluid fills universe; moves on geodesics
$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$
$T^{\alpha\beta} = \rho U^\alpha U^\beta + p\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$

Coordinate free cosmology

fluid fills universe & moves on geodesics

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$$

$$T^{\alpha\beta} = \rho U^\alpha U^\beta + p\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$T^{\alpha\beta}{}_{;\beta} = 0$$

Coordinate-free cosmology

fluid fills universe & moves on geodesics

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$$

$$T^{\alpha\beta} = \rho U^\alpha U^\beta + p\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$T^{\alpha\beta}{}_{;\beta} = 0$$

$$U^\alpha{}_{;\beta} U^\beta = 0$$

Coordinate-free cosmology

fluid fills universe, moves on geodesics

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$$

$$T^{\alpha\beta} = \rho U^\alpha U^\beta + p\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$\left.\begin{aligned} T^{\alpha\beta}{}_{;\beta} &= 0 \\ U^\alpha{}_{;\beta} U^\beta &= 0 \end{aligned}\right\} \quad p = w\rho$$

Coordinate-free cosmology

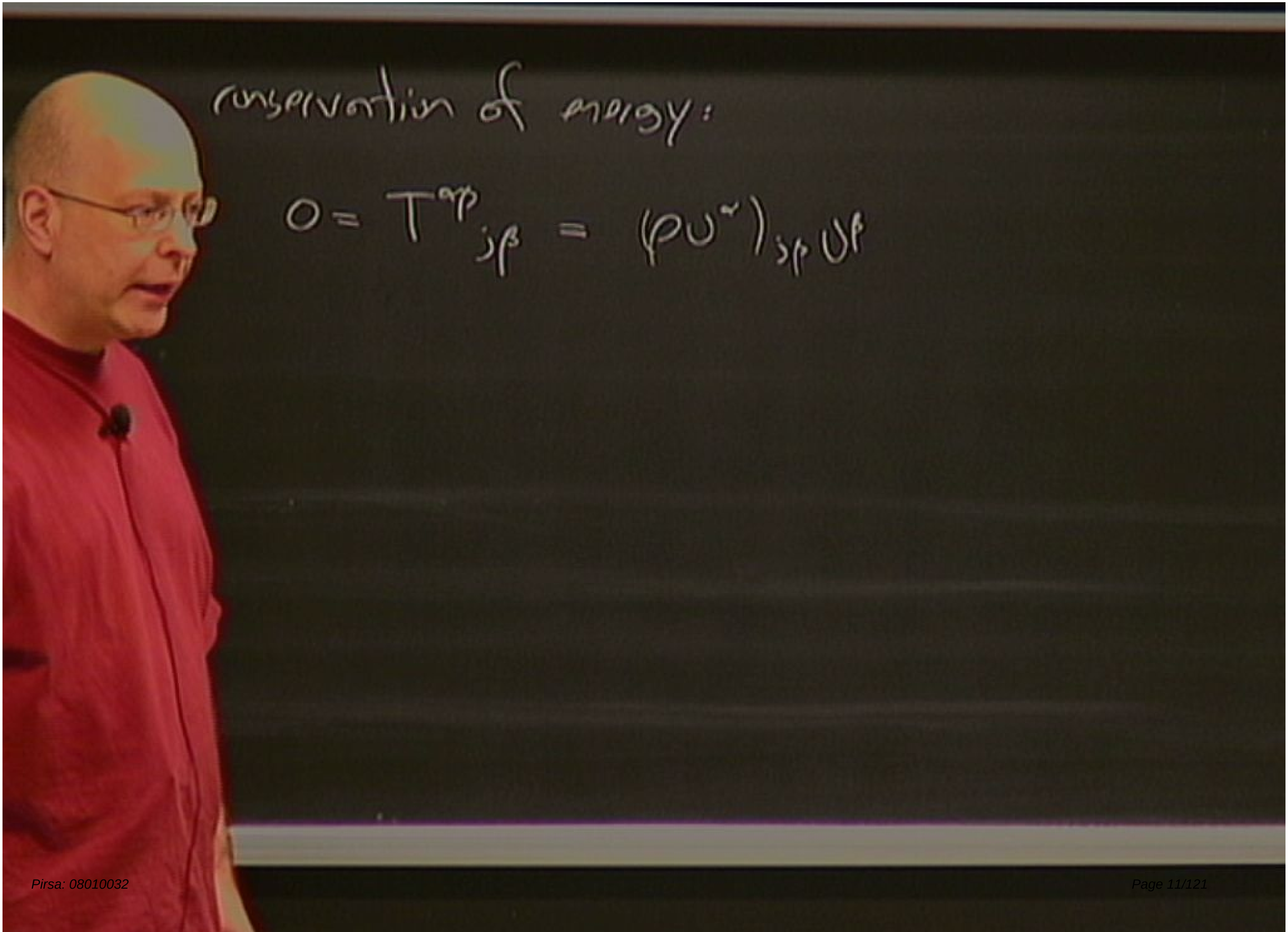
fluid fills universe & moves on geodesics

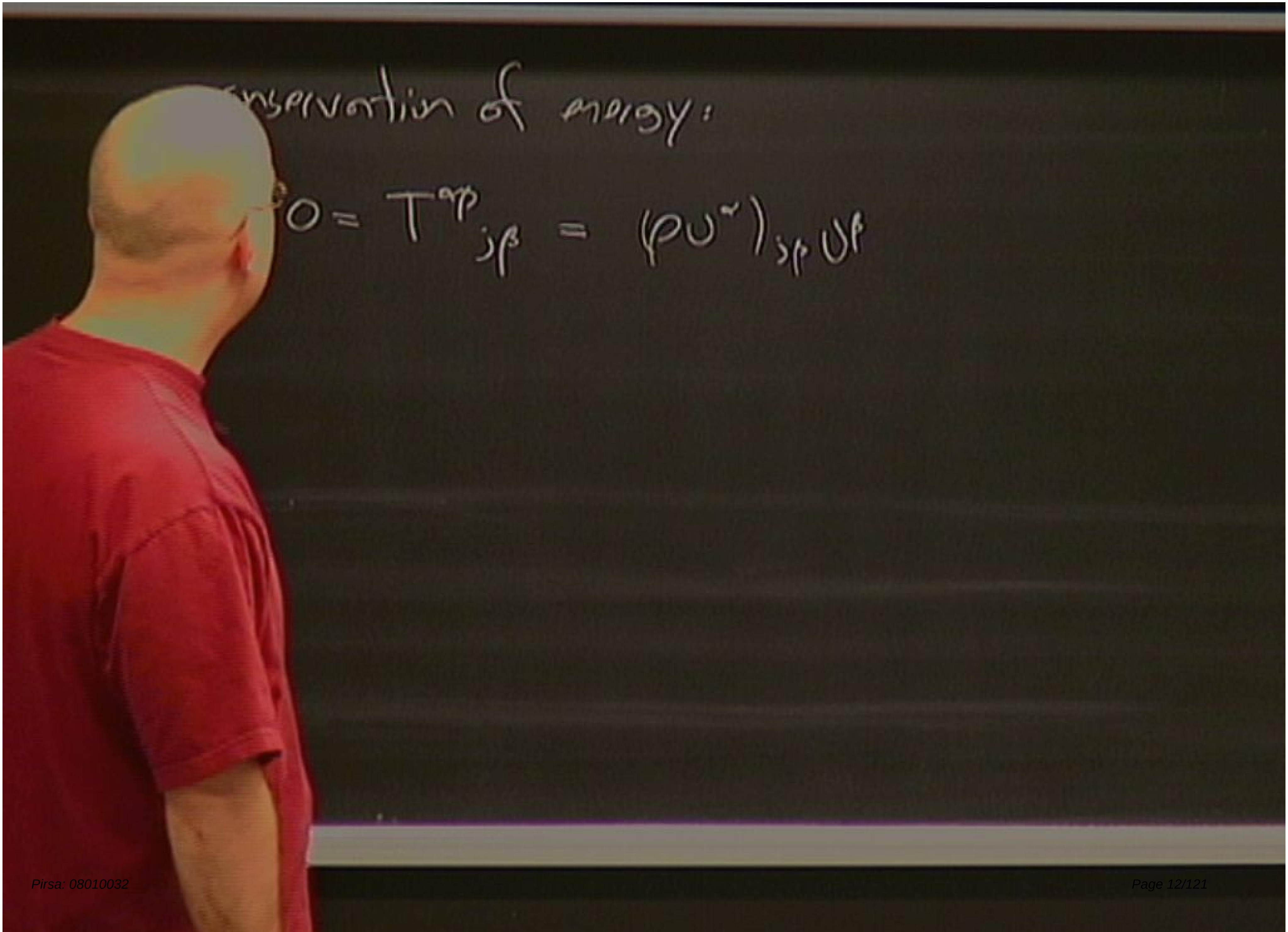
$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta$$

$$T^{\alpha\beta} = \rho U^\alpha U^\beta + p\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$\left.\begin{array}{l} T^{\alpha\beta}{}_{;\beta} = 0 \\ U^\alpha{}_{;\beta} U^\beta = 0 \end{array}\right\} \quad p = w\rho$$

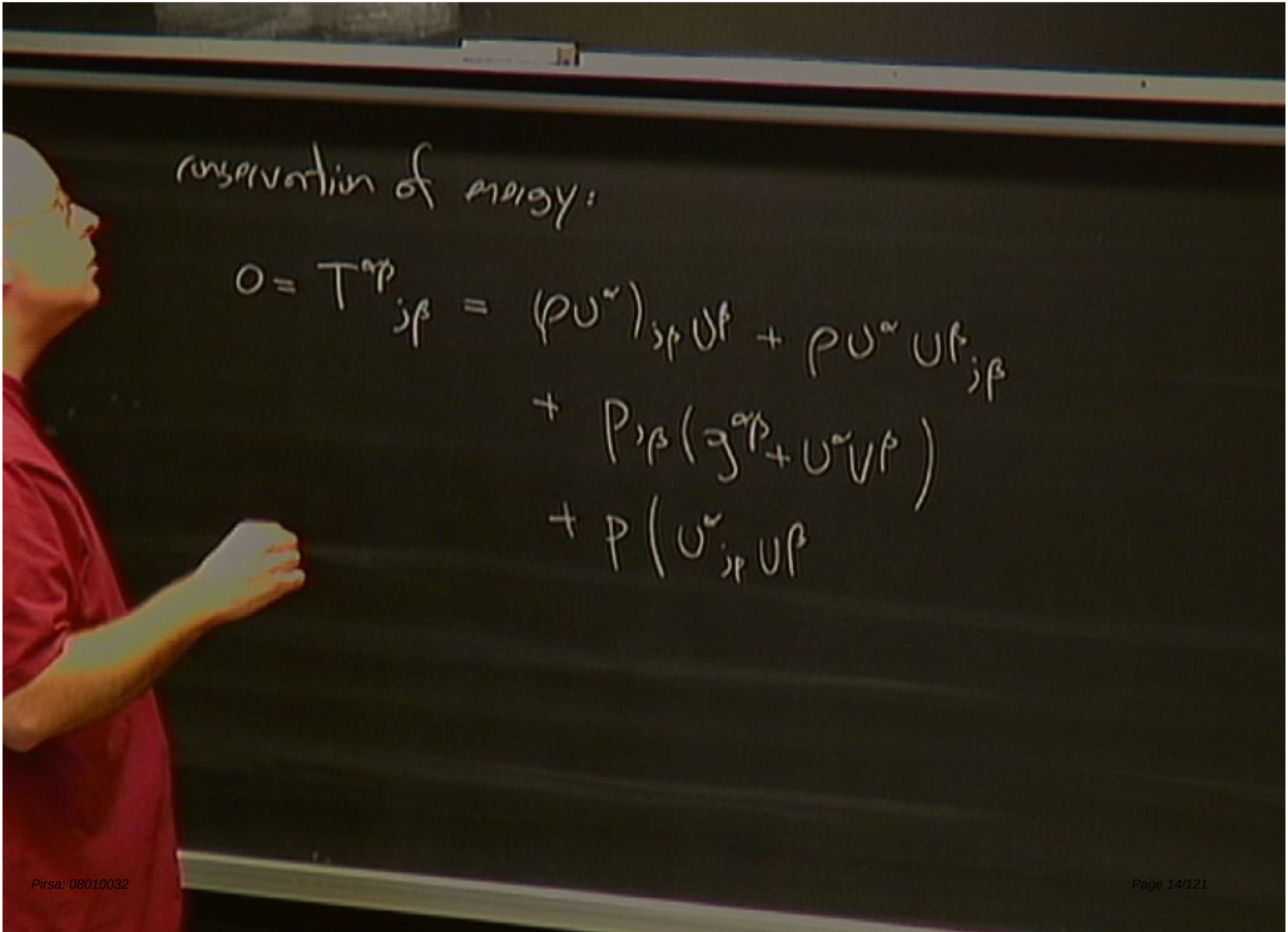
(w → constant)


conservation of energy:
0 = T^{αβ}_{;β} = (ρu^α)_{;β} u^β


nservation of energy:
$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^{\alpha})_{;\beta} U^{\beta}$

conservation of energy:

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^\alpha)_{;\beta} U^\beta + \rho U^\alpha U^\beta{}_{;\beta}$$
$$+ P_{;\beta}(g^{\alpha\beta} + U^\alpha U^\beta)$$


conservation of energy:
$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^\alpha)_{;\beta} U^\beta + \rho U^\alpha U^\beta{}_{;\beta}$
$+ P_{,\beta}(g^{\alpha\beta} + U^\alpha U^\beta)$
$+ P(U^\alpha{}_{;\beta} U^\beta$

conservation of energy:

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^\alpha)_{;\beta} U^\beta + \rho U^\alpha U^\beta{}_{;\beta} + P_{,\beta}\left(g^{\alpha\beta} + U^\alpha U^\beta\right) + P\left(U^\alpha{}_{;\beta} U^\beta + U^\alpha U^\beta{}_{;\beta}\right)$$

conservation of energy:

$$0 = T^{\alpha\beta}{}_{;\beta} = \overbrace{(\rho u^\alpha)_{;\beta}}^{\rho_{;\beta} u^\beta u^\alpha + \rho u^\alpha{}_{;\beta} u^\beta} u^\beta + \rho u^\alpha u^\beta{}_{;\beta}$$

$$+ p_{;\beta}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)$$

$$+ p\left(u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta}\right)$$

conservation of energy:

$$\rho_{;\beta} U^\beta U^\alpha + \cancel{\rho U^\alpha{}_{;\beta} U^\beta}$$

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^\alpha)_{;\beta} U^\beta + \rho U^\alpha U^\beta{}_{;\beta}$$

$$+ p_{;\beta}\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$+ p\left(U^\alpha{}_{;\beta} U^\beta + U^\alpha U^\beta{}_{;\beta}\right)$$

conservation of energy:

$$\overbrace{(\rho u^\alpha)_{;\beta} u^\beta}^{\rho_{;\beta} u^\beta u^\alpha + \cancel{\rho u^\alpha{}_{;\beta} u^\beta}}$$

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho u^\alpha)_{;\beta} u^\beta + \rho u^\alpha u^\beta{}_{;\beta}$$

$$+ \; p_{;\beta}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)$$

$$+ \; p\left(\cancel{u^\alpha{}_{;\beta} u^\beta} + u^\alpha u^\beta{}_{;\beta}\right)$$

conservation of energy:

$$\rho_{;\beta}U^\beta U^\alpha + \cancel{\rho U^\alpha_{;\beta}U^\beta}$$

$$0 = T^{\alpha\beta}{}_{;\beta} = \underbrace{(\rho U^\alpha)_{;\beta}U^\beta} + \rho U^\alpha U^\beta{}_{;\beta}$$

$$+ P_{,\beta}(g^{\alpha\beta} + U^\alpha U^\beta)$$

$$+ P\left(\cancel{U^\alpha{}_{;\beta}U^\beta} + U^\alpha U^\beta{}_{;\beta}\right)$$

$$= U^\alpha\left(\frac{d\rho}{d\tau} + \rho\Theta\right.$$

conservation of energy:

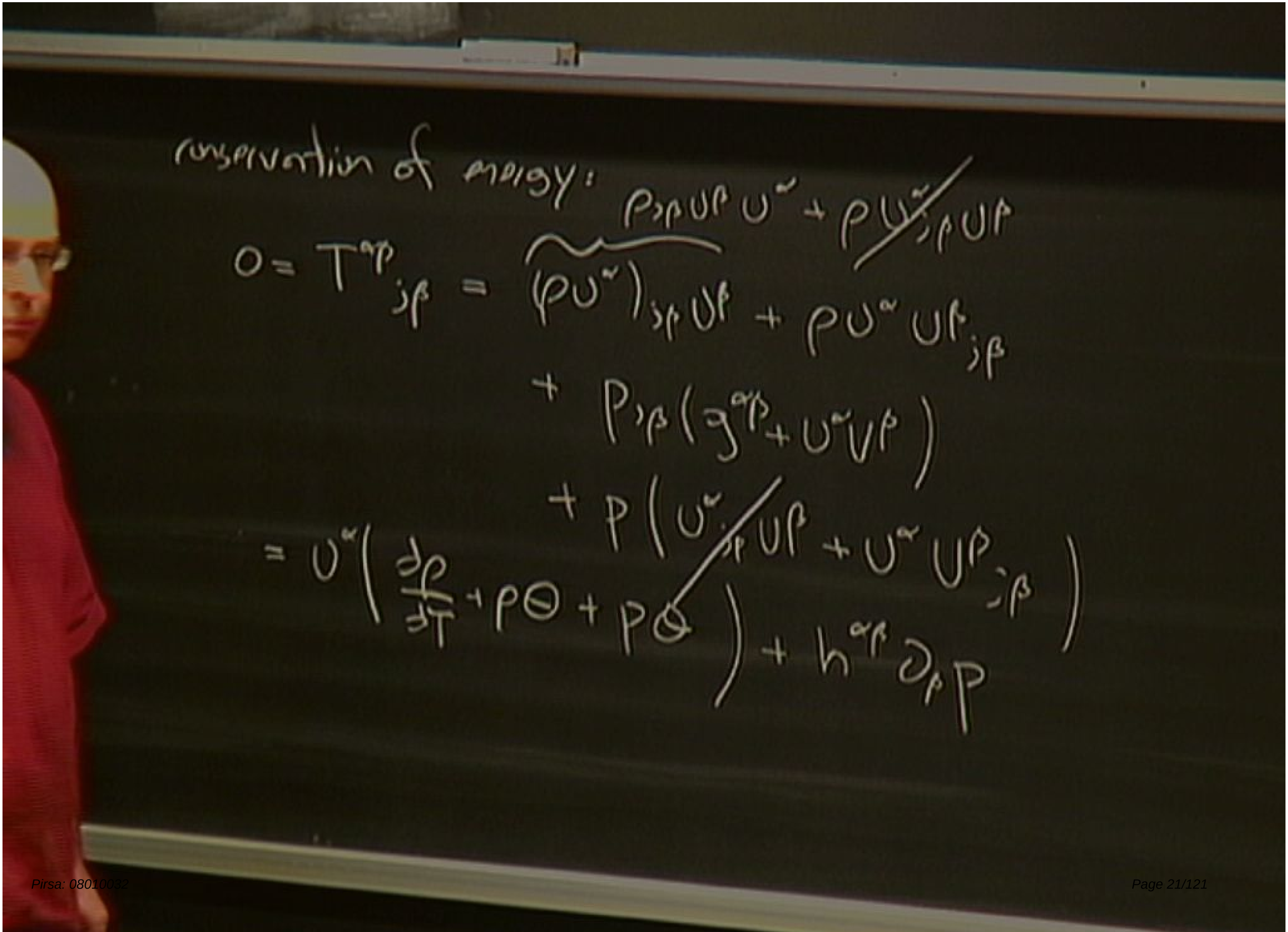
$$\rho_{,\beta} u^\beta u^\alpha + \rho u^\alpha_{;\beta} u^\beta$$

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho u^\alpha)_{;\beta} u^\beta + \rho u^\alpha u^\beta{}_{;\beta}$$

$$+ p_{,\beta}(g^{\alpha\beta} + u^\alpha u^\beta)$$

$$+ p\left(u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta}\right)$$

$$= u^\alpha\left(\frac{d\rho}{d\tau} + \rho\Theta + p\Theta\right)$$


conservation of energy:
$\rho_{,\beta} u^\beta u^\alpha + \rho u^\alpha{}_{;\beta} u^\beta$
$0 = T^{\alpha\beta}{}_{;\beta} = (\rho u^\alpha)_{;\beta} u^\beta + \rho u^\alpha u^\beta{}_{;\beta}$
$+ p_{,\beta} (g^{\alpha\beta} + u^\alpha u^\beta)$
$+ p (u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta})$
$= u^\alpha \left(\frac{d\rho}{d\tau} + \rho\Theta + p\Theta \right) + h^{\alpha\beta} \partial_\beta p$

conservation of energy:

$$\rho_{;\beta} U^\beta U^\alpha + \rho U^\alpha{}_{;\beta} U^\beta$$

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho U^\alpha)_{;\beta} U^\beta + \rho U^\alpha U^\beta{}_{;\beta}$$

$$+ P_{;\beta}\left(g^{\alpha\beta} + U^\alpha U^\beta\right)$$

$$+ P\left(U^\alpha{}_{;\beta} U^\beta + U^\alpha U^\beta{}_{;\beta}\right)$$

$$= U^\alpha\left(\frac{\partial\rho}{\partial\tau} + \rho\Theta + P\Theta\right) + h^{\alpha\beta}\partial_\beta P$$

conservation of energy:

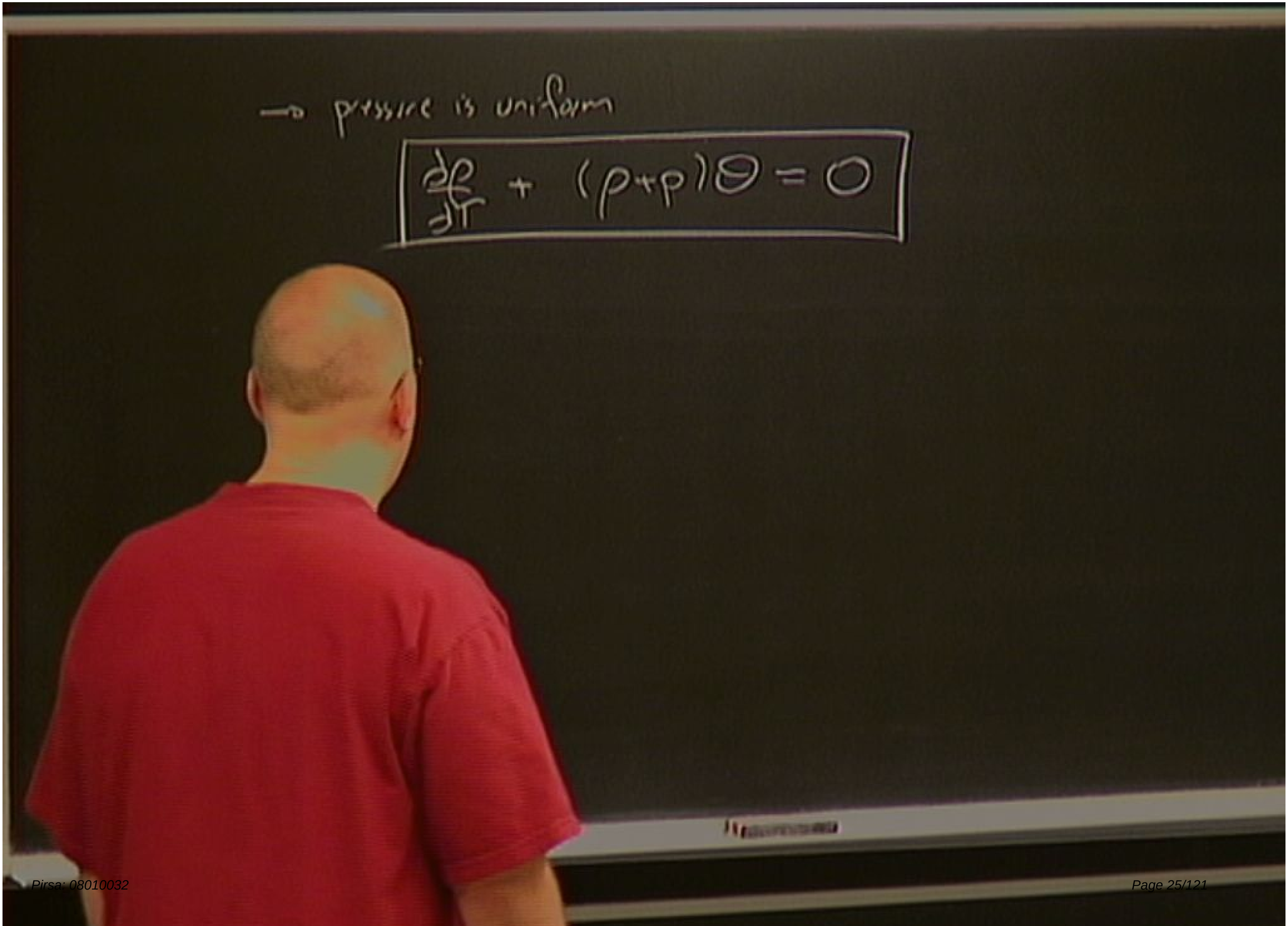
$$\underbrace{(\rho u^\alpha)_{;\beta} u^\beta}_{\rho_{;\beta} u^\beta u^\alpha + \rho u^\alpha{}_{;\beta} u^\beta}$$

$$0 = T^{\alpha\beta}{}_{;\beta} = (\rho u^\alpha)_{;\beta} u^\beta + \rho u^\alpha u^\beta{}_{;\beta} + p_{;\beta}(g^{\alpha\beta} + u^\alpha u^\beta) + p(u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta})$$

$$= u^\alpha \Big(\underbrace{\frac{\partial \rho}{\partial \tau} + \rho\Theta + p\Theta}_{0} \Big) + \underbrace{h^{\alpha\beta}\partial_\beta p}_{0}$$

→ pressure is uniform

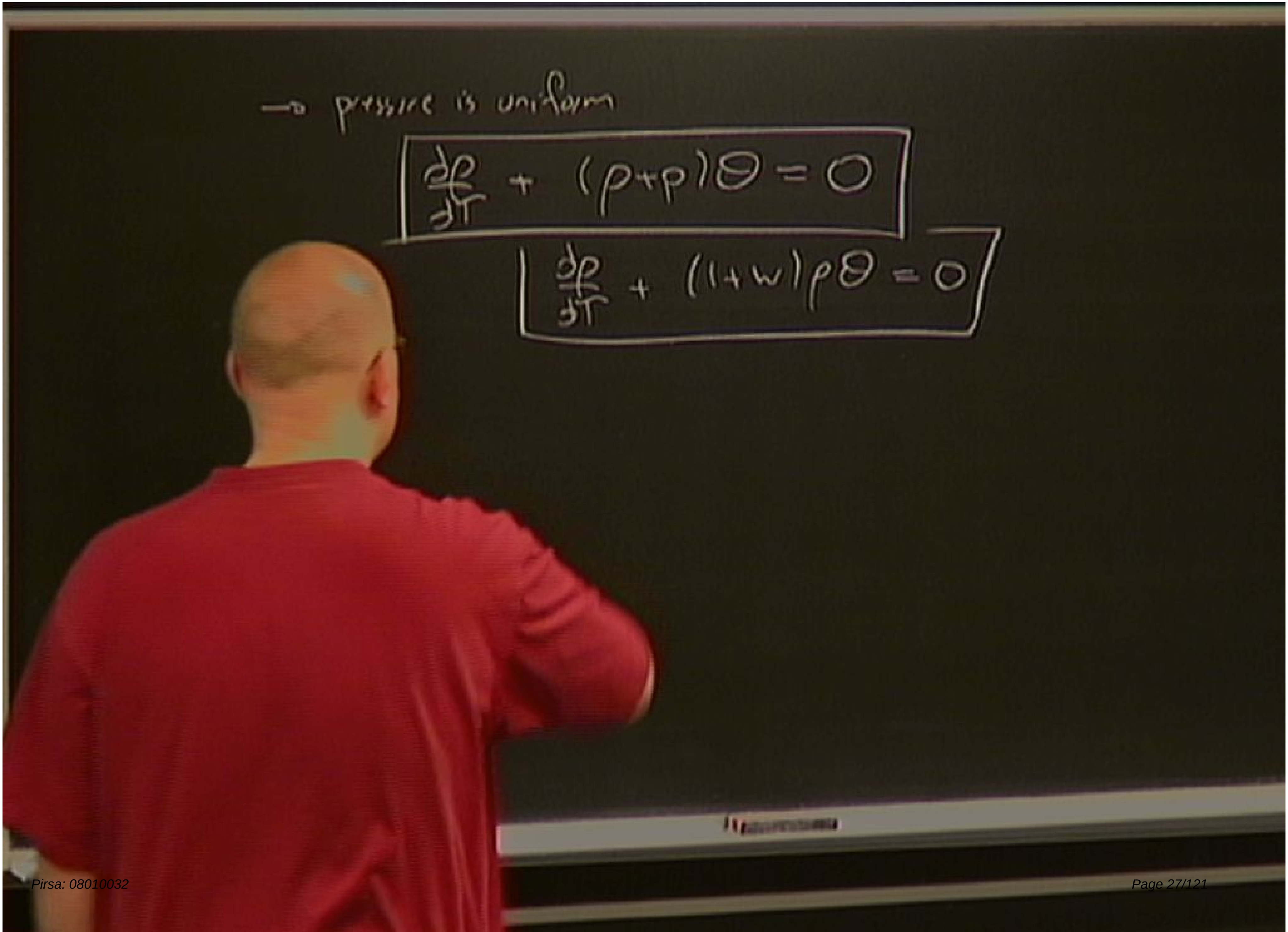
$$\frac{d\rho}{d\tau} + (\rho + p)\theta = 0$$


→ pressure is uniform
dρ/dτ + (ρ+p)θ = 0

→ pressure is uniform

$$\boxed{\frac{d\rho}{d\tau} + (\rho + p)\theta = 0}$$

$$\frac{d\rho}{d\tau} + (1+w)\rho\theta = 0$$


→ pressure is uniform
$\frac{d\rho}{d\tau} + (\rho + p)\Theta = 0$
$\frac{d\rho}{d\tau} + (1 + w)\rho\Theta = 0$

→ pressure is uniform

$$\frac{d\rho}{d\tau} + (\rho + p)\theta = 0$$

$$\frac{d\rho}{d\tau} + (1+w)\rho\theta = 0$$

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V$$

→ pressure is uniform

$$\frac{d\rho}{d\tau} + (\rho + p)\theta = 0$$

$$\frac{d\rho}{d\tau} + (1+w)\rho\theta = 0$$

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V \quad \rightarrow \quad \delta V\,\dot{\rho} + (\rho + p)\,\delta\dot{V} = 0$$

→ pressure is uniform

$$\frac{d\rho}{d\tau} + (\rho + p)\theta = 0$$

$$\frac{d\rho}{d\tau} + (1+w)\rho\theta = 0$$

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V \quad \to \quad \delta V\,\dot{\rho} + (\rho + p)\,\delta\dot{V} = 0$$

$$(\rho\,\delta V)^{\cdot}$$

→ pressure is uniform

$$\boxed{\frac{\partial\rho}{\partial\tau} + (\rho+p)\Theta = 0}$$

$$\boxed{\frac{\partial\rho}{\partial\tau} + (1+w)\rho\Theta = 0}$$

$$\Theta = \frac{1}{\delta V}\frac{\partial}{\partial\tau}\delta V \quad \rightarrow \quad \delta V\,\dot\rho + (\rho+p)\,\delta\dot V = 0$$

$$(\rho\,\delta V)^{\cdot} + p\,(\delta V)^{\cdot} = 0$$

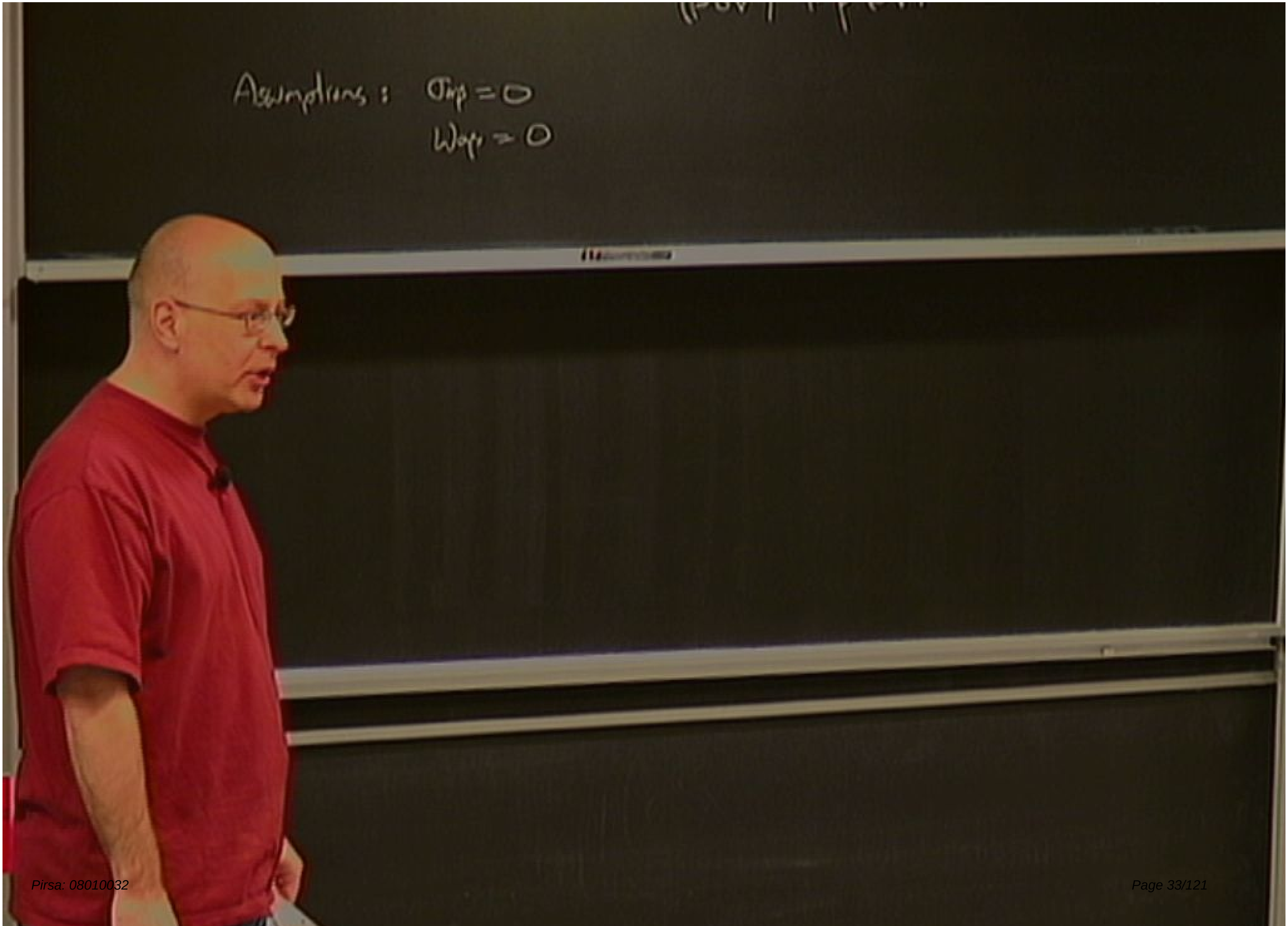
→ pressure is uniform

$$\frac{d\rho}{d\tau} + (\rho + p)\Theta = 0$$

$$\frac{d\rho}{d\tau} + (1+w)\rho\Theta = 0$$

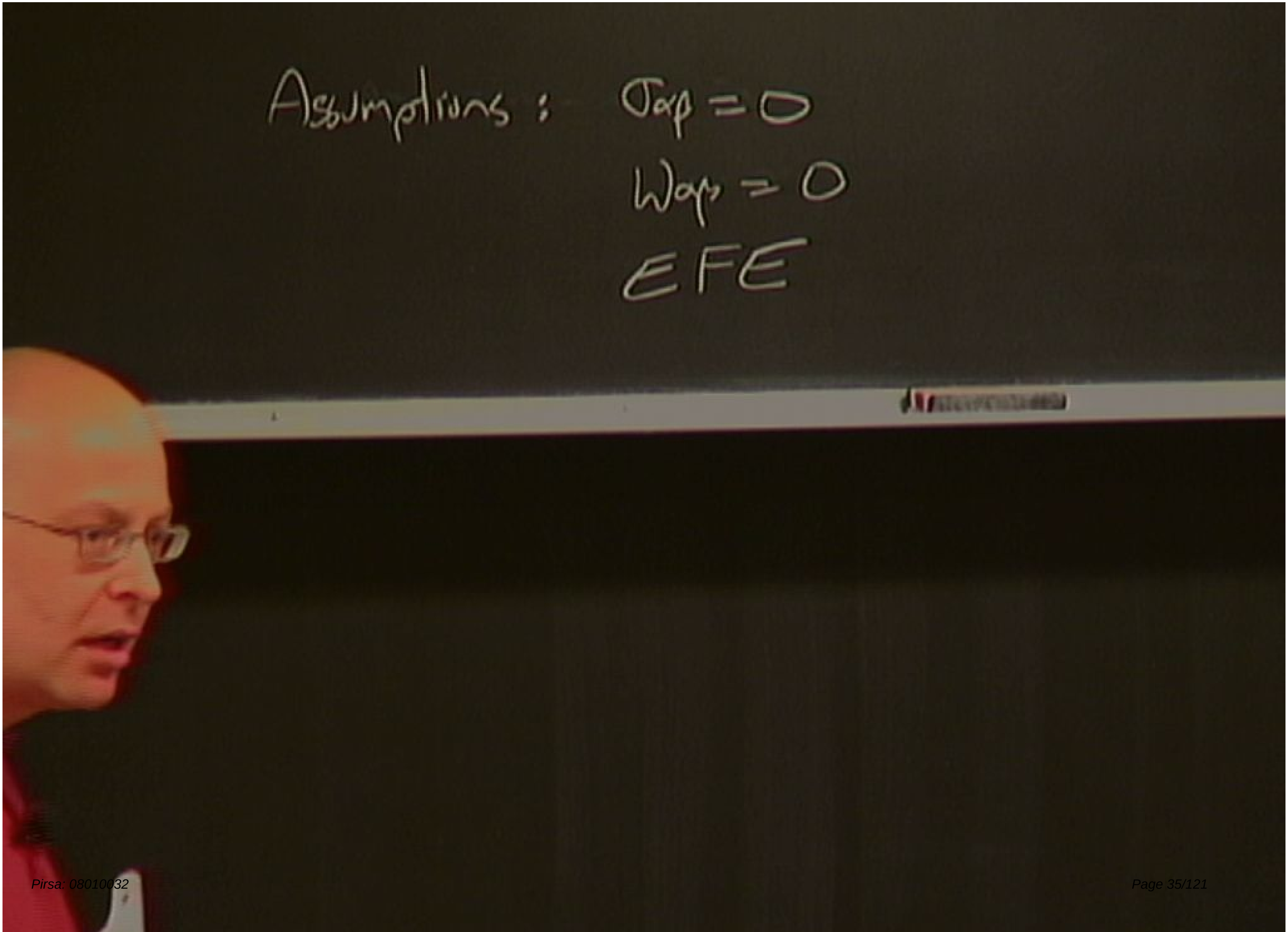
$$\Theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V \rightarrow \delta V\dot{\rho} + (\rho + p)\,\delta\dot{V} = 0$$

$$(\rho\,\delta V)^{\cdot} + p\,(\delta V)^{\cdot} = 0 \quad \text{(first law)}$$


Assumptions :

Assumptions: $\sigma_{\alpha\beta} = 0$

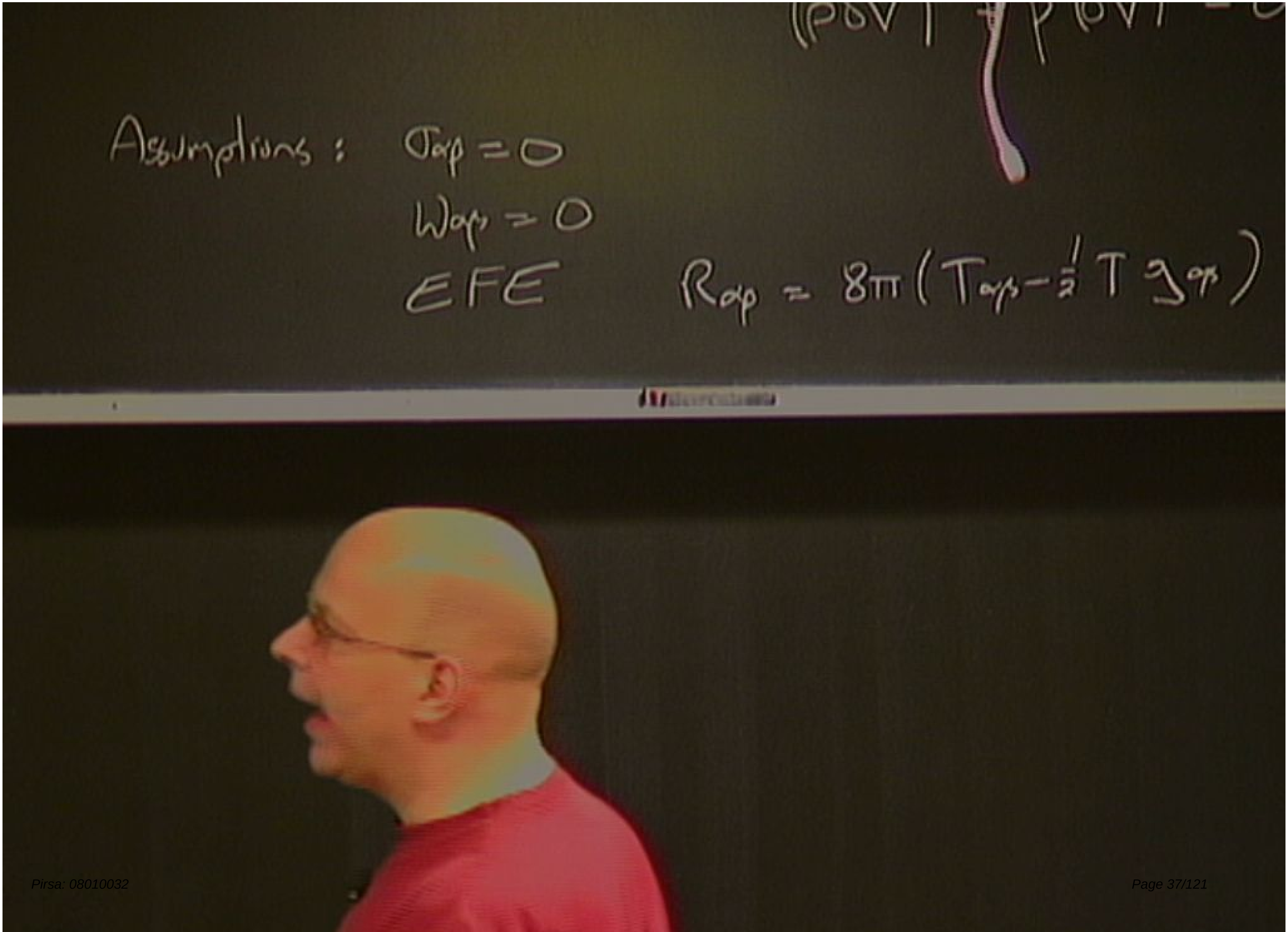
$W_{\alpha\beta} = 0$


Assumptions : $\sigma_{\alpha\beta} = 0$
$W_{\alpha\beta} = 0$
EFE

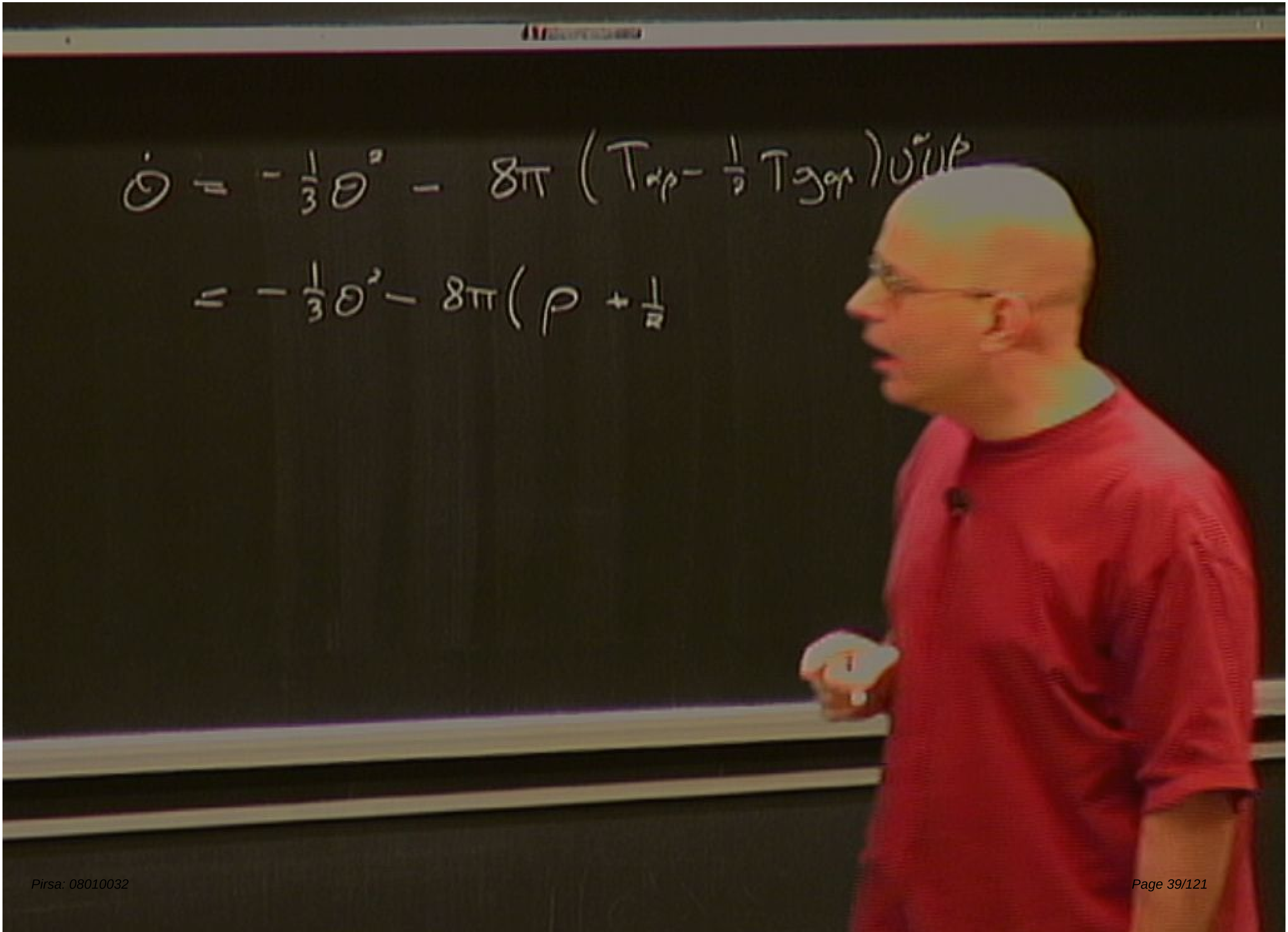
Assumptions: $\sigma_{\alpha\beta} = 0$

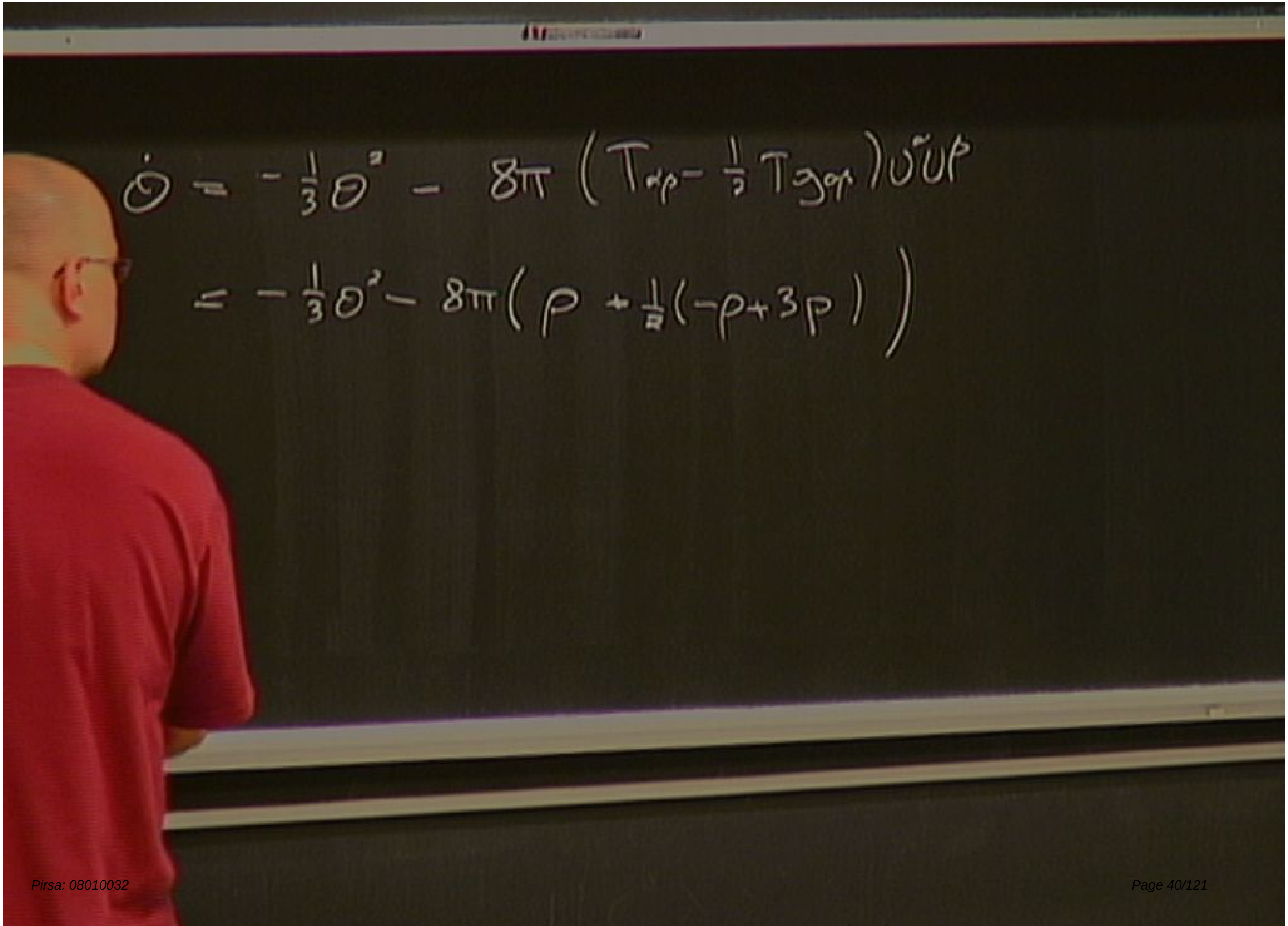
$\omega_{\alpha\beta} = 0$

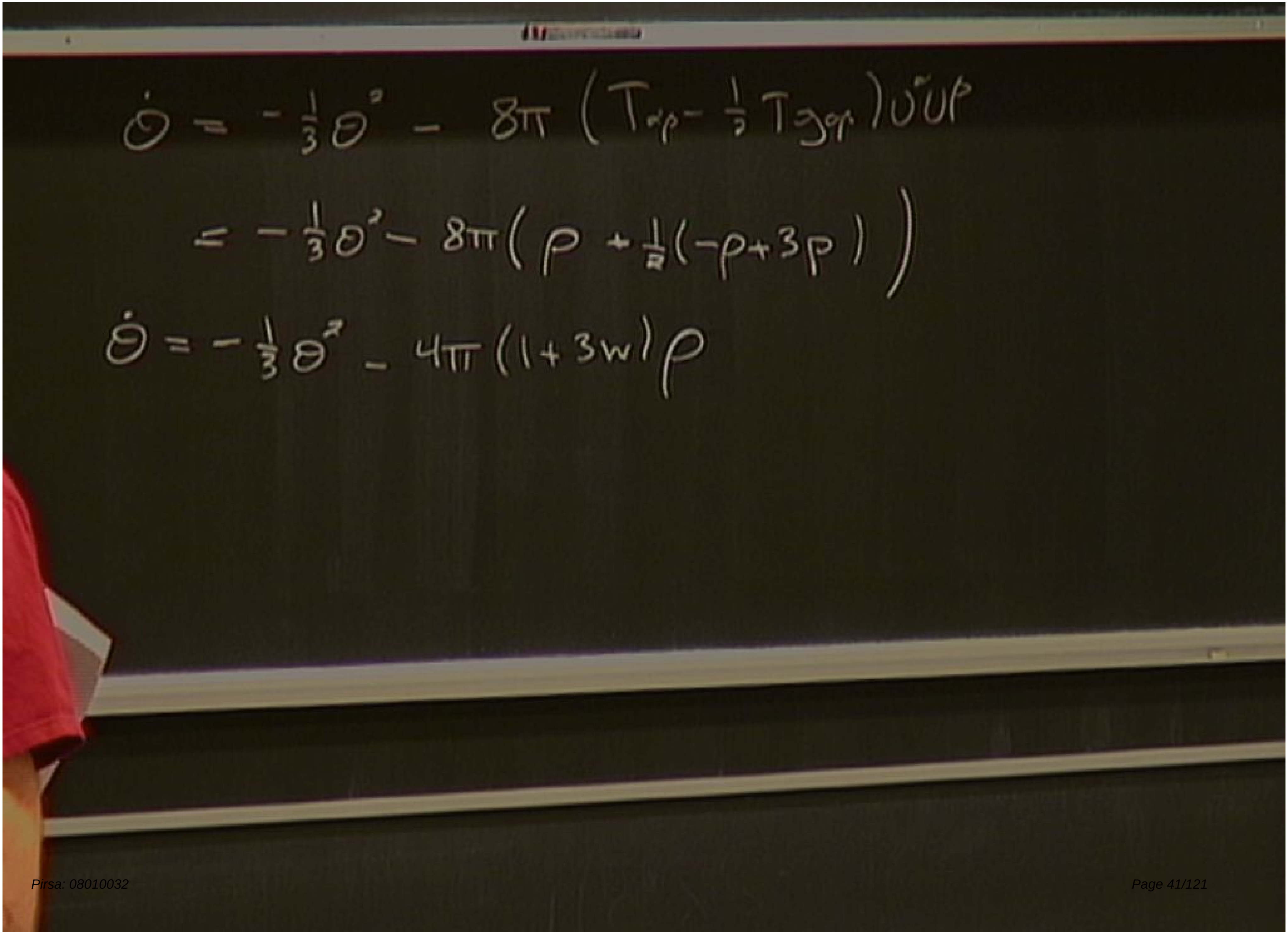
EFE $\quad R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right)$


Assumptions: $\sigma_{\alpha\beta} = 0$
$\omega_{\alpha\beta} = 0$
EFE $R_{\alpha\beta} = 8\pi (T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta})$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)u^\alpha u^\beta$$


$\dot{\theta} = -\frac{1}{3}\theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)u^\alpha u^\beta$
$= -\frac{1}{3}\theta^2 - 8\pi\left(\rho + \frac{1}{2}\right.$


$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)U^\alpha U^\beta$
$= -\frac{1}{3}\Theta^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$


$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) U^\alpha U^\beta$
$= -\frac{1}{3}\Theta^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$
$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 4\pi(1+3w)\rho$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)U^\alpha U^\beta$$

$$= -\frac{1}{3}\Theta^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 4\pi(1+3w)\rho$$

$$\dot{\rho} = -(1+w)\Theta$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)u^\alpha u^\beta$$

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Define scale factor

EFE $\quad R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta})$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right)u^\alpha u^\beta$$

$$= -\frac{1}{3}\Theta^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 4\pi(1+3w)\rho$$

$$\dot{\rho} = -(1+w)\Theta$$

Define scale factor: $\Theta \equiv \frac{1}{a^3}\frac{d}{d\tau}a^3$

EFE $R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta})$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)u^\alpha u^\beta$$

$$= -\frac{1}{3}\Theta^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right)$$

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Define scale factor: $\Theta = \frac{1}{a^3}\frac{d}{d\tau}a^3$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)u^\alpha u^\beta$$

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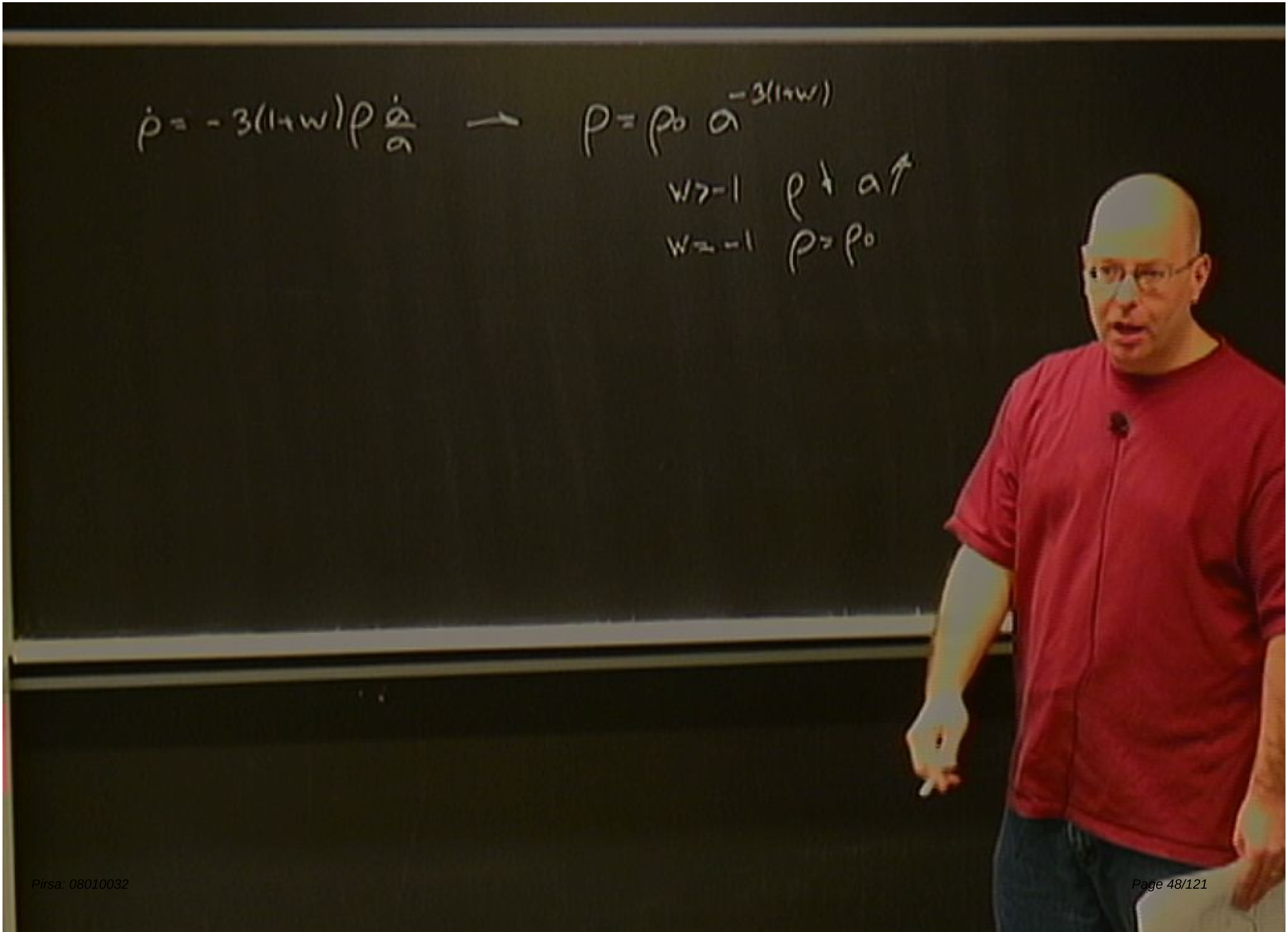
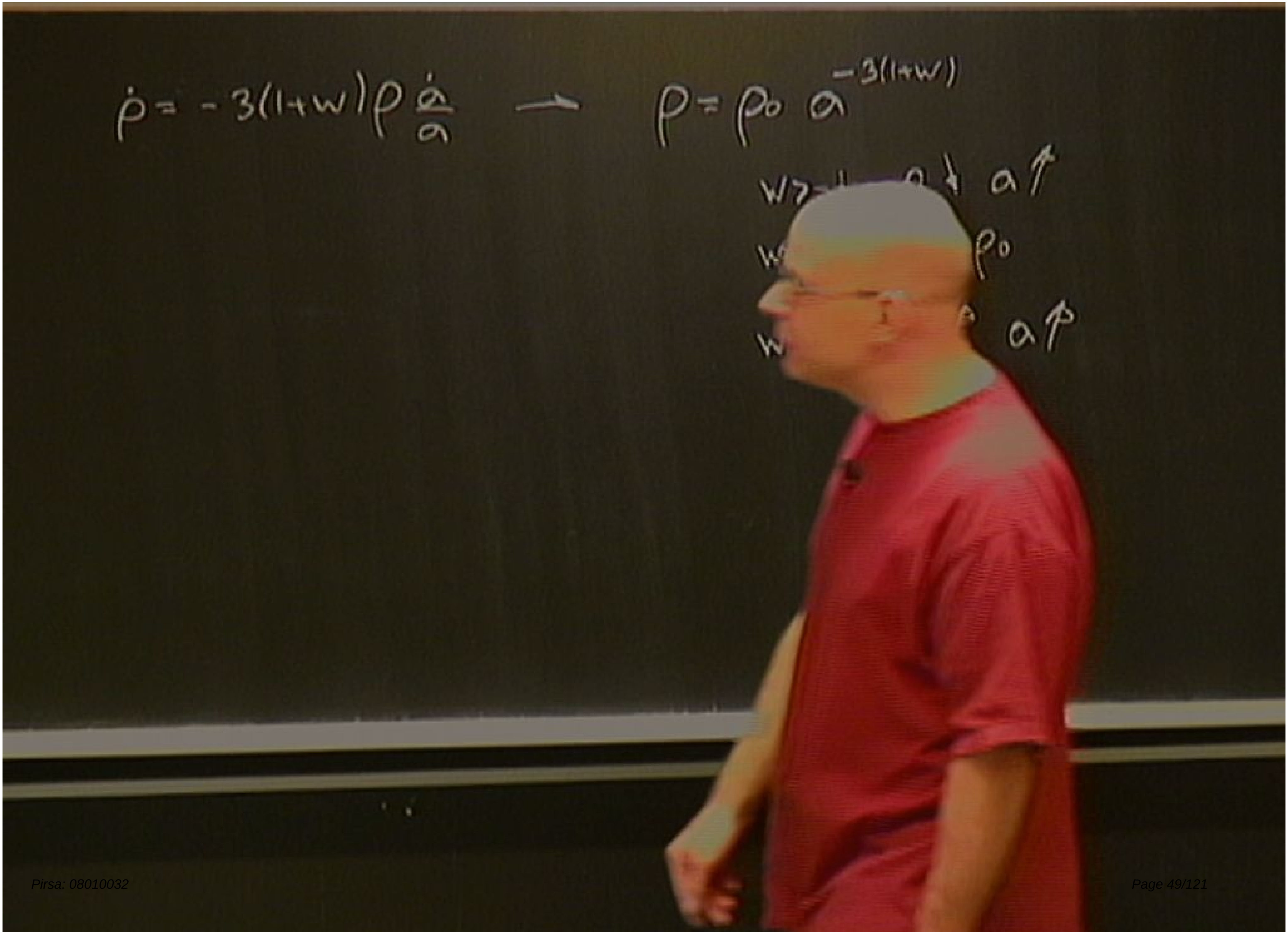
Define scale factor: $\Theta = \frac{1}{a^3}\frac{d}{d\tau}a^3$

$$\Theta = \frac{3}{a}\dot{a}$$


$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \rightarrow \rho = \rho_0 a^{-3(1+w)}$
$w > -1 \quad \rho \downarrow \quad a \uparrow$
$w = -1 \quad \rho = \rho_0$

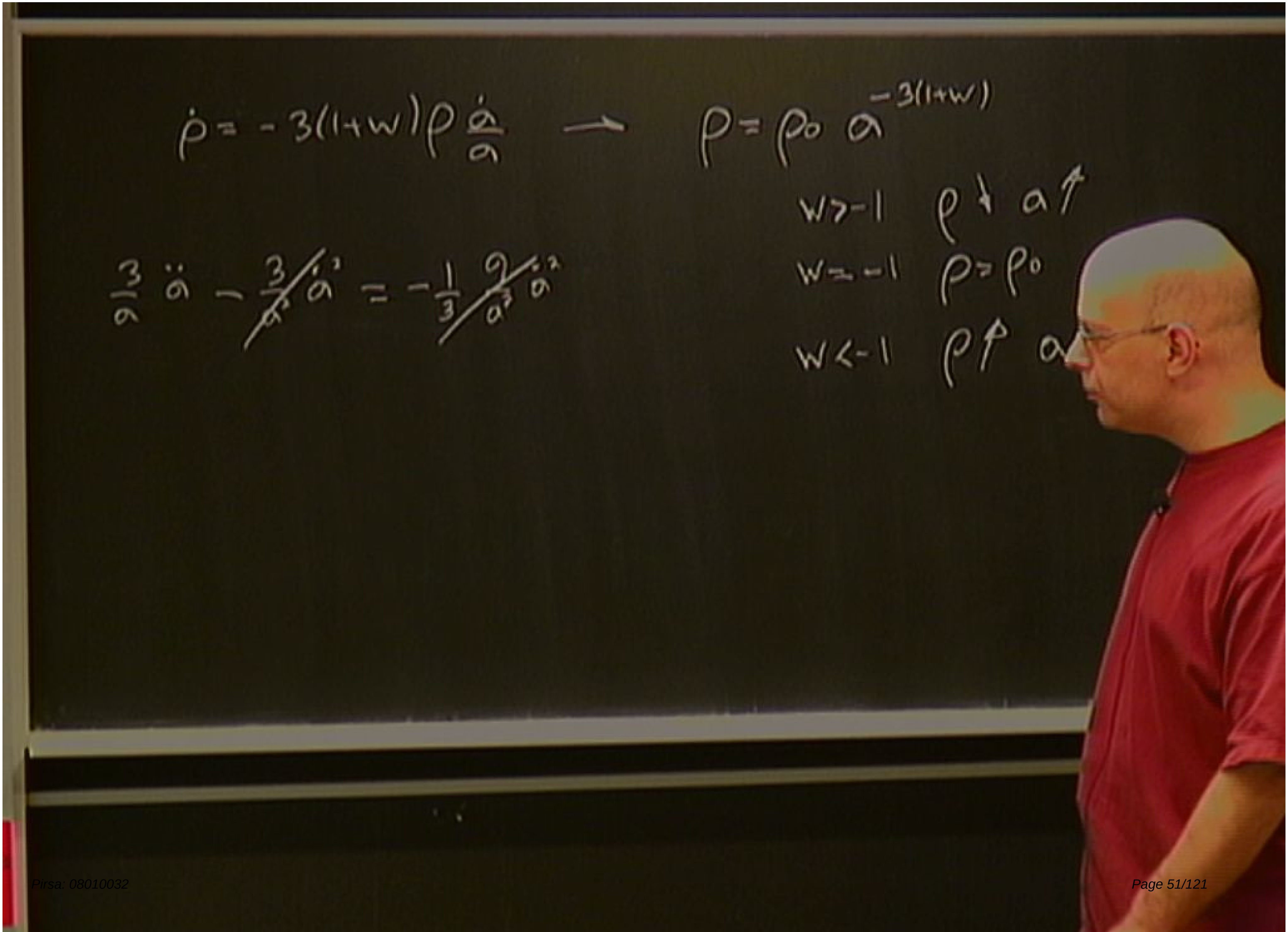

$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \rightarrow \rho = \rho_0 a^{-3(1+w)}$
$a \uparrow$
ρ_0
$a \uparrow$

$$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \longrightarrow \rho = \rho_0\, a^{-3(1+w)}$$

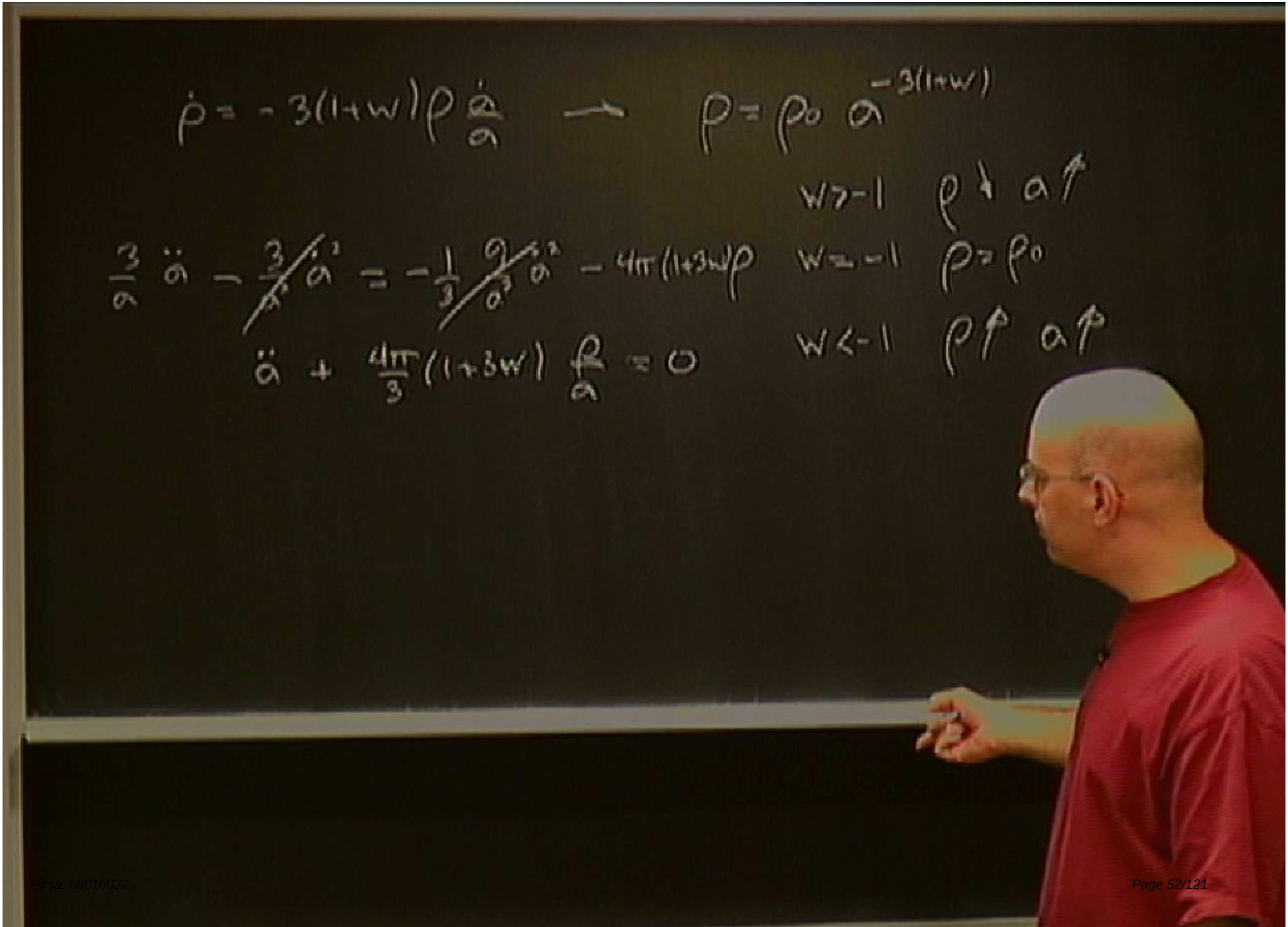
$w > -1$ $\quad \rho \downarrow \quad a \uparrow$

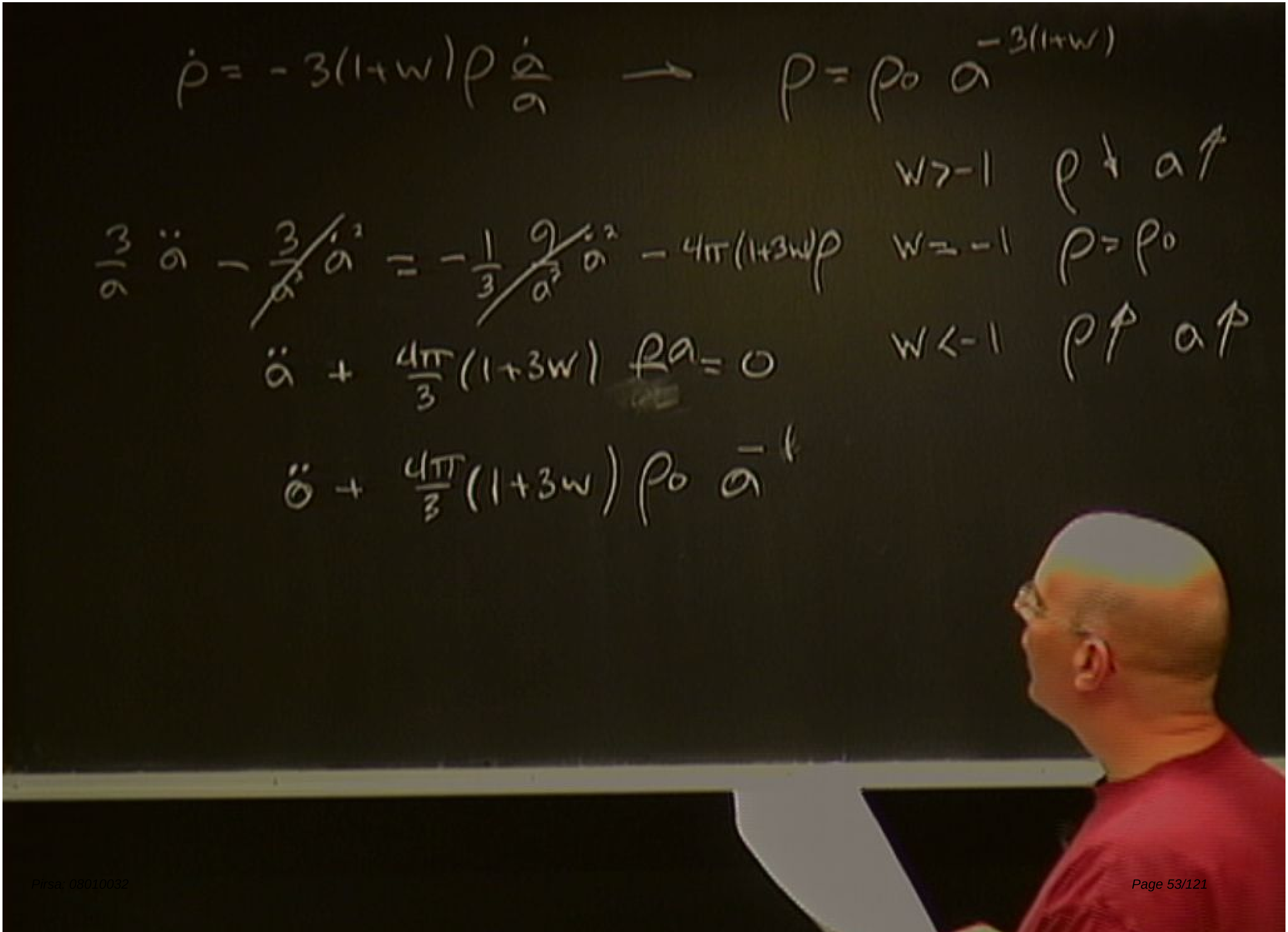
$w = -1$ $\quad \rho = \rho_0$

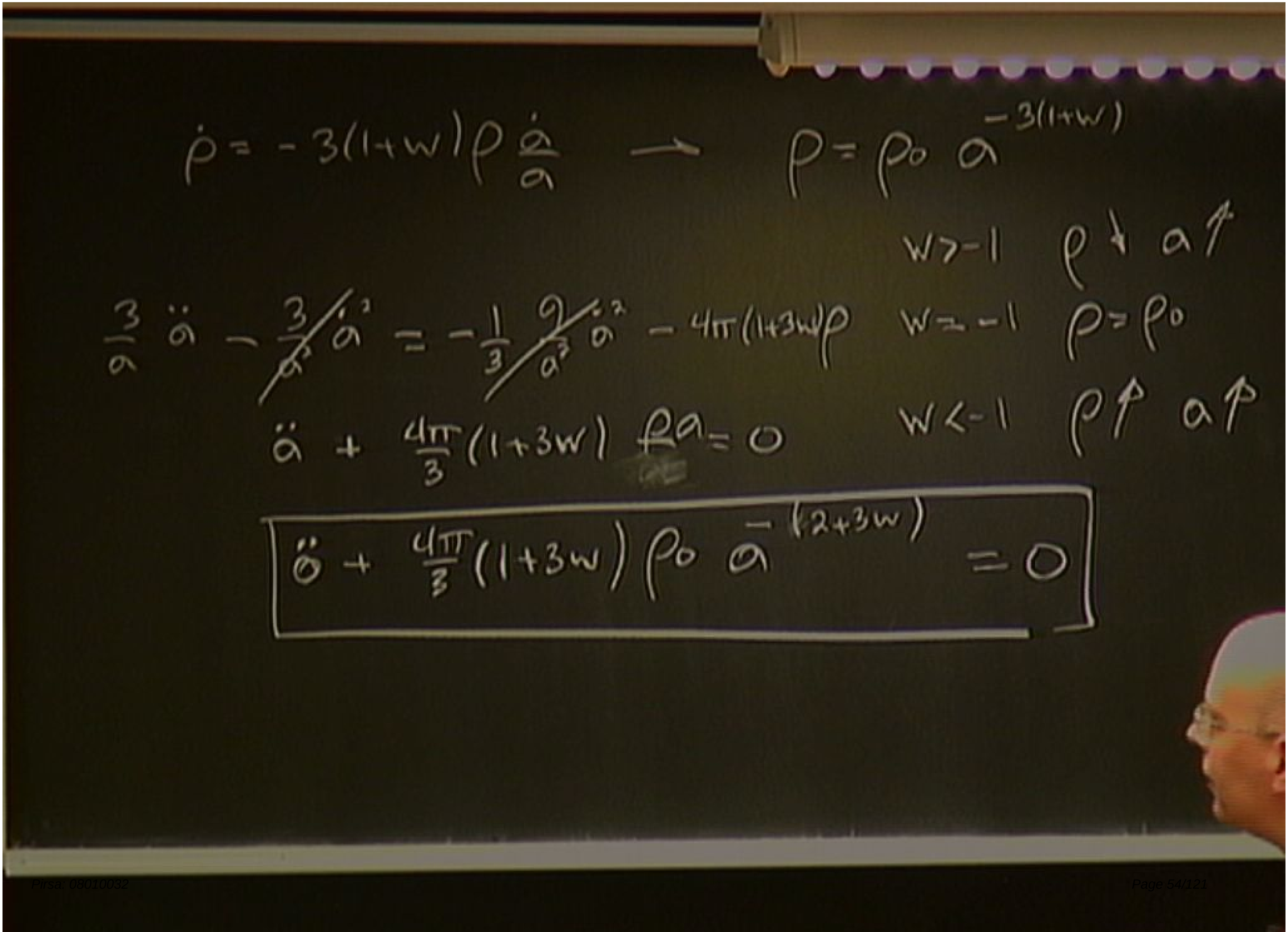
$w < -1$ $\quad \rho \uparrow \quad a \uparrow$

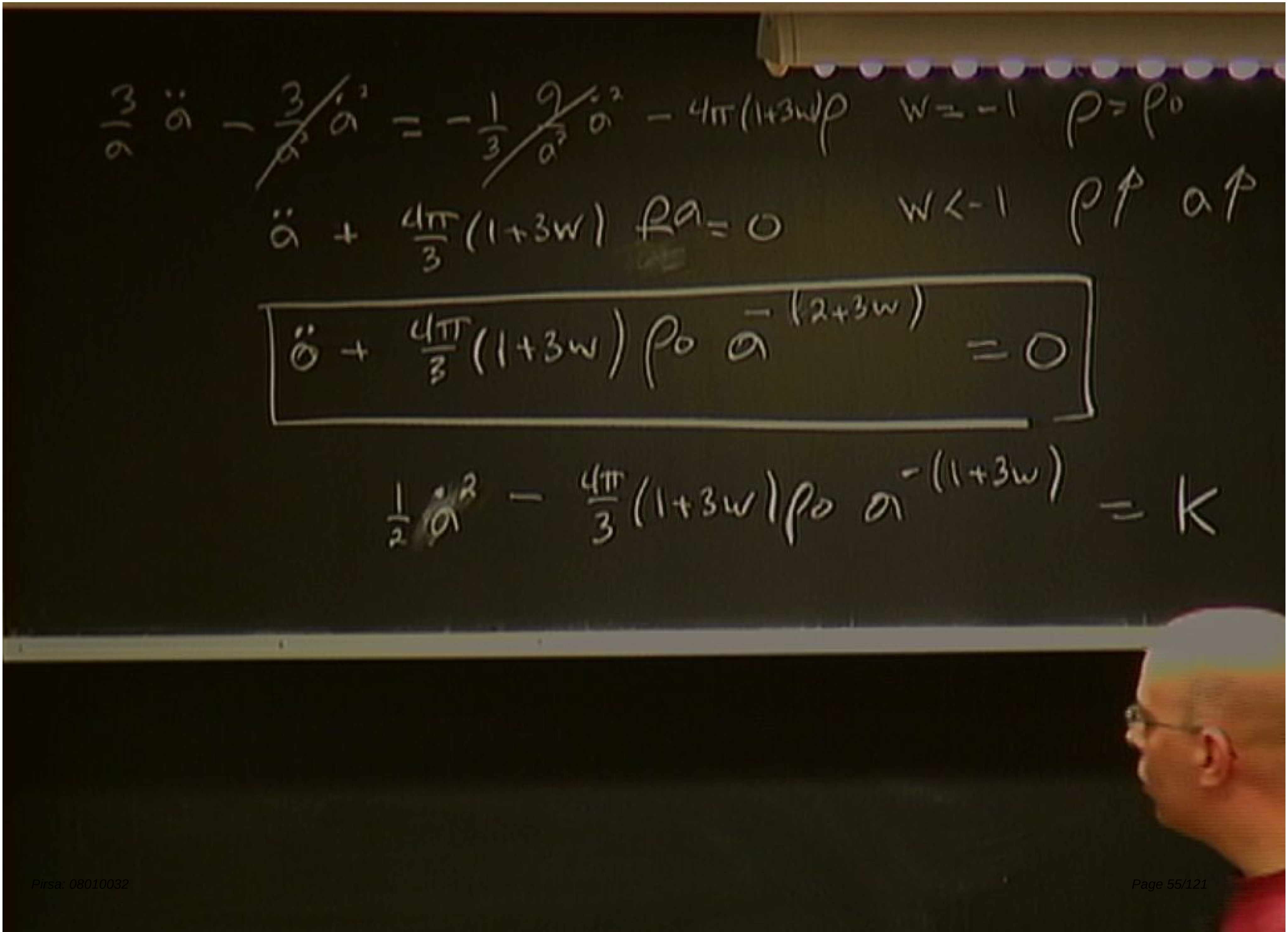


$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \rightarrow \rho = \rho_0 a^{-3(1+w)}$
$w > -1 \quad \rho \downarrow \quad a \uparrow$
$w = -1 \quad \rho = \rho_0$
$w < -1 \quad \rho \uparrow \quad a$
$\frac{3}{a}\ddot{a} - \frac{3}{a^2}\dot{a}^2 = -\frac{1}{3}\frac{\rho}{a^3}\dot{a}^2$

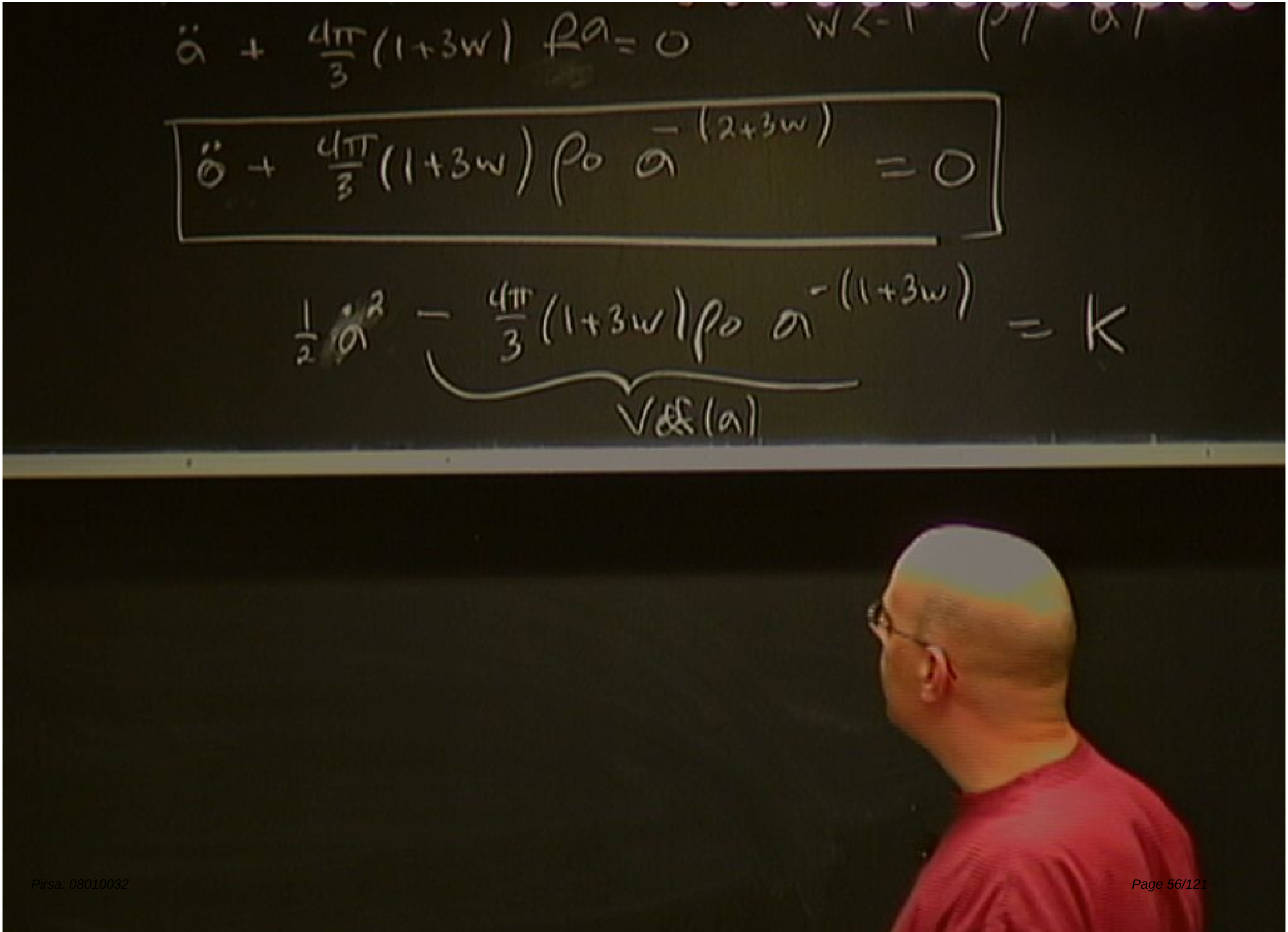

$\dot\rho = -3(1+w)\rho\frac{\dot a}{a} \rightarrow \rho = \rho_0 a^{-3(1+w)}$
$w > -1 \quad \rho\downarrow \quad a\uparrow$
$w = -1 \quad \rho = \rho_0$
$w < -1 \quad \rho\uparrow \quad a\uparrow$
$\frac{3}{a}\ddot a - \frac{3}{a^2}\dot a^2 = -\frac{1}{3}\frac{9}{a^2}\dot a^2 - 4\pi(1+3w)\rho$
$\ddot a + \frac{4\pi}{3}(1+3w)\frac{\rho}{a} = 0$


$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \longrightarrow \rho = \rho_0 a^{-3(1+w)}$
$w > -1 \quad \rho \downarrow \; a \uparrow$
$w = -1 \quad \rho = \rho_0$
$w < -1 \quad \rho \uparrow \; a \uparrow$
$\frac{3}{a}\ddot{a} - \frac{3}{a^2}\dot{a}^2 = -\frac{1}{3}\frac{9}{a^2}\dot{a}^2 - 4\pi(1+3w)\rho$
$\ddot{a} + \frac{4\pi}{3}(1+3w)\rho a = 0$
$\ddot{a} + \frac{4\pi}{3}(1+3w)\rho_0 a^{-1}$


$\dot\rho = -3(1+w)\rho\frac{\dot a}{a} \longrightarrow \rho = \rho_0 a^{-3(1+w)}$
$w > -1 \quad \rho\downarrow \; a\uparrow$
$\frac{3}{a}\ddot a - \frac{3}{a^2}\dot a^2 = -\frac{1}{3}\frac{9}{a^2}\dot a^2 - 4\pi(1+3w)\rho$
$w = -1 \quad \rho = \rho_0$
$w < -1 \quad \rho\uparrow \; a\uparrow$
$\ddot a + \frac{4\pi}{3}(1+3w)\rho a = 0$
$\ddot a + \frac{4\pi}{3}(1+3w)\rho_0 a^{-(2+3w)} = 0$



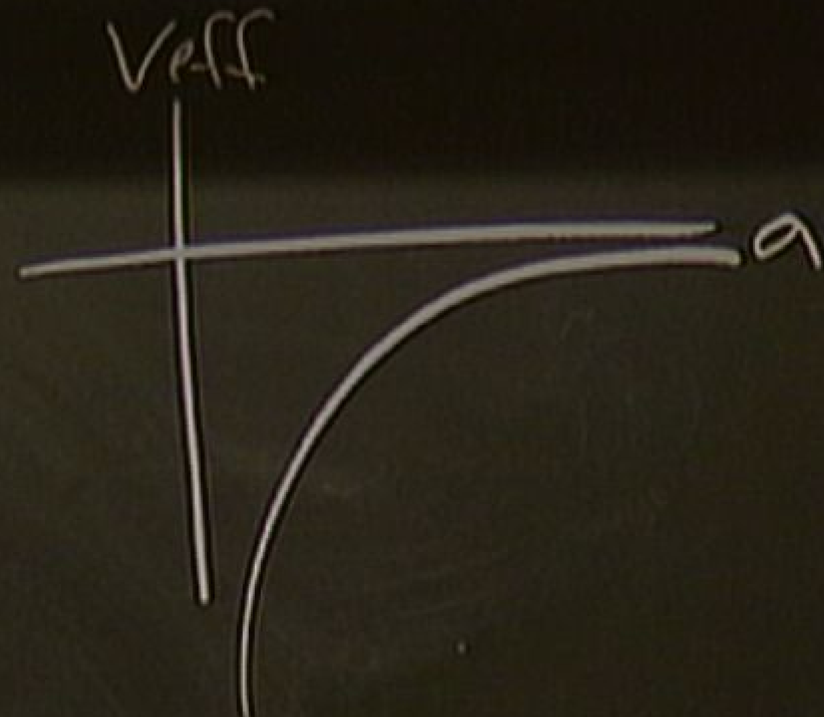
$\frac{3}{a}\ddot{a} - \frac{3}{a^2}\dot{a}^2 = -\frac{1}{3}\frac{9}{a^2}\dot{a}^2 - 4\pi(1+3w)\rho$
$w = -1$ $\rho = \rho_0$
$\ddot{a} + \frac{4\pi}{3}(1+3w)\rho a = 0$
$w < -1$ $\rho \uparrow$ $a \uparrow$
$\ddot{a} + \frac{4\pi}{3}(1+3w)\rho_0 a^{-(2+3w)} = 0$
$\frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}(1+3w)\rho_0 a^{-(1+3w)} = K$

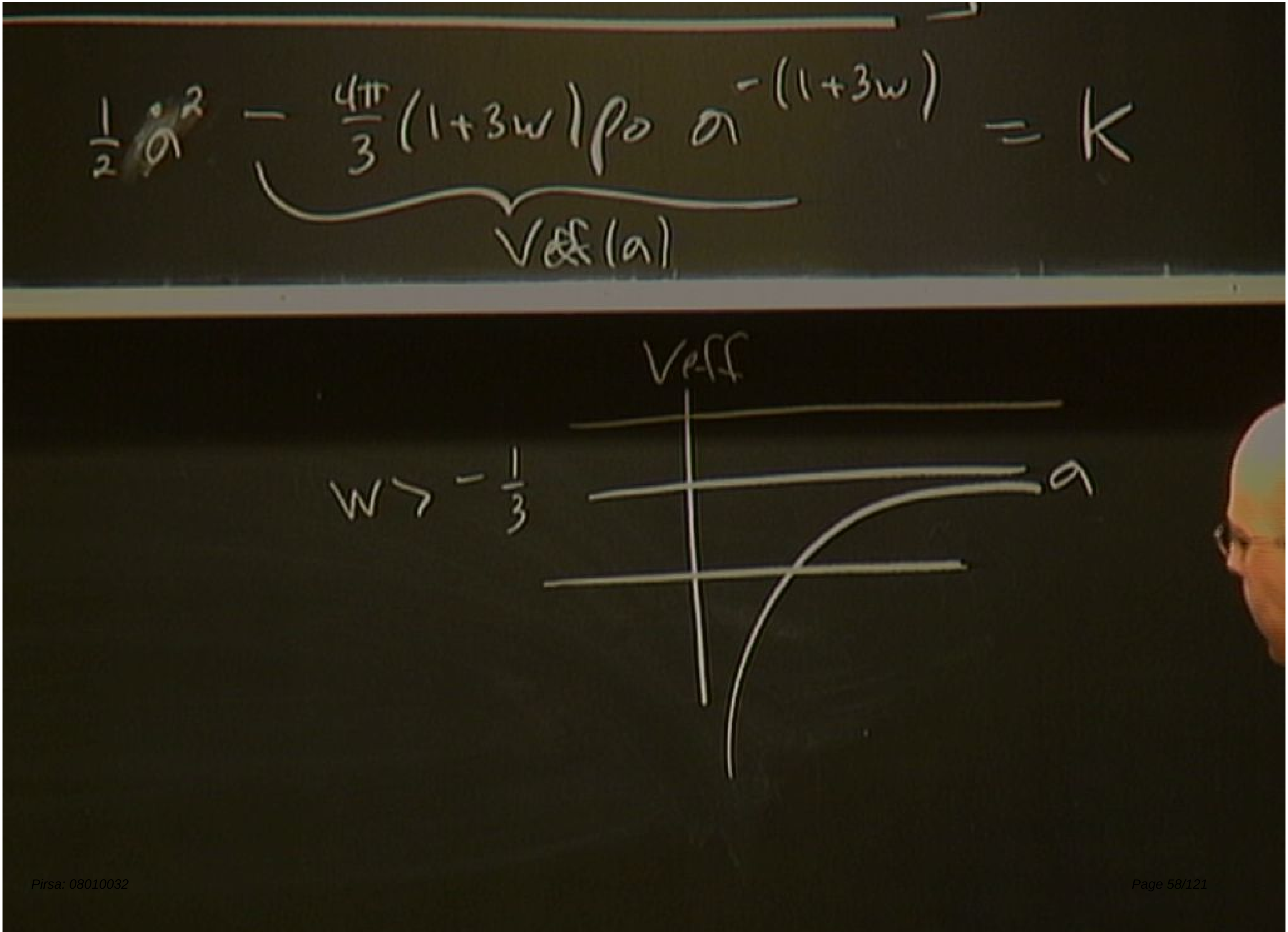


ä + 4π/3 (1+3w) ρa = 0
w < -1
ä + 4π/3 (1+3w) ρo a^-(2+3w) = 0
1/2 ȧ^2 − 4π/3 (1+3w) ρo a^-(1+3w) = K
Veff(a)

$$\frac{1}{2}\dot{a}^2 \underbrace{- \frac{4\pi}{3}(1+3w)\rho_0 \, a^{-(1+3w)}}_{V_{eff}(a)} = K$$

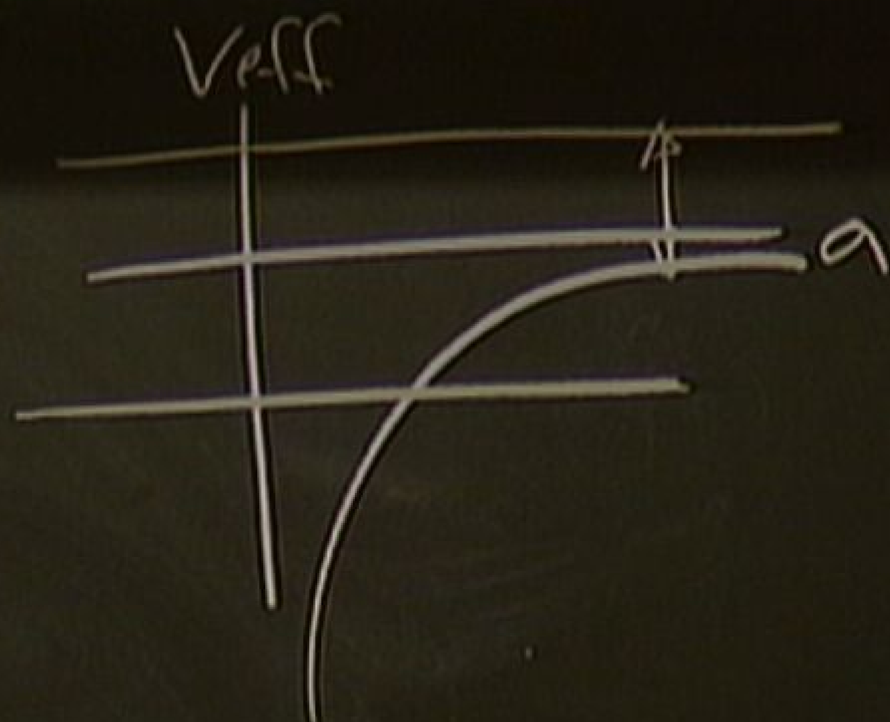
$w > -\frac{1}{3}$

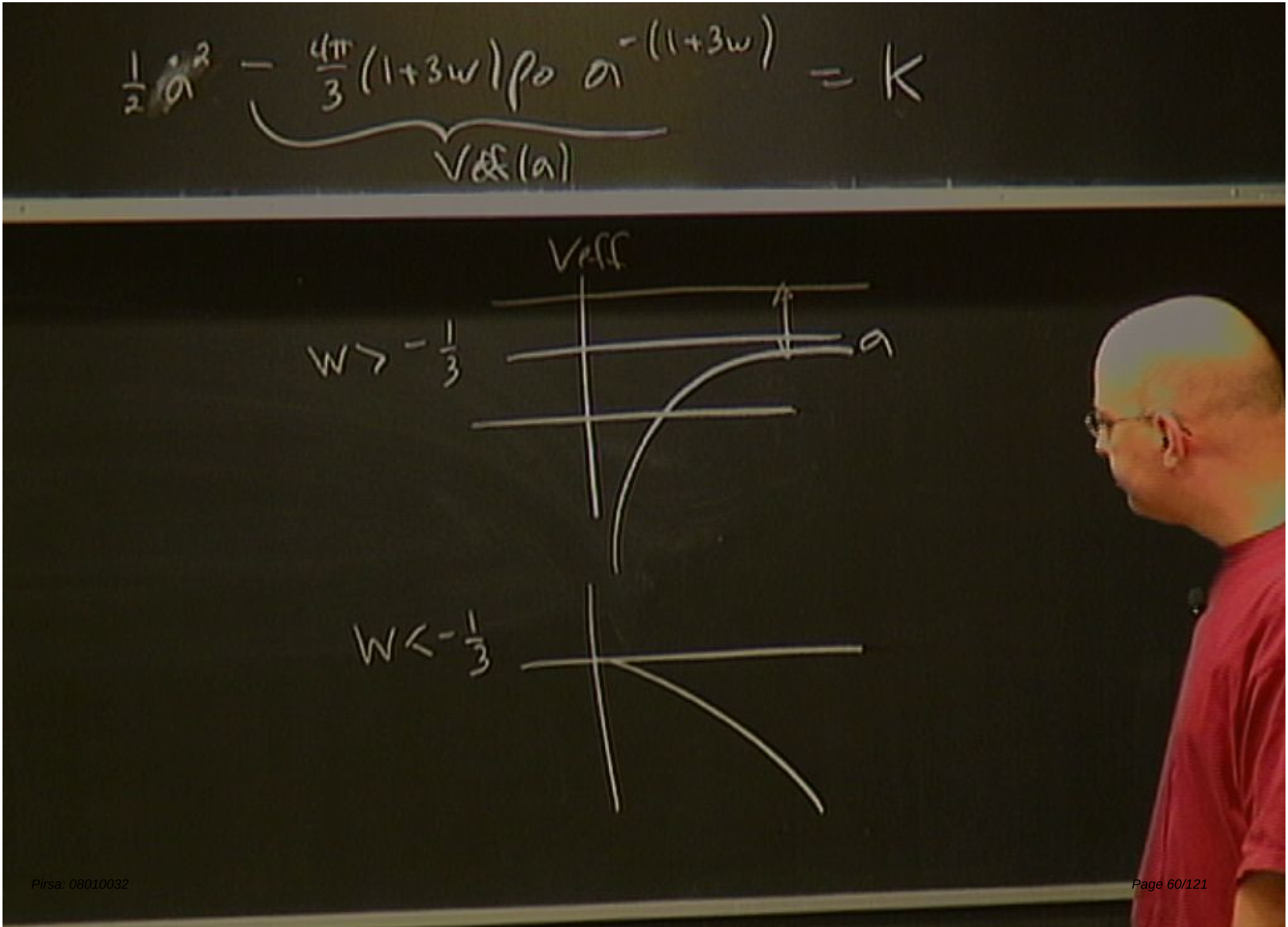



$\frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}(1+3w)\rho_0\, a^{-(1+3w)} = K$
Veff(a)
Veff
$w > -\frac{1}{3}$
a

$$\underbrace{\frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}(1+3w)\rho_0\, a^{-(1+3w)}}_{V_{eff}(a)} = K$$

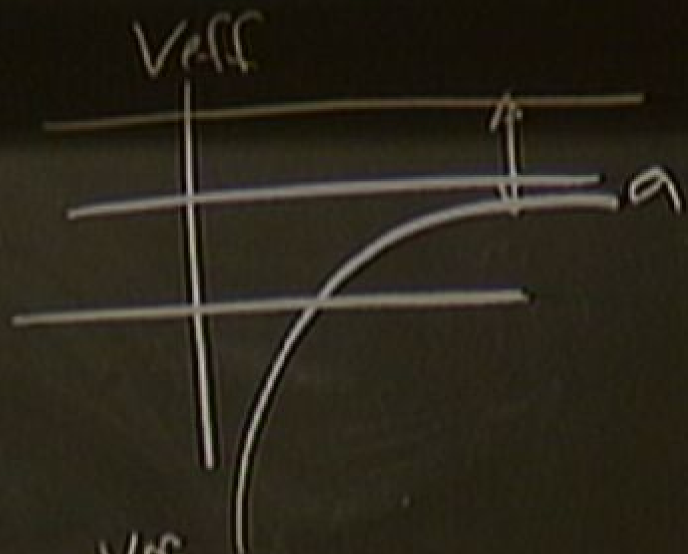
$w > -\frac{1}{3}$




1/2 ȧ² − 4π/3 (1+3w) ρ0 a^−(1+3w) = K
Veff(a)
Veff
w > −1/3
a
w < −1/3

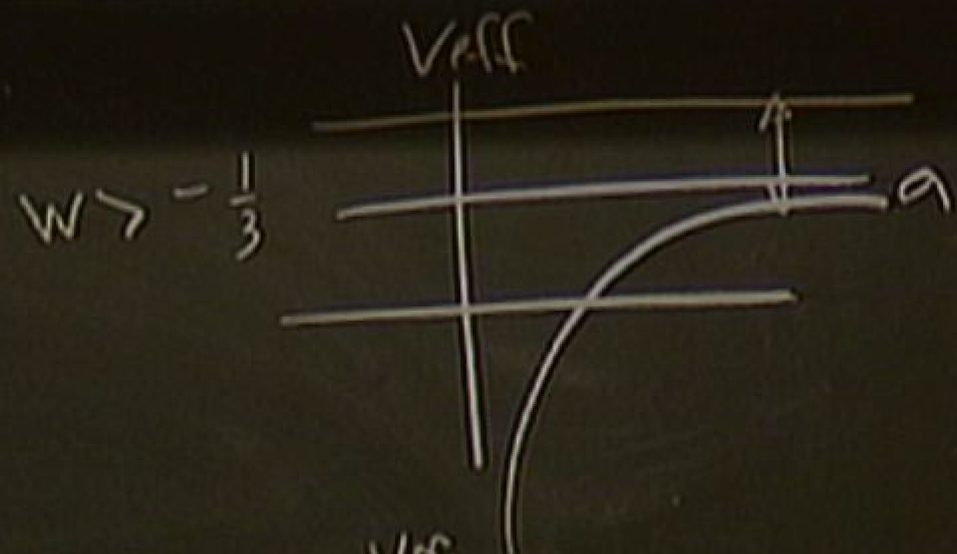
$$\frac{1}{2}\dot{a}^2 \underbrace{- \frac{4\pi}{3}(1+3w)\rho_0\, a^{-(1+3w)}}_{V_{eff}(a)} = K$$

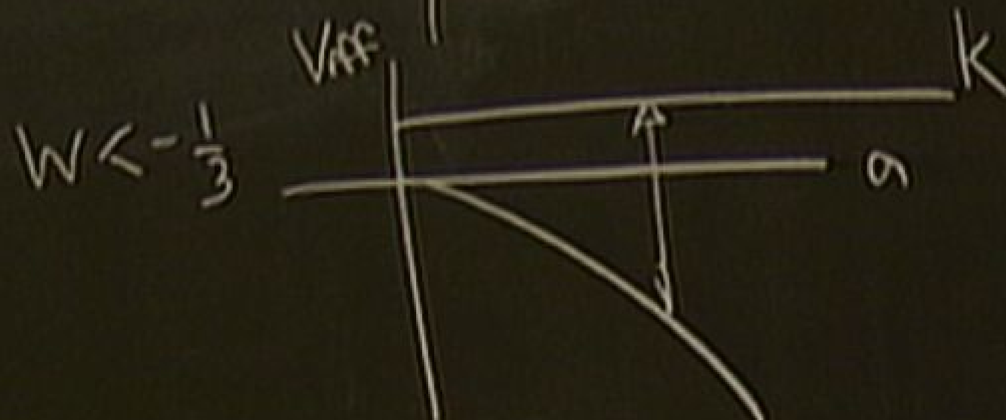
$w > -\frac{1}{3}$



$w < -\frac{1}{3}$

$$\frac{1}{2}\dot{a}^2 \underbrace{- \frac{4\pi}{3}(1+3w)\rho_0\, a^{-(1+3w)}}_{V_{eff}(a)} = K$$





$\rightarrow \delta V \dot{\rho} + (\rho + p)\, \dot{\delta V} = 0$

$d(\rho \delta V) + p\, d(\delta V) = 0 \quad ($

...mptions: $\sigma_{\alpha\beta} = 0$

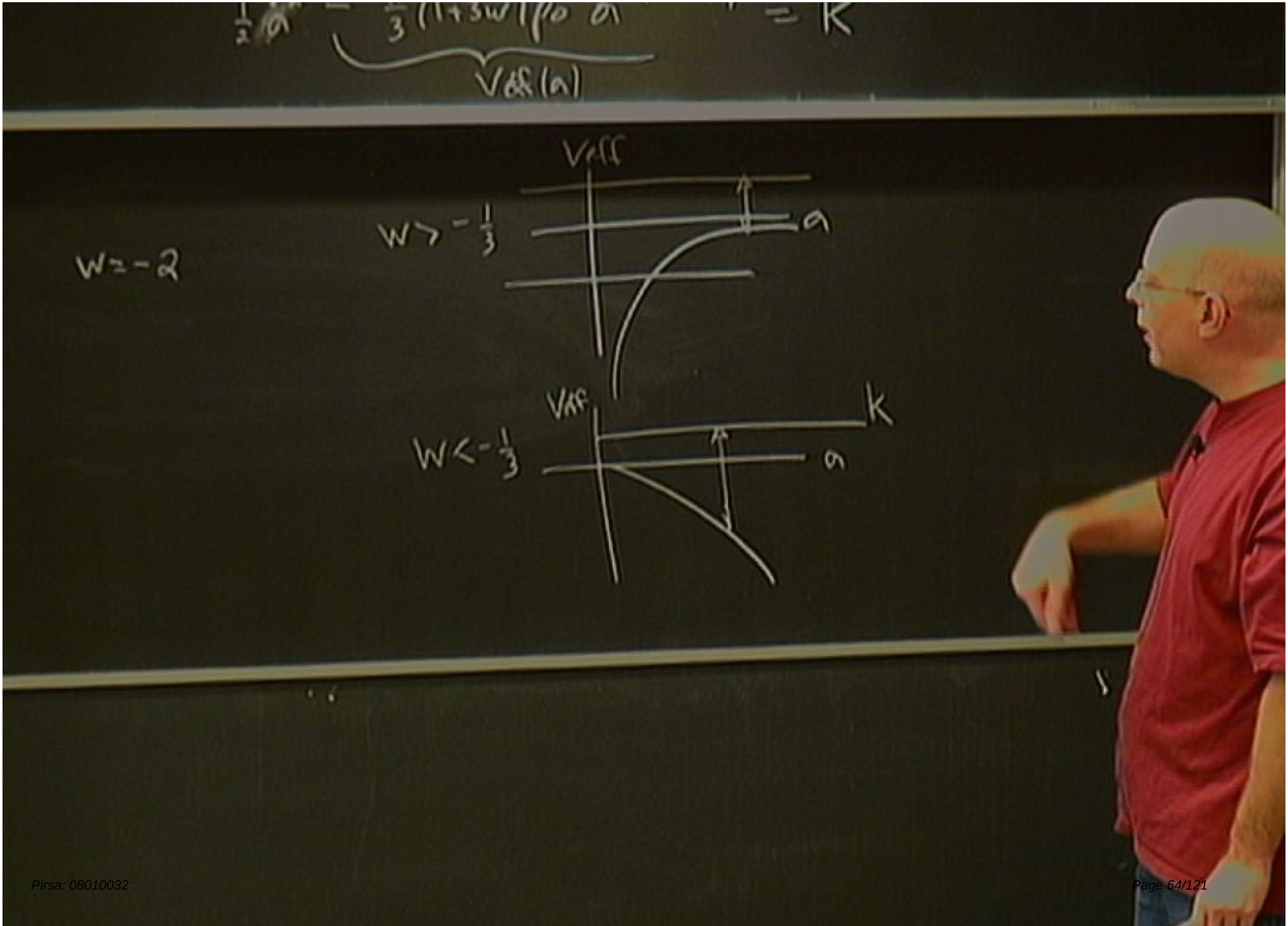
$\omega_{\alpha\beta} = 0$

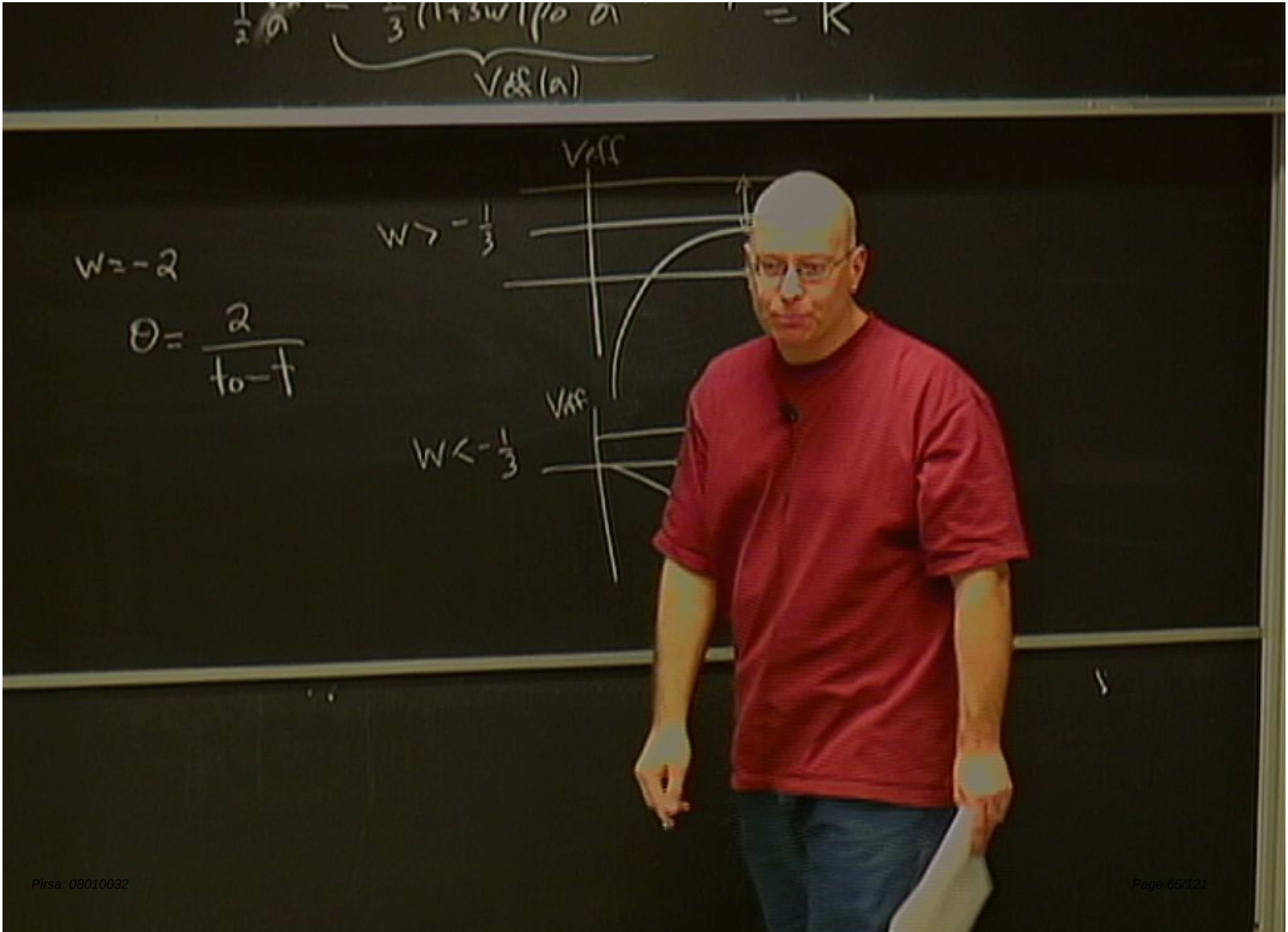
EFE $\qquad R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right)$

$\dot{\theta} = -\frac{1}{3}\theta^2 - 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) u^\alpha u^\beta$

$= -\frac{1}{3}\theta^2 - 8\pi \left(\rho + \frac{1}{2}(-\rho + 3p) \right)$

$\dot{\theta} = -\frac{1}{3}\theta^2 - 4\pi (1 + 3w)\rho$ $\qquad$ Define scale factor:


= K
Veff(a)
Veff
w > -1/3
a
w = -2
Veff
K
w < -1/3
a


= K
Veff(a)
Veff
w > -1/3
w = -2
θ = 2/(t0 - t)
Veff
w < -1/3

Big Rip:

$w \simeq -2$

$\theta = \dfrac{2}{t_0 - t}$

$w > -\frac{1}{3}$

$w < -\frac{1}{3}$

Example – timelike congruence

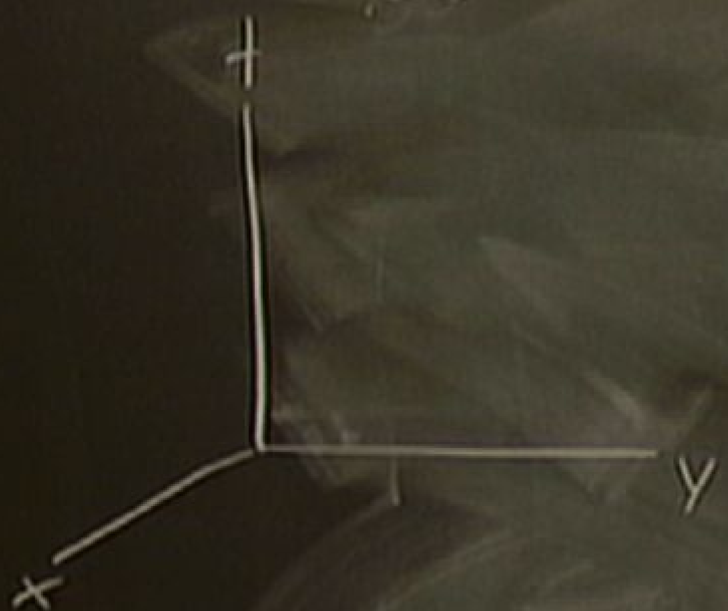
flat spacetime.

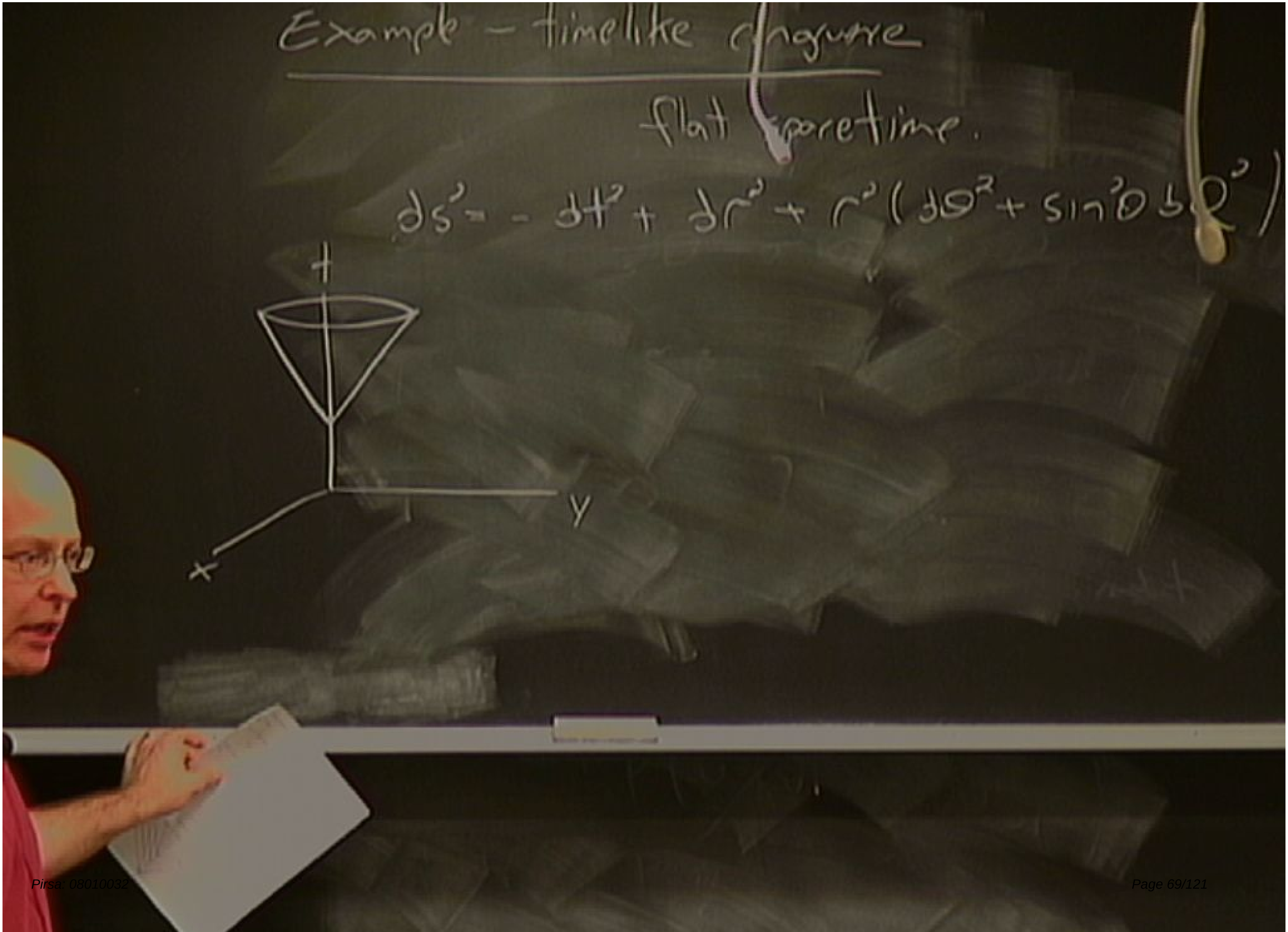
$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

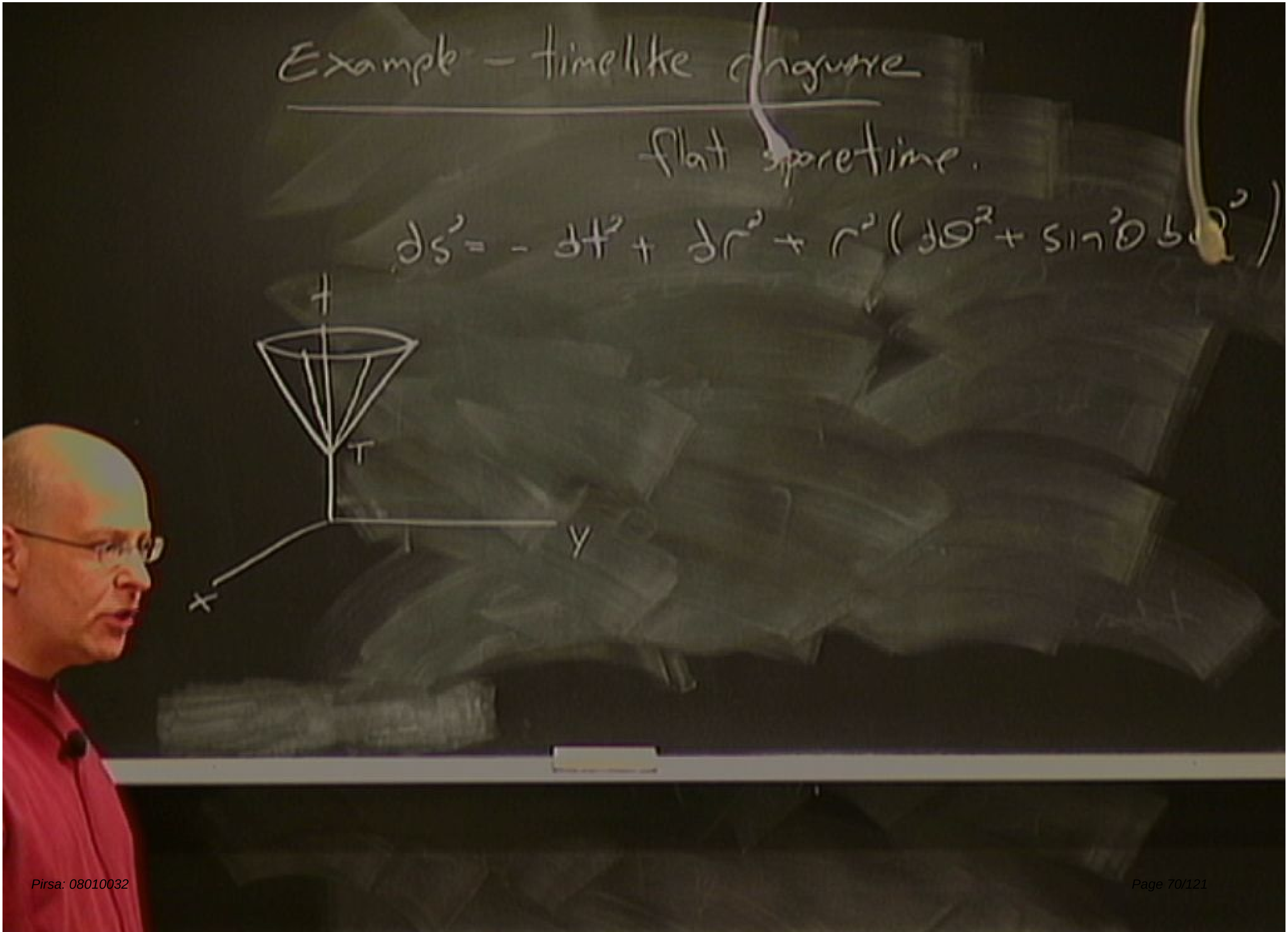
Example – timelike congruence

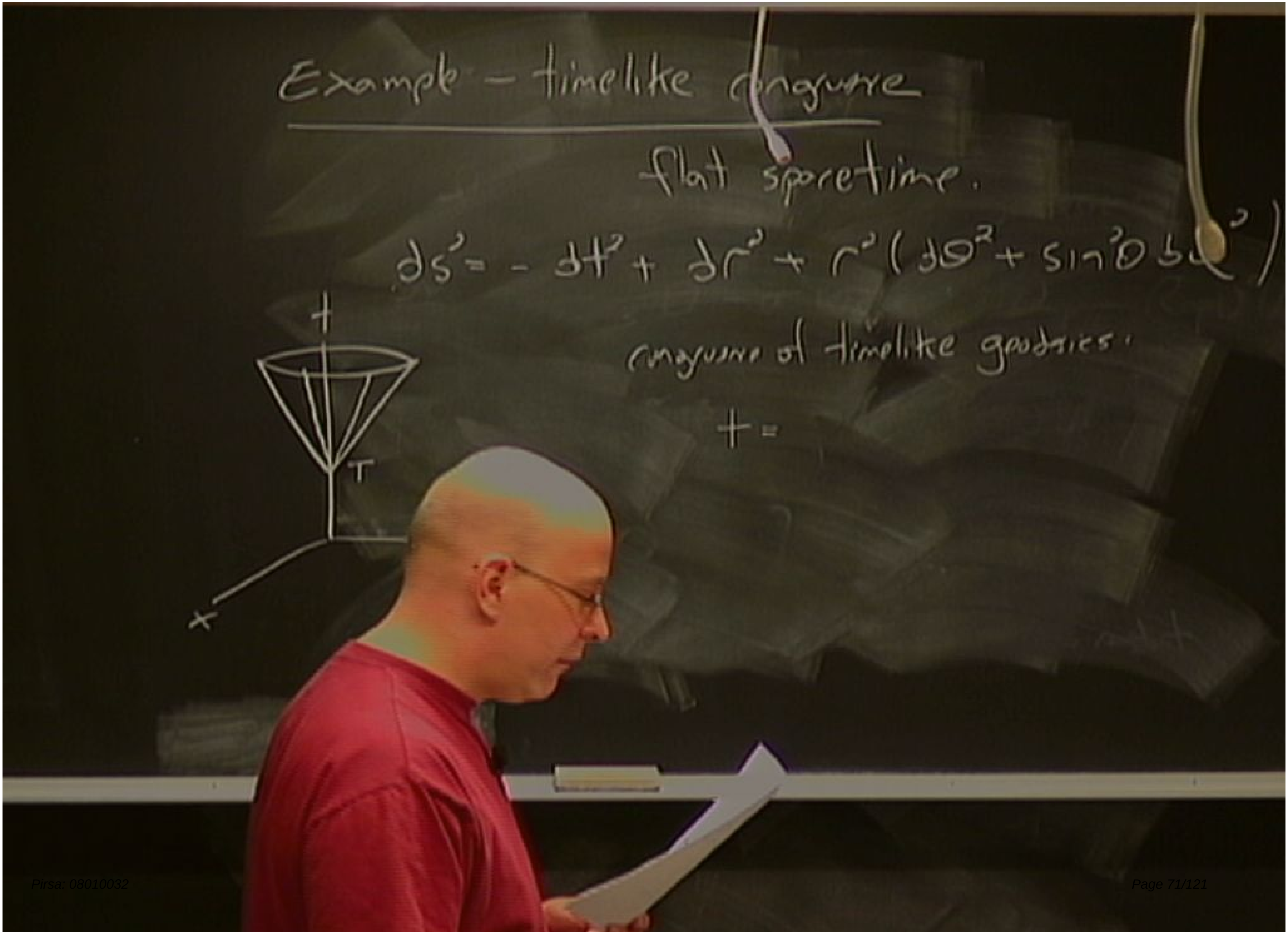
flat spacetime.

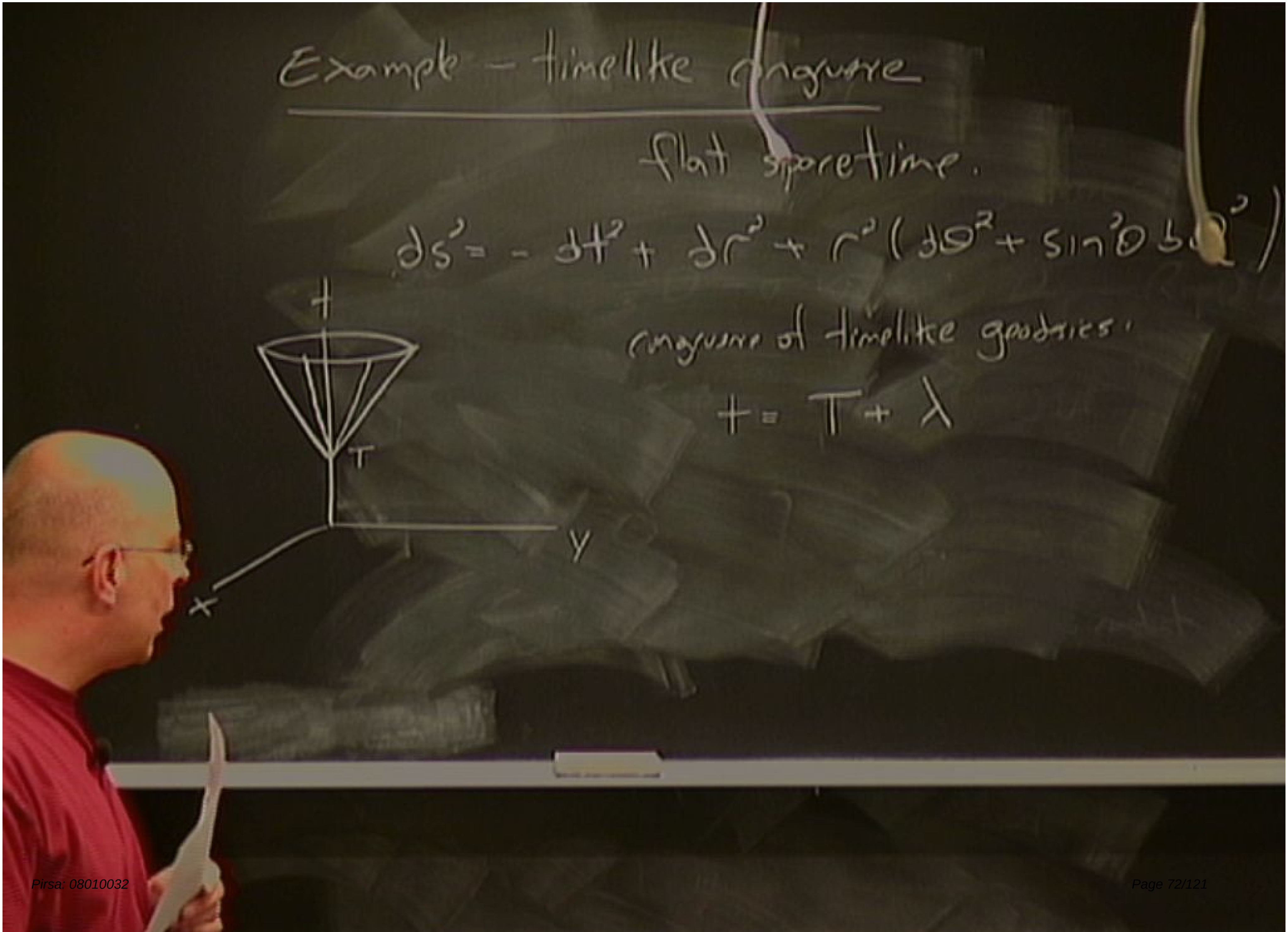
$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

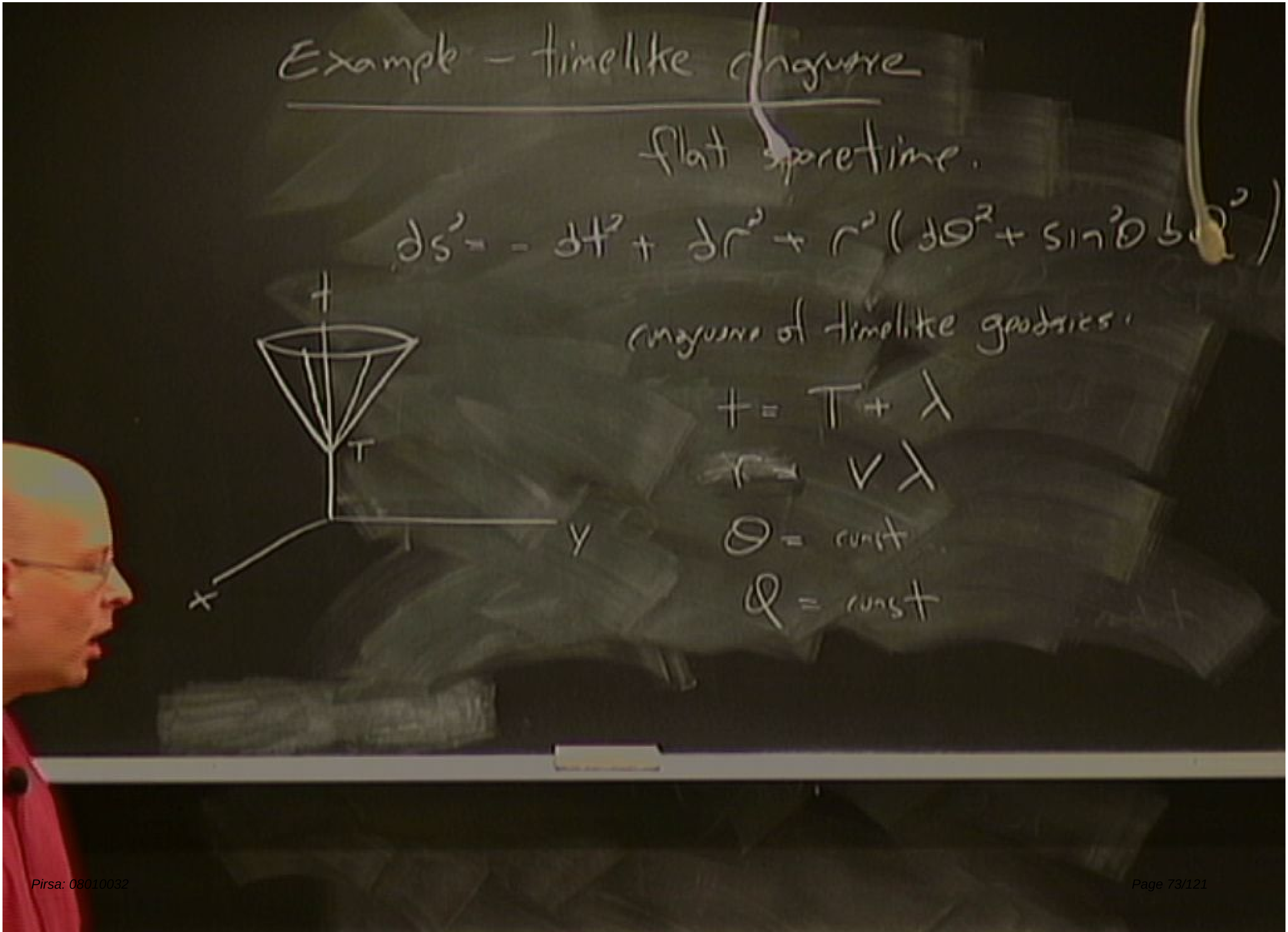


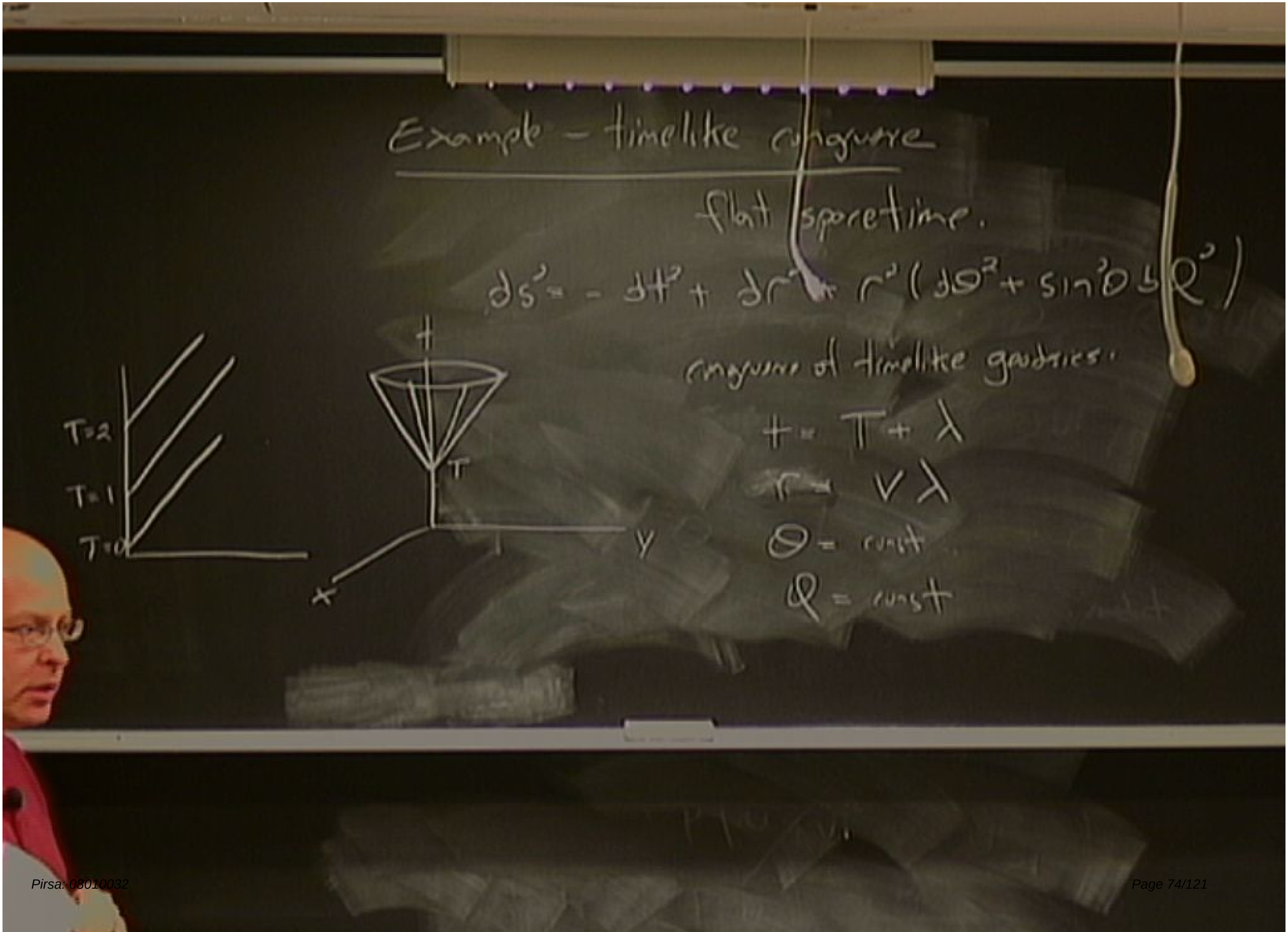

Example – timelike congruence
flat spacetime.
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
t
y
x

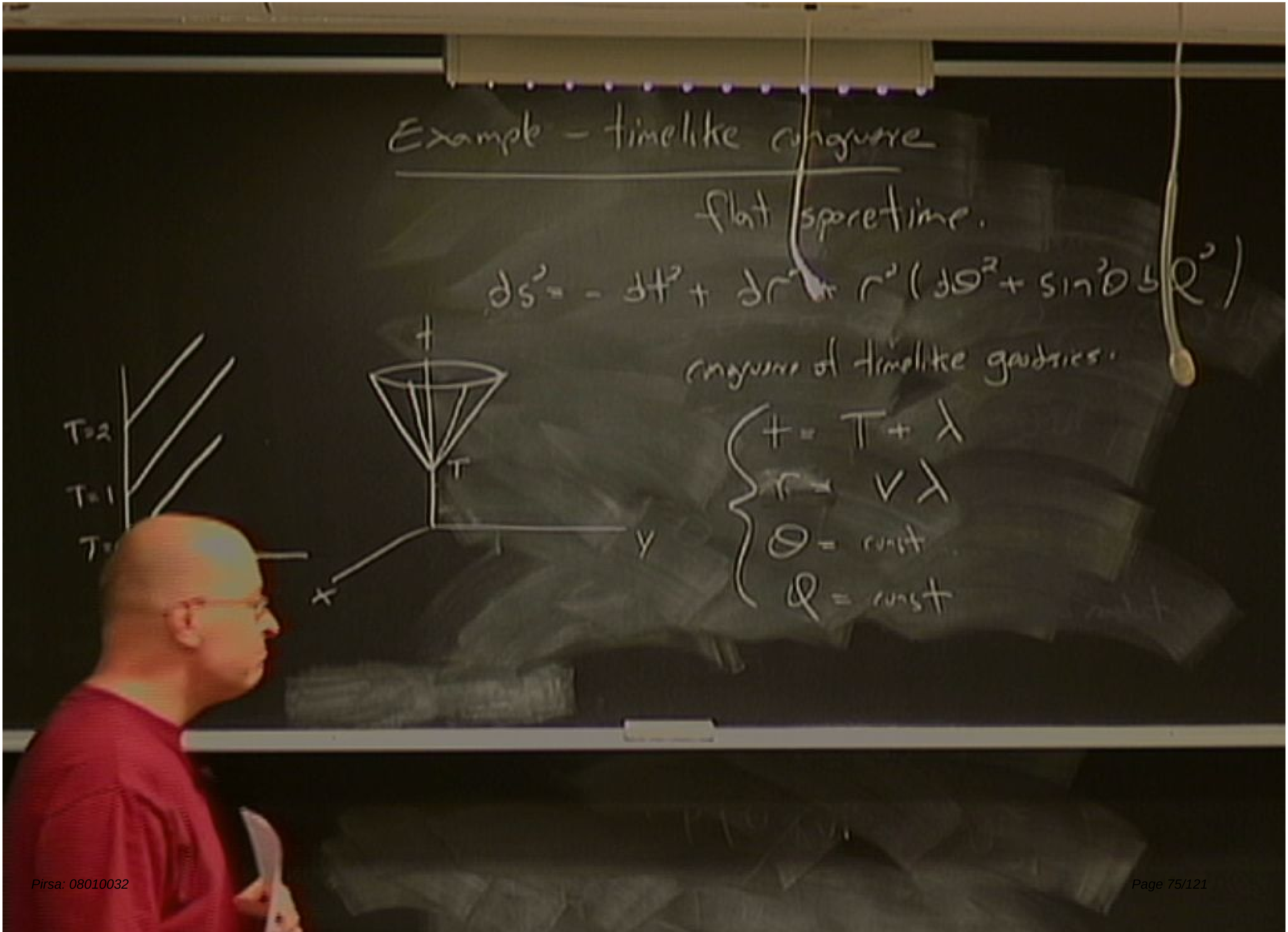

Example – timelike
flat spacetime.
ds² = – dt² + dr² + r²(dθ² + sin²θ dφ²)
t
T
y
x


Example - timelike congruence
flat spacetime.
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
congruence of timelike geodesics.
t =
t
T
x


Example - timelike congruence
flat spacetime.
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
t
congruence of timelike geodesics:
$t = T + \lambda$
T
y
x


Example – timelike congruence
flat spacetime.
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
t
T
y
x
congruence of timelike geodesics:
$t = T + \lambda$
$r = v\lambda$
$\theta = \text{const}$
$\varphi = \text{const}$


Example – timelike congruence
flat spacetime.
ds² = – dt² + dr² + r²(dθ² + sin²θ dφ²)
congruence of timelike geodesics.
t = T + λ
r = vλ
θ = const
φ = const
T=2
T=1
T=0
t
T
x
y



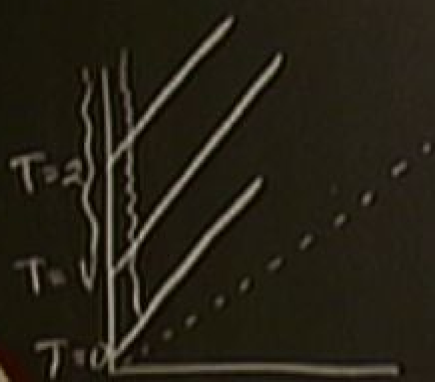
Example - timelike congruence
flat spacetime.
ds² = - dt² + dr² + r²(dθ² + sin²θ dφ²)
congruence of timelike geodesics.
t = T + λ
r = vλ
θ = const
φ = const
t
T
y
x
T=2
T=1

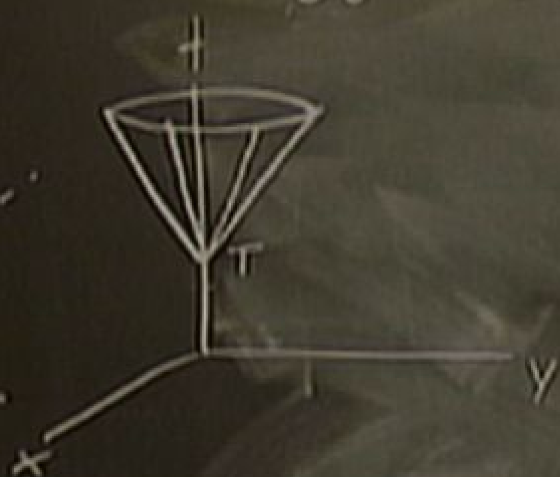
Example – timelike congruence

flat spacetime.

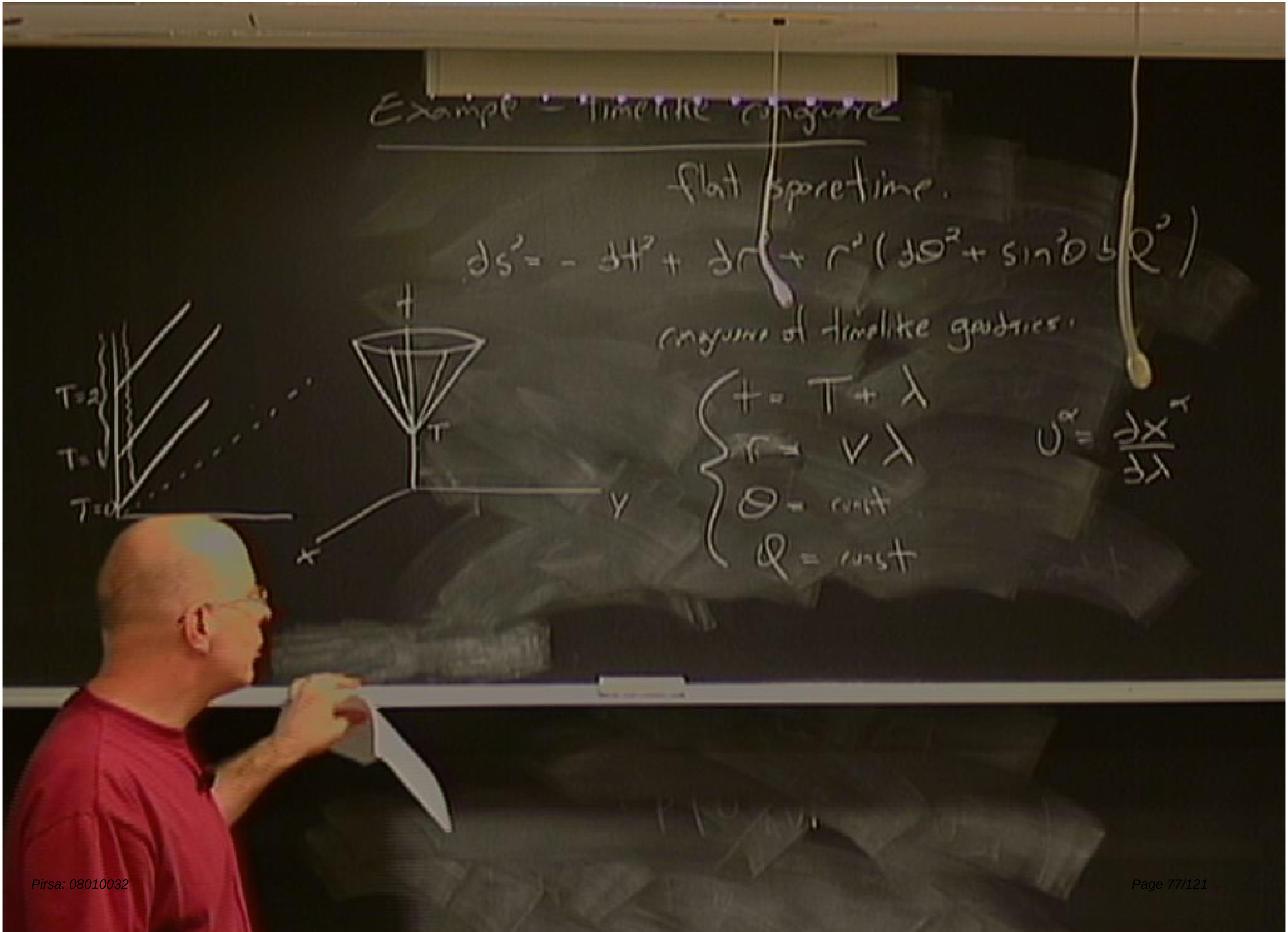
$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

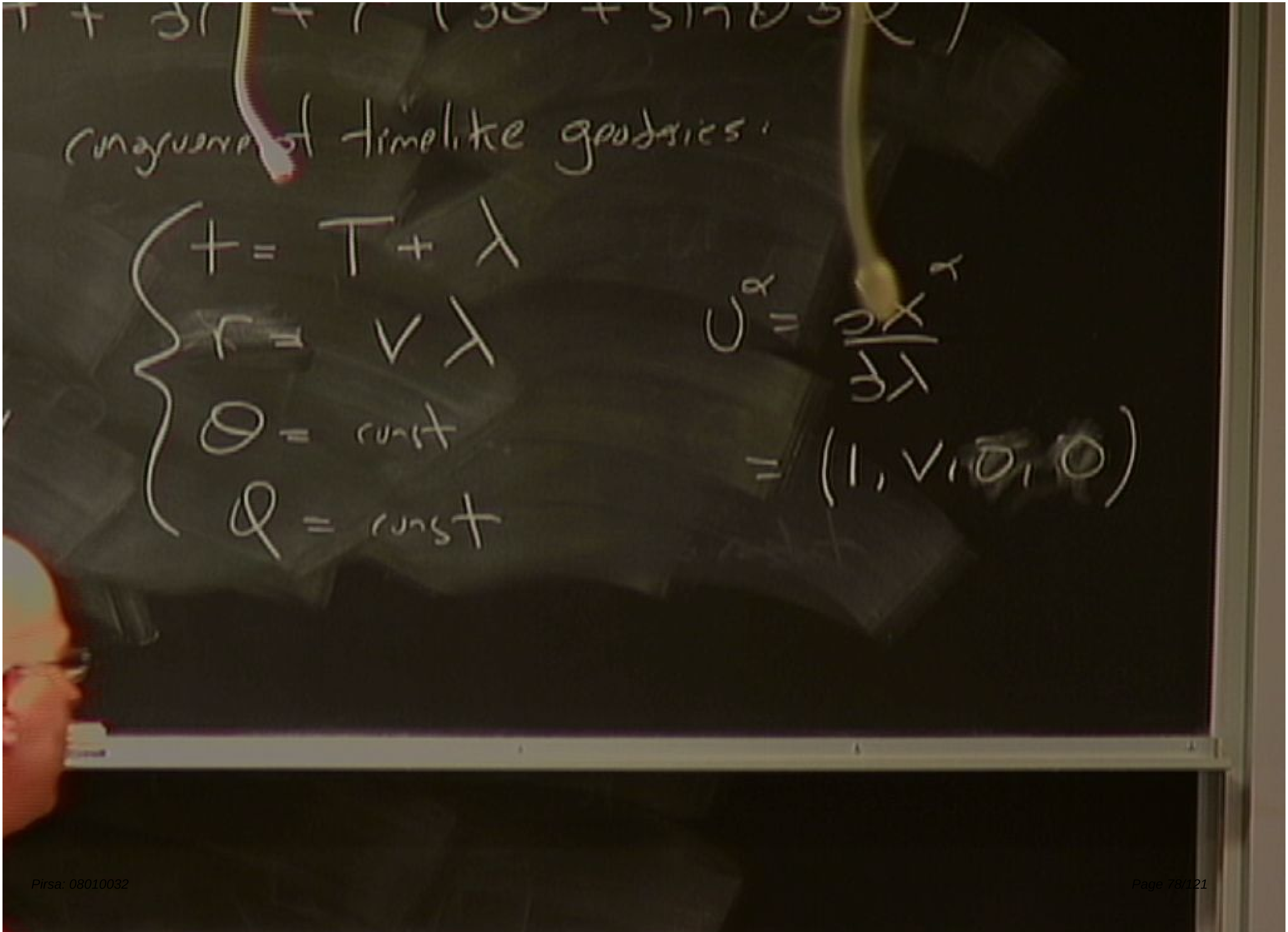
congruence of timelike geodesics.

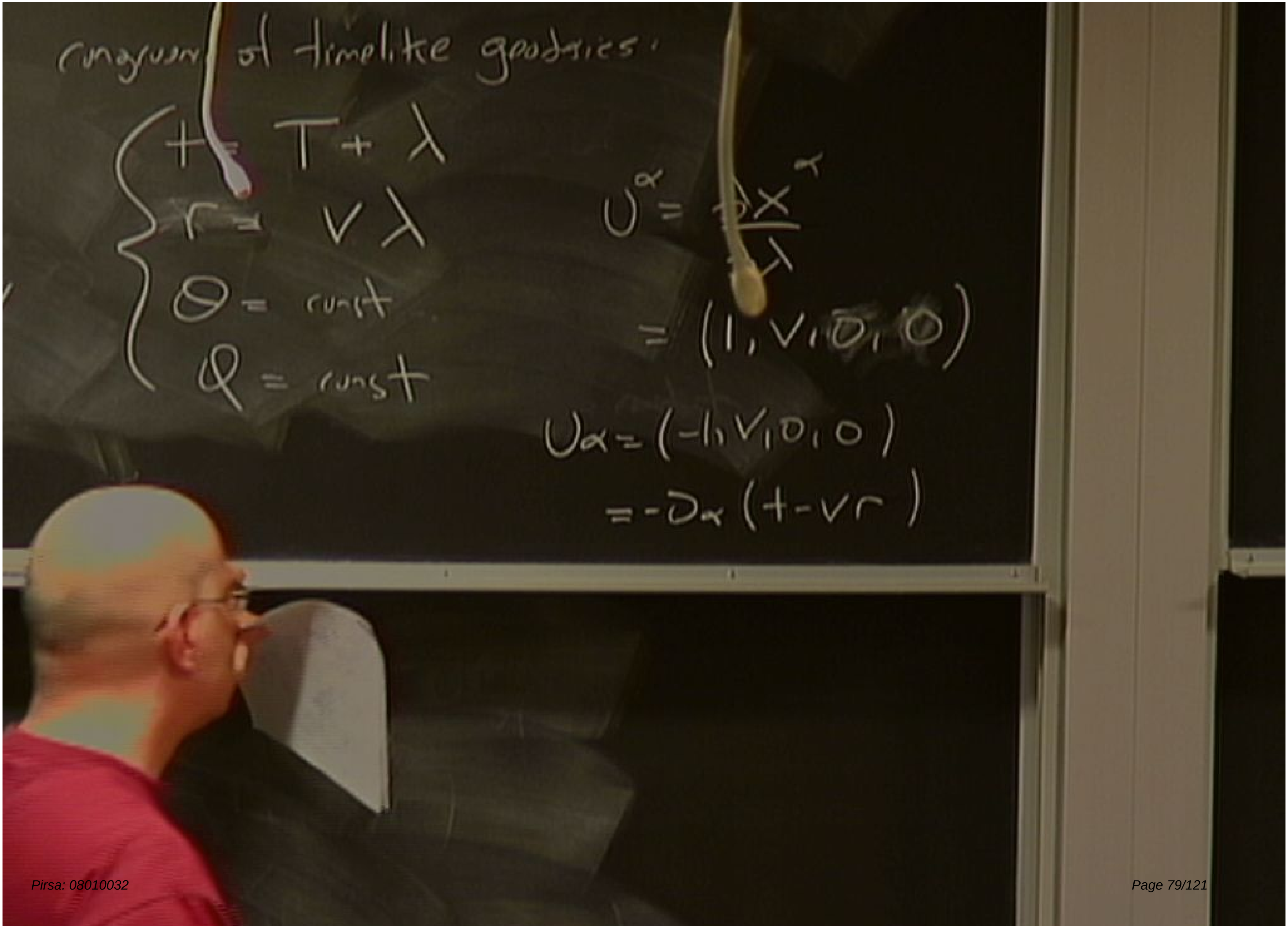


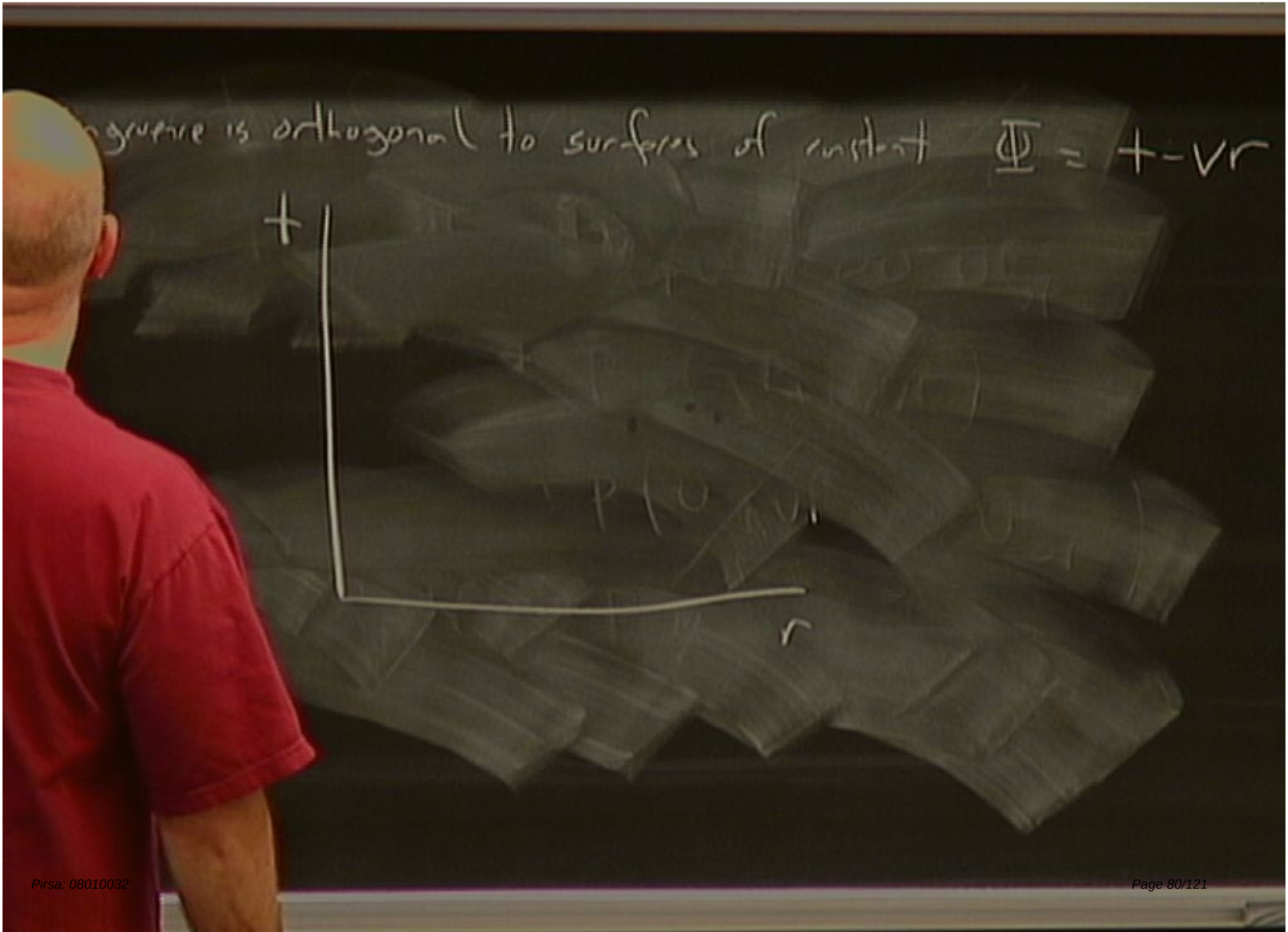


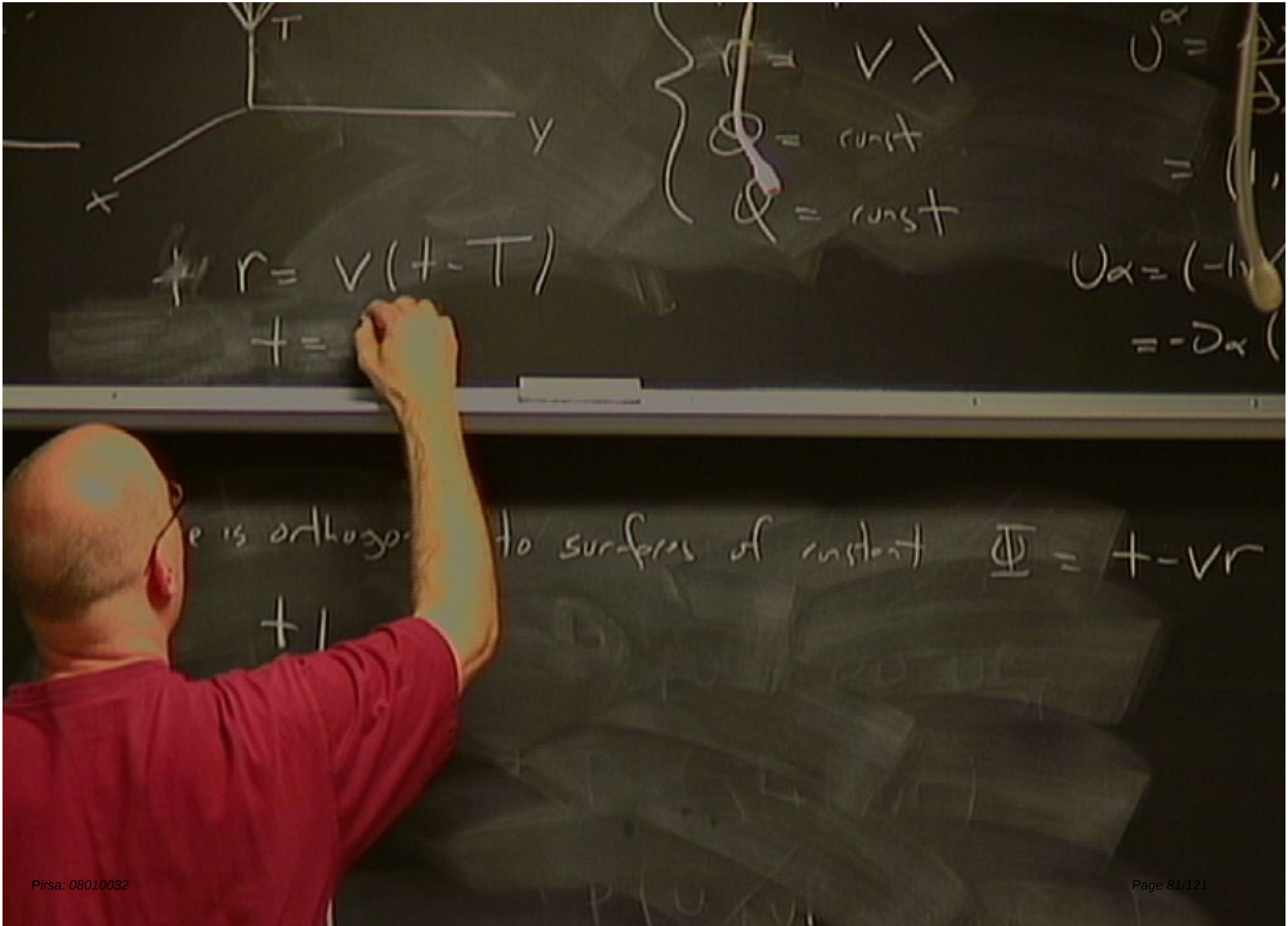
$$\begin{cases} t = T + \lambda \\ r = v\lambda \\ \theta = \text{const} \\ \varphi = \text{const} \end{cases}$$


Example - timelike congruence
flat spacetime.
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
congruence of timelike geodesics:
$t = T + \lambda$
$r = v\lambda$
$\theta = const$
$\varphi = const$
$U^\alpha = \frac{dx^\alpha}{d\lambda}$
T=2
T=1
T=0
t
T
y
x


+ dr² + r²(dθ² + sin²θ dφ²)
congruence of timelike geodesics:
t = T + λ
r = vλ
θ = const
φ = const
u^α = ∂x^α/∂λ
= (1, v, 0, 0)


of timelike geodesics:
t = T + λ
r = v λ
θ = const
φ = const
U^α = dx^α / dλ
= (1, v, 0, 0)
U_α = (-1, v, 0, 0)
= -∂_α (t - vr)


ngruence is orthogonal to surfaces of constant $\Phi = t - vr$
t
r


T
y
x
r = vλ
θ = const
φ = const
r = v(t - T)
t =
U^α =
Uα = (-1
= -∂α (
e is orthogo to surfaces of constant Φ = t - vr

$$\left\{ \begin{array}{l} r = v\lambda \\ \theta = \text{const} \\ \phi = \text{const} \end{array} \right.$$

$U^\alpha =$ …

$= (1,$ …

$r = v(t - T)$

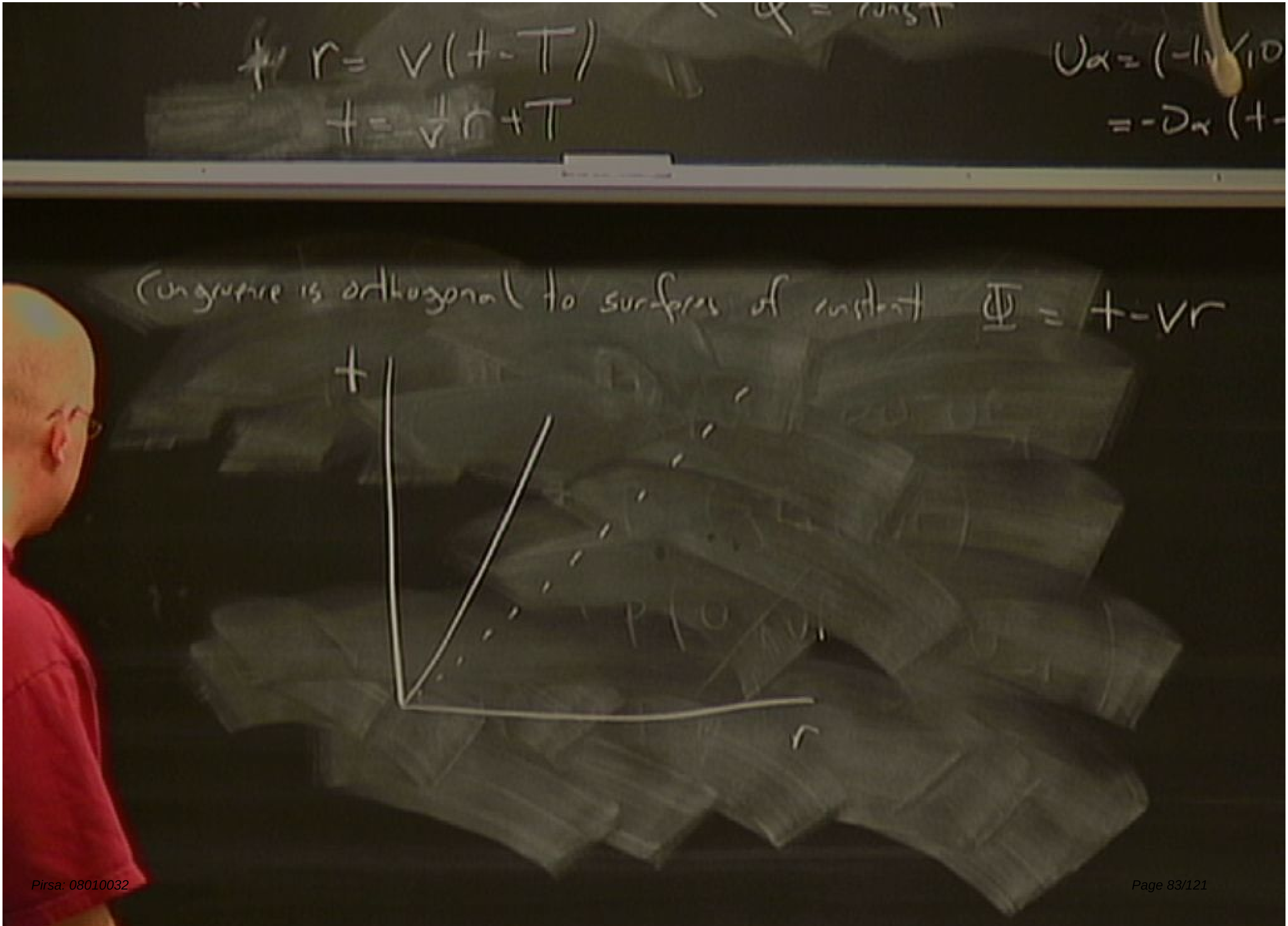
$t = \frac{1}{v} r + T$

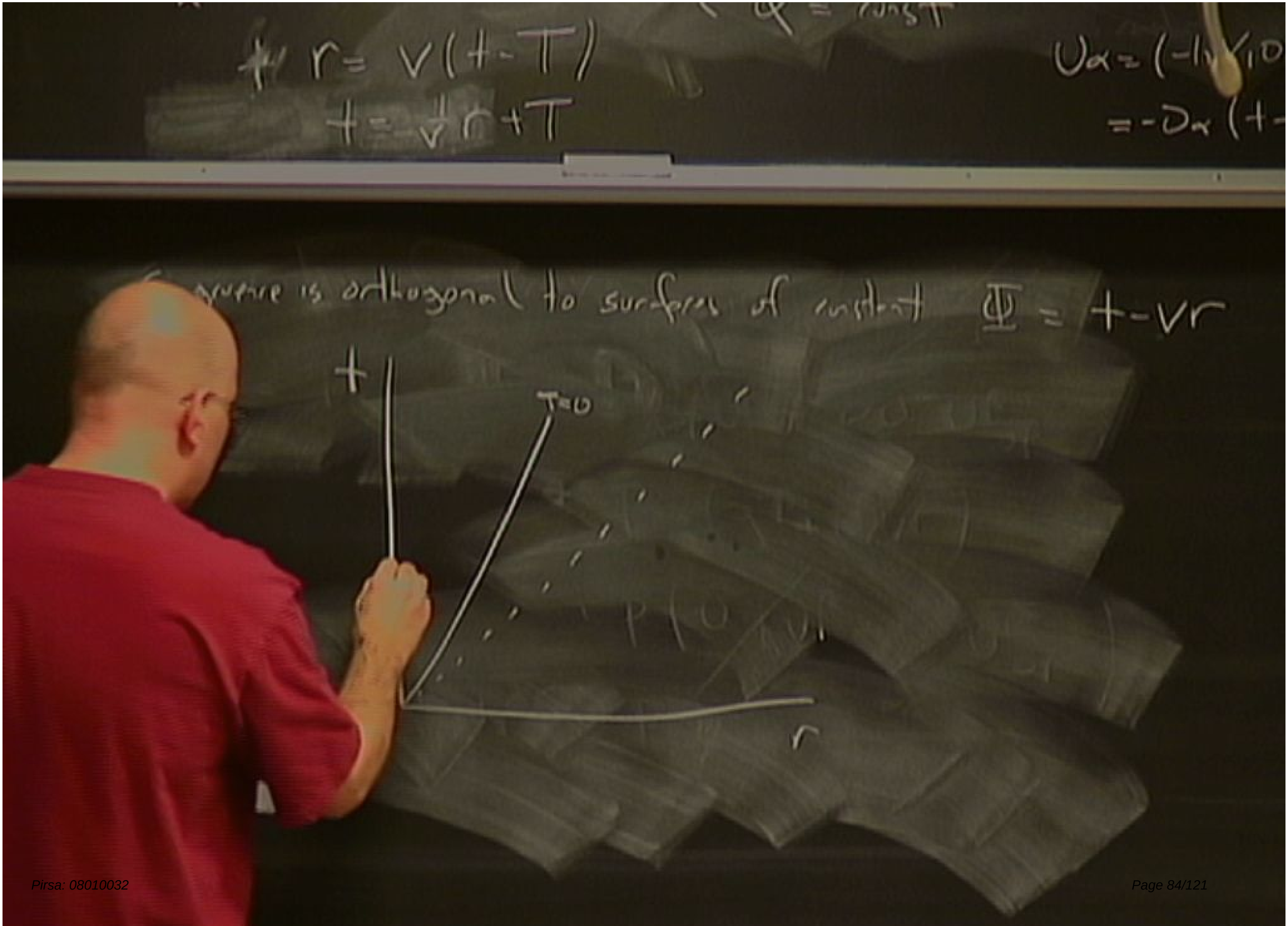
$U_\alpha = (-1,$ …

$= -\partial_\alpha ($ …

Congruence is orthogonal to surfaces of constant $\Phi = t - vr$

t


r = v(t - T)
t = (1/v) r + T
Congruence is orthogonal to surfaces of constant Φ = t - vr
t
r


r = v(t - T)
t = 1/v r + T
orthogonal to surfaces of constant Φ = t - vr
t
T=0
r

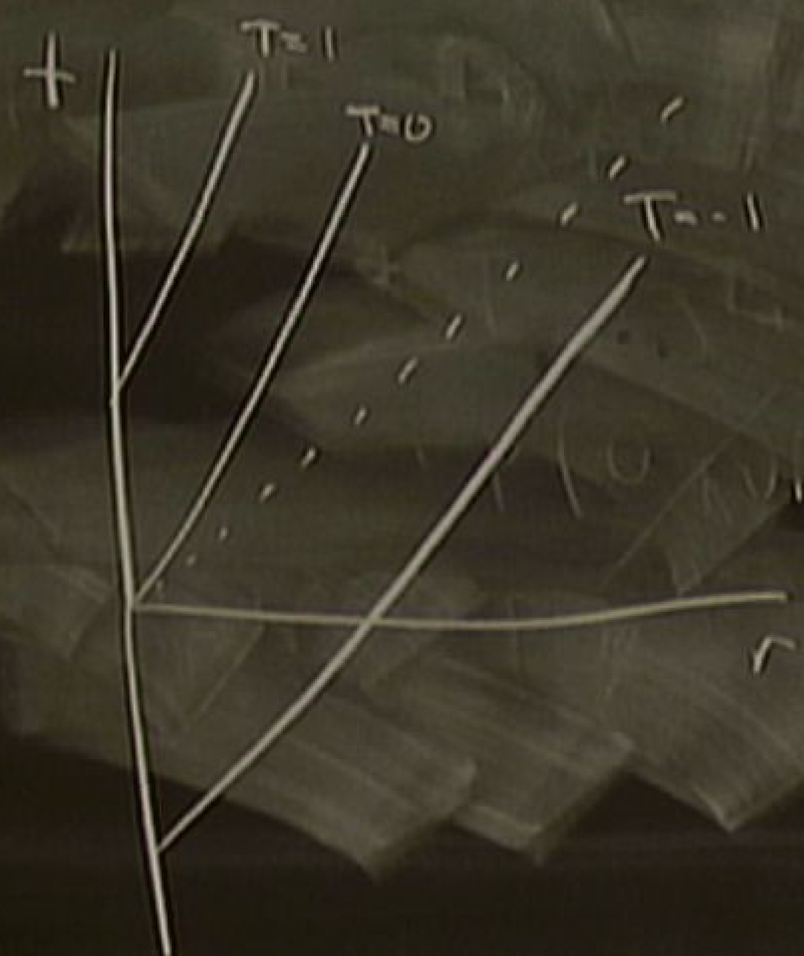
$$r = v(t - T)$$

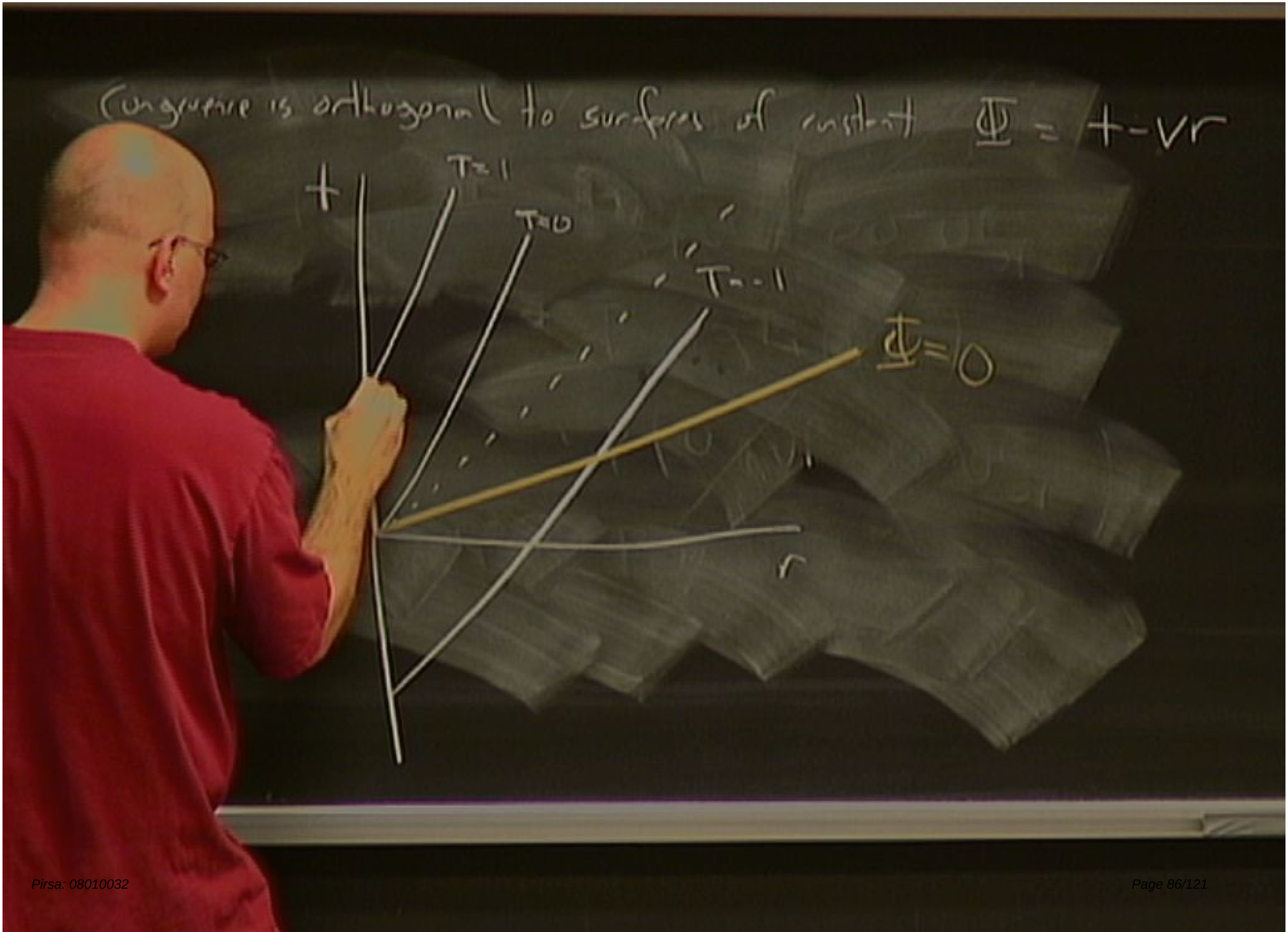
$$t = \frac{1}{v} r + T$$

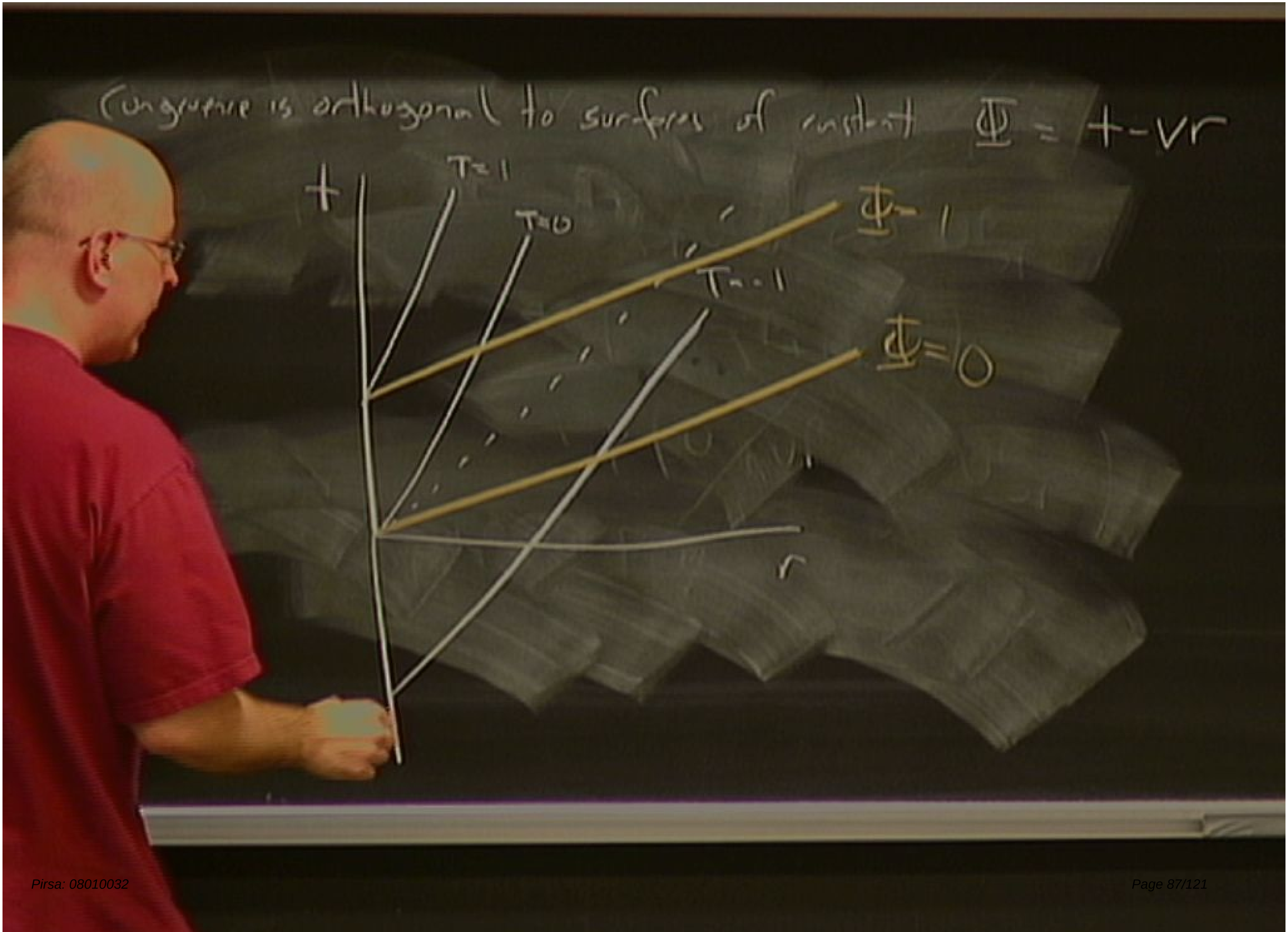
$U_\alpha = (-1, \ldots$

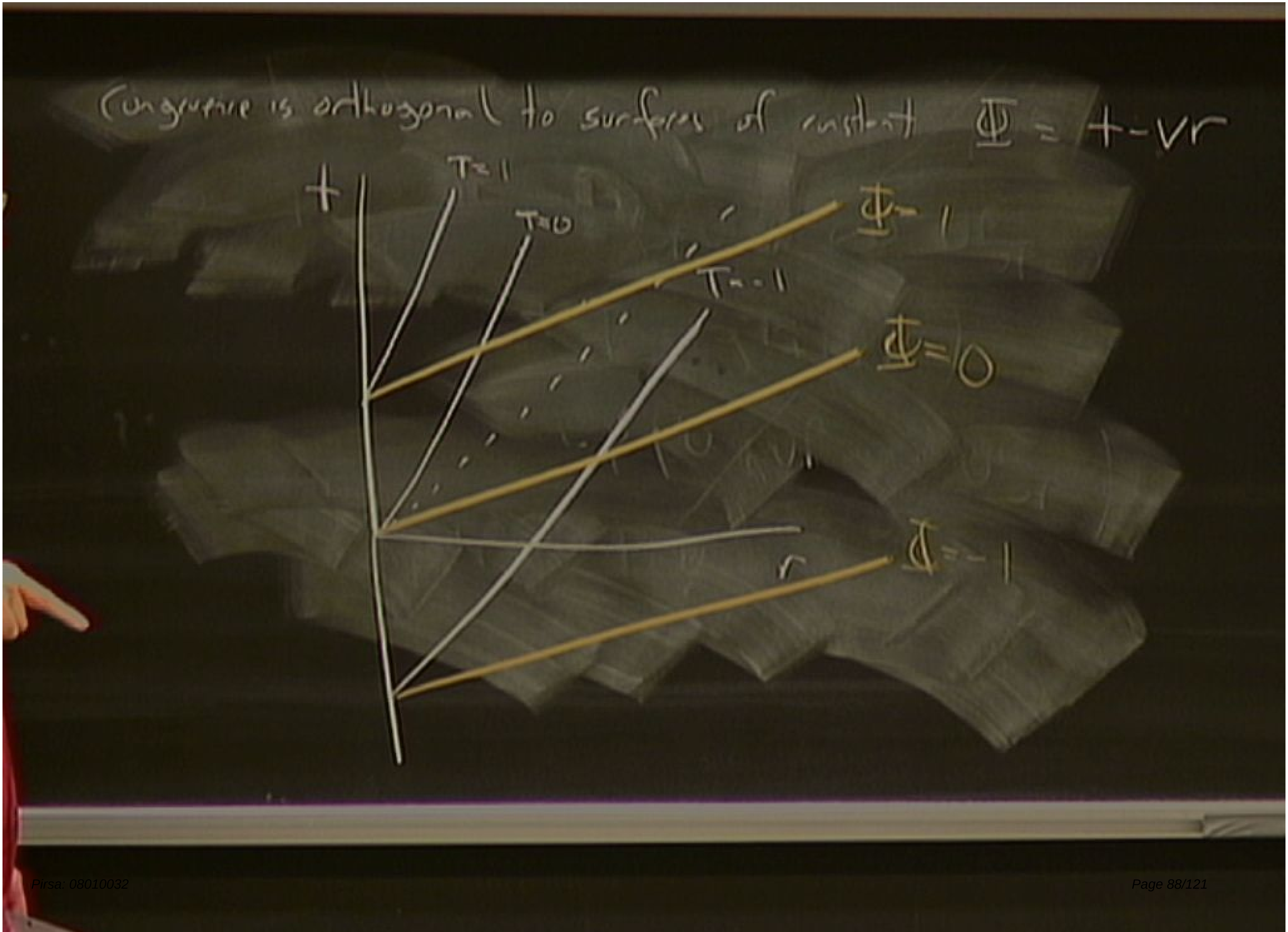
$= -\partial_\alpha (t - \ldots$

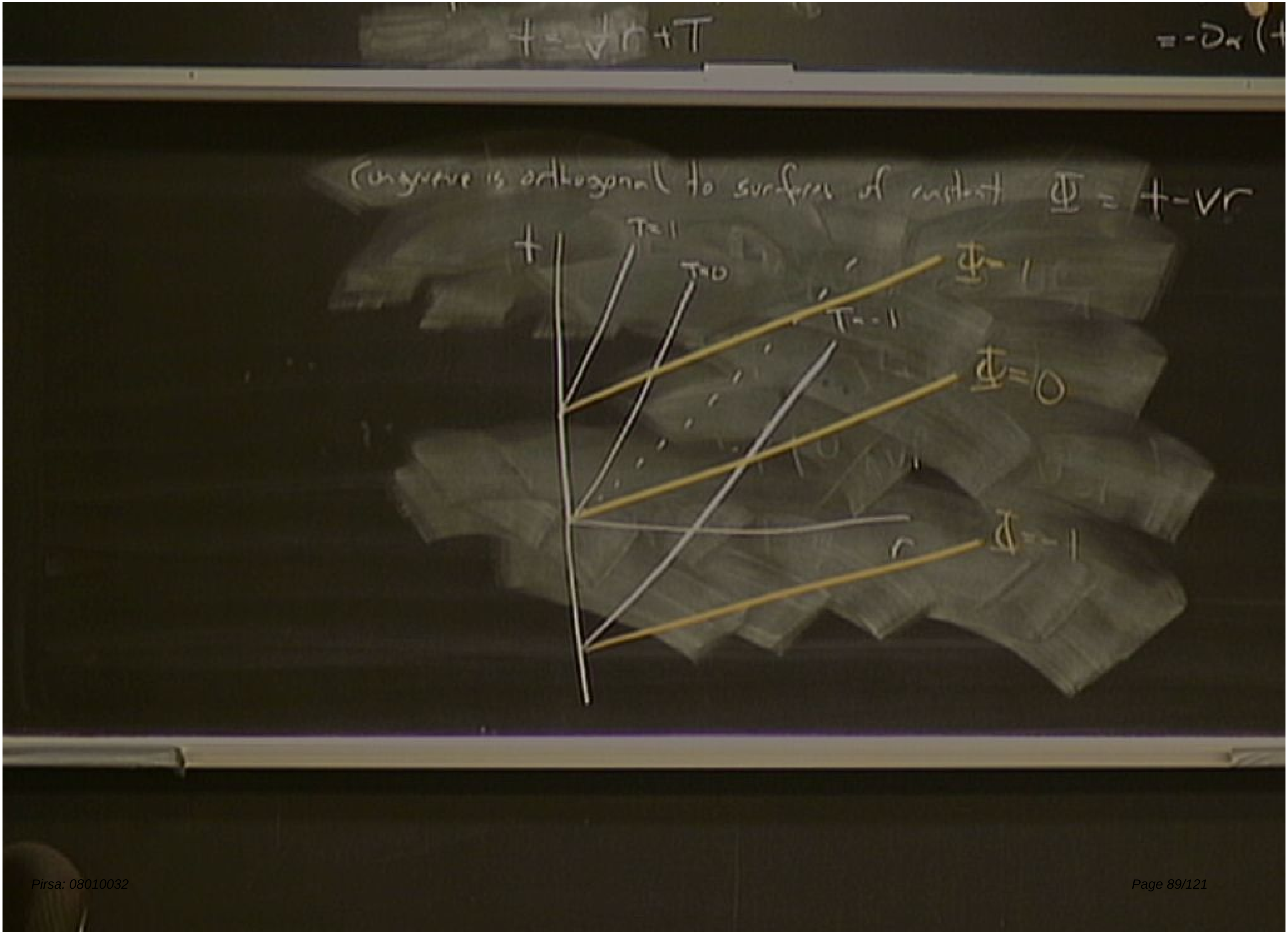
Congruence is orthogonal to surfaces of constant $\Phi = t - vr$

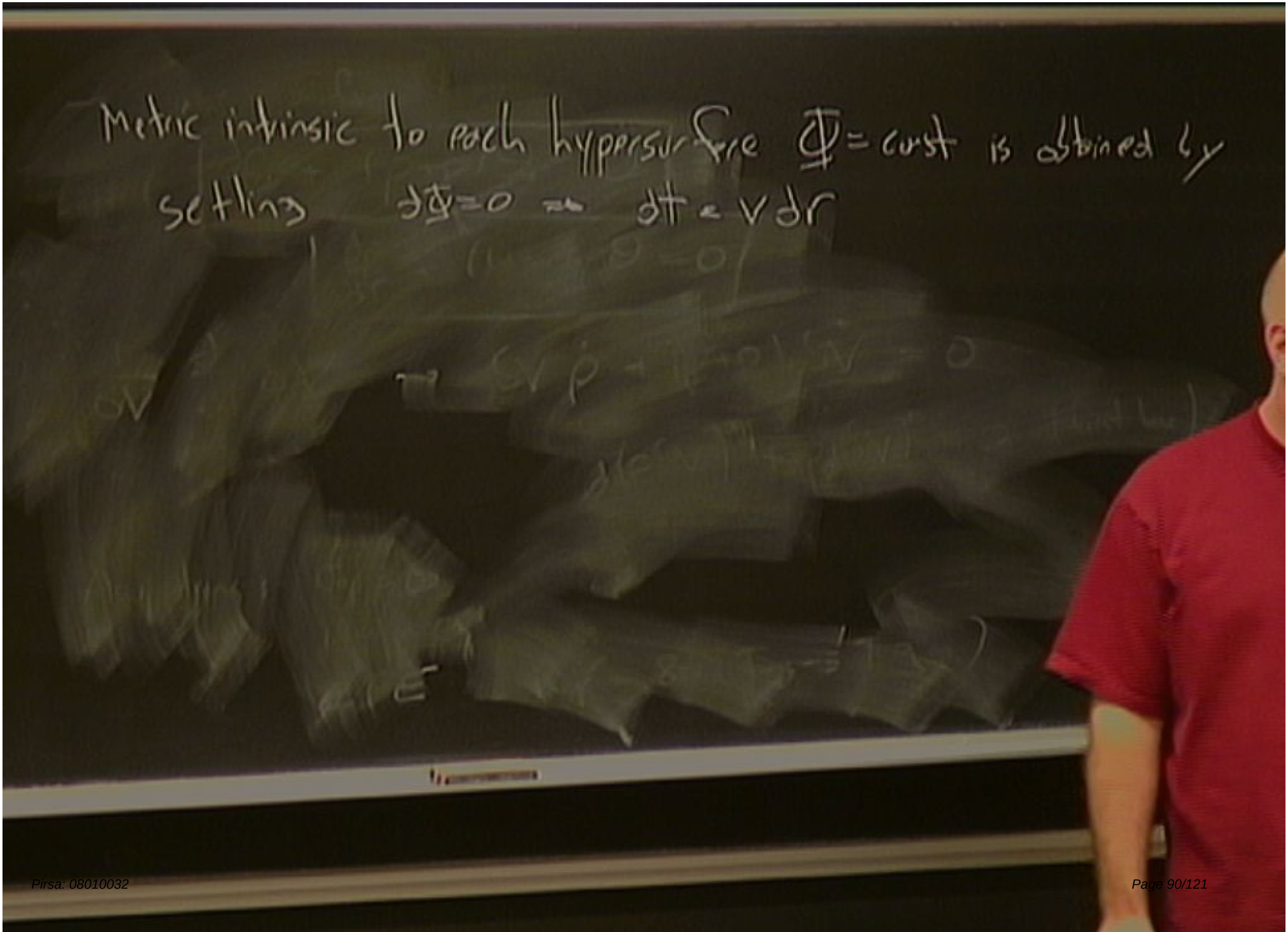



Congruence is orthogonal to surfaces of constant Φ = t - vr
t
T=1
T=0
T=-1
Φ=0
r


Congruence is orthogonal to surfaces of constant Φ = t - vr
t
T=1
T=0
T=-1
Φ=1
Φ=0
r


Congruence is orthogonal to surfaces of constant Φ = t-vr
t
T=1
T=0
T=-1
Φ=1
Φ=0
Φ=-1
r

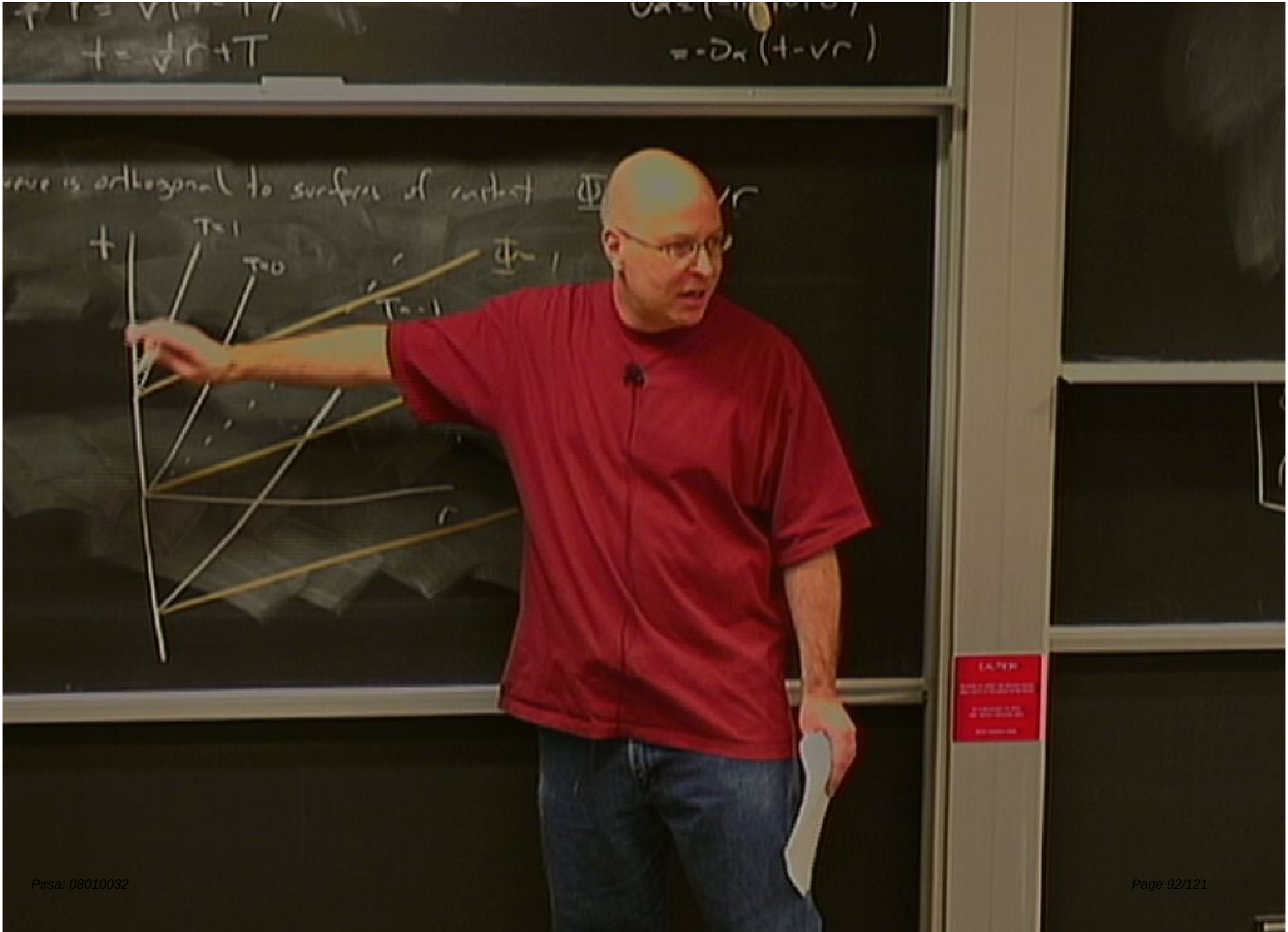

t = -vr + T
Congruence is orthogonal to surfaces of constant Φ = t - vr
t
T=1
T=0
T=-1
Φ=1
Φ=0
Φ=-1
r

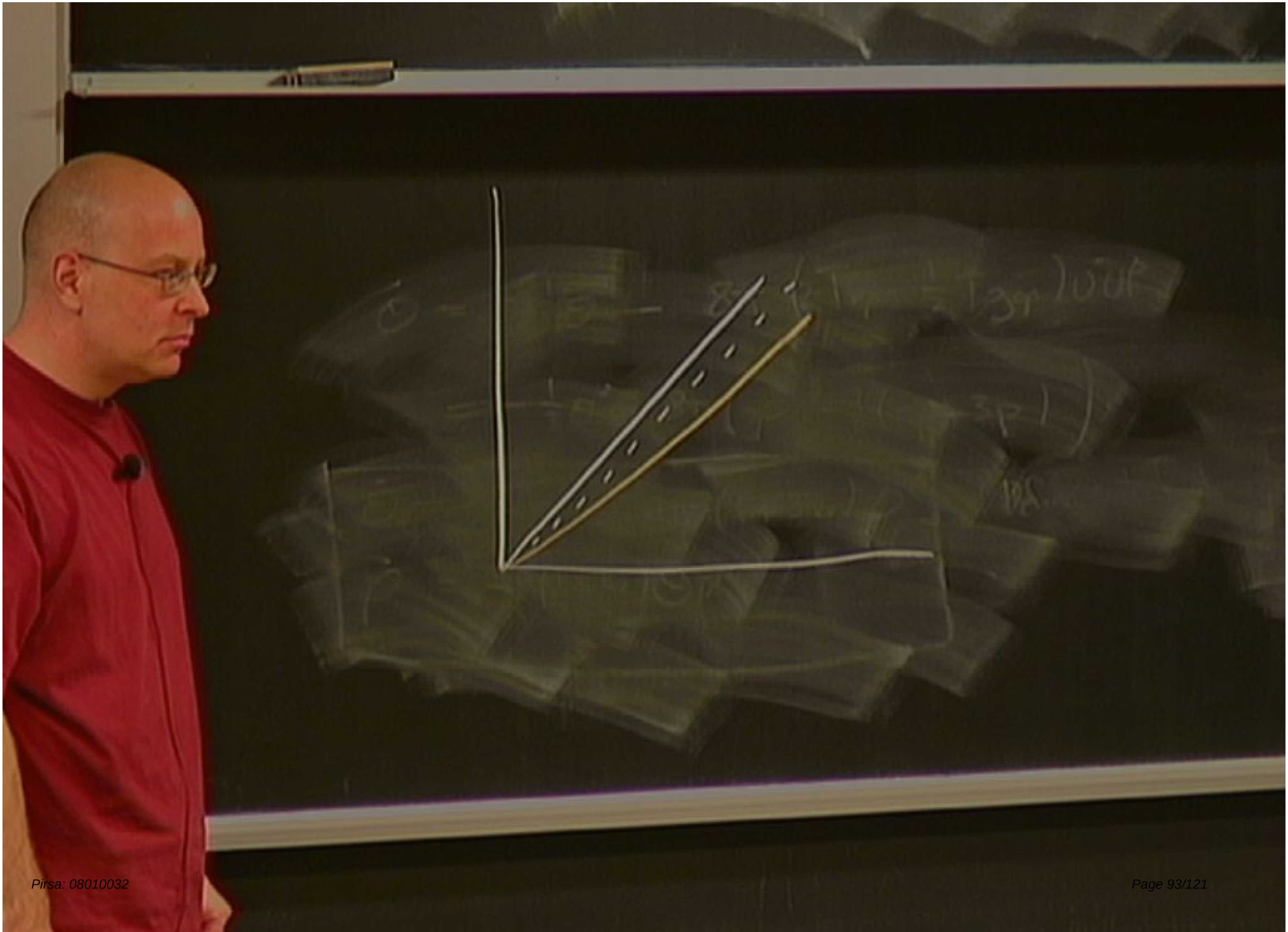

Metric intrinsic to each hypersurface Φ = const is obtained by
setting dΦ = 0 ⇒ dt = v dr

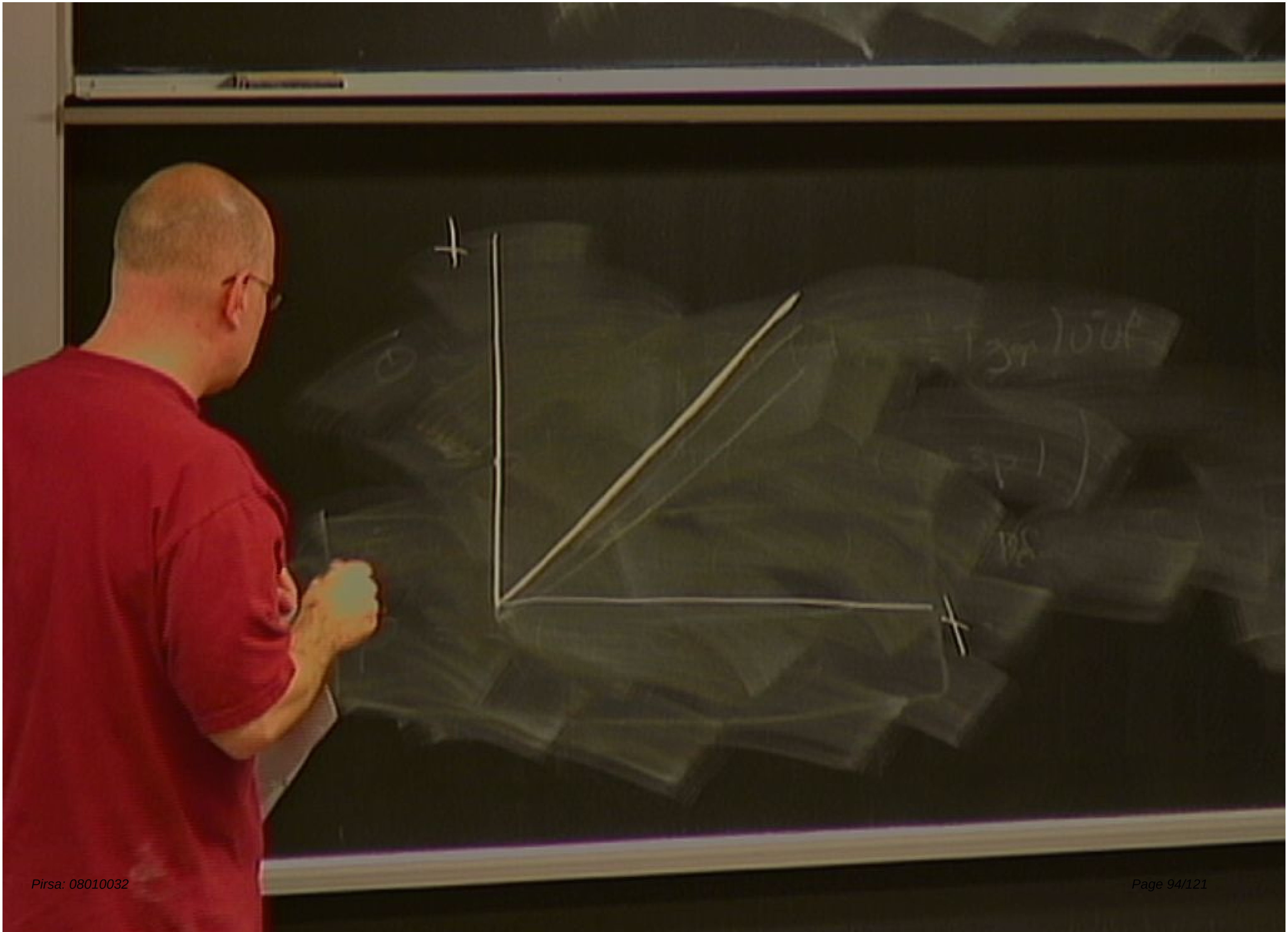
Metric intrinsic to each hypersurface $\Phi = \text{const}$ is obtained by setting $d\Phi = 0 \Rightarrow dt = v\,dr$

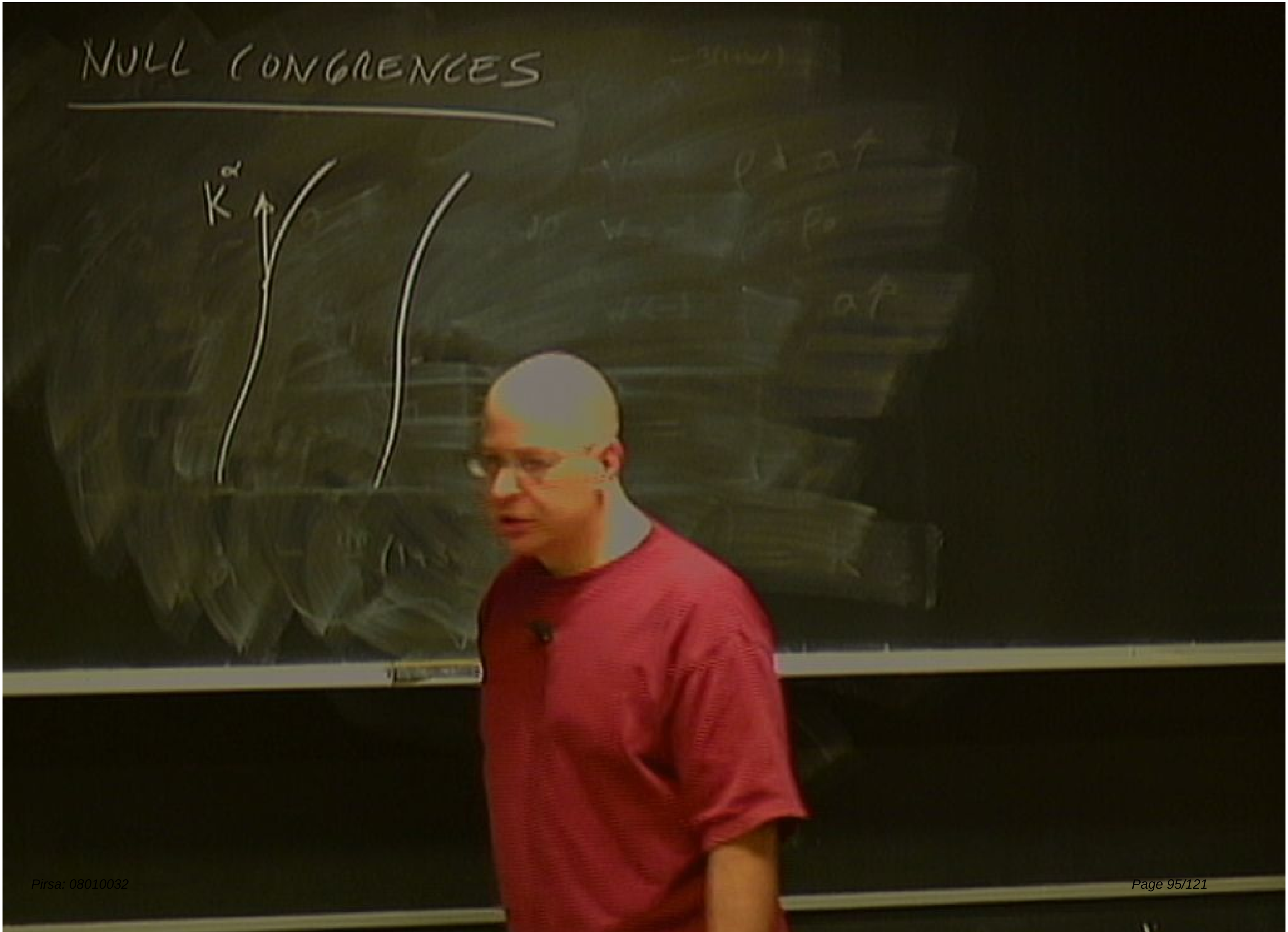
$$ds^2 = -v^2 dr^2 + dr^2 + r^2 d\Omega^2$$

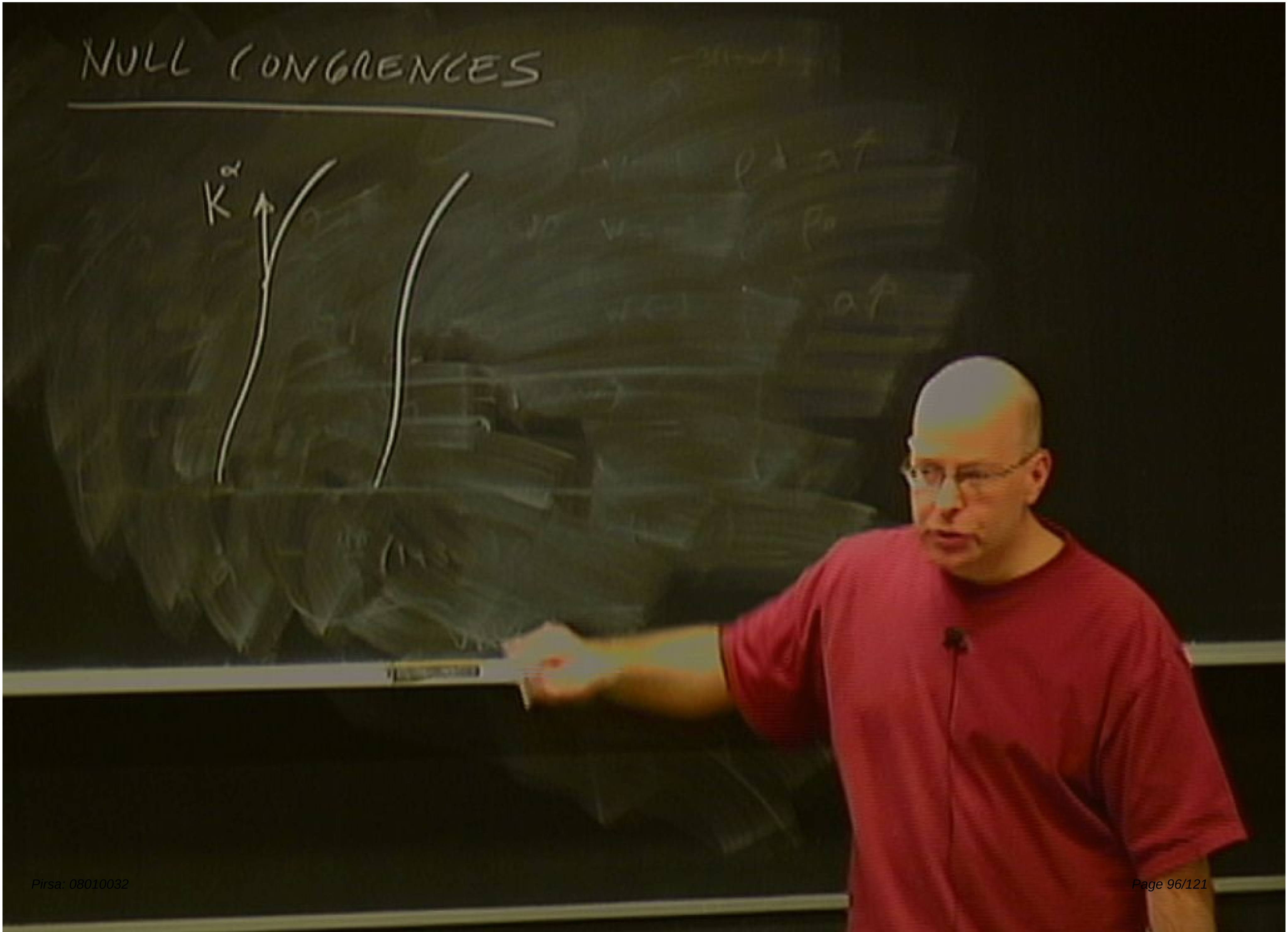
$$ds^2_{\Phi} = (1 - v^2)\,dr^2 + r^2 d\Omega^2$$

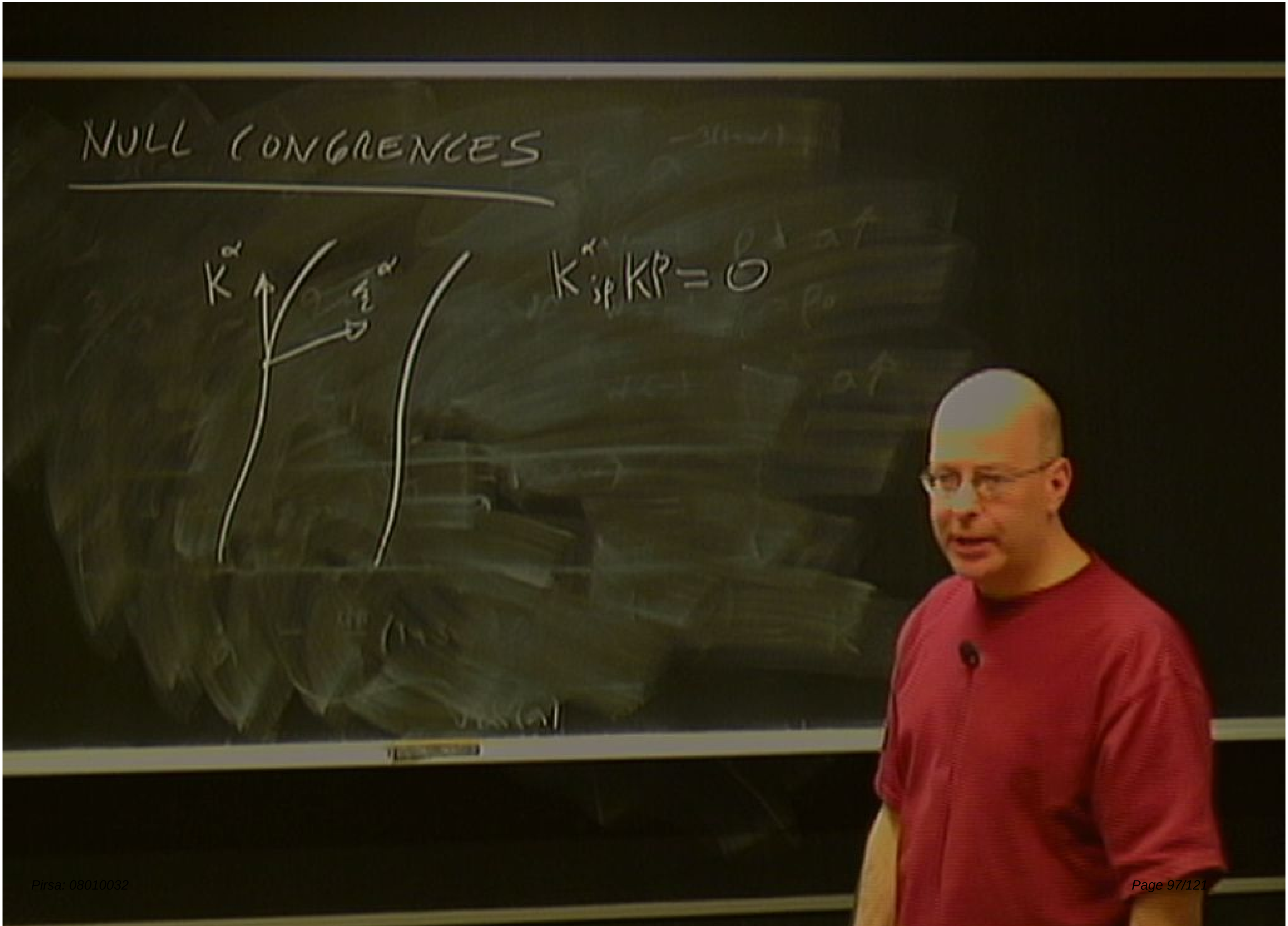

is orthogonal to surfaces of constant
T=1
T=0
T=-1

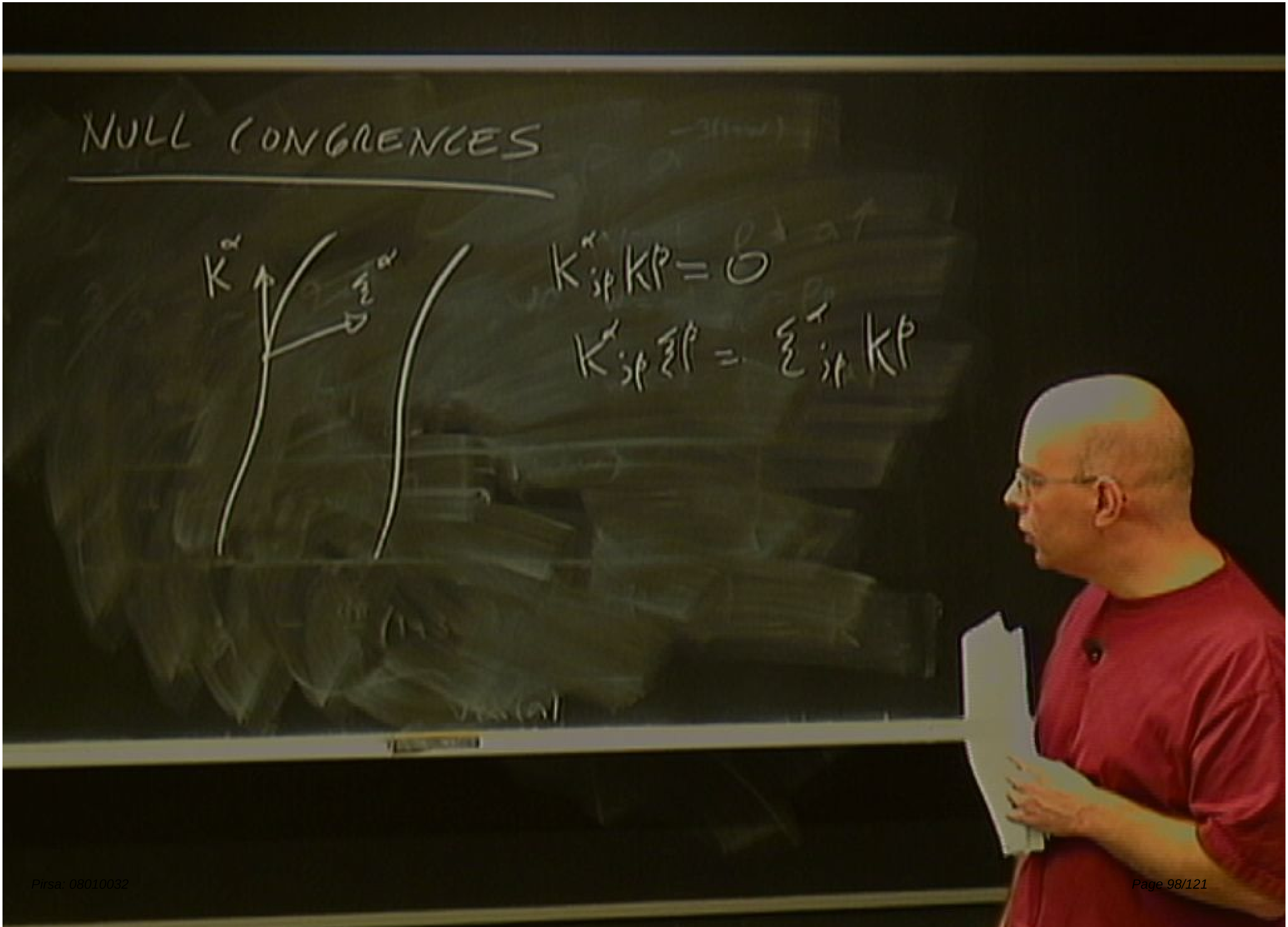


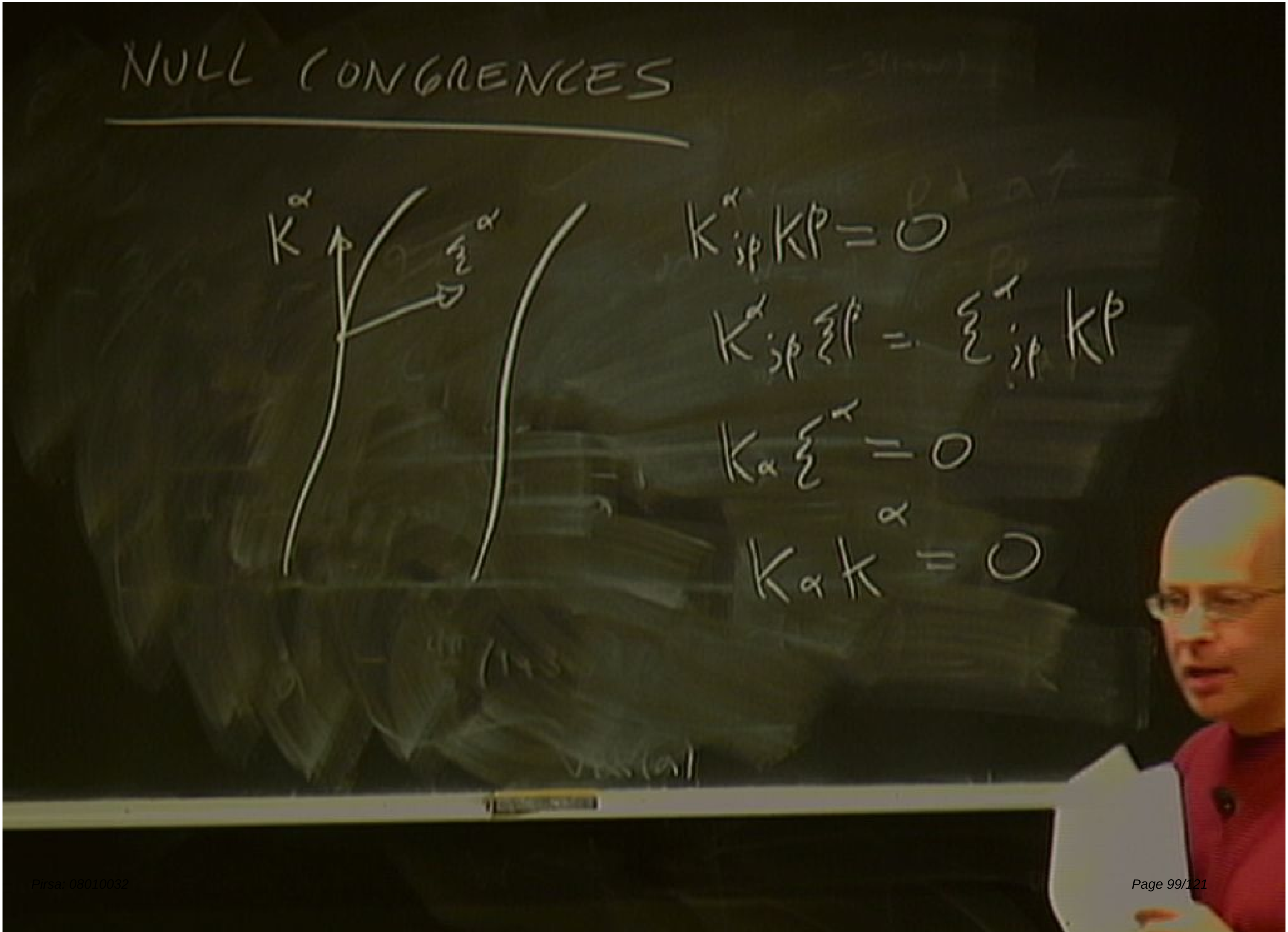


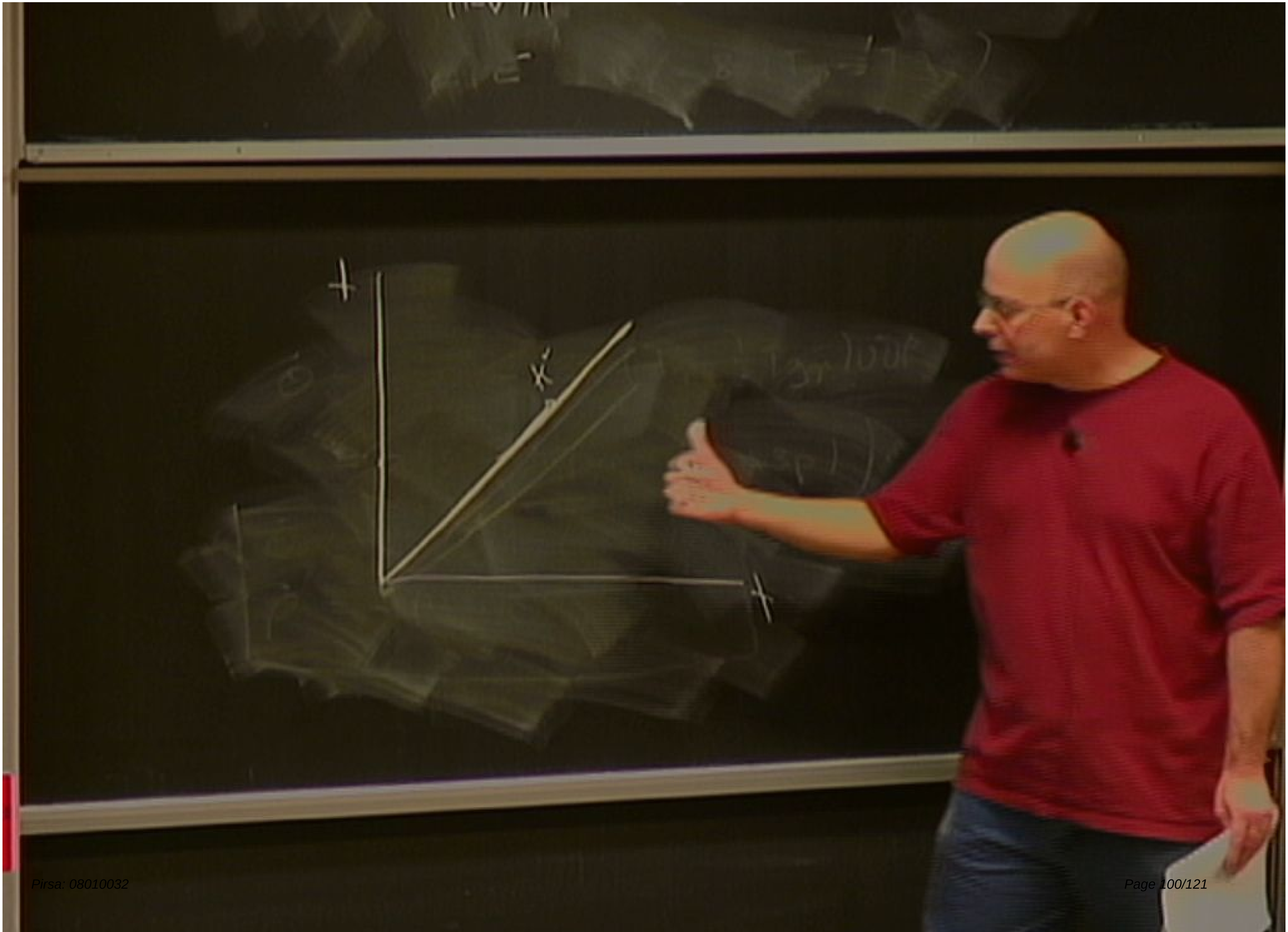

NULL CONGRENCES
K^α

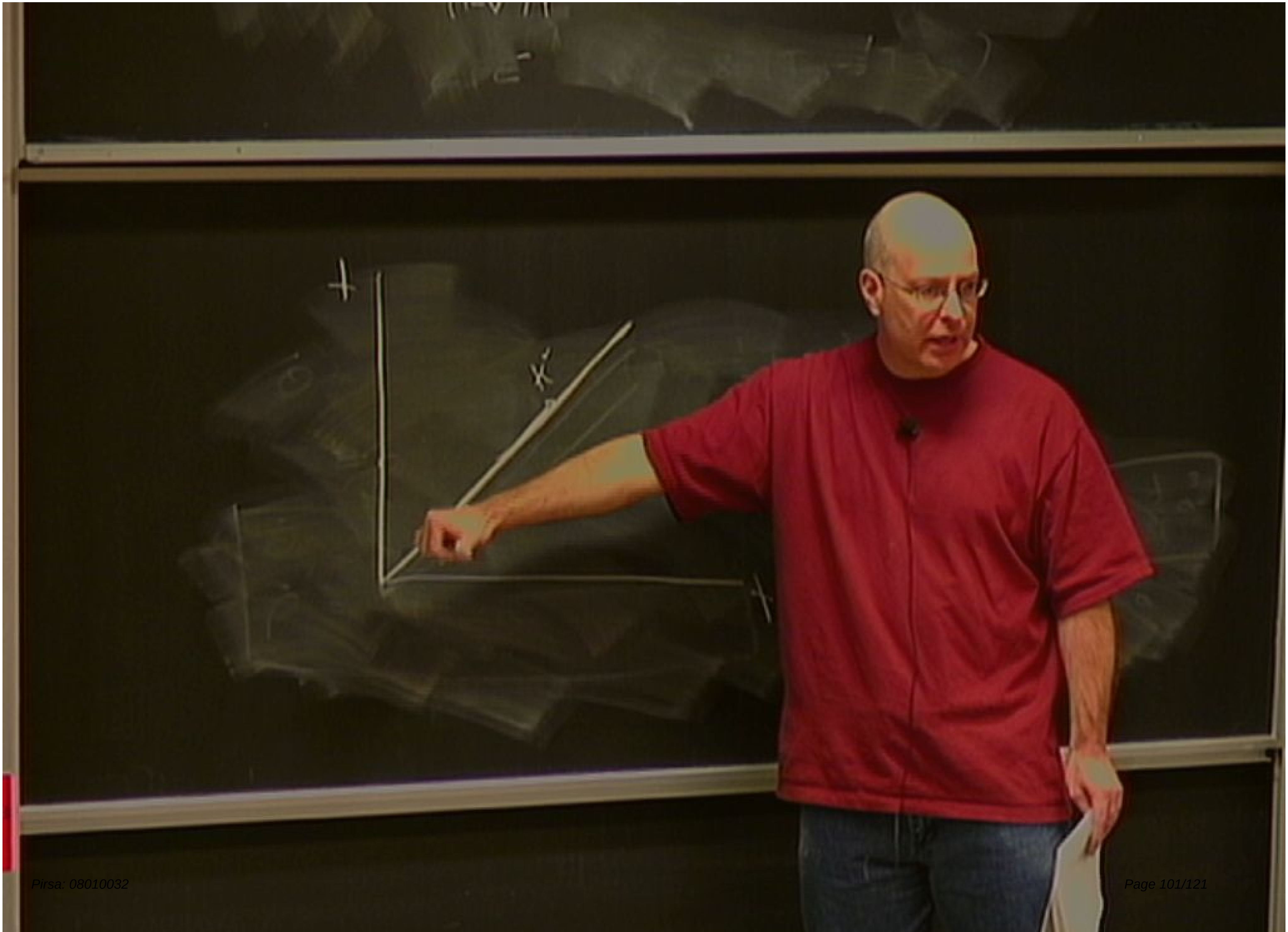

NULL CONGRENCES
K^α

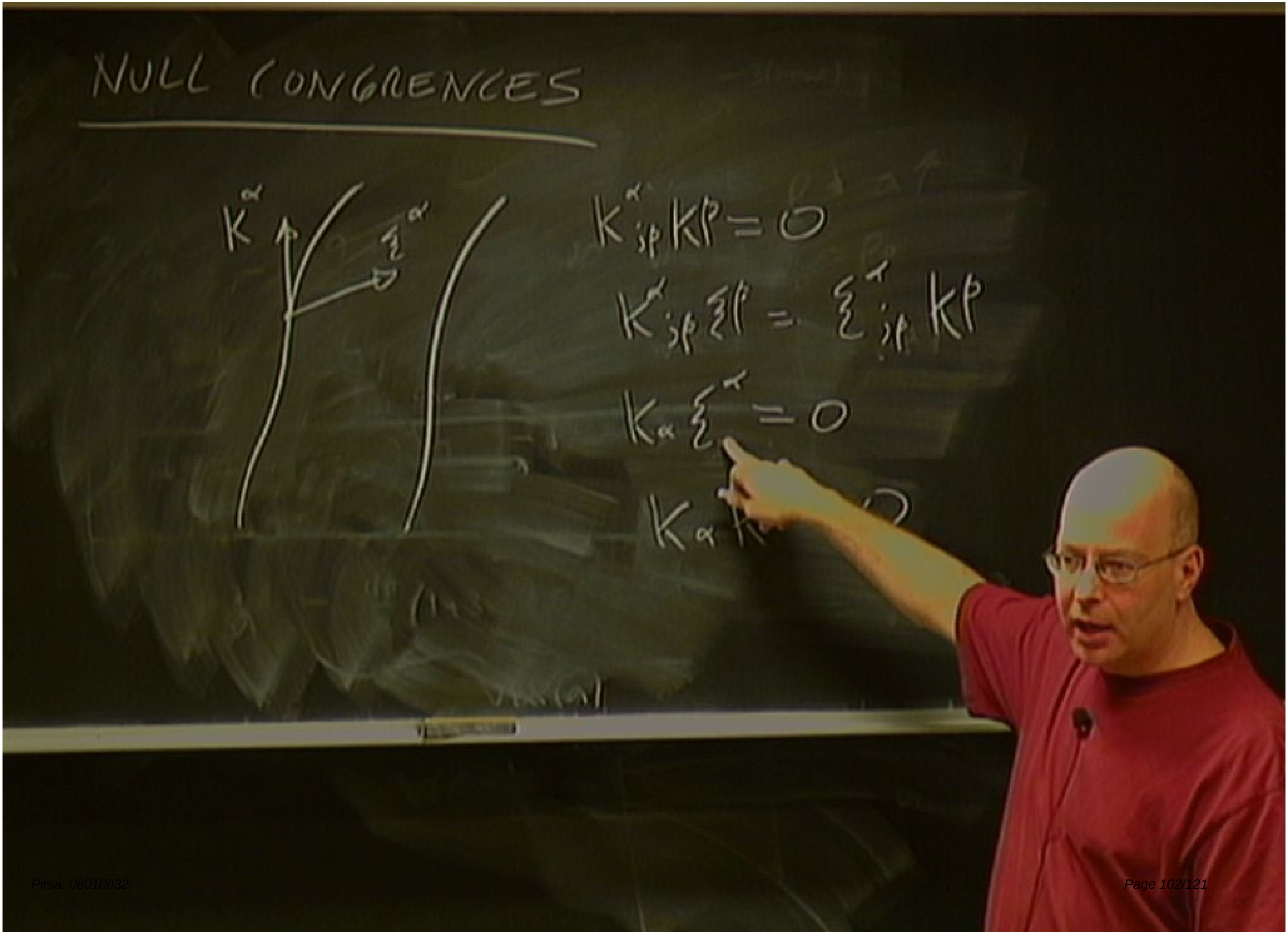

NULL CONGRENCES
K^{α}
ξ^{α}
$K^{\alpha}{}_{;\beta} K^{\beta} = 0$

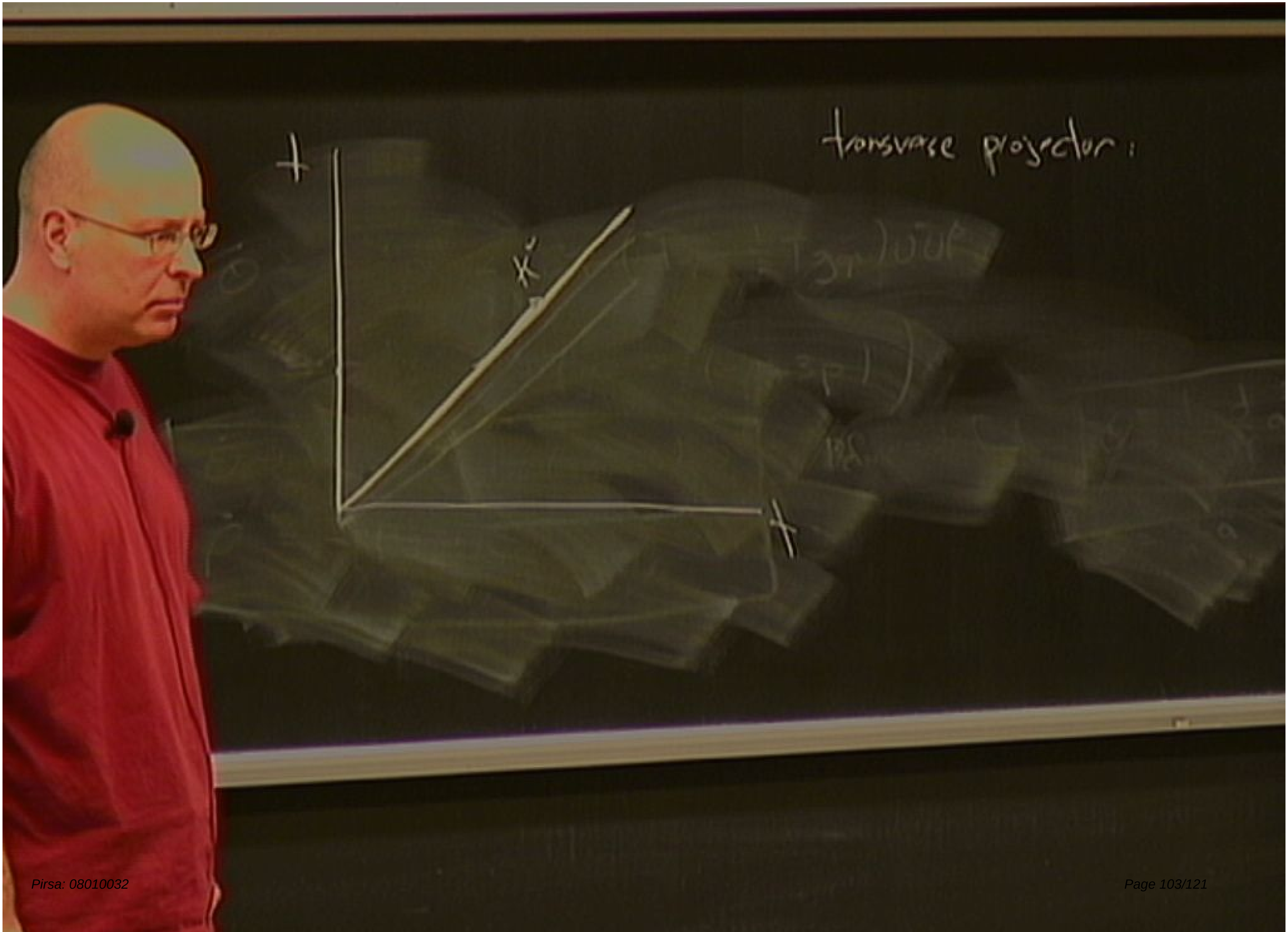

NULL CONGRENCES
K^α
ξ^α
$K^\alpha{}_{;\beta} K^\beta = 0$
$K^\alpha{}_{;\beta} \xi^\beta = \xi^\alpha{}_{;\beta} K^\beta$

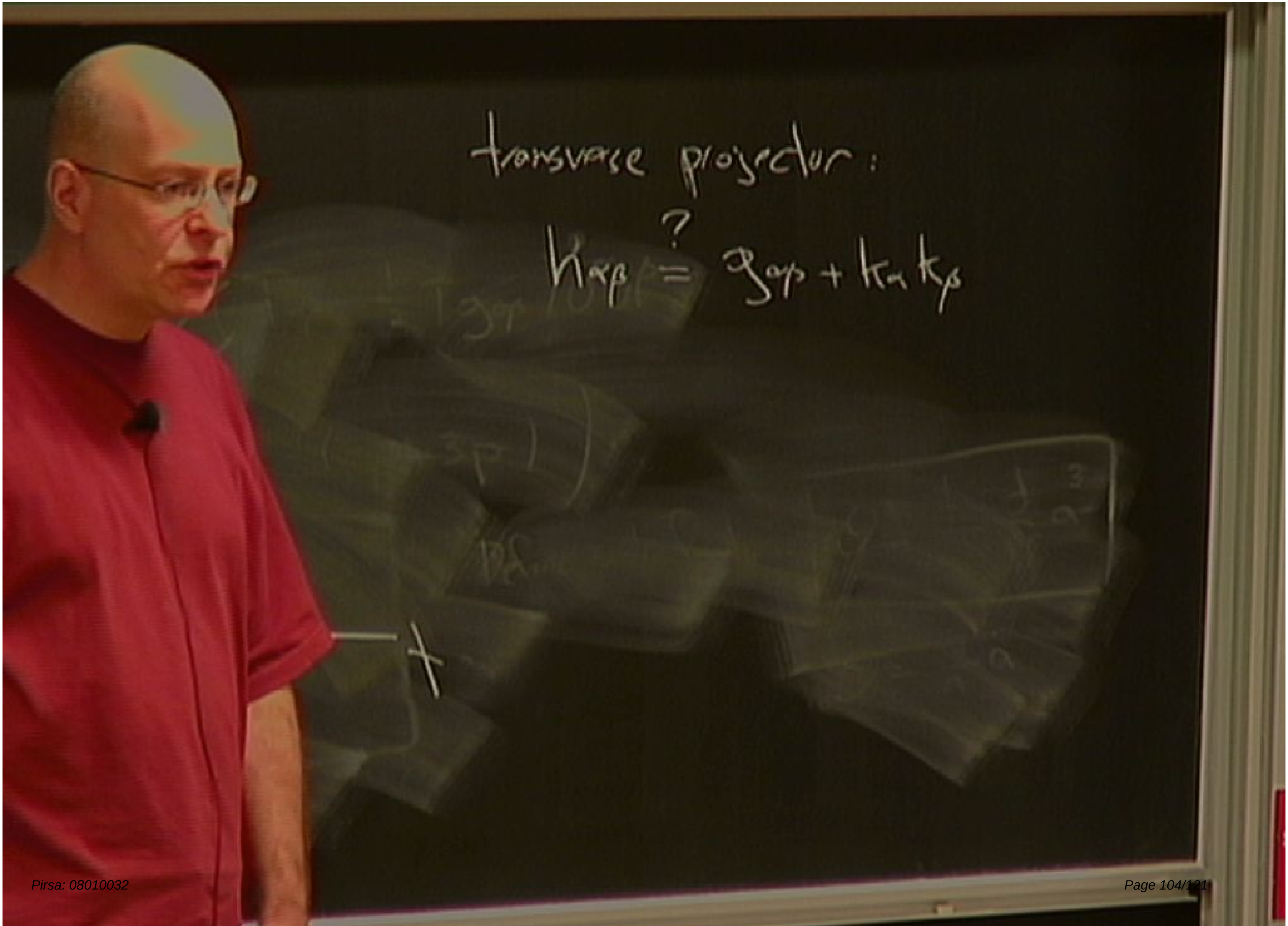

NULL CONGRENCES
K^α
ξ^α
K^α_{;β} K^β = 0
K^α_{;β} ξ^β = ξ^α_{;β} K^β
K_α ξ^α = 0
K_α K^α = 0

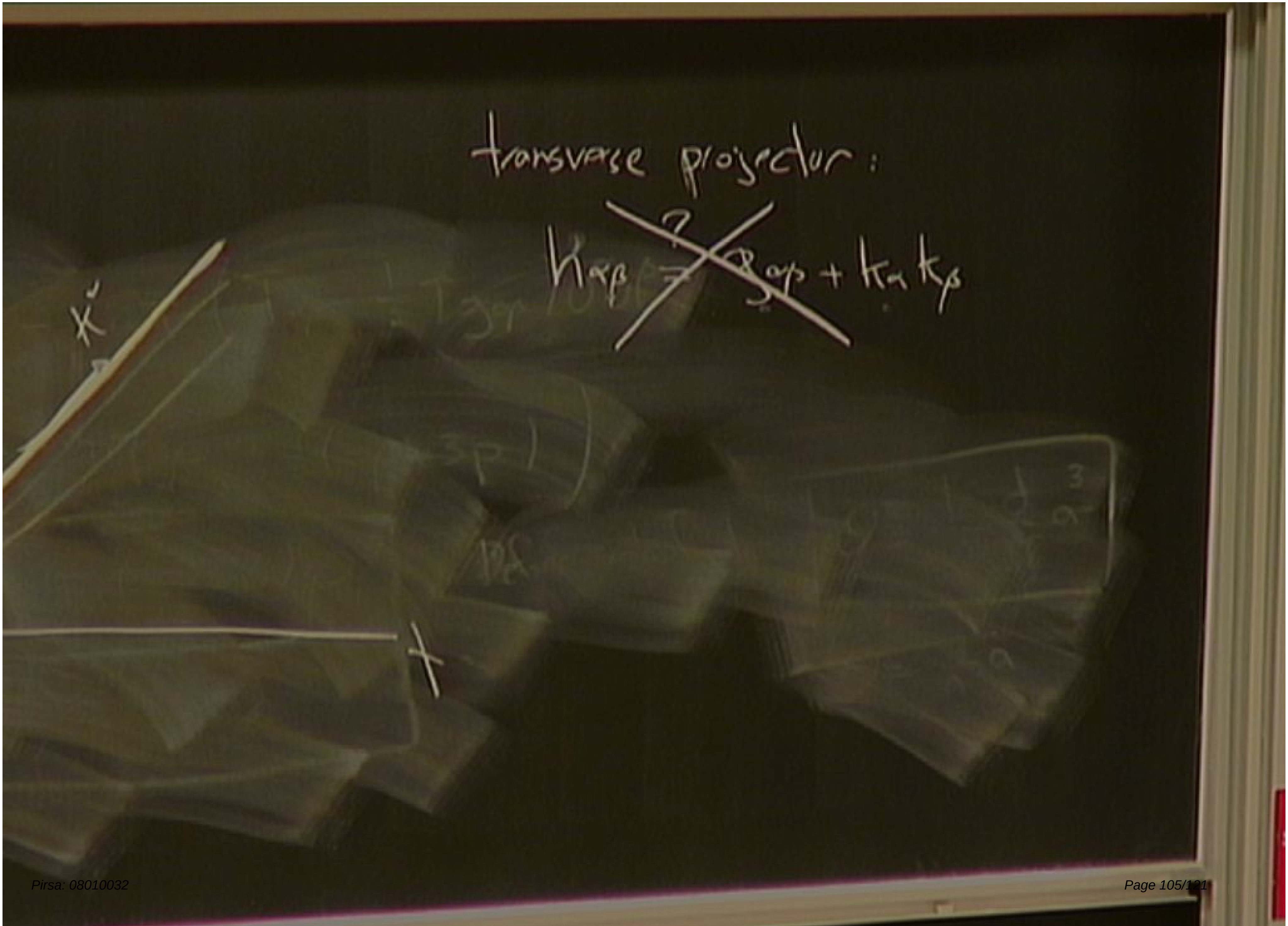

t
x'
x

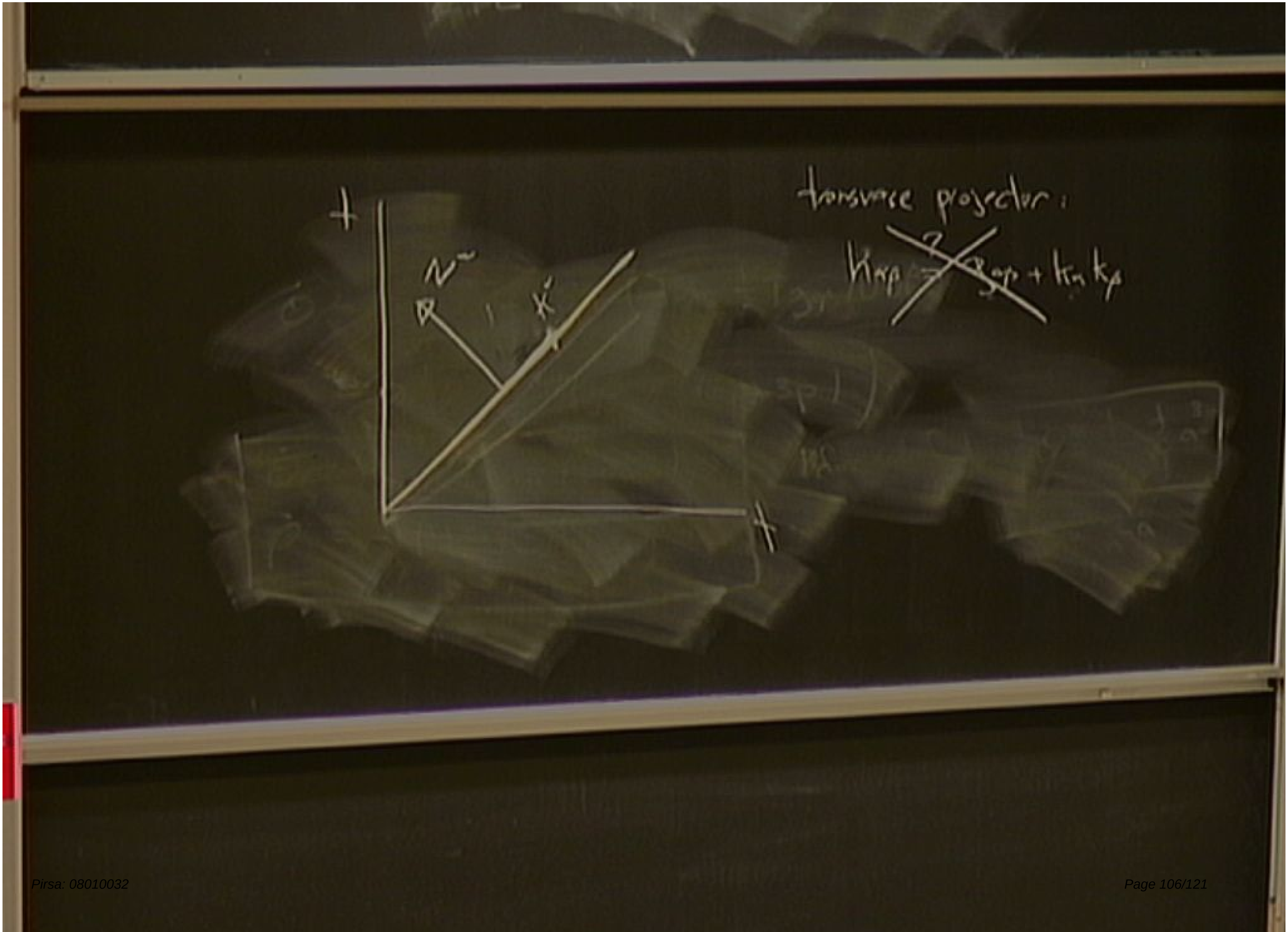



NULL CONGRENCES
K^α
ξ^α
$K^\alpha{}_{;\beta} K^\beta = 0$
$K^\alpha{}_{;\beta} \xi^\beta = \xi^\alpha{}_{;\beta} K^\beta$
$K_\alpha \xi^\alpha = 0$

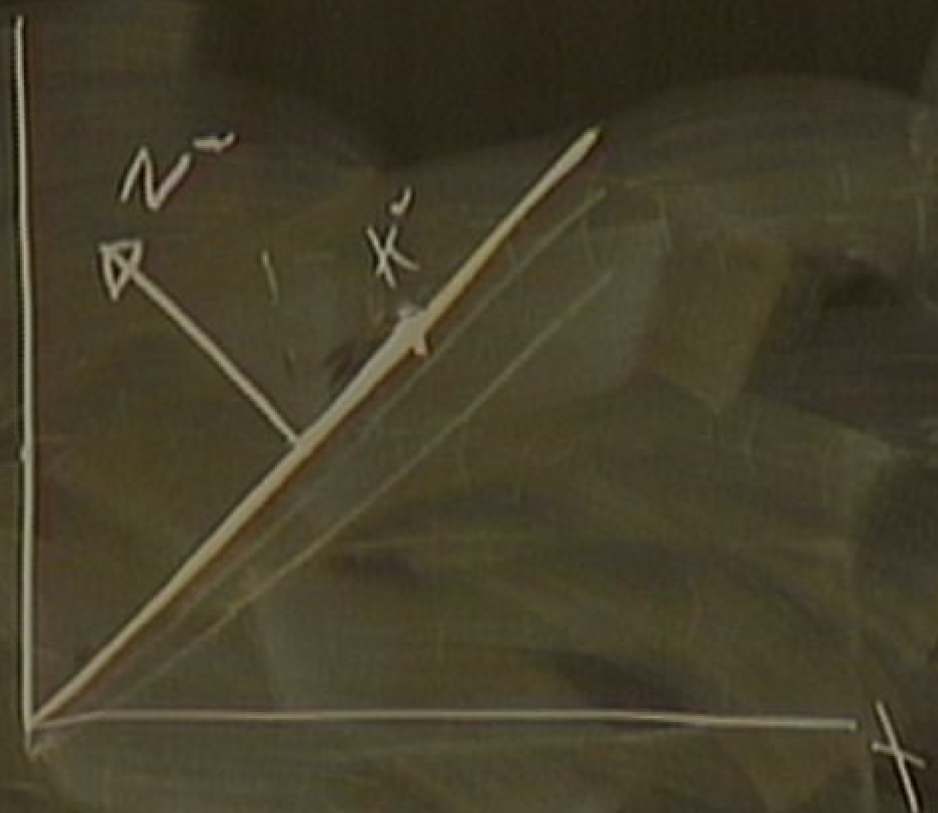

transverse projector :

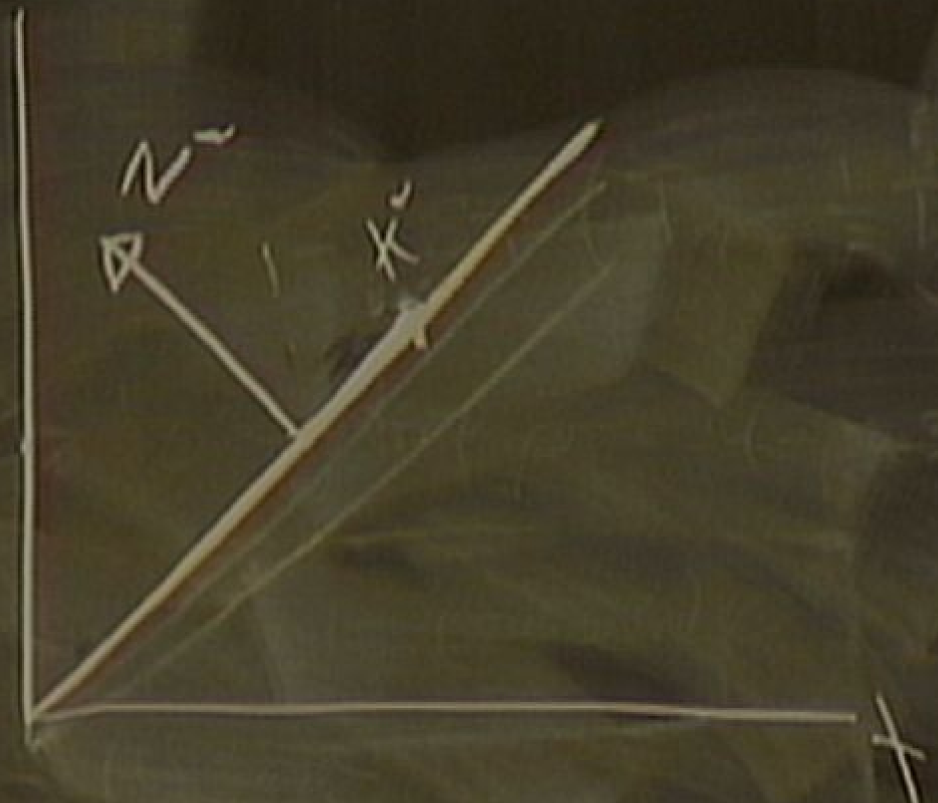

transverse projector :
$h_{\alpha\beta} \overset{?}{=} g_{\alpha\beta} + k_\alpha k_\beta$


transverse projector :
hαβ ?= gαβ + kα kβ


transverse projector :

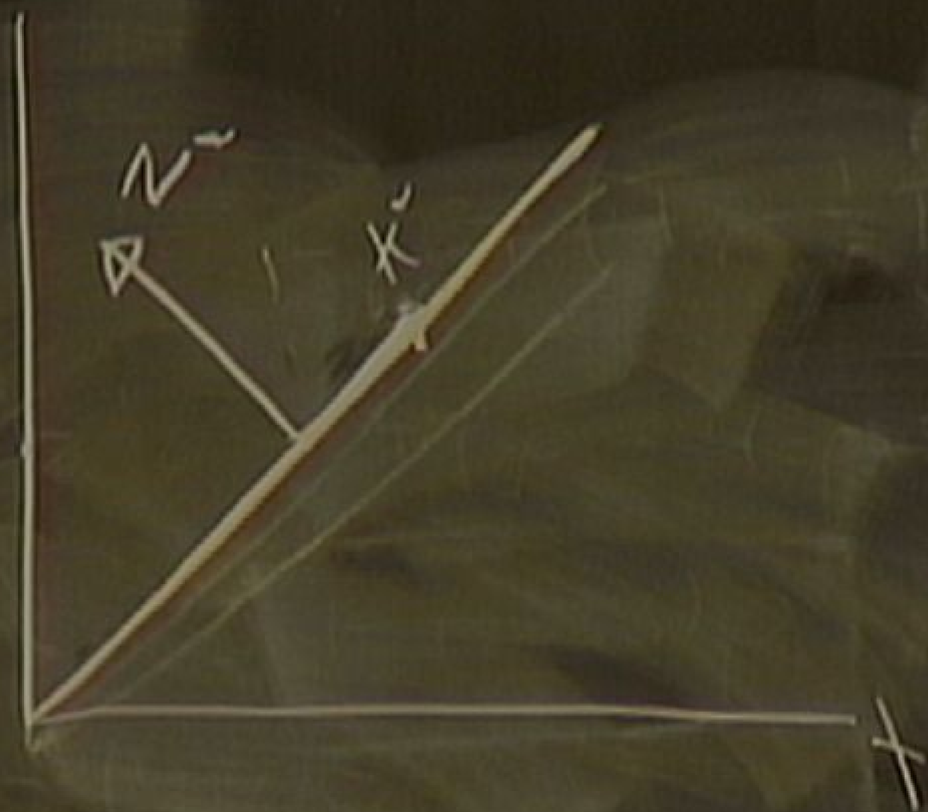
Introduce a new null vector field N^α;





Introduce a new null vector field N^α;

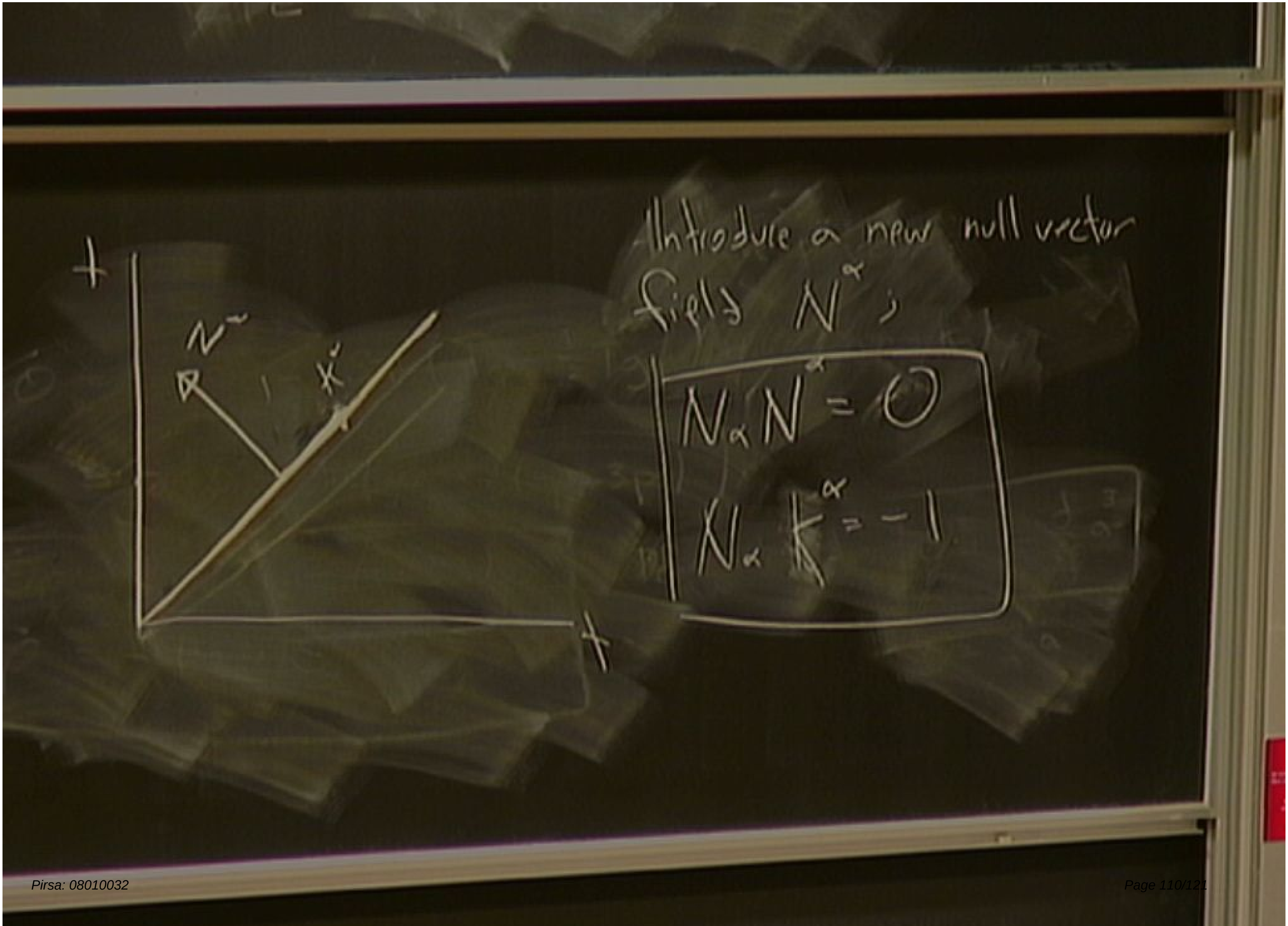
$$N_\alpha N^\alpha = 0$$

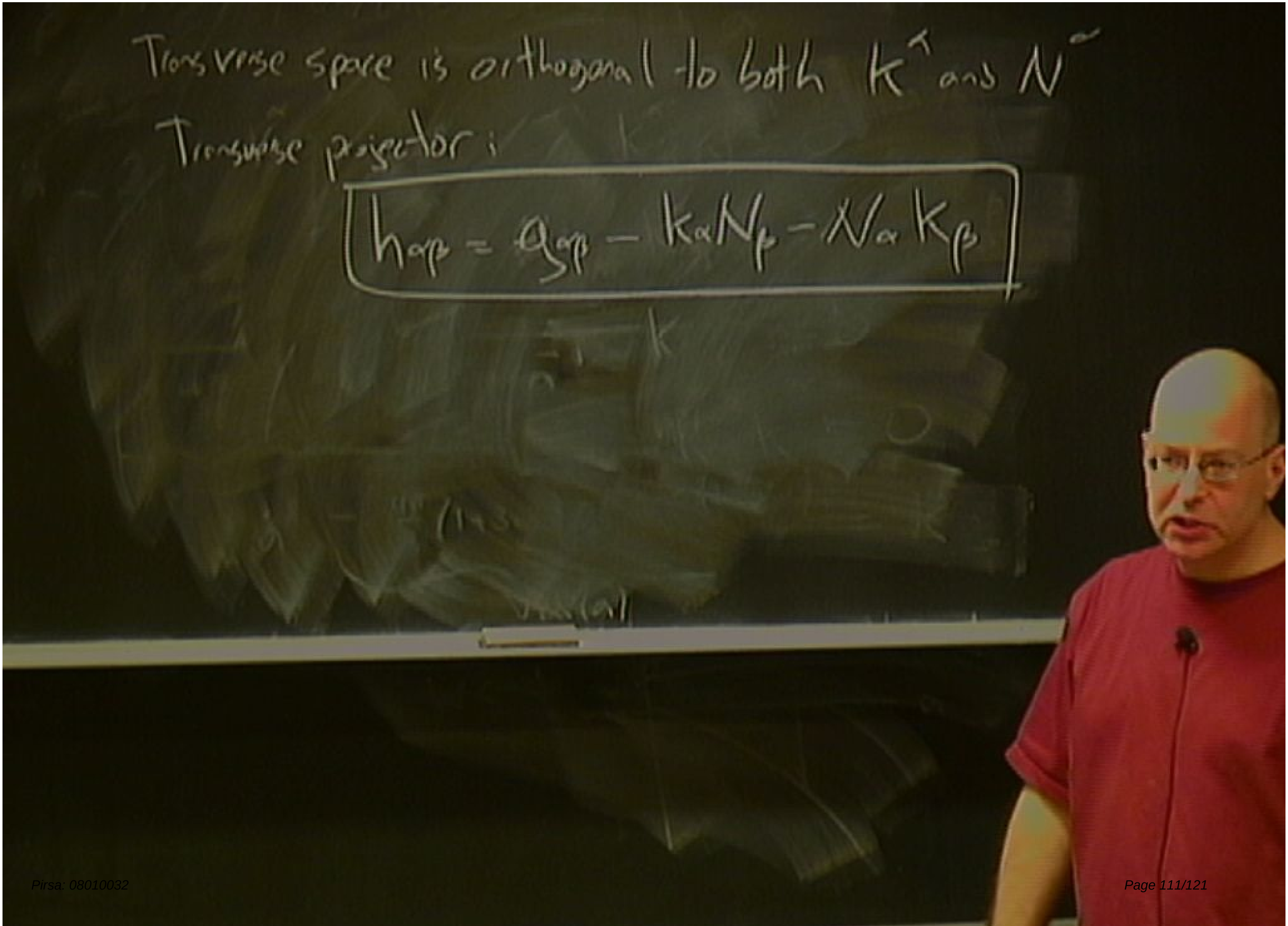


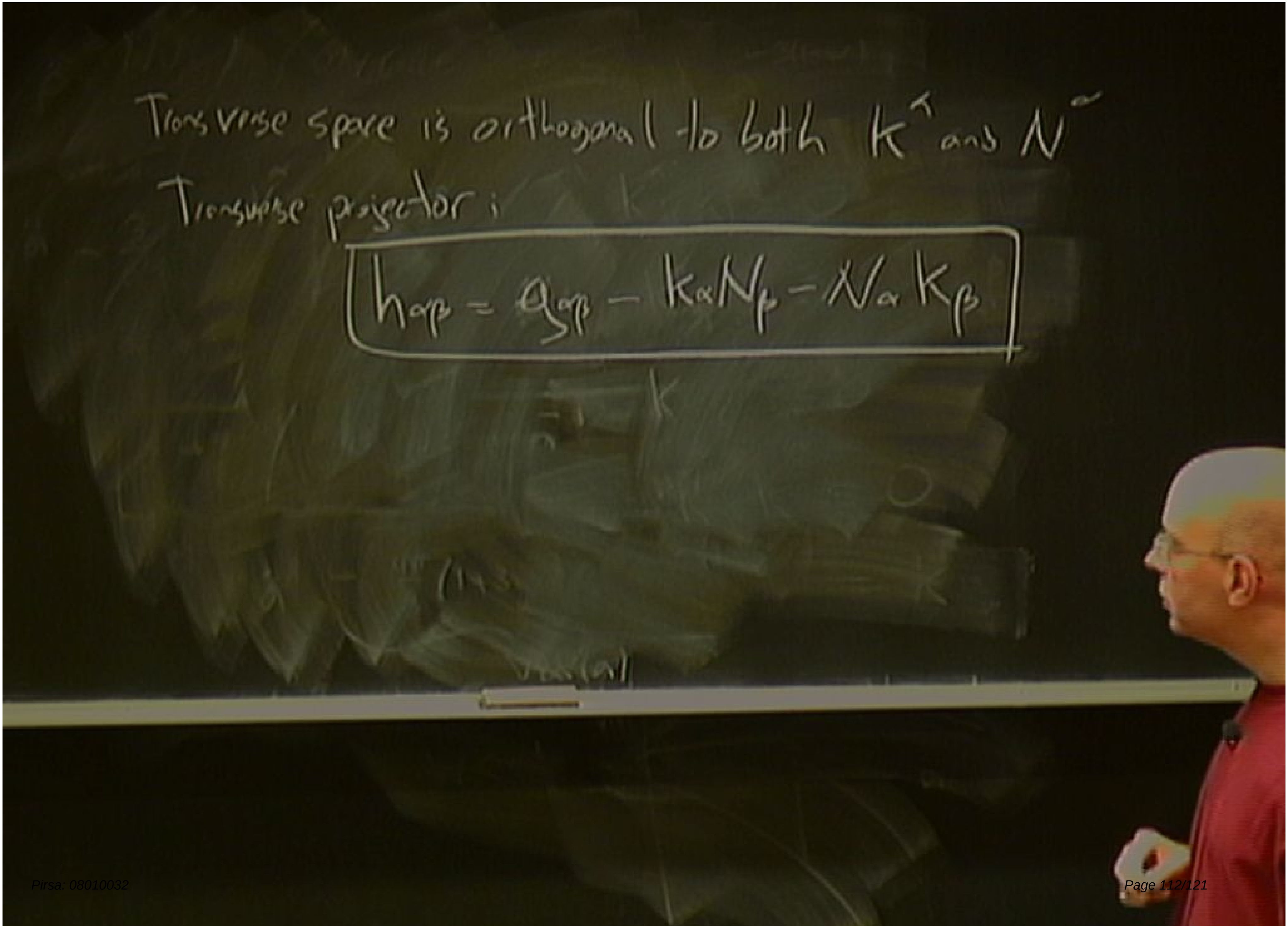
Introduce a new null vector field N^α;

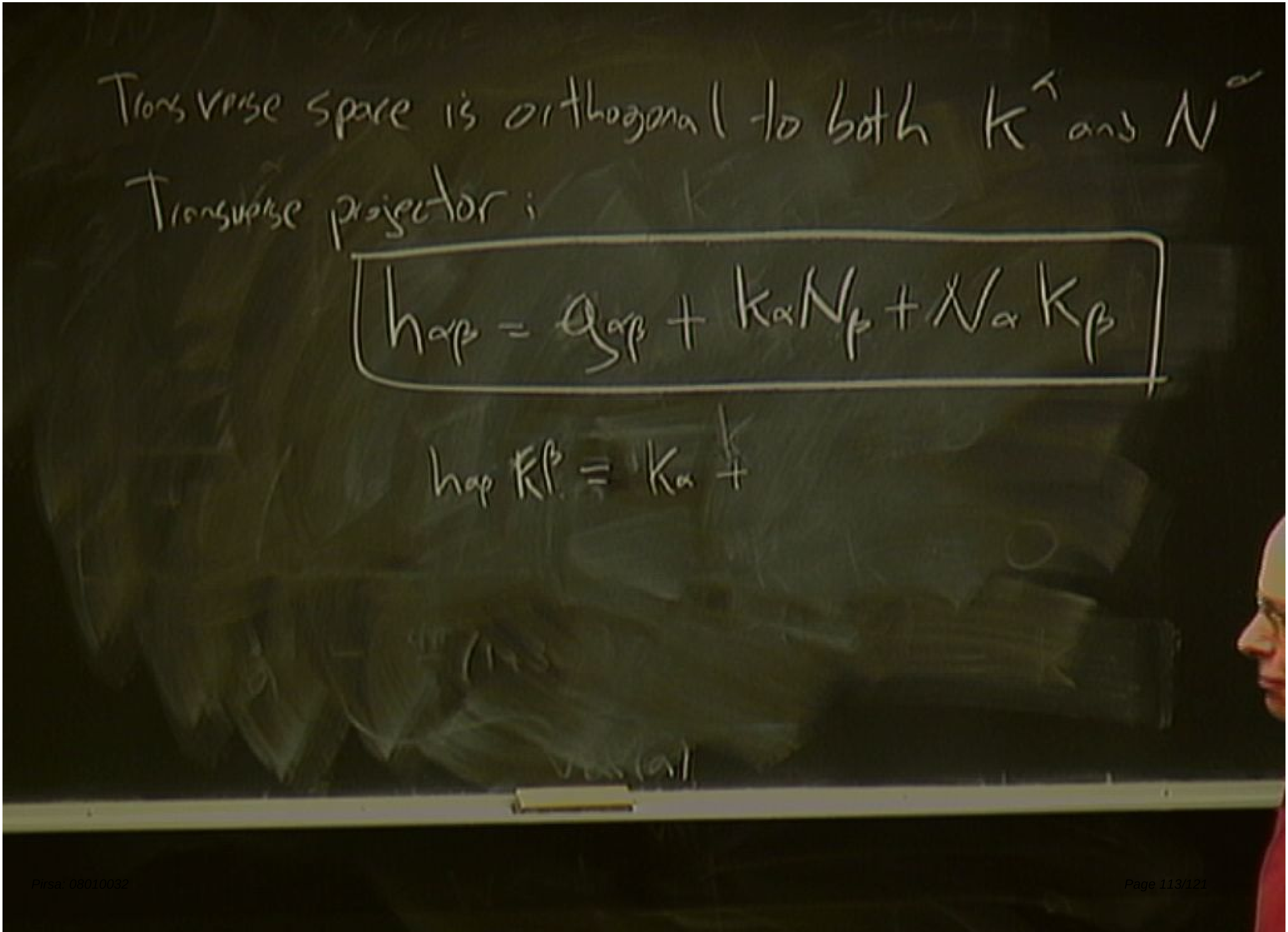
$$N_\alpha N^\alpha = 0$$

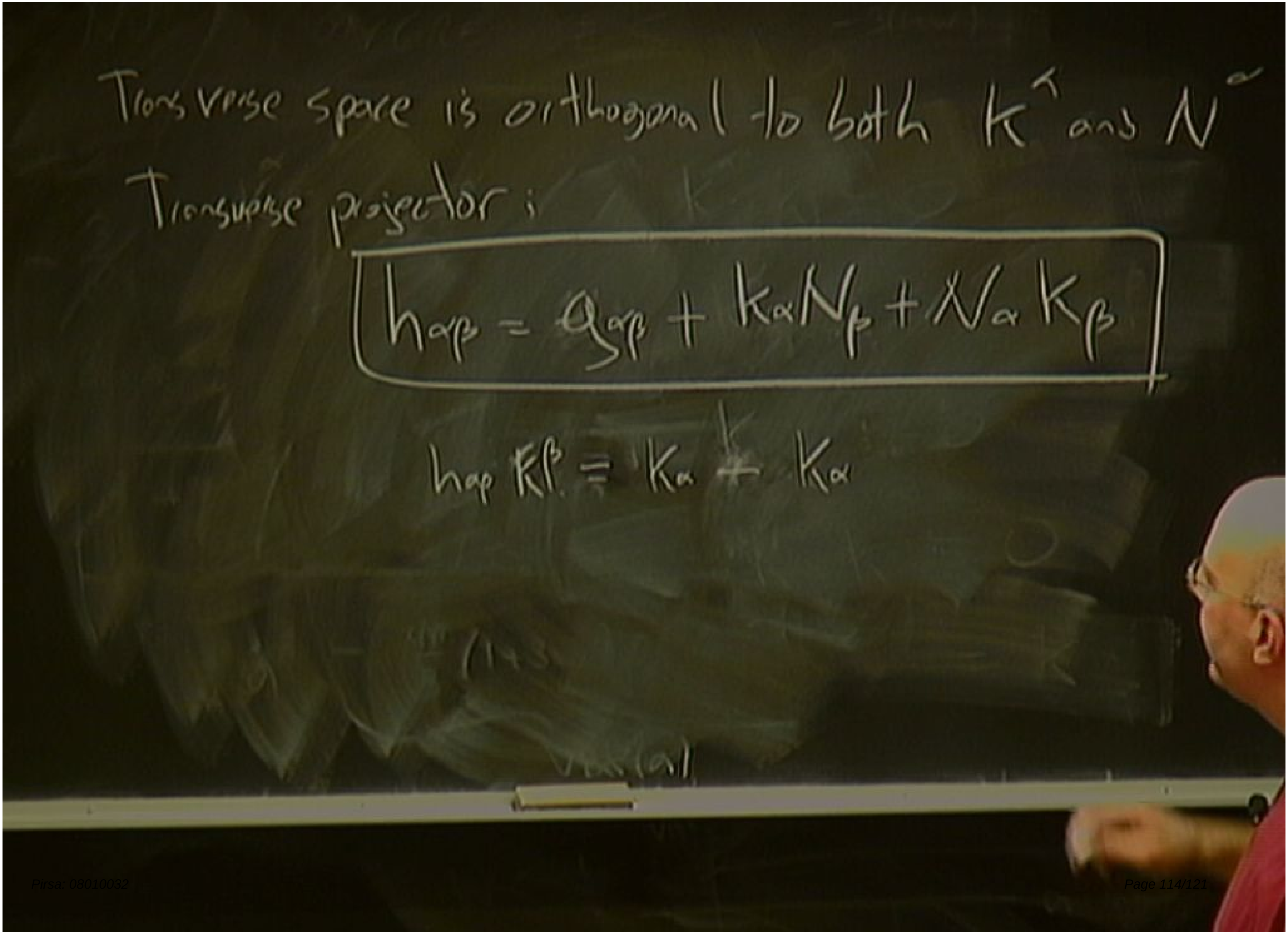
$$N_\alpha \xi^\alpha = -1$$


Introduce a new null vector
field N^α ;
N_α N^α = 0
N_α k^α = -1
t
x
N^α
k^α


Transverse space is orthogonal to both K^α and N^α
Transverse projector:
$h_{\alpha\beta} = g_{\alpha\beta} - K_\alpha N_\beta - N_\alpha K_\beta$


Transverse space is orthogonal to both k^α and N^α
Transverse projector:
$h_{\alpha\beta} = g_{\alpha\beta} - k_\alpha N_\beta - N_\alpha k_\beta$


Transverse space is orthogonal to both K^α and N^α
Transverse projector:
$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta + N_\alpha K_\beta$
$h_{\alpha\beta} K^\beta = K_\alpha +$


Transverse space is orthogonal to both k^α and N^α
Transverse projector:
$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$h_{\alpha\beta} k^\beta = k_\alpha + k_\alpha$

Transverse space is orthogonal to both k^α and N^α

Transverse projector:

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$h_{\alpha\beta} k^\beta = k_\alpha - k_\alpha = 0$$

$$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha = 0$$

Transverse space is orthogonal to both k^α and N^α

Transverse projector:

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$h_{\alpha\beta} k^\beta = k_\alpha - k_\alpha = 0 \qquad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$$

$$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha = 0$$

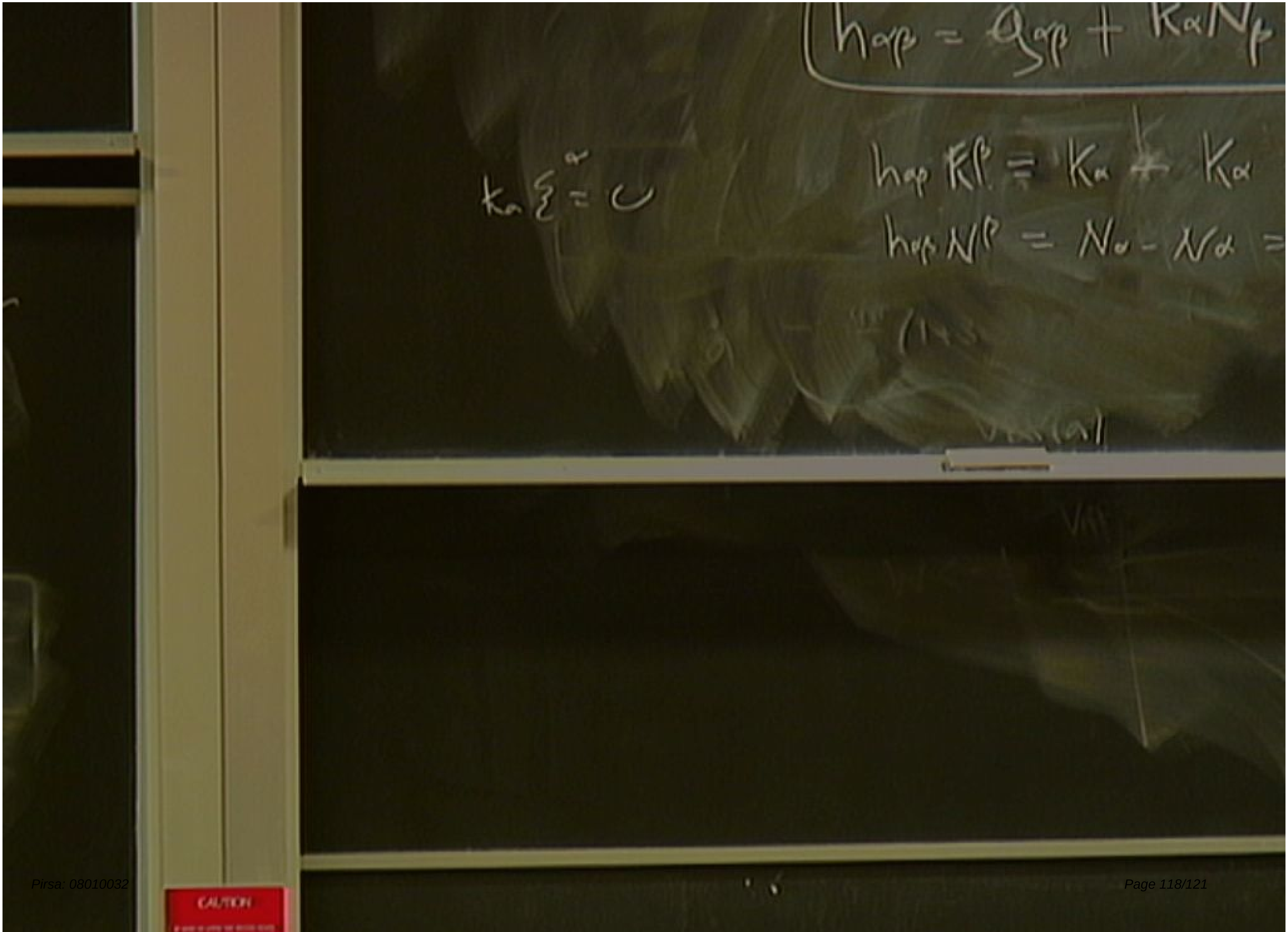
Transverse space is orthogonal to both k^α and N^α

Transverse projector:

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$h_{\alpha\beta} k^\beta = k_\alpha - k_\alpha = 0 \qquad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$$

$$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha = 0 \qquad h^\alpha{}_\alpha = 2$$


$h_{\alpha\beta} = g_{\alpha\beta} + K_\alpha N_\beta$
$K_\alpha \xi^\alpha = C$
$h_{\alpha\beta} K^\beta = K_\alpha - K_\alpha$
$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha =$
CAUTION

Transverse space is orthogonal to both k^α and N^α

Transverse projector:

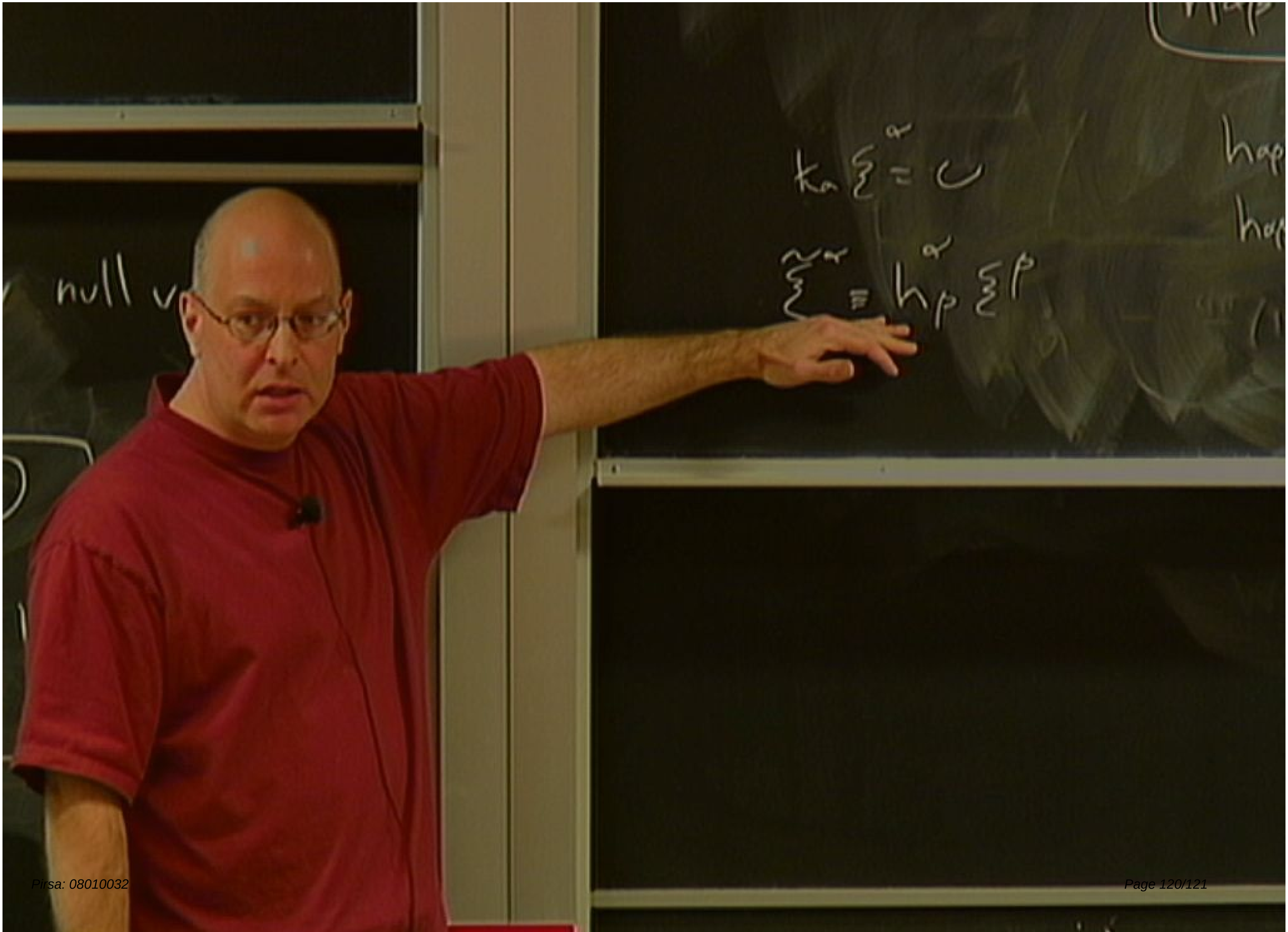
$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

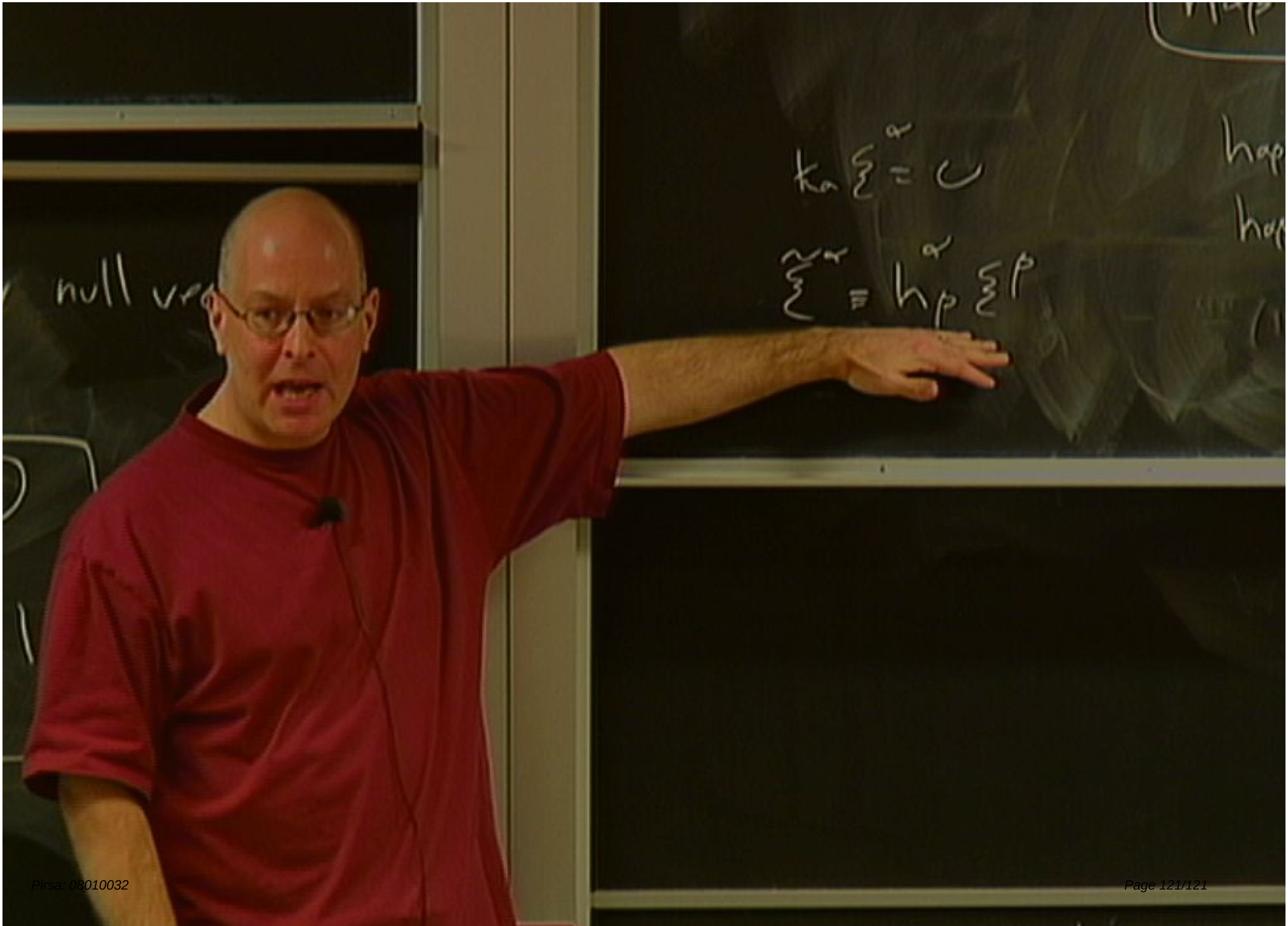
$$h_{\alpha\beta} k^\beta = k_\alpha - k_\alpha = 0 \qquad h^\alpha{}_\mu h^\mu{}_\beta = h^\alpha{}_\beta$$

$$h_{\alpha\beta} N^\beta = N_\alpha - N_\alpha = 0 \qquad h^\alpha{}_\alpha = 2$$

$$k_\alpha \xi^\alpha = 0$$

$$\tilde{\xi}^\alpha = h^\alpha{}_\beta \xi^\beta$$


null v


null
ka ξ^α = 0
ξ̃^α = h^α_β ξ^β