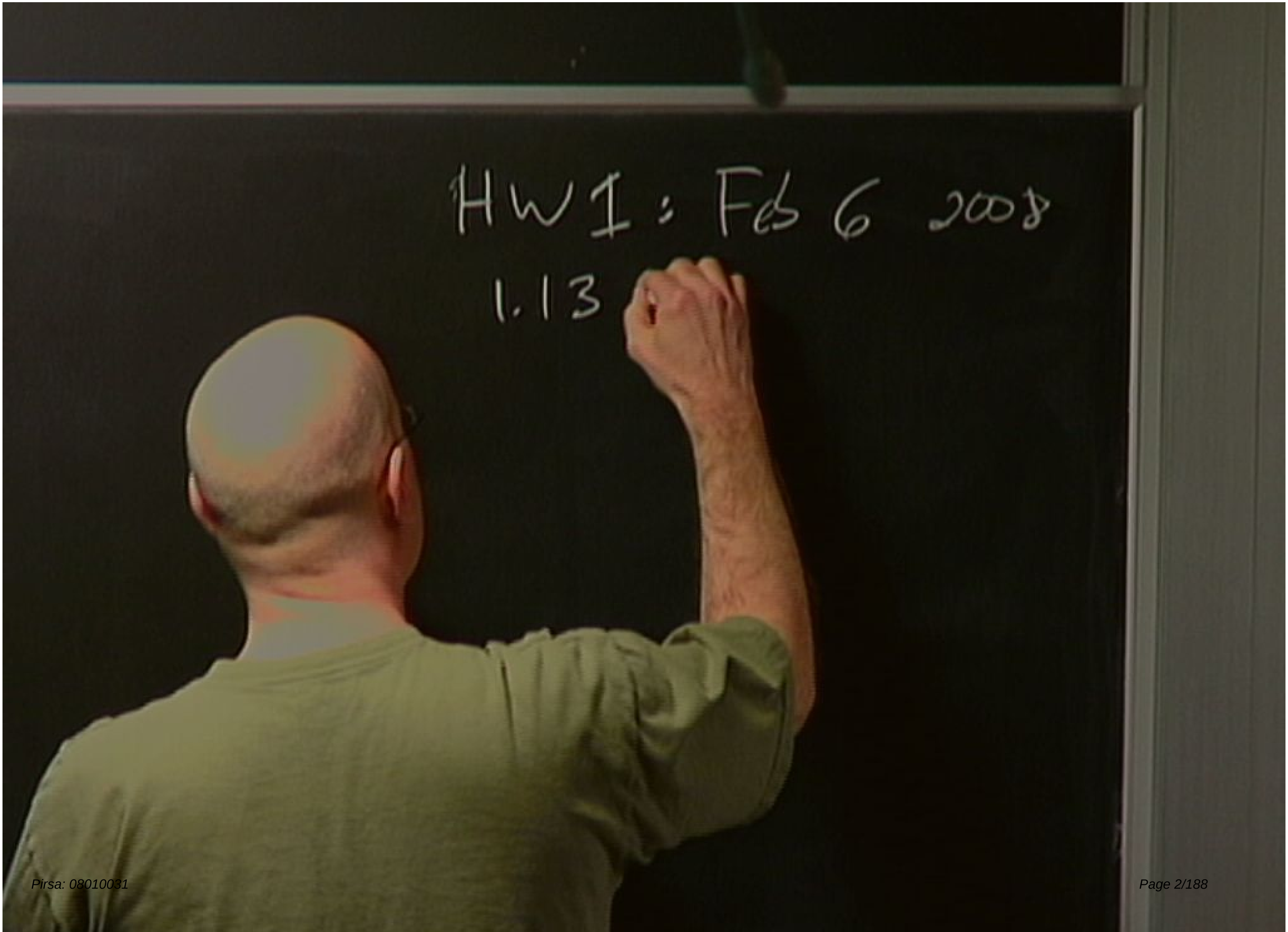
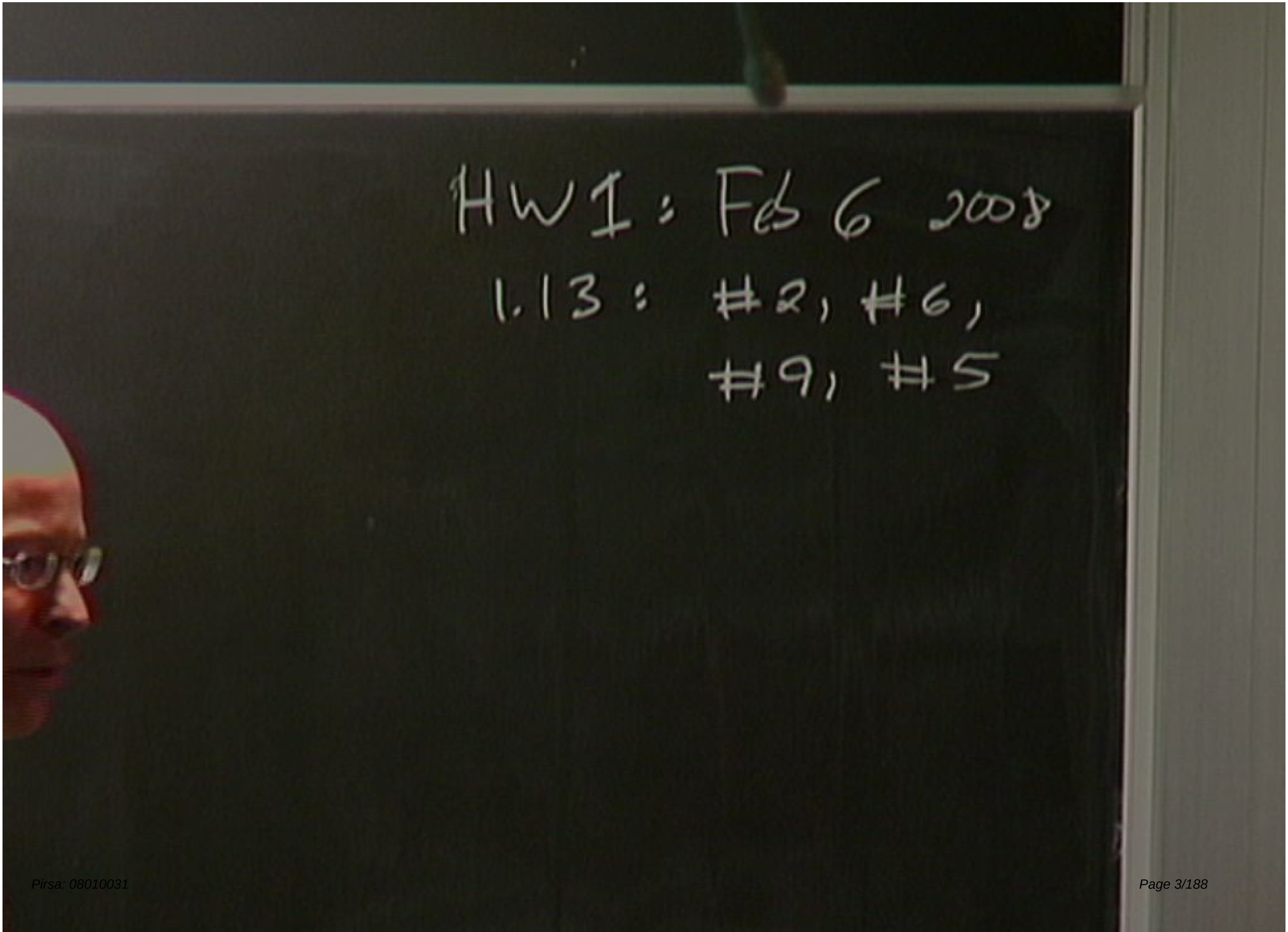
Title: Advanced General Relativity - Lecture 3B

Date: Jan 23, 2008 04:00 PM

URL: http://pirsa.org/08010031

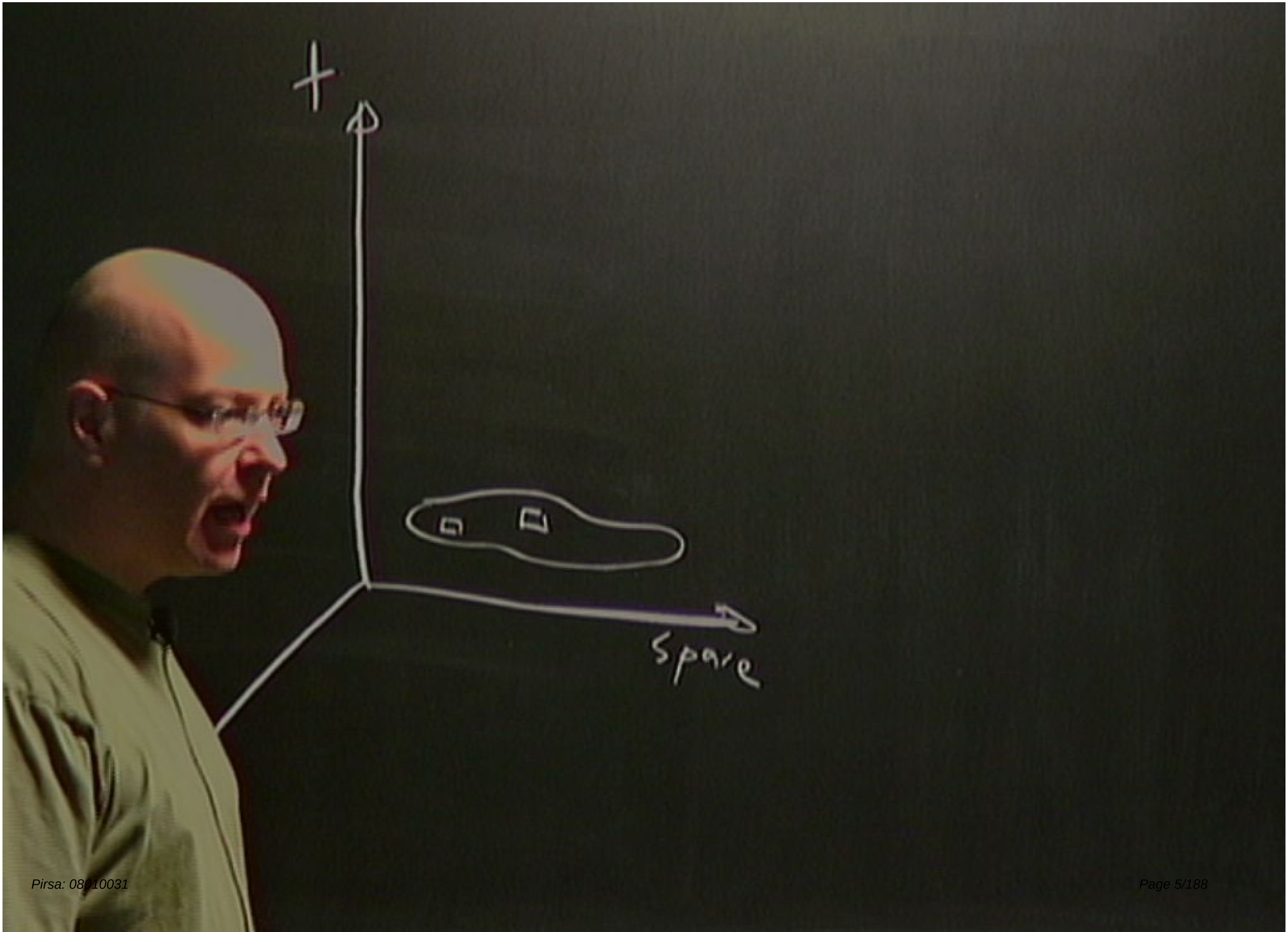
Abstract: Advanced General Relativity

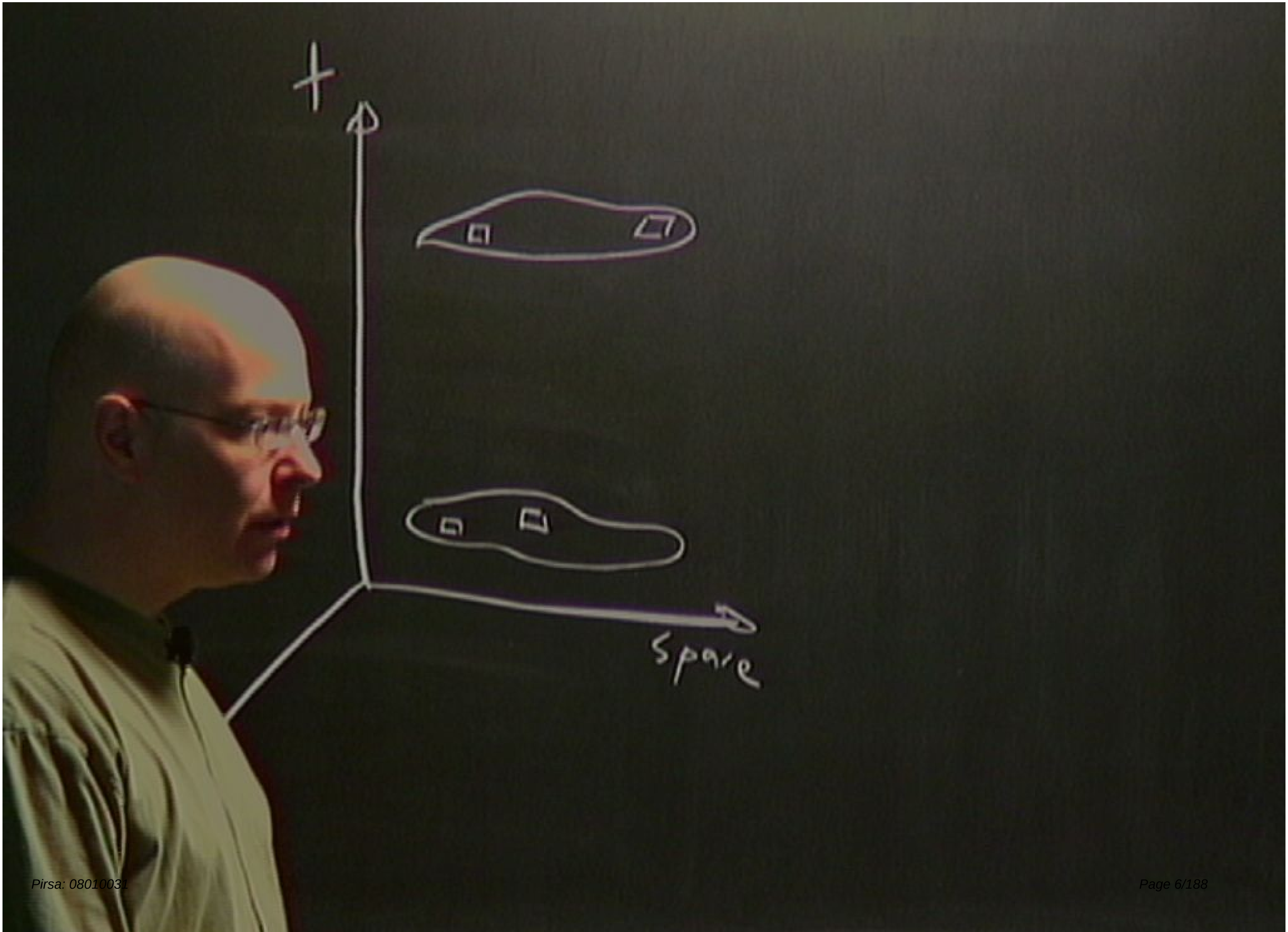

HW1: Feb 6 2008
1.13

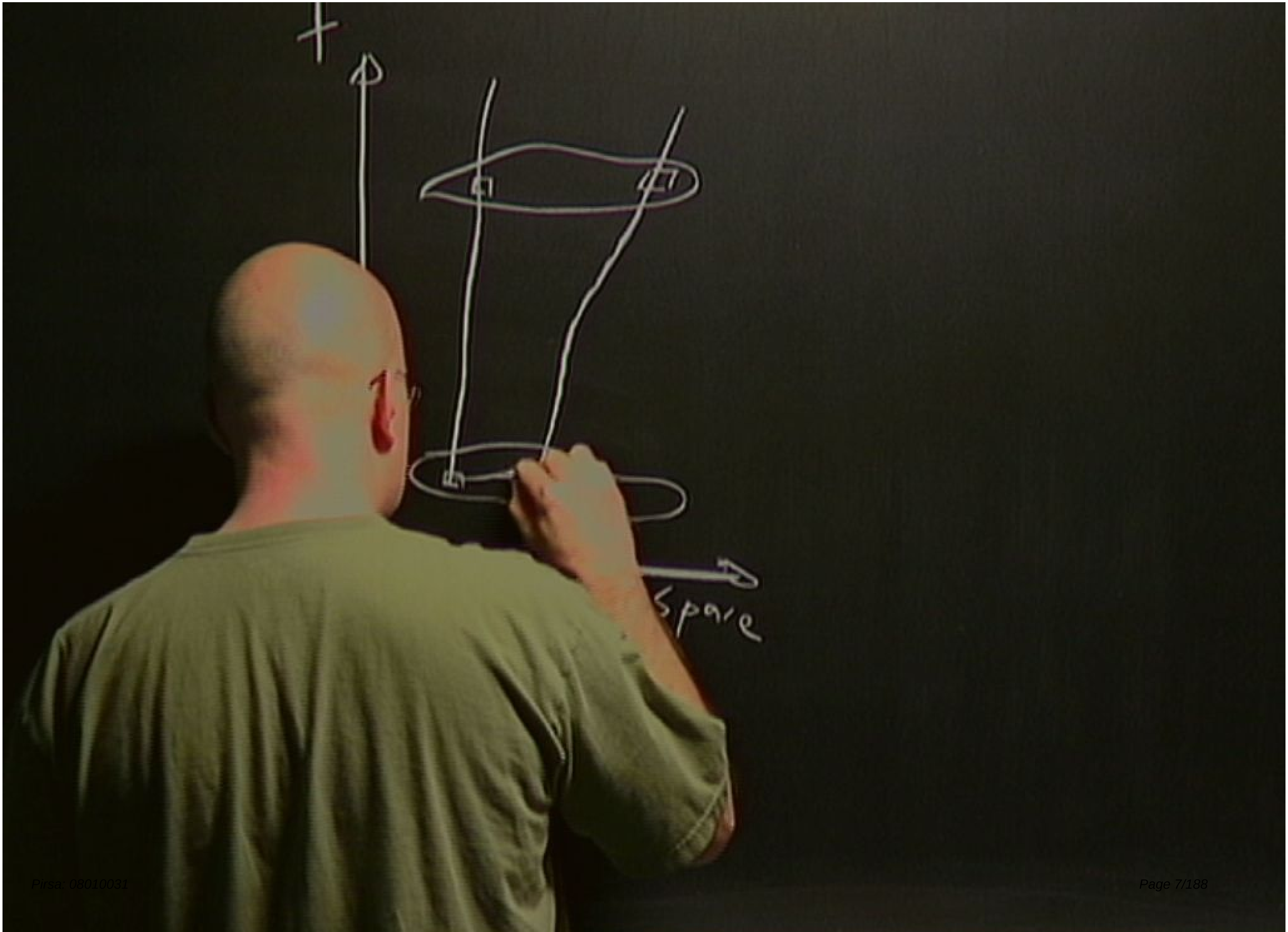

HW1: Feb 6 2008
1.13: #2, #6,
#9, #5

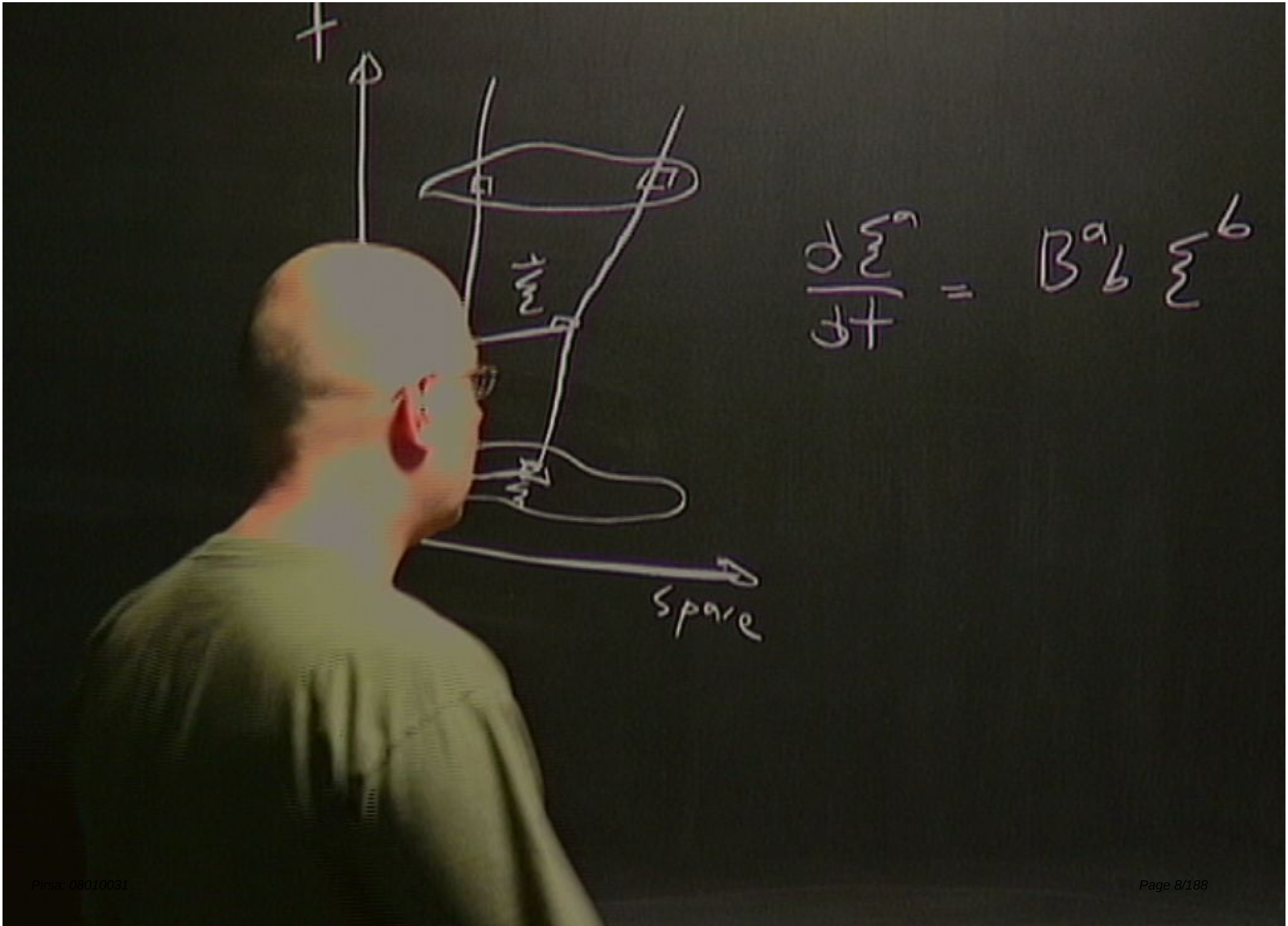
HW 1: Feb 6 2008

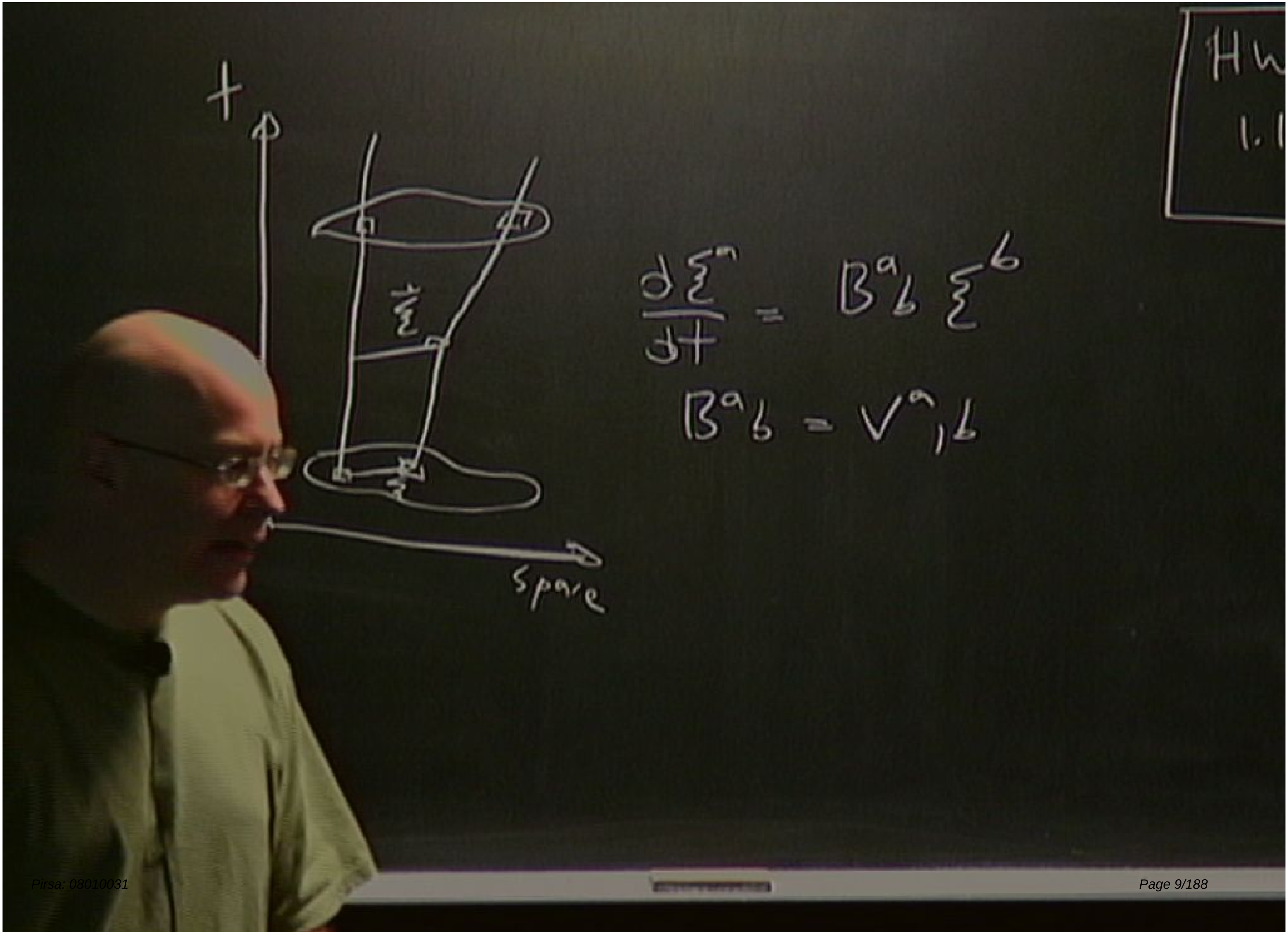
1.13: #2, #6,
#9, #5

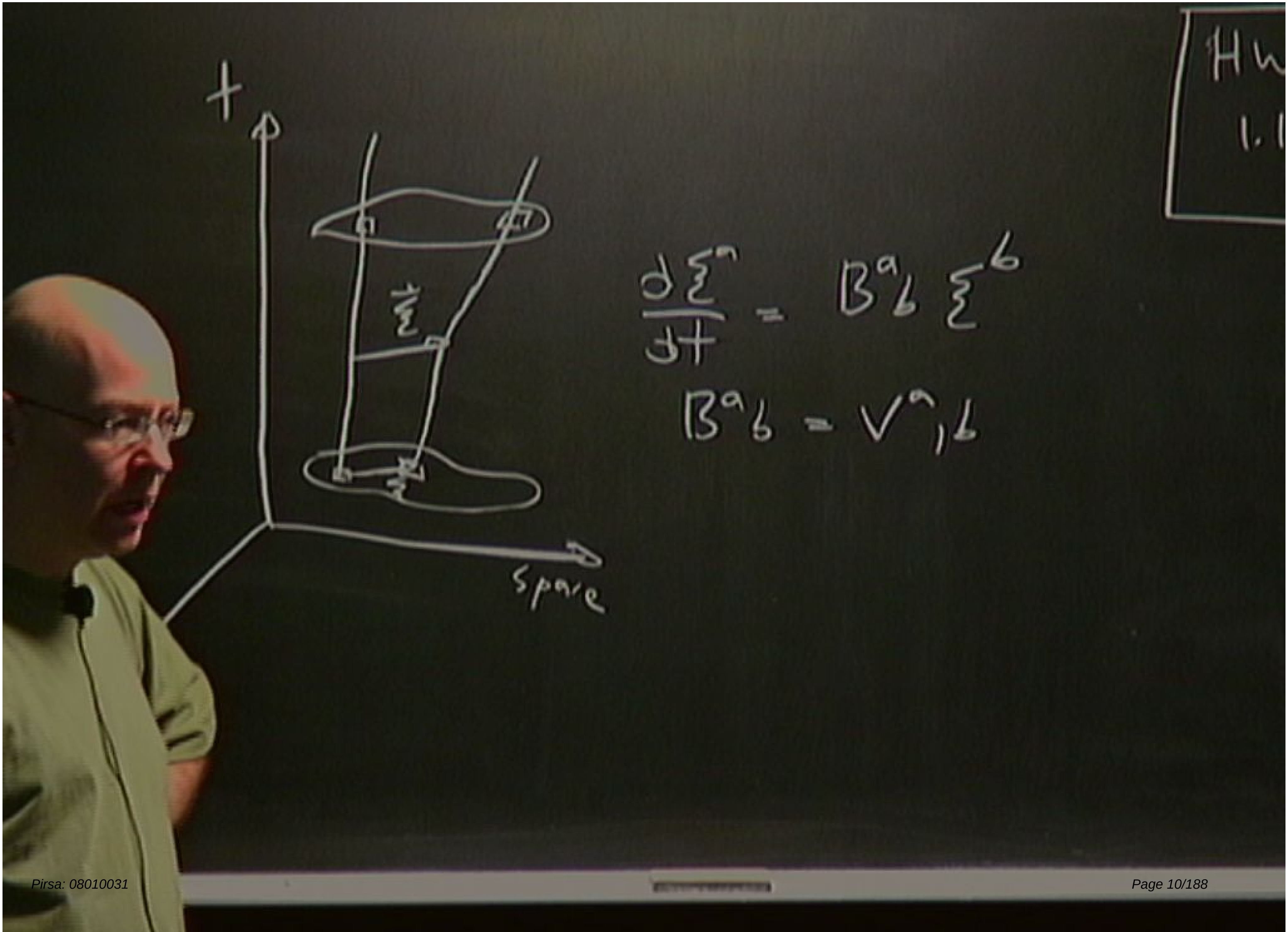

t
space

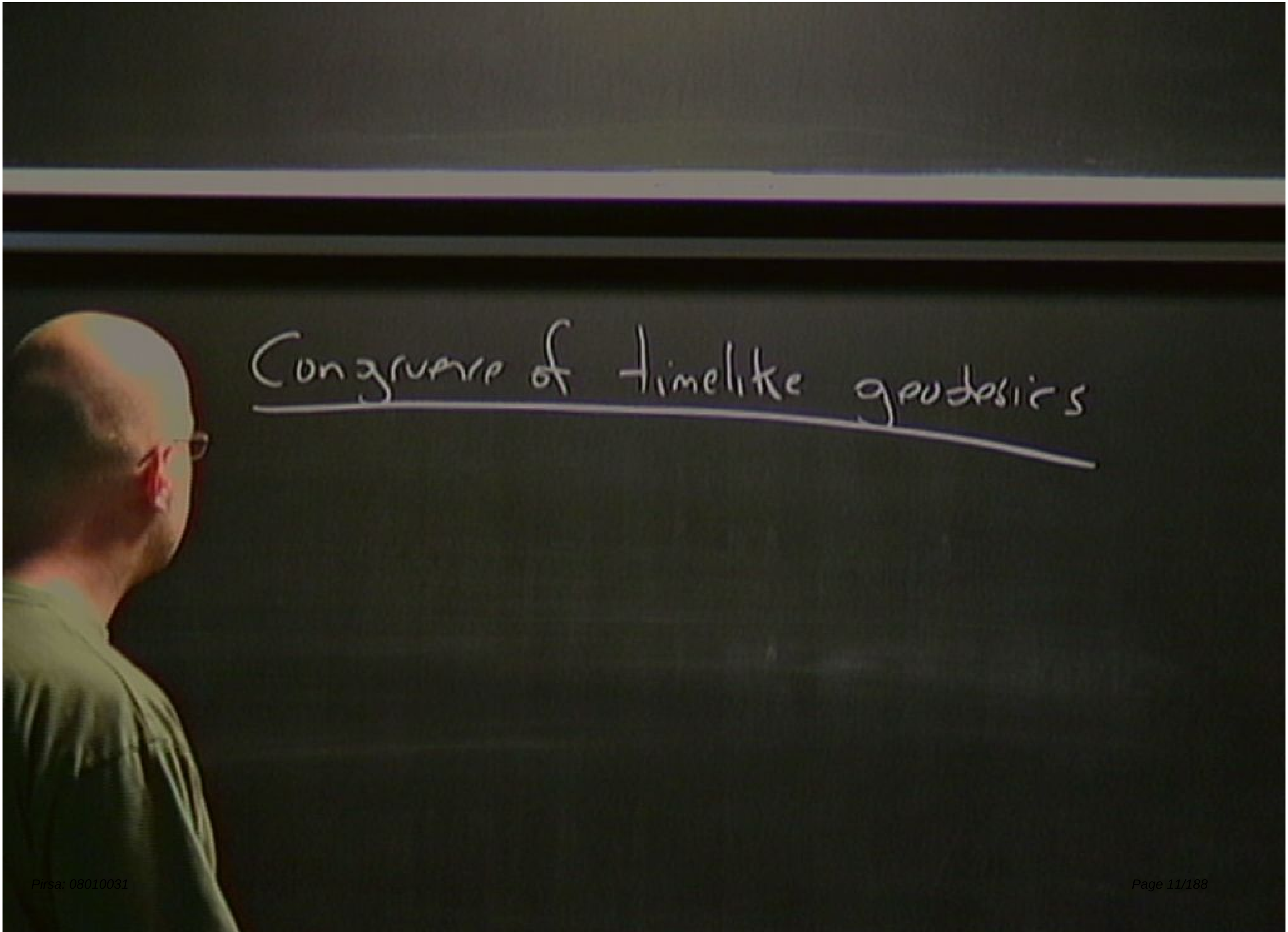

t
Space

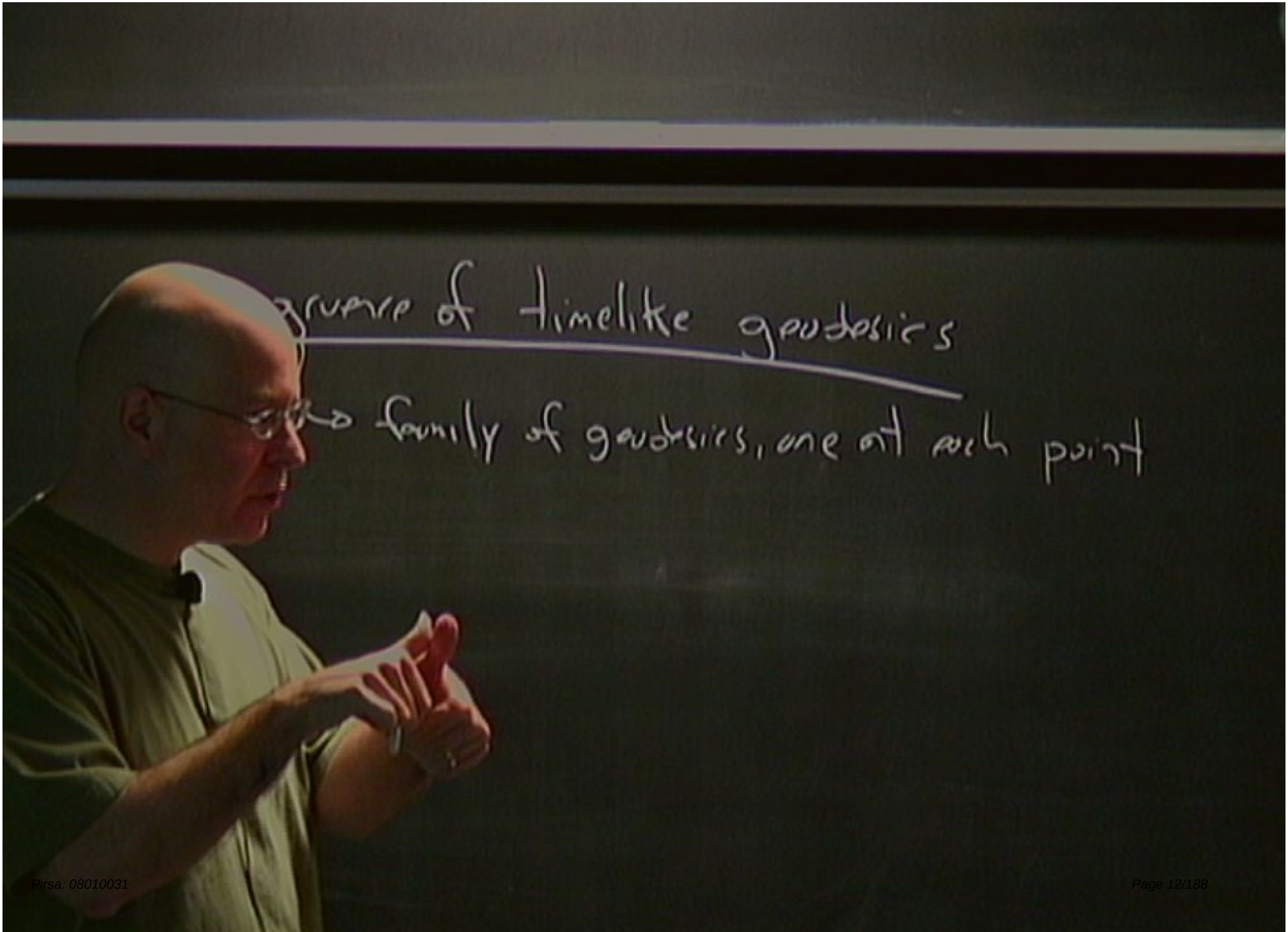

t
space

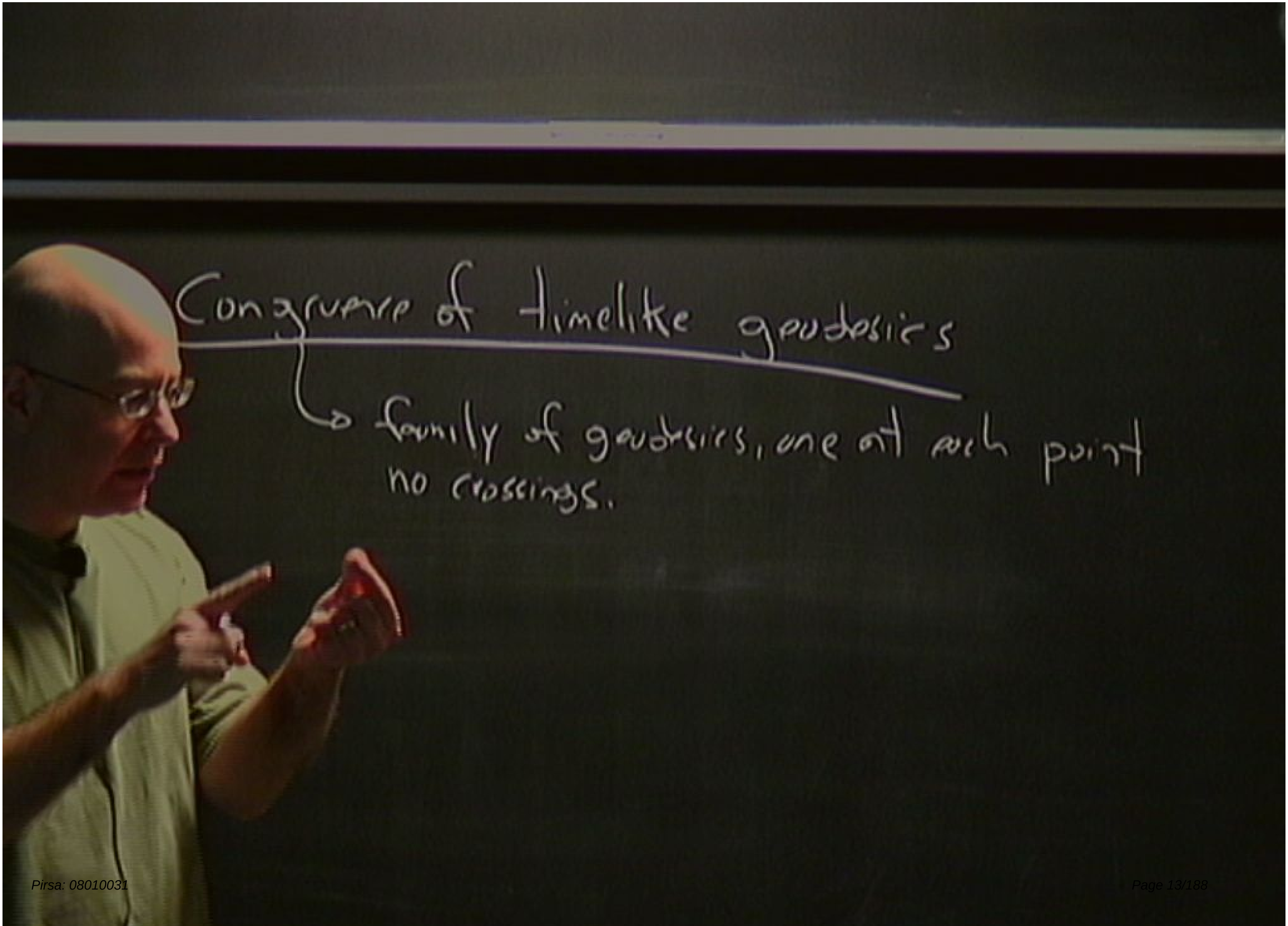

t
ξ
$\frac{d\xi^a}{dt} = B^a{}_b\,\xi^b$
space

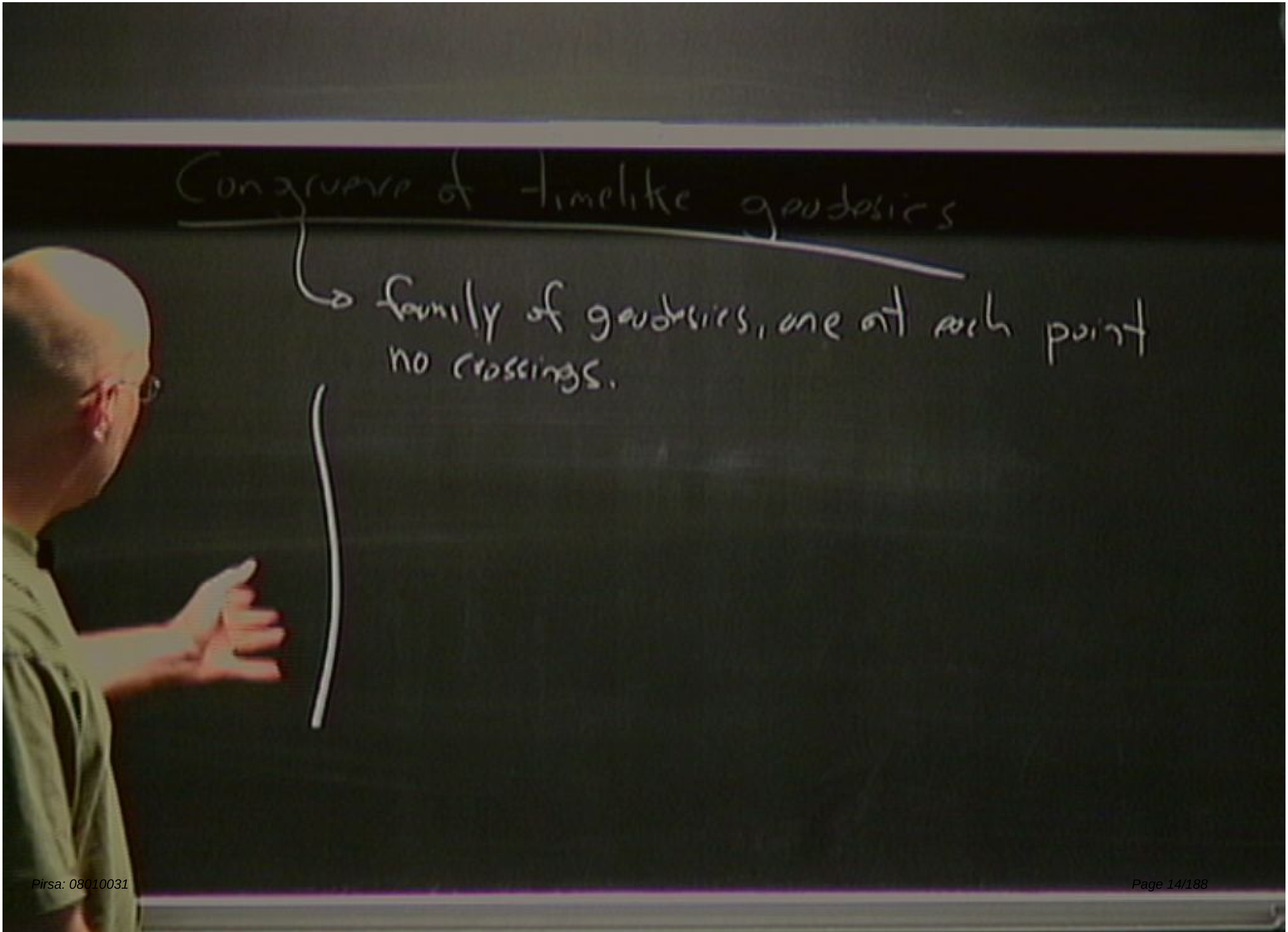

t
space
$\frac{d\xi^a}{dt} = B^a{}_b\,\xi^b$
$B^a{}_b = v^a{}_{,b}$
HW
1.1

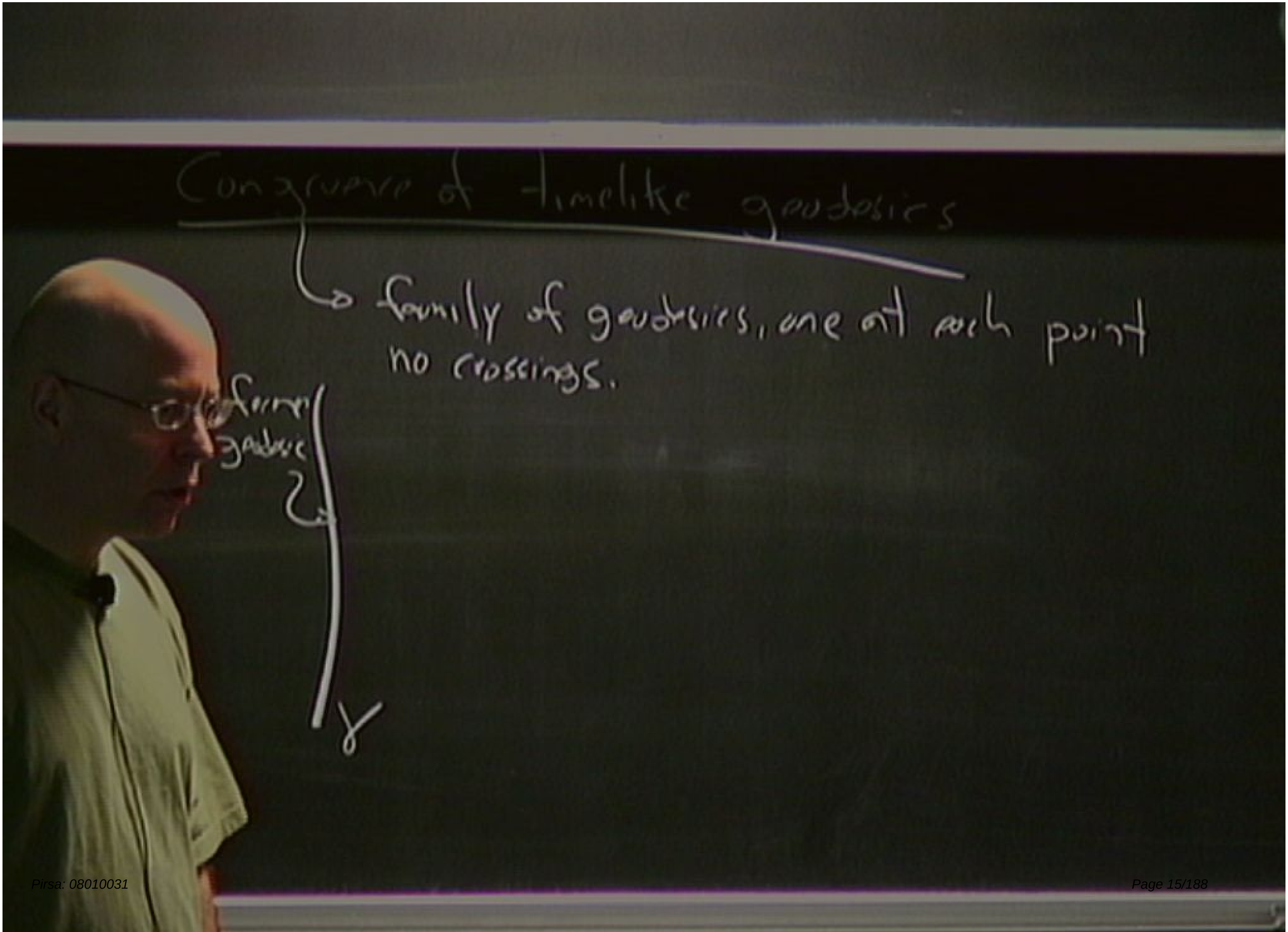

t
space
$\frac{d\xi^a}{dt} = B^a{}_b\, \xi^b$
$B^a{}_b = V^a{}_{;b}$

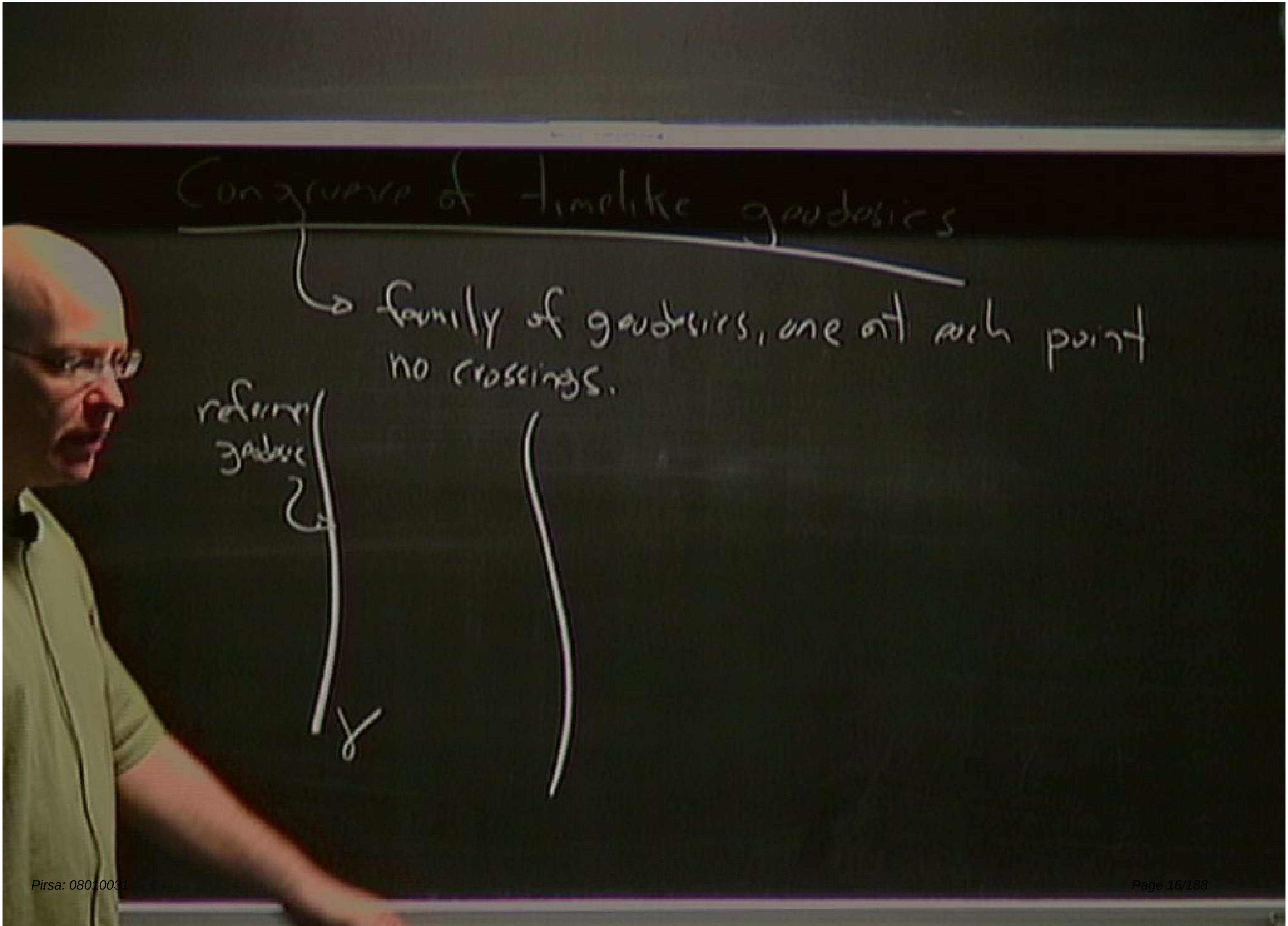

Congruence of timelike geodesics

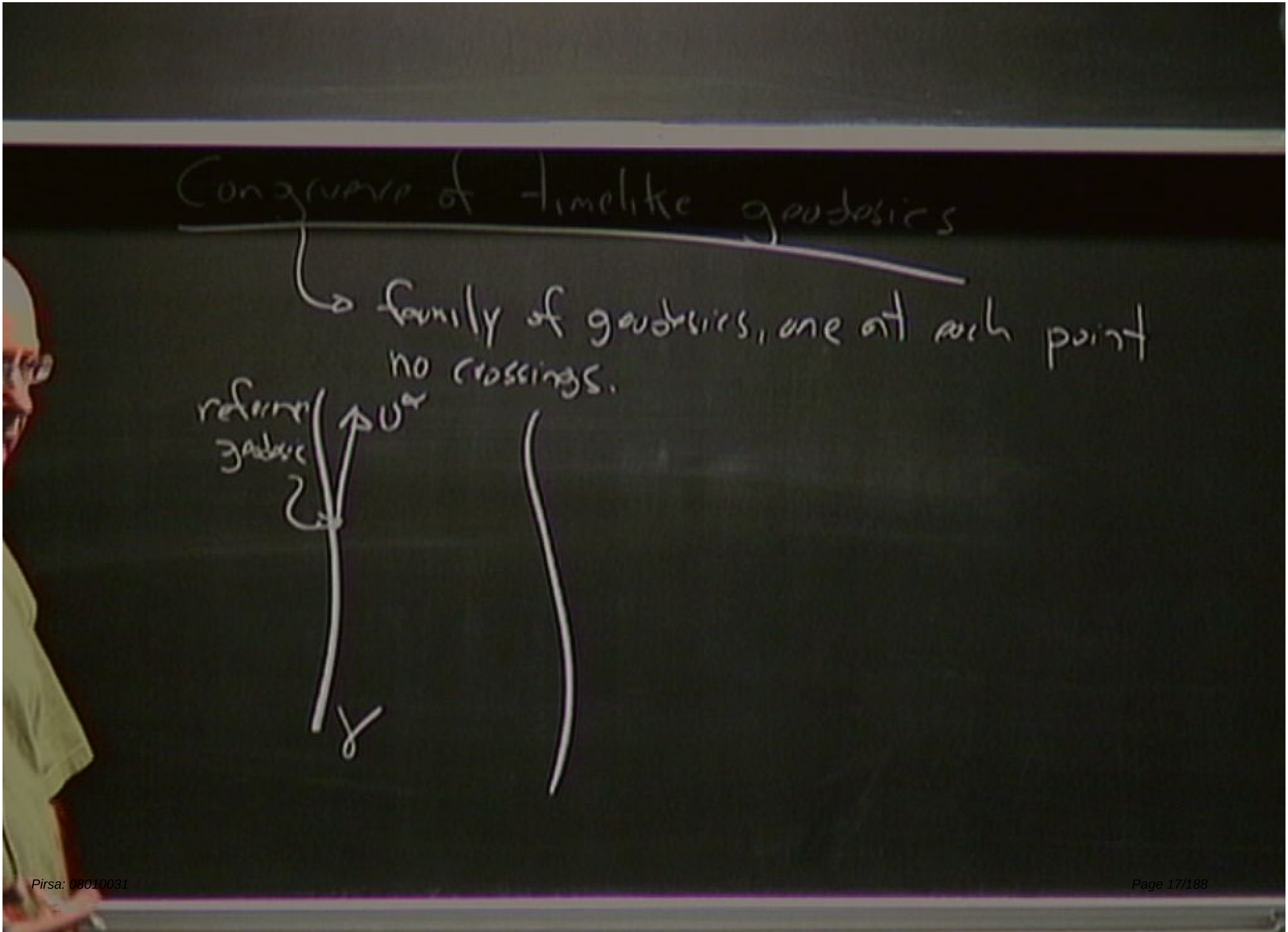

gruence of timelike geodesics
→ family of geodesics, one at each point

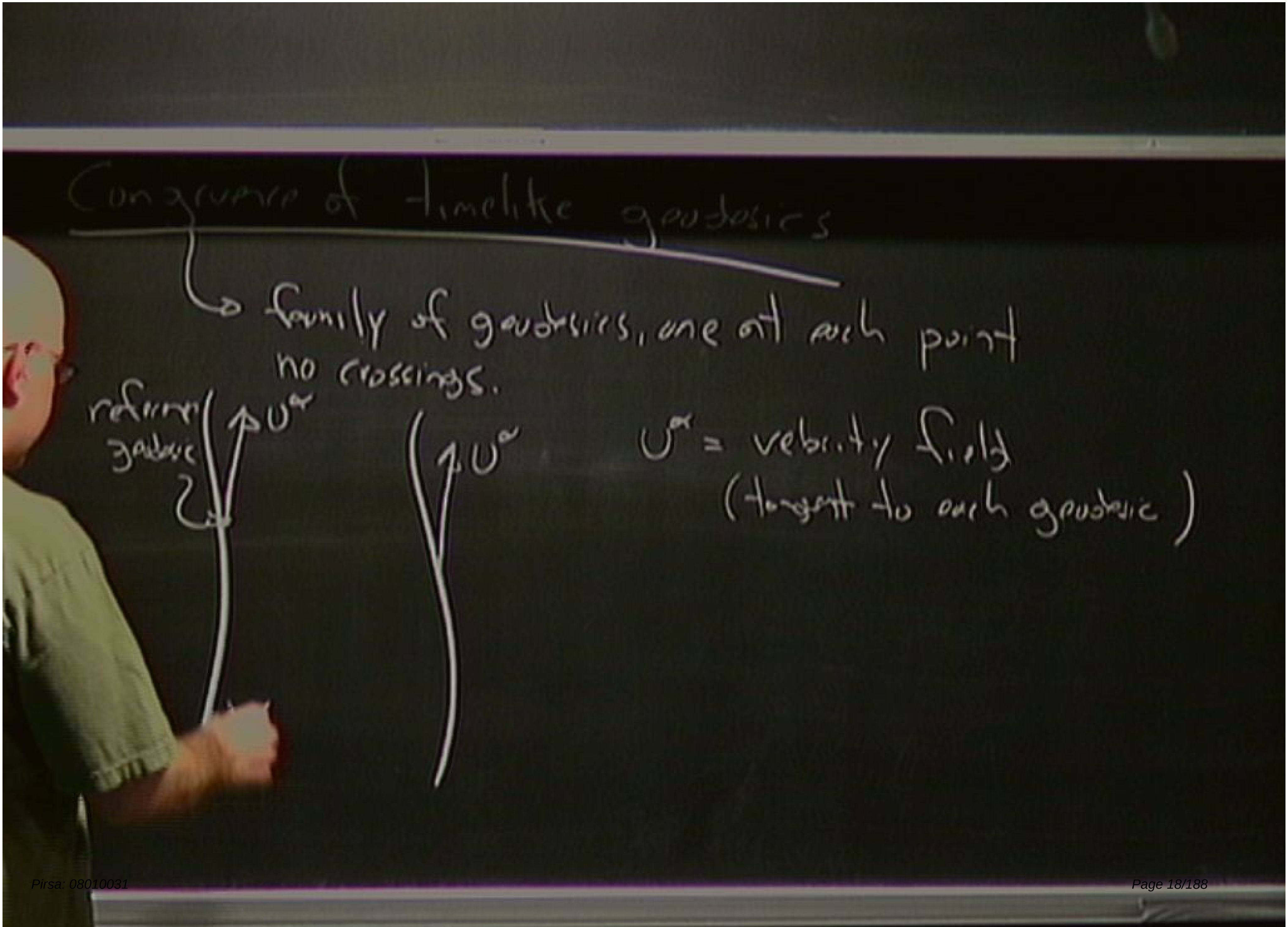

Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.


Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.


Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.
geodesic
γ

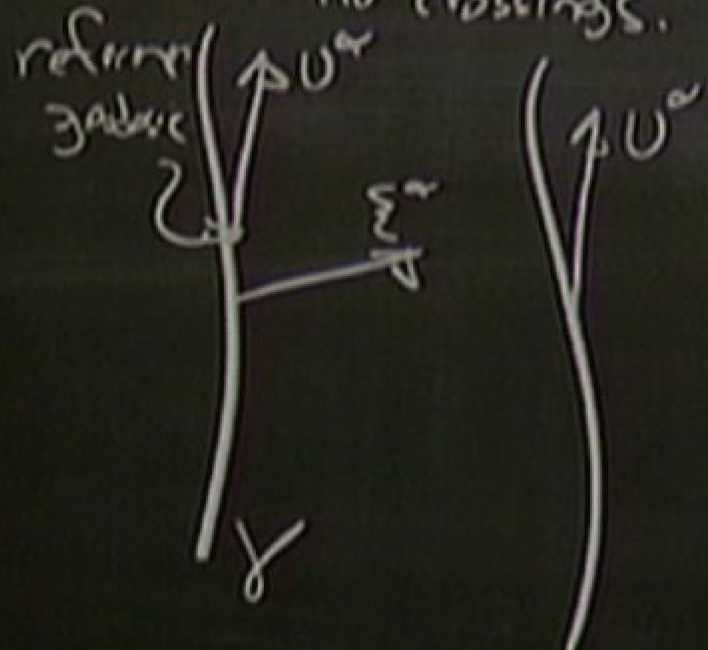

Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.
reference geodesic
γ


Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.
reference geodesic
u^α
γ

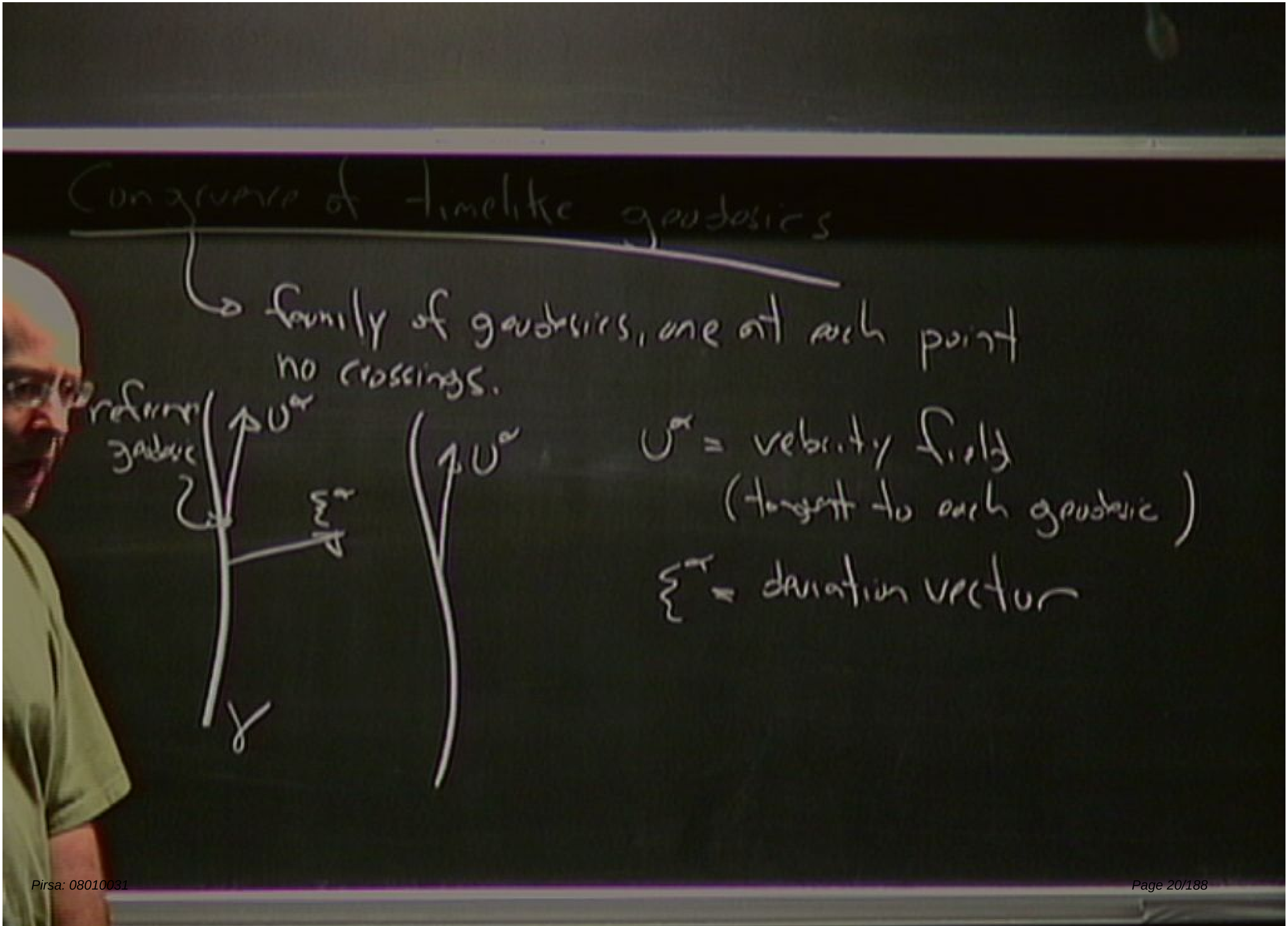

Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.
U^α
U^α
U^α = velocity field
(tangent to each geodesic)

Congruence of timelike geodesics

→ family of geodesics, one at each point
no crossings.

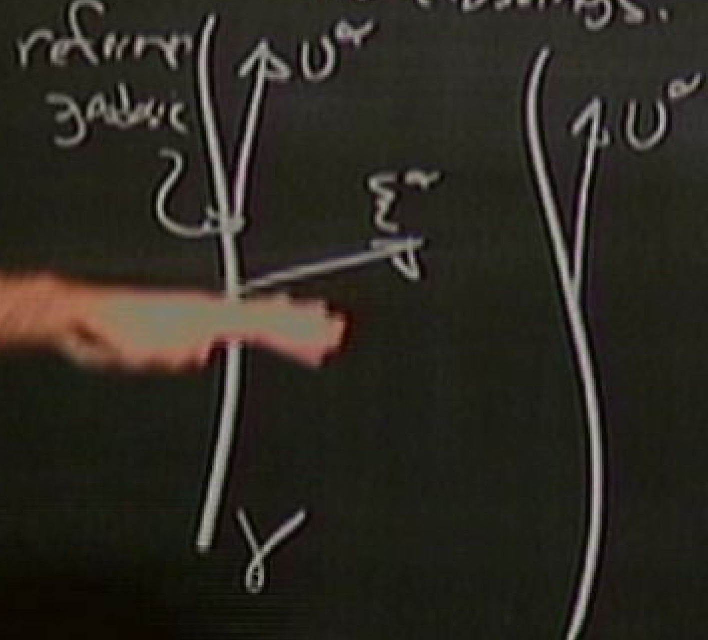


U^α = velocity field
(tangent to each geodesic)


Congruence of timelike geodesics
family of geodesics, one at each point
no crossings.
reference geodesic
u^α
ξ^α
γ
u^α
u^α = velocity field
(tangent to each geodesic)
ξ^α = deviation vector

Congruence of timelike geodesics

→ family of geodesics, one at each point
no crossings.

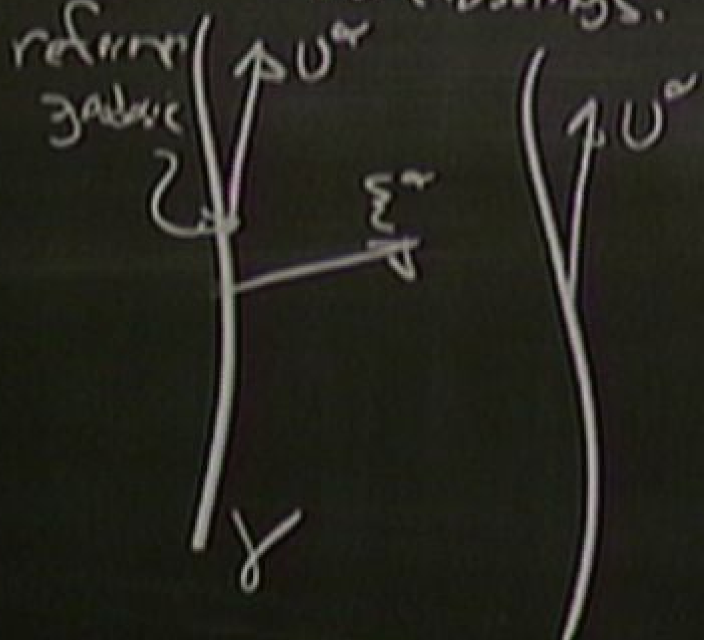


u^α = velocity field
(tangent to each geodesic)

ξ^α = deviation vector

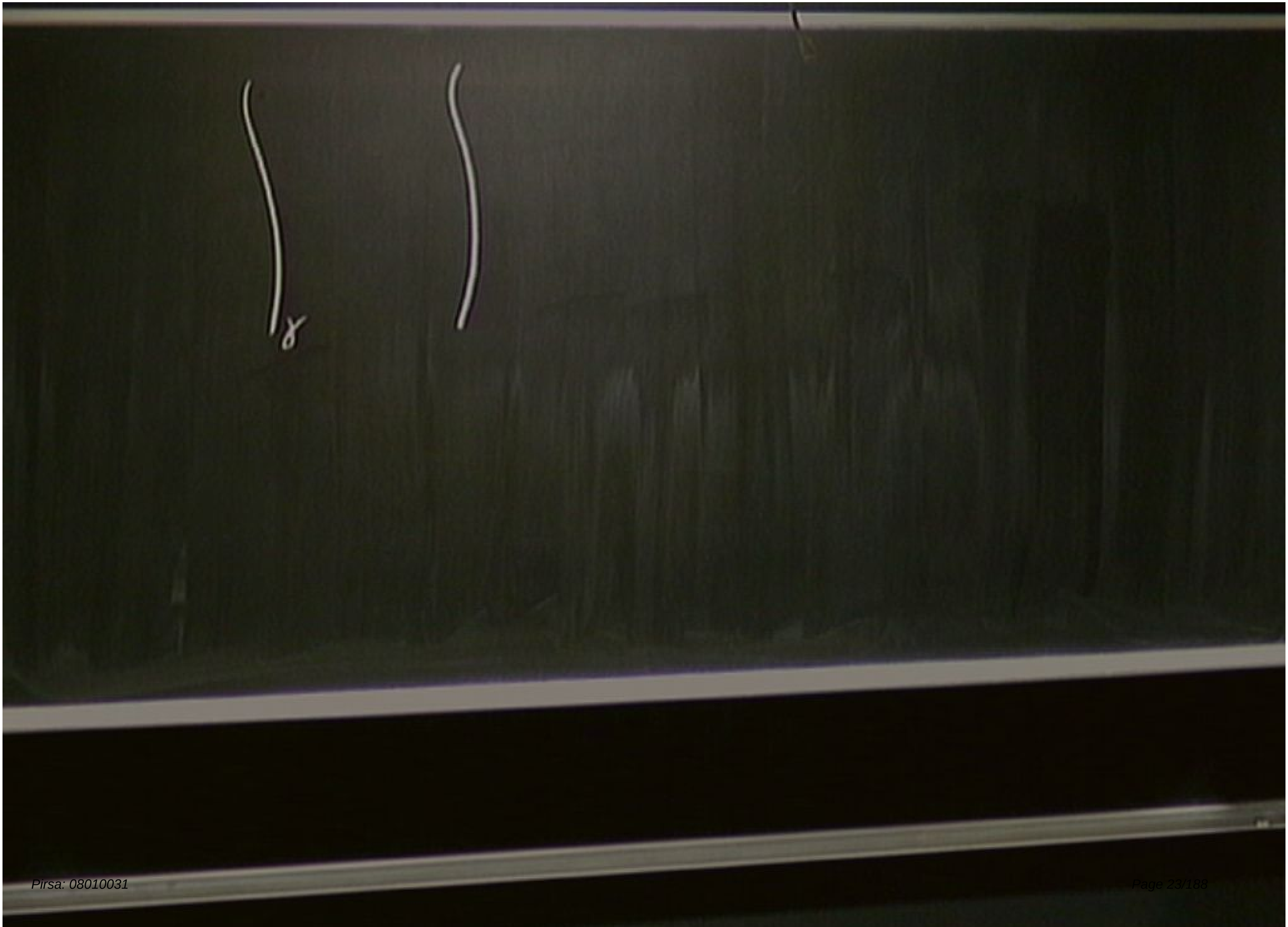
Congruence of timelike geodesics

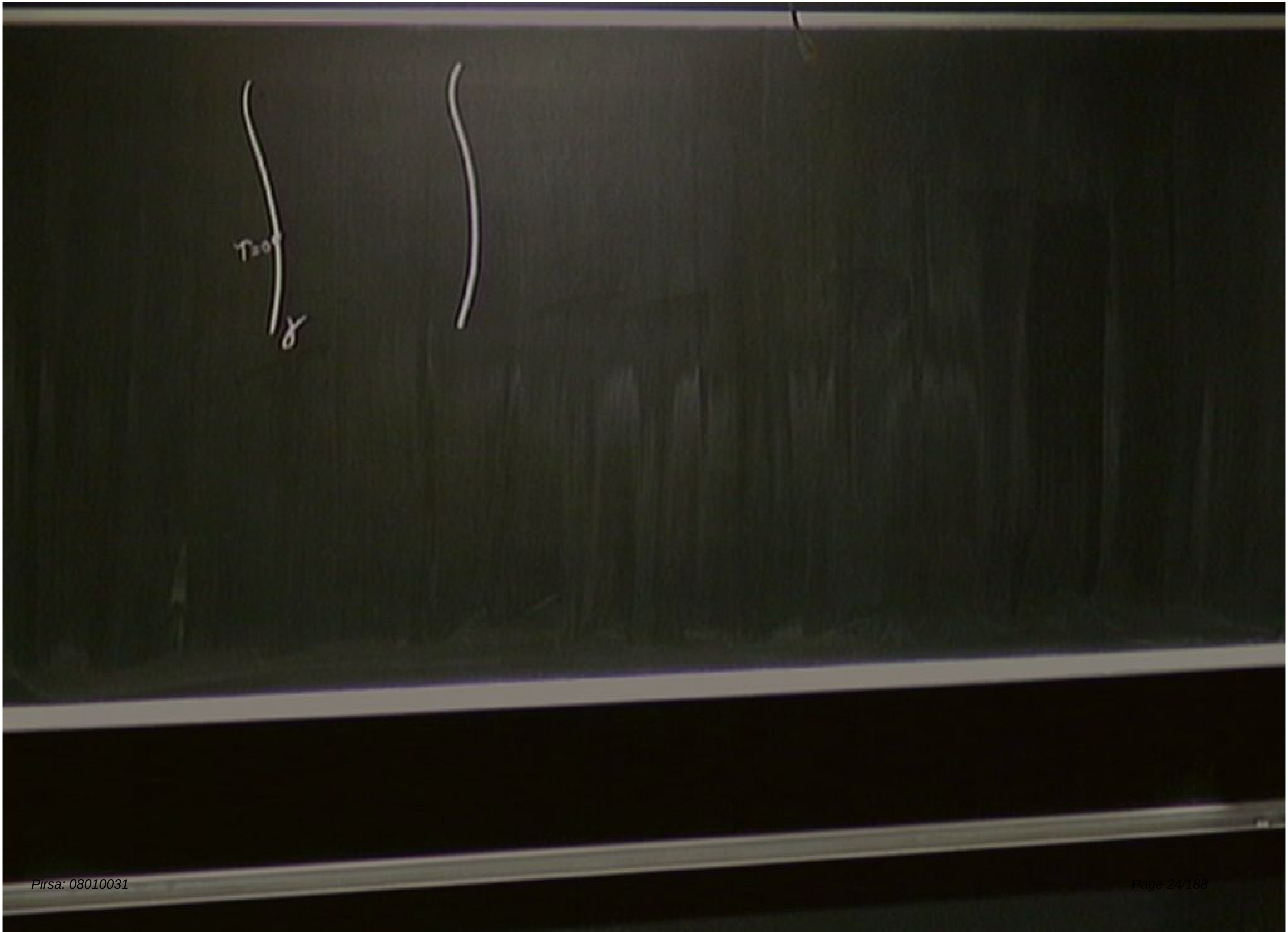
→ family of geodesics, one at each point
no crossings.

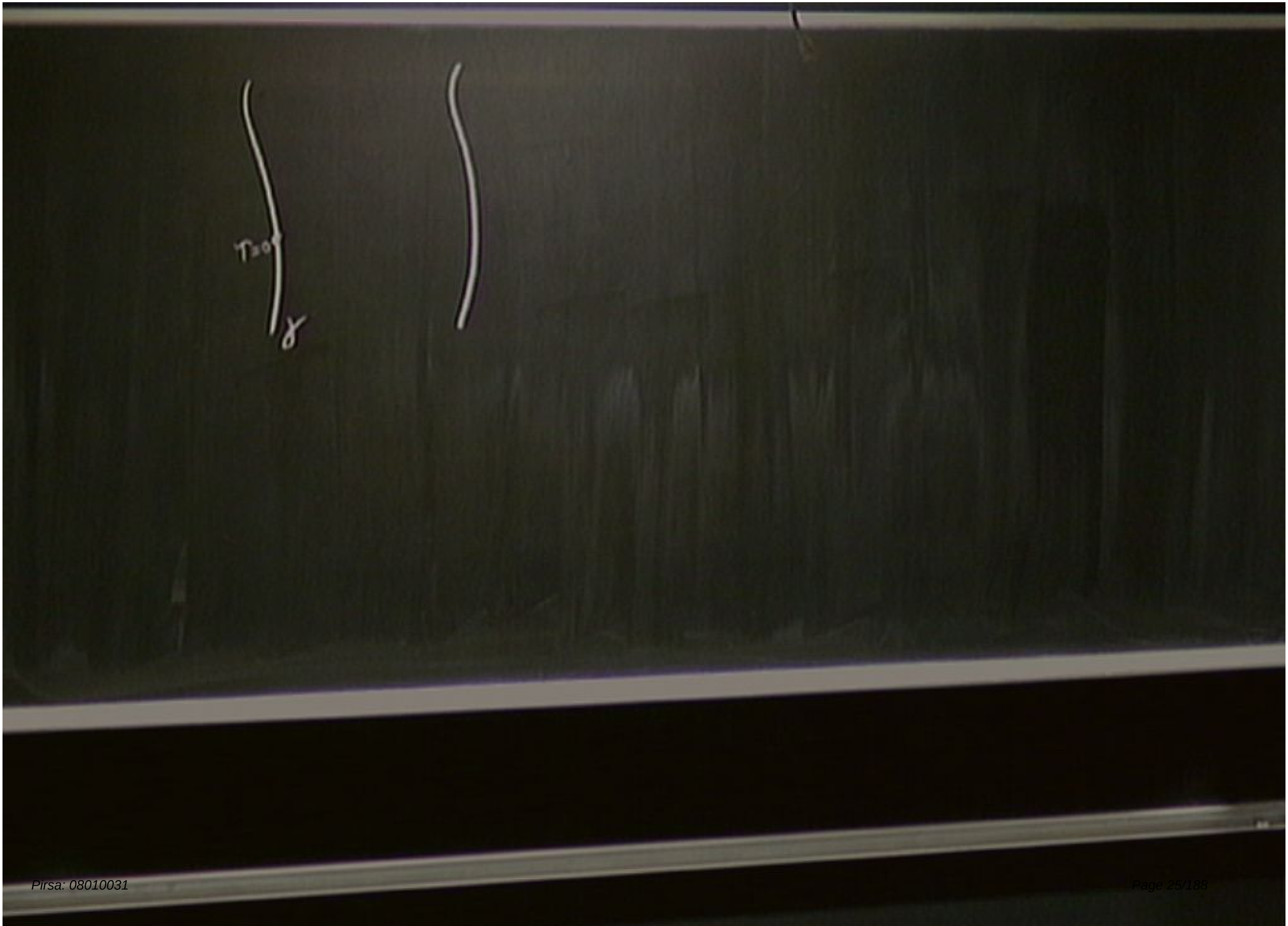


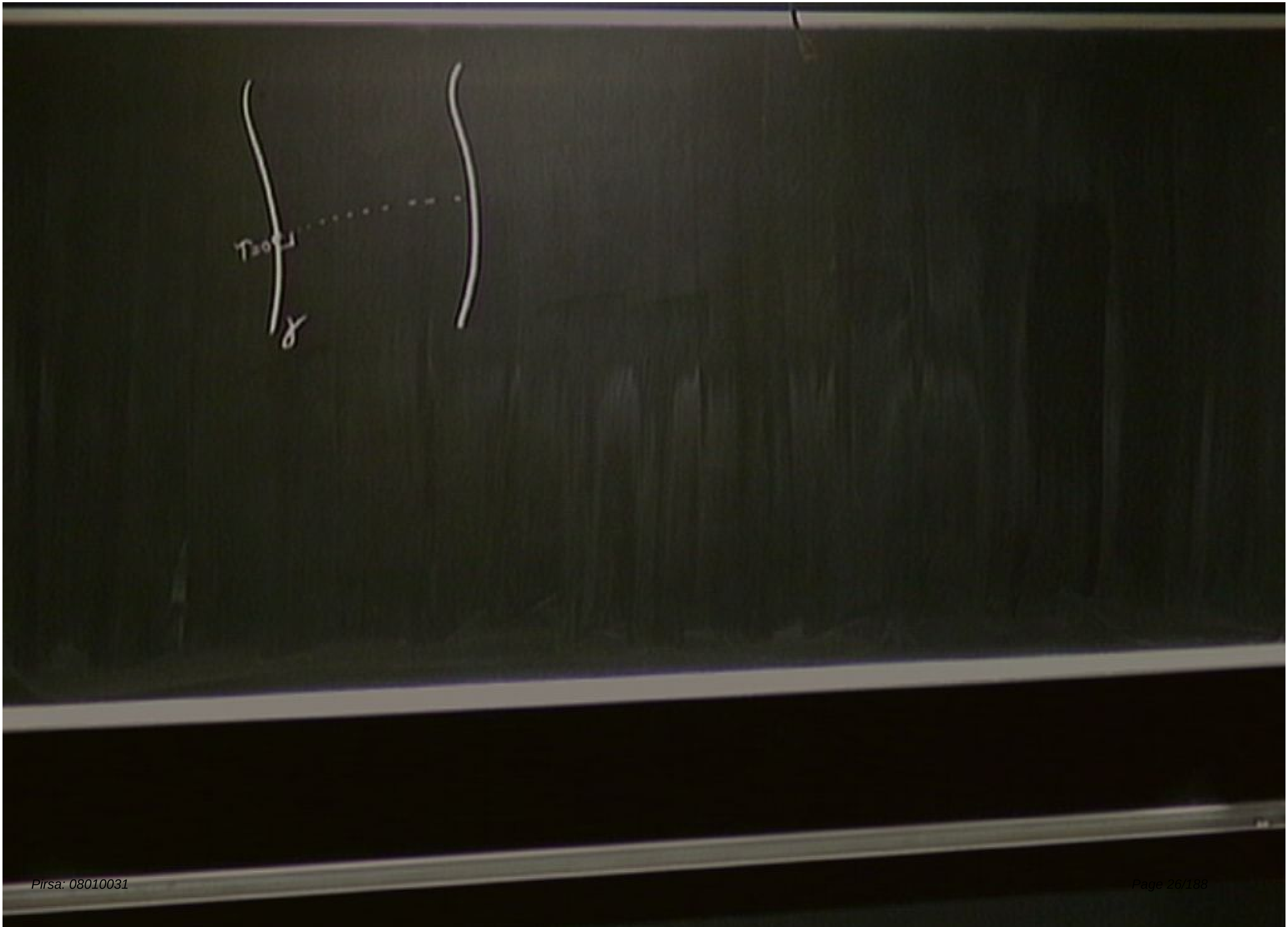
U^α = velocity field
(tangent to each geodesic)

ξ^α = deviation vector

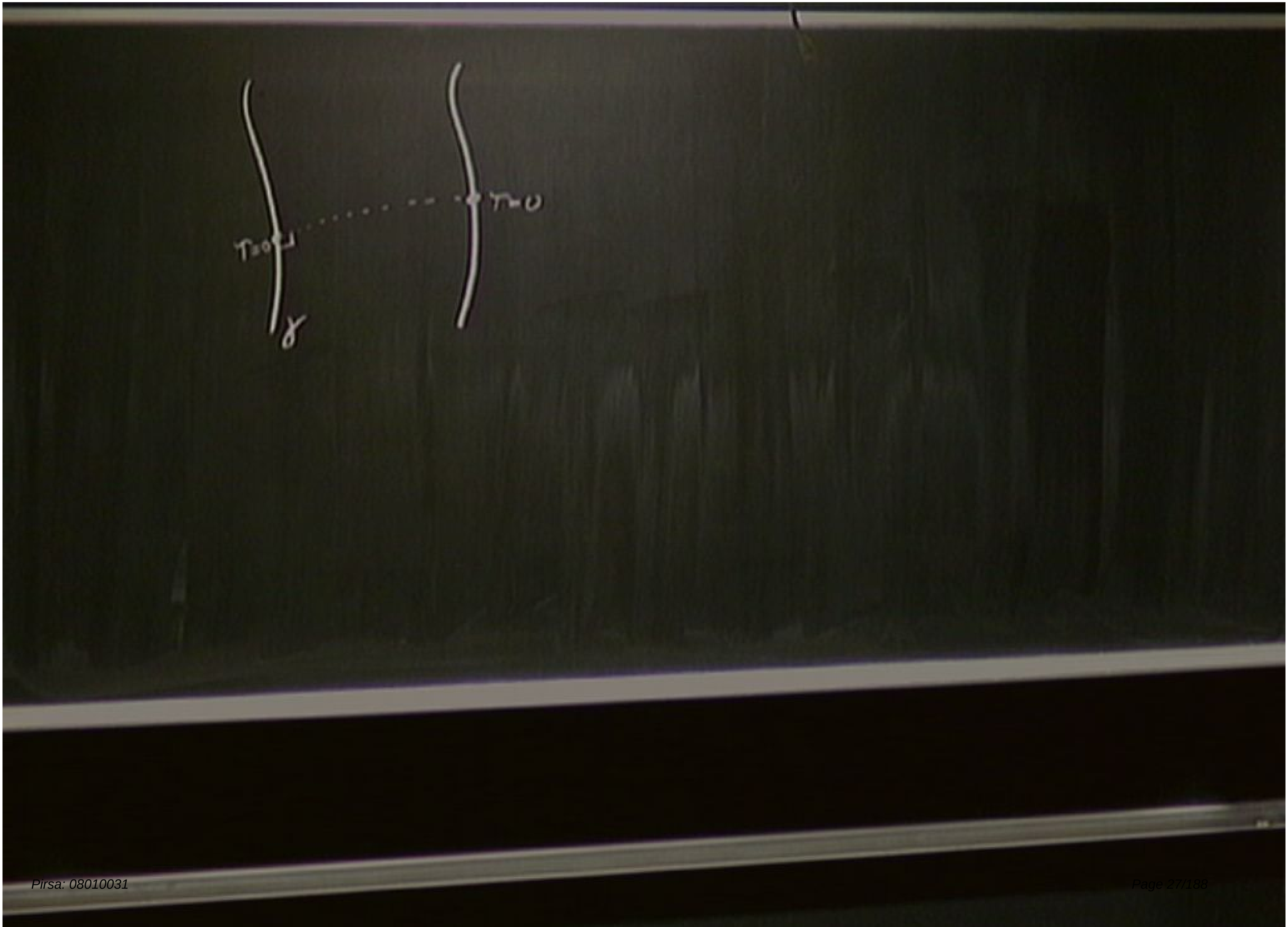

γ

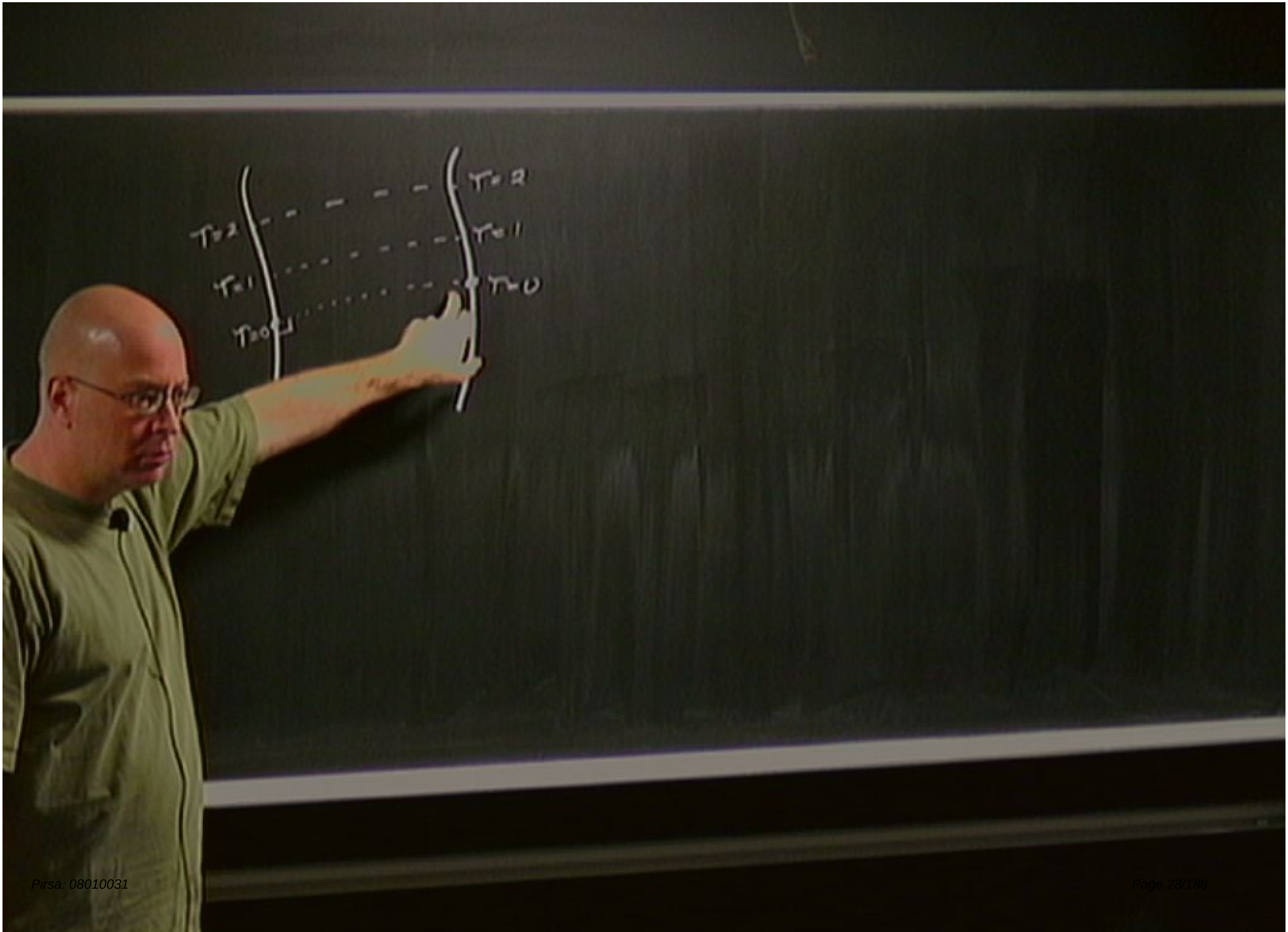

τ=0
γ

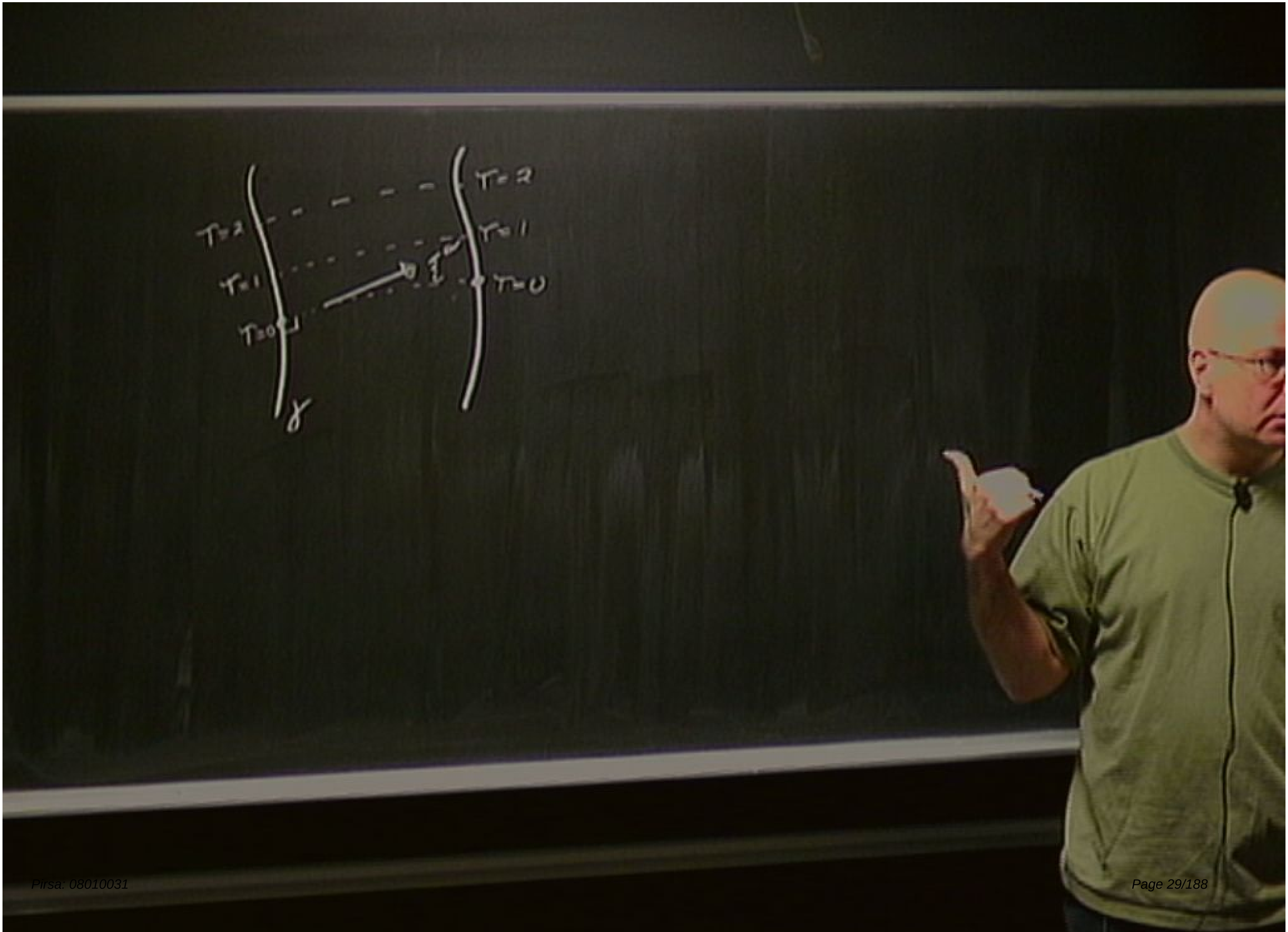

τ=0
γ

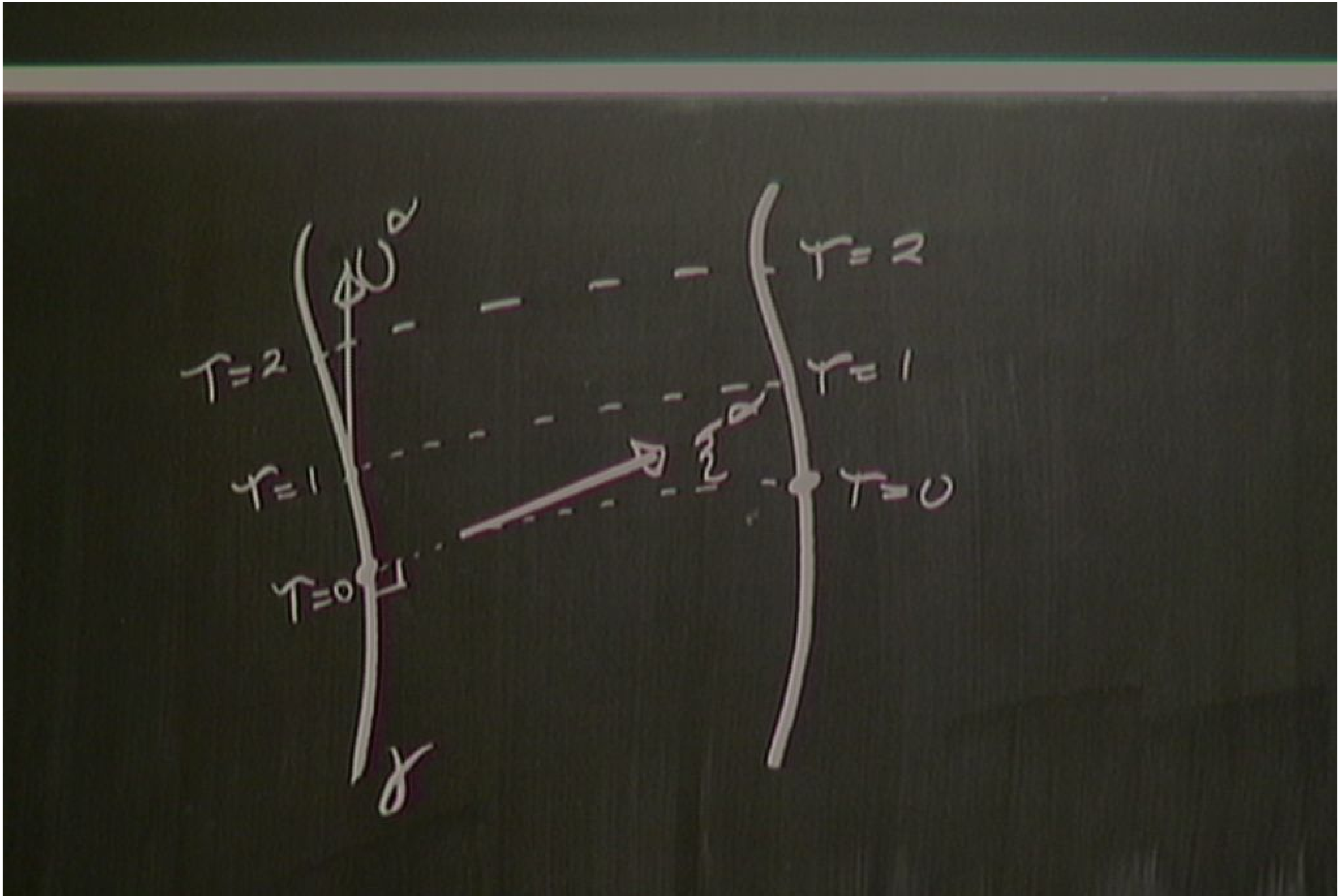


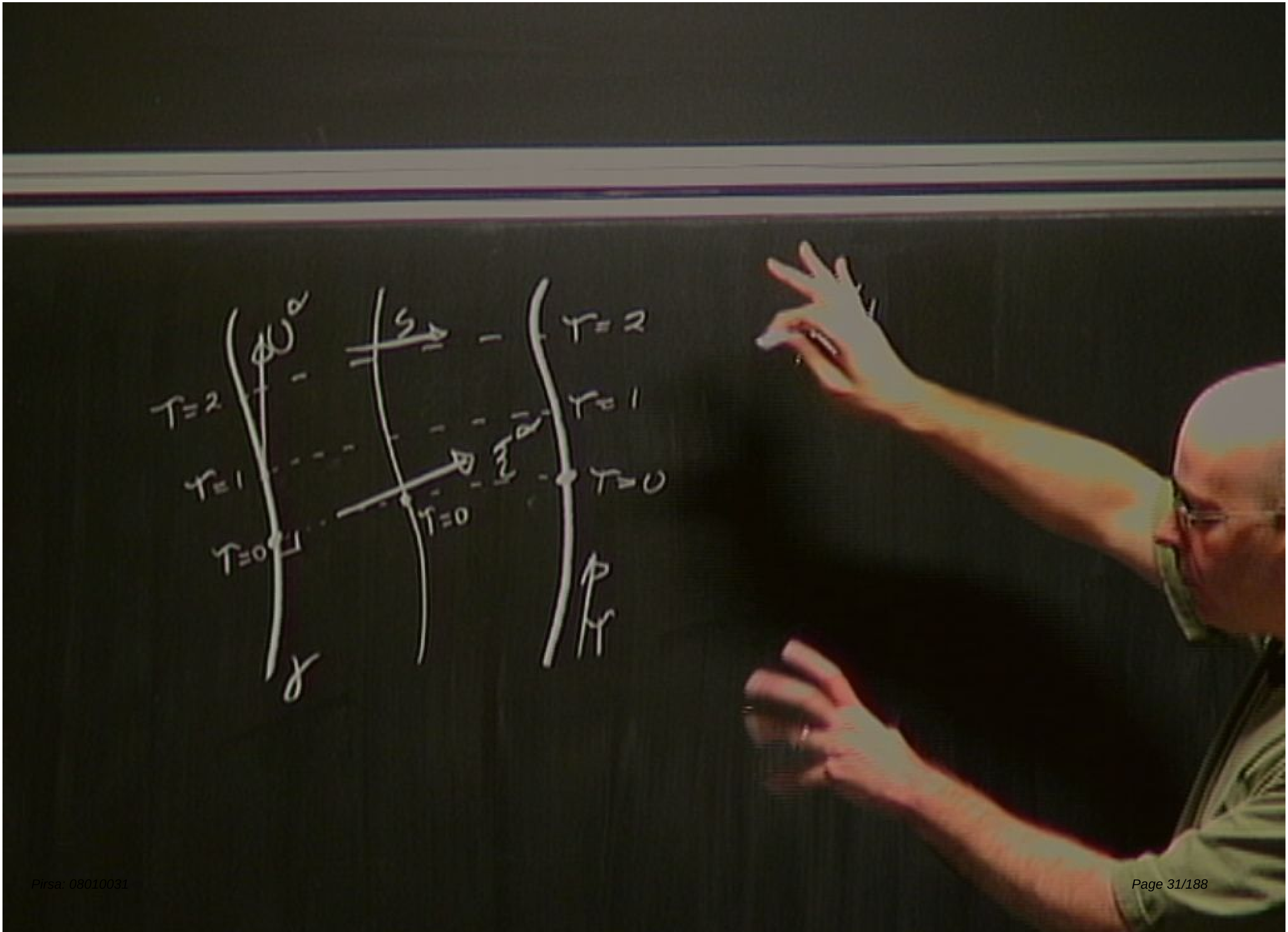
τ=0
γ

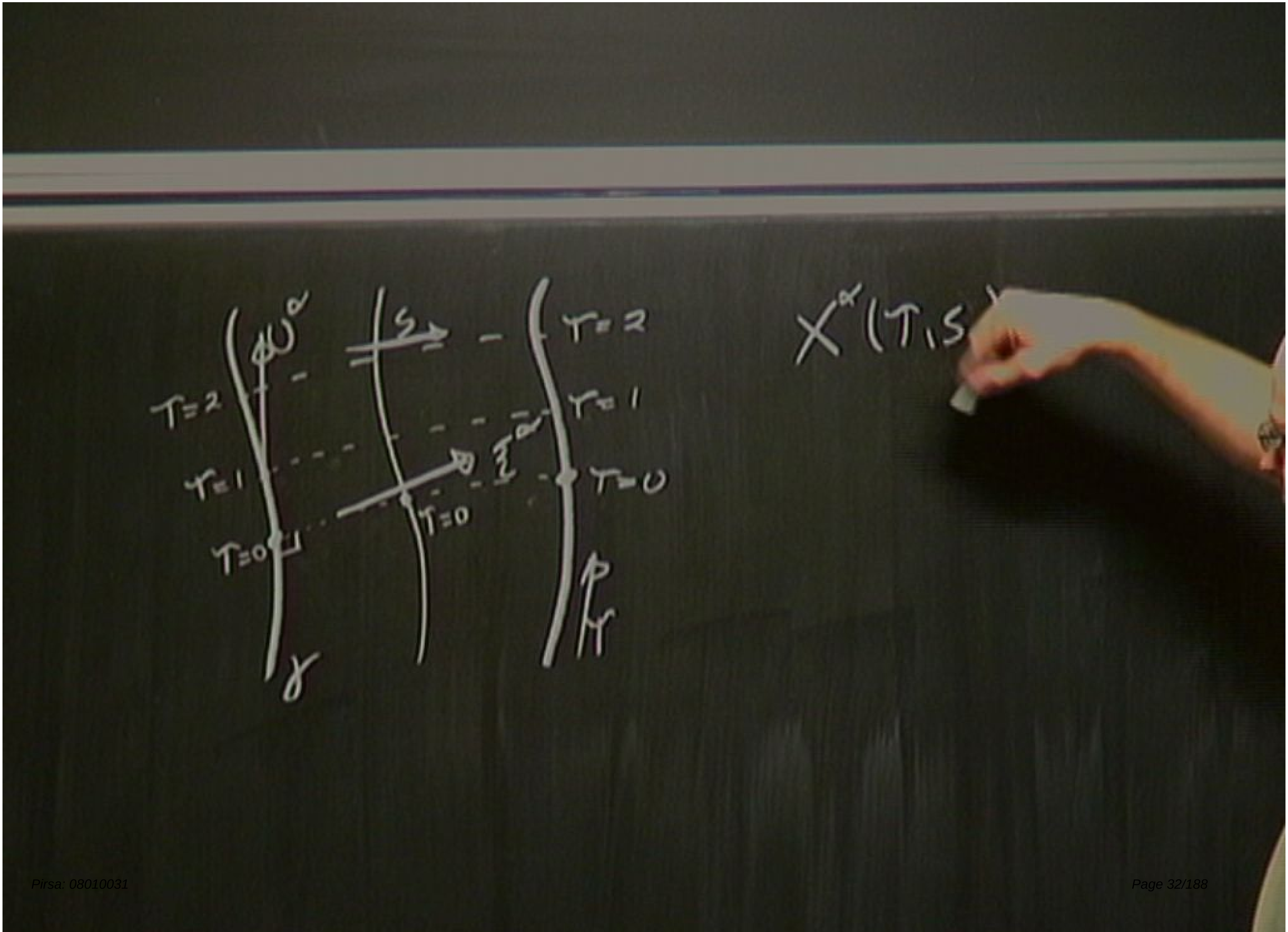

T=0
T=0
γ

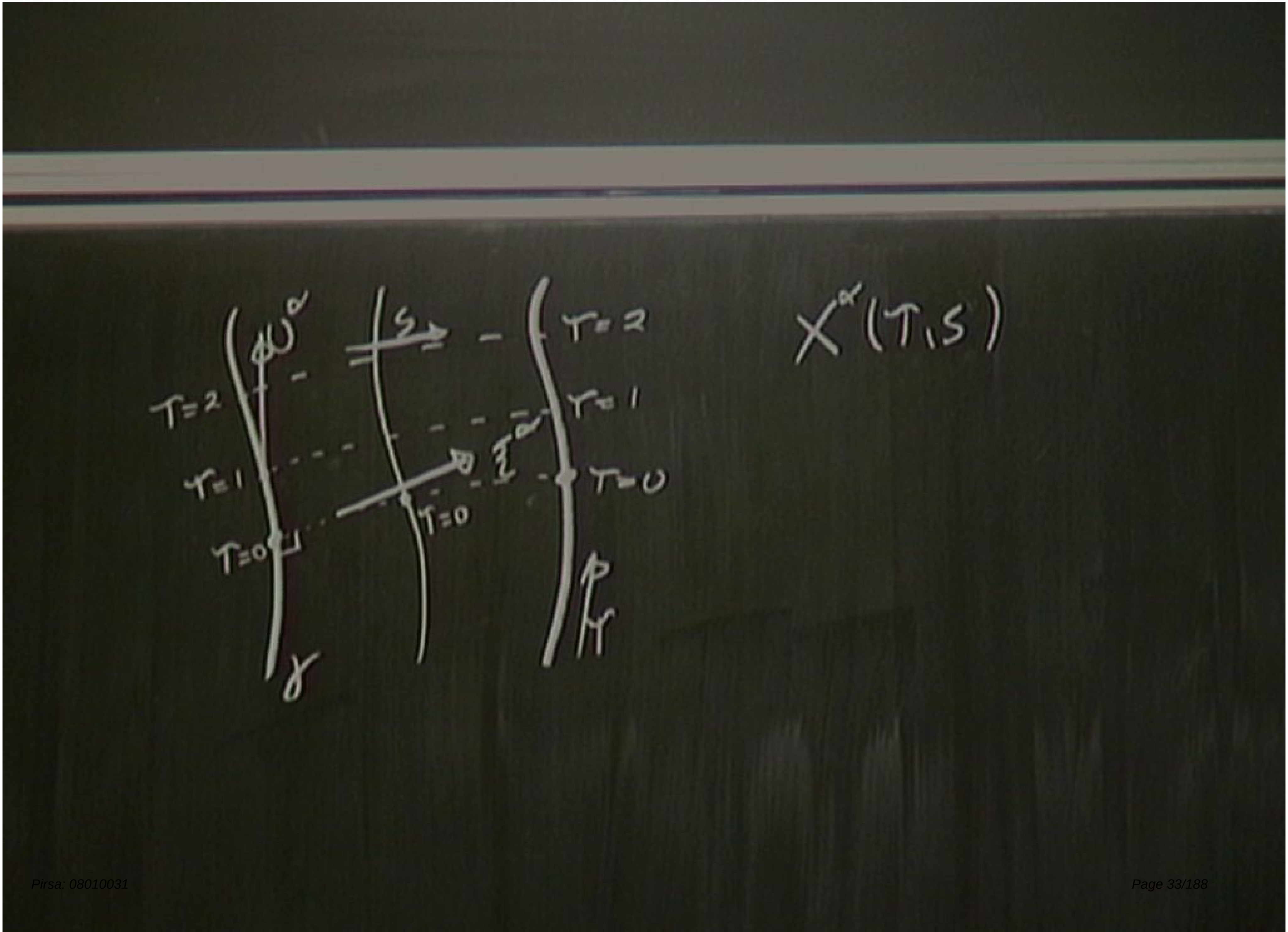

T=2
T=1
T=0
T=2
T=1
T=0

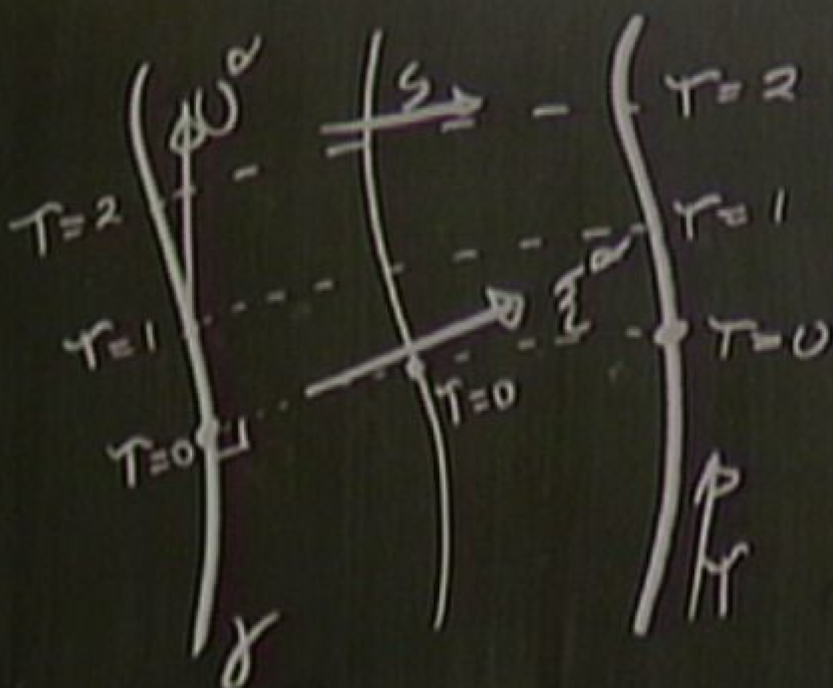

T=2
T=1
T=0
T=2
T=1
T=0
γ


U^a
$\tau=2$
$\tau=1$
$\tau=0$
ξ^a
$\tau=2$
$\tau=1$
$\tau=0$
γ


τ=2
τ=1
τ=0
τ=0
τ=2
τ=1
τ=0

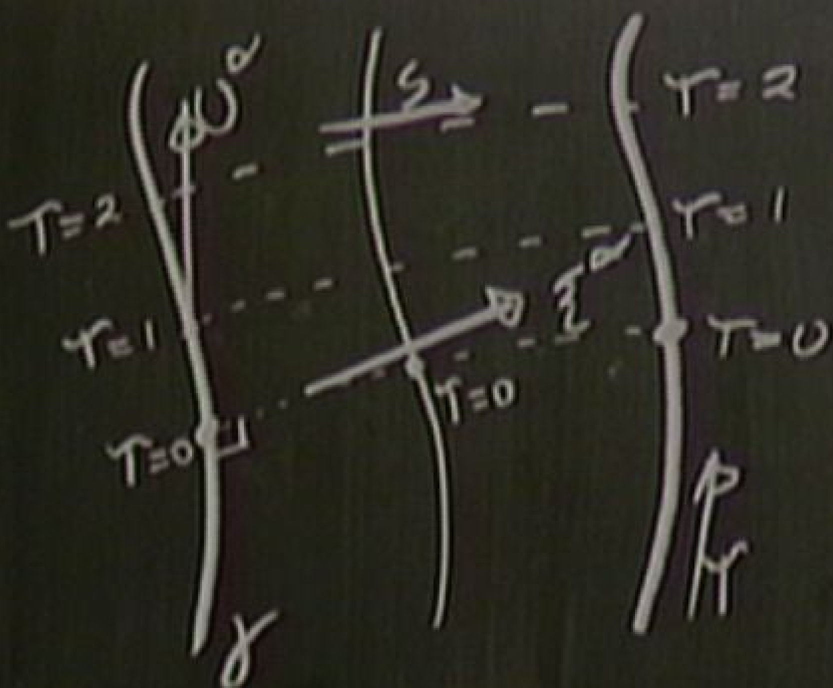

u^α
S
$\tau=2$
$\tau=1$
$\tau=0$
ξ^α
γ
$X^\alpha(\tau, S)$


$X^\alpha(\tau, s)$
u^α
s
ξ^α
$\tau=2$
$\tau=1$
$\tau=0$
$\tau=0$
$\tau=2$
$\tau=1$
$\tau=0$
γ



$$X^\alpha(\tau, s)$$

$$\frac{\partial X^\alpha}{\partial \tau}$$



$$X^\alpha(\tau, s)$$

$$\frac{\partial X^\alpha}{\partial \tau} = u^\alpha$$

$$x^\alpha(\tau, s)$$

$$\frac{\partial x^\alpha}{\partial \tau} = u^\alpha \qquad \frac{\partial x^\alpha}{\partial s} = \xi^\alpha$$

$$\frac{\partial X^\alpha}{\partial T} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial S} = \xi^\alpha$$

$$\frac{\partial U^\alpha}{\partial S} = \frac{\partial \xi^\alpha}{\partial T}$$

$$\frac{\partial x^\alpha}{\partial T} = u \qquad\qquad \frac{\partial x^\alpha}{\partial S} = \xi$$

$$\frac{\partial u^\alpha}{\partial S} = \frac{\partial \xi^\alpha}{\partial T}$$

$$u^\alpha{}_{\beta}\, \xi^\beta$$

$$\frac{\partial x^\alpha}{\partial T} = U \qquad \frac{\partial x^\alpha}{\partial S} = \xi$$

$$\frac{\partial U^\alpha}{\partial S} = \frac{\partial \xi^\alpha}{\partial T}$$

$$U^\alpha{}_{;\beta}\, \xi^\beta = \xi^\alpha{}_{;\beta}\, U^\beta$$

$$\frac{\quad}{\partial S} = \frac{\quad}{\partial T}$$

$$U^{\alpha}{}_{;\beta}\,\xi^{\beta} = \xi^{\alpha}{}_{;\beta}\,U^{\beta}$$

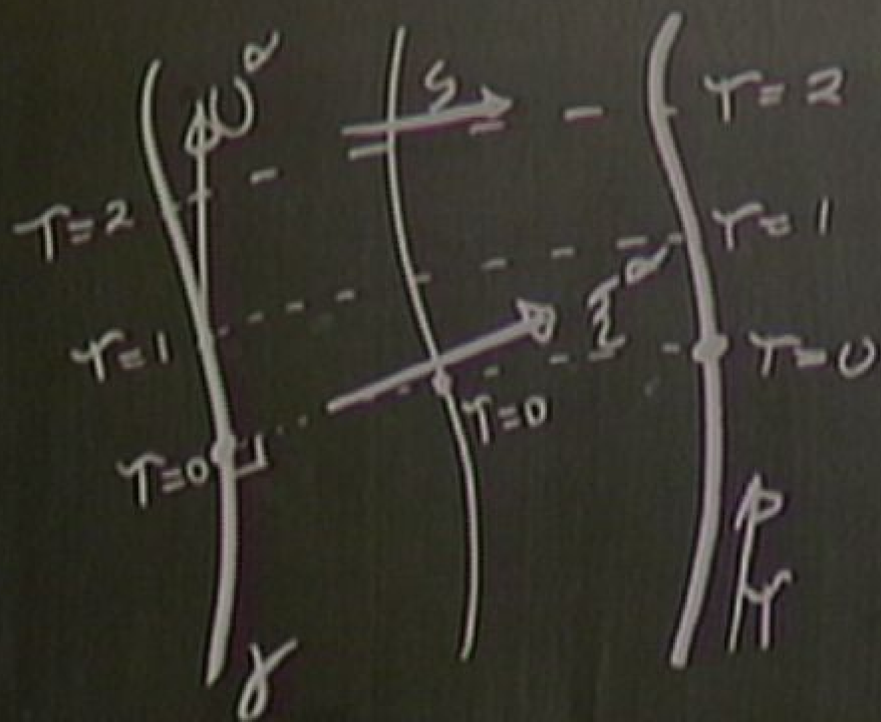
$$\mathcal{L}_U\,\xi^{\alpha} = 0$$

$$\partial S \qquad \partial T$$

$$U^\alpha{}_{;\beta}\,\xi^\beta = \xi^\alpha{}_{;\beta}\,U^\beta$$

$$\mathcal{L}_U\,\xi^\alpha = 0 = \mathcal{L}_\xi\,U^\alpha$$

$$\xi^{\alpha}{}_{j\beta}\, U^{\beta} = U^{\alpha}{}_{j\beta}\, \xi^{\beta}$$



$$\boxed{X^\alpha(\tau, S)}$$

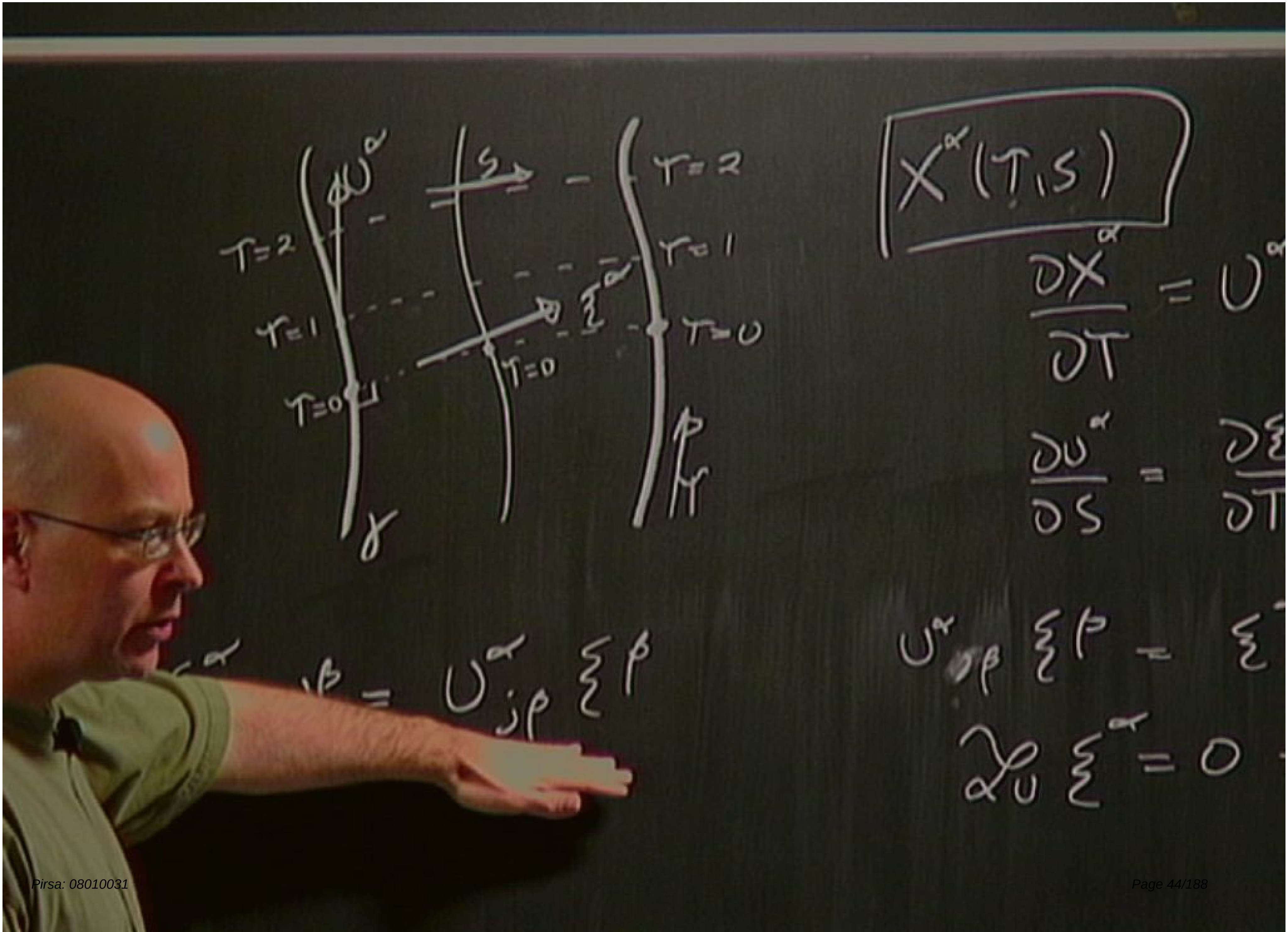
$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha$$

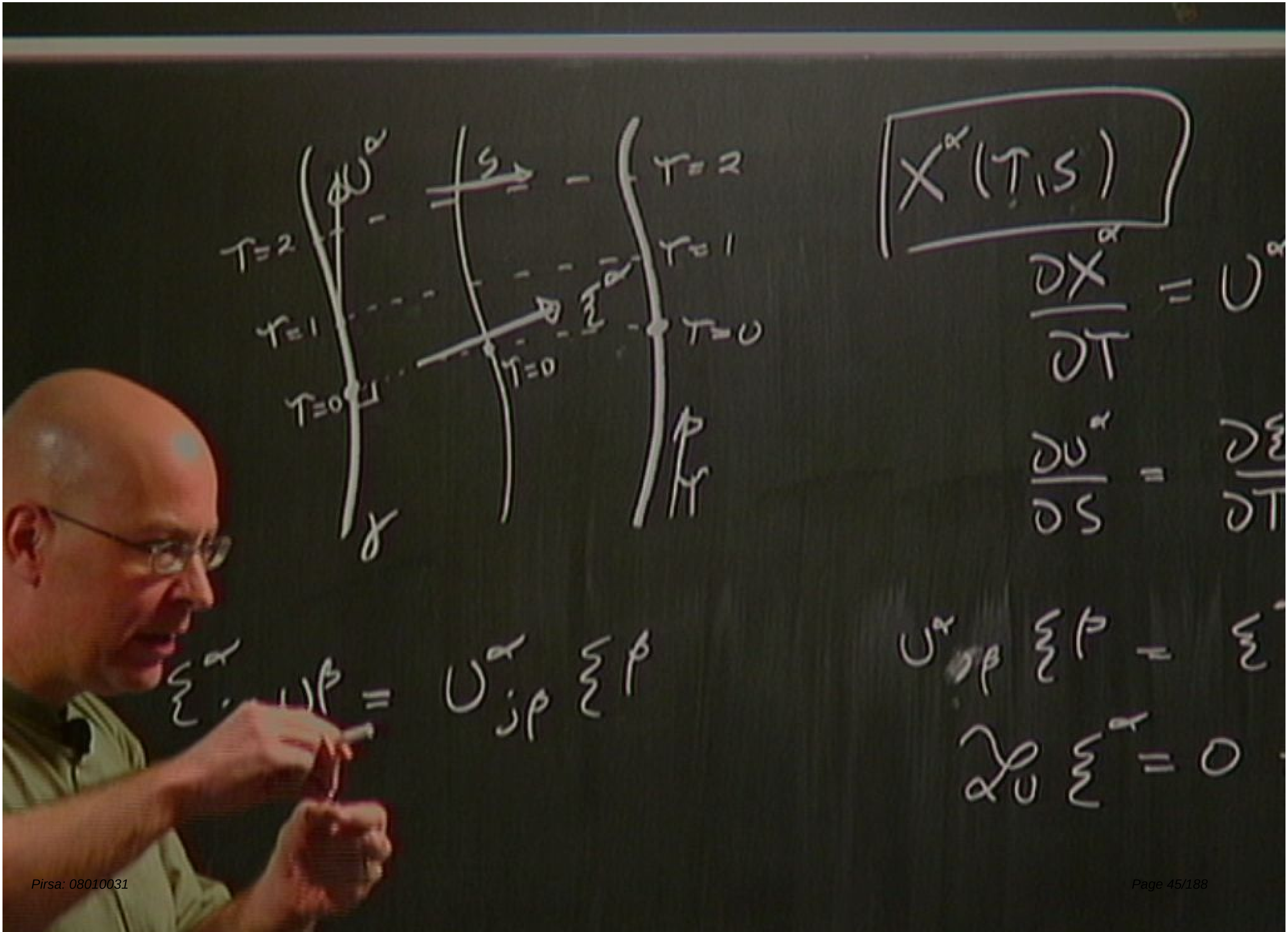
$$\frac{\partial U^\alpha}{\partial S} = \frac{\partial \xi}{\partial \tau}$$

$$\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta}\, \xi^\beta$$

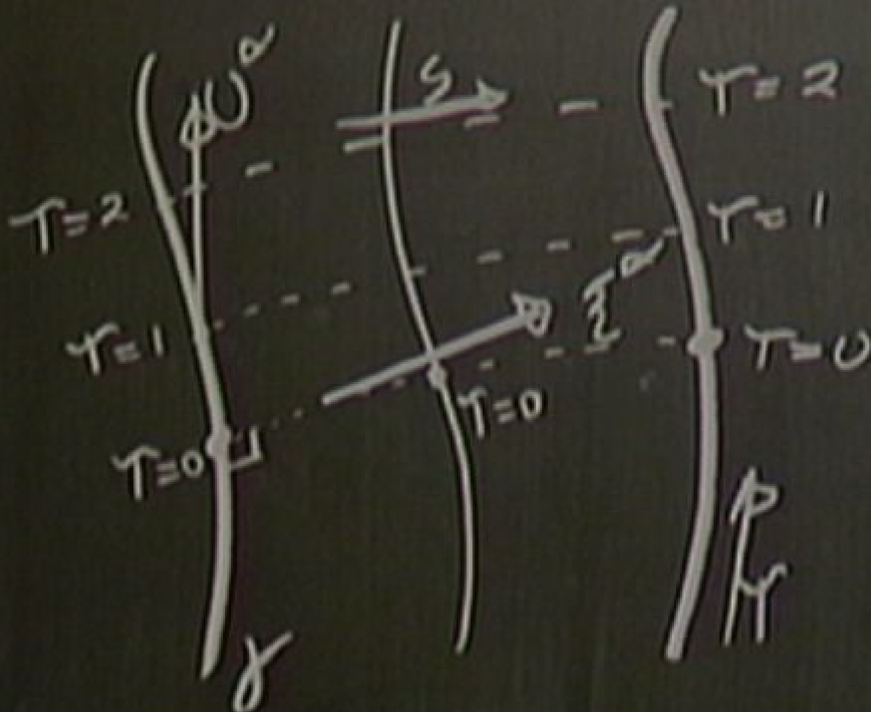
$$U^\alpha{}_{;\beta}\, \xi^\beta = \xi$$

$$\mathcal{L}_U \xi^\alpha = 0$$


X^α(T, S)
∂X^α / ∂T = U^α
∂U^α / ∂S =
U^α ;β ξ^β =
ℒ_U ξ^α = 0
T=2
T=1
T=0
ξ^α
U^α
S
γ



$X^\alpha(\tau, s)$
$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha$
$\frac{\partial U^\alpha}{\partial s} =$
$\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta} \xi^\beta$
$U^\alpha{}_{;\beta} \xi^\beta =$
$\mathcal{L}_U \xi^\alpha = 0$
$\tau=2$
$\tau=1$
$\tau=0$
U^α
ξ^α
γ



$$\boxed{X^\alpha(\tau, S)}$$

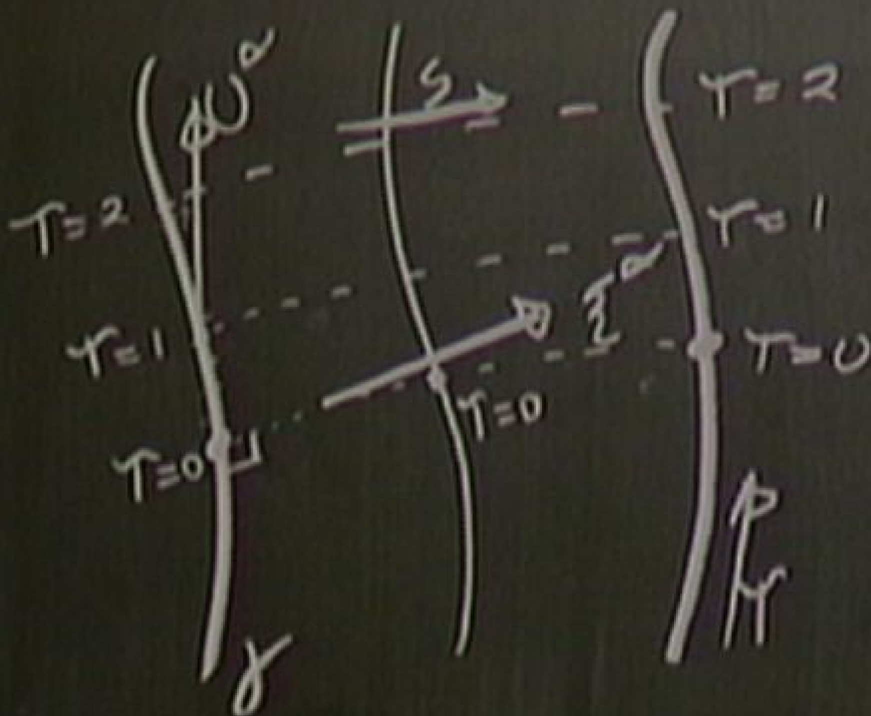
$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha$$

$$\frac{\partial U^\alpha}{\partial S} = \frac{\partial \xi}{\partial \tau}$$

$$\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta}\, \xi^\beta$$

$$U^\alpha{}_{;\beta}\, \xi^\beta = \xi$$

$$\mathcal{L}_U\, \xi^\alpha = 0$$



$$\boxed{x^\alpha(\tau, s)}$$

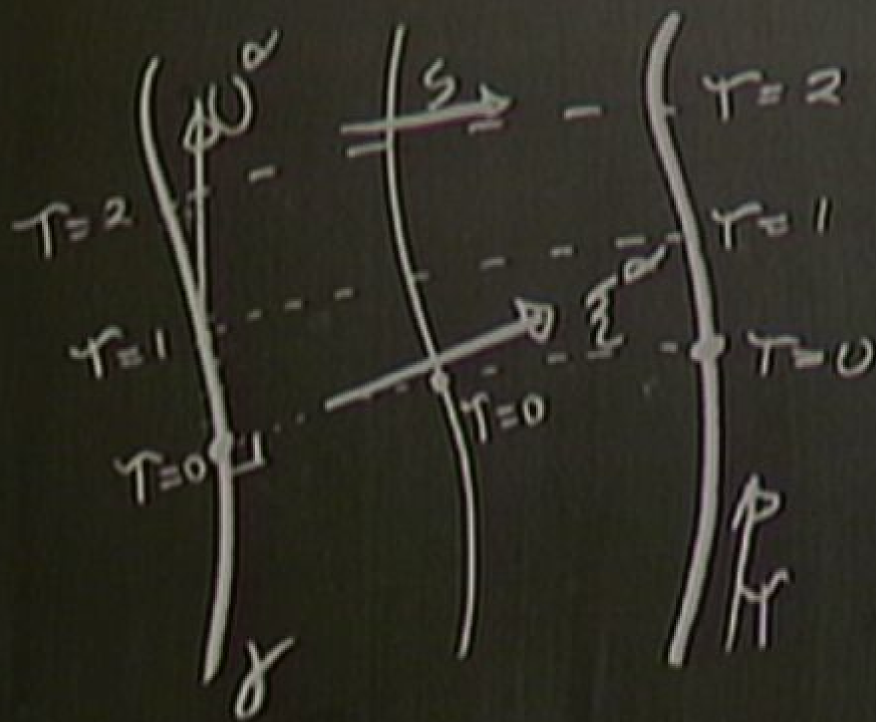
$$\frac{\partial x^\alpha}{\partial \tau} = U^\alpha$$

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi}{\partial \tau}$$

$$\boxed{\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta}\, \xi^\beta}$$

$$U^\alpha{}_{;\beta}\, \xi^\beta = \xi$$

$$\mathcal{L}_U \xi^\alpha = 0$$



$$\boxed{X^\alpha(\tau, s)}$$

$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha$$

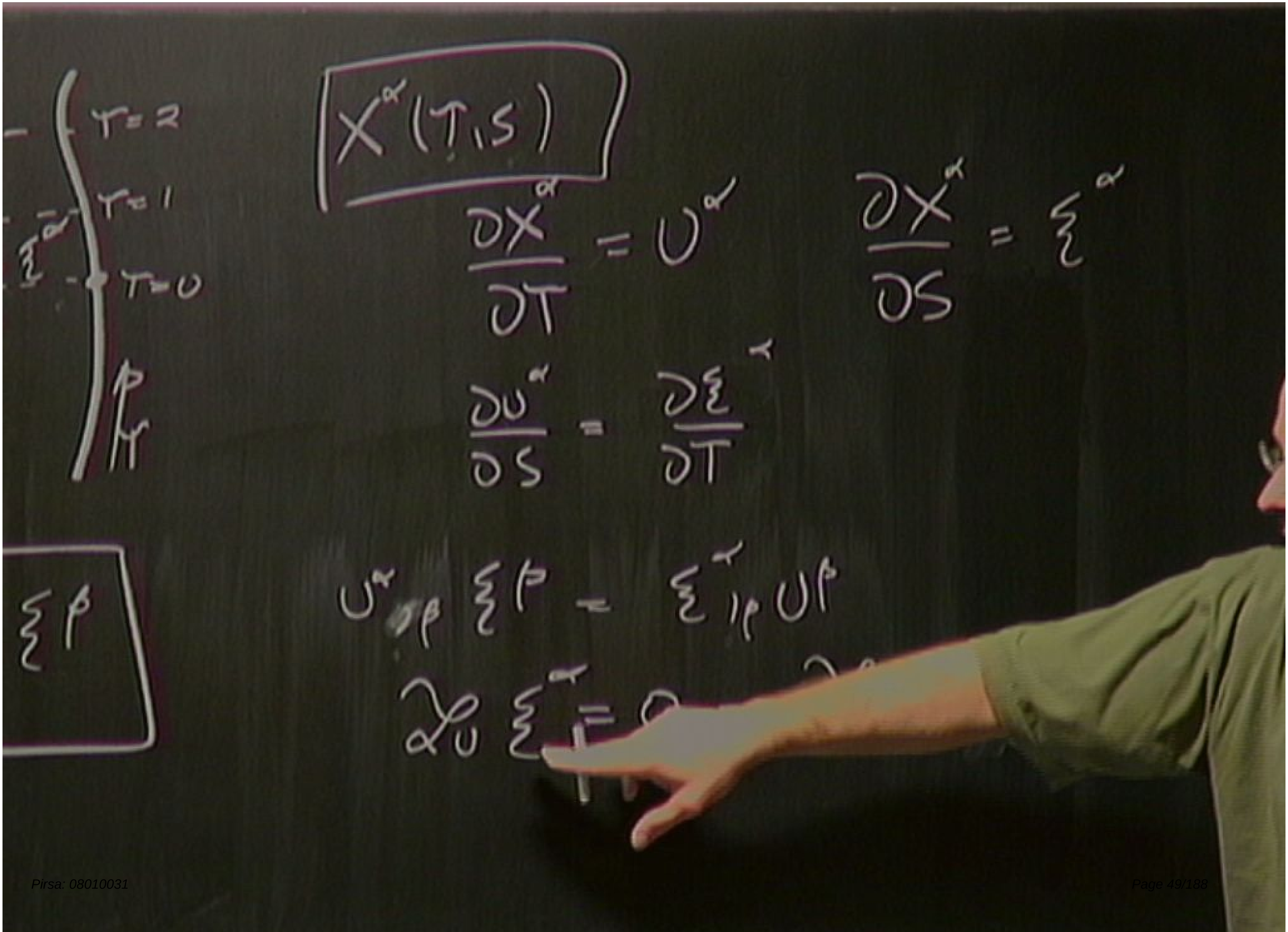
$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi}{\partial \tau}$$

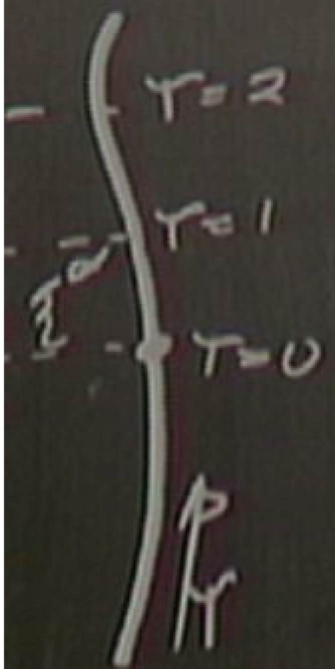
$$U^\alpha{}_{;\beta}\,\xi^\beta = \xi$$

$$\mathcal{L}_U \xi^\alpha = 0$$

$$\boxed{\xi^\alpha{}_{;\beta}\,U^\beta = U^\alpha{}_{;\beta}\,\xi^\beta}$$

$\boxed{\text{on } \gamma}$


$\tau = 2$
$\tau = 1$
$\tau = 0$
ξ^α
ξ^β
$X^\alpha(\tau, s)$
$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha$
$\frac{\partial X^\alpha}{\partial s} = \xi^\alpha$
$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$
$U^\alpha{}_{;\beta}\, \xi^\beta = \xi^\alpha{}_{;\beta}\, U^\beta$
$\mathcal{L}_U \xi^\alpha = 0$



$$\boxed{X^\alpha(\tau, s)}$$

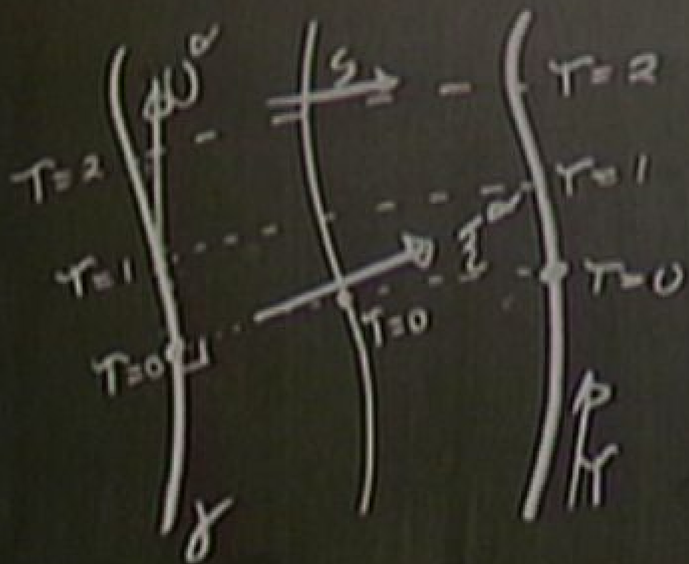
$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial s} = \xi^\alpha$$

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$$

$$\boxed{\xi^\beta}$$

$$U^\alpha{}_{;\beta}\,\xi^\beta = \xi^\alpha{}_{;\beta}\,U^\beta$$

$$\mathcal{L}_U \xi^\alpha = 0 = \mathcal{L}_\xi U^\alpha$$



$$\boxed{X^\alpha(\tau, s)}$$

$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial s} = \xi^\alpha$$

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$$

$$U^\alpha{}_{;\beta}\,\xi^\beta = \xi^\alpha{}_{;\beta}\,U^\beta$$

$$\mathcal{L}_U \xi^\alpha = 0 = \mathcal{L}_\xi U^\alpha$$

$$\boxed{\xi^\alpha{}_{;\beta}\,U^\beta = U^\alpha{}_{;\beta}\,\xi^\beta}$$

on γ

At $T=0$ $\quad U^{\tilde{\alpha}} \xi_\alpha = 0$

At $T=0$ $\quad U^{\alpha}\xi_{\alpha} = 0$

At $\tau = 0$ $\quad U^{-}\xi_\alpha = 0$

$$\frac{\partial}{\partial \tau}\left(U^{-}\xi_\alpha\right)$$

At $\tau = 0$ $\quad U^\alpha \xi_\alpha = 0$

$$\frac{d}{d\tau}(U^\alpha \xi_\alpha) = (U^\alpha \xi_\alpha)_{;\beta} U^\beta$$

At $\tau = 0$ $\quad U^\alpha \xi_\alpha = 0$

$$\frac{d}{d\tau}\left(U^\alpha \xi_\alpha\right) = \left(U^\alpha \xi_\alpha\right)_{;\beta} U^\beta = U^\alpha{}_{;\beta} U^\beta \xi_\alpha + U^\alpha \xi_{\alpha;\beta} U^\beta$$

At $\tau = 0$ $\quad U^\alpha \xi_\alpha = 0$

$$\frac{d}{d\tau}(U^\alpha \xi_\alpha) = (U^\alpha \xi_\alpha)_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta} U^\beta}_{0} \xi_\alpha + U^\alpha \xi_{\alpha;\beta} U^\beta$$

At $\tau = 0$ $\quad U^\alpha \xi_\alpha = 0$

$$\frac{d}{d\tau}\left(U^\alpha \xi_\alpha\right) = \left(U^\alpha \xi_\alpha\right)_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta} U^\beta}_{0} \xi_\alpha + U^\alpha \underbrace{\xi_{\alpha;\beta} U^\beta}$$

At $T=0 \quad U^\alpha \xi_\alpha = 0$

$$(U^\alpha \xi_\alpha) = (U^\alpha \xi_\alpha)_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta} U^\beta}_{0} \xi_\alpha + U^\alpha \underbrace{\xi_{\alpha;\beta} U^\beta}_{U_{\alpha;\beta}\xi^\beta}$$

$$= U^\alpha U_{\alpha;\beta} \xi^\beta$$

0 $\quad$ $U_{\alpha;\beta}\,\xi^\beta$

$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$

$$= \tfrac{1}{2}\left(U^\alpha U_\alpha\right)_{;\beta}$$

0 $\quad$ $U_{\alpha;\beta}\xi^{\beta}$

$$= U^{\alpha} U_{\alpha;\beta}\, \xi^{\beta}$$

$$= \tfrac{1}{2}\left(U^{\alpha} U_{\alpha}\right)_{;\beta}\, \xi^{\beta}$$

0 $\qquad\qquad$ $U_{\alpha;\beta}\,\xi^\beta$

$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$

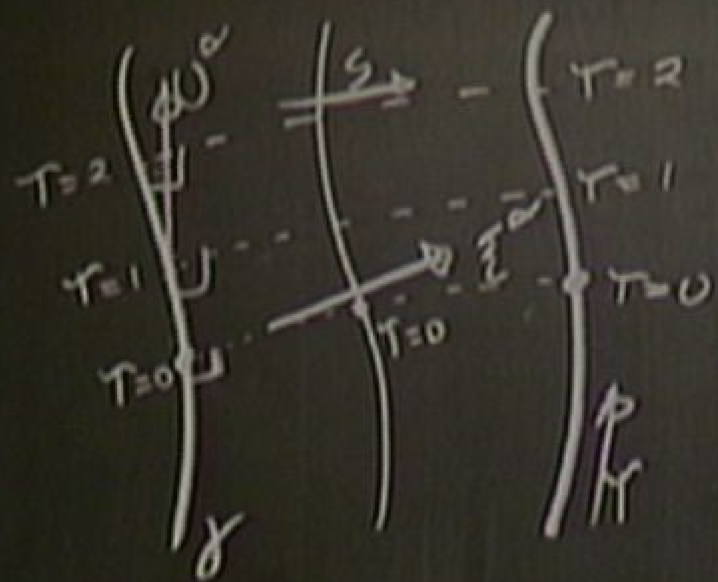
$$= \frac{1}{2}\big(\underbrace{U^\alpha U_\alpha}_{-1}\big)_{;\beta}\,\xi^\beta$$

0 $\quad U_{\alpha;\beta}\xi^\beta$

$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$

$$= \tfrac{1}{2}\big(\underbrace{U^\alpha U_\alpha}_{-1}\big)_{;\beta}\,\xi^\beta$$

$$= 0$$



$$\boxed{X^\alpha(\tau, s)}$$

$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial s} = \xi^\alpha$$

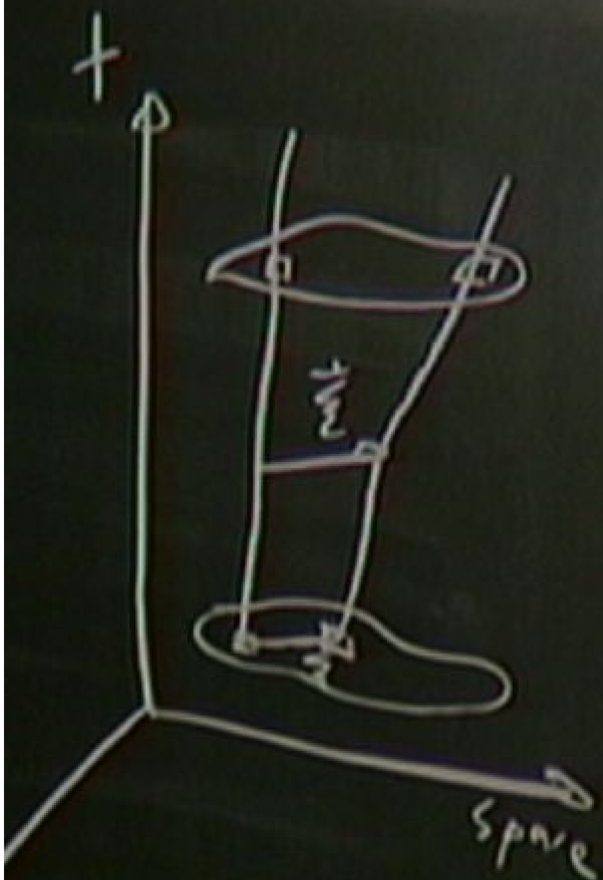
$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$$

$$\boxed{\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta} \xi^\beta}$$

$$\boxed{\text{on } \gamma}$$

$$U^\alpha{}_{,\beta} \xi^\beta - \xi^\alpha{}_{,\beta} U^\beta$$

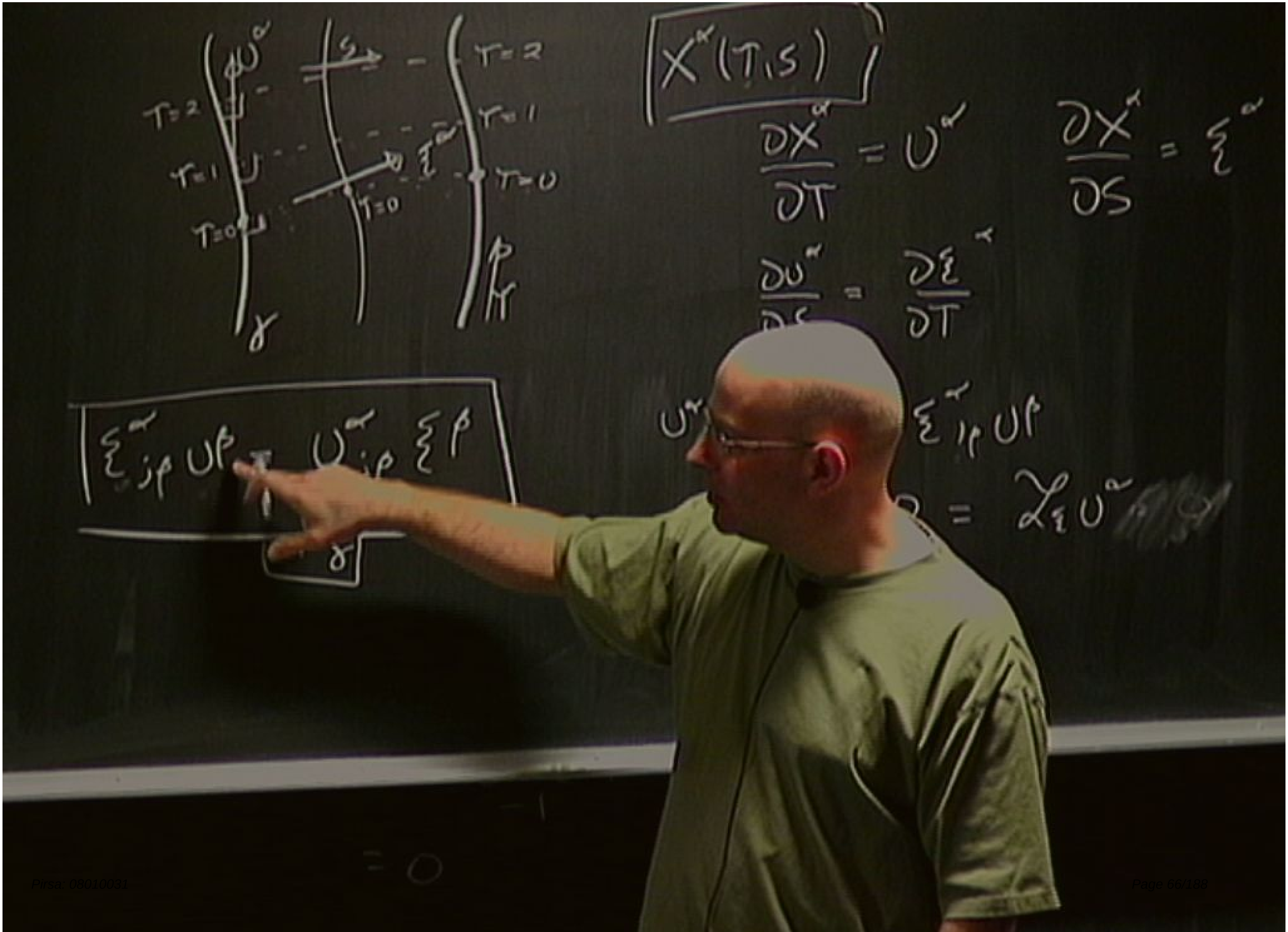
$$\mathcal{L}_U \xi^\alpha = 0 = \mathcal{L}_\xi U^\alpha$$

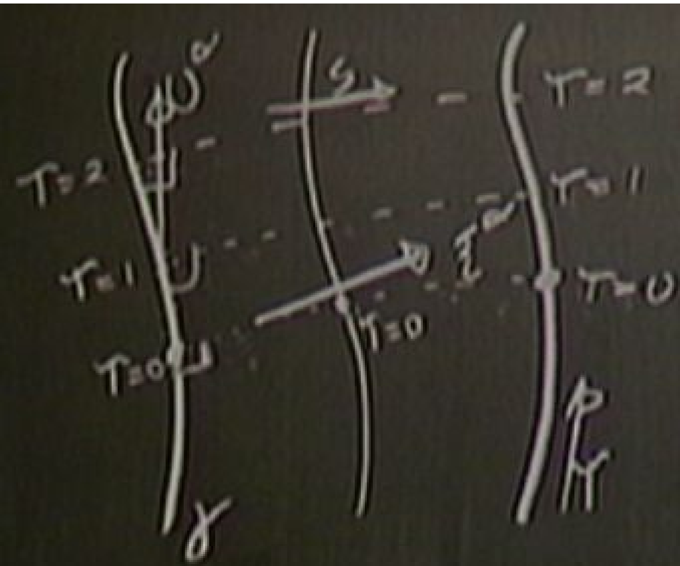


$$\frac{d\xi^a}{dt} = B^a{}_b\, \xi^b$$

$$B^a{}_b = V^a{}_{,b}$$

HW1: Feb 6 .
1.13: #2, #6
#9, #


X^α(T,S)
∂X^α/∂T = U^α
∂X^α/∂S = ξ^α
∂U^α/∂S = ∂ξ^α/∂T
T=2
T=1
T=0
ξ^α ;β U^β = U^α ;β ξ^β



$$\boxed{X^\alpha(\tau, s)}$$

$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial s} = \xi^\alpha$$

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$$

$$\boxed{\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta} \xi^\beta}$$

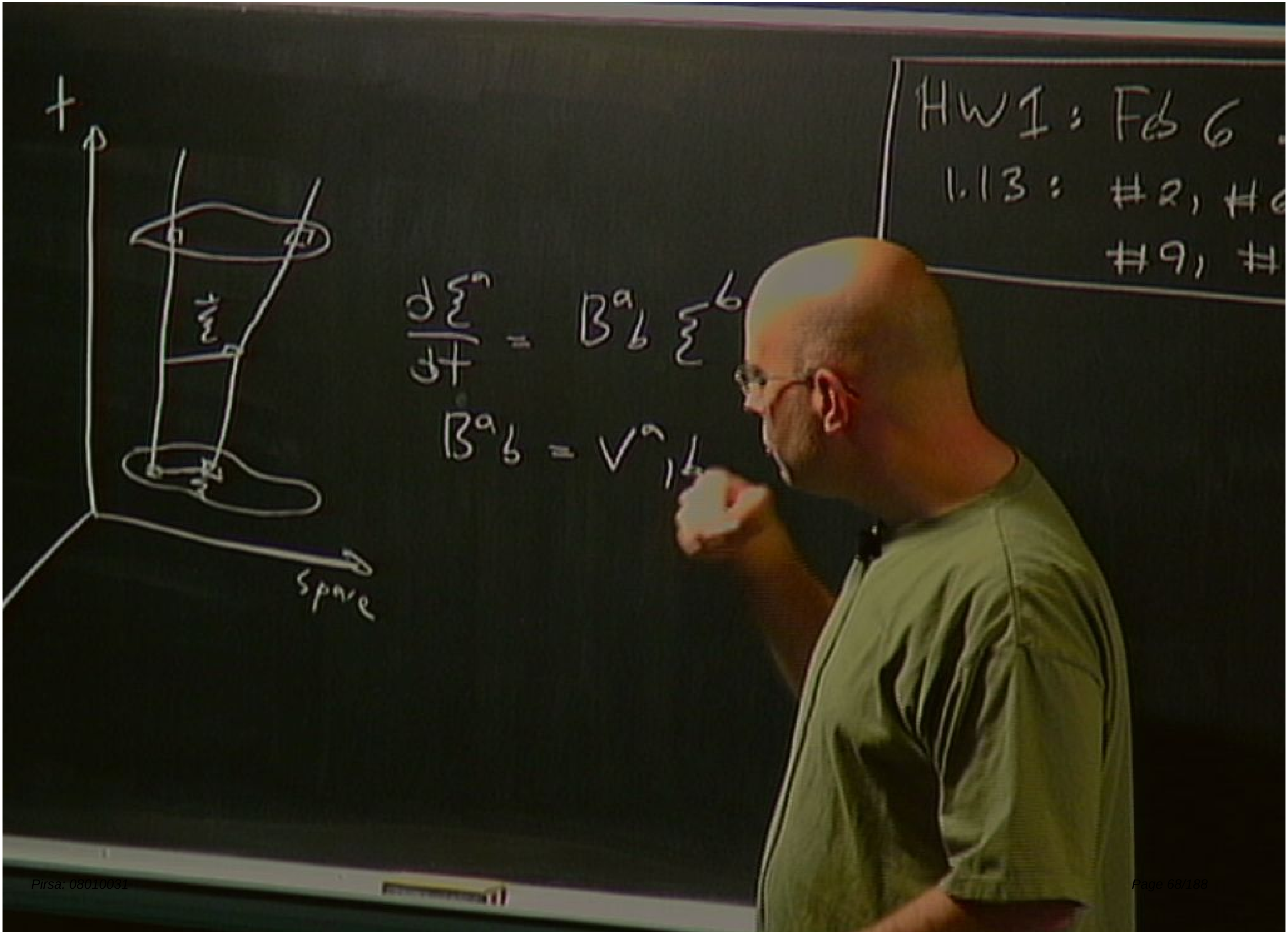
$$\boxed{\text{on } \gamma}$$

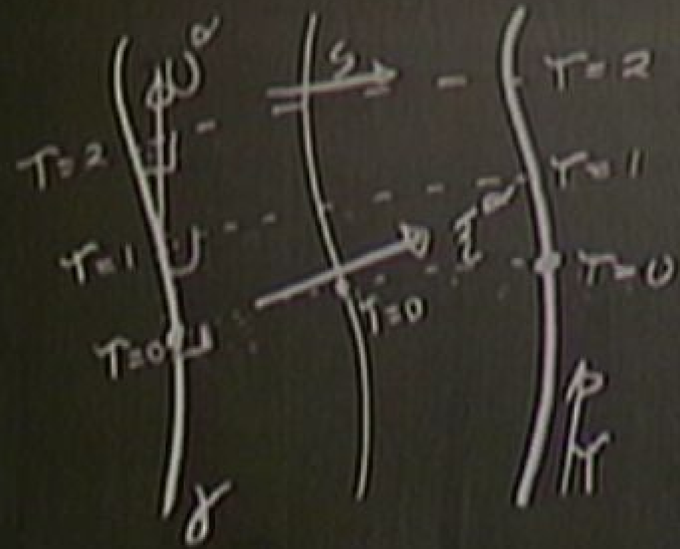
$$U^\alpha{}_{;\beta} \xi^\beta - \xi^\alpha{}_{;\beta} U^\beta$$

$$\mathcal{L}_U \xi^\alpha = 0 = \mathcal{L}_\xi U^\alpha$$

-1

$= 0$


t
space
HW 1: Feb 6
1.13: #2, #6
#9, #
$\frac{d\xi^a}{dt} = B^a{}_b \xi^b$
$B^a{}_b = V^a{}_{,b}$



$$\boxed{X^\alpha(\tau, s)}$$

$$\frac{\partial X^\alpha}{\partial \tau} = U^\alpha \qquad \frac{\partial X^\alpha}{\partial s} = \xi^\alpha$$

$$\frac{\partial U^\alpha}{\partial s} = \frac{\partial \xi^\alpha}{\partial \tau}$$

$$\boxed{\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta} \xi^\beta}$$

$$\boxed{\text{on } \gamma}$$

$$U^\alpha{}_{;\beta} \xi^\beta = \xi^\alpha{}_{;\beta} U^\beta$$

$$\mathcal{L}_U \xi^\alpha = 0 = \mathcal{L}_\xi U^\alpha$$

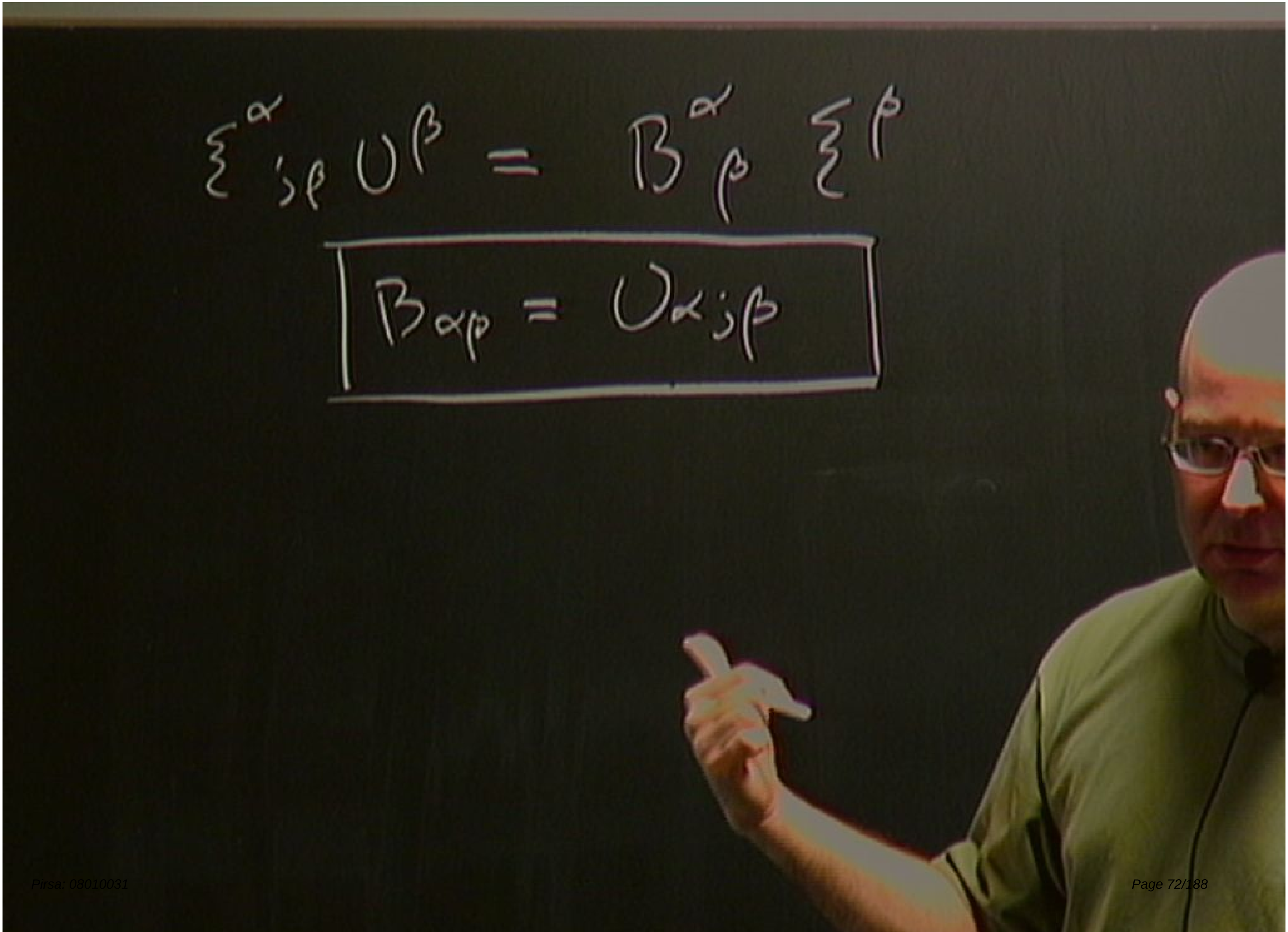
-1

$= 0$

$$\xi^{\alpha}{}_{;\beta} U^{\beta} = B^{\alpha}{}_{\beta} \xi^{\beta}$$

$$\xi^{\alpha}{}_{;\beta}\, U^{\beta} = B^{\alpha}{}_{\beta}\, \xi^{\beta}$$

$$B_{\alpha\beta} = U_{\alpha;\beta}$$


$\xi^{\alpha}{}_{;\beta} U^{\beta} = B^{\alpha}{}_{\beta} \xi^{\beta}$
$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$

$$\xi^{\alpha}{}_{;\beta} U^{\beta} = B^{\alpha}{}_{\beta}\, \xi^{\beta}$$

$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse"

$$\xi^{\alpha}{}_{;\beta} U^{\beta} = B^{\alpha}{}_{\beta} \xi^{\beta}$$

$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$U_{\alpha} B^{\alpha}{}_{\beta}$$

$$\xi^{\alpha}{}_{;\beta}\, U^{\beta} = B^{\alpha}{}_{\beta}\, \xi^{\beta}$$

$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$U_{\alpha} B^{\alpha}{}_{\beta} =$$

$$\xi^{\alpha}{}_{;\beta}\, u^{\beta} = B^{\alpha}{}_{\beta}\, \xi^{\beta}$$

$$\boxed{B_{\alpha\beta} = u_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$u_{\alpha} B^{\alpha}{}_{\beta} = u_{\alpha} u^{\alpha}{}_{;\beta} = \frac{1}{2}\left(u_{\alpha} u^{\alpha}\right)_{;\beta} = 0$$

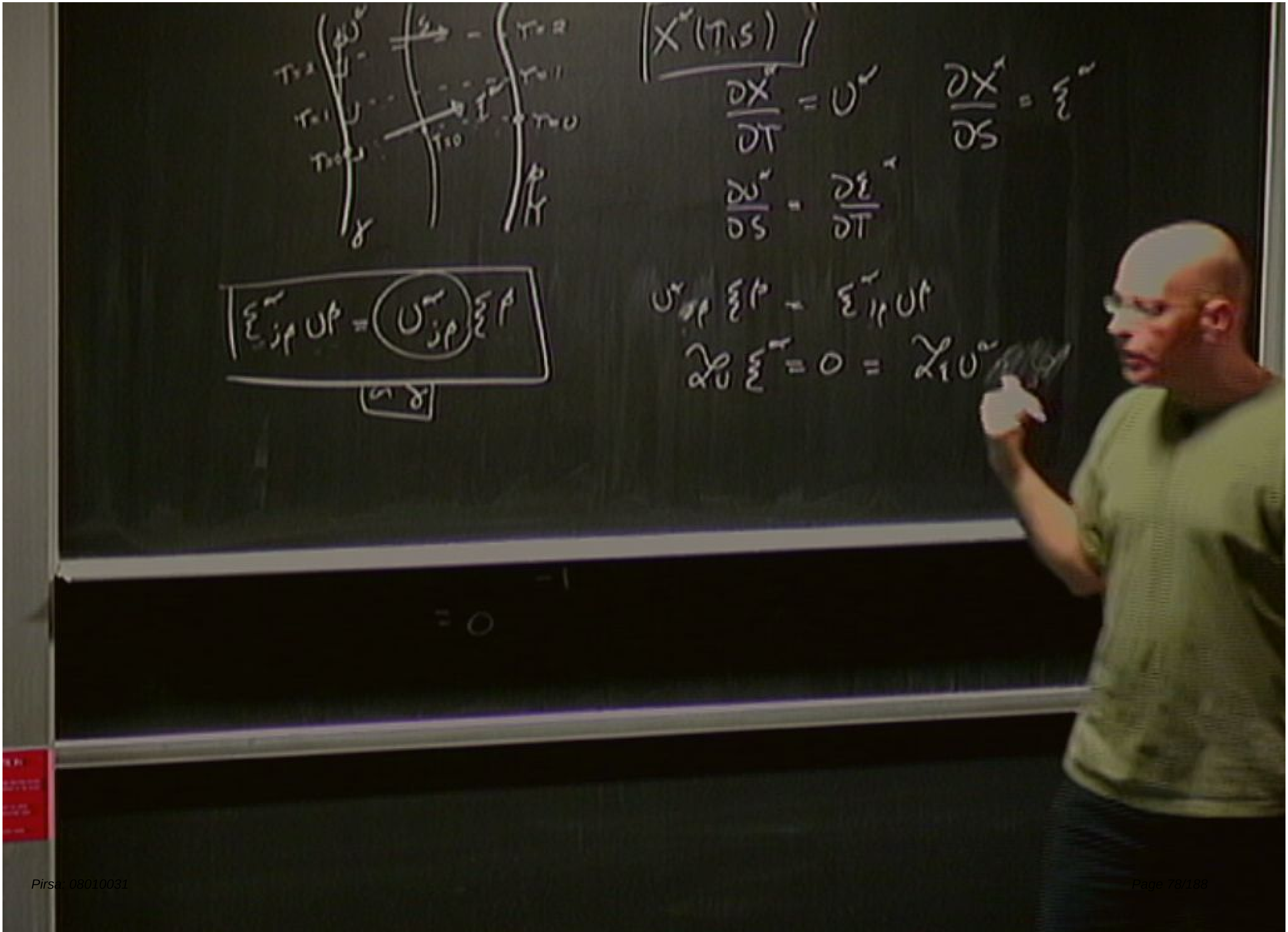
$$\xi^{\alpha}{}_{;\beta} U^{\beta} = B^{\alpha}{}_{\beta} \xi^{\beta}$$

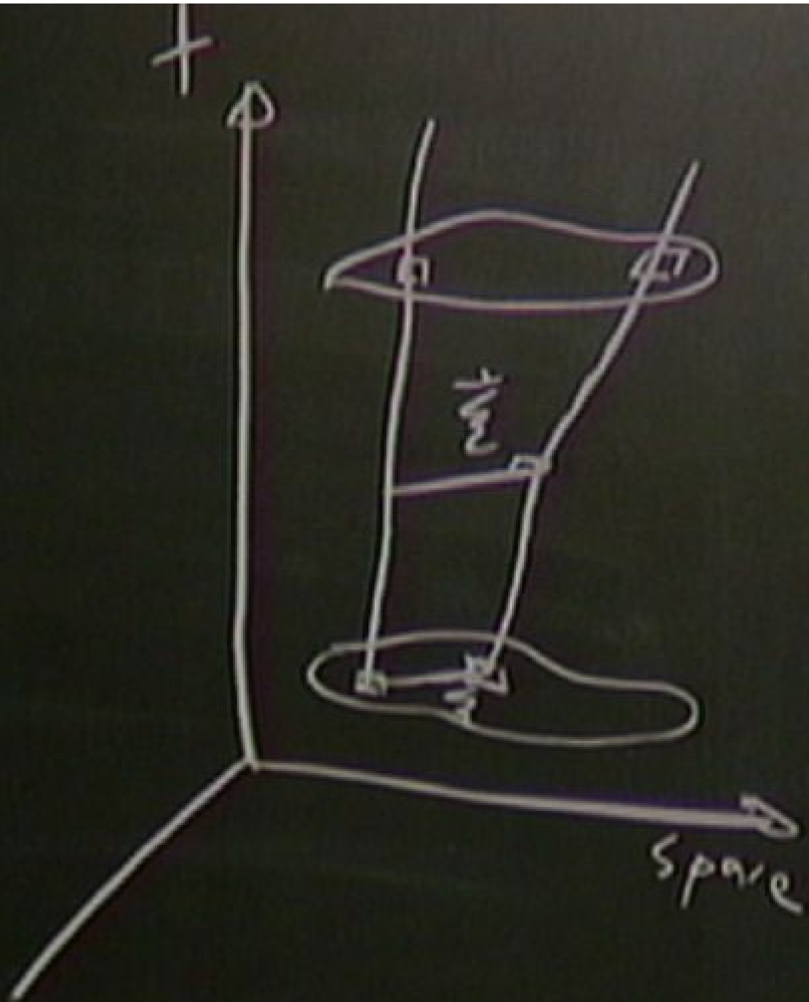
$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$U_{\alpha} B^{\alpha}{}_{\beta} = U_{\alpha} U^{\alpha}{}_{;\beta} = \frac{1}{2}(U_{\alpha}U^{\alpha})_{;\beta} = 0$$

$$B_{\alpha\beta} U^{\beta} = U_{\alpha;\beta} U^{\beta} = 0$$


$X^\mu(\tau, s)$
$\frac{\partial X^\mu}{\partial \tau} = U^\mu$
$\frac{\partial X^\mu}{\partial s} = \xi^\mu$
$\frac{\partial U^\mu}{\partial s} = \frac{\partial \xi^\mu}{\partial \tau}$
$\xi^\mu{}_{;\rho} U^\rho = U^\mu{}_{;\rho} \xi^\rho$
$U^\mu{}_{;\rho} \xi^\rho - \xi^\mu{}_{;\rho} U^\rho$
$\mathcal{L}_U \xi^\mu = 0 = \mathcal{L}_\xi U^\mu$



$$\frac{d\xi^a}{dt} = B^a{}_b\, \xi^b$$

$$B^a{}_b = v^a{}_{,b}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

$B_{\alpha\beta}$ is spatial, or transverse :

$$u_\alpha B^\alpha{}_\beta = u_\alpha u^\alpha{}_{;\beta} = \frac{1}{2}(u_\alpha u^\alpha)_{;\beta} = 0$$

$$B_{\alpha\beta} u^\beta = u_{\alpha;\beta} u^\beta = 0$$

Introduce

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$$

$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$U_\alpha B^\alpha{}_\beta = U_\alpha U^\alpha{}_{;\beta} = \frac{1}{2}(U_\alpha U^\alpha)_{;\beta} = 0$$

$$B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$$

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \quad \text{(purely transverse)}$$

$$\boxed{B_{\alpha\beta} = U_{\alpha;\beta}}$$

$B_{\alpha\beta}$ is "spatial", or "transverse":

$$U_\alpha B^\alpha{}_\beta = U_\alpha U^\alpha{}_{;\beta} = \frac{1}{2}(U_\alpha U^\alpha)_{;\beta} = 0$$

$$B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$$

Introduce

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \quad \text{(purely transverse)}$$

$$U^\alpha h_{\alpha\beta} = U_\beta - U_\beta = 0$$

$B_{\alpha\beta}$ is "spatial", or "transverse"

$$U_\alpha B^\alpha{}_\beta = U_\alpha U^\alpha{}_{;\beta} = \frac{1}{2}(U_\alpha U^\alpha)_{;\beta} = 0$$

$$B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$$

Introduce

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \quad \text{(purely transverse)}$$

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$B_{\alpha\beta}$ is "spatial", or "transvers

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$$B_{\alpha\beta}U^\beta = U_{\alpha;\beta}U^\beta = 0$$

Introduce

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \quad \text{(purely transverse)}$$

$$U^\alpha h_{\alpha\beta} = U_\beta - U_\beta = 0$$

$$g_{\alpha\beta} = -U_\alpha U_\beta + h_{\alpha\beta}$$

$B_{\alpha\beta}$ is "spatial", or "transverse"

$$U_\alpha B^\alpha{}_\beta = U_\alpha U^\alpha{}_{;\beta} = \frac{1}{2}(U_\alpha U^\alpha)_{;\beta} = 0$$

$$B_{\alpha\beta}U^\beta = U_{\alpha;\beta}U^\beta = 0$$

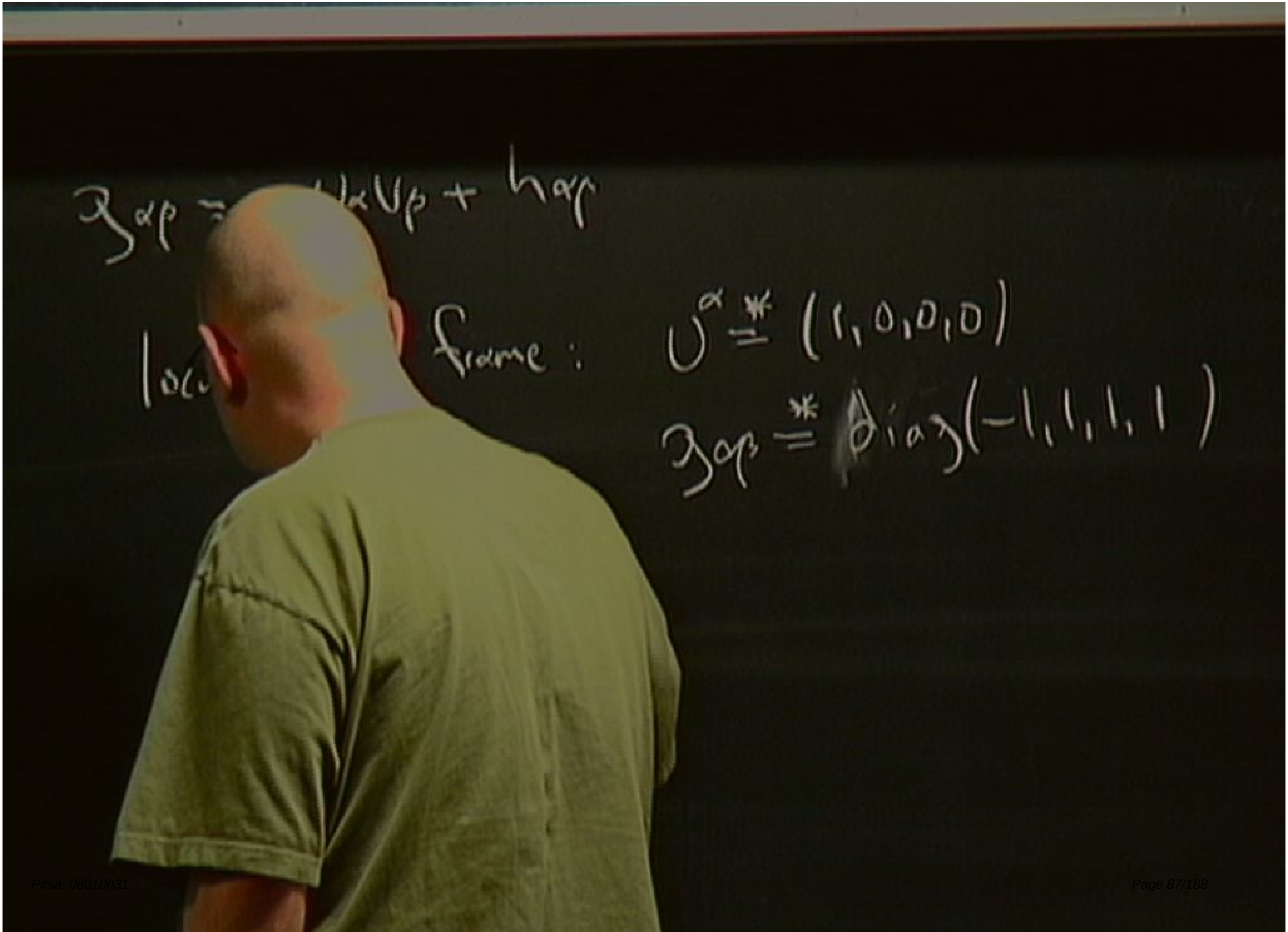
Introduce

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \quad \text{(purely transverse)}$$

$$U^\alpha h_{\alpha\beta} = U_\beta - U_\beta = 0$$

$$g_{\alpha\beta} = -U_\alpha U_\beta + h_{\alpha\beta}$$

local Lorentz frame: $U^\alpha \overset{*}{=} (1,0,0,0)$



$g_{\alpha\beta} = \ldots u_\alpha u_\beta + h_{\alpha\beta}$
loc... frame :
$u^\alpha \overset{*}{=} (1,0,0,0)$
$g_{\alpha\beta} \overset{*}{=} \mathrm{diag}(-1,1,1,1)$

$$g_{\alpha\beta} = -U_\alpha U_\beta + h_{\alpha\beta}$$

local Lorentz frame: $U^\alpha \overset{*}{=} (1,0,0,0)$

$$g_{\alpha\beta} \overset{*}{=} \mathrm{diag}(-1,1,1,1)$$

$$h_{\alpha\beta} \overset{*}{=} \mathrm{diag}(0,1,1,1)$$

$$g_{\alpha\beta} = -U_\alpha U_\beta + h_{\alpha\beta}$$

local Lorentz frame: $U^\alpha \overset{*}{=} (1,0,0,0)$

$$g_{\alpha\beta} \overset{*}{=} \mathrm{diag}(-1,1,1,1)$$

$$h_{\alpha\beta} \overset{*}{=} \mathrm{diag}(0,1,1,1)$$

Introduce

$$B_{\alpha\beta}U^{\beta} = U_{\alpha;\beta}U^{\beta} = 0$$

$$h_{\alpha\beta} = g_{\alpha\beta} + U_{\alpha}U_{\beta} \quad \text{(purely transverse)}$$

Spatial metric

$$U^{\alpha}h_{\alpha\beta} = U_{\beta} - U_{\beta} = 0$$

$$g_{\alpha\beta} = -U_{\alpha}U_{\beta} + h_{\alpha\beta}$$

local Lorentz frame: $U^{\alpha} \overset{*}{=} (1,0,0,0)$

$$g_{\alpha\beta} \overset{*}{=} \mathrm{diag}(-1,1,1,1)$$

$$h_{\alpha\beta} \overset{*}{=} \mathrm{diag}(0,1,1,1)$$

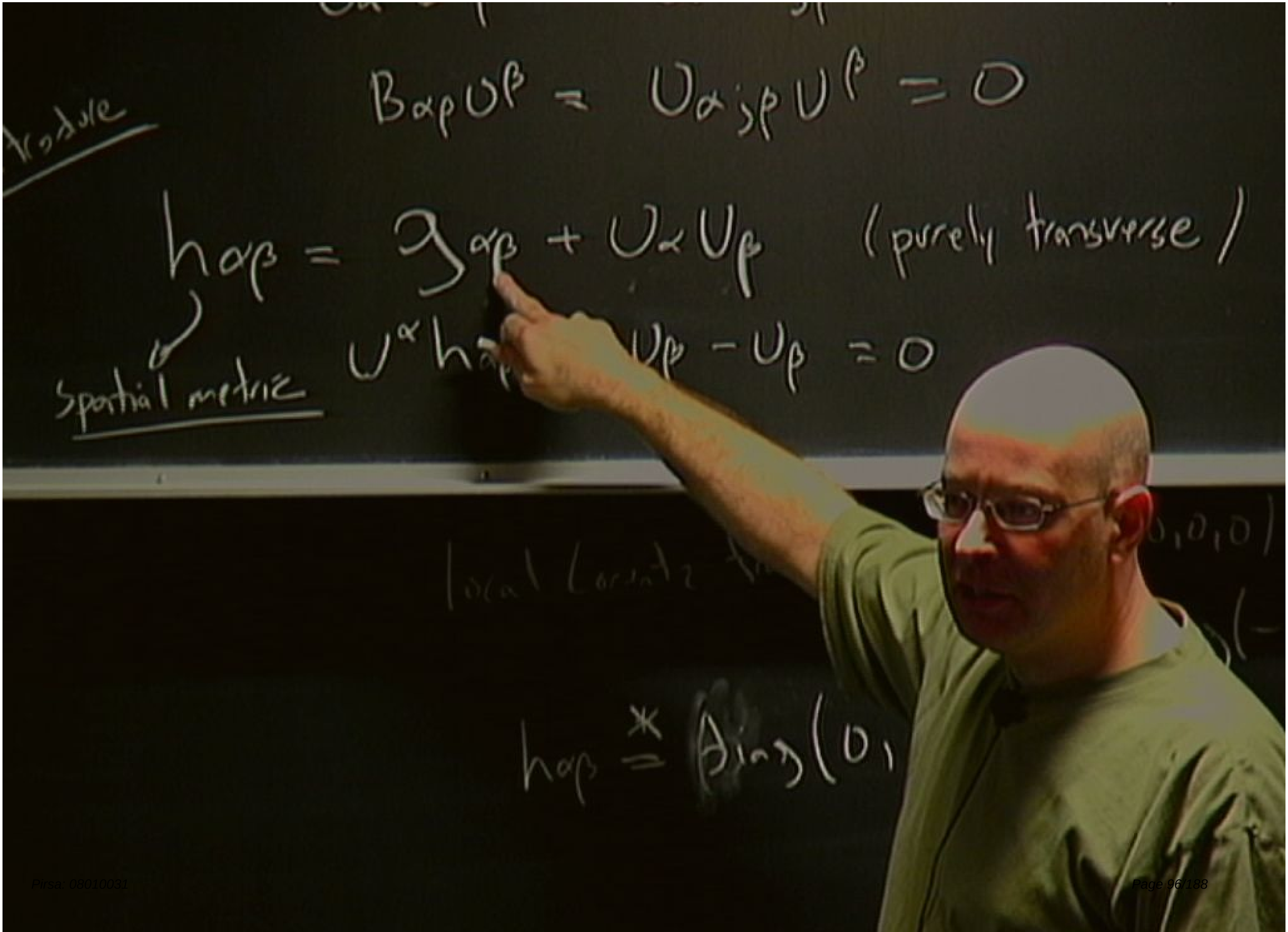
$$B_{\alpha\beta} =$$

$$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta}$$

$$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$B_{\alpha\beta} = \frac{1}{3}\underbrace{\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}$$

$$B_{\alpha\beta} = \underbrace{\frac{1}{3}\Theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}$$

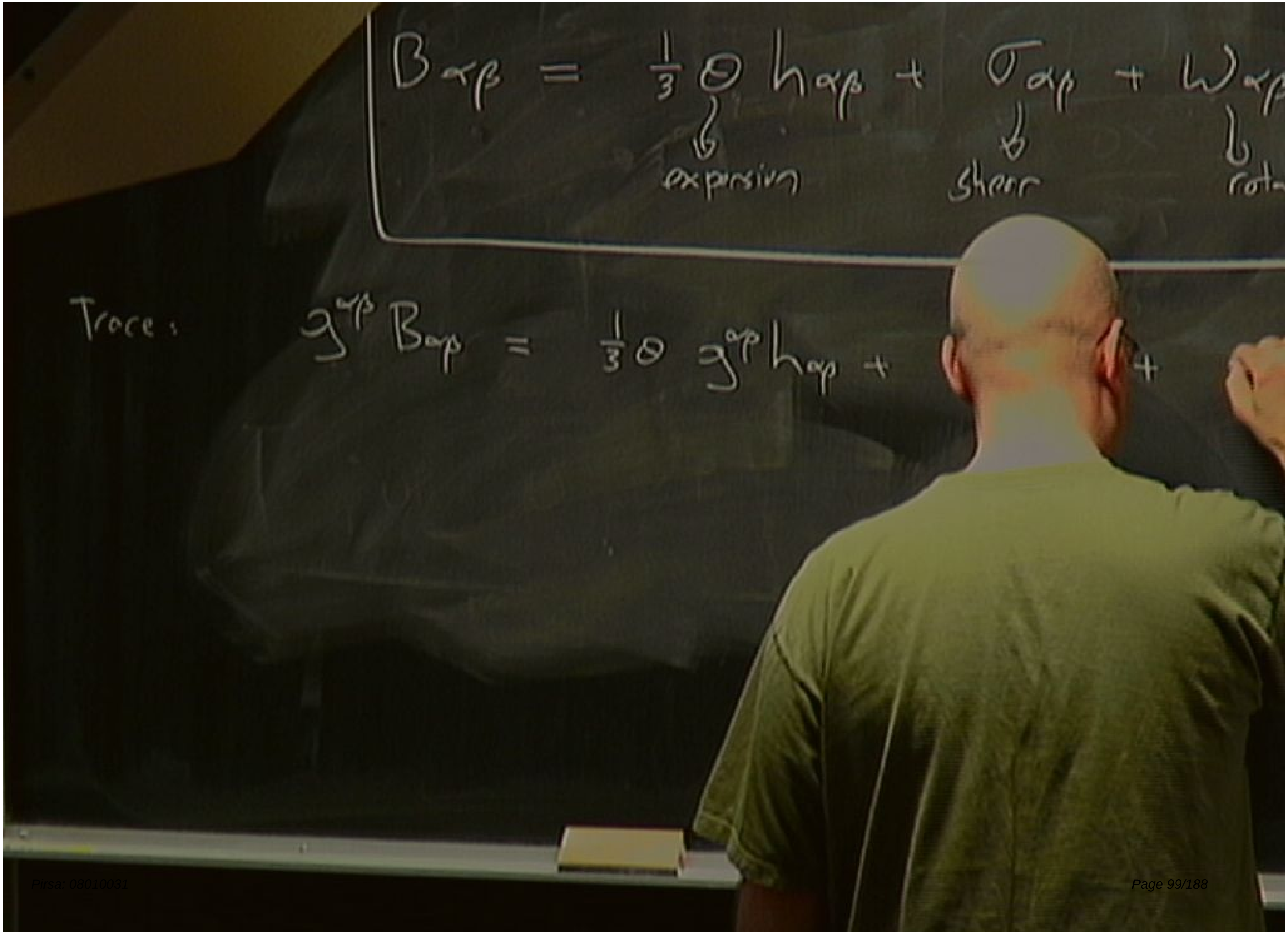


BαβUβ = Uα;βUβ = 0
hαβ = gαβ + UαUβ (purely transverse)
Spatial metric
local Lorentz
hαβ =* diag(0,

$$\boxed{B_{\alpha\beta} = \frac{1}{3}\underbrace{\Theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}}$$

$$\boxed{B_{\alpha\beta} = \frac{1}{3}\underbrace{\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}}$$

$$g^{\alpha\beta} B_{\alpha\beta}$$



$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
expansion
shear
Trace:
$g^{\alpha\beta}B_{\alpha\beta} = \frac{1}{3}\Theta g^{\alpha\beta}h_{\alpha\beta} +$
$+$

expansion shear rotation

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\theta\, g^{\alpha\beta} h_{\alpha\beta} + g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\alpha\beta} \omega_{\alpha\beta}$$

$$\boxed{= \underbrace{\tfrac{1}{3}\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \tfrac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + g^{\alpha\beta} \sigma_{\alpha\beta} + g^{\alpha\beta} \omega_{\alpha\beta}$$

$$= \frac{1}{3}\underset{\text{expansion}}{\theta}\, h_{\alpha\beta} + \underset{\text{shear}}{\sigma_{\alpha\beta}} + \underset{\text{rotation}}{\omega_{\alpha\beta}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\theta\, \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{g^{\alpha\beta}\left(g_{\alpha\beta} + U_\alpha U_\beta\right)} + g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\alpha\beta}\omega_{\alpha\beta}$$

$$= \underbrace{\tfrac{1}{3}\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \tfrac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3} + g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\alpha\beta}\omega_{\alpha\beta}$$

$$= \frac{1}{3}\underbrace{\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}} + g^{\alpha\beta}\omega_{\alpha\beta}$$

$$= \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion shear rotation

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{g^{\alpha\beta}(g_{\alpha\beta} + U_\alpha U_\beta) = 3} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}$$

$$\boxed{B_{\alpha\beta} = \underbrace{\tfrac{1}{3}\theta}_{\text{expansion}}\, h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation}}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \tfrac{1}{3}\theta\, \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}\left(g_{\alpha\beta} + u_\alpha u_\beta\right) = 3$$

$$= \frac{1}{3}\underbrace{\Theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\substack{\text{shear,}\\ \text{STF}}} + \underbrace{\omega_{\alpha\beta}}_{\substack{\text{rotation}\\ \text{antisym.}}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\Theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

STF

antisym.

$$\rho = \underbrace{\tfrac{1}{3}\theta\, g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta} \omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}(g_{\alpha\beta} + U_\alpha U_\beta) = 3$$

$$\theta = g^{\alpha\beta} B_{\alpha\beta}$$

$$= U^\alpha U_{\alpha;\beta}\, \xi^\beta$$

STF antisym.

$$\rho = \frac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}\left(g_{\alpha\beta} + U_\alpha U_\beta\right) = 3$$

$$\theta = g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta}$$

$$= U^\alpha U_{\alpha;\beta}\xi^\beta$$

$$= \tfrac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}\left(g_{\alpha\beta} + u_\alpha u_\beta\right) = 3$$

$$\theta = g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^{\alpha}{}_{;\alpha}$$

$$= u^\alpha u_{\alpha;\beta}\, \xi^\beta$$

$$= \frac{1}{3}\theta \, \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$\theta = g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^{\alpha}{}_{;\alpha}$$

$$= \frac{1}{\delta V}\frac{d}{d\tau}\delta V$$

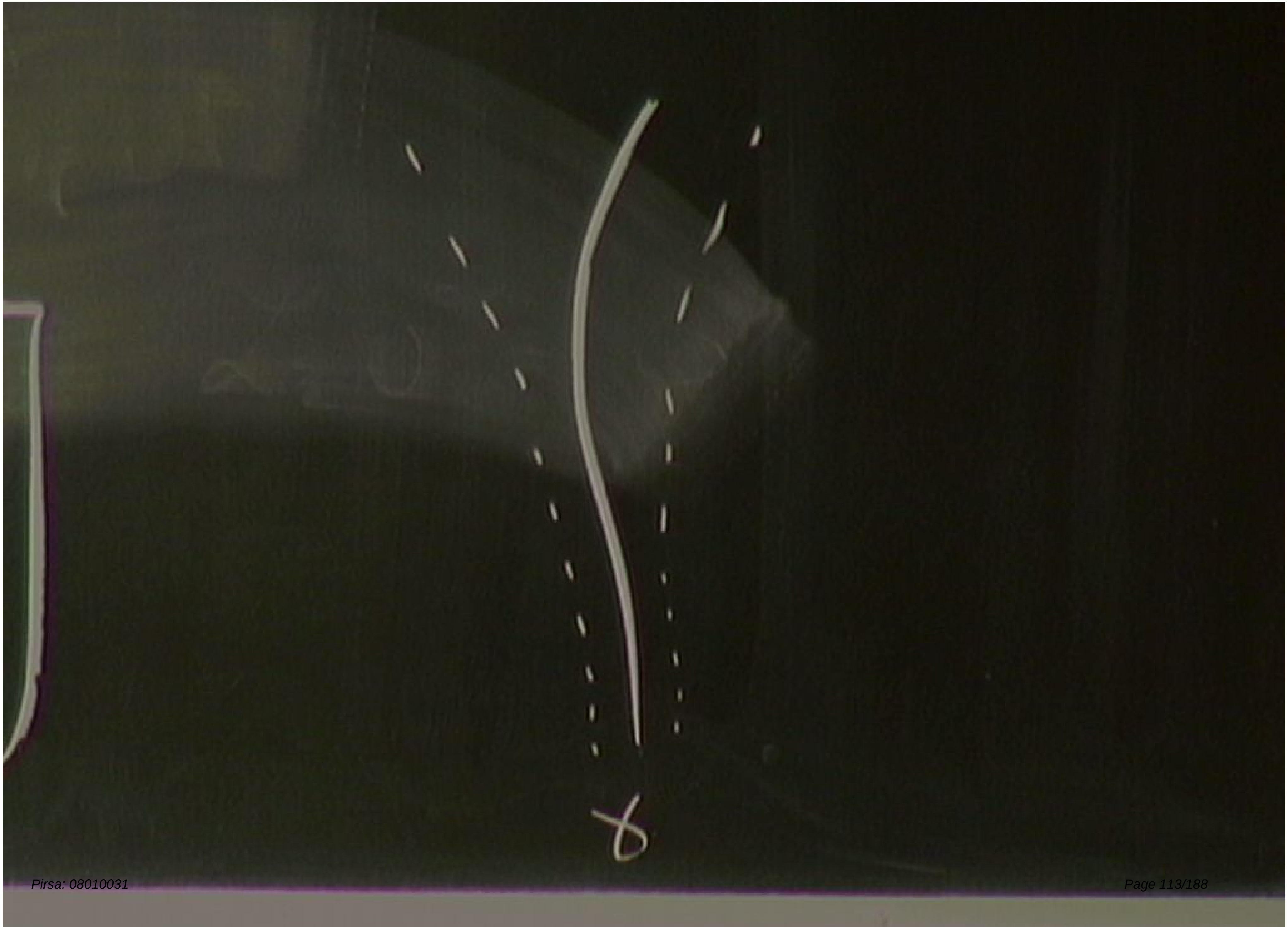
$$= u^{\alpha} u_{\alpha;\beta} \xi^{\beta}$$

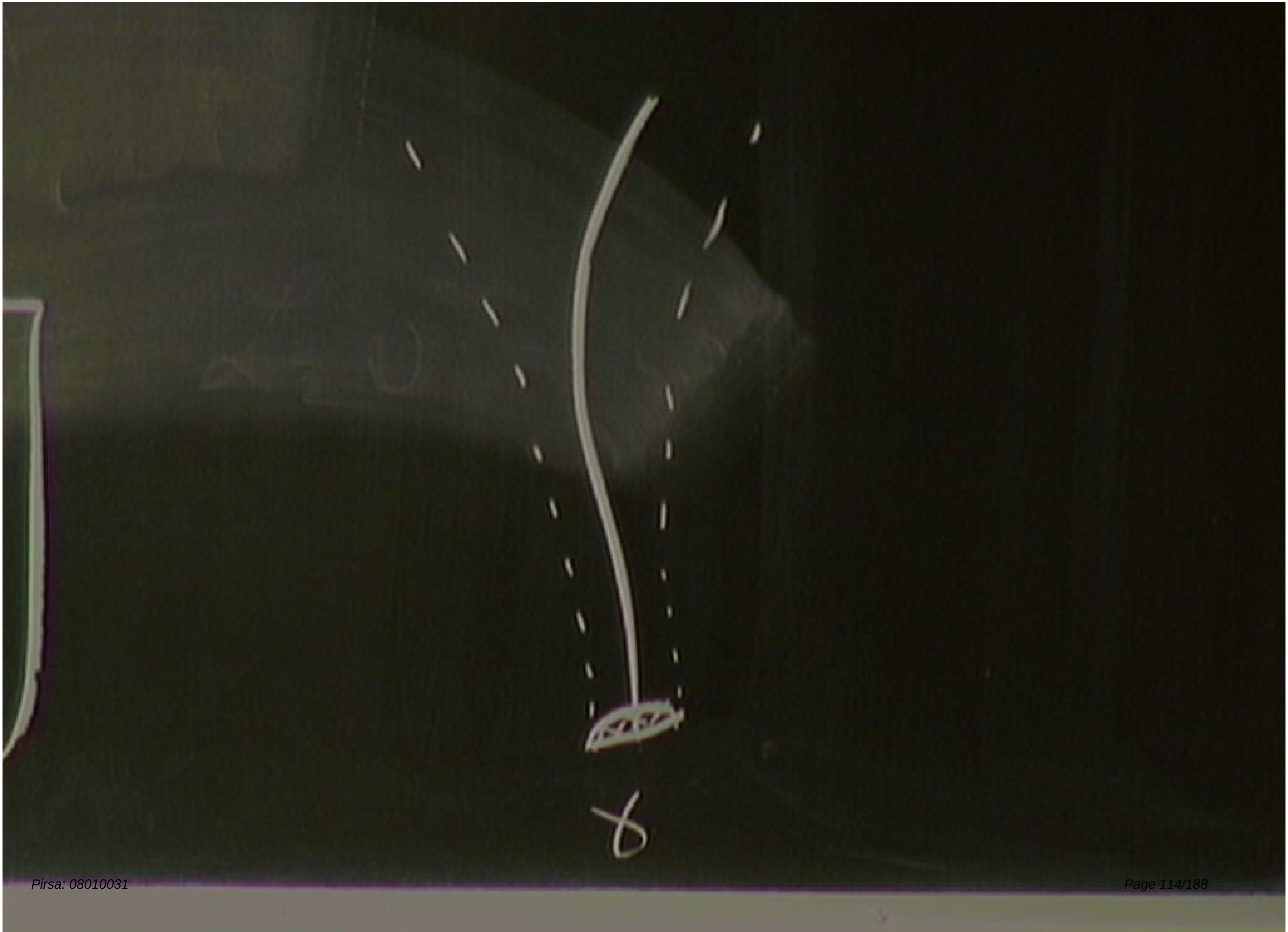
$$= \frac{1}{3}\Theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}}_{} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta} \omega_{\alpha\beta}}_{0}$$

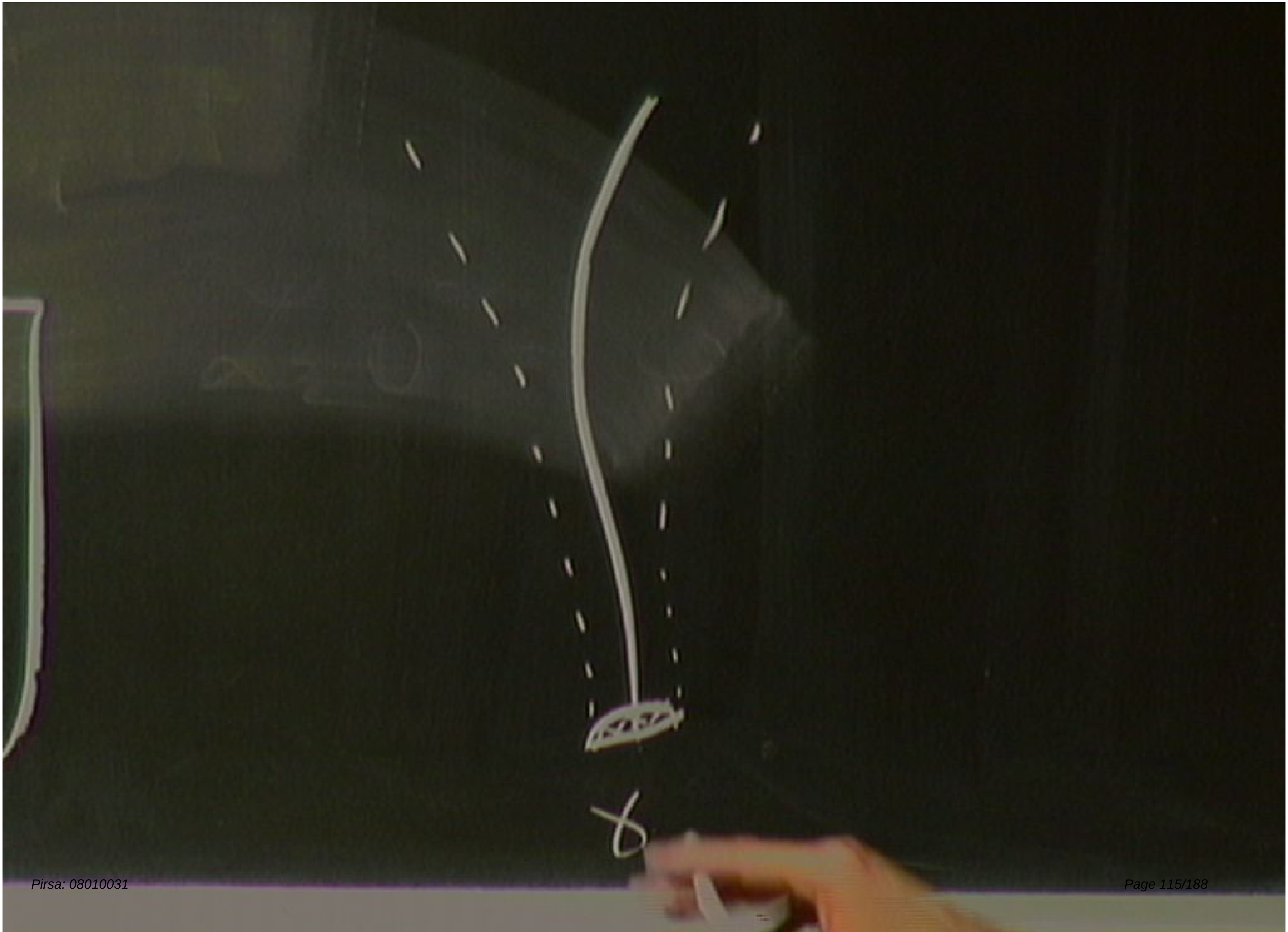
$$g^{\alpha\beta}(g_{\alpha\beta} + U_\alpha U_\beta) = 3$$

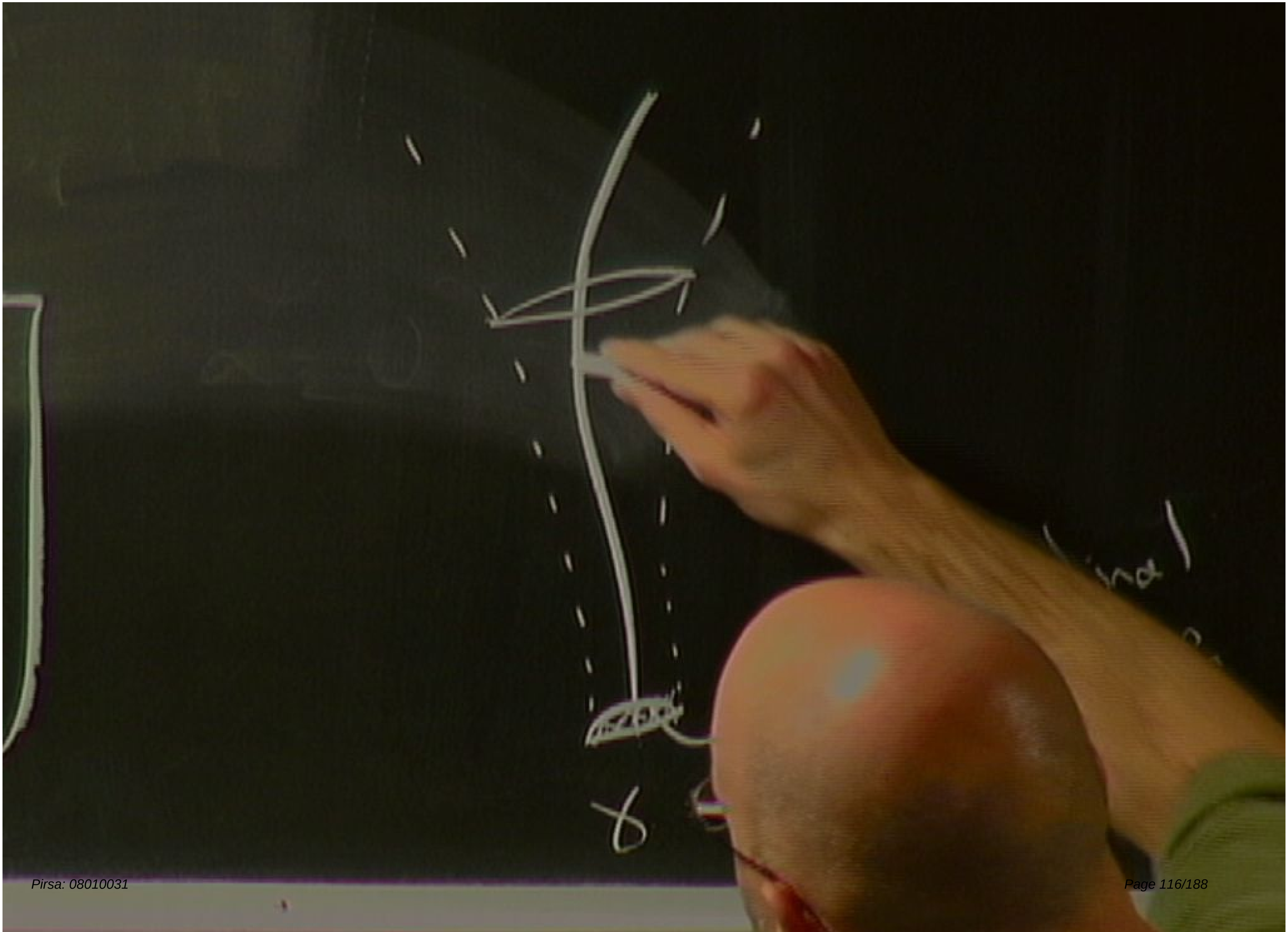
$$\boxed{\begin{aligned}\Theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = U^\alpha{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$

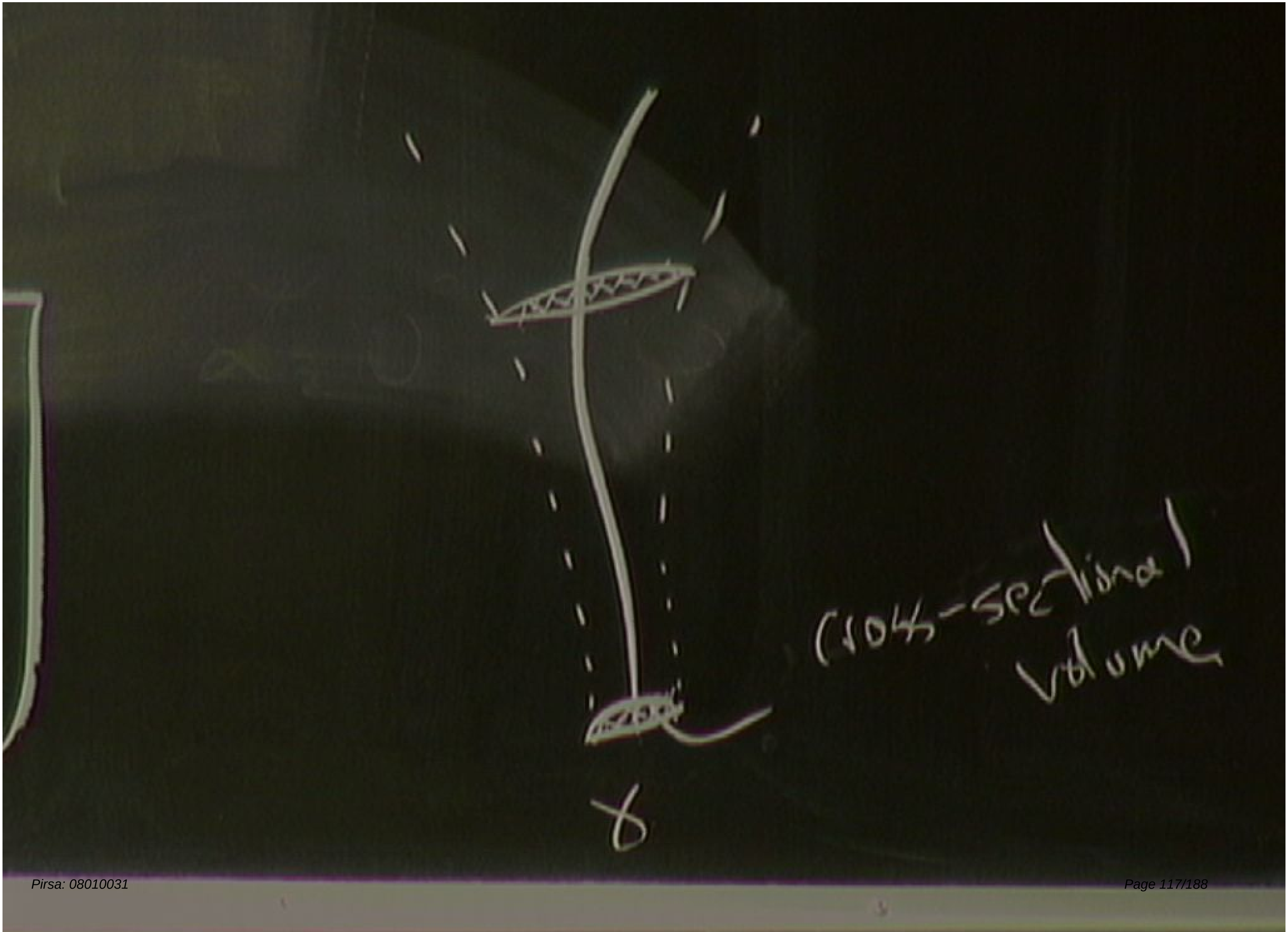
$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$


δ





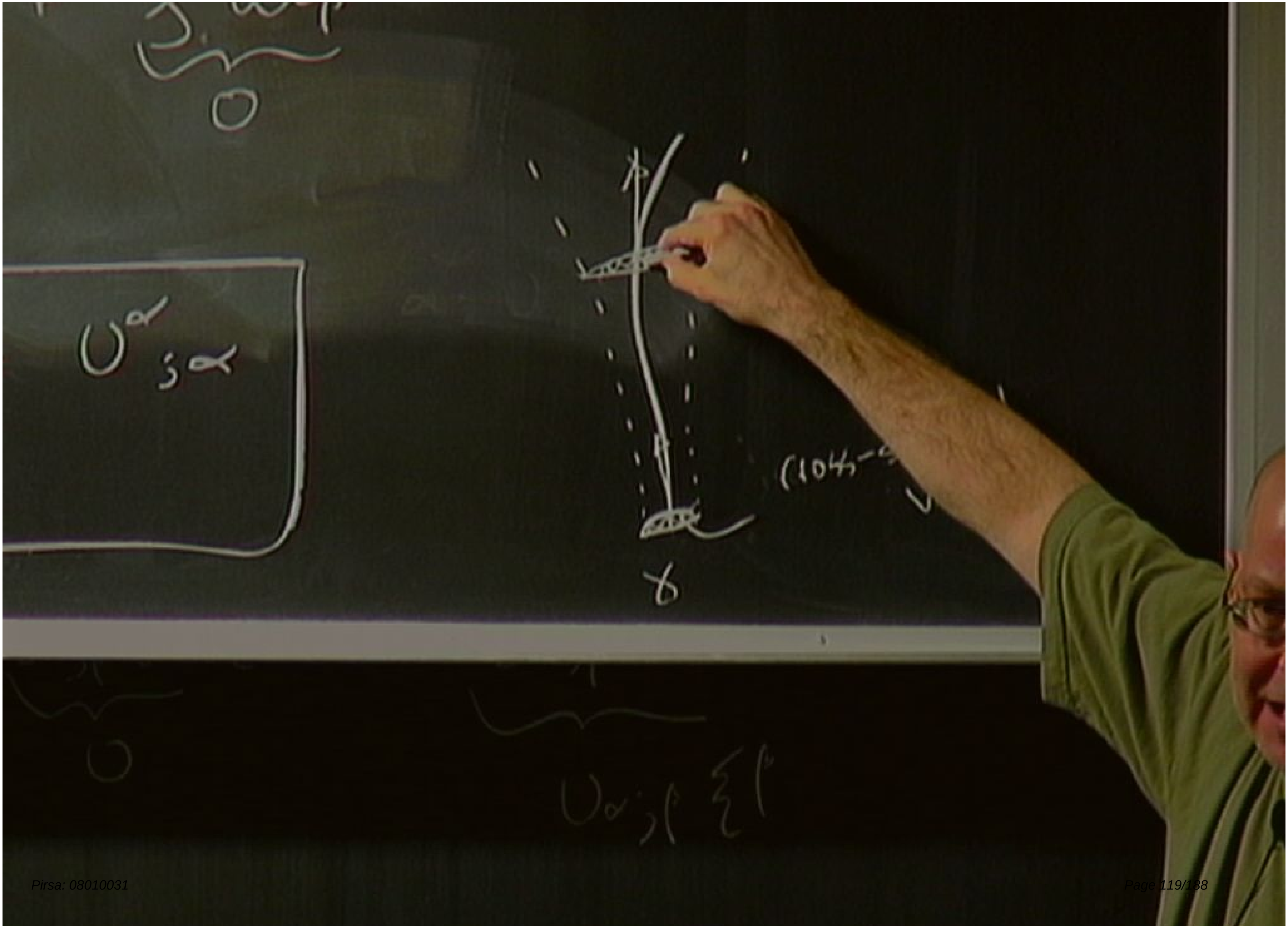


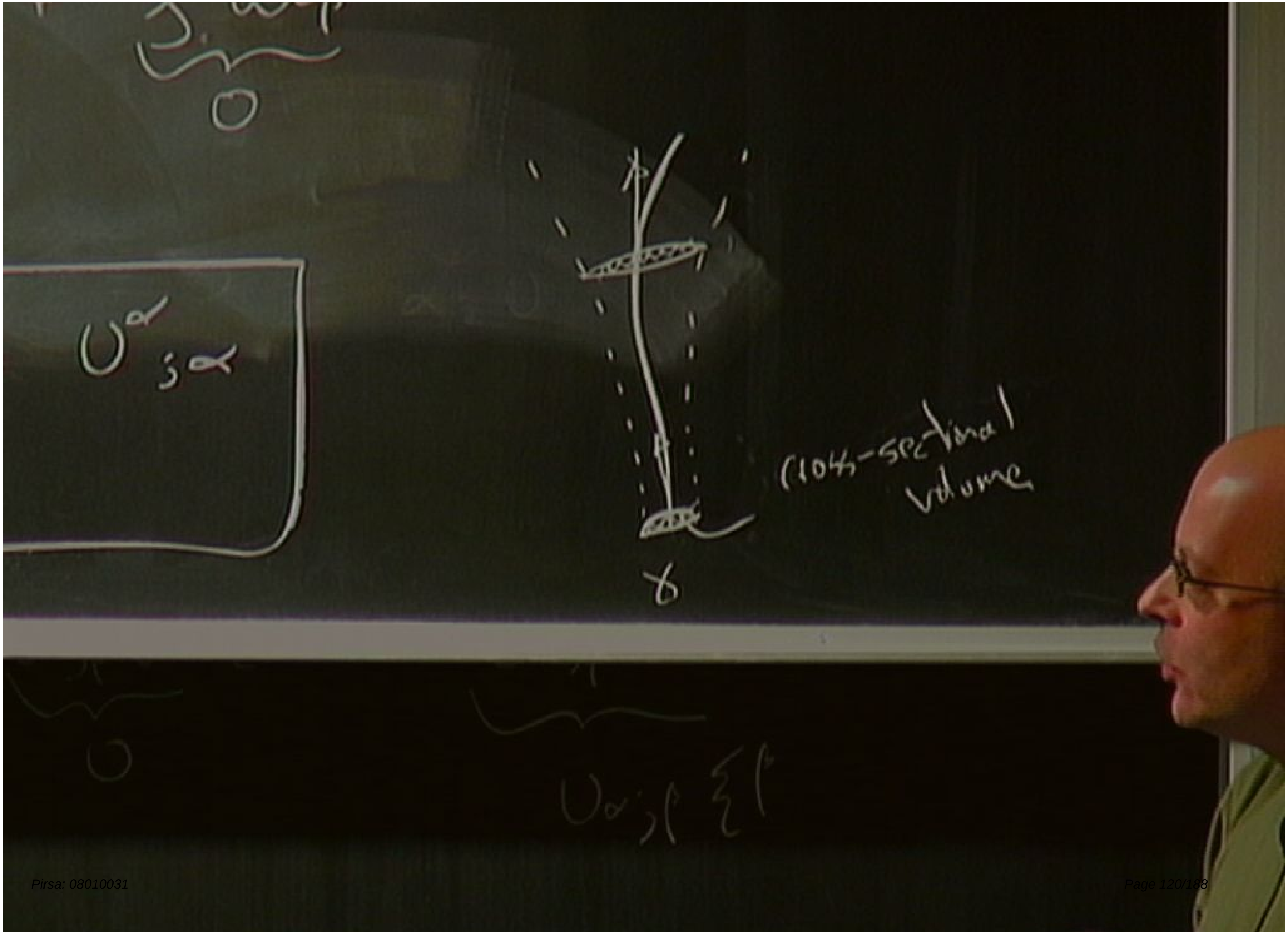

cross-sectional
volume
γ

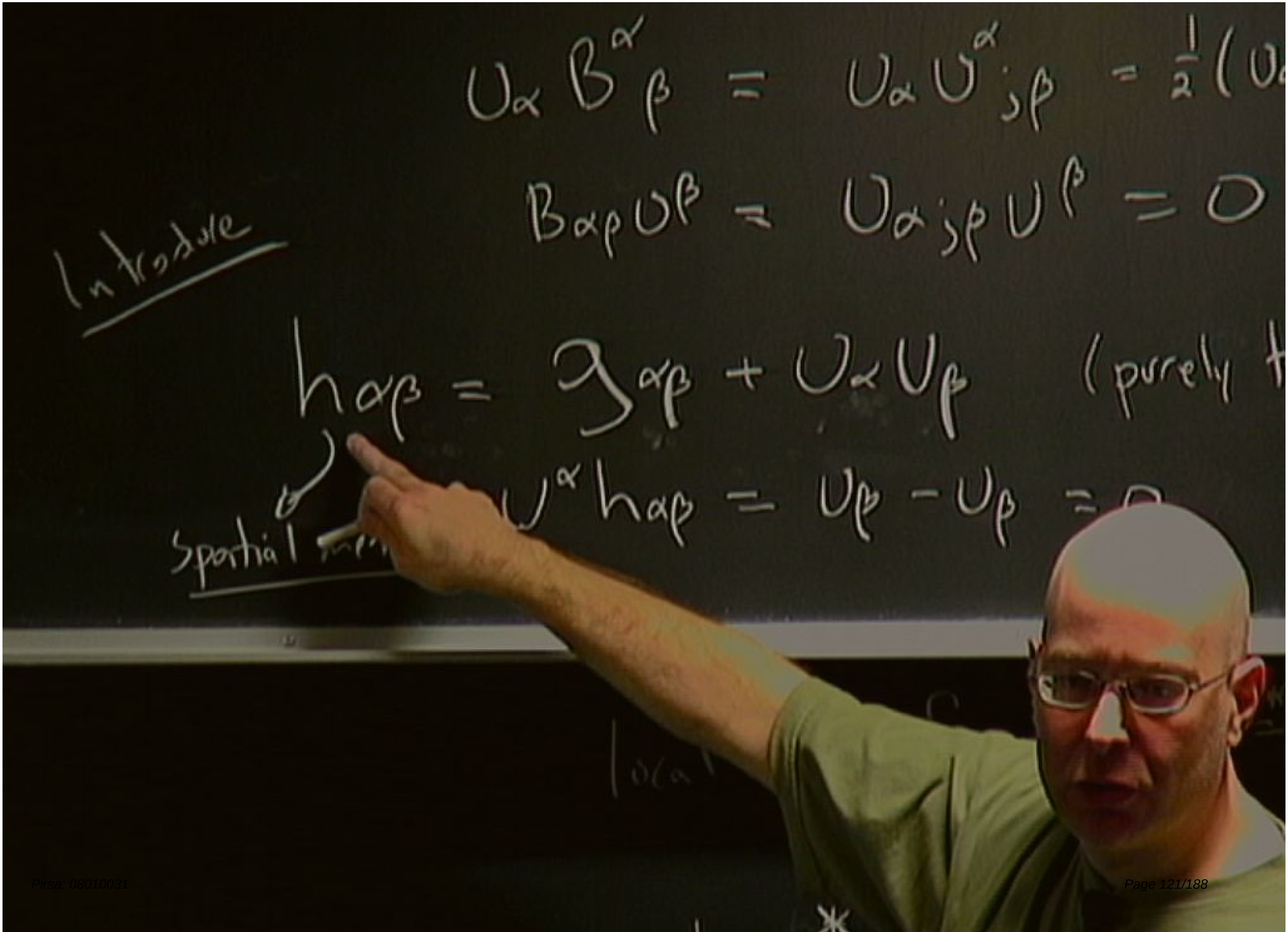
$$= \frac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

$$\boxed{\begin{aligned}\theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^{\alpha}{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$


$U^{\alpha}{}_{;\alpha}$


Cross-sectional
volume
γ

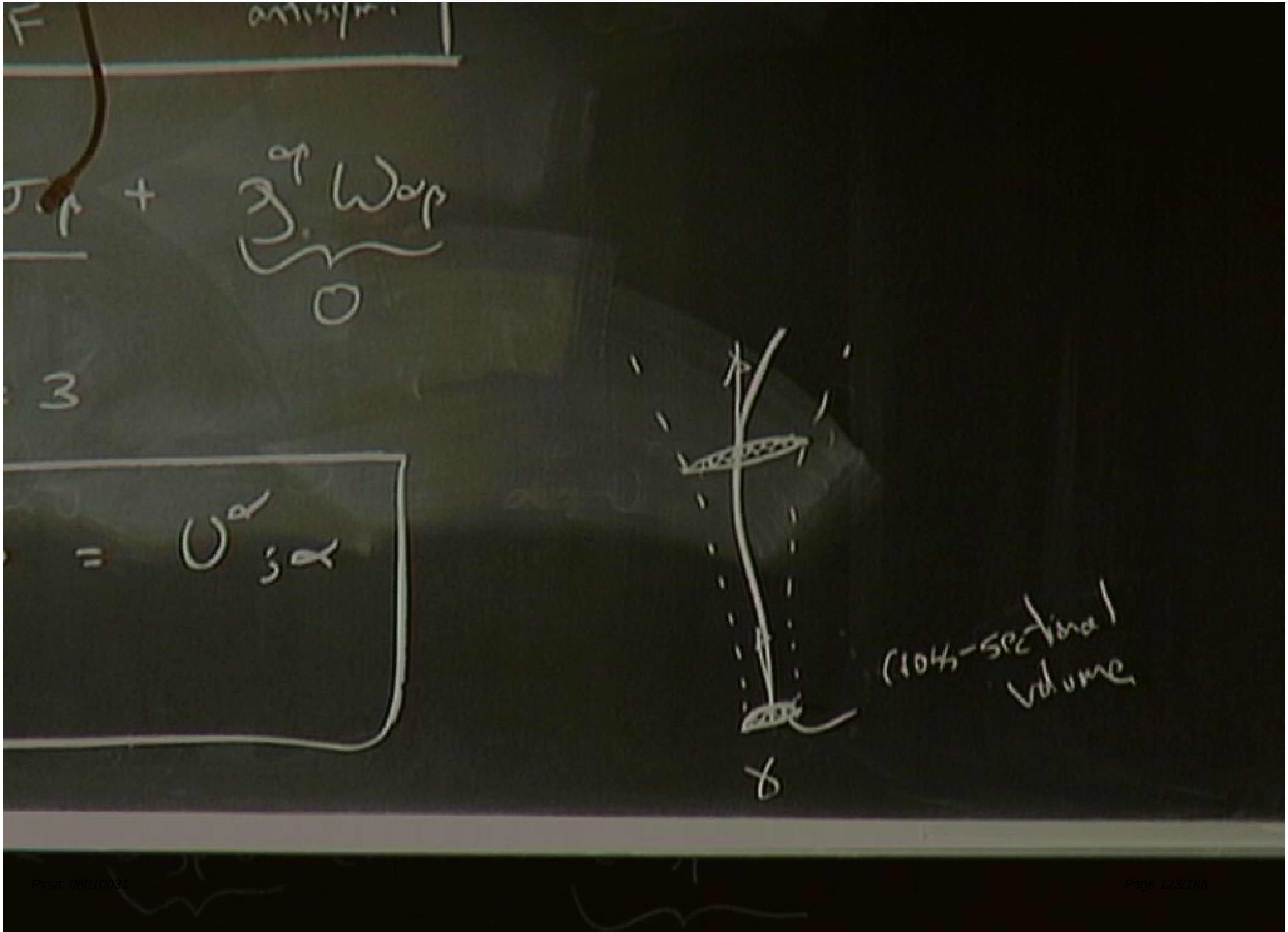


$U_\alpha B^\alpha{}_\beta = U_\alpha U^\alpha{}_{;\beta} = \frac{1}{2}($
$B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$
Introduce
$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$
(purely
$U^\alpha h_{\alpha\beta} = U_\beta - U_\beta$
Spatial

$$= \frac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta} \omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

$$\boxed{\begin{aligned} &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^\alpha{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V \end{aligned}}$$


antisym.
cross-sectional
volume

$$\ldots\beta = \underbrace{\tfrac{1}{3}\theta\, g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}\left(g_{\alpha\beta} + U_\alpha U_\beta\right) = 3$$

$$\boxed{\begin{aligned}\Theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$

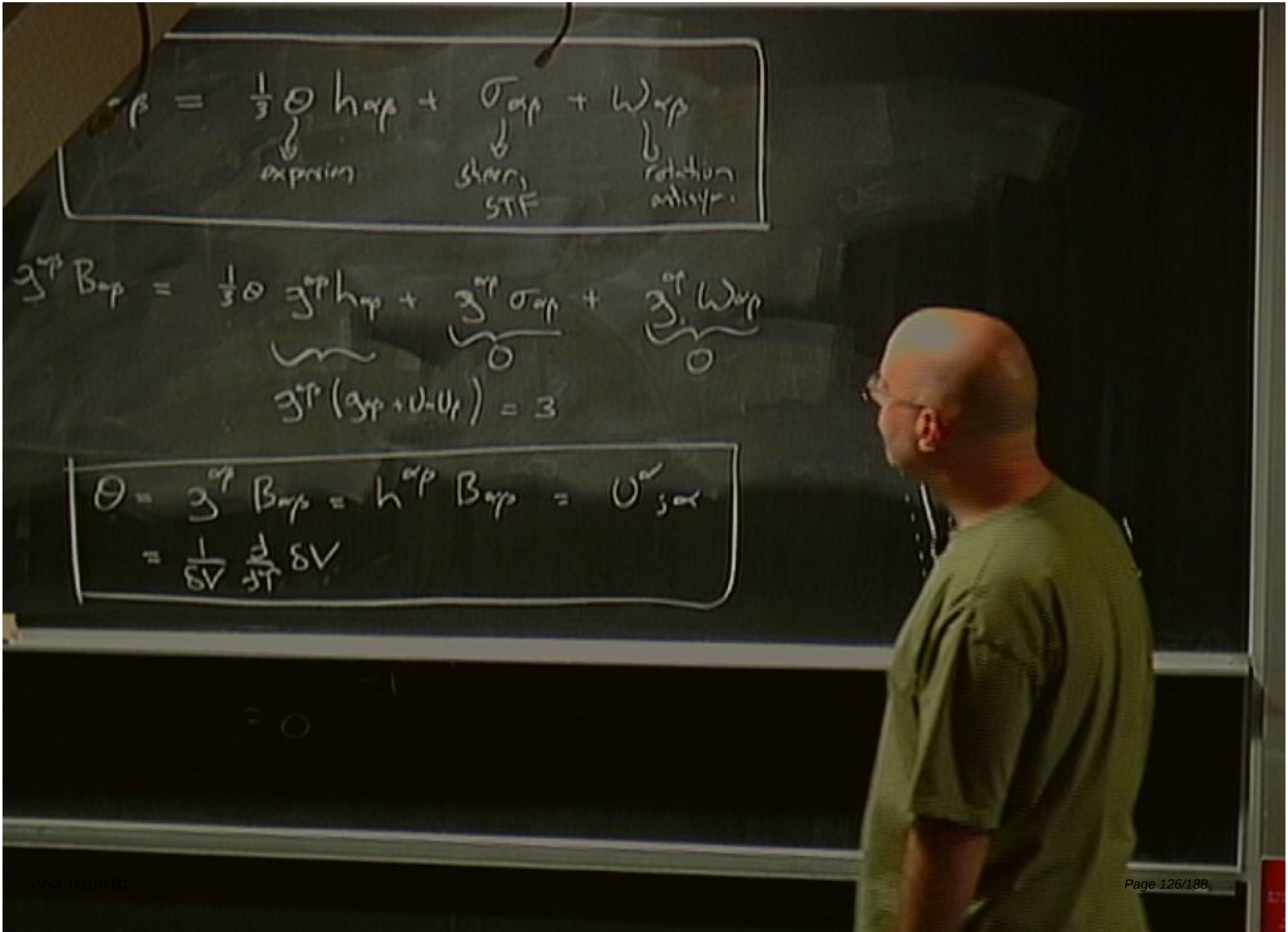
$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$

$$\rho = \frac{1}{3}\theta\, \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}\left(g_{\alpha\beta} + U_\alpha U_\beta\right) = 3$$

$$\boxed{\begin{aligned}\theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = U^{\alpha}{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$

$$= U^\alpha U_{\alpha;\beta}\,\xi^\beta$$


$\beta = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
expansion
shear, STF
rotation antisym.
$g^{\alpha\beta}B_{\alpha\beta} = \frac{1}{3}\theta \, g^{\alpha\beta}h_{\alpha\beta} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$
$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$
$\theta = g^{\alpha\beta}B_{\alpha\beta} = h^{\alpha\beta}B_{\alpha\beta} = u^{\alpha}{}_{;\alpha}$
$= \frac{1}{\delta V}\frac{d}{d\tau}\delta V$

$$\ldots_{\alpha\beta} = \underset{\text{expansion}}{\frac{1}{3}\theta}\, h_{\alpha\beta} + \underset{\text{shear, STF}}{\sigma_{\alpha\beta}} + \underset{\text{rotation antisym.}}{\omega_{\alpha\beta}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \frac{1}{3}\theta\, \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + g^{\alpha\beta}\ldots$$

$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

$$\boxed{\theta = g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^{\alpha}{}_{;\alpha} = \frac{1}{\delta V}\frac{d}{d\tau}\delta V}$$

$$\boxed{B_{\alpha\beta} = \underbrace{\tfrac{1}{3}\theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear, STF}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation antisym.}}}$$

$$g^{\alpha\beta} B_{\alpha\beta} = \tfrac{1}{3}\theta \underbrace{g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

$$\boxed{\begin{aligned}\theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^\alpha{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$

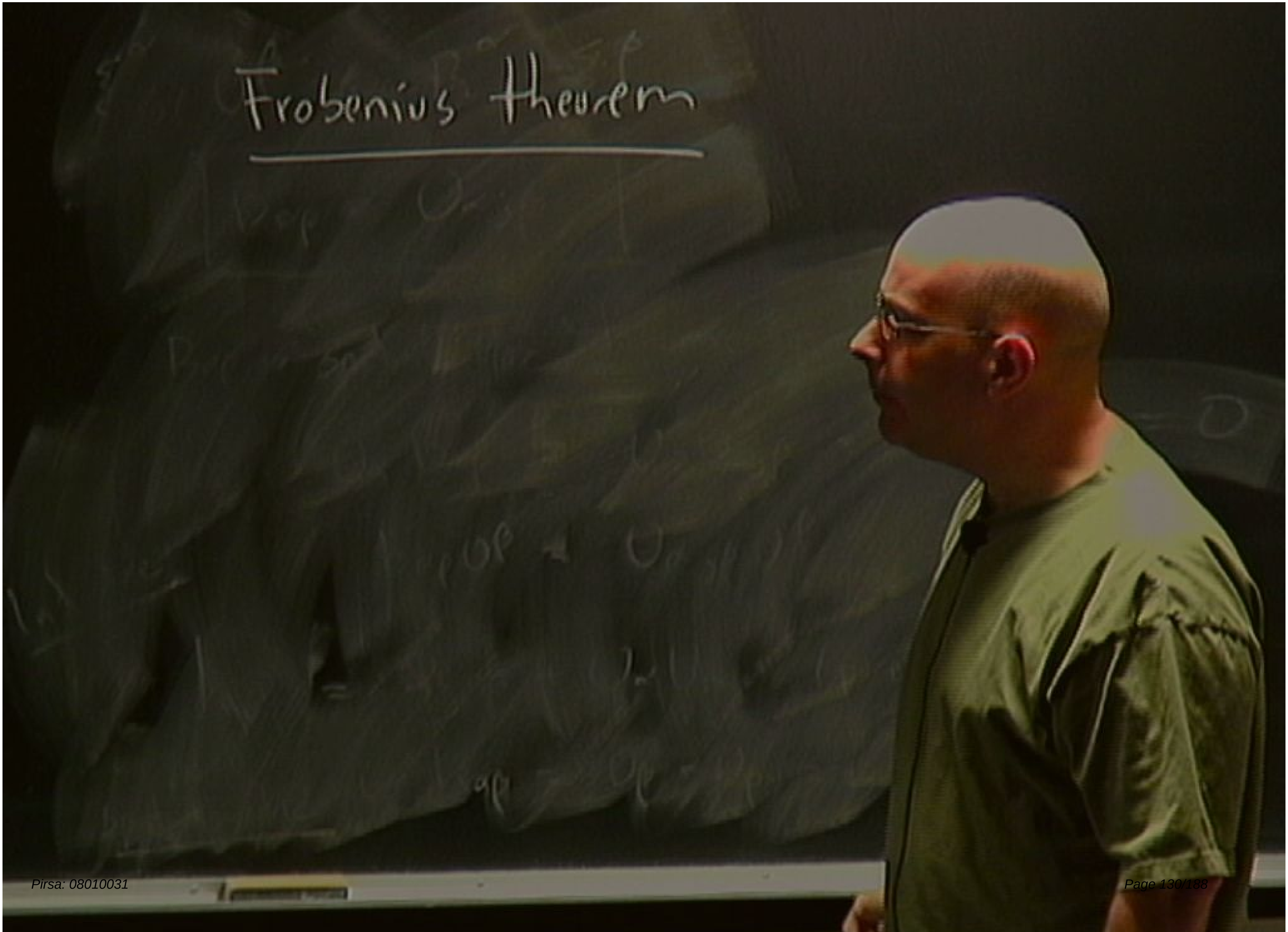
expansion — shear, STF — rotation antisym.

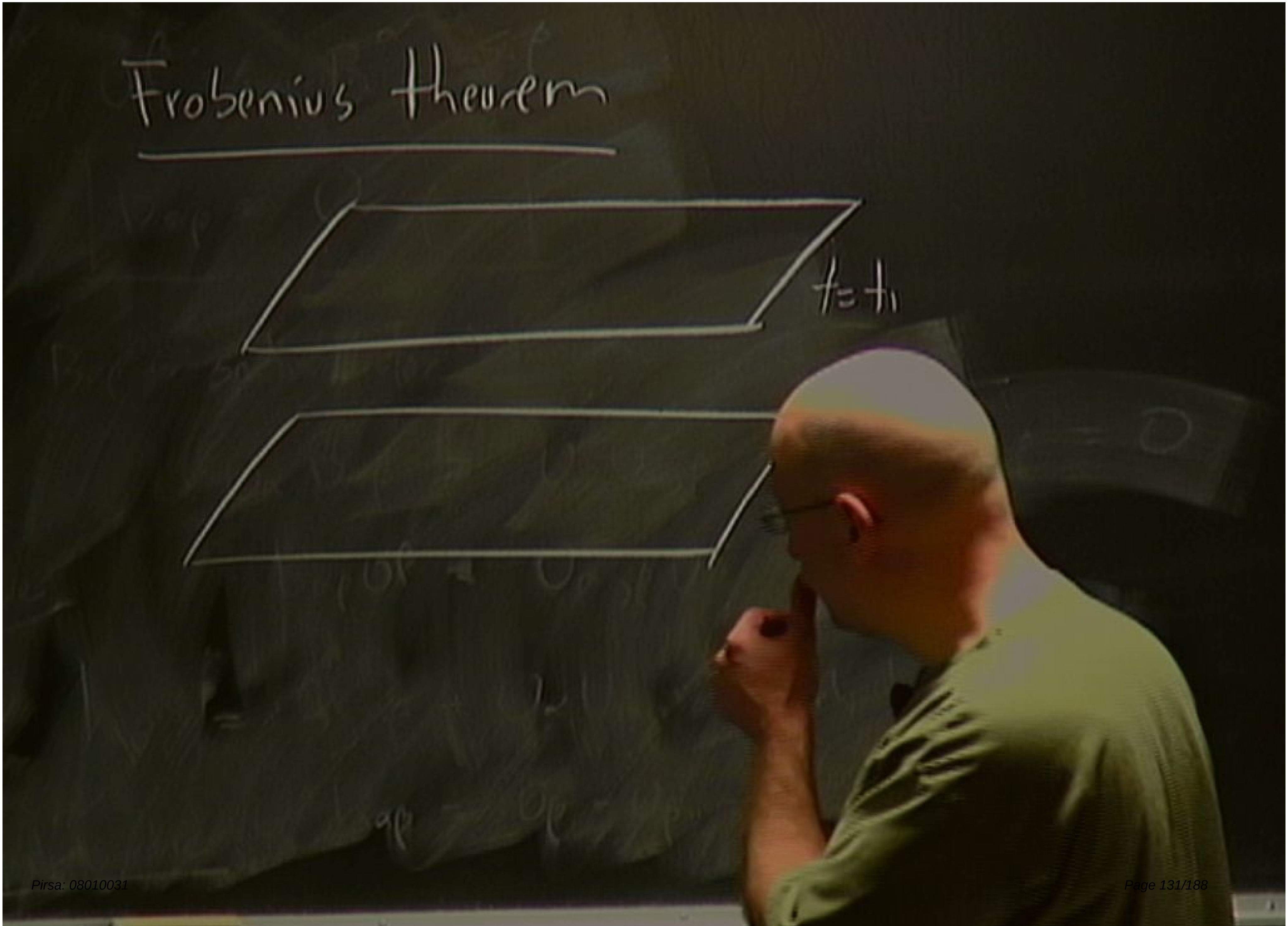
$$g^{\alpha\beta} B_{\alpha\beta} = \underbrace{\tfrac{1}{3}\theta\, g^{\alpha\beta} h_{\alpha\beta}} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$$

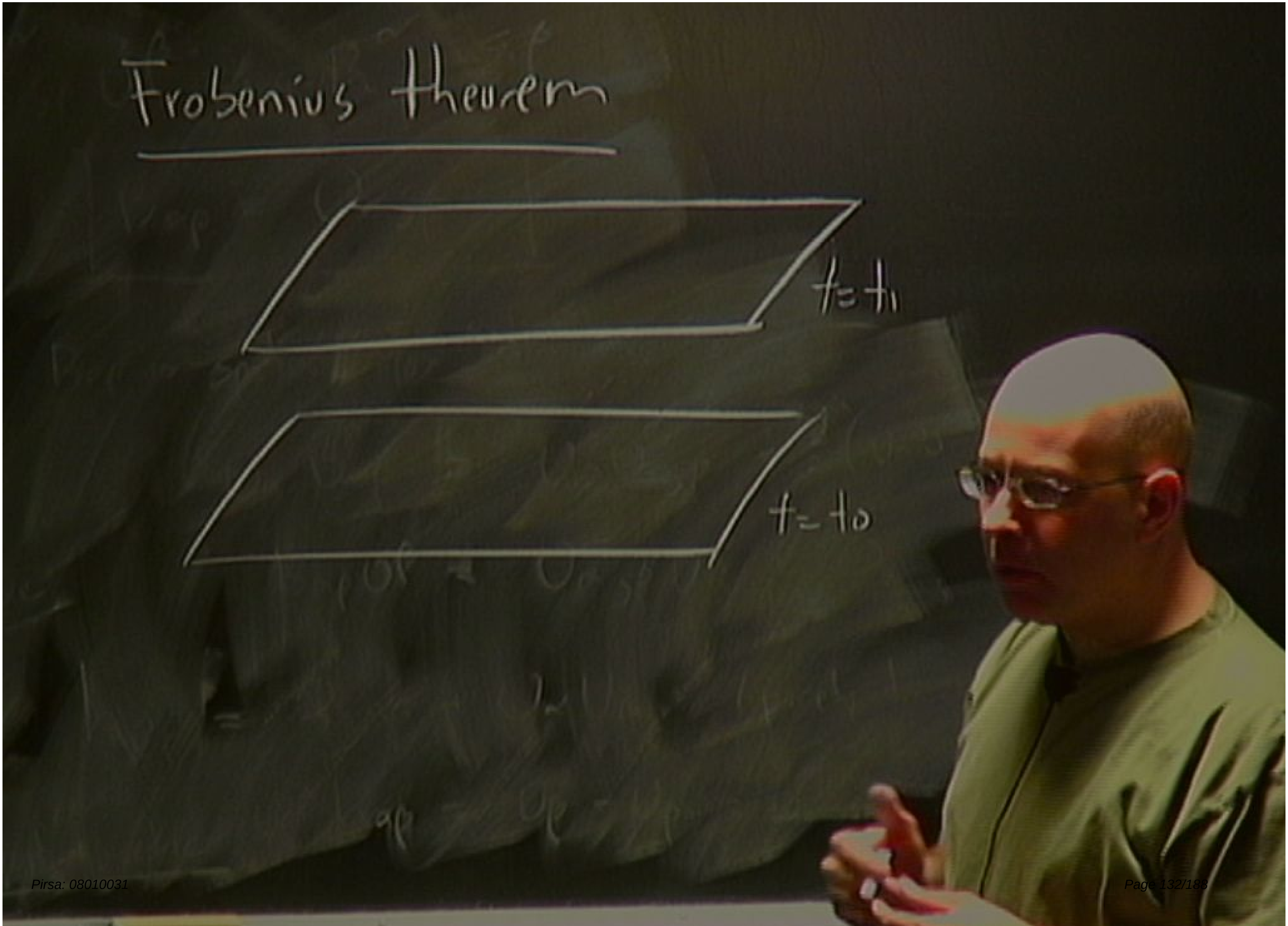
$$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$$

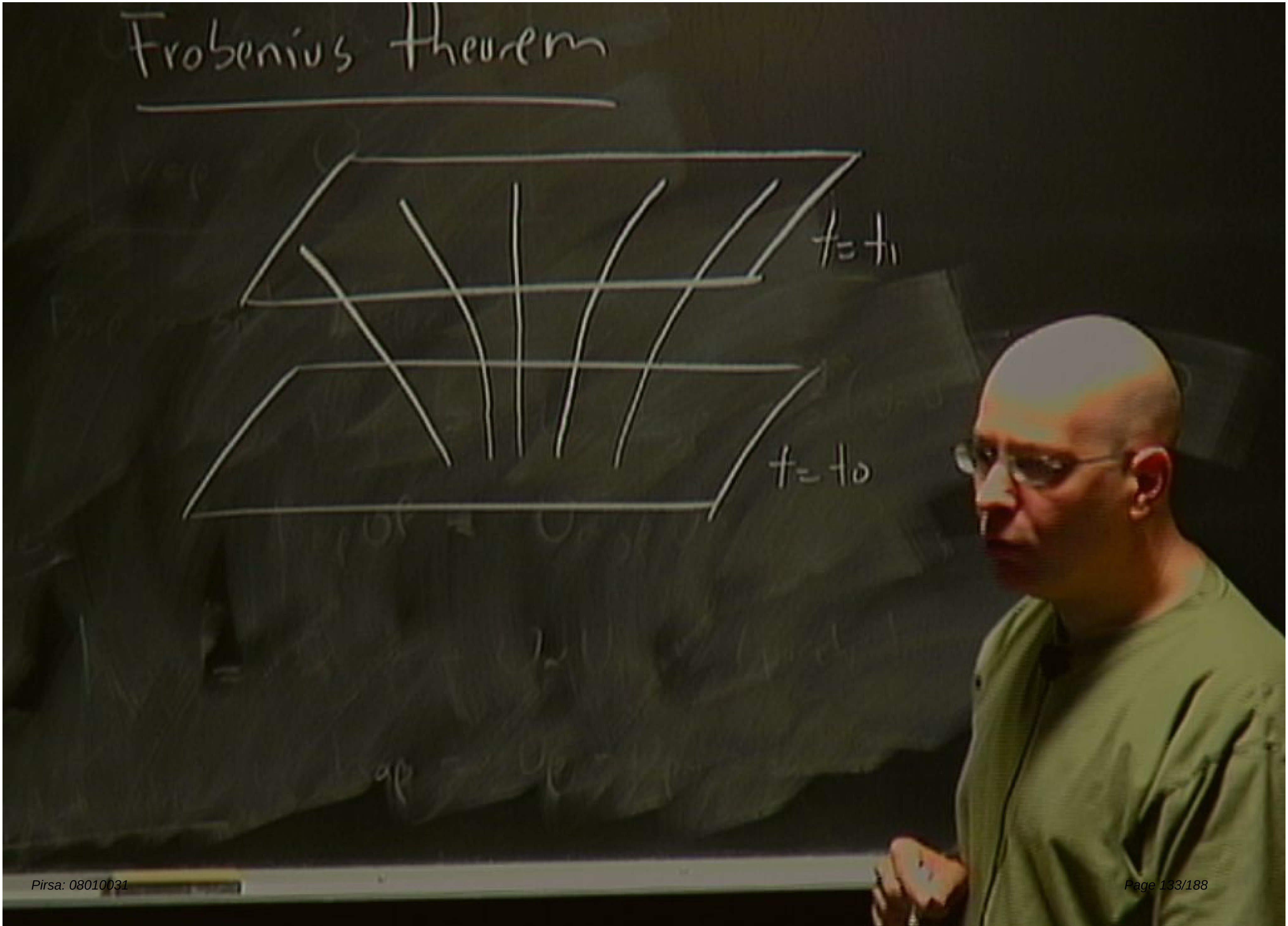
$$\boxed{\begin{aligned}\theta &= g^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = u^\alpha{}_{;\alpha} \\ &= \frac{1}{\delta V}\frac{d}{d\tau}\delta V\end{aligned}}$$

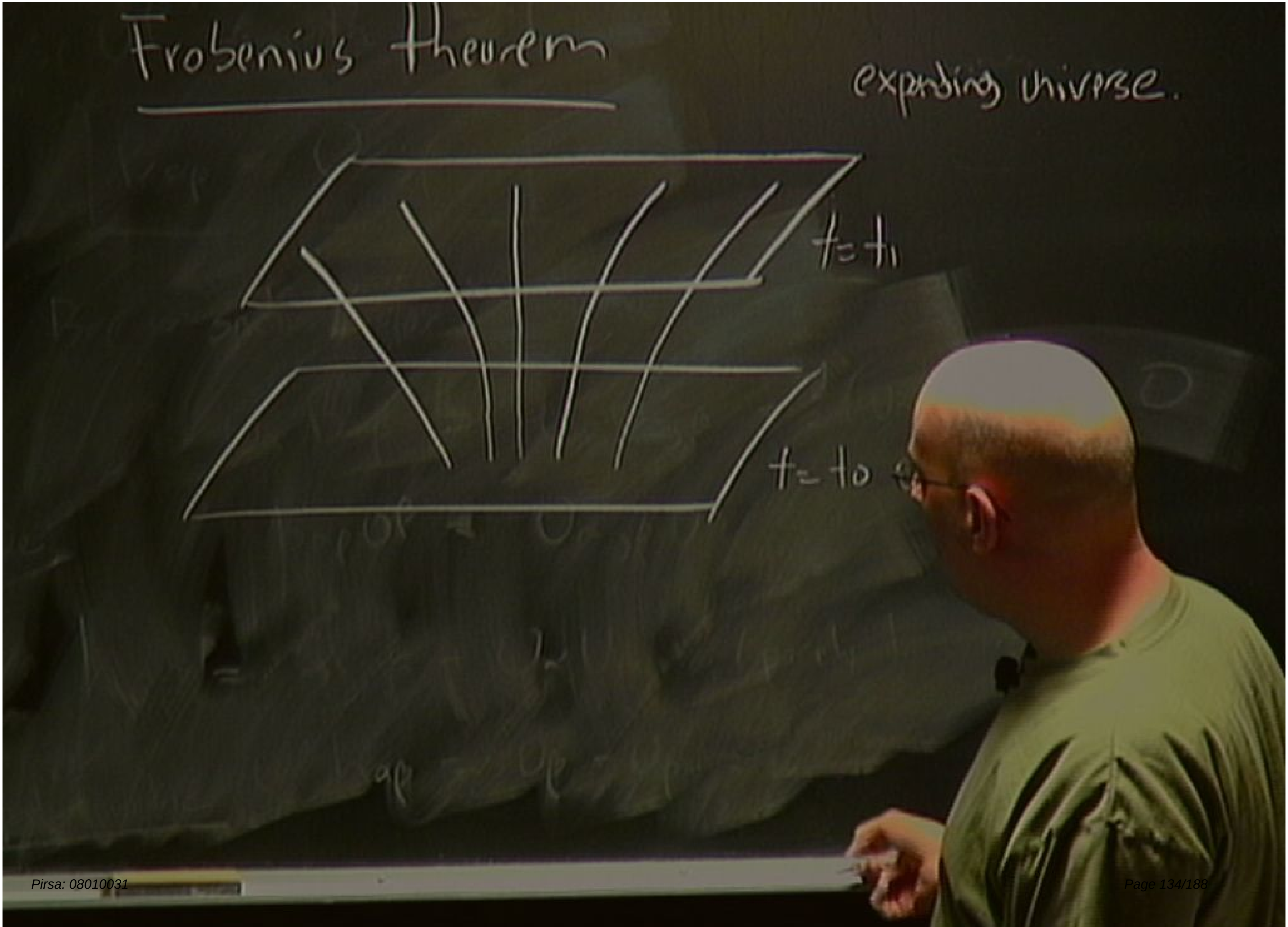
$$= \frac{1}{2}\underbrace{(u^\alpha u_\alpha)}_{-1}{}_{;\beta}\, \xi^\beta$$

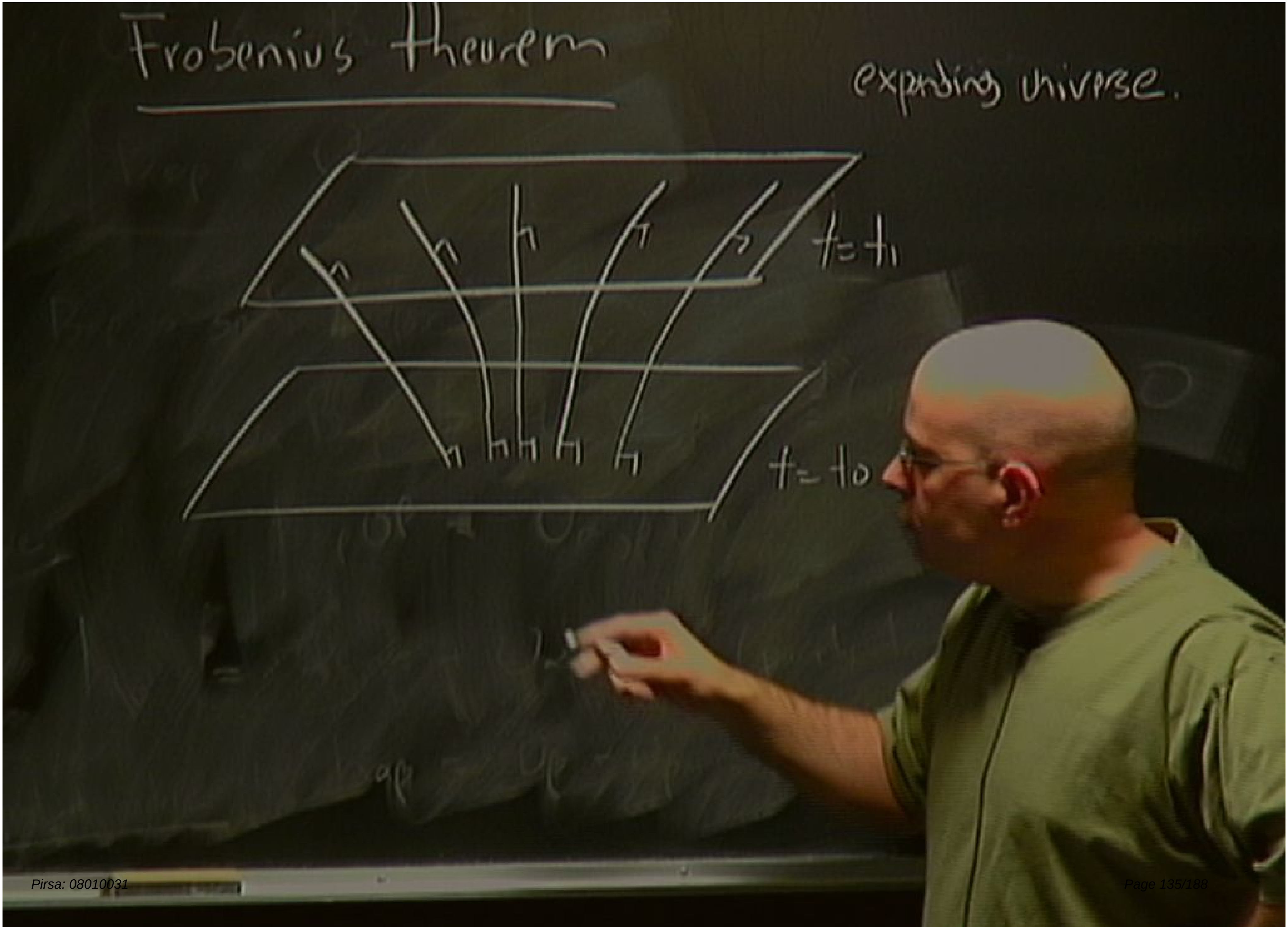

Frobenius theorem

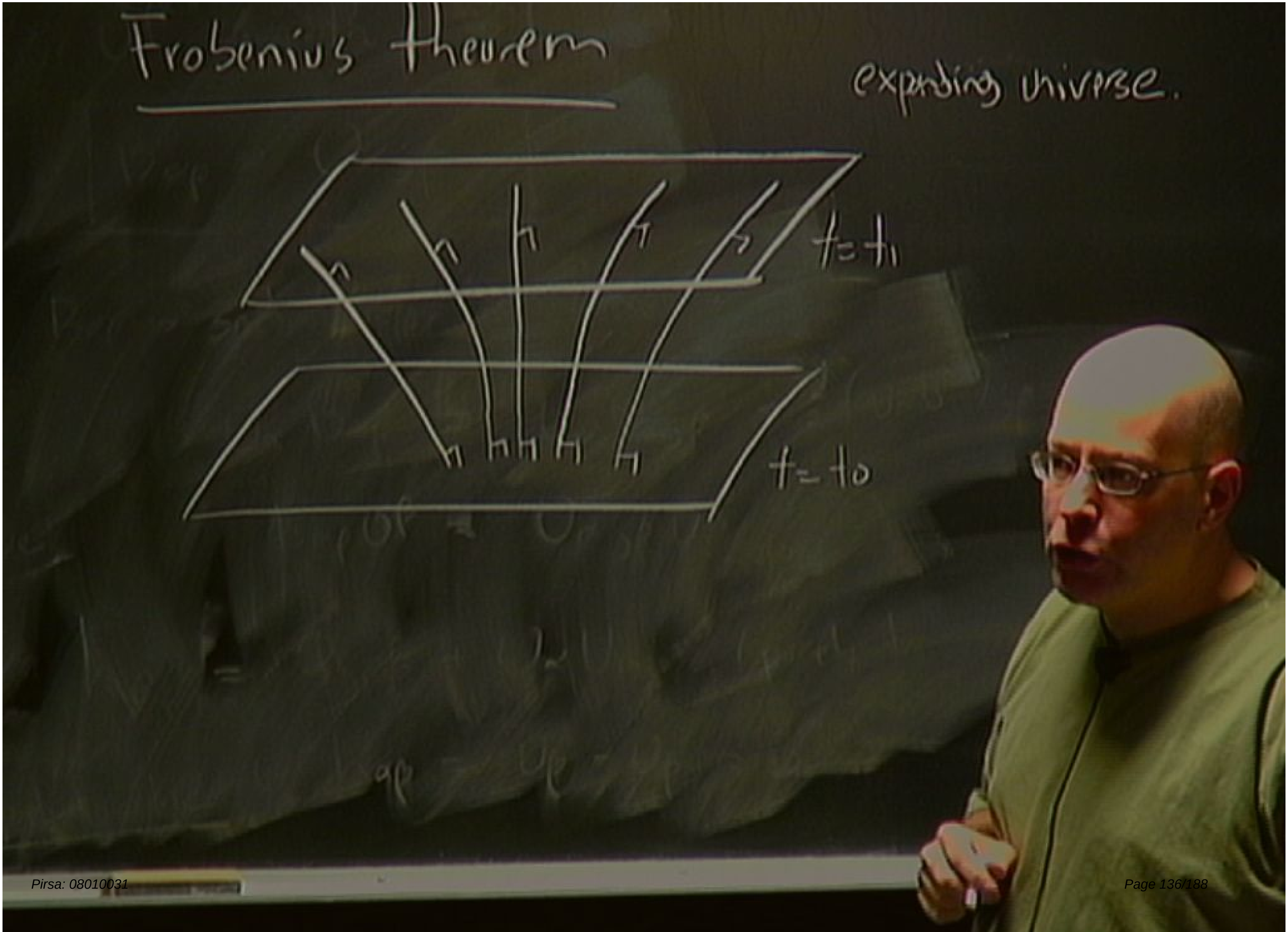

Frobenius theorem
t=t1


Frobenius theorem
t=t1
t=t0

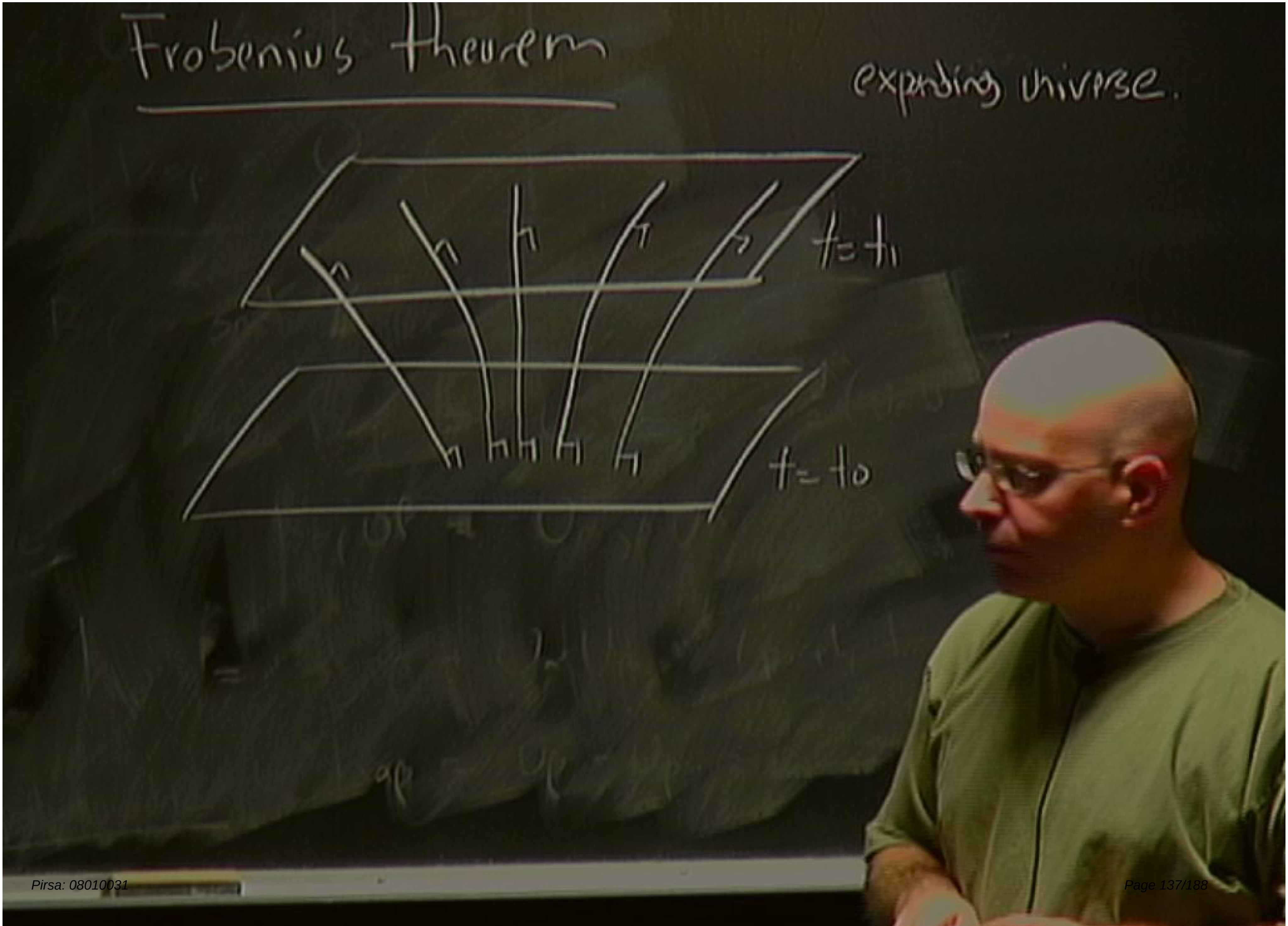

Frobenius theorem
t = t1
t = t0


Frobenius theorem
expanding universe.
t=t1
t=t0


Frobenius theorem
expanding universe.
t=t1
t=t0

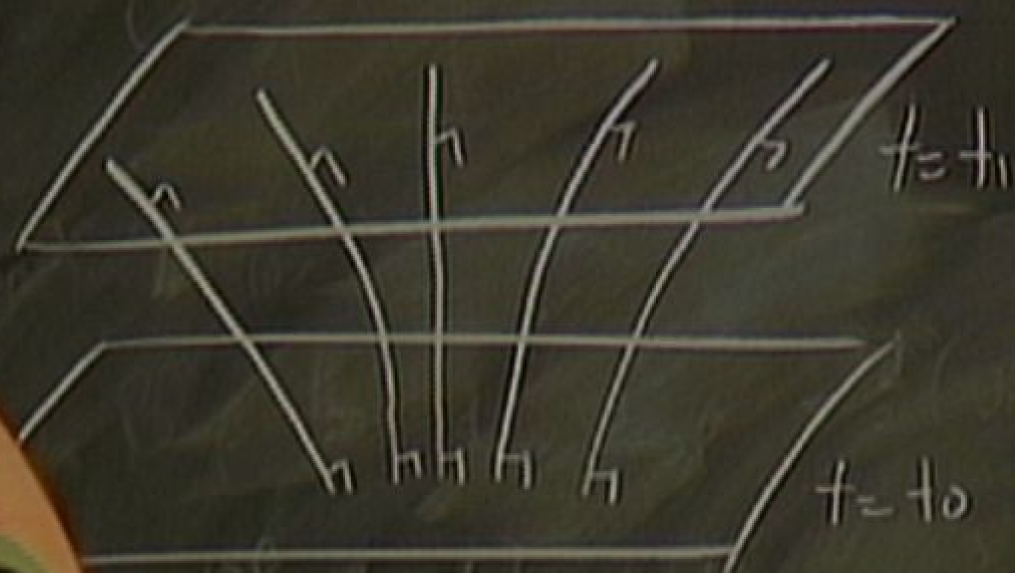


Frobenius theorem
expanding universe.
t = t1
t = t0


Frobenius theorem
expanding universe.
t=t1
t= t0

Frobenius theorem

expanding universe.

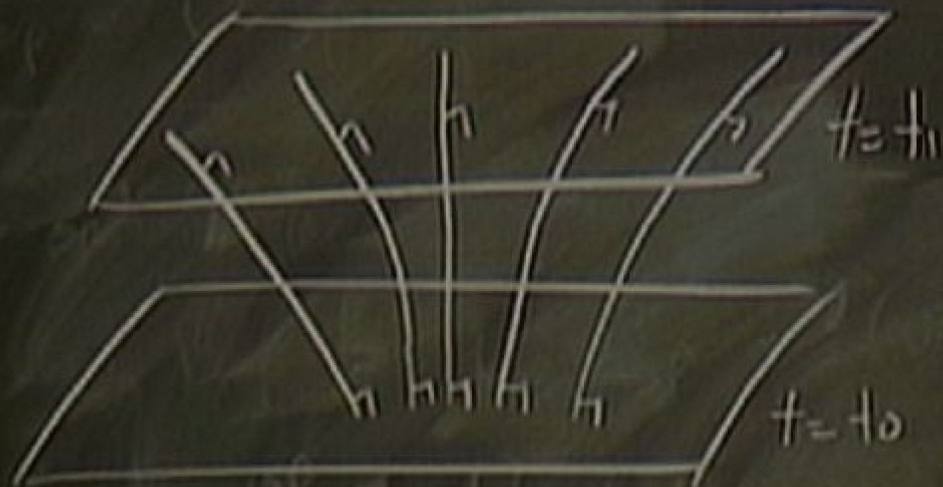


congruence is hypersurface orthogonal.

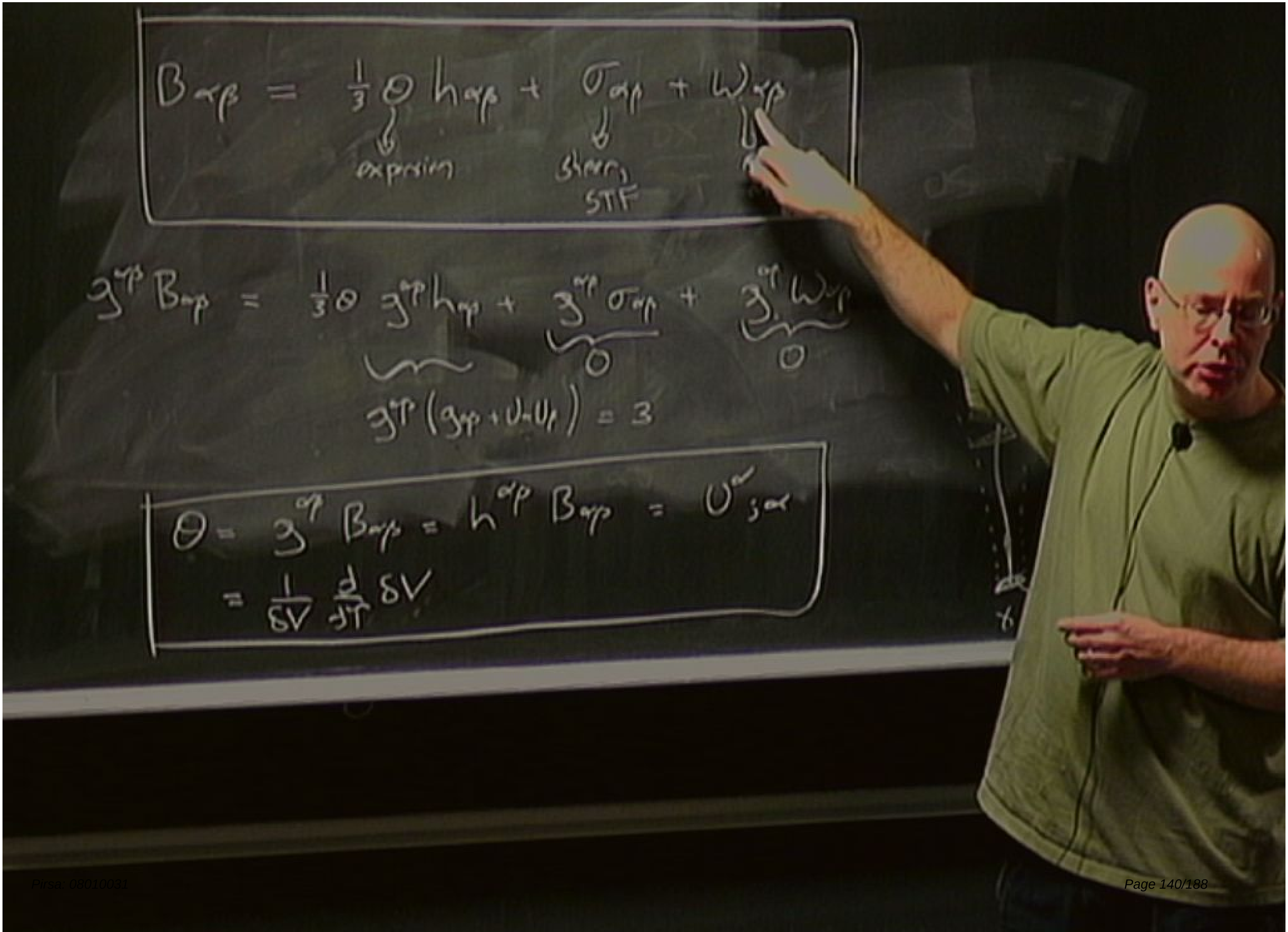
Frobenius theorem

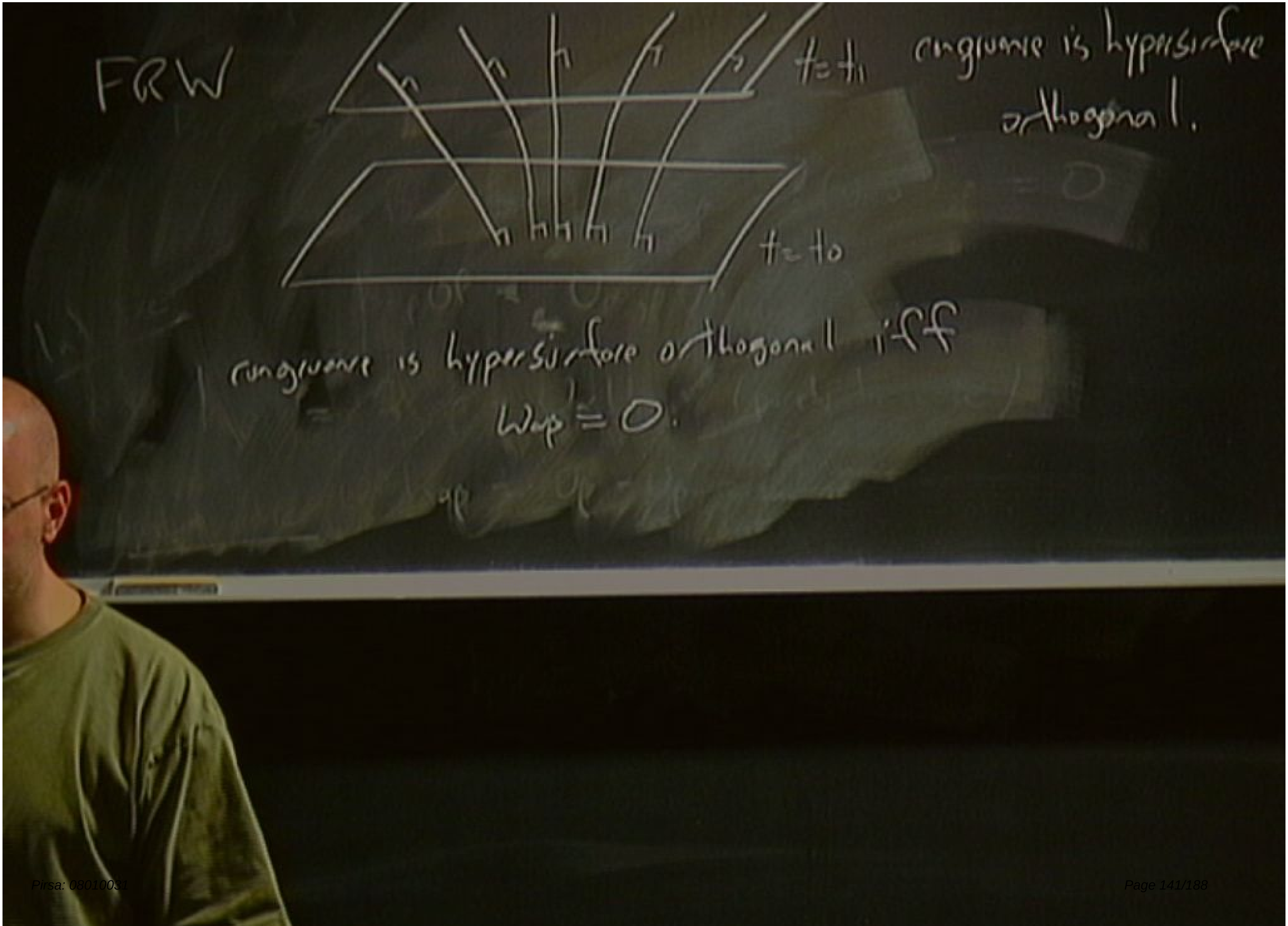
expanding universe.

FRW



congruence is hypersurface orthogonal.

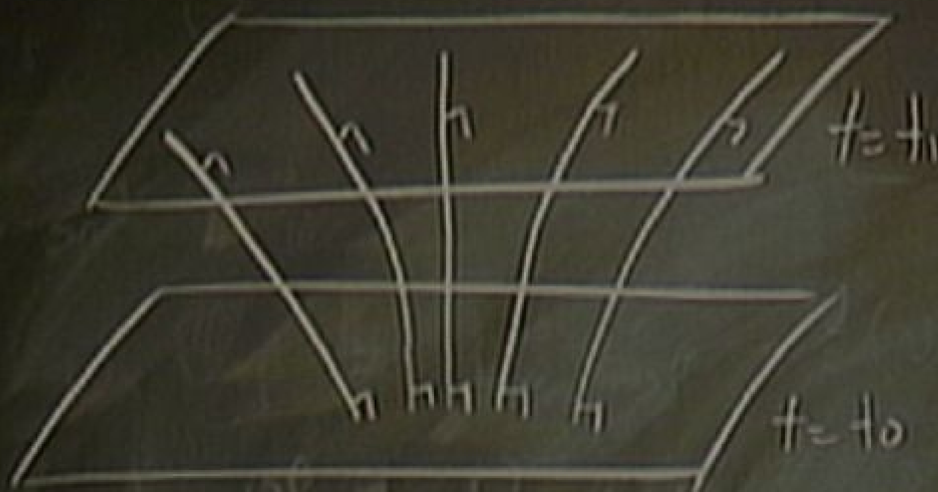

$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
expasion
shear, STF
$g^{\alpha\beta}B_{\alpha\beta} = \frac{1}{3}\theta\, g^{\alpha\beta}h_{\alpha\beta} + \underbrace{g^{\alpha\beta}\sigma_{\alpha\beta}}_{0} + \underbrace{g^{\alpha\beta}\omega_{\alpha\beta}}_{0}$
$g^{\alpha\beta}(g_{\alpha\beta} + u_\alpha u_\beta) = 3$
$\theta = g^{\alpha\beta}B_{\alpha\beta} = h^{\alpha\beta}B_{\alpha\beta} = u^{\alpha}{}_{;\alpha}$
$= \frac{1}{\delta V}\frac{d}{d\tau}\delta V$


FRW
t=t1
t=t0
congruence is hypersurface orthogonal.
congruence is hypersurface orthogonal iff
ωab = 0.

Frobenius theorem

expanding universe.

FRW



congruence is hypersurface orthogonal.

congruence is hypersurface orthogonal iff

$$\omega_{\alpha\beta} = 0.$$

$$\boxed{B_{\alpha\beta} = \frac{1}{3}\underbrace{\Theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\substack{\text{shear,}\\ \text{STF}}} + \underbrace{\omega_{\alpha\beta}}_{\substack{\text{rotation}\\ \text{antisym.}}}}$$

hypersurface :

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

hypersurface :

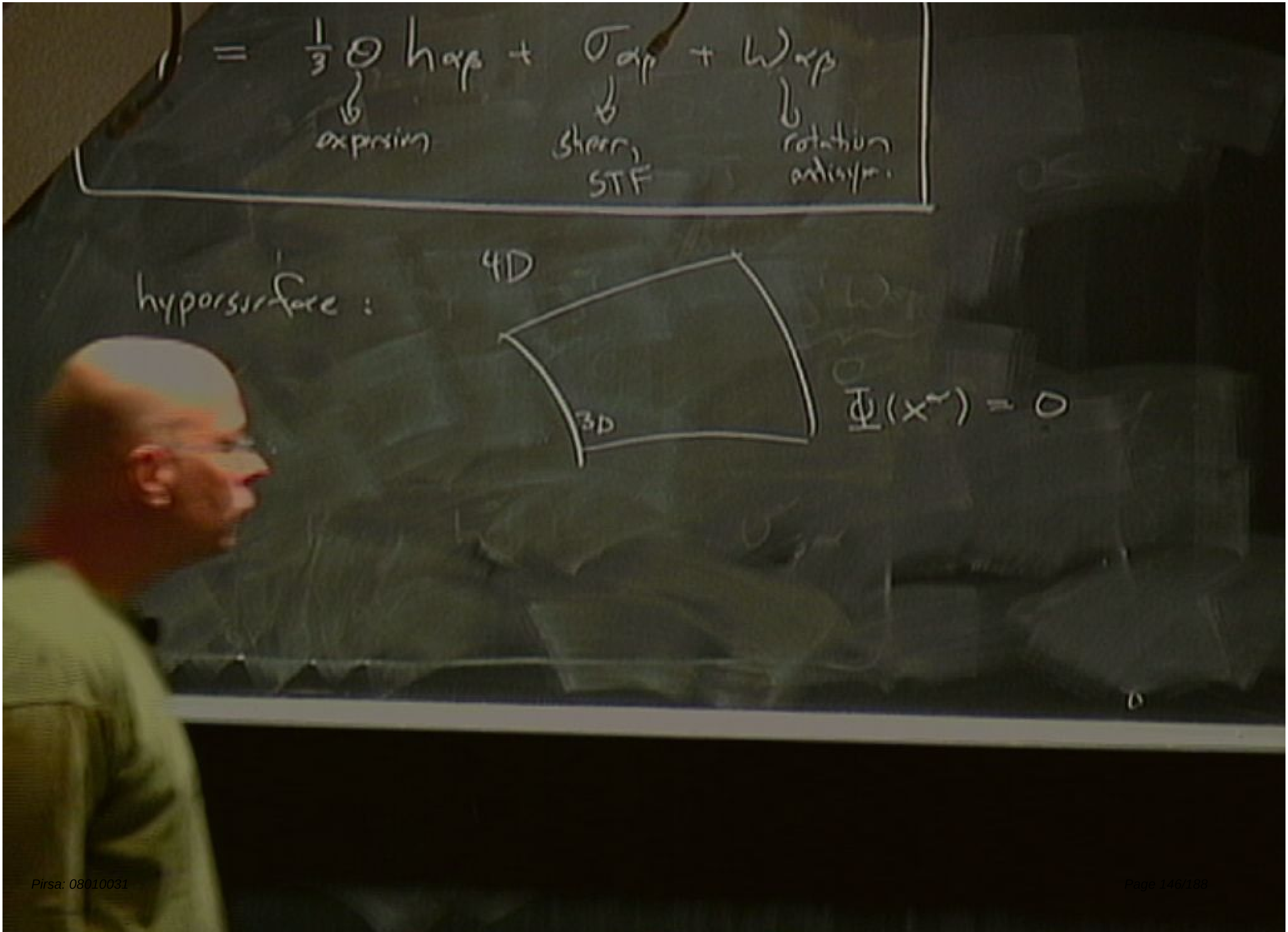
4D

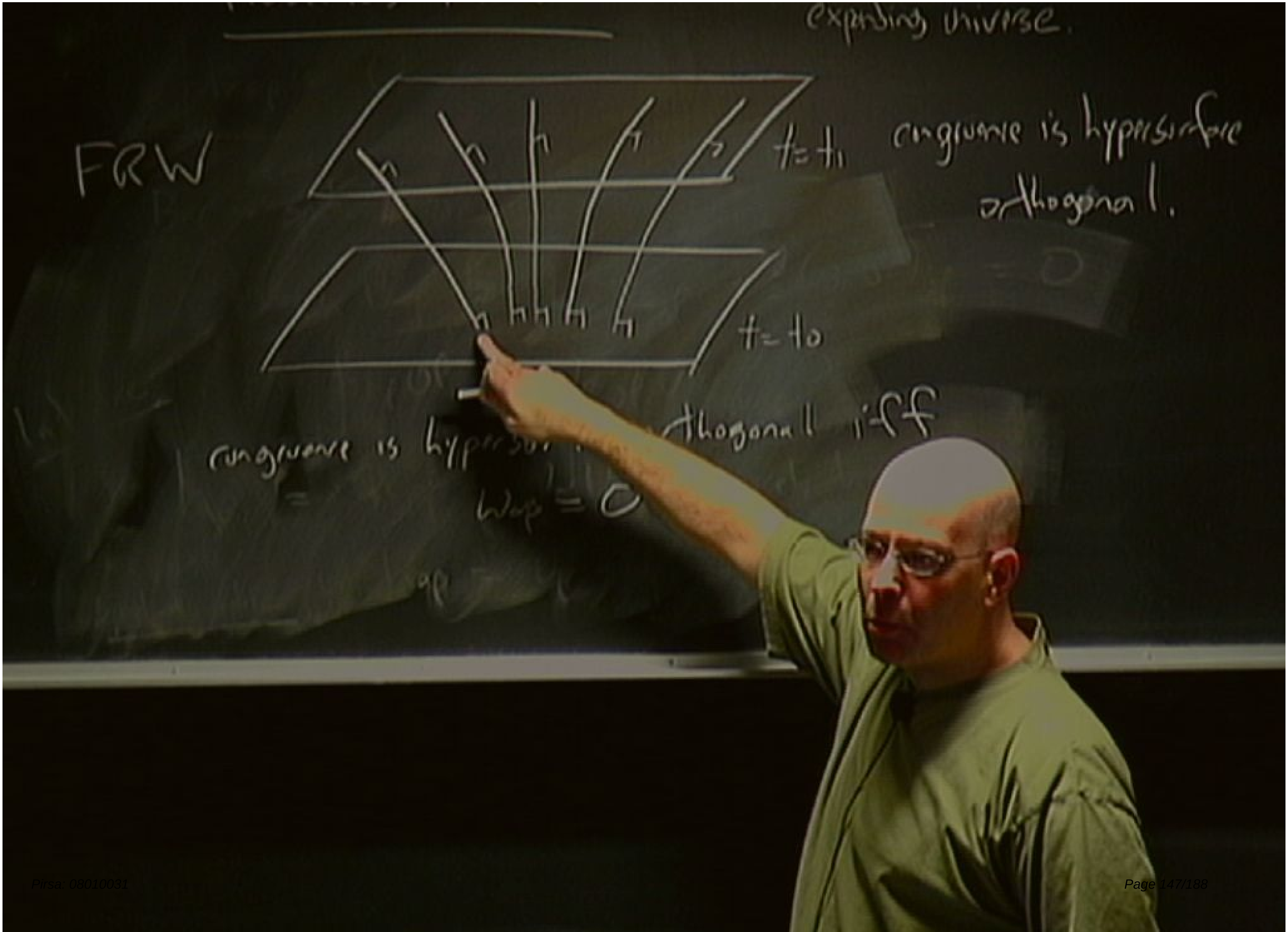
$\Phi(x^\mu) = 0$

$$= \underbrace{\tfrac{1}{3}\Theta h_{\alpha\beta}}_{\text{expansion}} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear, STF}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation antisym.}}$$

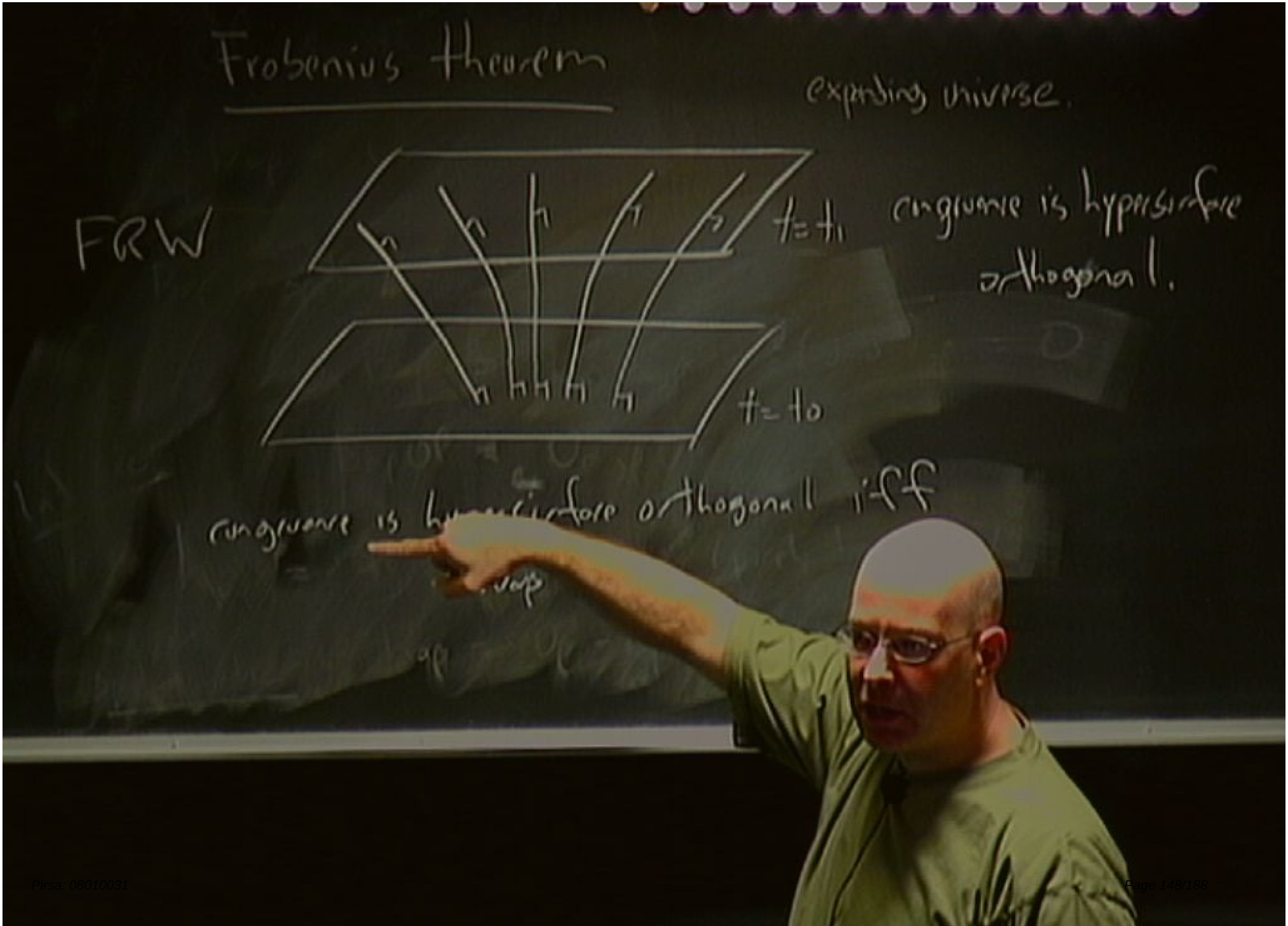
hypersurface :

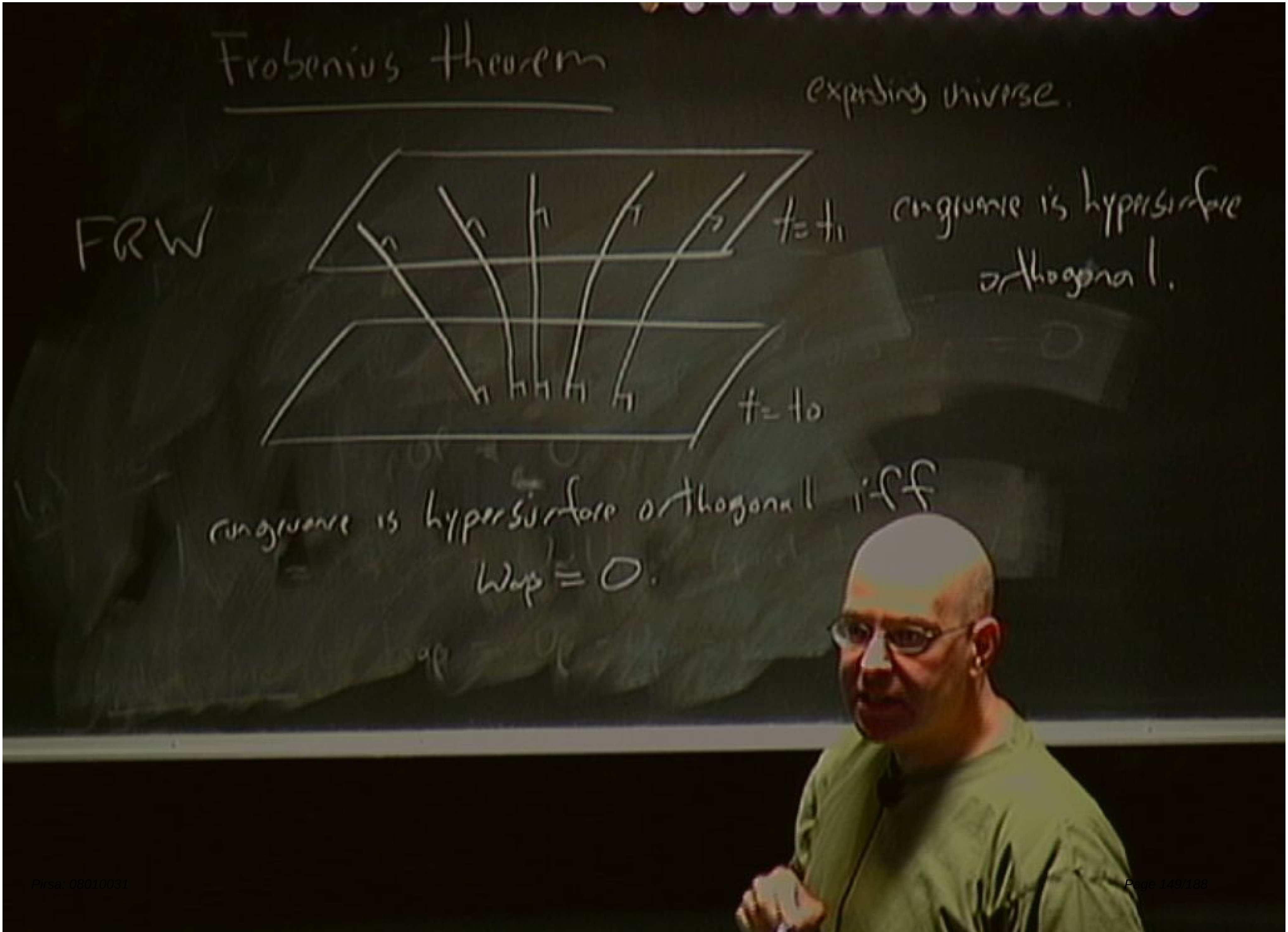
$\Phi(x^\mu) = 0$


$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
expansion
shear, STF
rotation antisym.
hypersurface :
4D
3D
$\Phi(x^\mu) = 0$



expanding universe.
FRW
t = t1
congruence is hypersurface orthogonal.
t = to
$\omega_{ab} = 0$

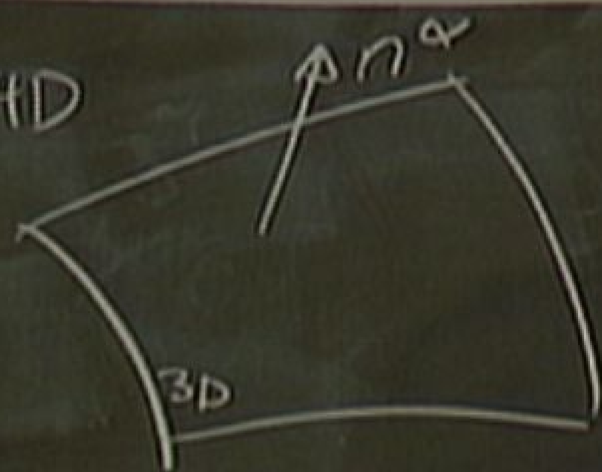

Frobenius theorem
expanding universe.
FRW
t = t1
t = t0
congruence is hypersurface orthogonal.


Frobenius theorem
expanding universe.
FRW
$t=t_1$
congruence is hypersurface orthogonal.
$t=t_0$
congruence is hypersurface orthogonal iff
$\omega_{\alpha\beta}=0$.

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

hypersurface :



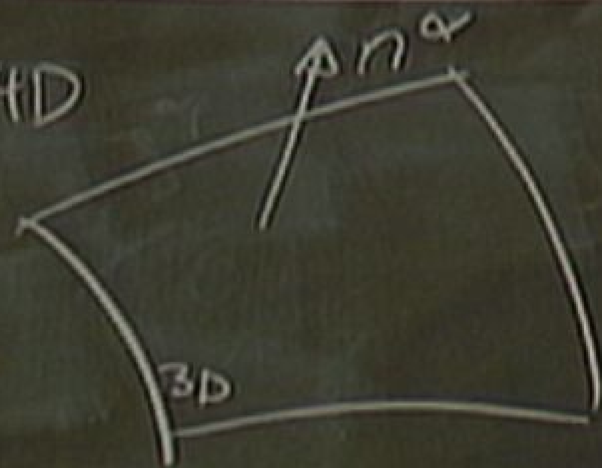
$$\Phi(x^\alpha) = 0$$

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

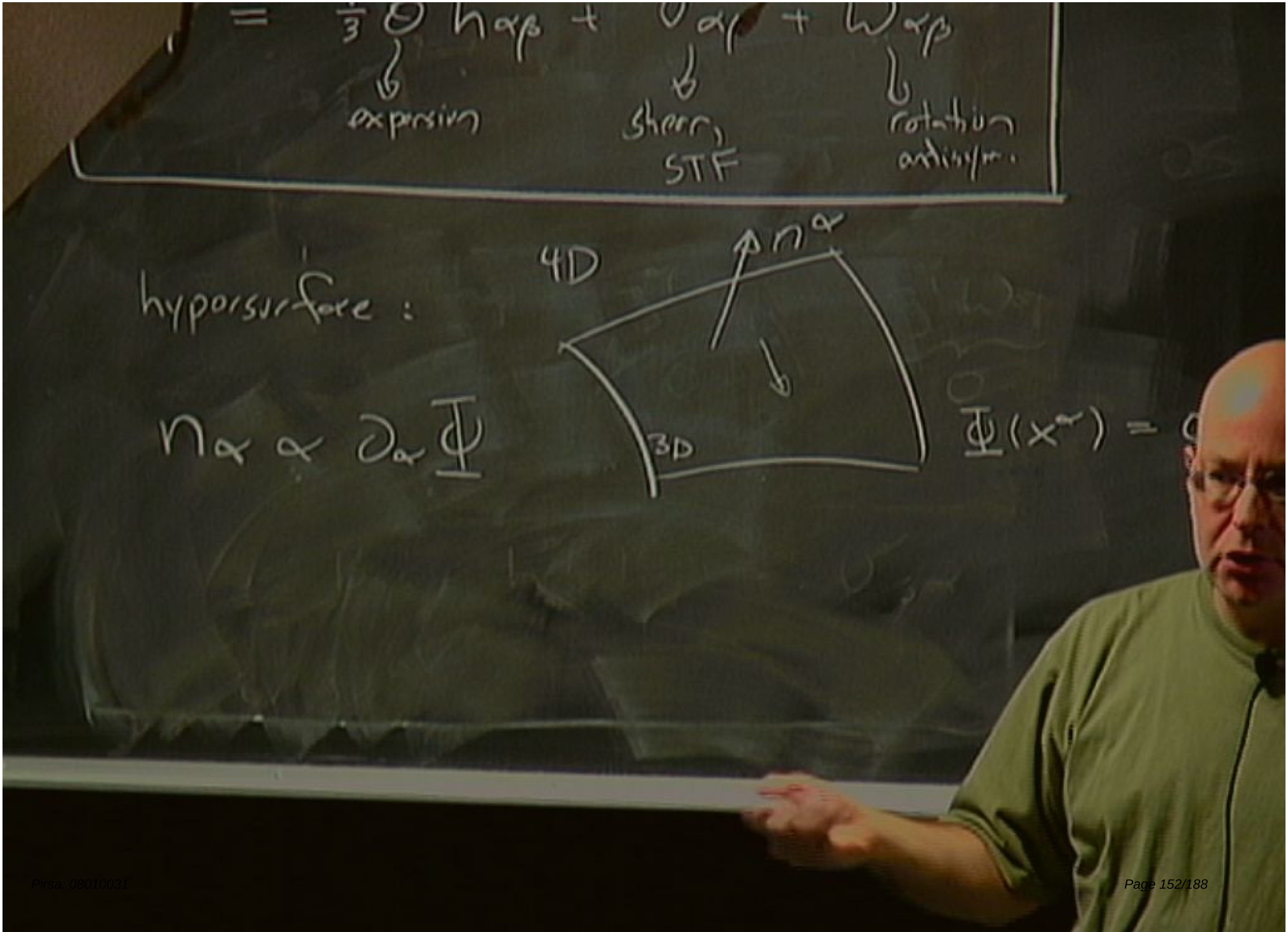
hypersurface :

4D



$n_\alpha \propto \partial_\alpha \Phi$

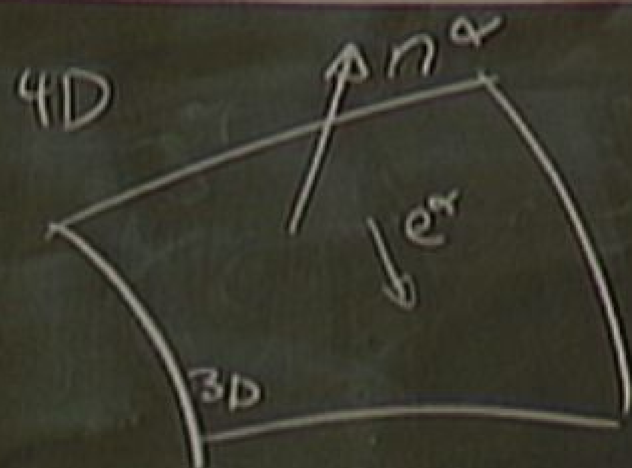
$\Phi(x^\alpha) = 0$


= 1/3 Θ h_αβ + σ_αβ + ω_αβ
expansion
shear,
STF
rotation
antisym.
hypersurface :
4D
n^α
3D
n_α ∝ ∂_α Φ
Φ(x^α) = 0

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

hypersurface :



$\Phi(x^\alpha) = 0$

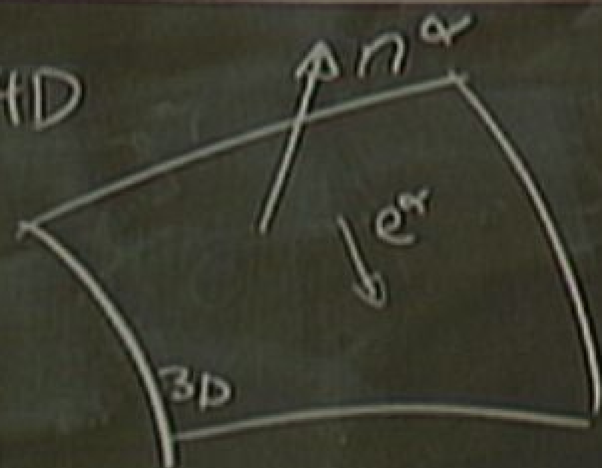
$$n_\alpha \propto \partial_\alpha \Phi$$

$$n_\alpha e^\alpha$$

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

hypersurface :



$\Phi(x^\alpha) = 0$

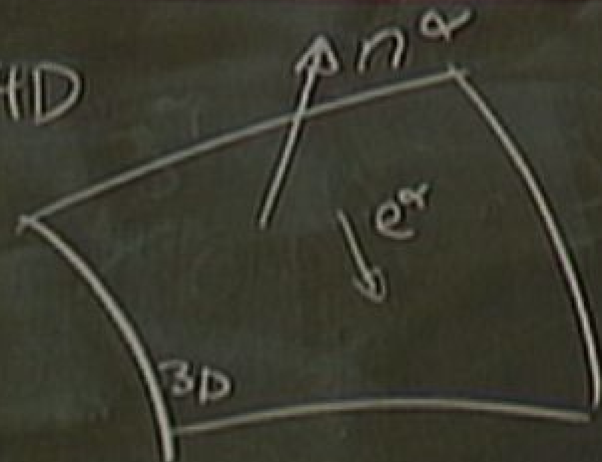
$$n_\alpha \propto \partial_\alpha \Phi$$

$$n_\alpha e^\alpha \propto \partial_\alpha \Phi\, e^\alpha$$

$$= \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

expansion — shear, STF — rotation antisym.

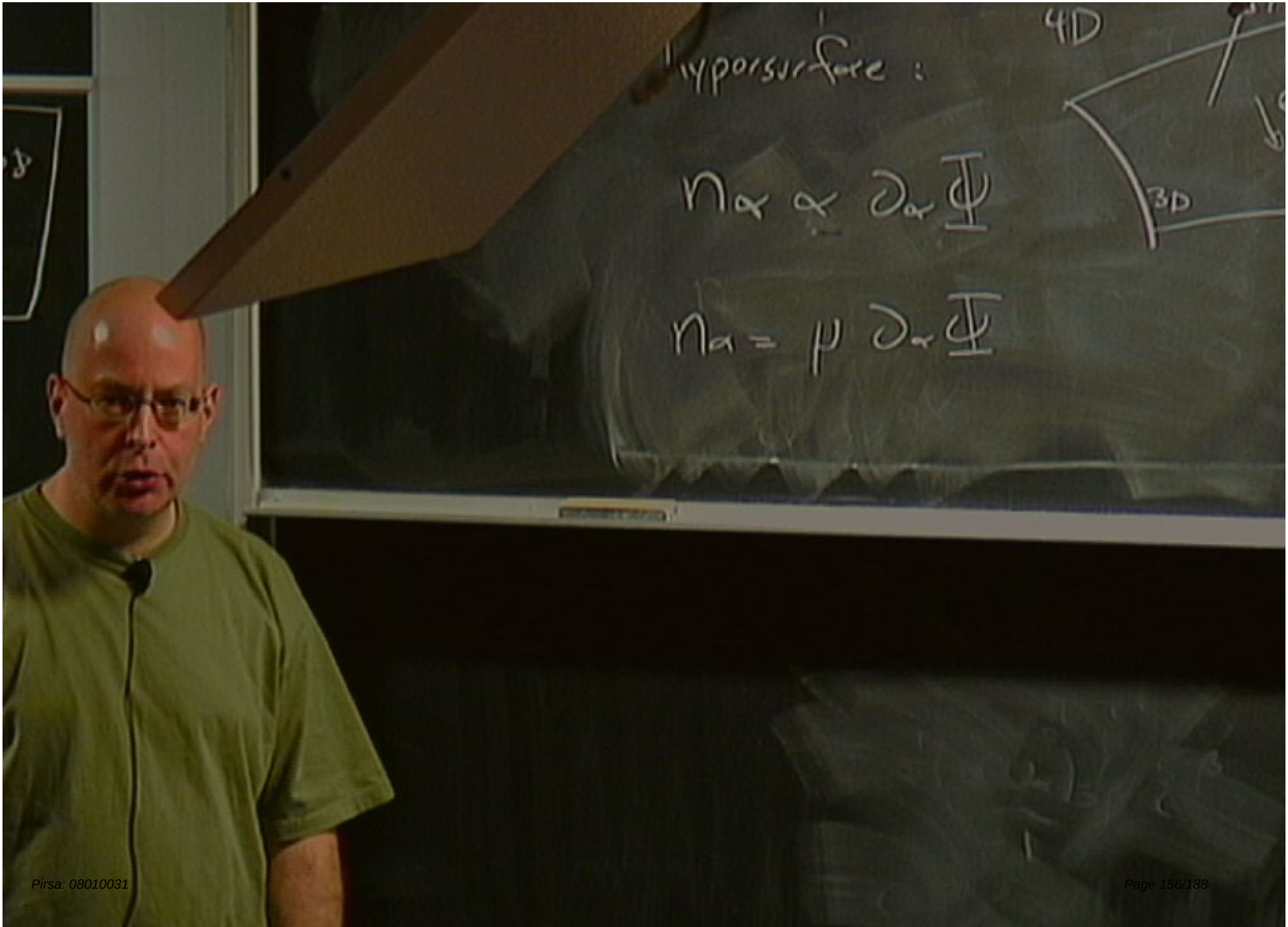
hypersurface :



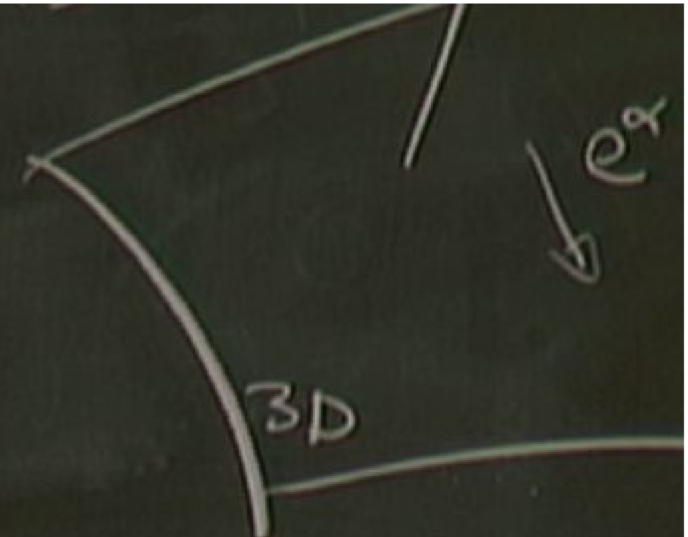
$$\Phi(x^\alpha) = 0$$

$$n_\alpha \propto \partial_\alpha \Phi$$

$$n_\alpha \underline{\underline{e^\alpha}} \propto \partial_\alpha \Phi\, e^\alpha = 0$$


Hypersurface :
4D
3D
$n_\alpha \propto \partial_\alpha \Phi$
$n_\alpha = \mu \, \partial_\alpha \Phi$

hypersurface :



$$n_\alpha \propto \partial_\alpha \Phi$$

$$\boxed{n_\alpha = \mu \, \partial_\alpha \Phi}$$

(normal to hypersurface.

$$n_\alpha \propto \partial_\alpha \Phi$$

$\Phi(x^\alpha) =$

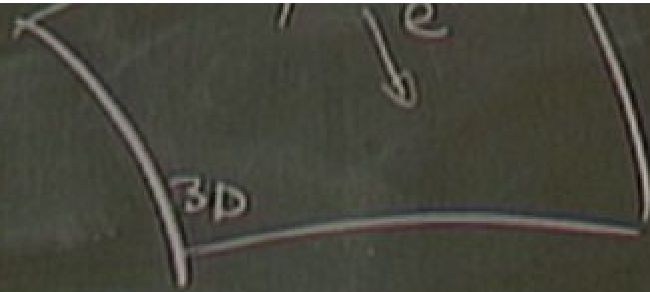
$$\boxed{n_\alpha = \mu\, \partial_\alpha \Phi}$$

normal to hypersurface.

μ ensures normalization:

$$n_\alpha n^\alpha = -1$$

$$n_\alpha \propto \partial_\alpha \Phi$$



$$\Phi(x^\alpha) = 0$$

$$\boxed{n_\alpha = \mu\, \partial_\alpha \Phi}$$

normal to hypersurface.

μ ensures normalization:

$$n_\alpha n^\alpha = -1$$

$$U_\alpha = n_\alpha$$

hypersurface orthogonal if $U_\alpha = n_\alpha$

hypersurface orthogonal if $v_\alpha = n_\alpha$ $\quad v_\alpha = \mu \partial_\alpha \Phi$

hypersurface orthogonal if $u_\alpha = n_\alpha$

$u_\alpha = \mu \, \partial_\alpha \Phi$

$u_{\alpha;\beta} = \mu \, \Phi_{;\alpha\beta}$

hypersurface orthogonal if $u_\alpha = n_\alpha$

$u_\alpha = \mu \partial_\alpha \Phi$

$u_{\alpha;\beta} = \mu \Phi_{;\alpha\beta} + \mu_{;\beta} \Phi_{,\alpha}$

hypersurface orthogonal if $u_\alpha = n_\alpha$

$$u_\alpha = \mu \partial_\alpha \Phi$$

$$u_{\alpha;\beta} = \mu \Phi_{;\alpha\beta} + \mu_{,\beta} \Phi_{,\alpha}$$

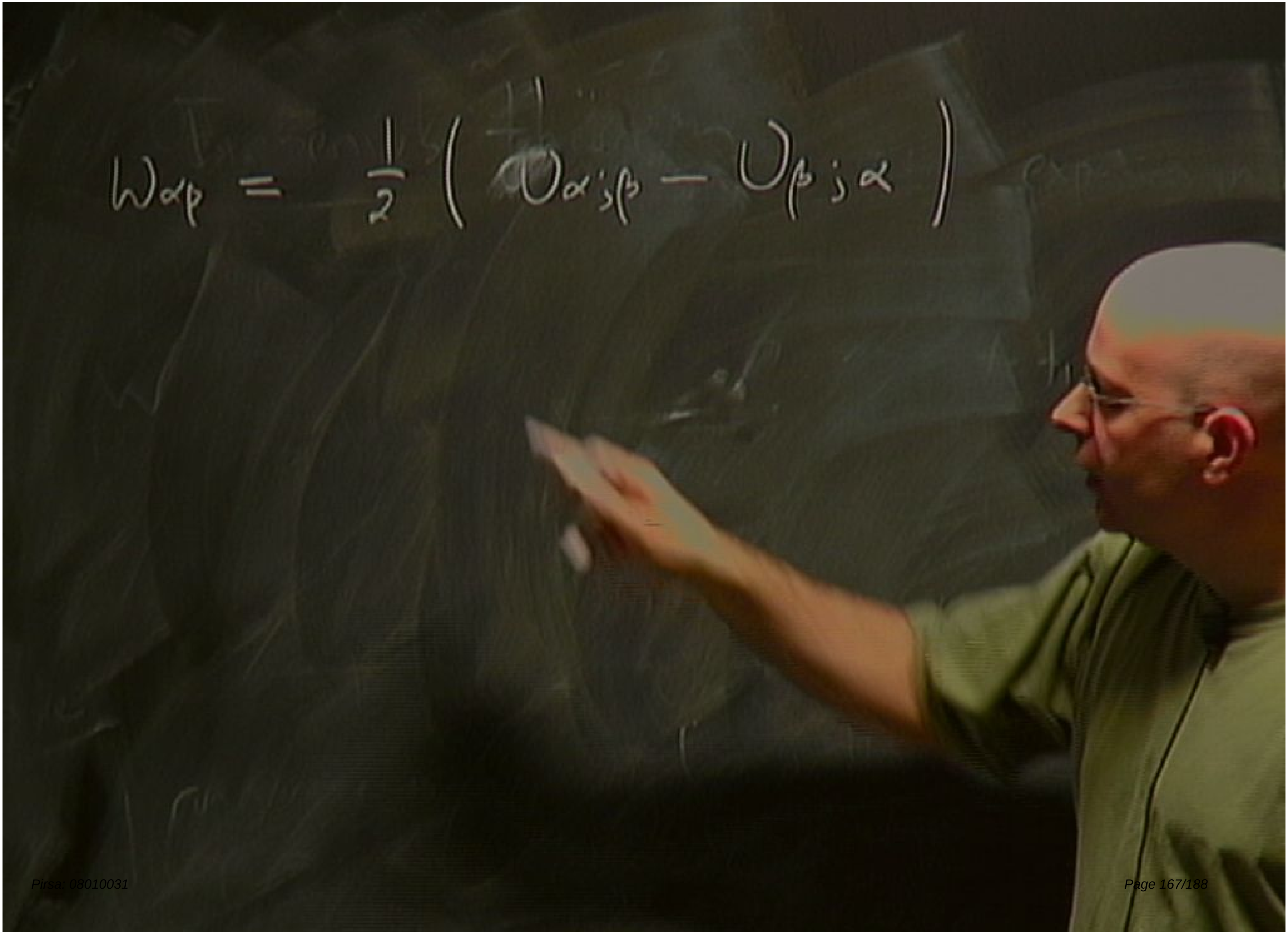
$$= \mu \Phi_{;\alpha\beta} + \frac{1}{\mu} \mu_{,\beta} u_\alpha$$

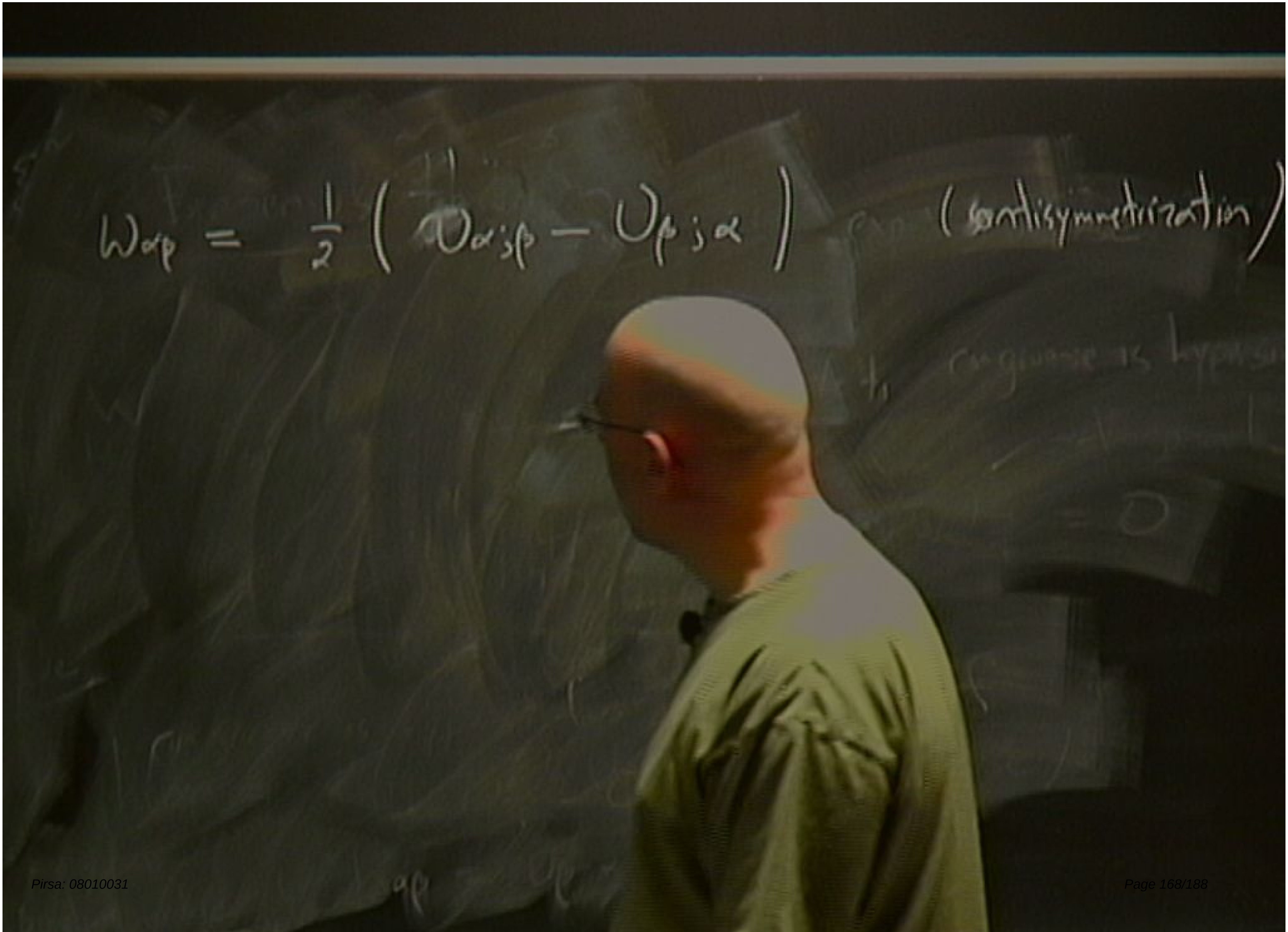
$= n_\alpha$

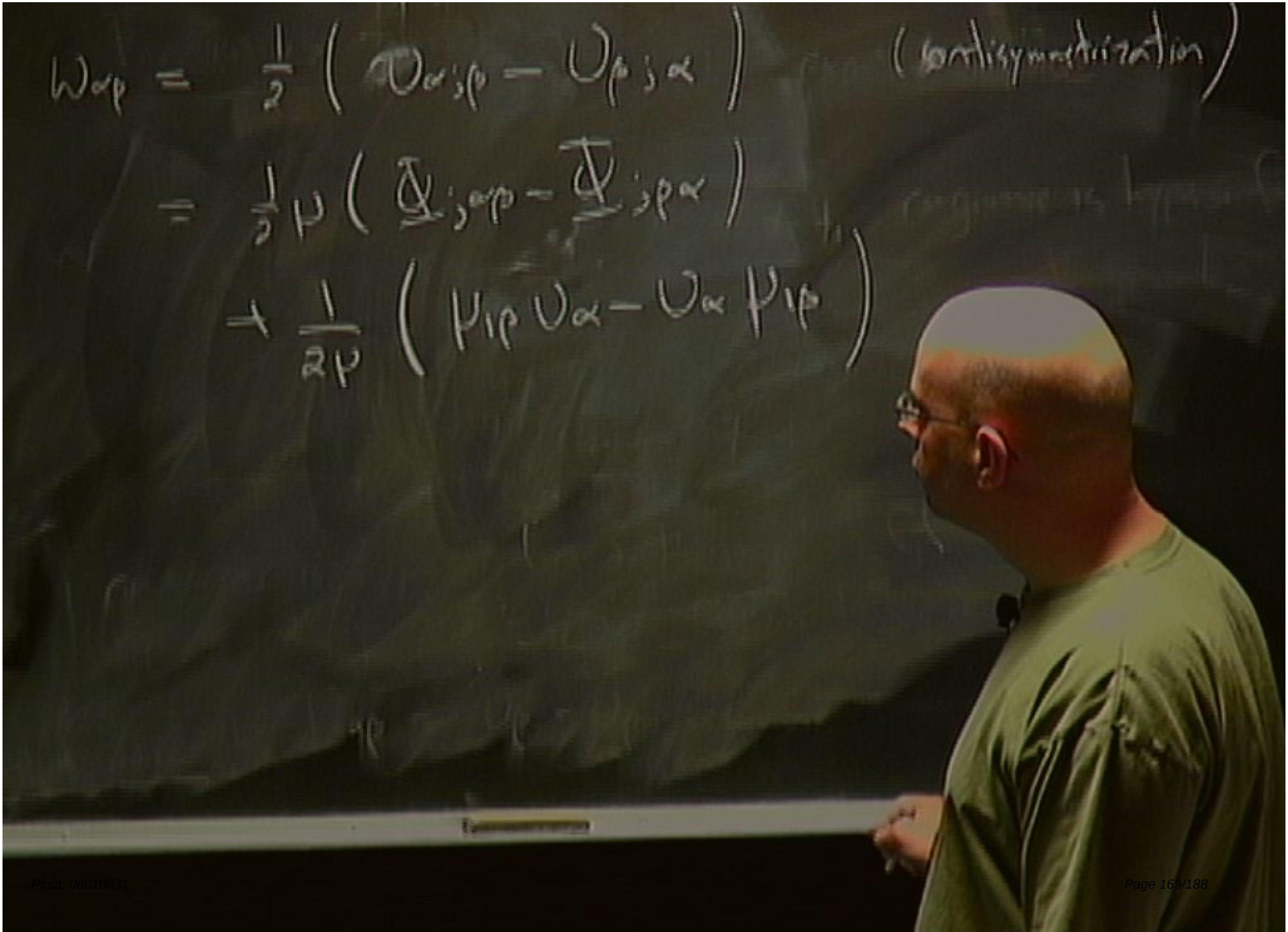
$$U_\alpha = \mu\, \partial_\alpha \Phi$$

$$U_{\alpha;\beta} = \mu\, \Phi_{;\alpha\beta} + \mu_{,\beta}\, \Phi_{,\alpha}$$

$$= \mu\, \Phi_{;\alpha\beta} + \frac{1}{\mu}\, \mu_{,\beta}\, U_\alpha$$


$\omega_{\alpha\beta} = \frac{1}{2} (U_{\alpha;\beta} - U_{\beta;\alpha})$


$\omega_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha;\beta} - v_{\beta;\alpha} \right)$ (antisymmetrization)


$W_{\alpha\beta} = \frac{1}{2} \left(v_{\alpha;\beta} - v_{\beta;\alpha} \right)$
(antisymmetrization)
$= \frac{1}{2}\mu \left(\Phi_{;\alpha\beta} - \Phi_{;\beta\alpha} \right)$
$+ \frac{1}{2\mu} \left(\mu_{,\beta} v_\alpha - v_\alpha \mu_{,\beta} \right)$

$$\omega_{\alpha\beta} = \frac{1}{2}\left(U_{\alpha;\beta} - U_{\beta;\alpha} \right) \quad \text{(antisymmetrization)}$$

$$= \frac{1}{2}\mu\left(\cancel{\Phi_{;\alpha\beta}} - \cancel{\Phi_{;\beta\alpha}} \right)$$

$$+ \frac{1}{2\mu}\left(\mu_{;\beta} U_\alpha - U_\alpha \mu_{;\beta} \right)$$

$$= \frac{1}{2\mu}\left(\mu_{;\beta} U_\alpha - U_\alpha \mu_{;\beta} \right)$$

$$\omega_{\alpha\beta} = \frac{1}{2}\left(U_{\alpha;\beta} - U_{\beta;\alpha} \right)$$

$$= \frac{1}{2}\mu\left(\cancel{\Phi_{;\alpha\beta}} - \cancel{\Phi_{;\beta\alpha}} \right)$$

$$+ \frac{1}{2\mu}\left(\mu_{,\beta} U_\alpha - \cancel{U_\alpha \mu_{,\beta}} \right)$$

$$= \frac{1}{2\mu}\left(\mu_{,\beta} U_\alpha - \mu_{,\alpha} U_\beta \right)$$

$$+ \frac{1}{2\mu}\left(\mu_{1\beta} v_\alpha - v_\alpha \mu_{1\beta} \right)$$

$$= \frac{1}{2\mu}\left(\mu_{1\beta} v_\alpha - \mu_{1\alpha} v_\beta \right)$$

$t_h \equiv$ longitudinal

$$= \frac{1}{2}\mu \left(\cancel{\Phi_{;\alpha\beta}} - \cancel{\Phi_{;\beta\alpha}} \right)$$

$$+ \frac{1}{2\mu} \left(\mu_{,\beta} U_\alpha - \cancel{U_\alpha} \mu_{,\beta} \right)$$

$$= \frac{1}{2\mu} \left(\mu_{,\beta} U_\alpha - \mu_{,\alpha} U_\beta \right)$$

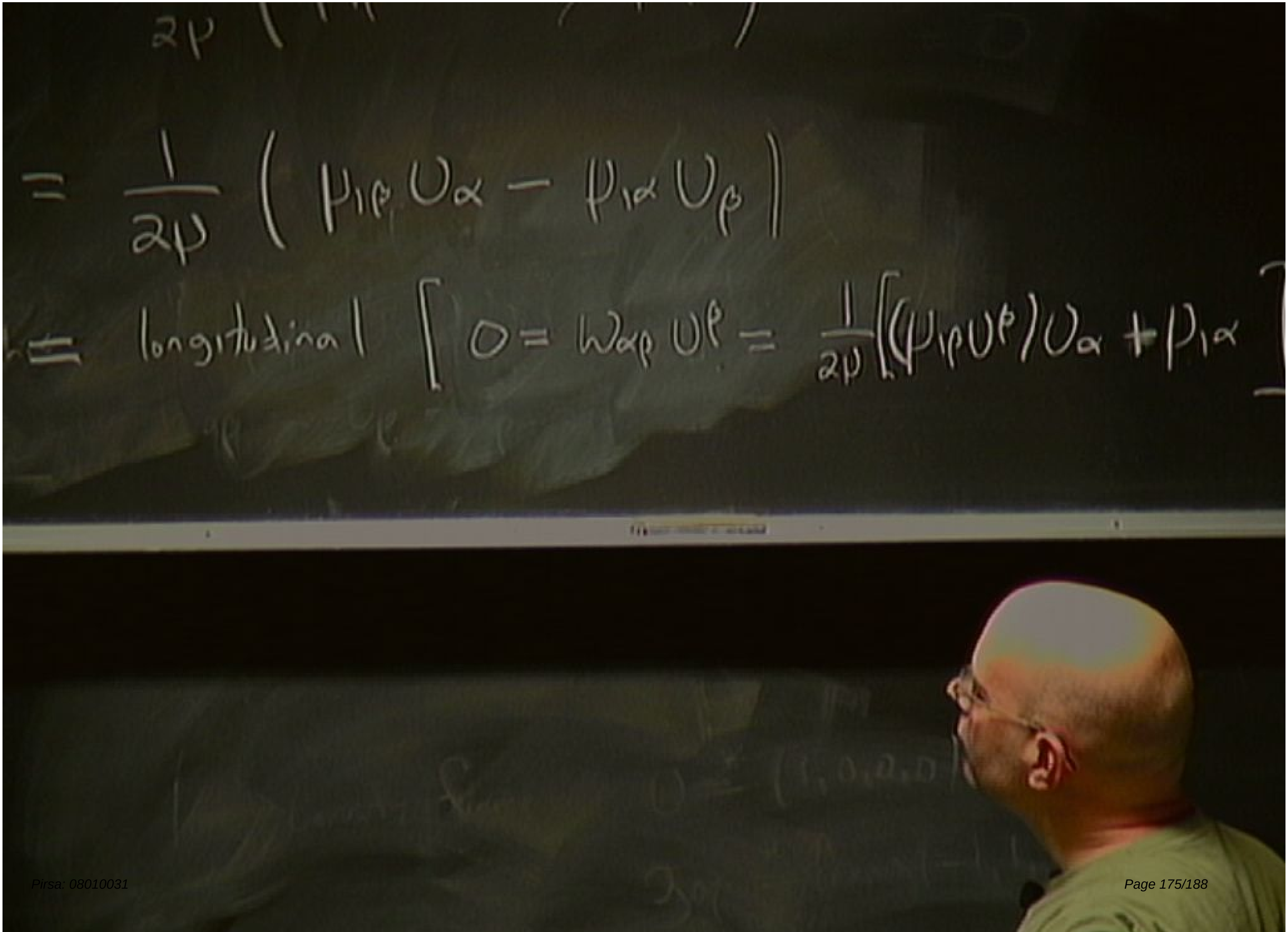
th = longitudinal $\quad 0 = W_{\alpha\beta} U^\beta =$

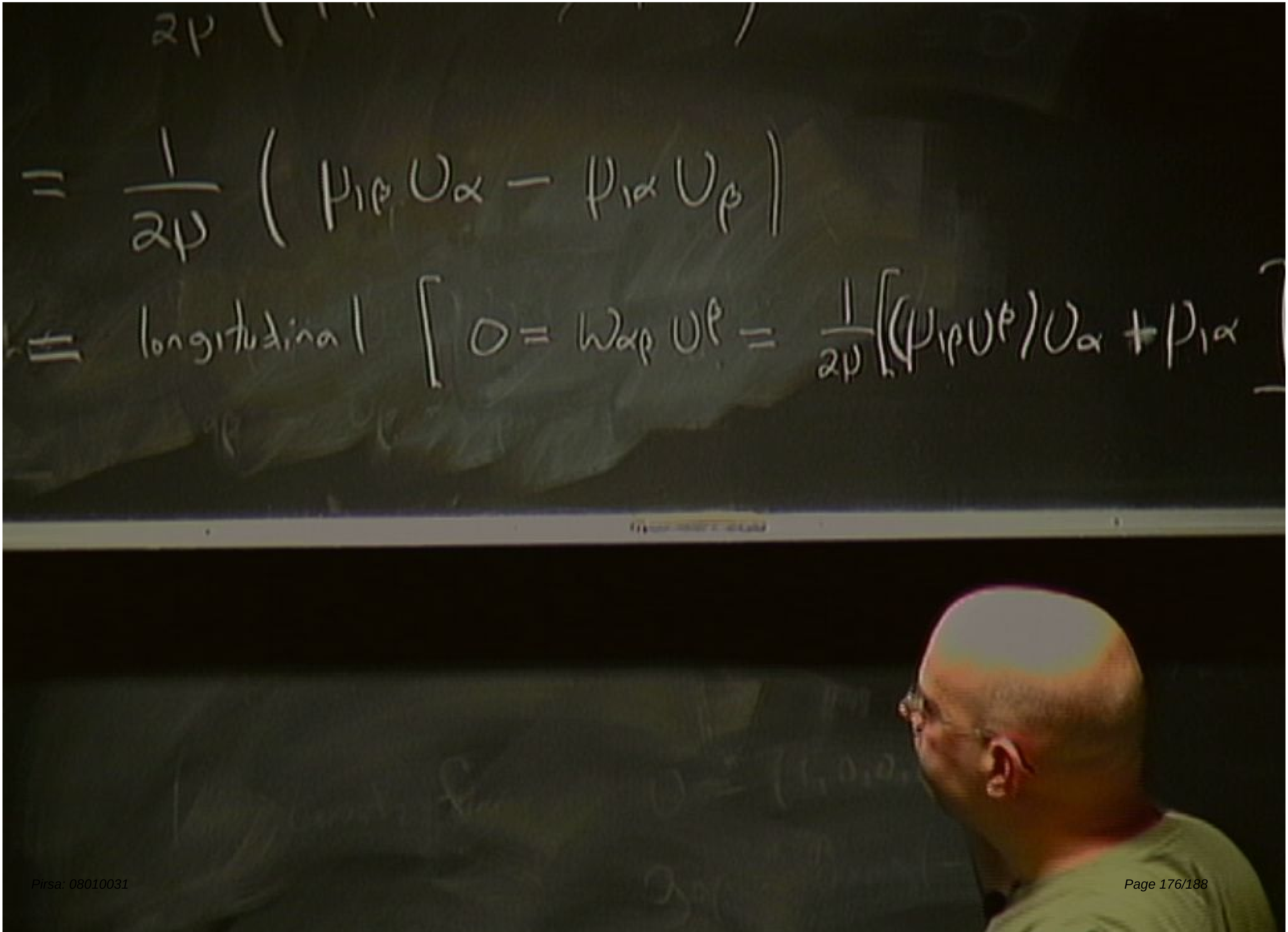
$$= \frac{1}{2}\mu \left(\cancel{\Phi_{;\alpha\beta}} - \cancel{\Phi_{;\beta\alpha}} \right)$$

$$+ \frac{1}{2\mu} \left(\mu_{;\beta} U_\alpha - \cancel{U_\alpha \mu_{;\beta}} \right)$$

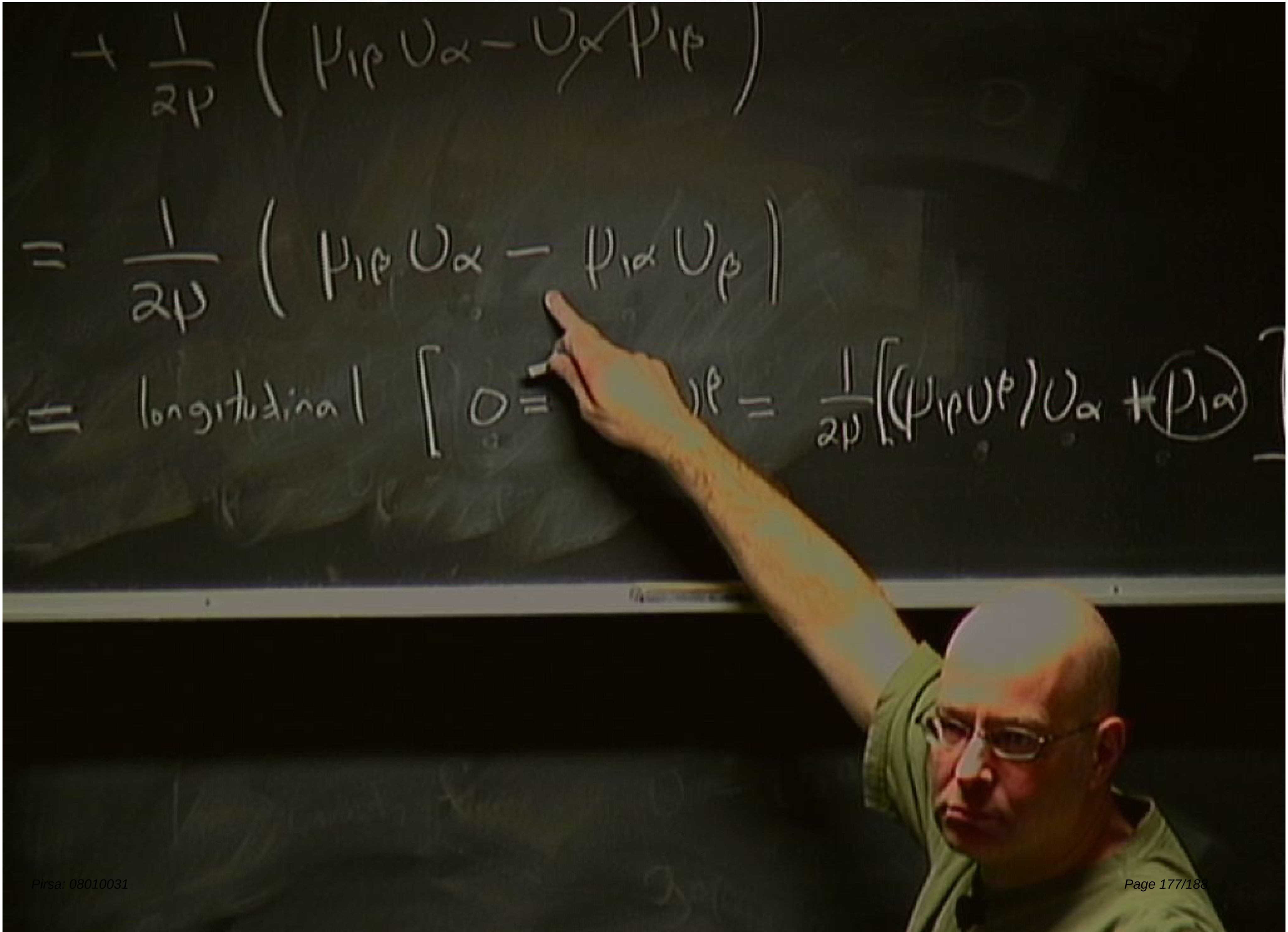
$$= \frac{1}{2\mu} \left(\mu_{;\beta} U_\alpha - \mu_{;\alpha} U_\beta \right)$$

th = longitudinal $\Big[\, 0 = \omega_{\alpha\beta} U^\beta = \frac{1}{2\mu} (\mu_{;\beta} U^\beta) U_\alpha$

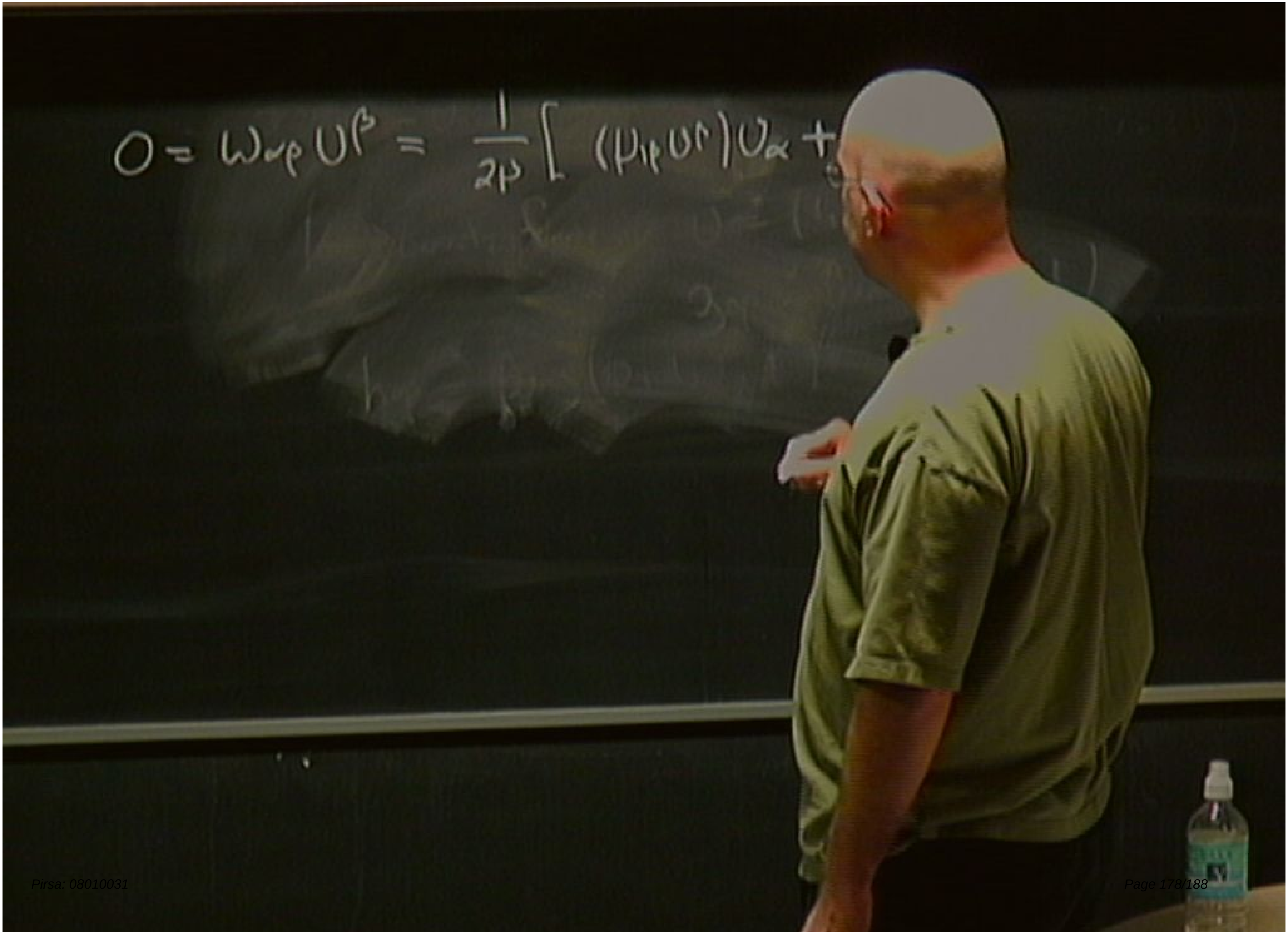

= 1/2μ (μ1β Uα − μ1α Uβ)
= longitudinal [0 = Wαβ U^β = 1/2μ [(μ1β U^β) Uα + μ1α

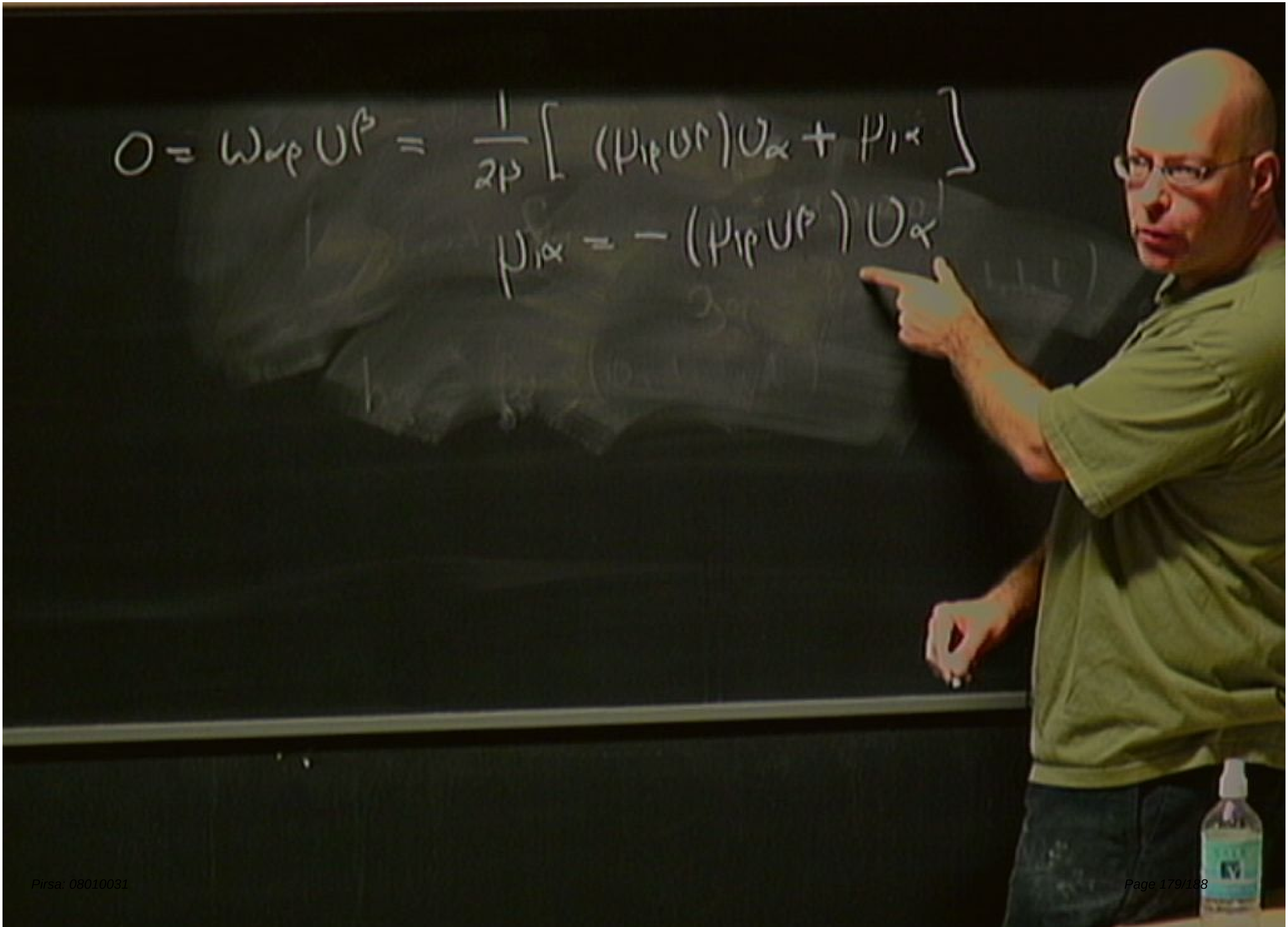


$= \frac{1}{2\mu} \left(\mu_{1\beta} U_\alpha - \mu_{1\alpha} U_\beta \right)$
$=$ longitudinal $\left[0 = W_{\alpha\beta} U^\beta = \frac{1}{2\mu} \left[(\mu_{1\beta} U^\beta) U_\alpha + \mu_{1\alpha} \right. \right.$



$+ \frac{1}{2\mu} \left(\mu_{1\beta} U_\alpha - U_\alpha \mu_{1\beta} \right)$
$= \frac{1}{2\mu} \left(\mu_{1\beta} U_\alpha - \mu_{1\alpha} U_\beta \right)$
longitudinal


$0 = \omega_{\alpha\beta} U^\beta = \frac{1}{2\beta} \Big[(\mu_{\gamma\beta} U^\gamma) U_\alpha +$


$0 = \omega_{\alpha\beta} U^\beta = \frac{1}{2p}\left[(\mu_{,\beta} U^\beta) U_\alpha + \mu_{,\alpha} \right]$
$\mu_{,\alpha} = -(\mu_{,\beta} U^\beta) U_\alpha$

$\omega_{\alpha\beta}$

rotation
antisym.

$$\sigma_{\alpha\beta} U^\beta = 0$$

$$\omega_{\alpha\beta} U^\beta = 0$$

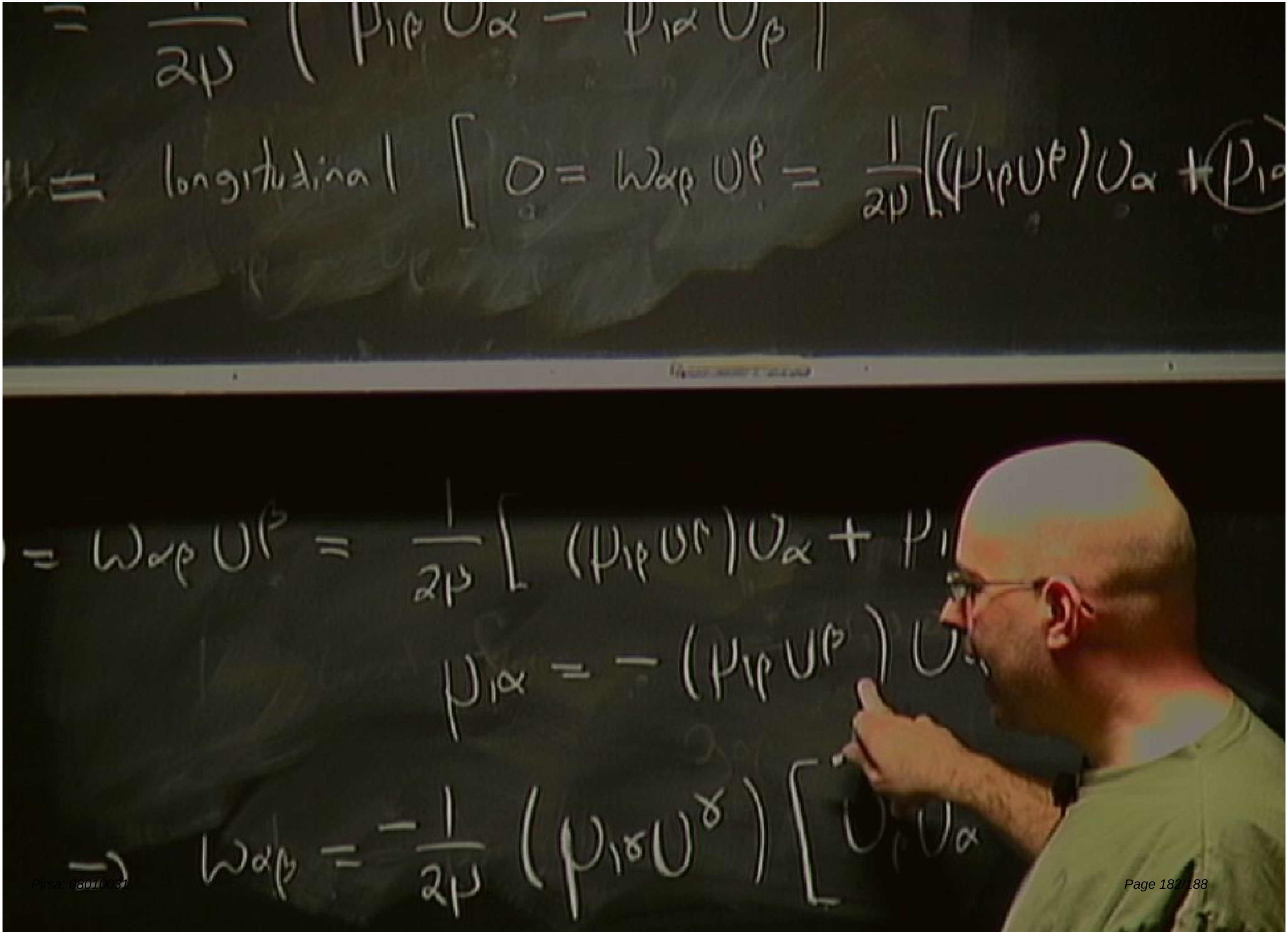
$n_\alpha = \mu\, \partial_\alpha \Phi$ $\qquad n_\alpha n^\alpha = -1$

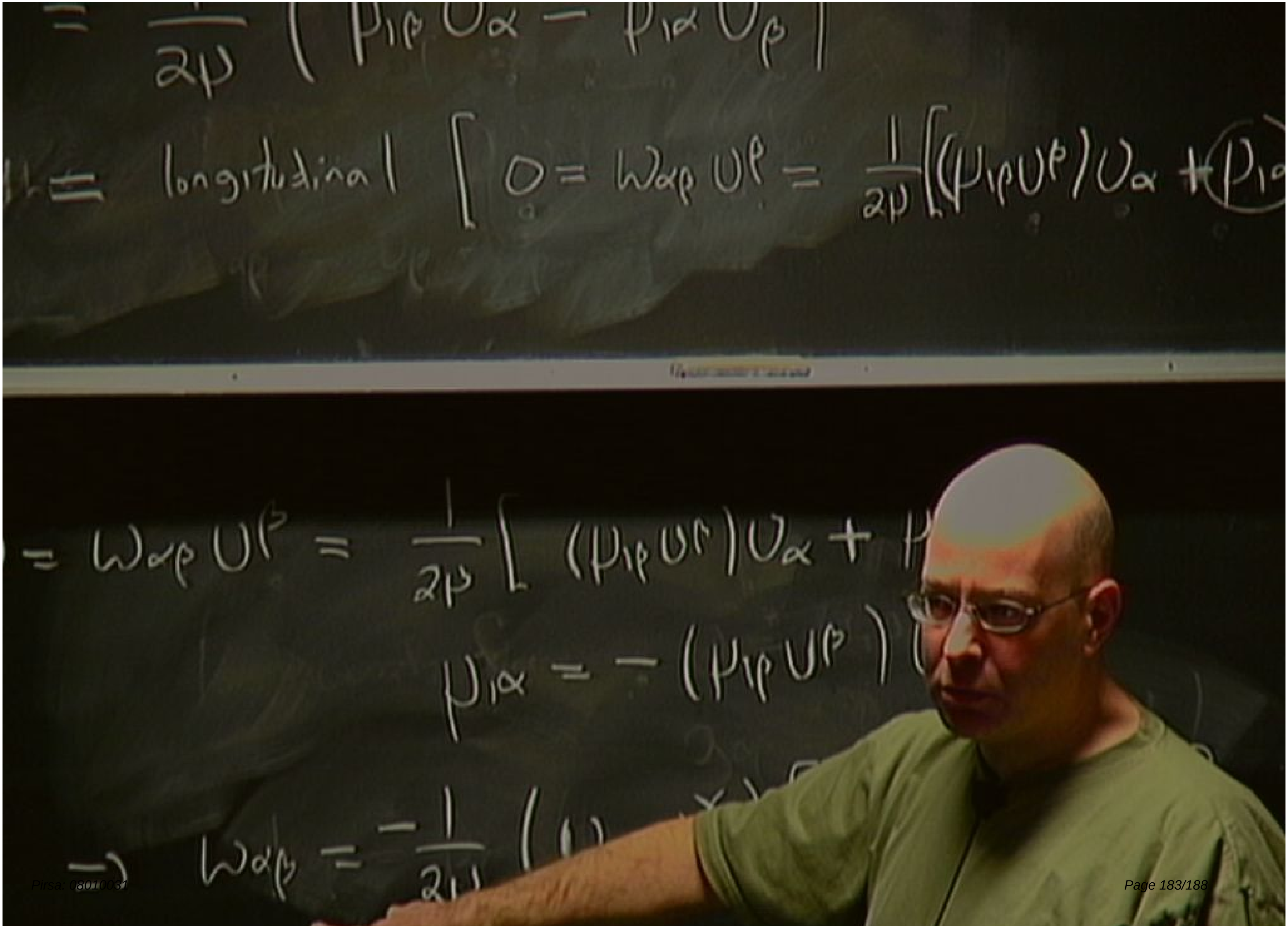
(normal to hypersurface.

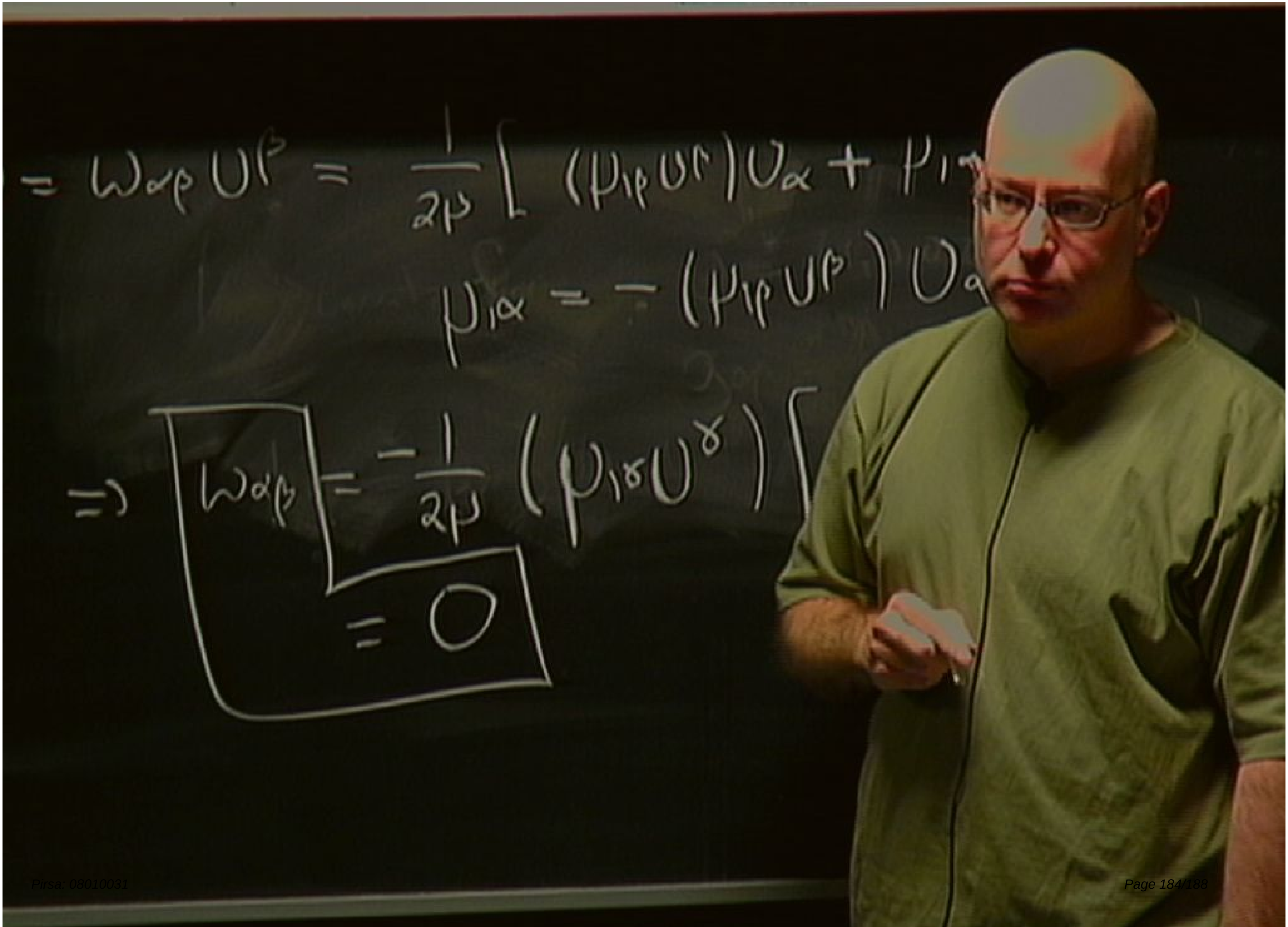
hypersurface orthogonal if $\boxed{u_\alpha = n_\alpha}$. $\quad \boxed{u_\alpha = \mu\, \partial_\alpha \Phi}$

$$u_{\alpha;\beta} = \mu\, \Phi_{;\alpha\beta} + \mu_{;\beta}\, \Phi_{,\alpha}$$

$$= \mu\, \Phi_{;\alpha\beta} + \frac{1}{\mu}\, \mu_{;\beta}\, u_\alpha$$


= 1/2μ (μ1β Uα − μ1α Uβ)
≡ longitudinal [0 = Wαβ Uβ = 1/2μ [(μ1β Uβ) Uα + μ1α
= Wαβ Uβ = 1/2μ [(μ1β Uβ) Uα + μ1
μ1α = − (μ1β Uβ) Uα
⇒ Wαβ = − 1/2μ (μ1γ Uγ) [Uβ Uα


$= \frac{1}{2\mu} \left(\mu_{1\beta} U_\alpha - \mu_{1\alpha} U_\beta \right)$
$\equiv$ longitudinal $\left[0 = W_{\alpha\beta} U^\beta = \frac{1}{2\mu} \left[(\mu_{1\beta} U^\beta) U_\alpha + \mu_{1\alpha}\right.\right.$
$= W_{\alpha\beta} U^\beta = \frac{1}{2\mu} \left[(\mu_{1\beta} U^\beta) U_\alpha + \right.$
$\mu_{1\alpha} = -(\mu_{1\beta} U^\beta)$
$\Rightarrow \quad W_{\alpha\beta} = -\frac{1}{2\mu}$

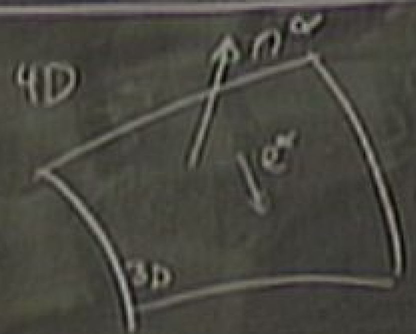

= ωαβ Uβ = 1/2μ [(μ,β Uβ) Uα + μ,
μ,α = − (μ,β Uβ) Uα
⇒ ωαβ = − 1/2μ (μ,γ Uγ) [
= 0

$$U_{\alpha;\beta} = \boxed{B_{\alpha\beta} = \tfrac{1}{3}\underbrace{\Theta}_{\text{expansion}} h_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear, STF}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation, antisym.}}}$$

$\sigma_{\alpha\beta}U^\beta = 0$

$\omega_{\alpha\beta}U^\beta = 0$

hypersurface :



$\Phi(x^\alpha) = 0$

$n_\alpha \propto \partial_\alpha \Phi$

$$\boxed{n_\alpha = \mu\, \partial_\alpha \Phi}$$

(normal to hypersurface)

μ ensures normalization:

$n_\alpha n^\alpha = -1$

$$\boxed{U_\alpha = \mu\, \partial_\alpha \Phi}$$

$\vec{U} = \vec{\nabla}\Phi$

$\vec{\nabla}\times\vec{U} = 0$

$$U_{\alpha;\beta} = \mu\, \Phi_{;\alpha\beta} + \mu_{,\beta}\, \Phi_{,\alpha}$$

$$= \mu\, \Phi_{;\alpha\beta} + \frac{1}{\mu}\, \mu_{,\beta}\, U_\alpha$$

$\alpha\beta$

otation
antisym.

$$\sigma_{\alpha\beta} U^{\beta} = 0$$

$$\omega_{\alpha\beta} U^{\beta} = 0$$

$$\omega_{\alpha\beta} = \frac{1}{2}\left(U_{\alpha;\beta} - U_{\beta;\alpha}\right)$$

$$\Phi(x^{\alpha}) = 0$$

alization:

$\sigma_{\alpha\beta} U^\beta = 0$

$W_{\alpha\beta} U^\beta = 0$

$$\boxed{W_{\alpha\beta} = \frac{1}{2}\left(U_{\alpha;\beta} - U_{\beta;\alpha}\right)}$$

$\alpha\beta$

rotation

antisym.

$\Phi(x^\alpha) = 0$

...lization: