Title: Advanced General Relativity - Lecture 1B

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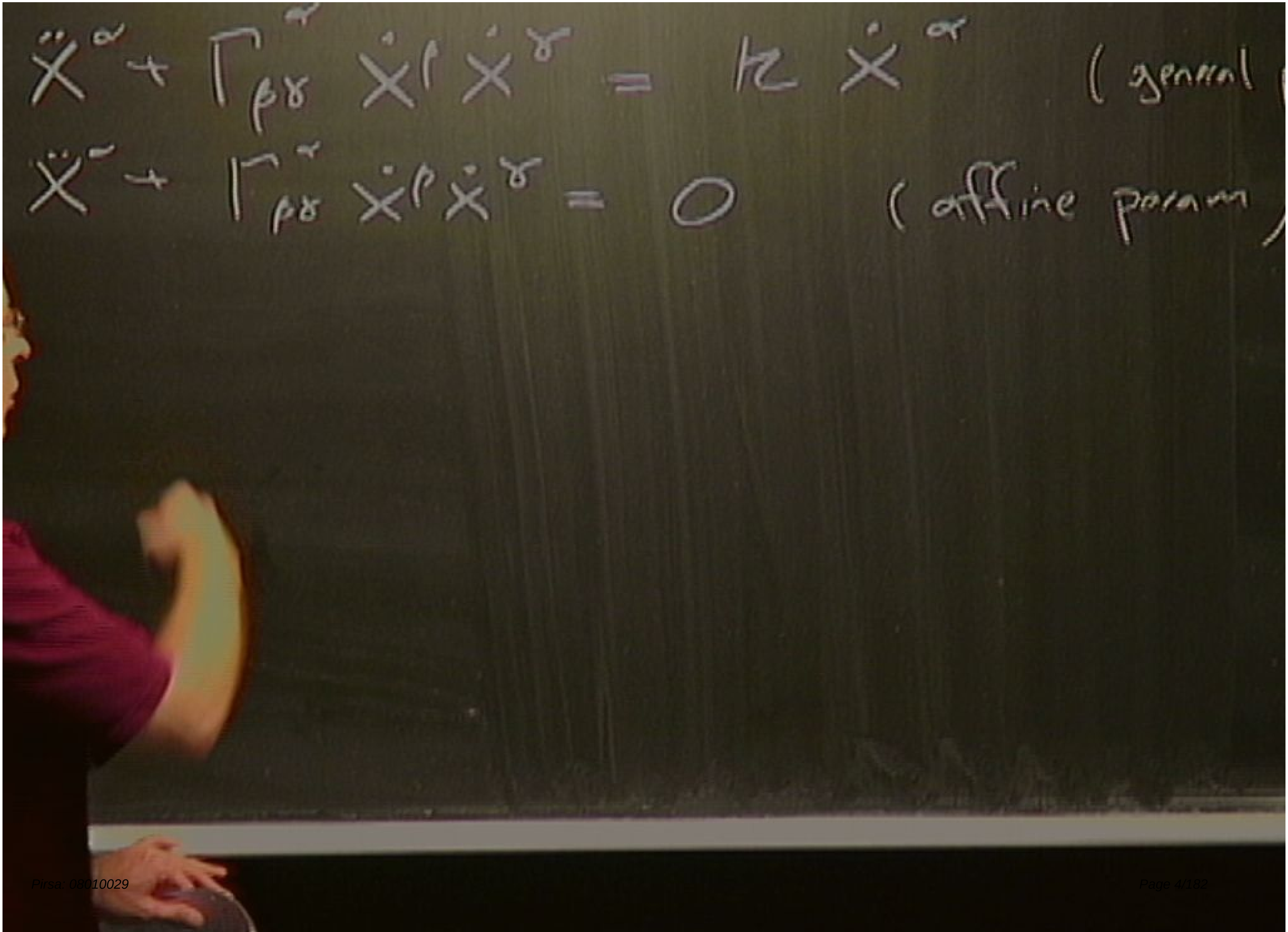
Abstract: Advanced General Relativity

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\,\dot{x}^{\beta}\dot{x}^{\gamma} = k\,\dot{x}^{\alpha}$$

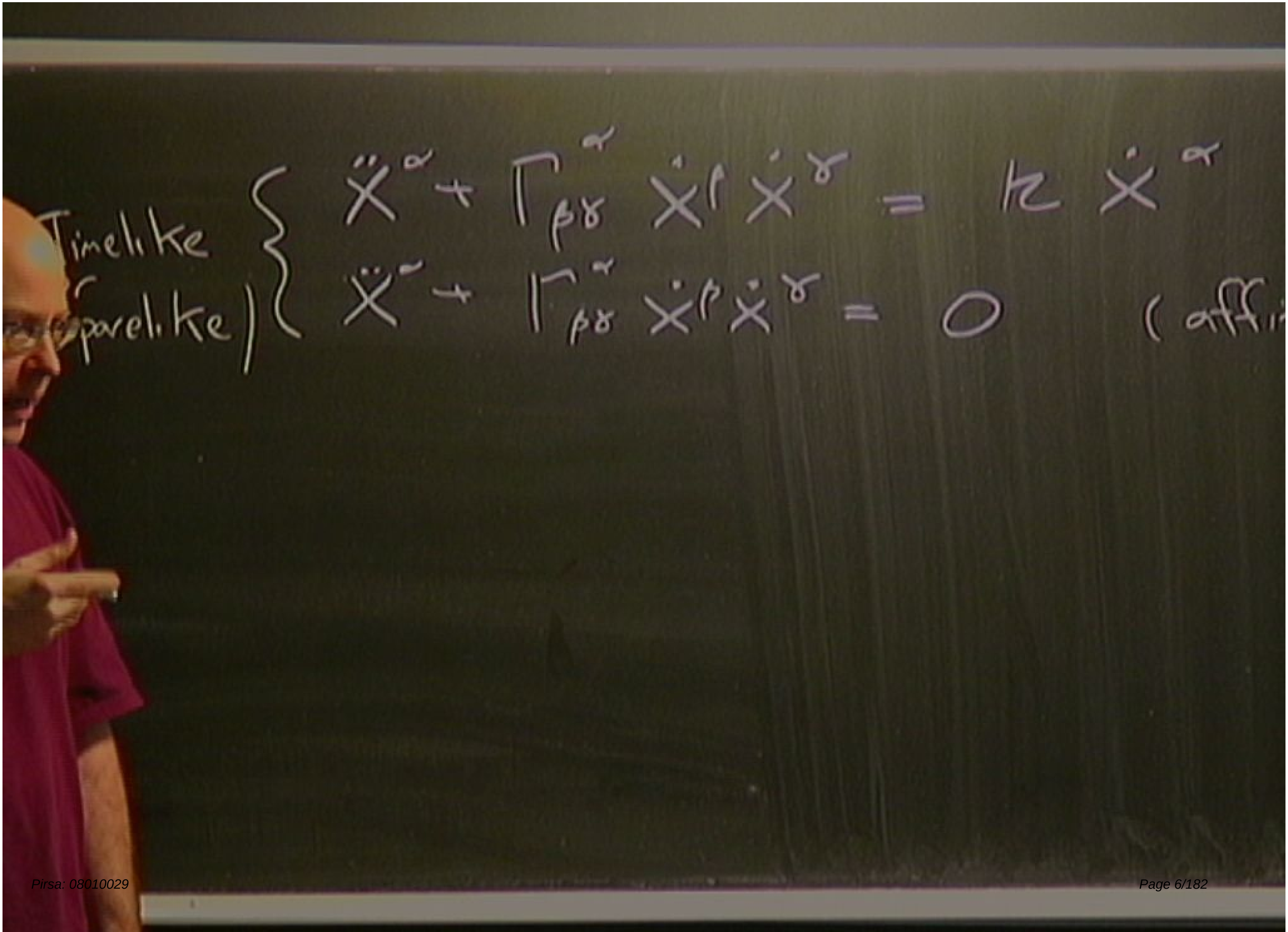
$$= k\,\dot{x}^{\alpha}$$

(general param)


$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = k\,\dot{x}^\alpha$ (general
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$ (affine param)

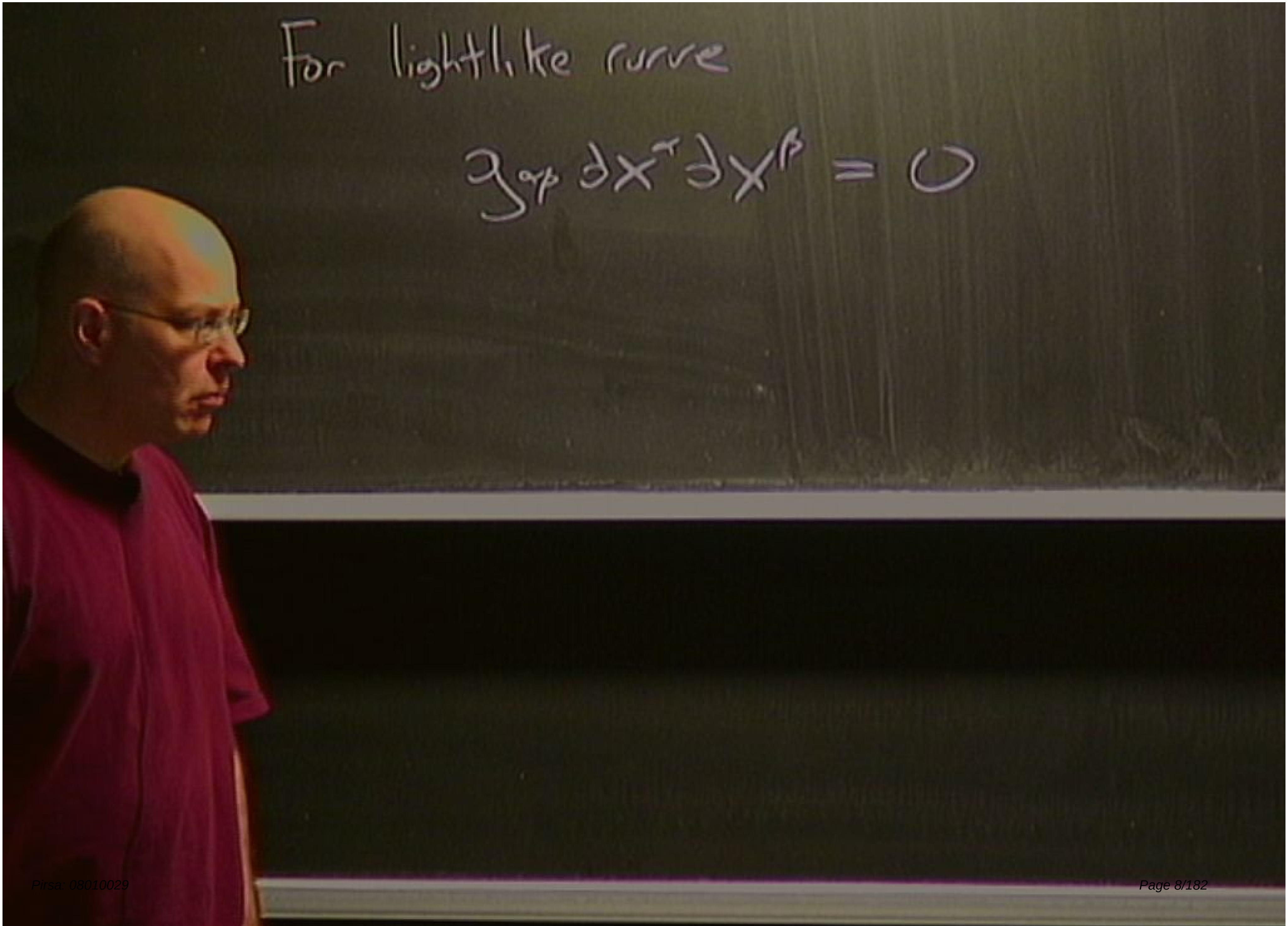
$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = k \dot{x}^\alpha \qquad \text{(general p}$$

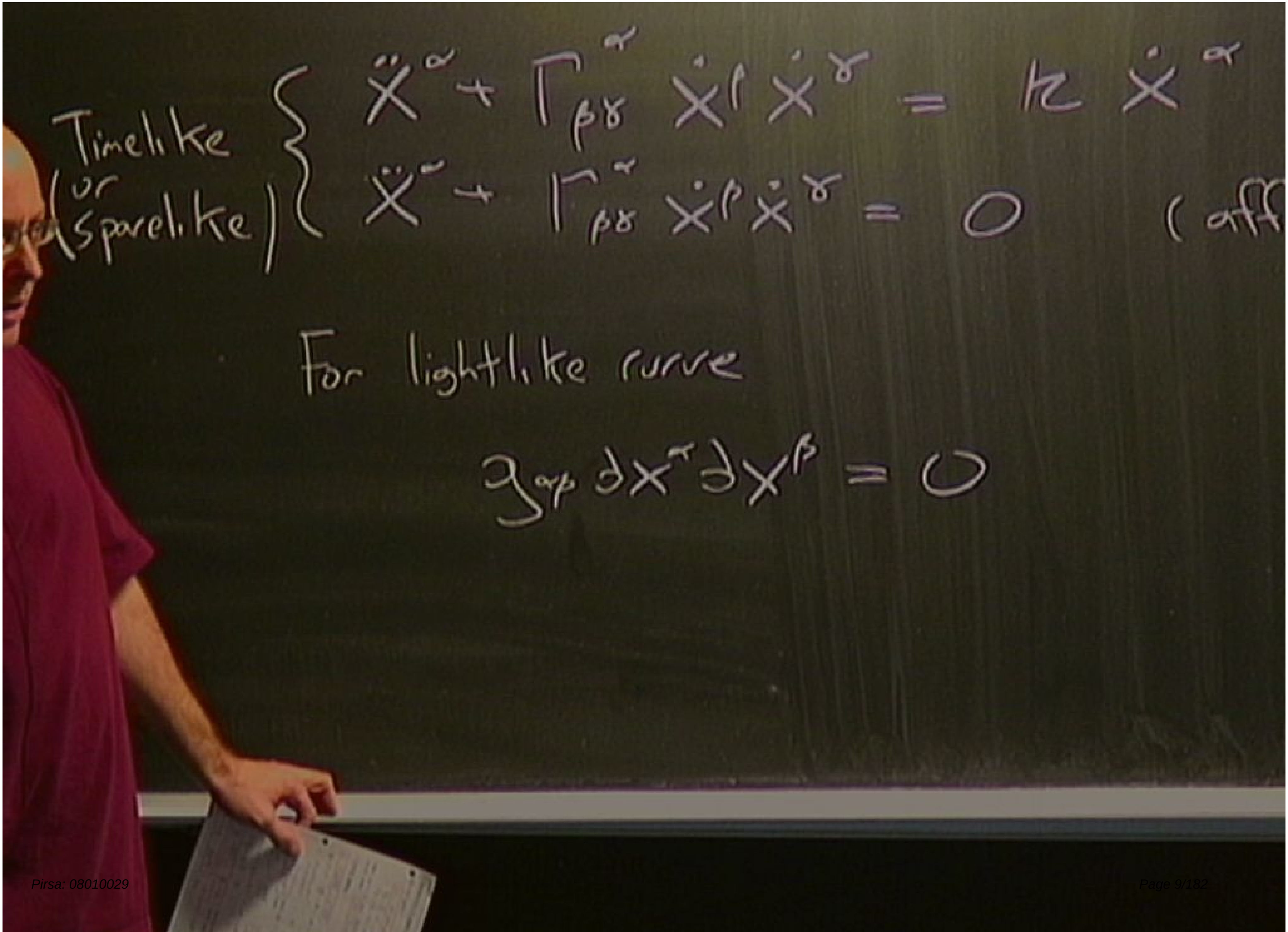
$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \qquad \text{(affine param)}$$


Timelike
(Spacelike)
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = k\,\dot{x}^\alpha$
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$

Timelike (or spacelike)

$$\begin{cases} \ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\,\dot{x}^{\beta}\dot{x}^{\gamma} = k\,\dot{x}^{\alpha} \\ \ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\,\dot{x}^{\beta}\dot{x}^{\gamma} = 0 \qquad (\text{affi} \end{cases}$$


For lightlike curve
$g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = 0$

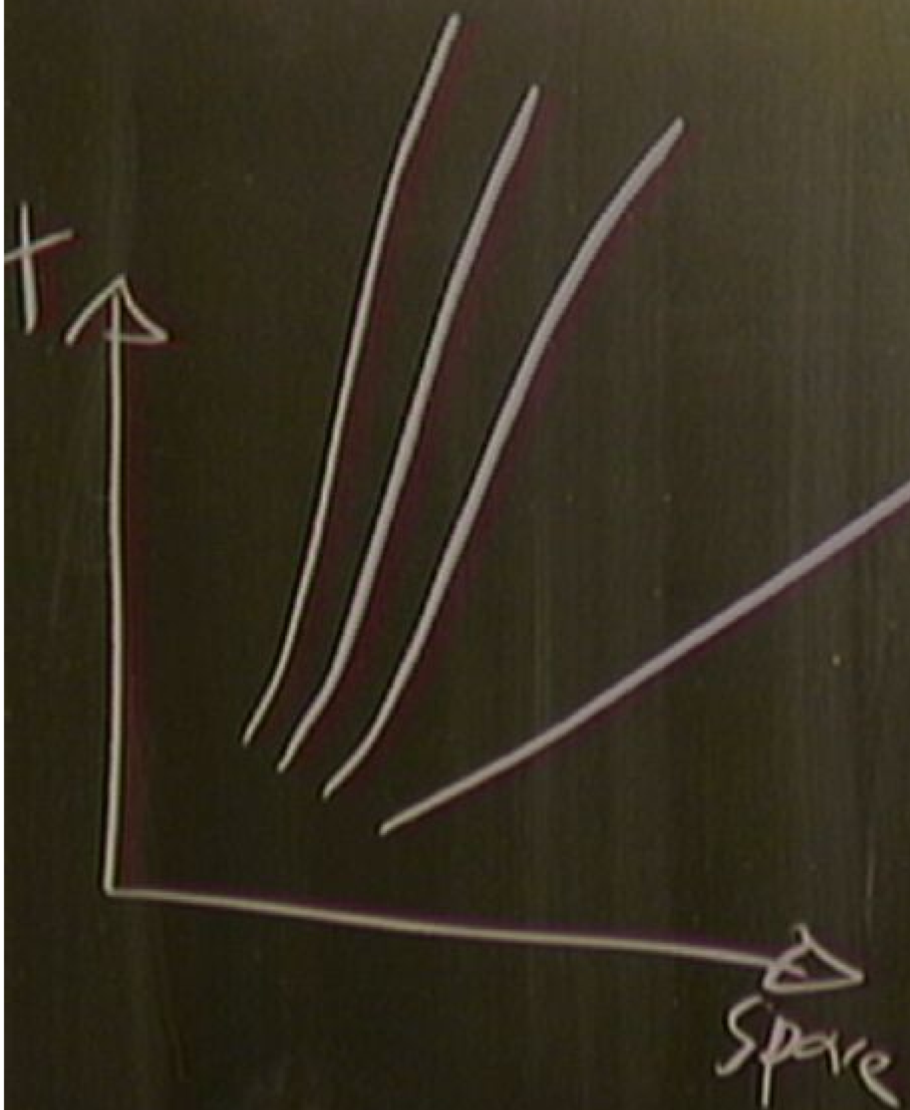

Timelike
(or spacelike)
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = k\dot{x}^\alpha$
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$
For lightlike curve
$g_{\alpha\beta}\, dx^\alpha dx^\beta = 0$

Timelike
arelike)

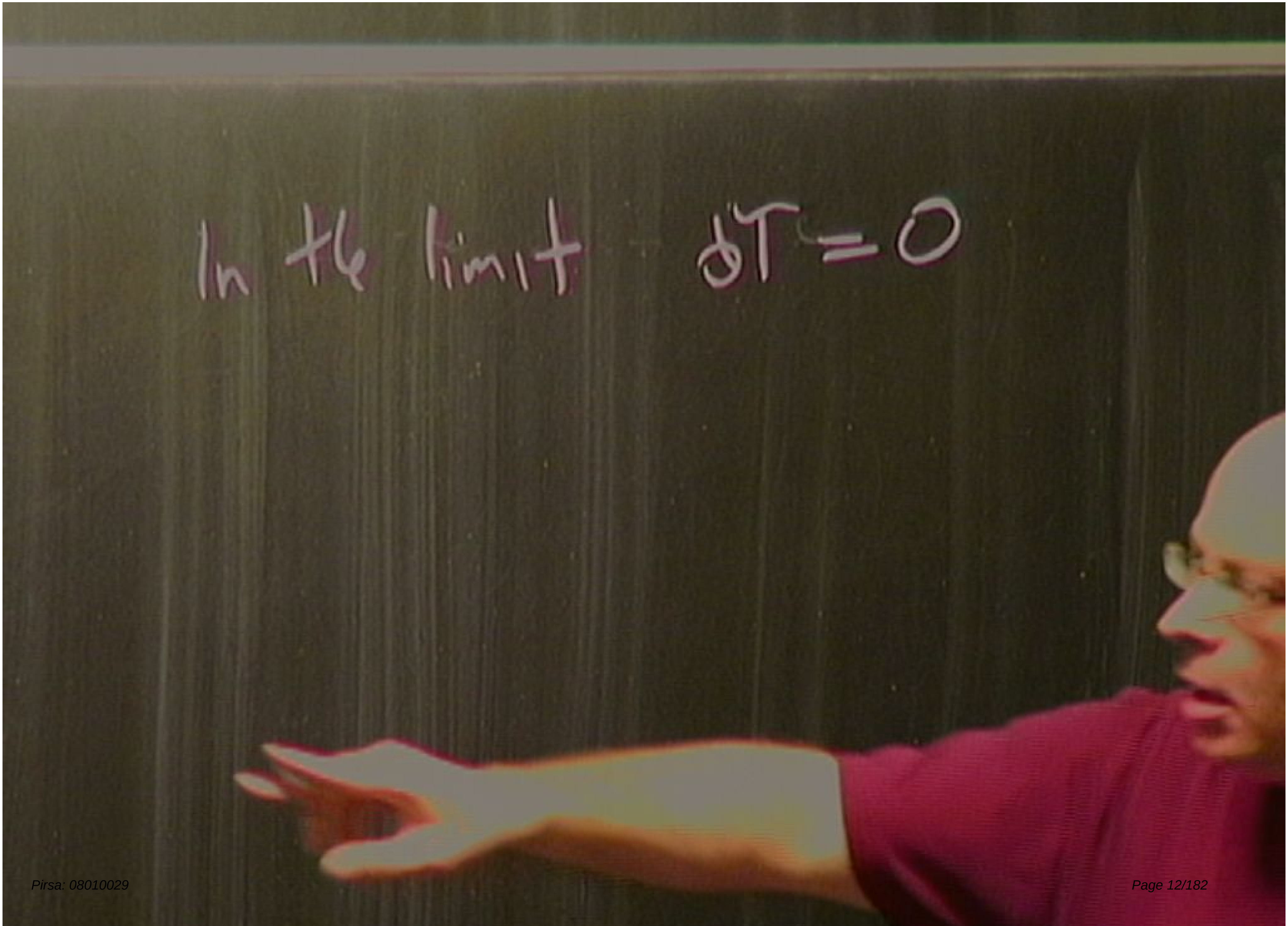
$$\begin{cases} \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma = k\,\dot{x}^\alpha \\ \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma = 0 \end{cases}$$ (aff

For lightlike curve

$$\boxed{g_{\alpha\beta}\,\partial x^\alpha\,\partial x^\beta = 0}$$



In the limi

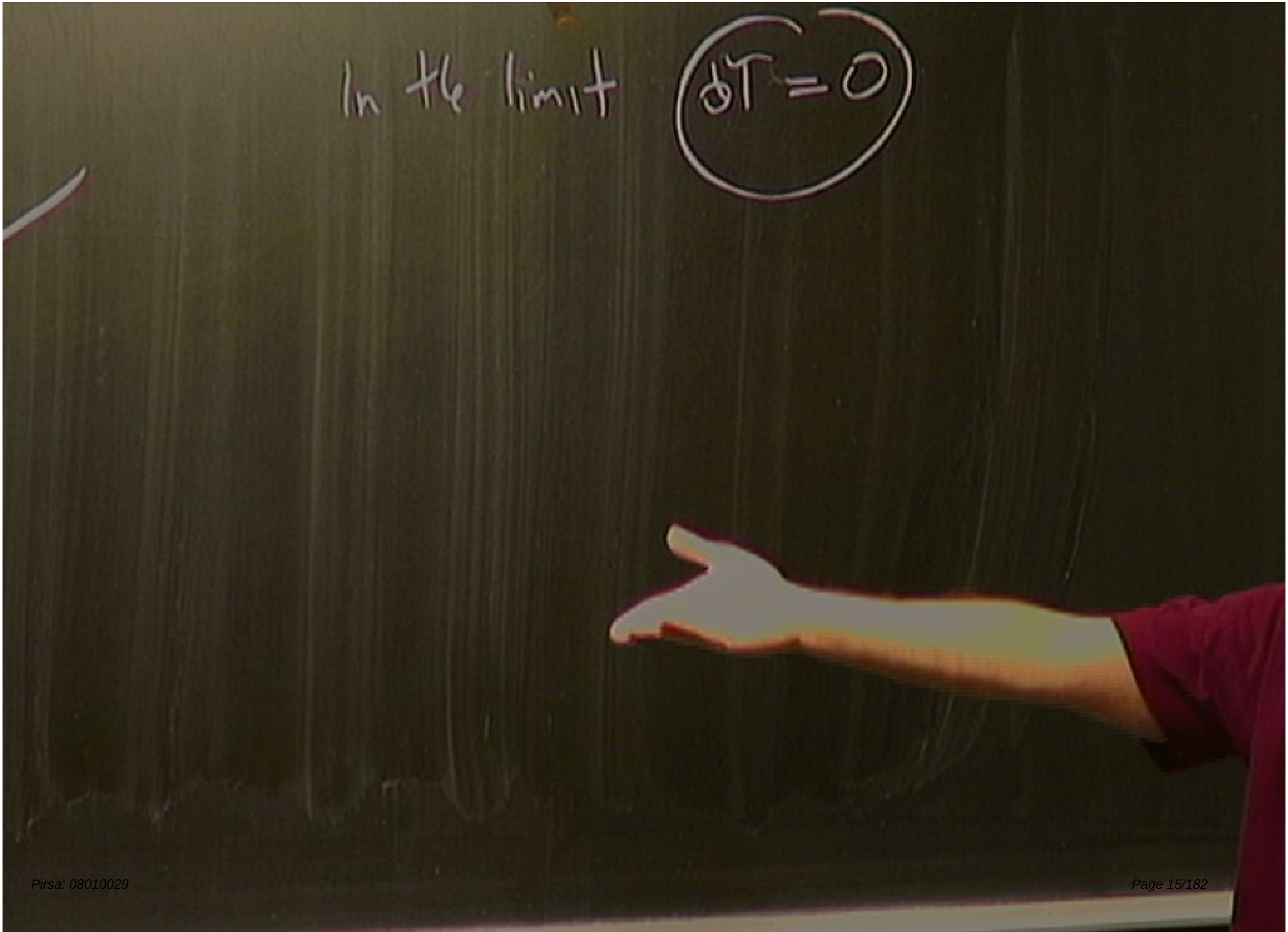

In the limit $dT = 0$

In the limit $\delta T = 0$

$$\left\{ \begin{array}{l} \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = k \dot{x}^\alpha \\ \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \quad (\end{array} \right.$$

For lightlike curve

$$\boxed{g_{\alpha\beta}\, \partial x^\alpha \partial x^\beta = 0}$$


In the limit
δT = 0

For null case, by limiting procedure:

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = k\dot{x}^\alpha.$$

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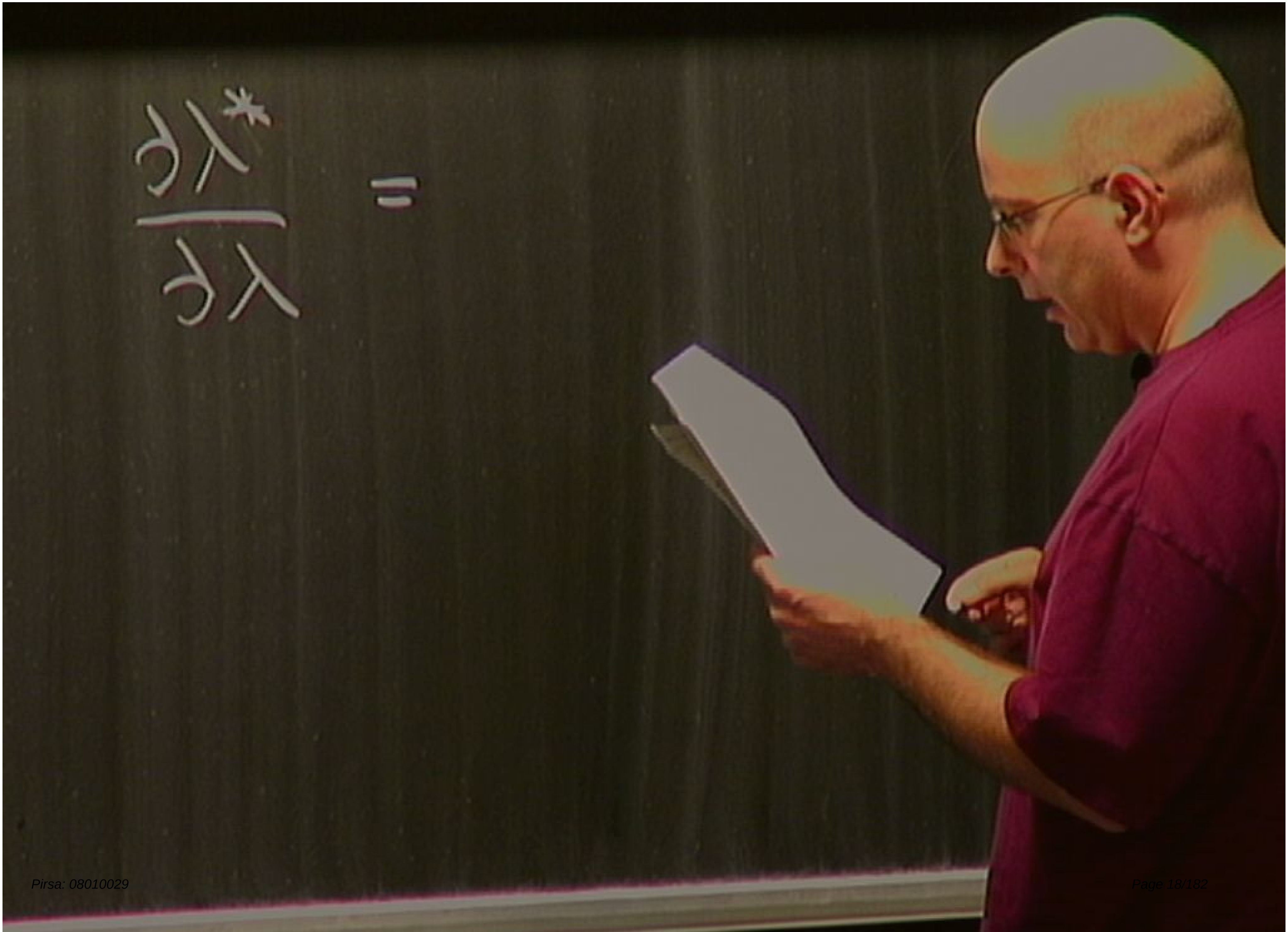
$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\, \dot{x}^{\beta} \dot{x}^{\gamma} = k \dot{x}^{\alpha}.$$

There is a transf. from old parameter λ to a new param λ^{*}


$\frac{\partial \lambda^*}{\partial \lambda} =$

$$\frac{\partial \lambda^*}{\partial \lambda} = \exp\left\{ \int^{*} k(\lambda')\, d\lambda' \right\}$$

$$\frac{\partial \lambda^*}{\partial \lambda} = \exp\left\{ \int^x k(x')\, dx' \right.$$

$$\frac{d^2 x^\alpha}{d\lambda^{*2}} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda^*} \frac{dx^\gamma}{d\lambda^*} = 0$$

$$\lambda^* \cancel{=} \lim T$$

$$\frac{\partial \lambda^*}{\partial \lambda}$$

$$\frac{\partial^2 x^*}{\partial \lambda^{*2}} + \Gamma$$

$$\frac{d\lambda^*}{d\lambda} = \exp\left\{ \int^{\lambda} k(x')\, d\lambda' \right.$$

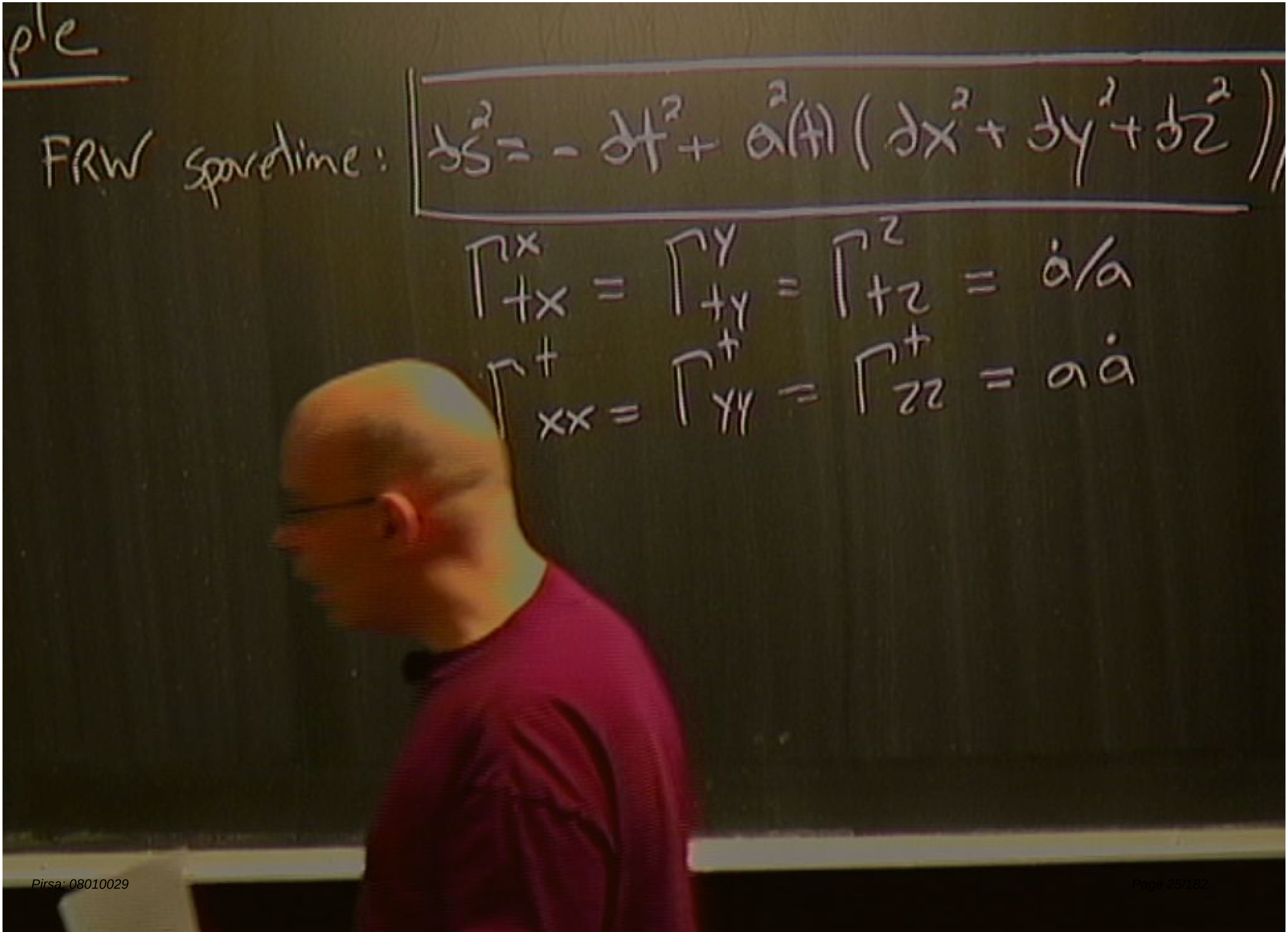
$$\boxed{\frac{d^2 x^\alpha}{d\lambda^{*2}} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda^*} \frac{dx^\gamma}{d\lambda^*} = 0}$$

Example

FRW

ple

FRW spacetime: $ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$


ple
FRW spacetime: $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$
$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \dot{a}/a$
$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}$

ple

FRW spacetime: $\boxed{ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)}$

$$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \dot{a}/a$$

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ple

FRW spacetime: $\boxed{ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)}$

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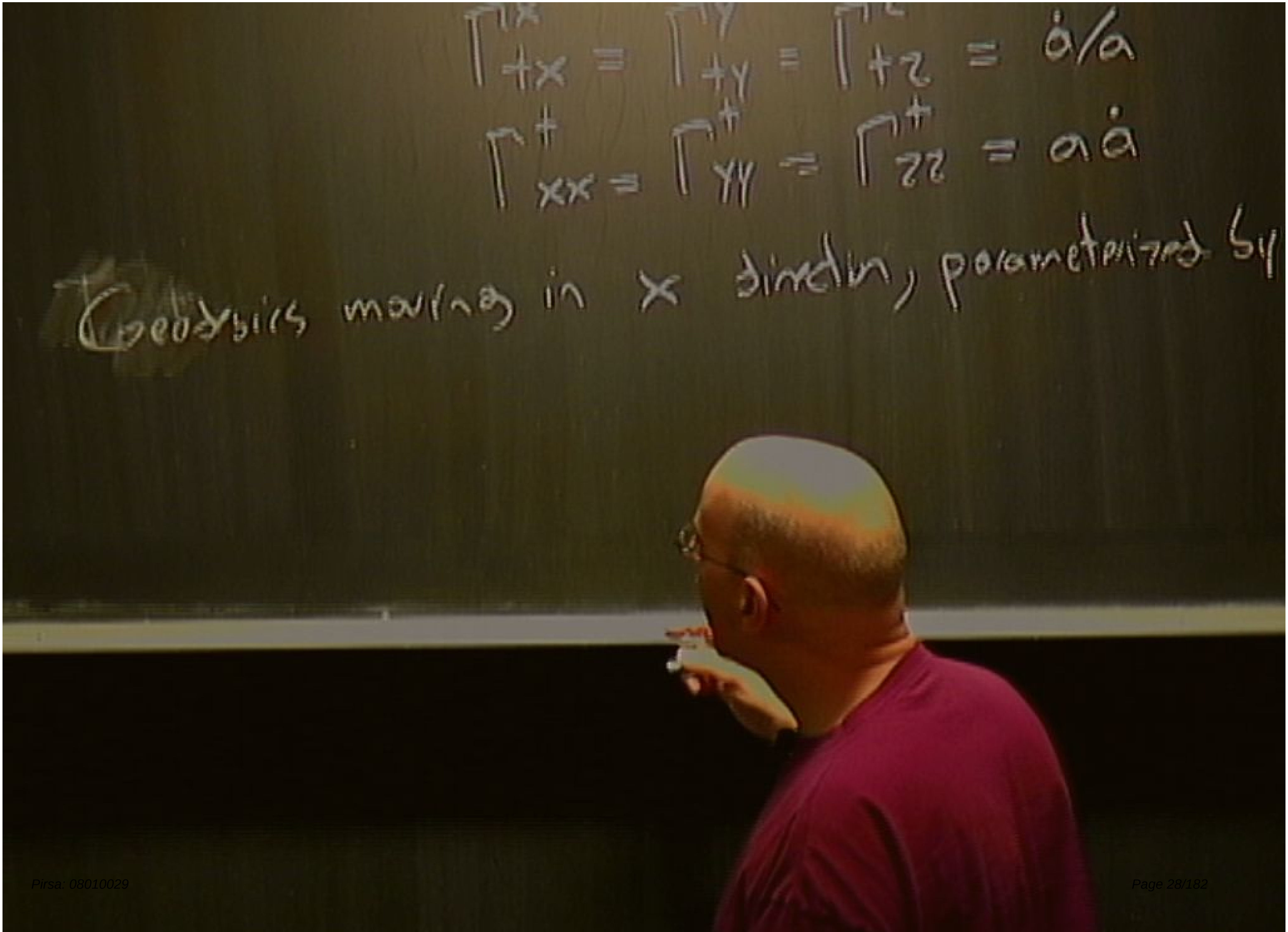
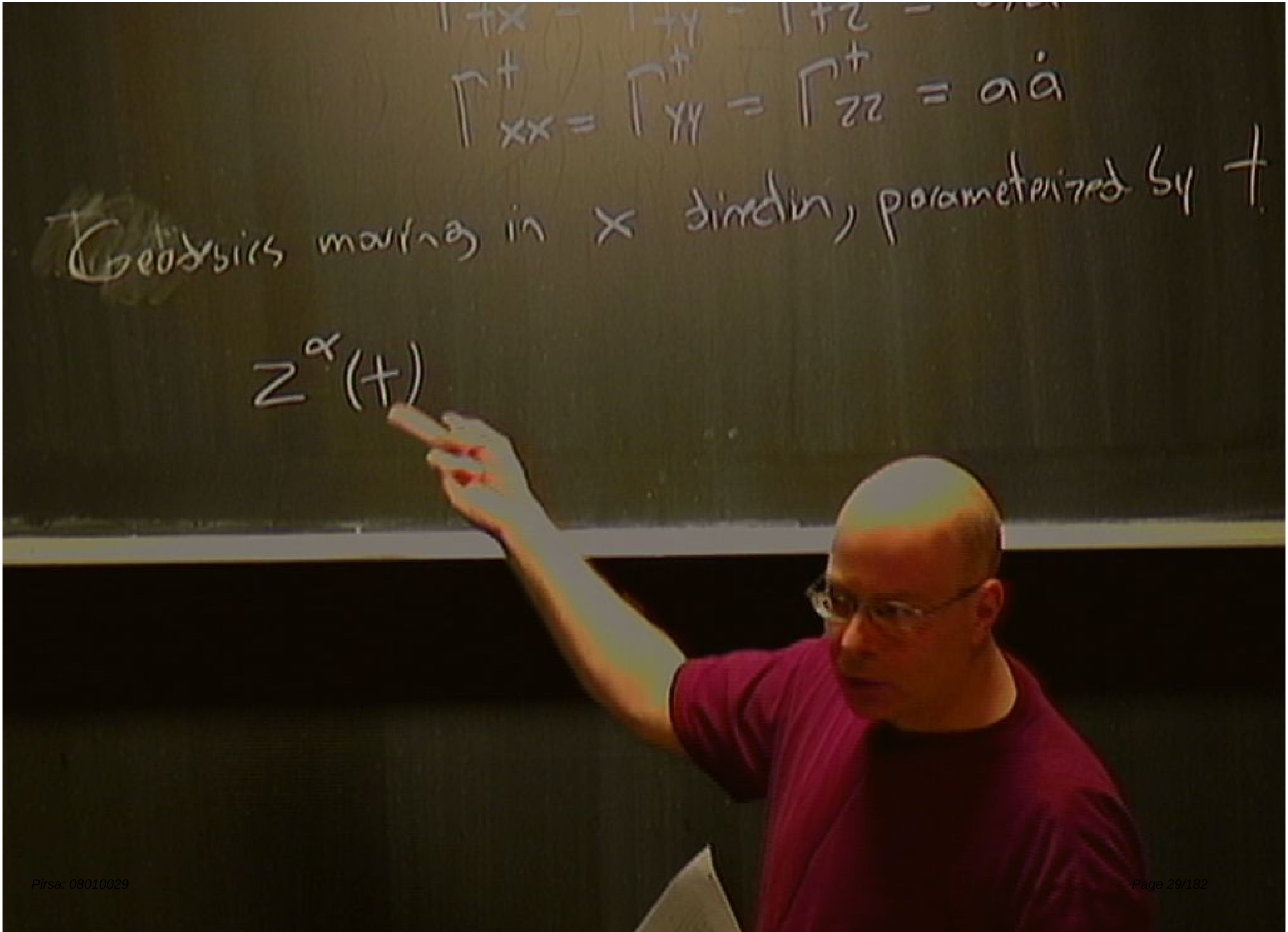

$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \dot{a}/a$
$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}$
Geodesics moving in x direction, parametrized by


$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}$
Geodesics moving in x direction, parameterized by t
$z^\alpha(t)$

$$Z^{\alpha}(t) = [t, x(t), 0, 0]$$

Timelike case

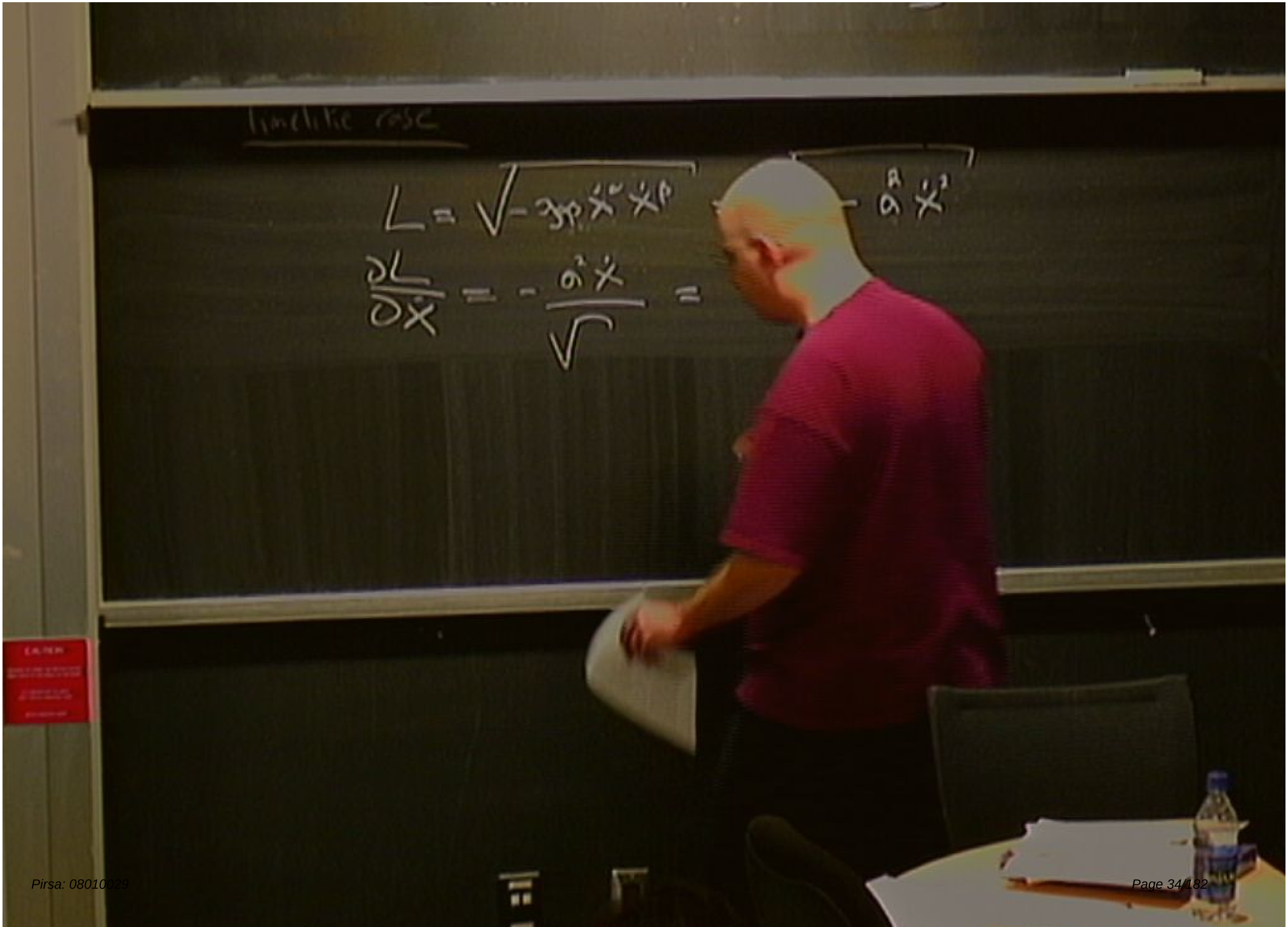
$$Z^{\alpha}(t) = [t, \;(t), \ldots$$

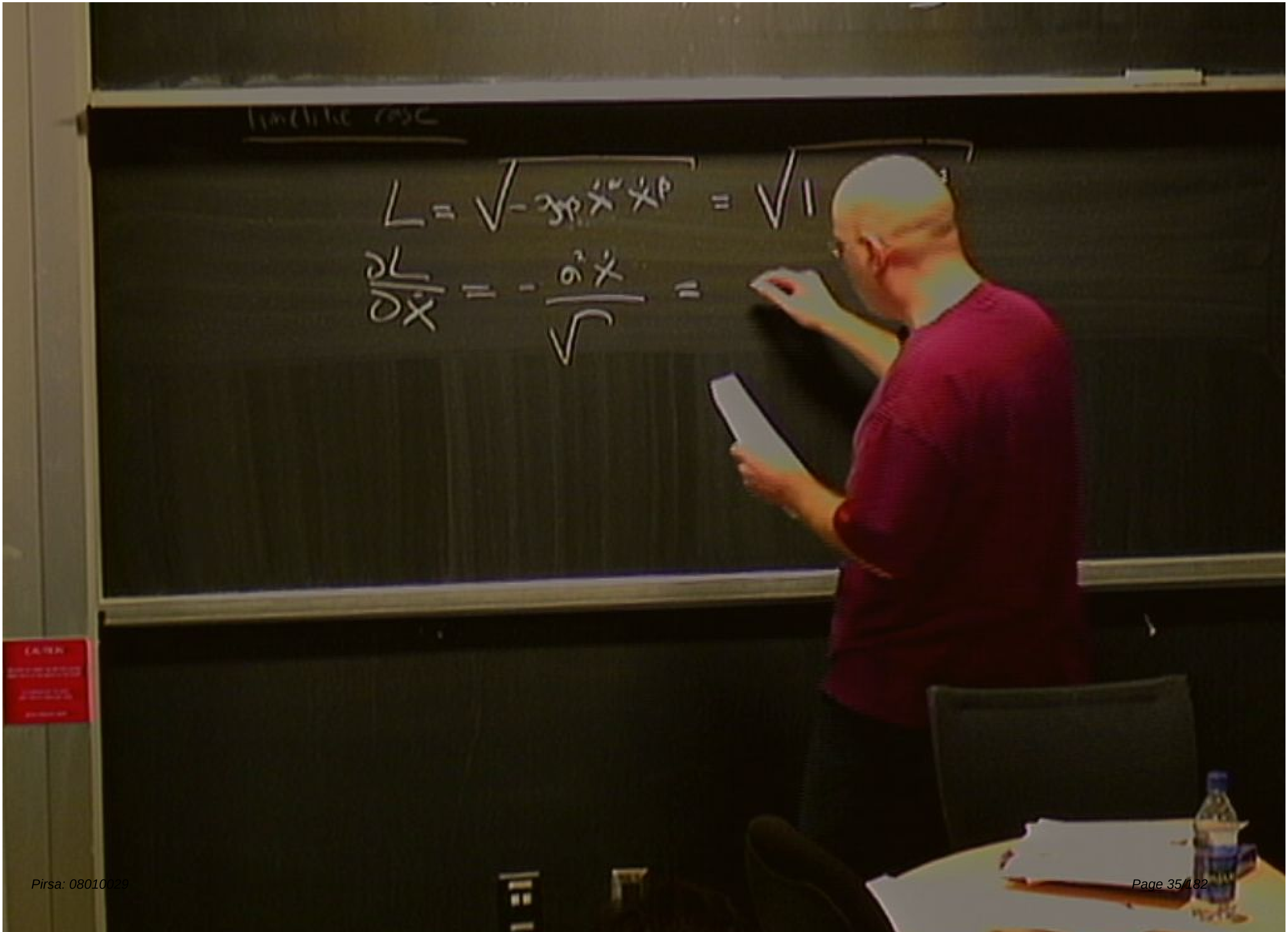
Timelike case

$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}$$

case

$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} = \sqrt{1 - a^{2}\dot{x}^{2}}$$


timelike case


timelike case
$L = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = \sqrt{1}$
$\frac{\partial L}{\partial \dot{x}} = -\frac{a^2 \dot{x}}{\sqrt{\;}} =$

$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} = \sqrt{1 - a^2\dot{x}^2}$$

$$\frac{\partial L}{\partial \dot{x}} = -\frac{a^2\dot{x}}{\sqrt{}} = -\frac{a^2\dot{x}}{L} =$$

...like case

$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} = \sqrt{1 - a^2\dot{x}^2}$$

$$\frac{\partial L}{\partial \dot{x}} = -\frac{a^2\dot{x}}{\sqrt{}} = -\frac{a^2\dot{x}}{L} = -P \equiv (a$$

$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} = \sqrt{1 - a^2 \dot{x}^2}$$

$$\frac{\partial L}{\partial \dot{x}} = -\frac{a^2 \dot{x}}{\sqrt{}} = -\frac{a^2 \dot{x}}{L} = -p \equiv \text{constant}$$

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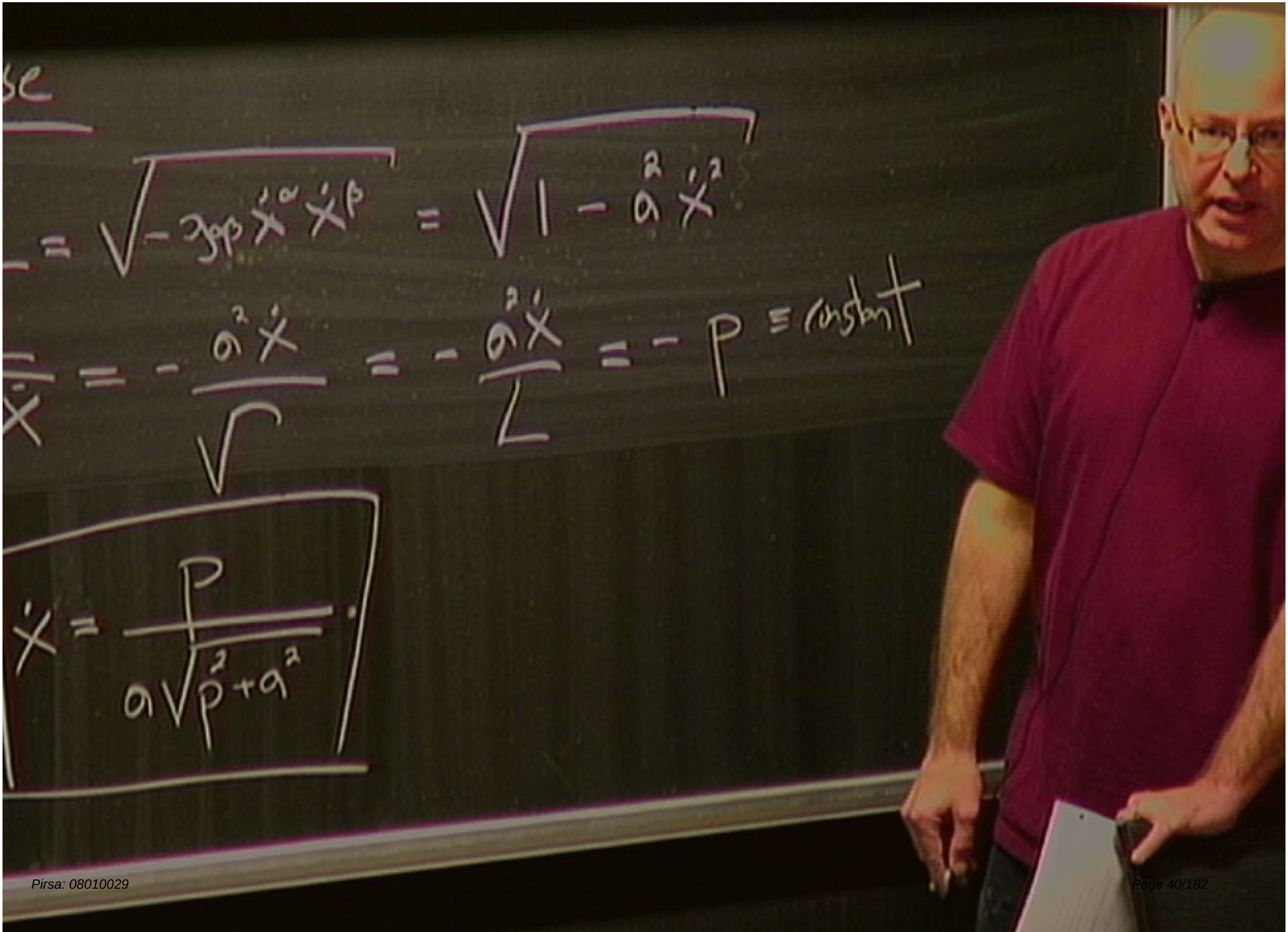
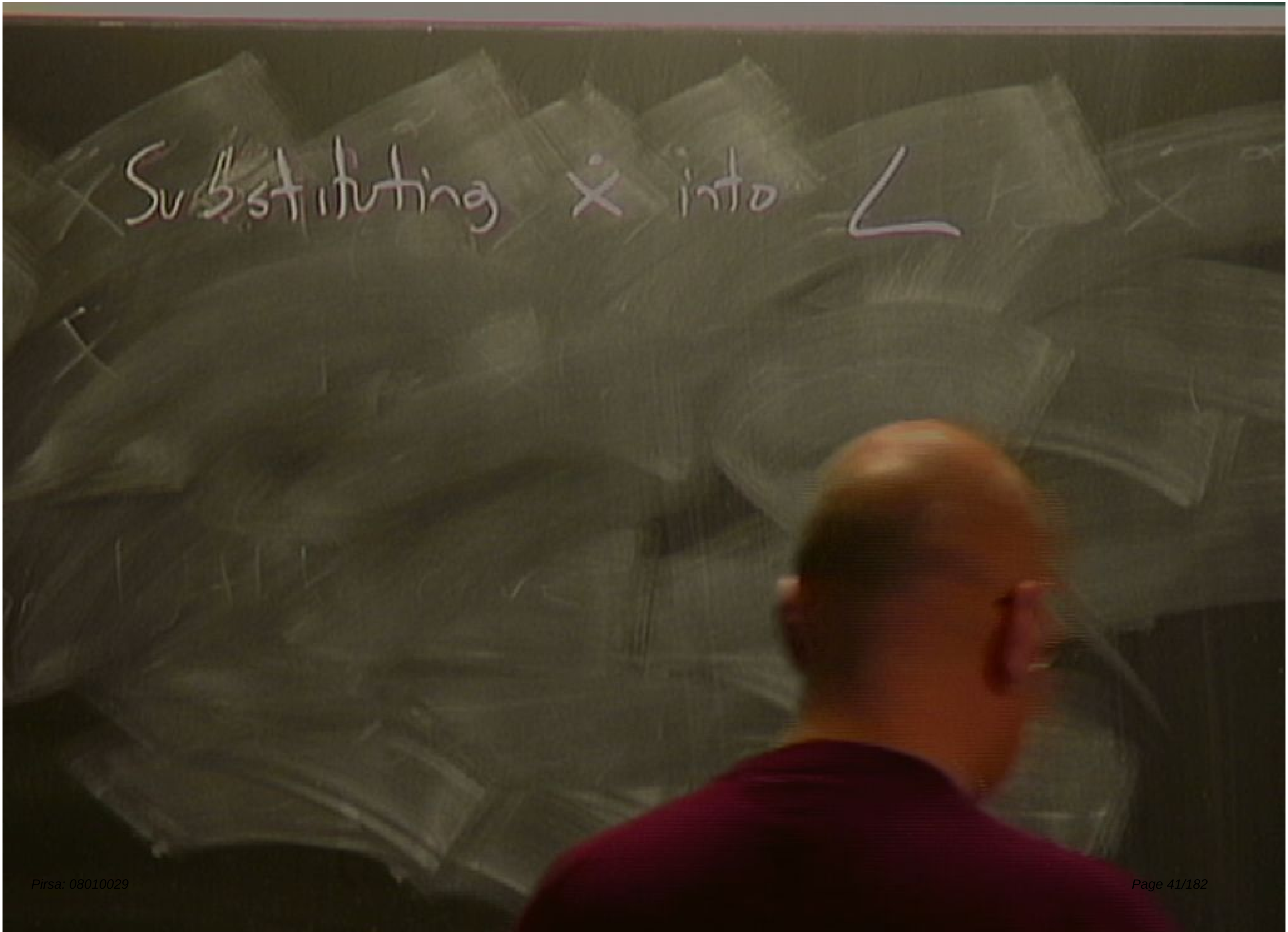
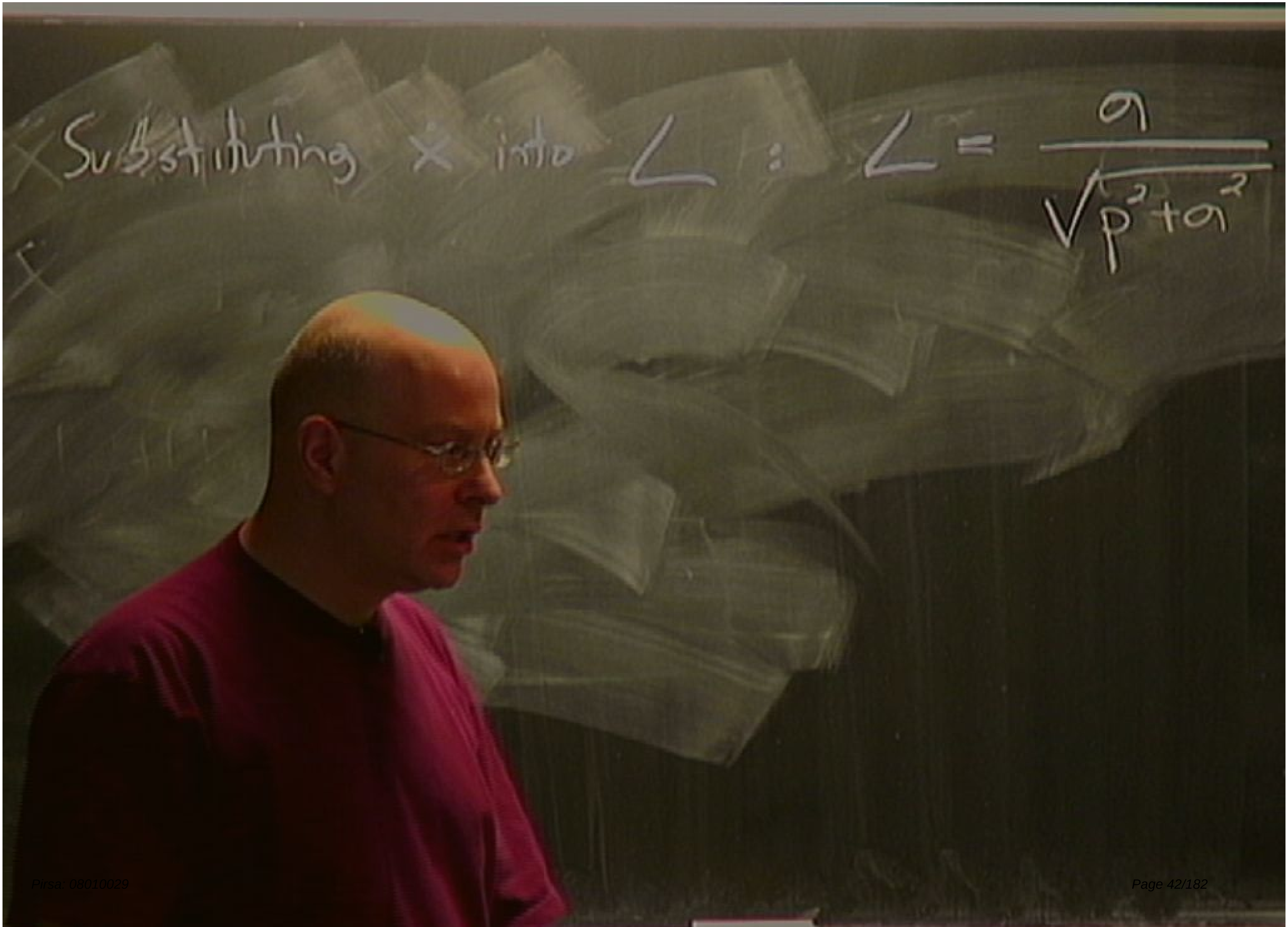
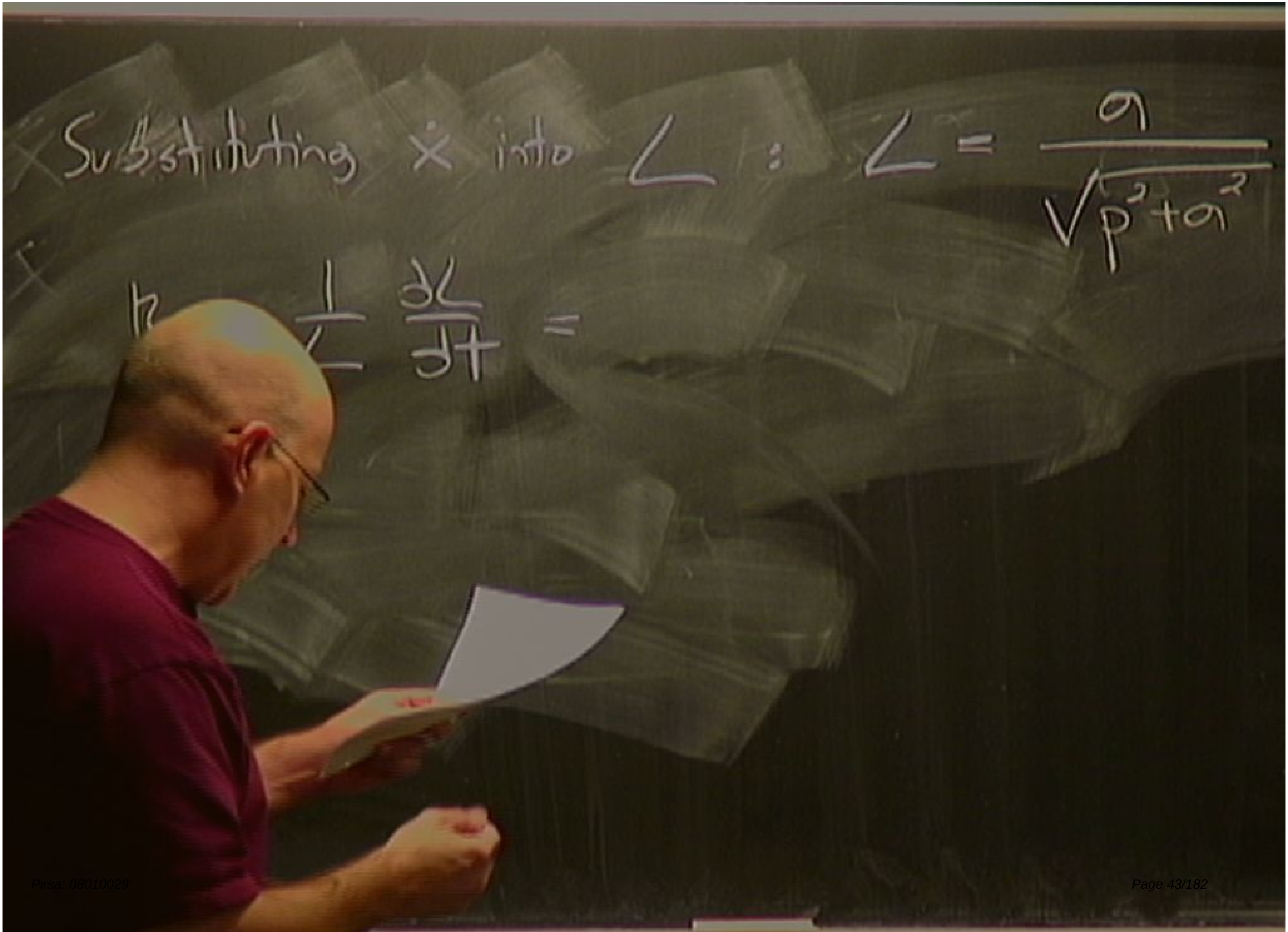
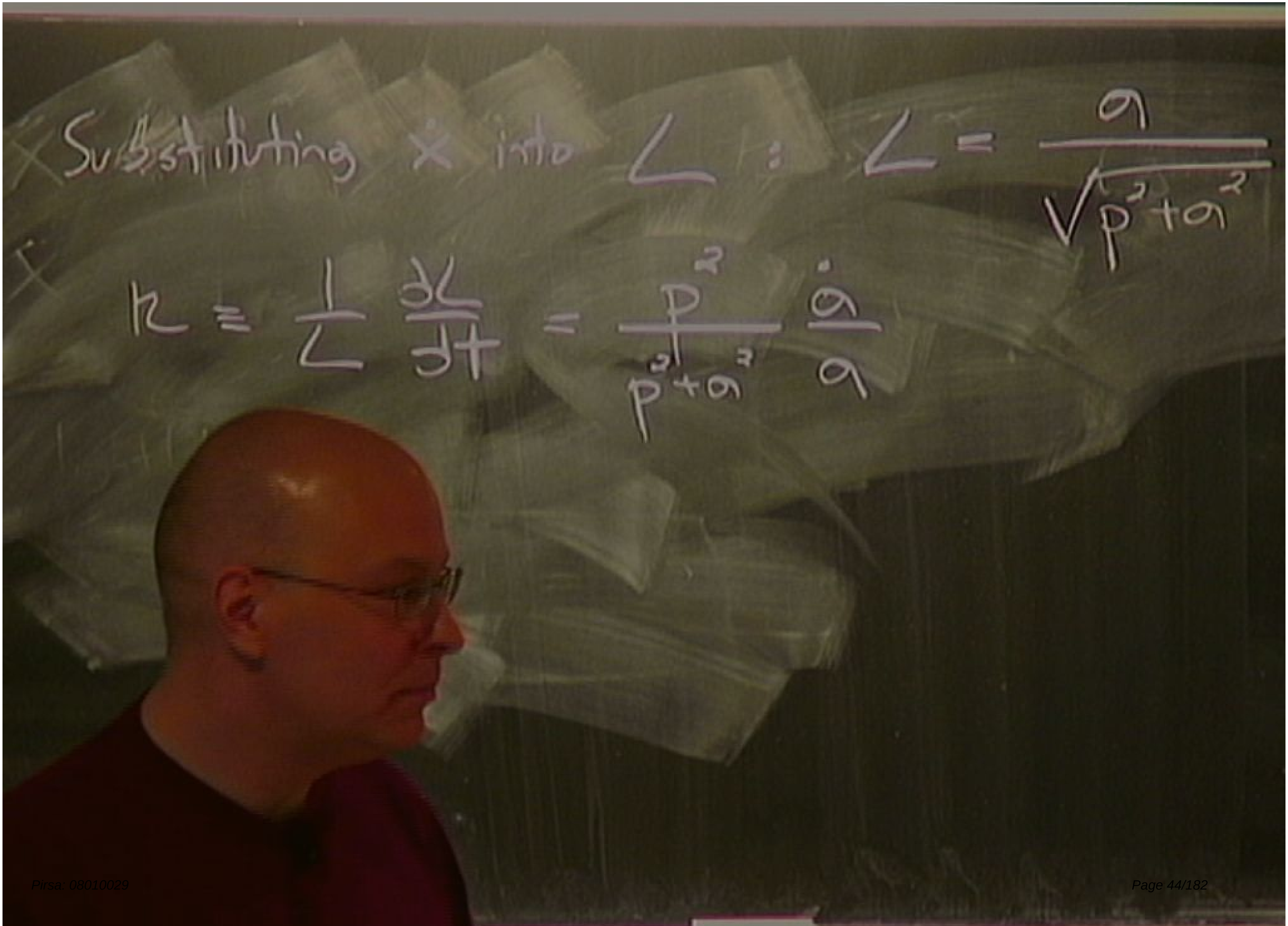
$$\boxed{\dot{x} = \frac{p}{a\sqrt{p^2 + a^2}}}$$


se
$= \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} = \sqrt{1 - a^2 \dot{x}^2}$
$= -\frac{a^2 \dot{x}}{\sqrt{\;}} = -\frac{a^2 \dot{x}}{L} = -p$ = constant
$\boxed{\dot{x} = \frac{p}{a\sqrt{p^2 + a^2}}}$


Substituting $\dot{x}$ into L


Substituting $\dot{x}$ into L : $L = \frac{a}{\sqrt{p^2+a^2}}$


Substituting $\dot{x}$ into L : $L = \frac{a}{\sqrt{p^2+a^2}}$
$p \quad \frac{1}{L}\frac{dL}{dt} =$

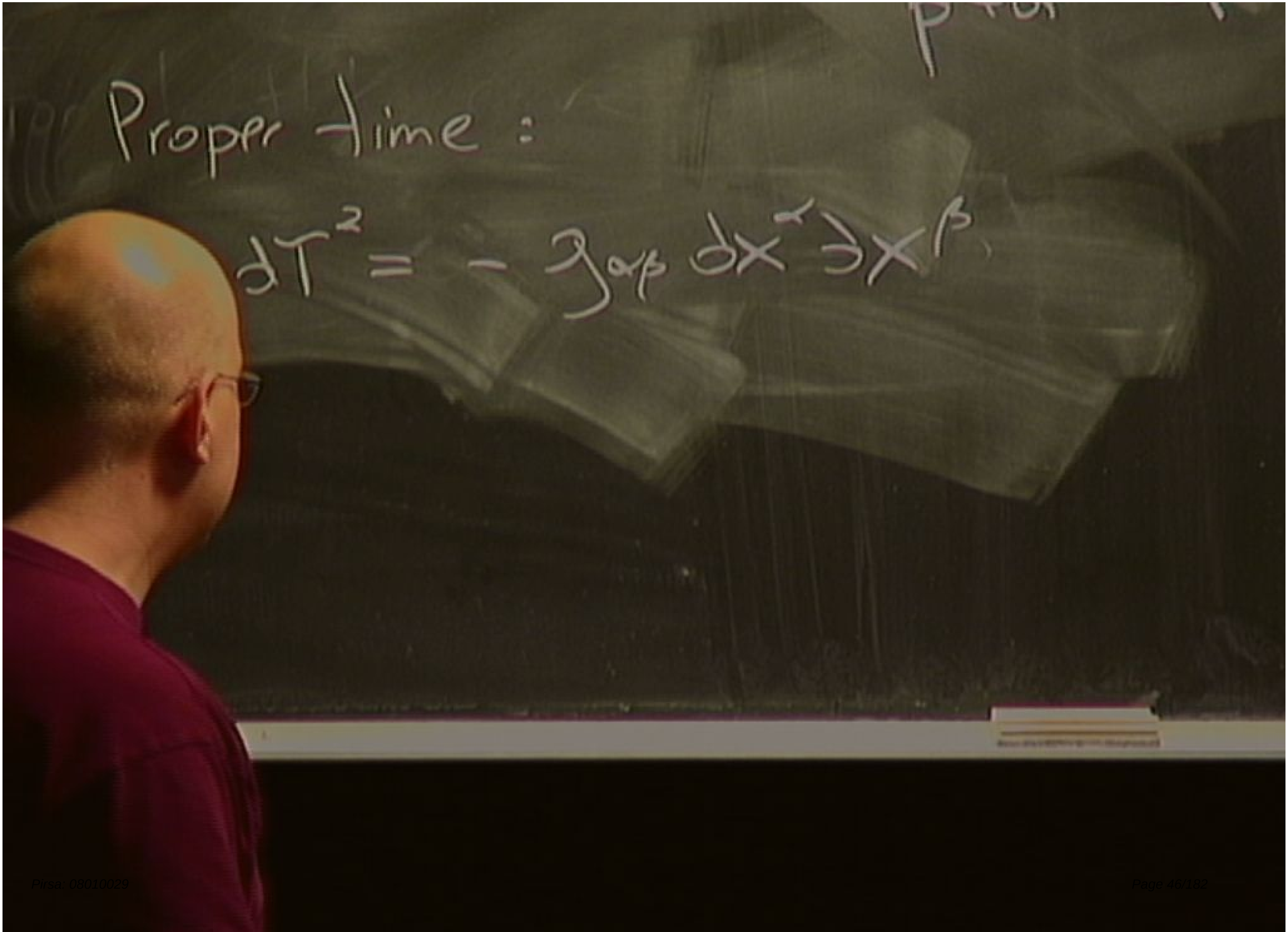

Substituting $\dot{x}$ into L : $L = \frac{a}{\sqrt{p^2+a^2}}$
$k = \frac{1}{L}\frac{dL}{dt} = \frac{p^2}{p^2+a^2}\frac{\dot{a}}{a}$

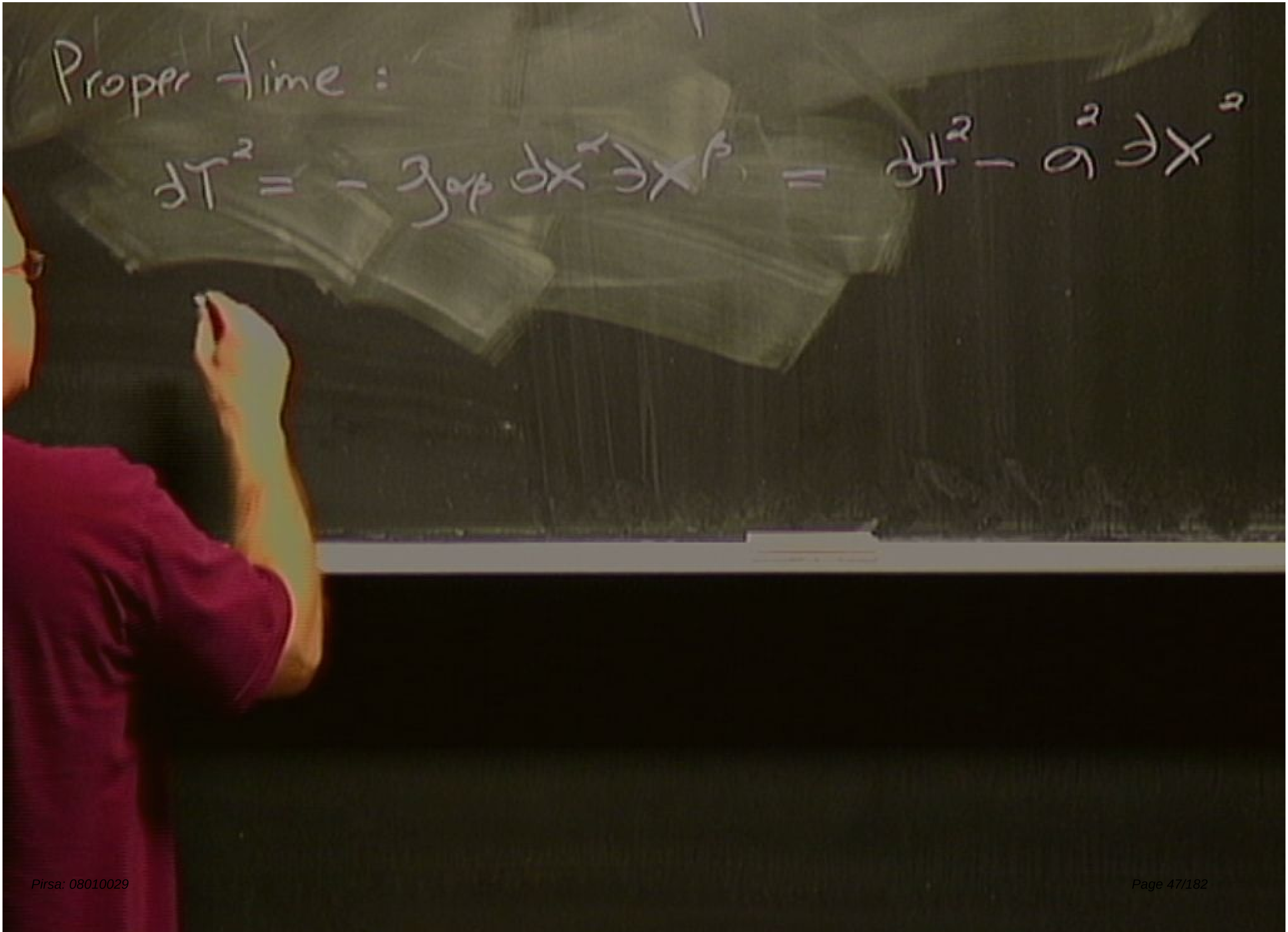
Timelike case

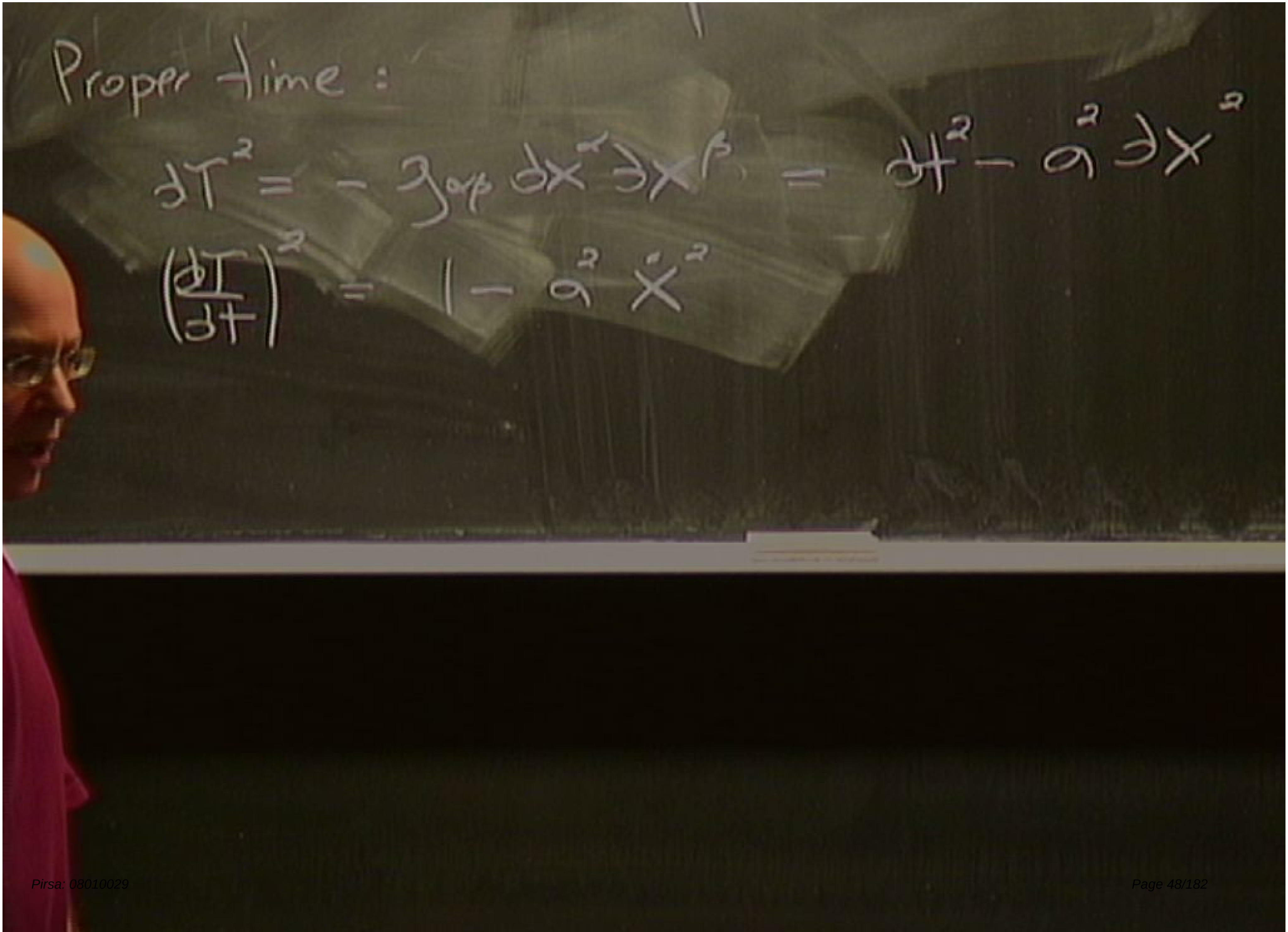
$$L = \sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} = \sqrt{1 - a^2\dot{x}^2}$$

$$\frac{\partial L}{\partial \dot{x}} = -\frac{a^2\dot{x}}{\sqrt{}} = -\frac{a^2\dot{x}}{L} = -p \equiv$$

$$\boxed{\dot{x} = \frac{p}{a\sqrt{p^2+a^2}}}.$$


Proper time :
$d\tau^2 = - g_{\alpha\beta}\, dx^\alpha dx^\beta$


Proper time :
$d\tau^2 = -g_{\alpha\beta}\, dx^\alpha dx^\beta = dt^2 - a^2 dx^2$


Proper time:
$d\tau^2 = -g_{\alpha\beta}\, dx^\alpha dx^\beta = dt^2 - a^2 dx^2$
$\left(\frac{d\tau}{dt}\right)^2 = 1 - a^2 \dot{x}^2$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - a^2 \dot{x}^2$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - a^2 \dot{x}^2}}$$

Proper time:

$$d\tau^2 = -g_{\alpha\beta}\,dx^\alpha dx^\beta = dt^2 - a^2 dx^2$$

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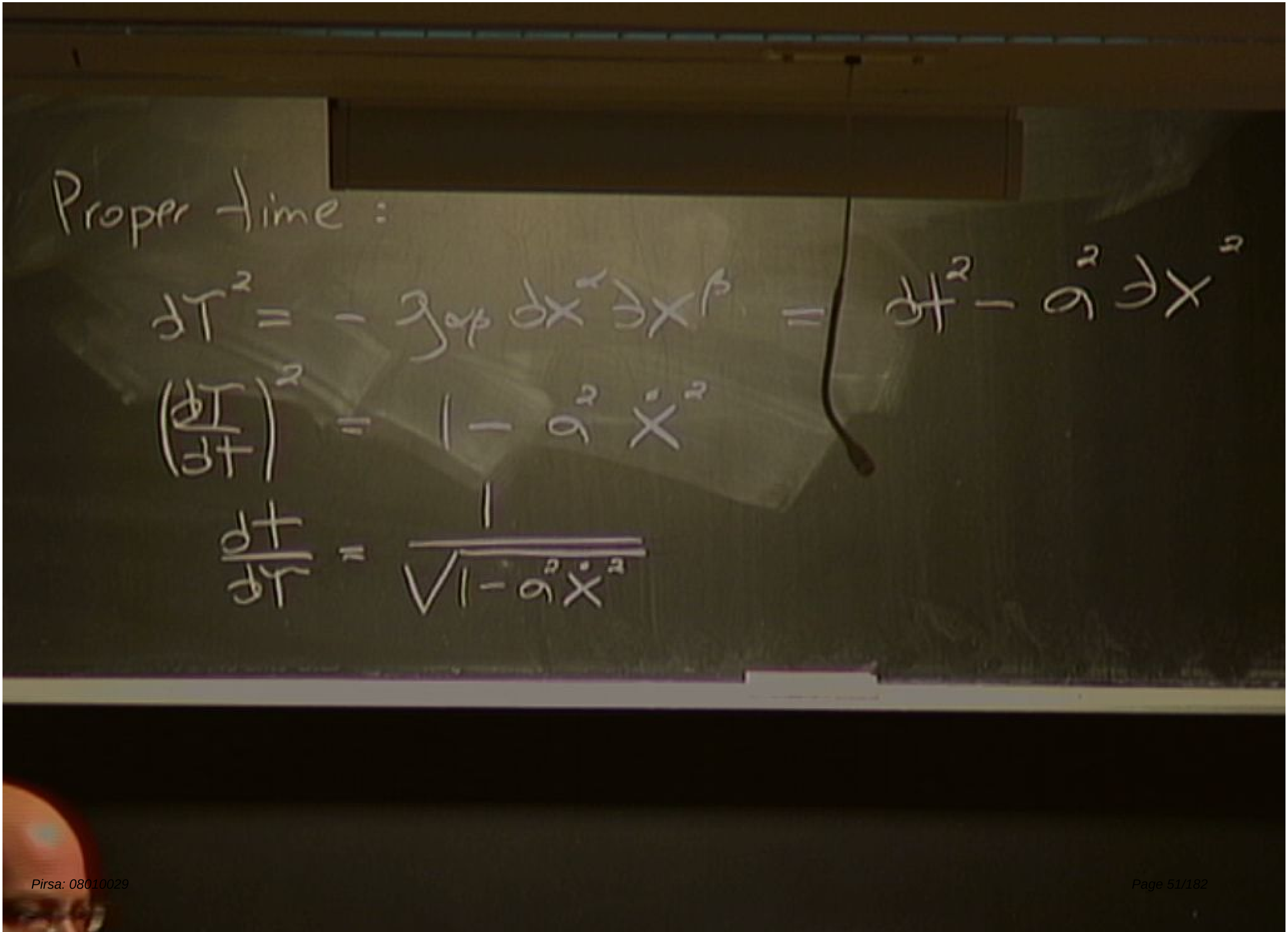
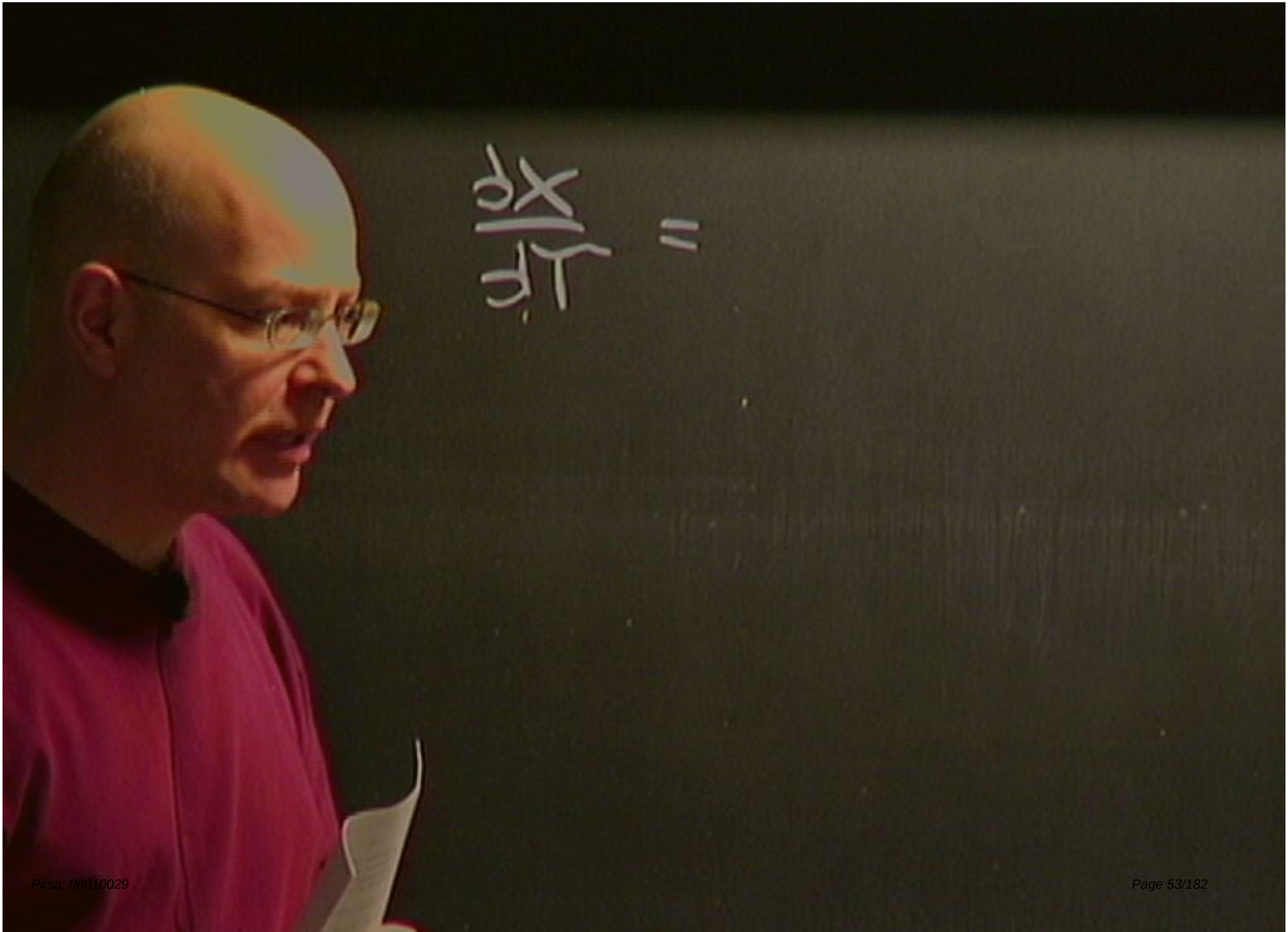
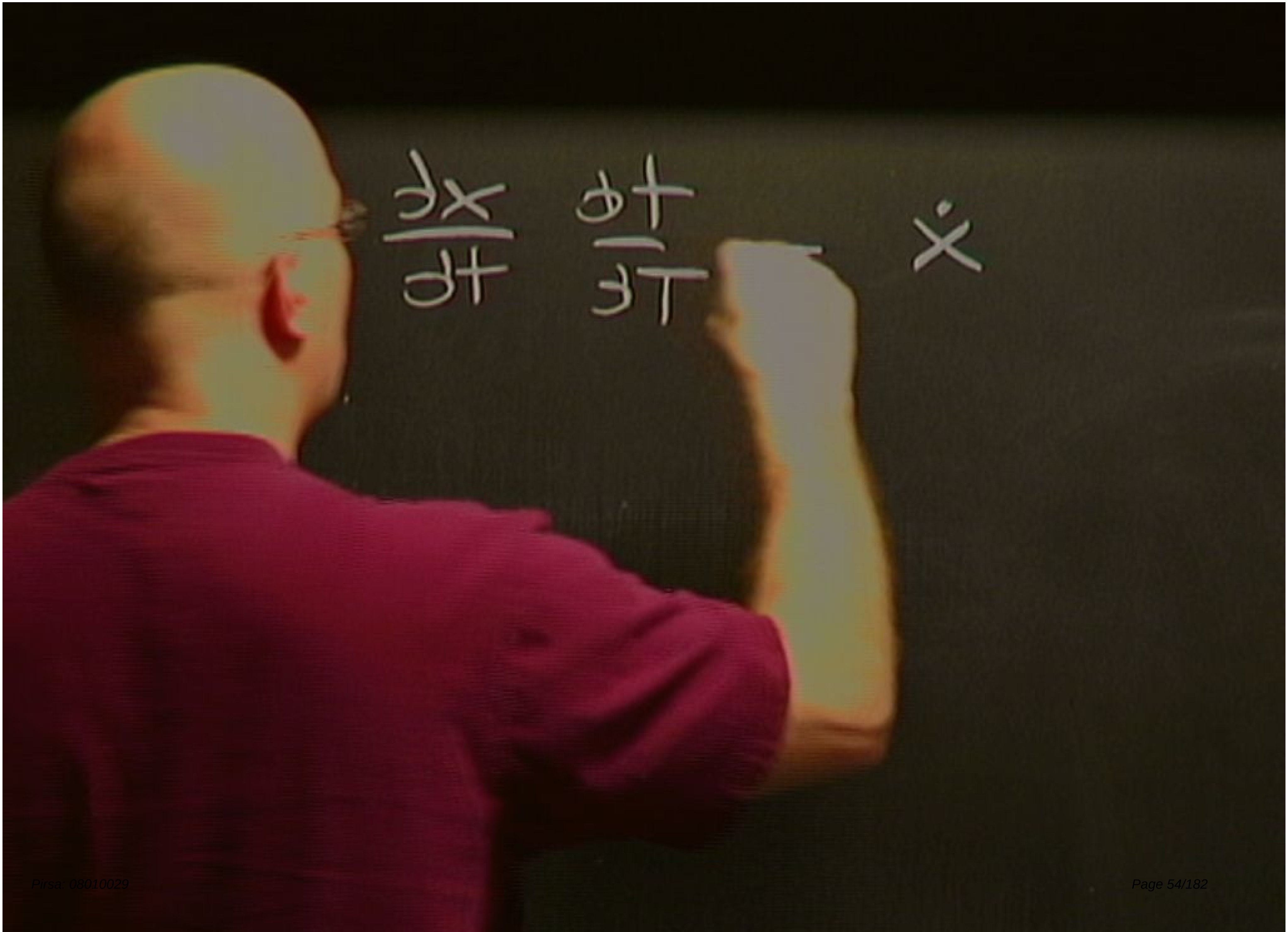
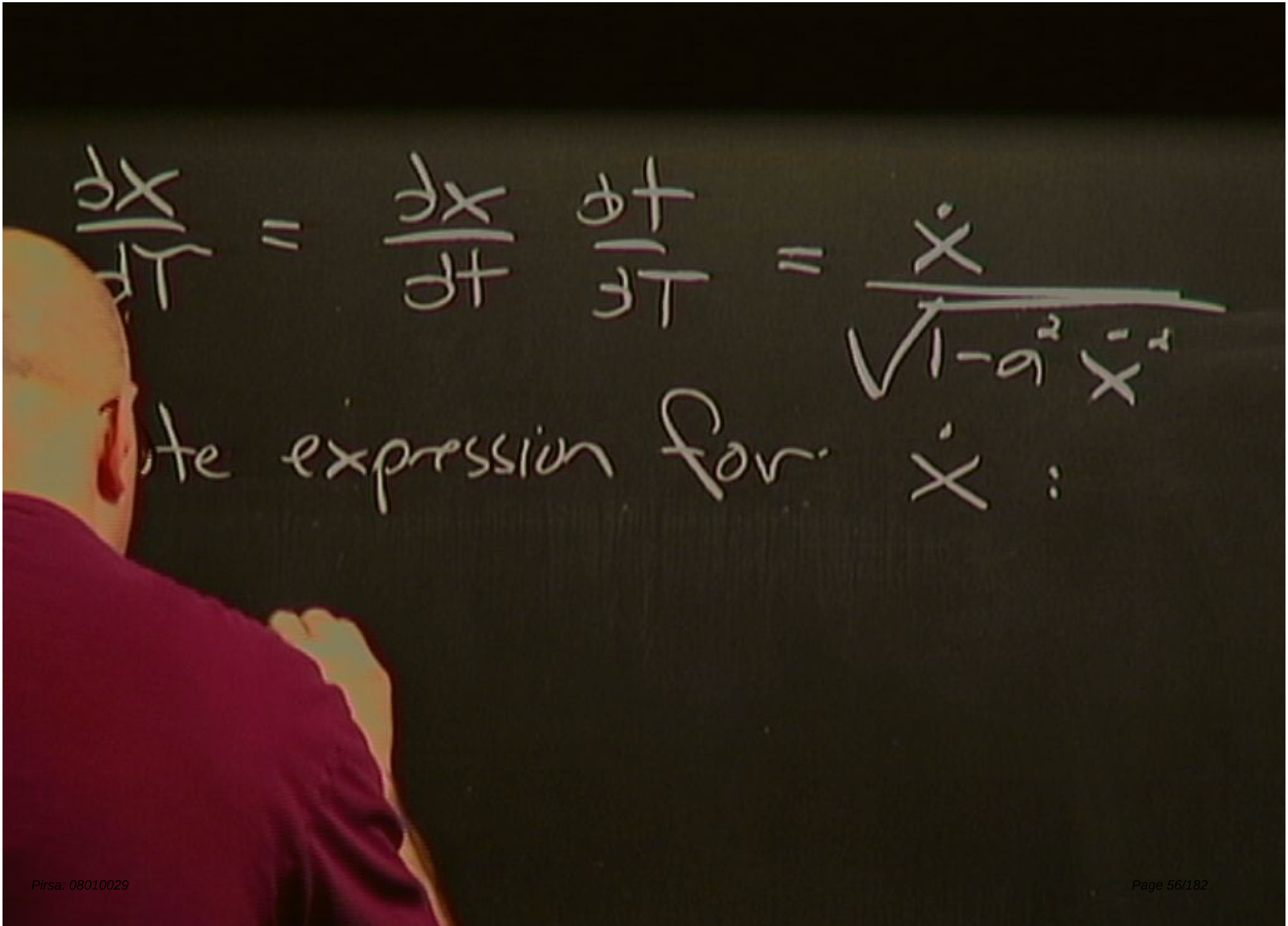

Proper time :
$d\tau^2 = -g_{\alpha\beta}\,dx^\alpha dx^\beta = dt^2 - a^2 dx^2$
$\left(\frac{d\tau}{dt}\right)^2 = 1 - a^2 \dot{x}^2$
$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - a^2 \dot{x}^2}}$

$$\frac{dx}{d\tau} =$$


$\frac{dx}{d\tau} =$


$\frac{dx}{dt}$
$\frac{dt}{dT}$
$\dot{x}$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\,\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^2\dot{x}^2}}$$

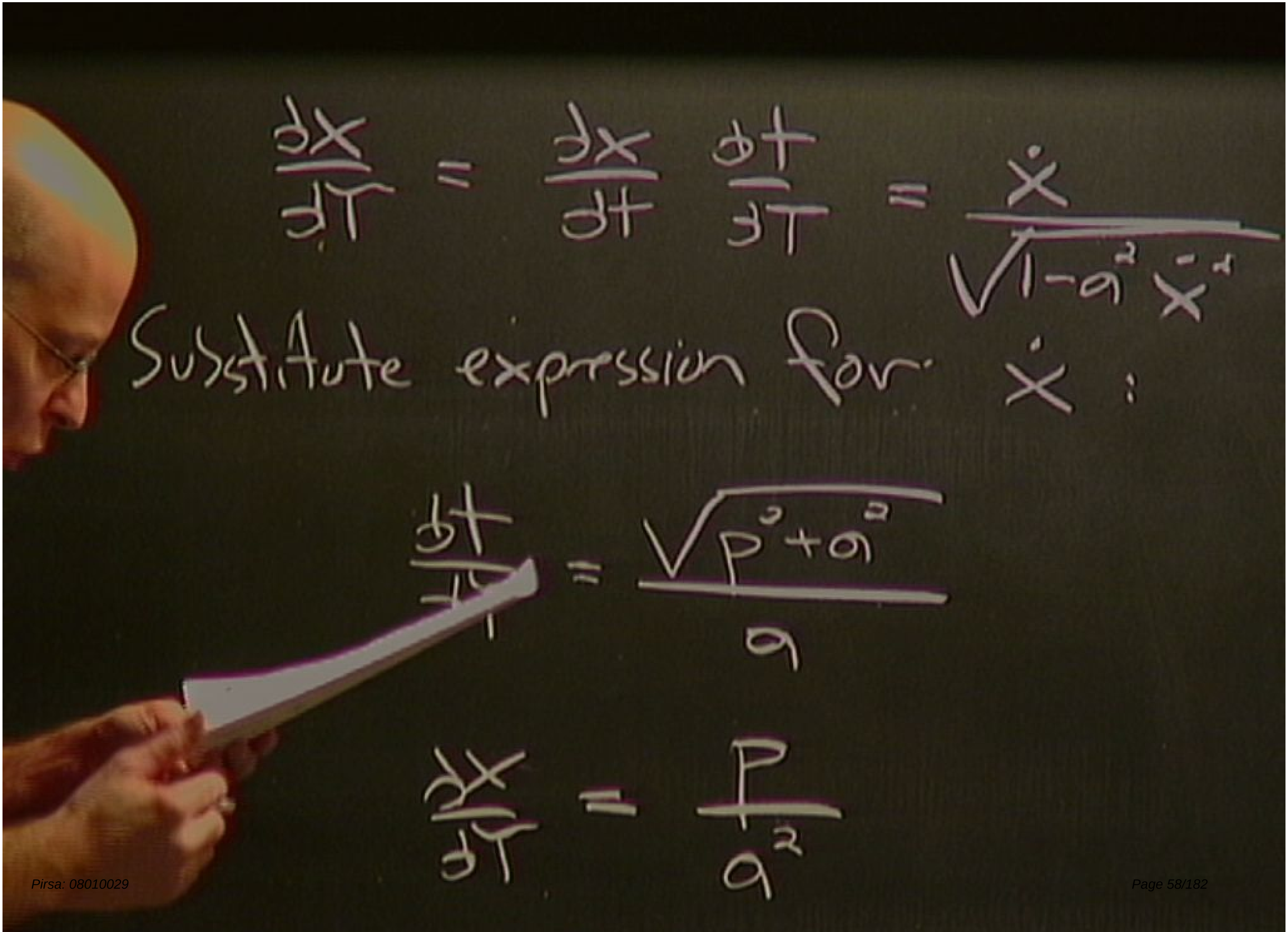


$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^2\dot{x}^2}}$
te expression for $\dot{x}$:

$$\frac{dx}{d\tau} = \frac{dx}{dt}\,\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^2\dot{x}^2}}$$

stitute expression for $\dot{x}$:

$$\frac{dt}{d\tau} =$$


$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^2x^{-2}}}$
Substitute expression for $\dot{x}$:
$\frac{dt}{d\tau} = \frac{\sqrt{p^2+a^2}}{a}$
$\frac{dx}{d\tau} = \frac{p}{a^2}$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\,\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^2\dot{x}^2}}$$

ubstitute expression for $\dot{x}$:

$$\frac{dt}{d\tau} = \frac{\sqrt{p^2+a^2}}{a}$$

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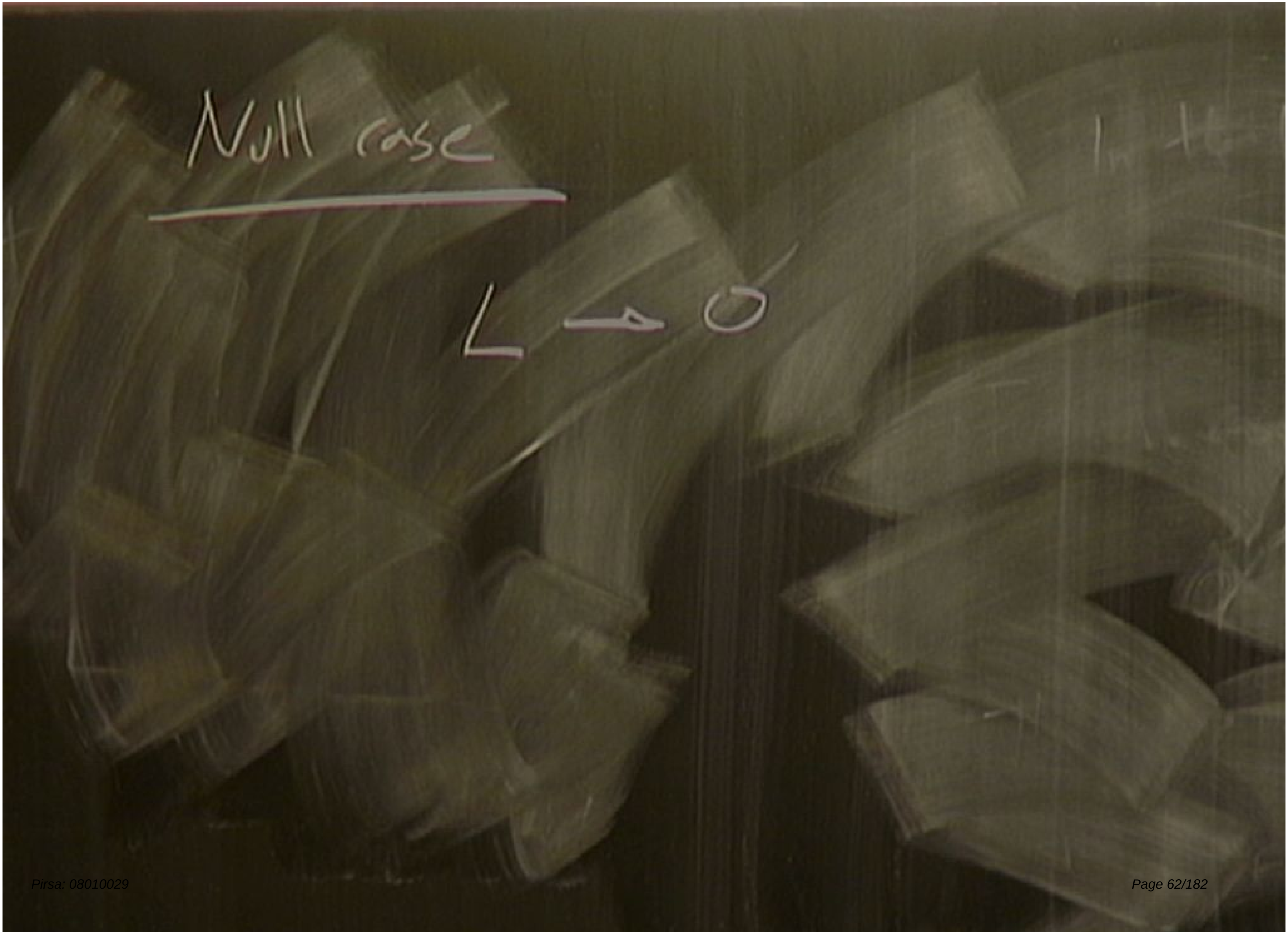
$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{\dot{x}}{\sqrt{1-a^{2}x^{-2}}}$$

Substitute expression for $\dot{x}$:

$$\frac{dt}{d\tau} = \frac{\sqrt{p^{2}+a^{2}}}{a} = U^{t}$$

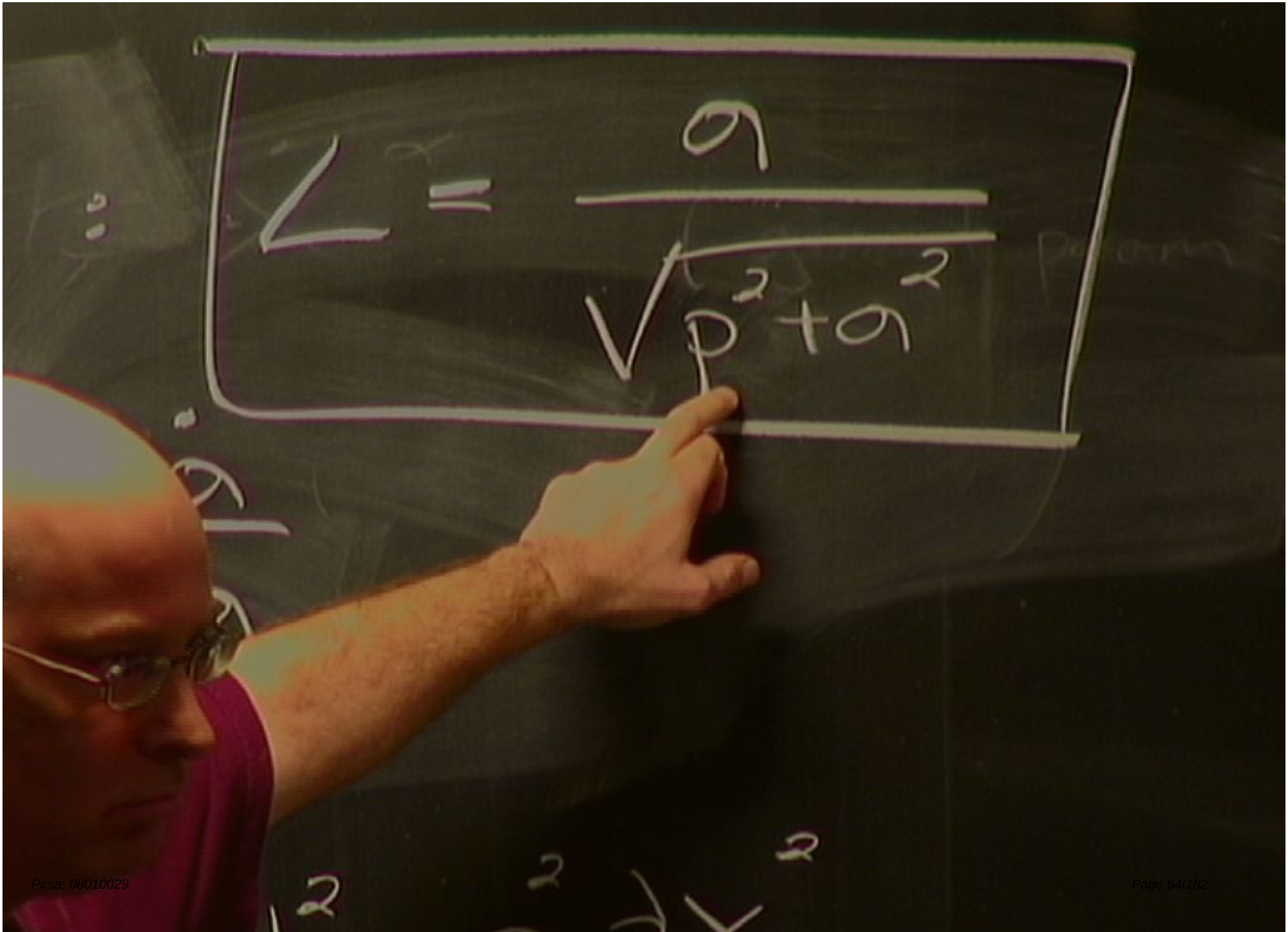
$$\frac{dx}{d\tau} = \frac{p}{a^{2}} = U^{x}$$

Null case


Null case
$L \to 0$

$$\boxed{L = \frac{a}{\sqrt{p^2 + a^2}}}$$

$$\frac{a}{a}$$


$L = \frac{a}{\sqrt{p^2+a^2}}$

se

$$L \to 0 \qquad (p \to \infty)$$

ute expression for x :

$$\boxed{\begin{aligned} \frac{dt}{d\tau} &= \frac{\sqrt{p^2+a^2}}{a} \quad U^t \\ \frac{dx}{d\tau} &= \frac{p}{a^2} = U^x \end{aligned}}$$

$$\boxed{\dot{x} = \frac{p}{a\sqrt{p^2+q^2}}}$$

$$L \to 0 \qquad (p -$$

$$x^a \to \frac{1}{a}$$

$\dot{x}$ into L : L

$$\frac{\partial L}{\partial t} = \frac{p^2}{p^2+q^2}\,\frac{\dot{q}}{q}$$

$$L \to 0 \quad (p \to$$

$$\dot{x} \to \frac{1}{a}$$

$$\hbar \to \dot{a}/a$$

Null case

$$L \to 0 \quad (p \to \infty)$$

$$\ddot{x} \to \frac{1}{a}$$

$$k \to \frac{\dot{a}}{a}$$

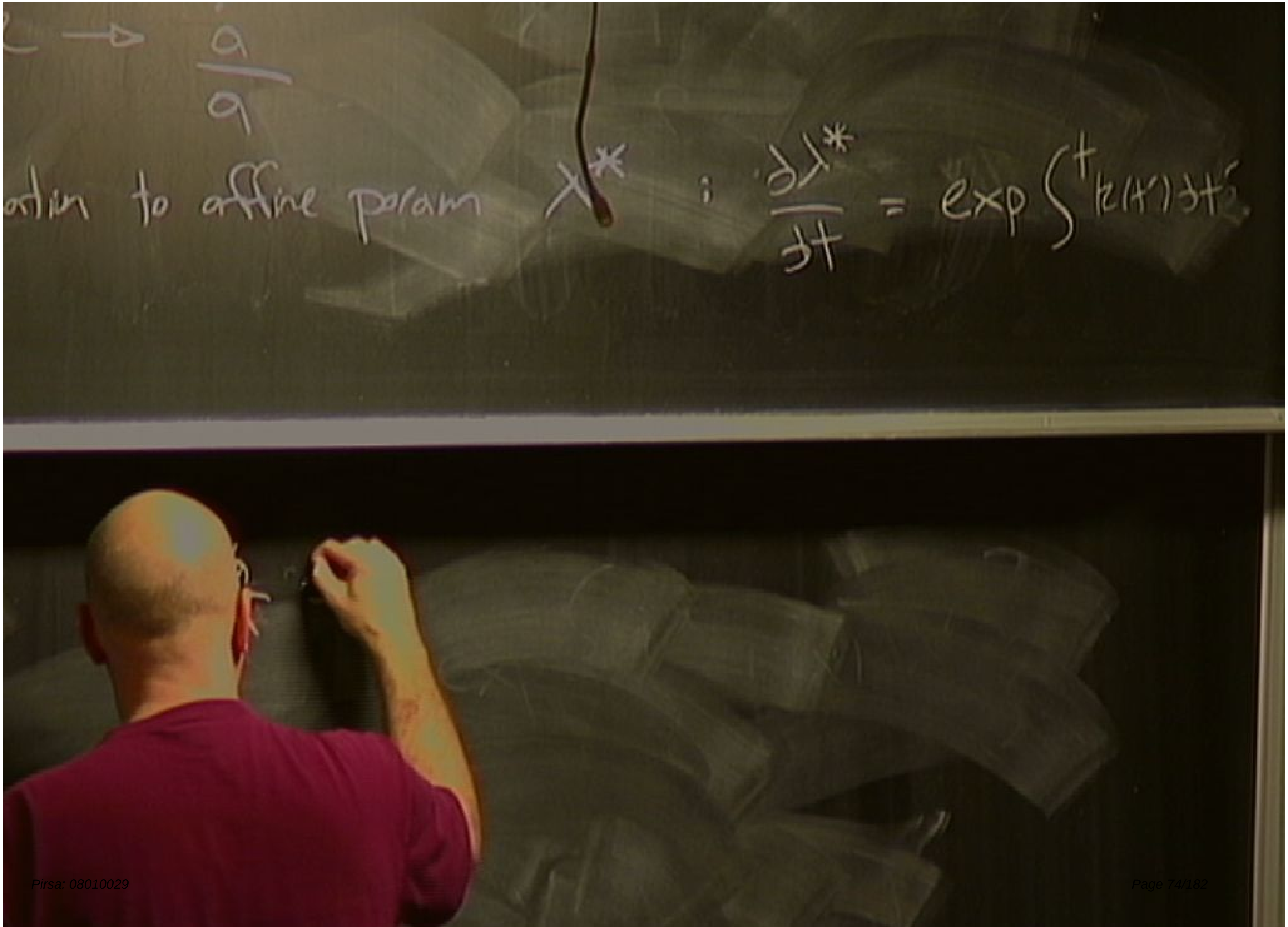
Null case

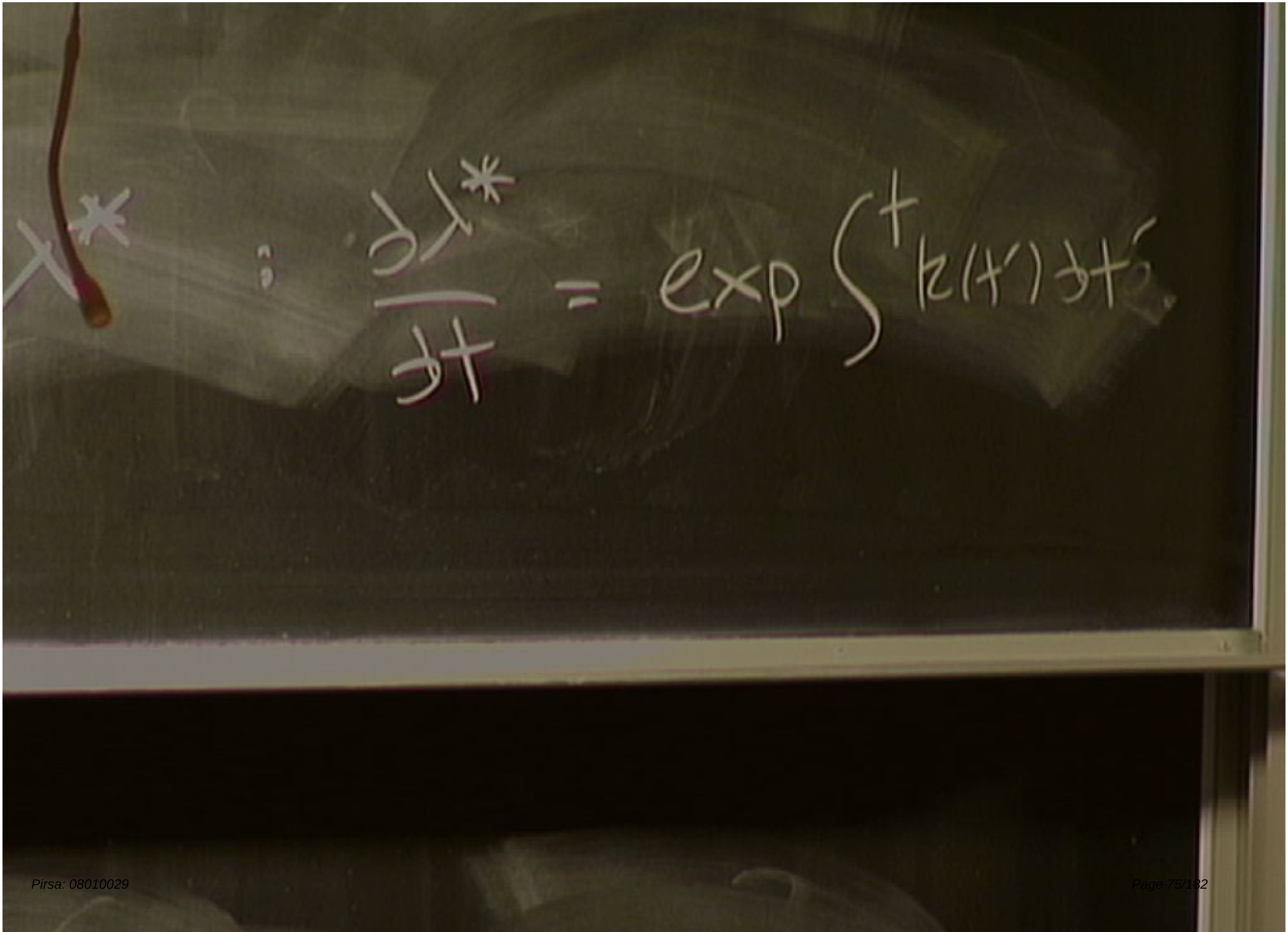
$$L \to 0 \quad (p \to \infty)$$

$$\ddot{x} \to \frac{1}{a}$$

$$k \to \frac{\dot{a}}{a}$$

Transformation to affine param λ^*


→ a/a
ation to affine param λ* ; dλ*/dt = exp ∫^t κ(t′) dt′


X^* : $\frac{\partial \lambda^*}{\partial t} = \exp \int^t k(t')\, dt'$

$$\int^{t} \frac{\dot{a}}{a}\, dt = \int \frac{da}{a} = \ln$$

$$\int^{t} \frac{\dot{a}}{a}\, dt = \int \frac{da}{a} = \ln a$$

$$\int^t \frac{\dot{a}}{a}\,dt = \int \frac{da}{a} = \ln$$

$$a(t) \longrightarrow \lambda^* = \int^t a(t')\,dt'$$

$$\int^{t} \frac{\dot{a}}{a}\, dt = \int \frac{da}{a}$$

$$\frac{d\lambda^{*}}{dt} = a(t) \longrightarrow \lambda^{*} = \int^{t} a(t')\, dt'$$

$$\int^t \frac{\dot a}{a}\, dt = \int \frac{da}{a}$$

$$\frac{d\lambda^*}{dt} = a(t) \rightarrow \boxed{\lambda^* = \int^t a(t')\, dt'}$$

$$\frac{\partial t}{\partial \lambda^*} = \frac{1}{a}$$

$$\frac{\partial x}{\partial \lambda^*} = \frac{1}{a^2}$$

$$\boxed{\begin{aligned} \frac{\partial t}{\partial \lambda^*} &= \frac{1}{a} \\ \frac{\partial x}{\partial \lambda^*} &= \frac{1}{a^2} \end{aligned}}$$

$$\frac{\dot{a}}{a}\,dt = \int \frac{da}{a} =$$

$$\boxed{\lambda^* = \int^t a(t')\,dt'}$$

$$\frac{\partial t}{\partial \lambda^*} = \frac{1}{a}$$

$$\frac{\partial x}{\partial \lambda^*} = \frac{1}{a^2}$$

To check that λ^* is affine,

calculate $\partial^2 t / \partial \lambda_*^2$

$\partial^2 x / \partial \lambda_*^2$

To check that λ^* is affine, calculate $\partial^2 t / \partial \lambda_*^2$, $\partial^2 x / \partial \lambda_*^2$, substitute in geodesic eqn $\Rightarrow$ RHS $= 0$

$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

$$g_{\alpha\beta} U^\alpha U^\beta$$

$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

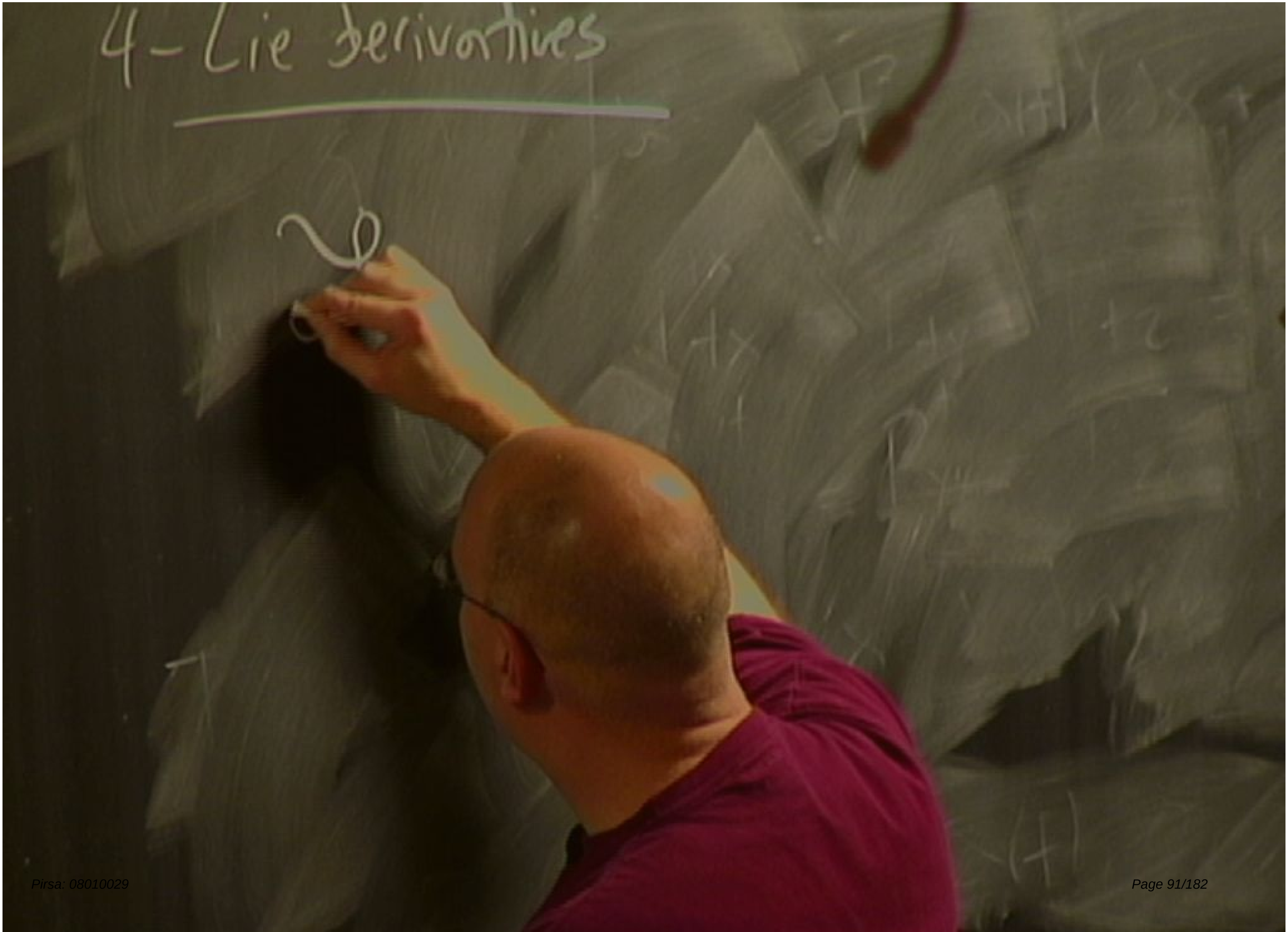
$$g_{\alpha\beta} U^\alpha U^\beta = \frac{g_{\alpha\beta}\, dx^\alpha dx^\beta}{d\tau^2}$$

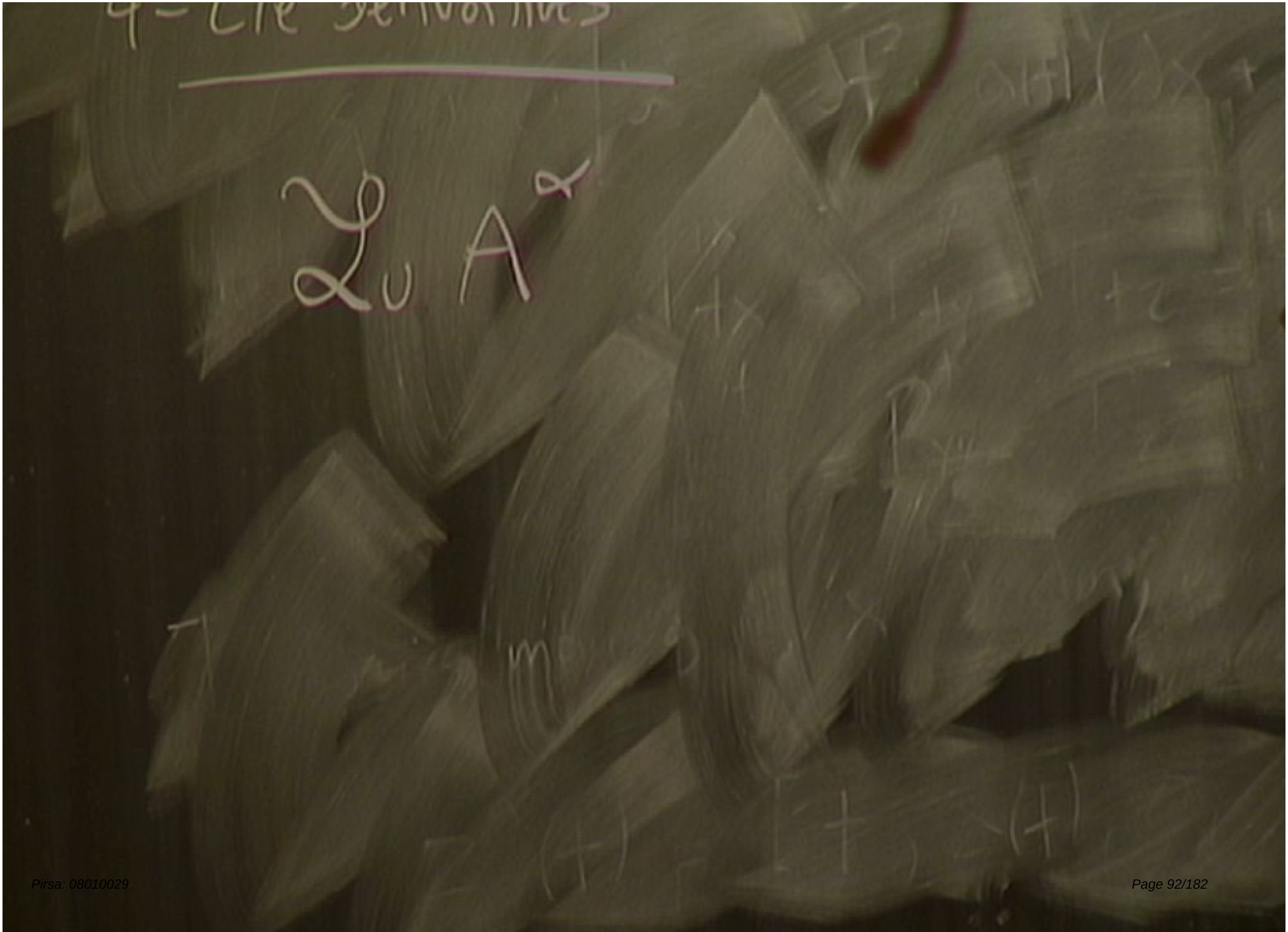
$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

$$g_{\alpha\beta} U^\alpha U^\beta = \frac{g_{\alpha\beta}\, dx^\alpha dx^\beta}{d\tau^2}$$

$$= -1$$

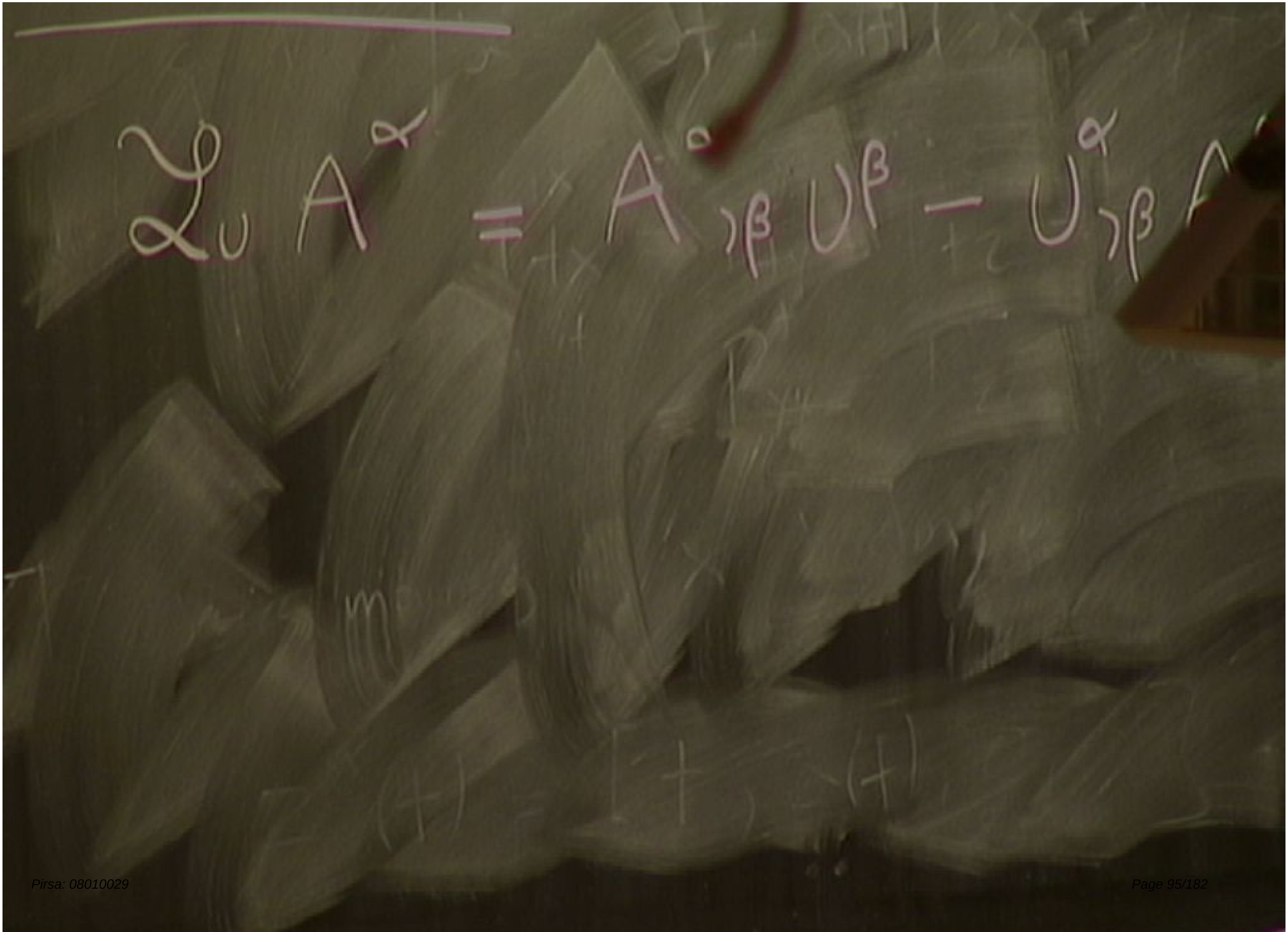
4 - Lie derivatives

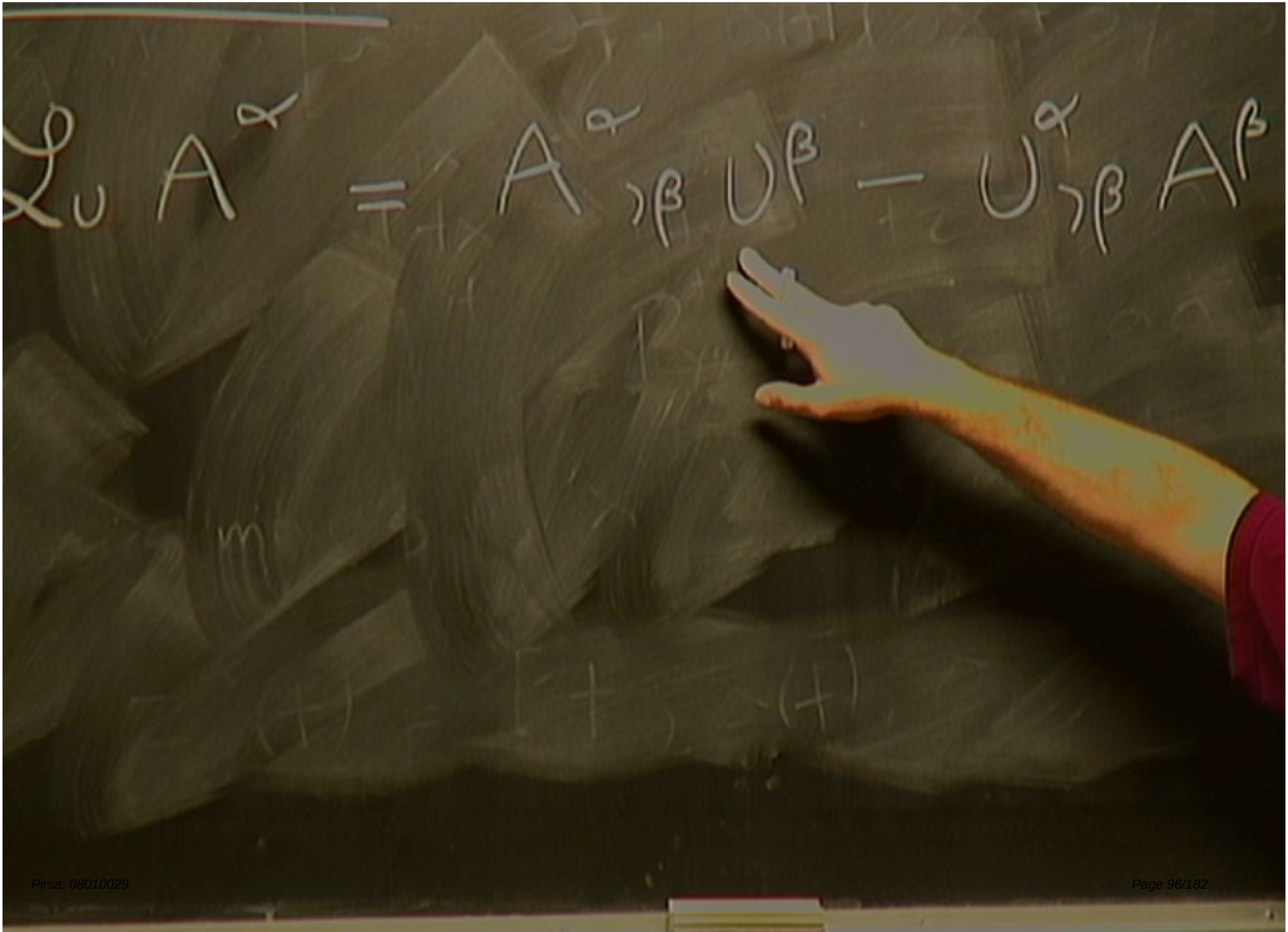

4 - Lie derivatives


q-Lie derivatives
$\mathcal{L}_U A^\alpha$

$$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta}$$

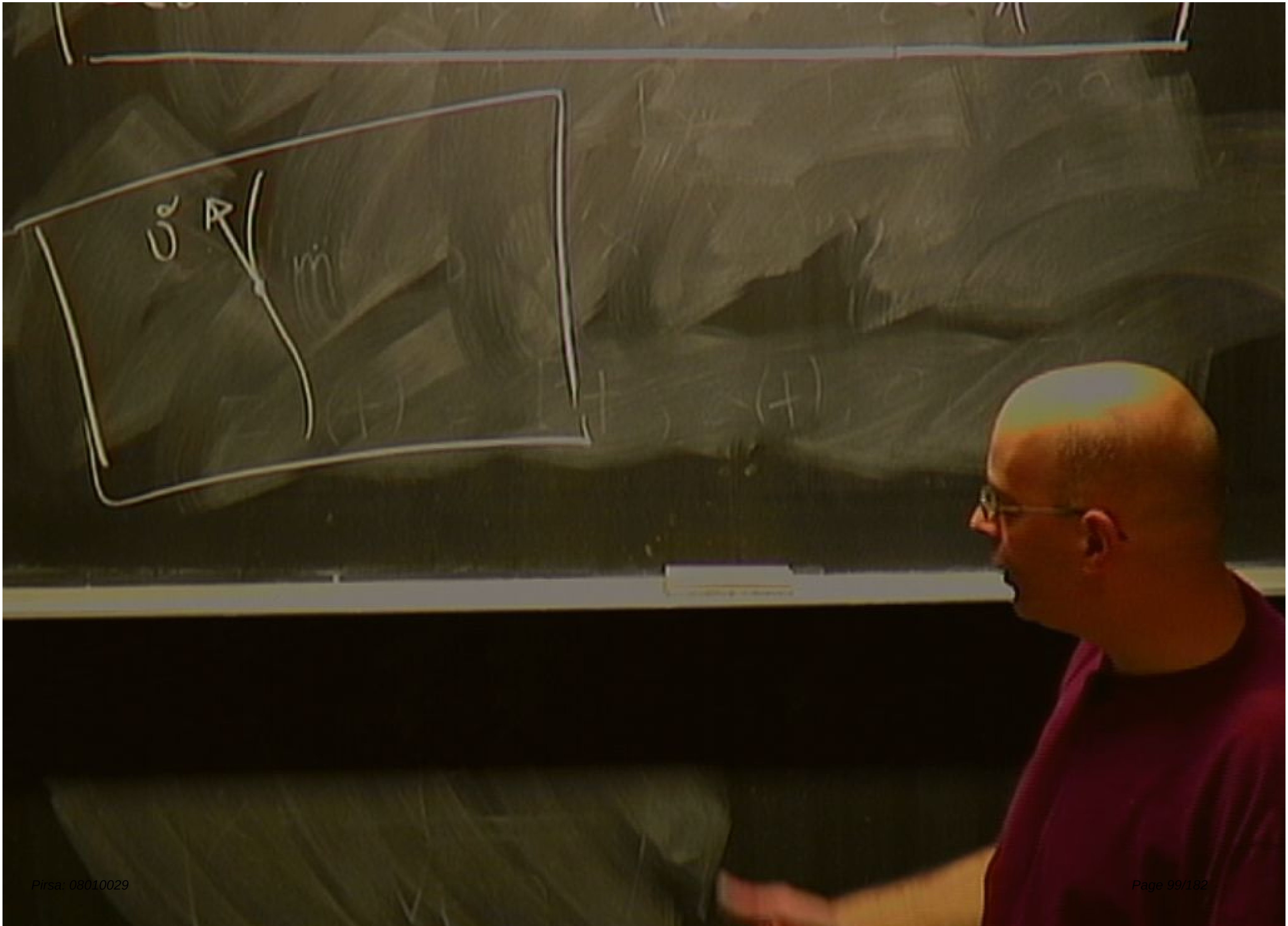
$$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta$$

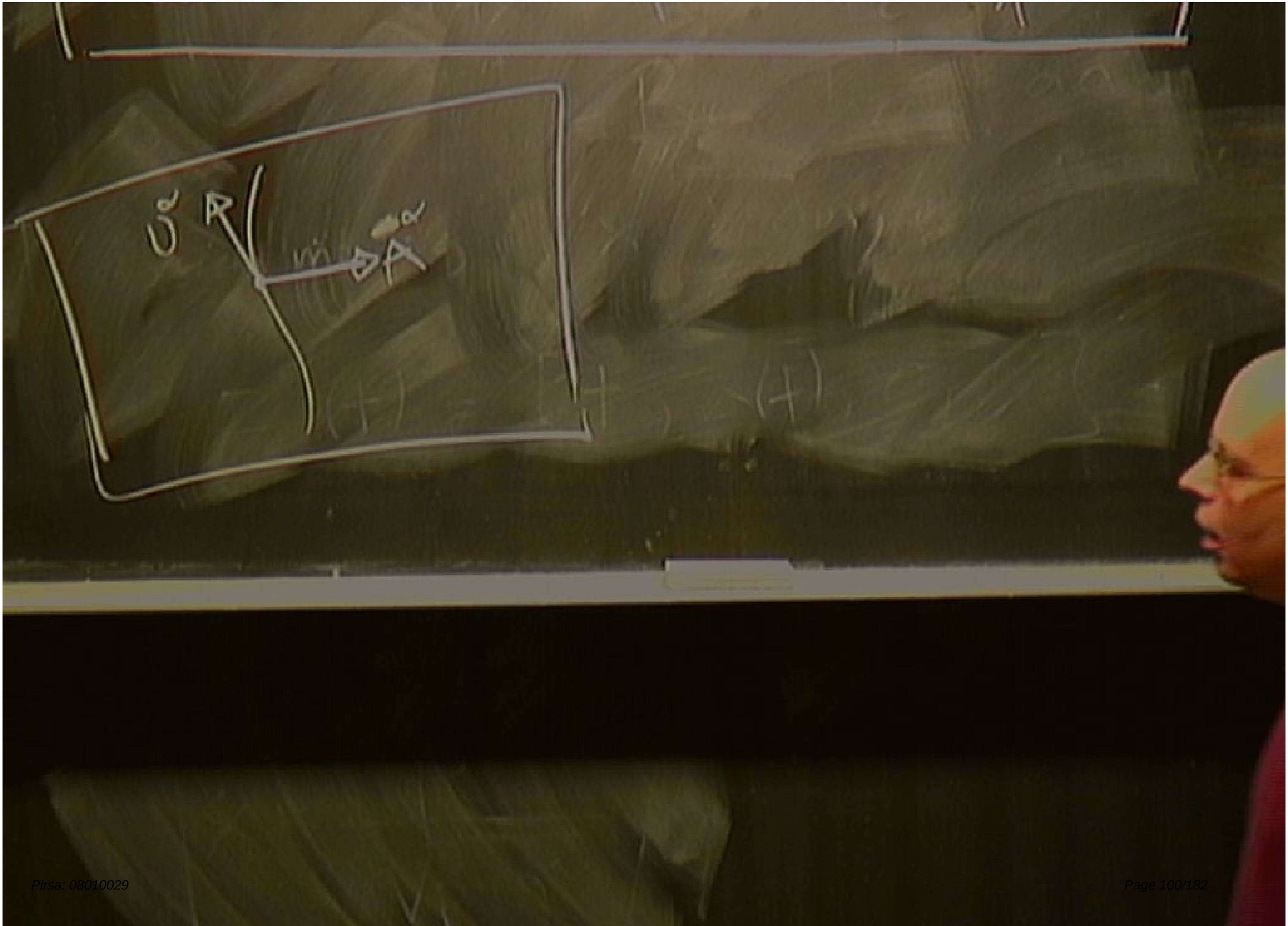

$\mathcal{L}_U A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A$

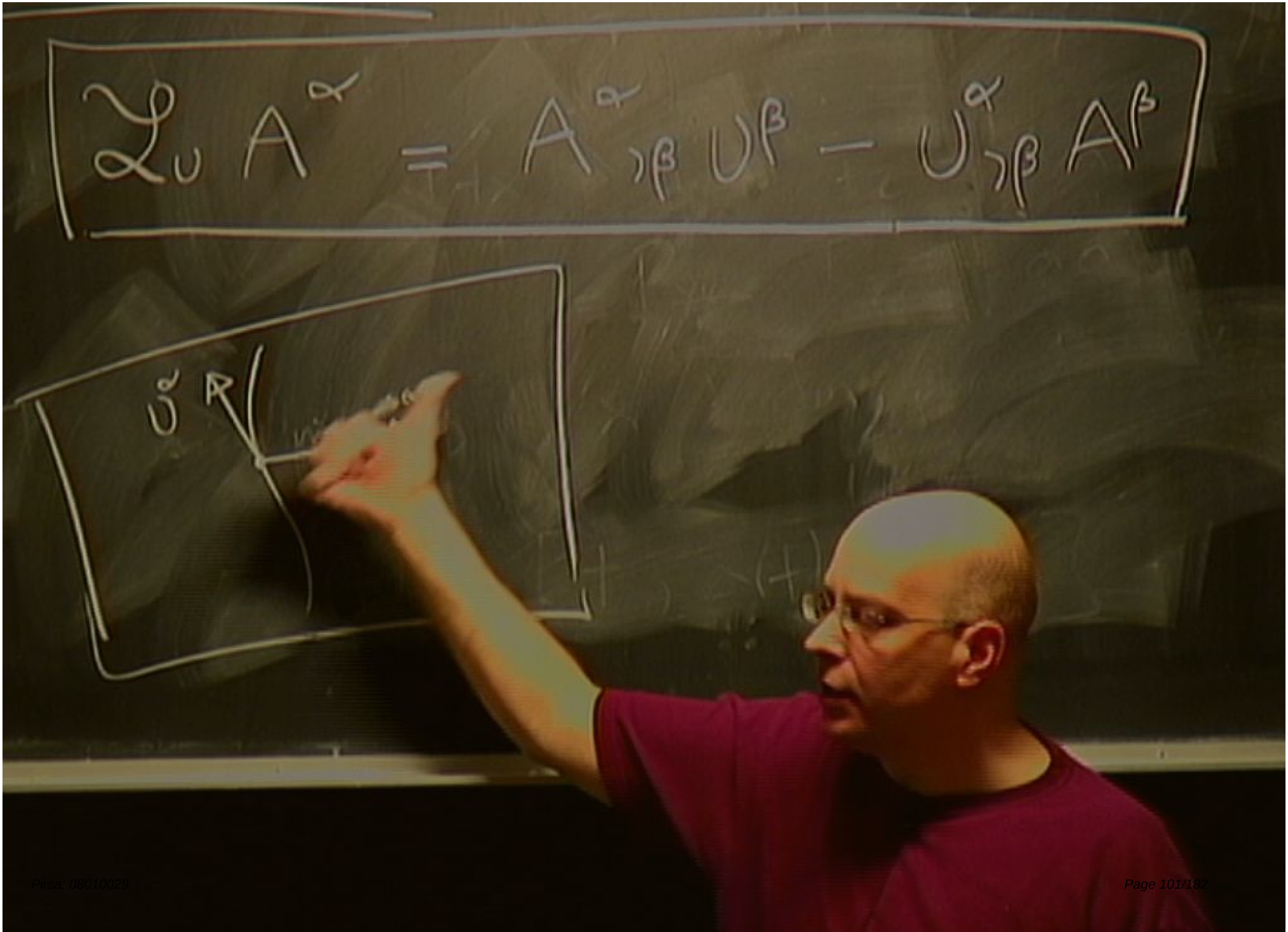

$\mathcal{L}_U A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta$

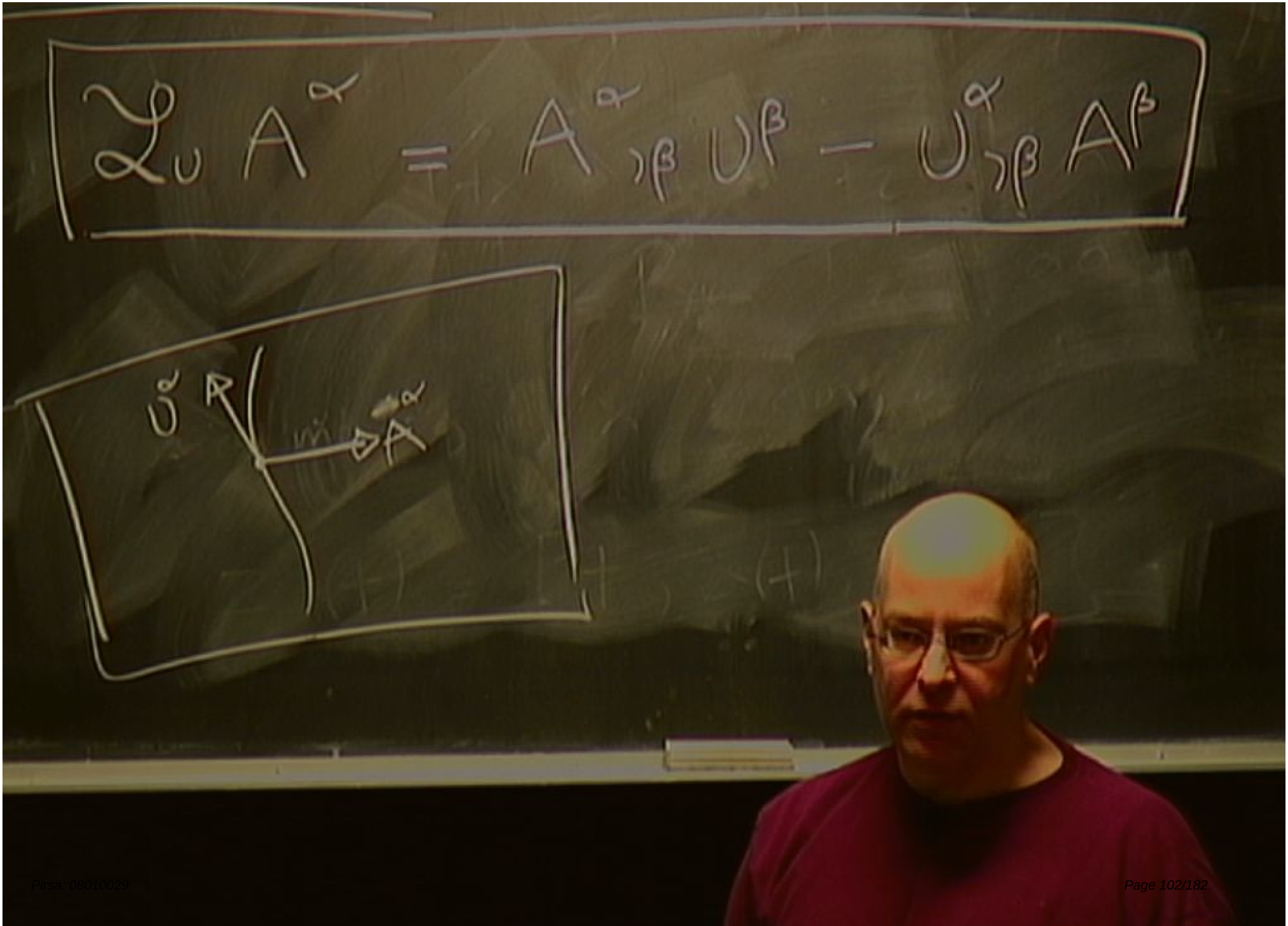
$$\mathcal{L}_U A^{\alpha} = A^{\alpha}{}_{;\beta} U^{\beta} - U^{\alpha}{}_{;\beta} A^{\beta}$$

$$\boxed{\mathcal{L}_U A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta}$$



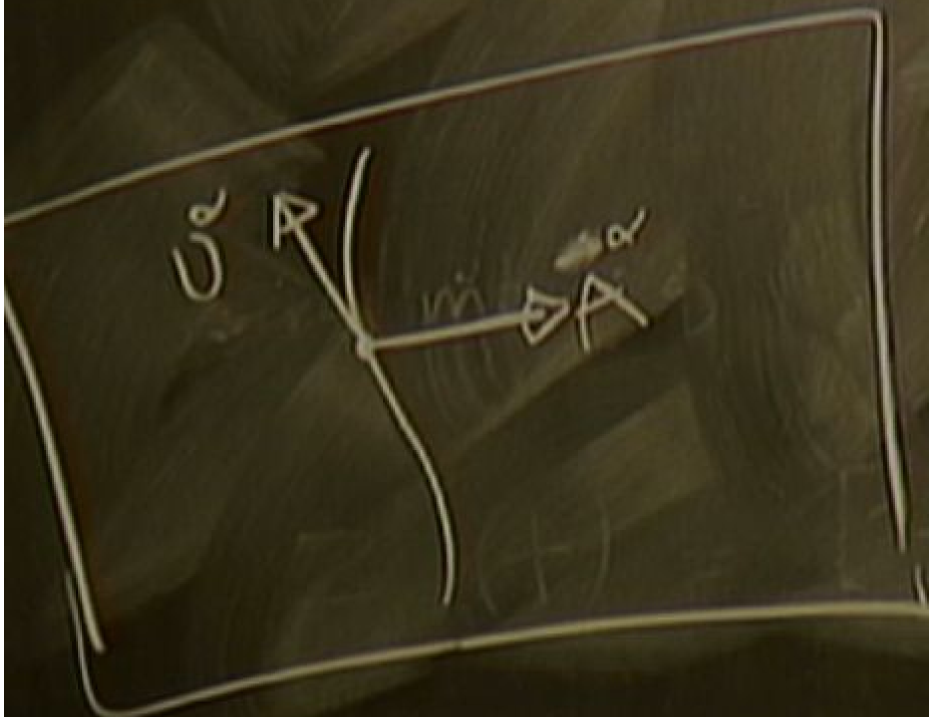


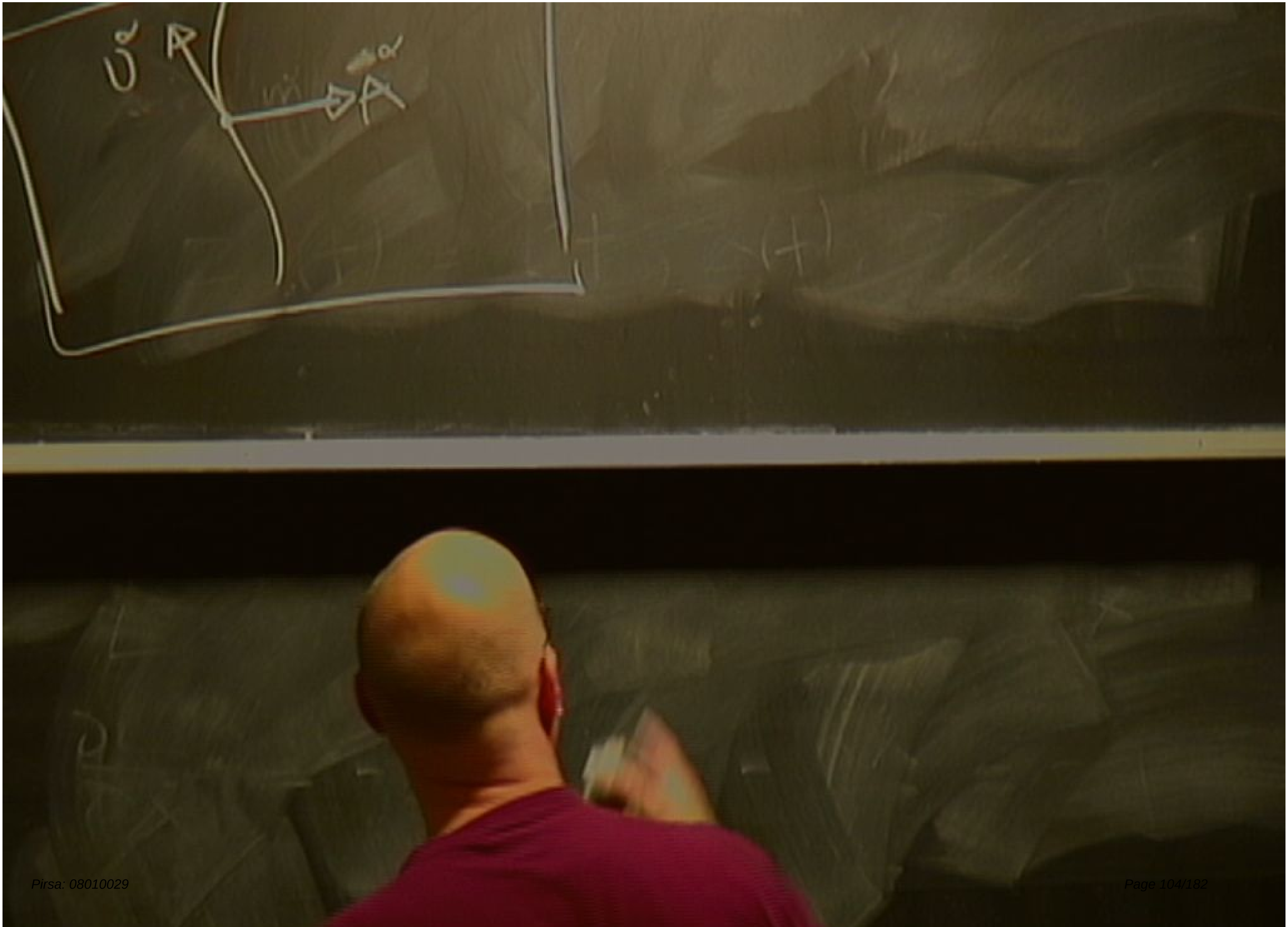

$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta$

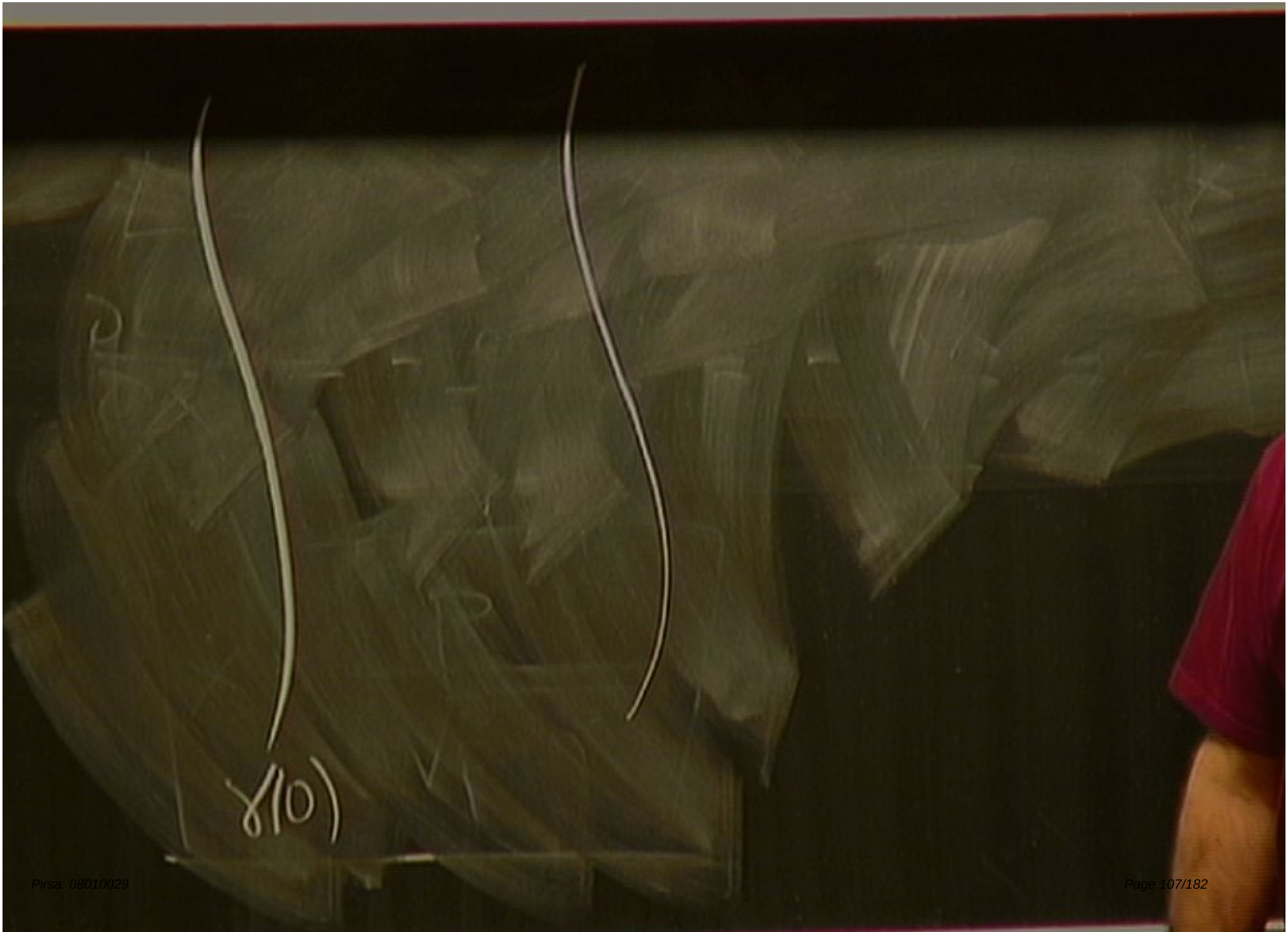


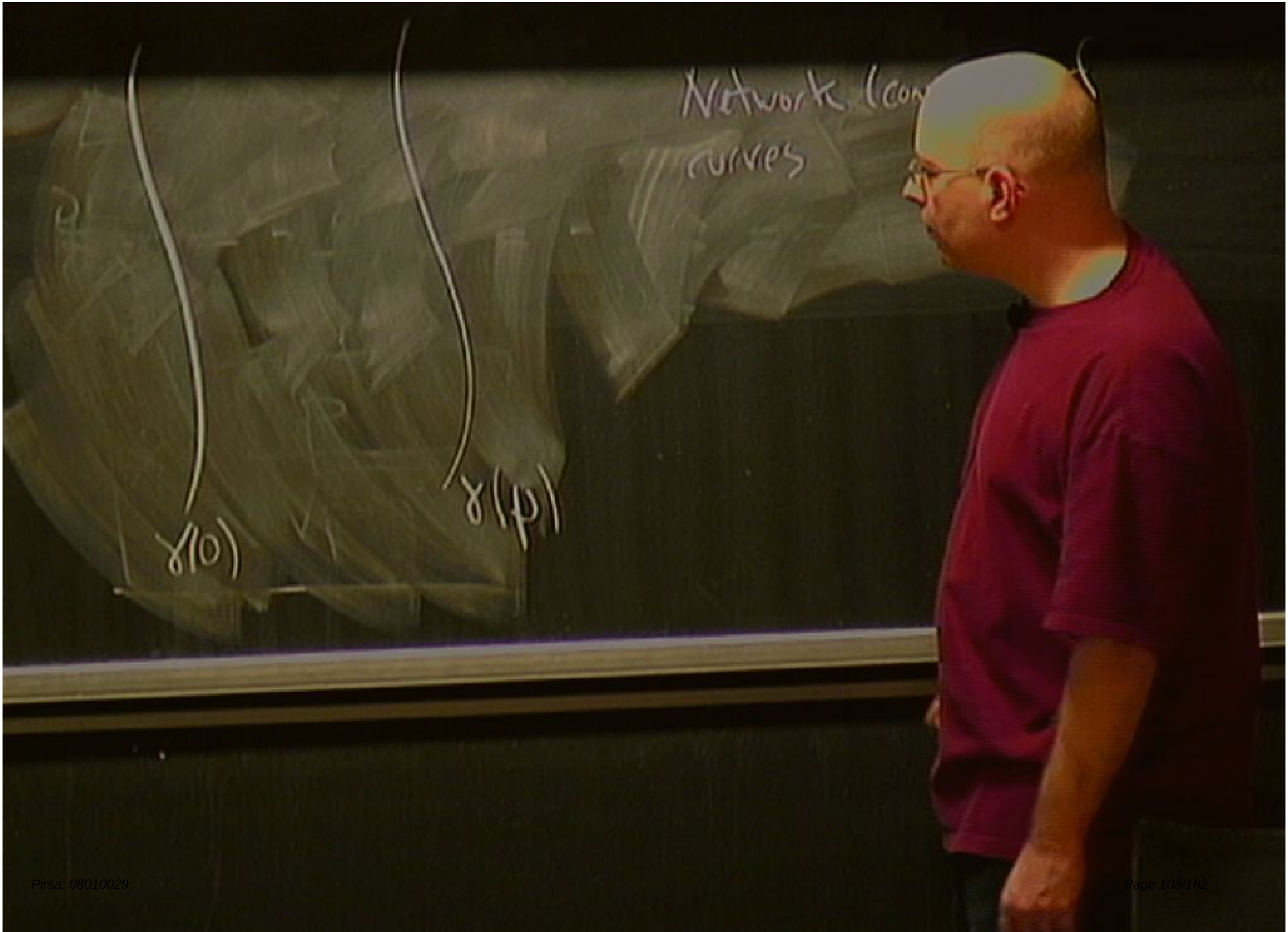
$\mathcal{L}_U A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta$
U^α
A^α

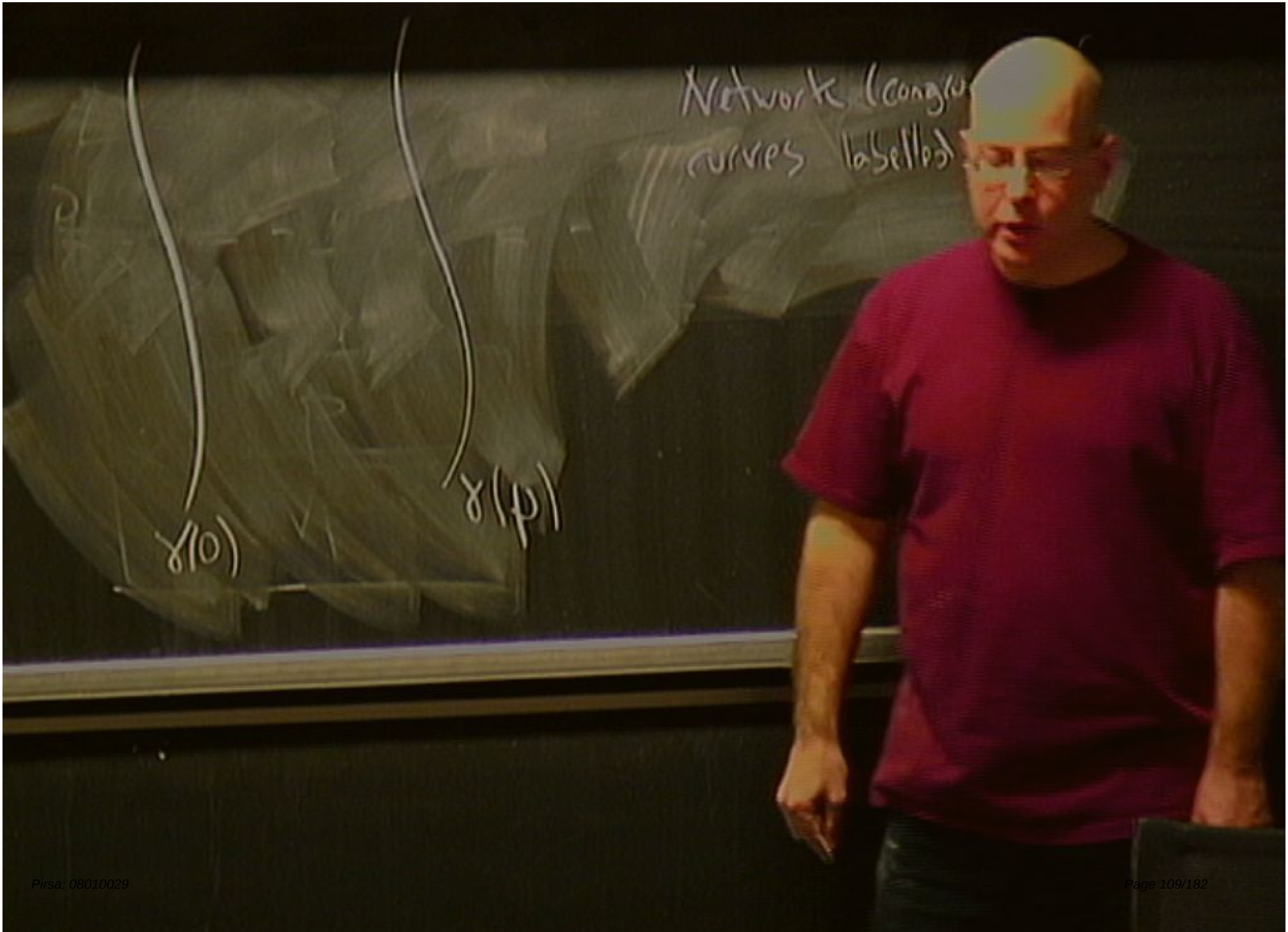
$$\boxed{\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta}$$

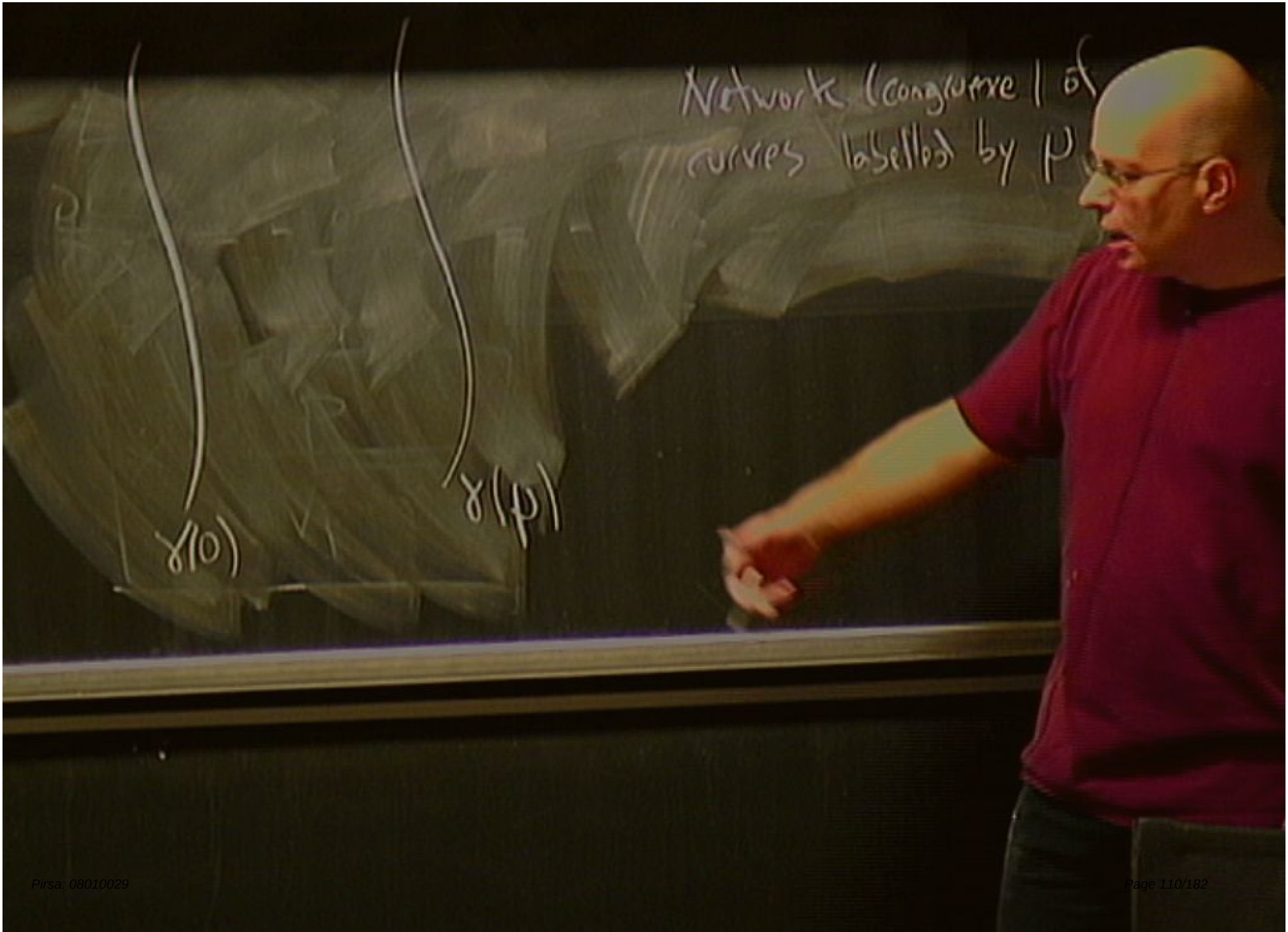


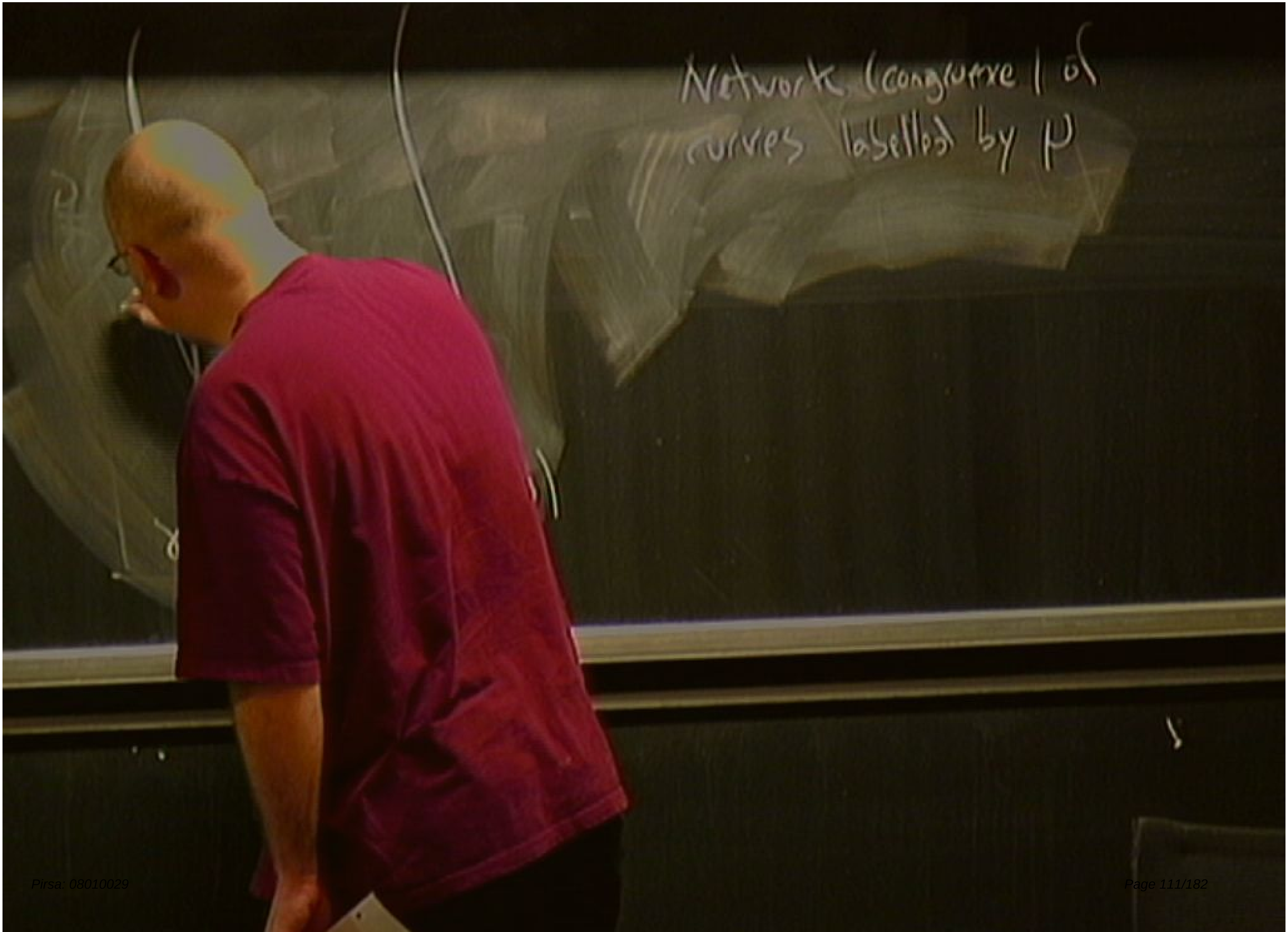



γ(0)

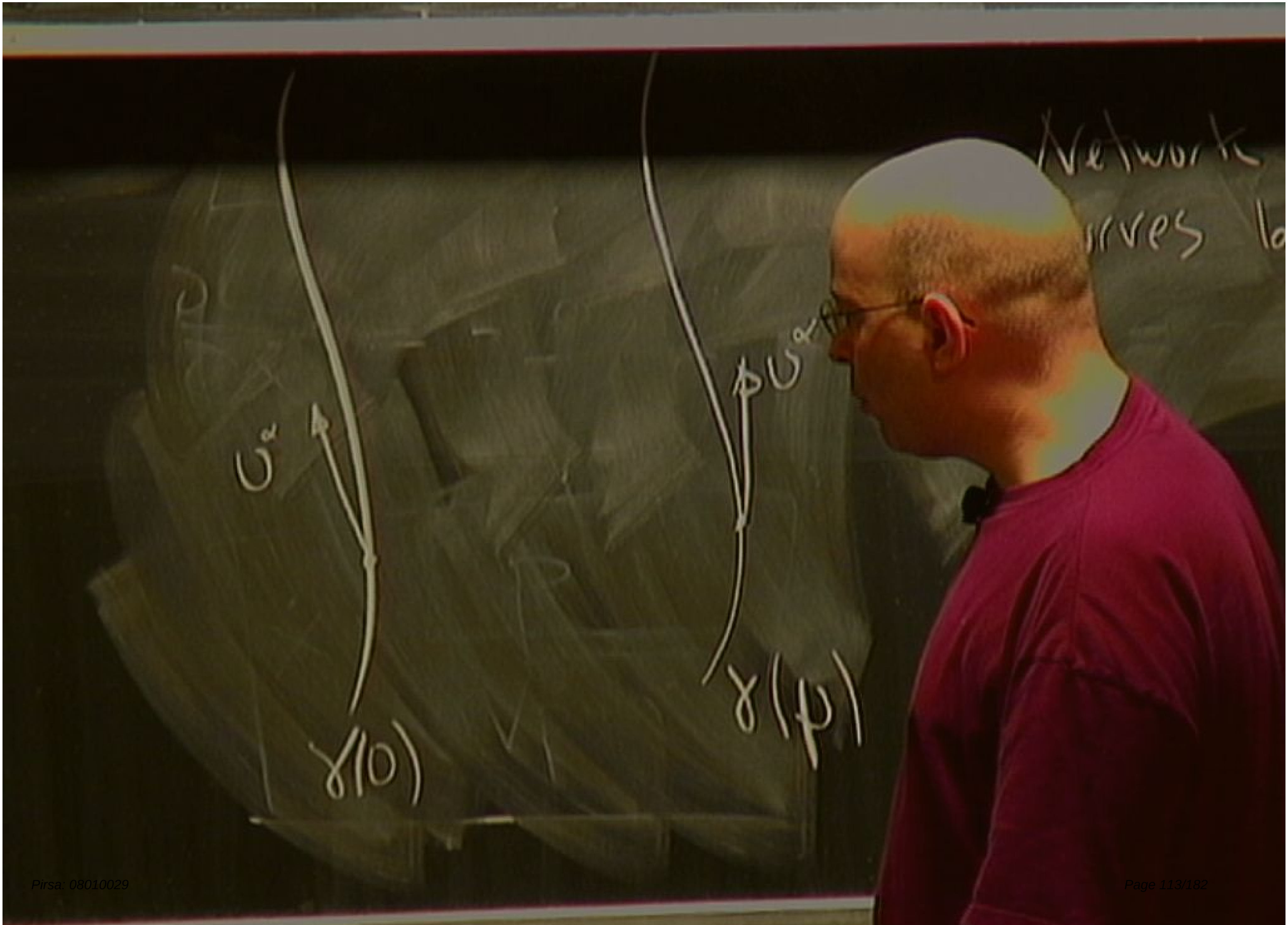

Network
curves
γ(0)
γ(p)

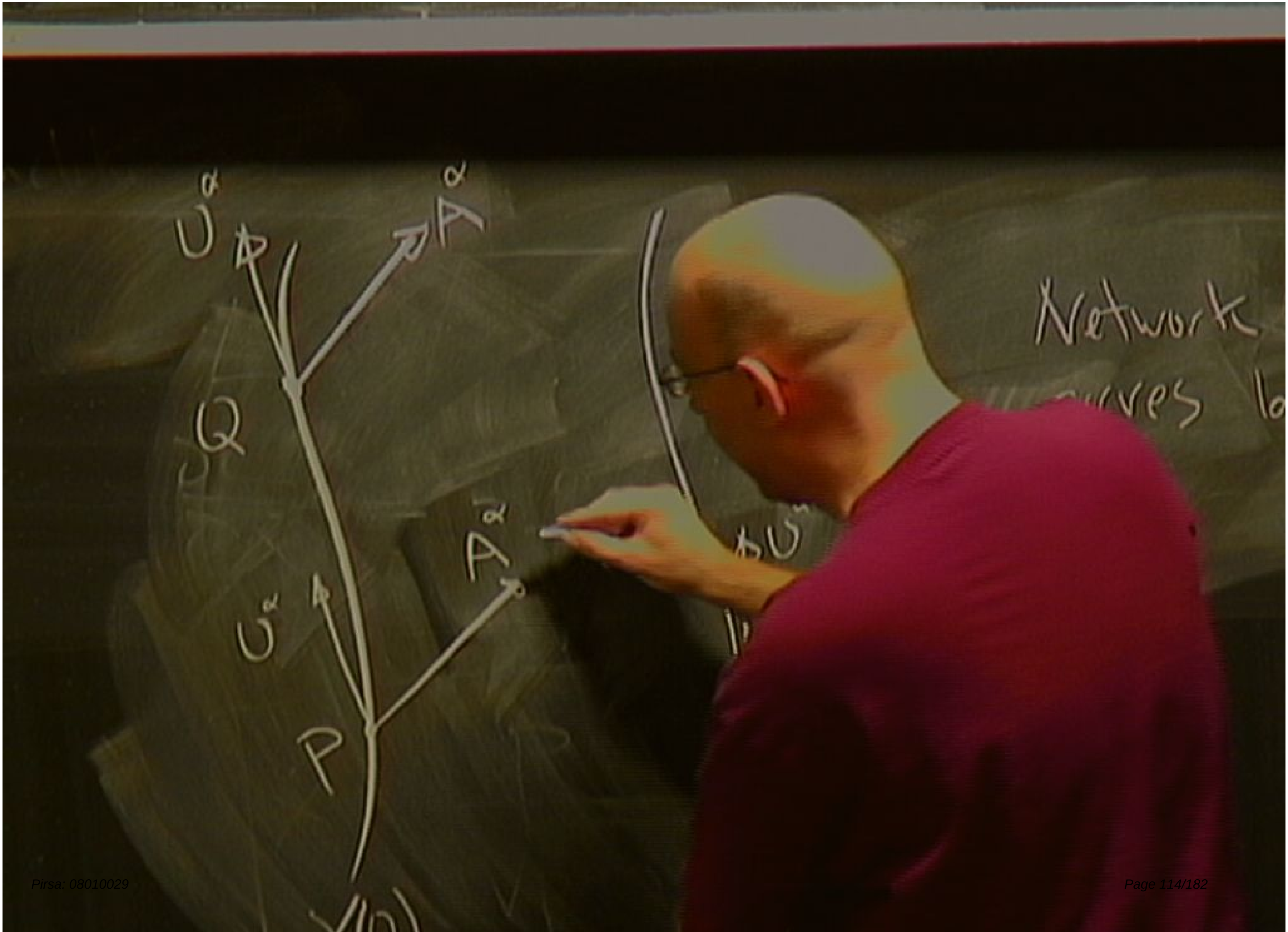

Network (congru
curves labelled
$\gamma(0)$
$\gamma(p)$

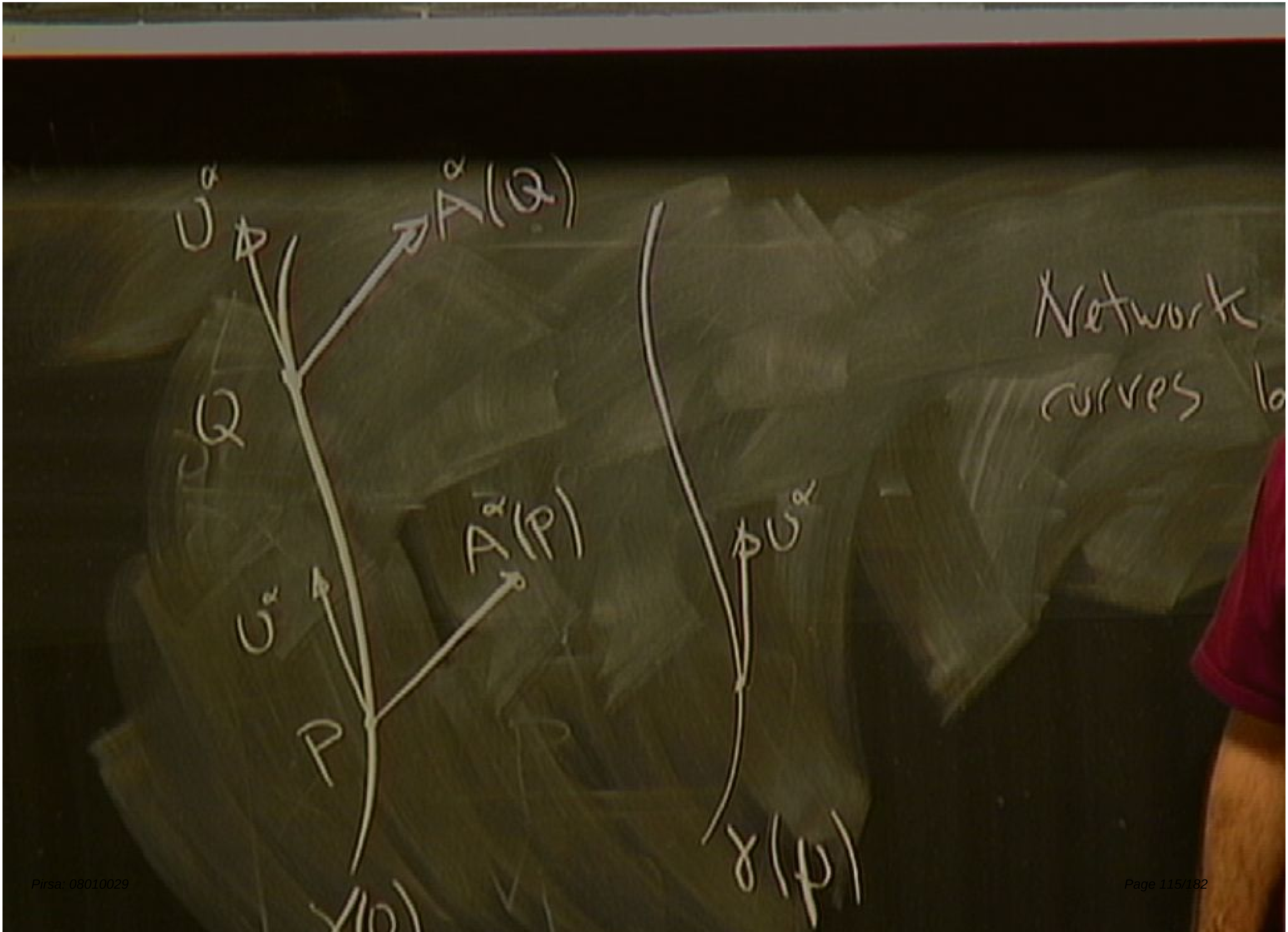

Network (congruence) of
curves labelled by p
γ(0)
γ(p)

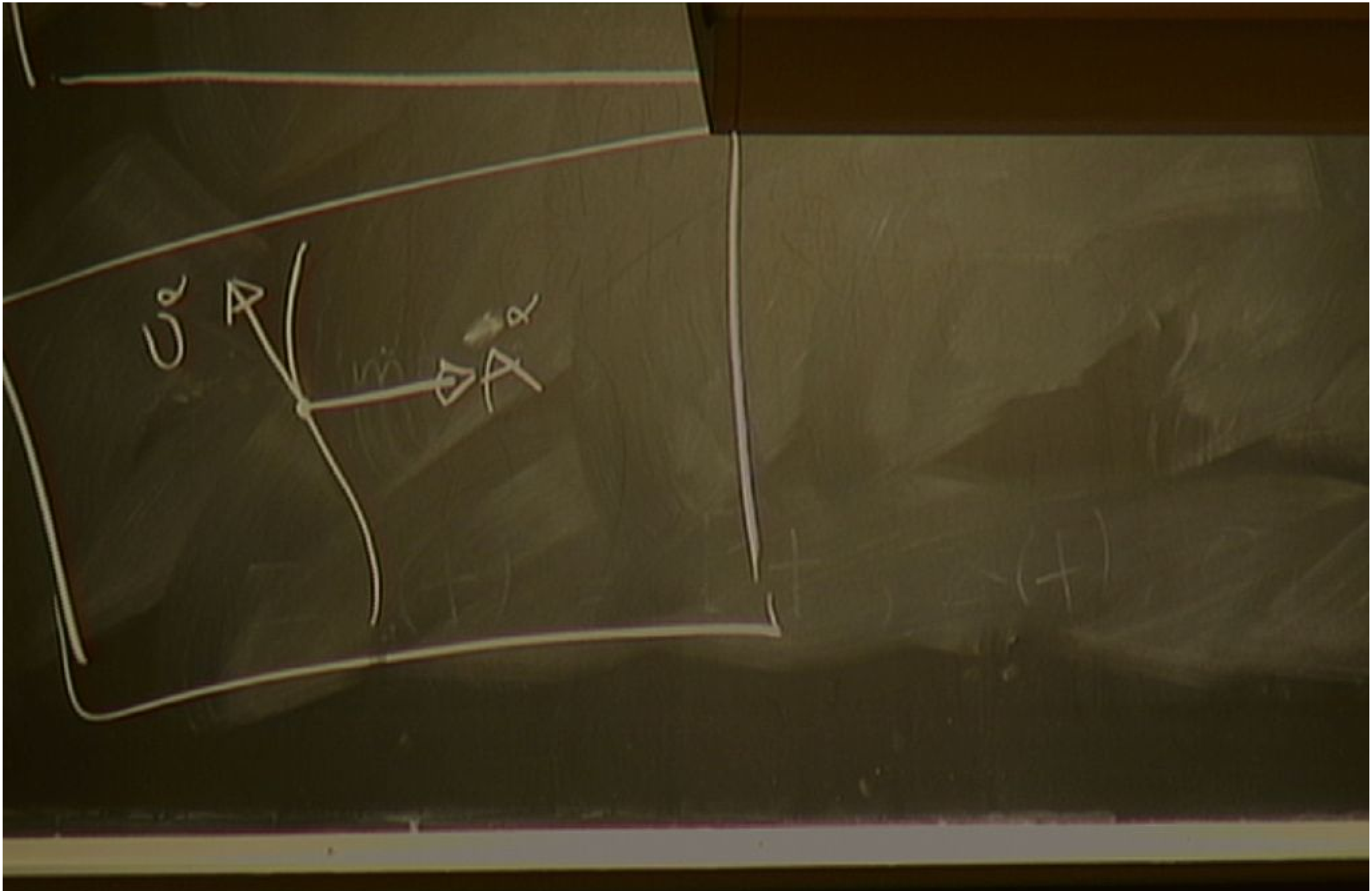

Network (congruence) of
curves labelled by

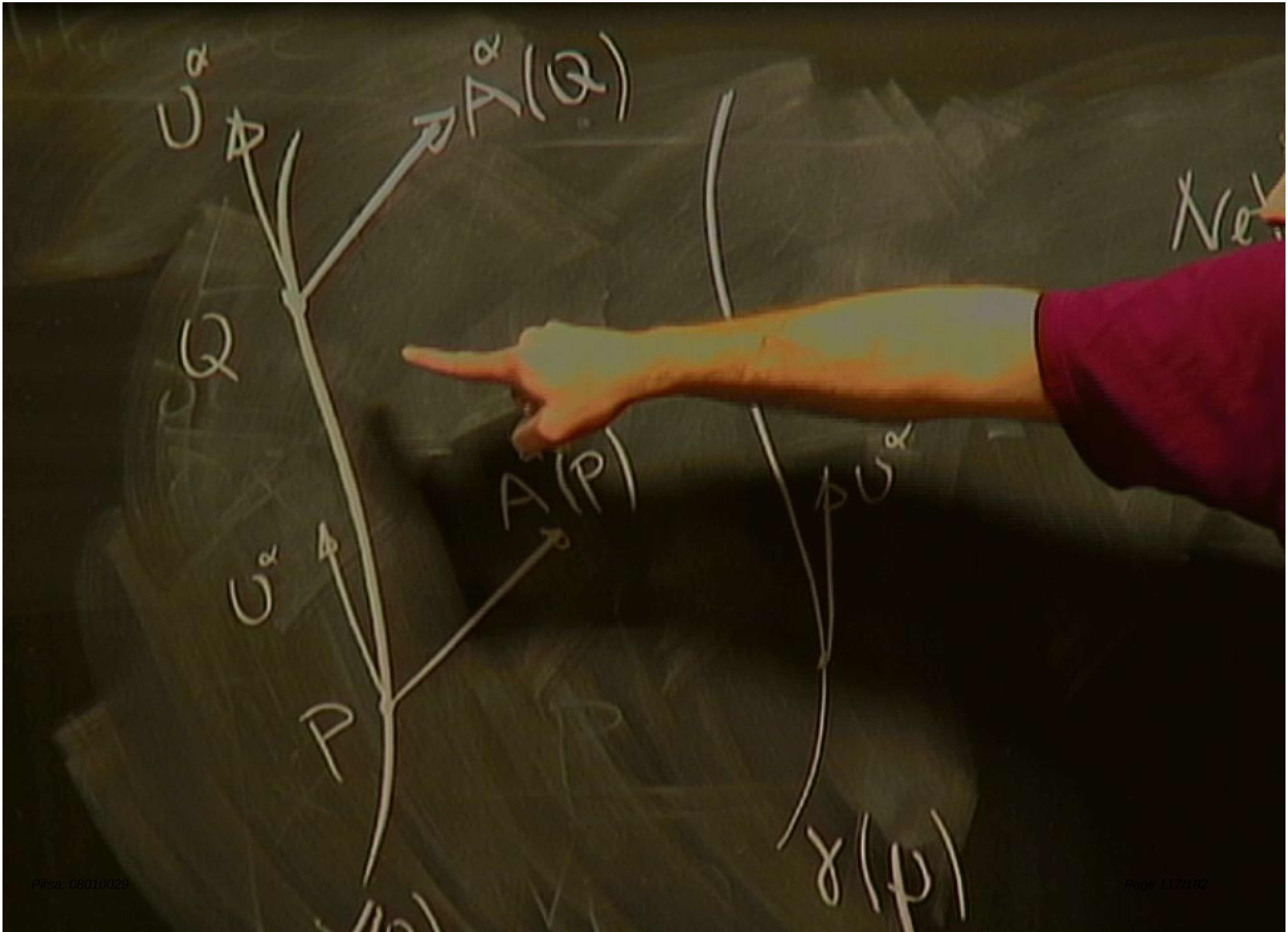
Network (congruence) of curves labelled by μ

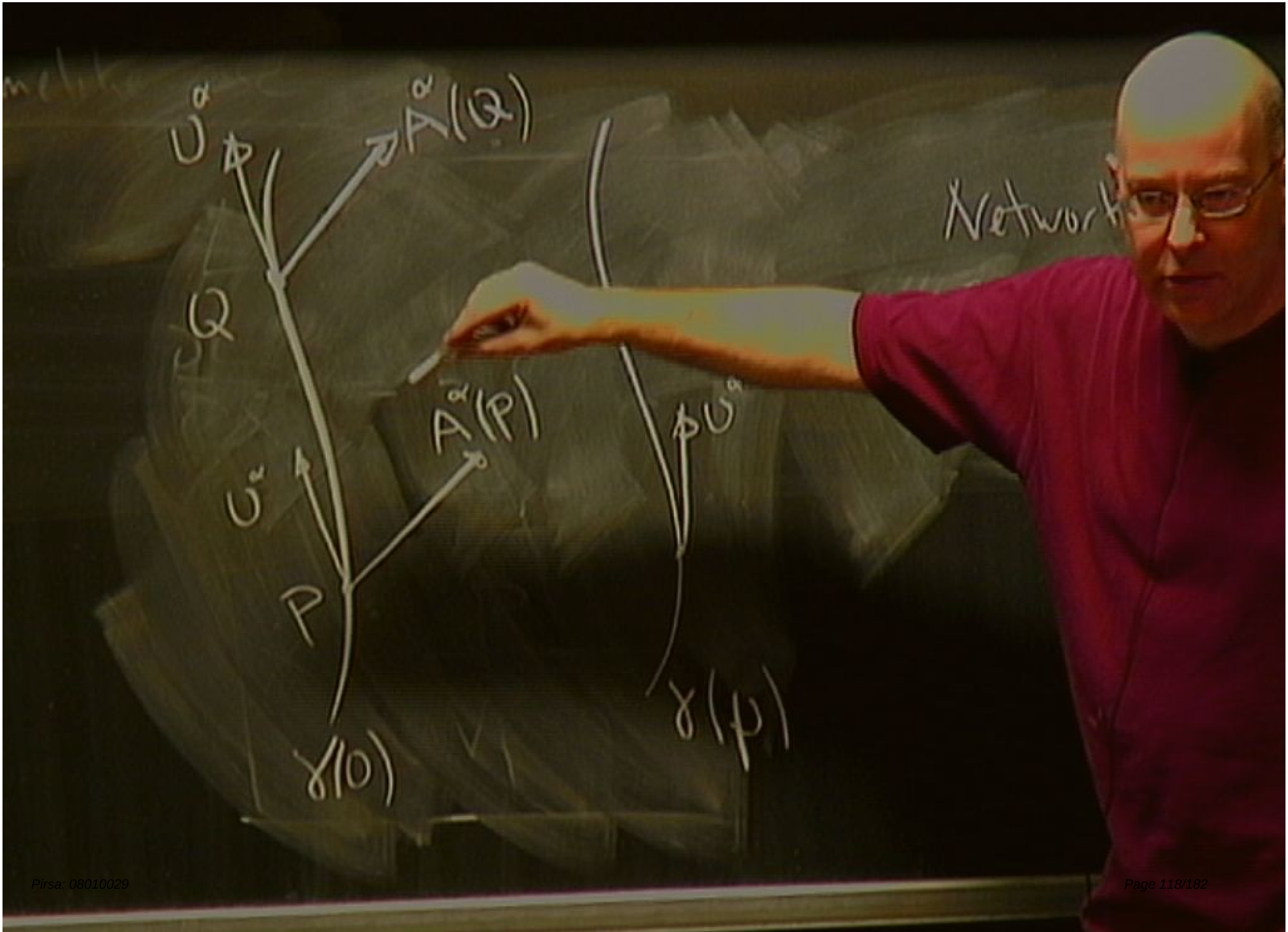

Network
γ(0)
γ(p)

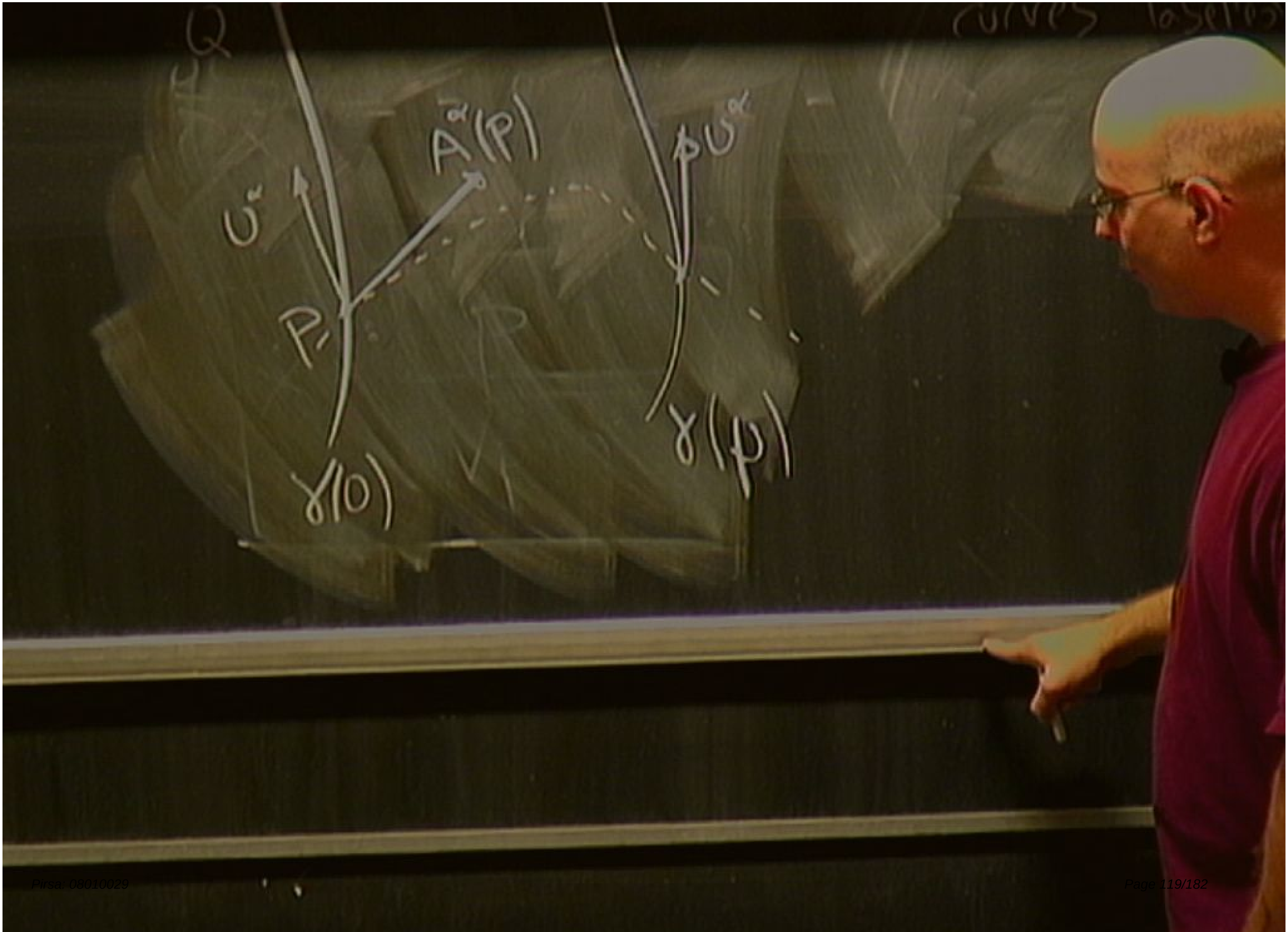

Network


U^α
A^α(Q)
Q
A^α(P)
U^α
P
U^α
γ(p)
Network
curves

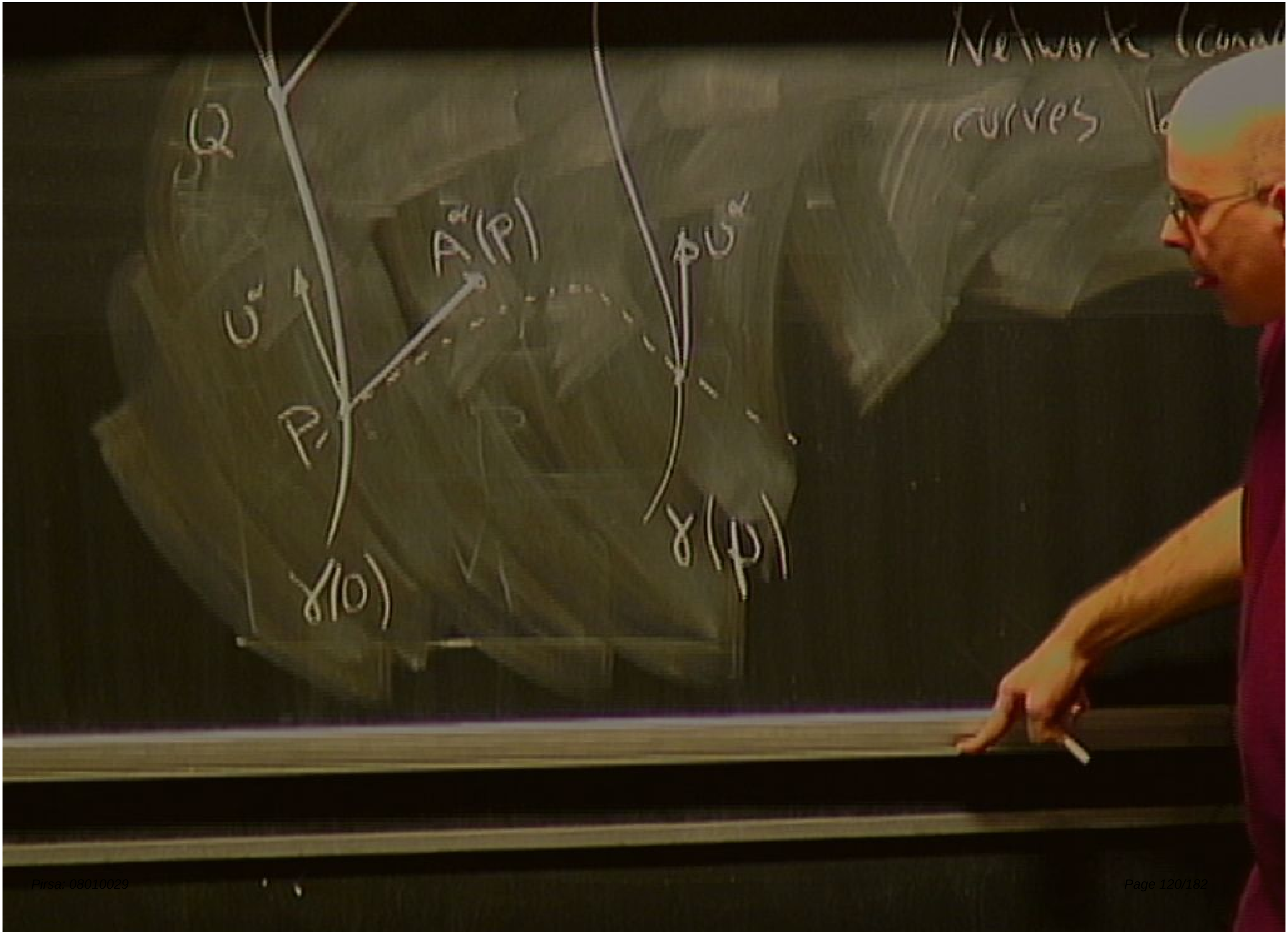


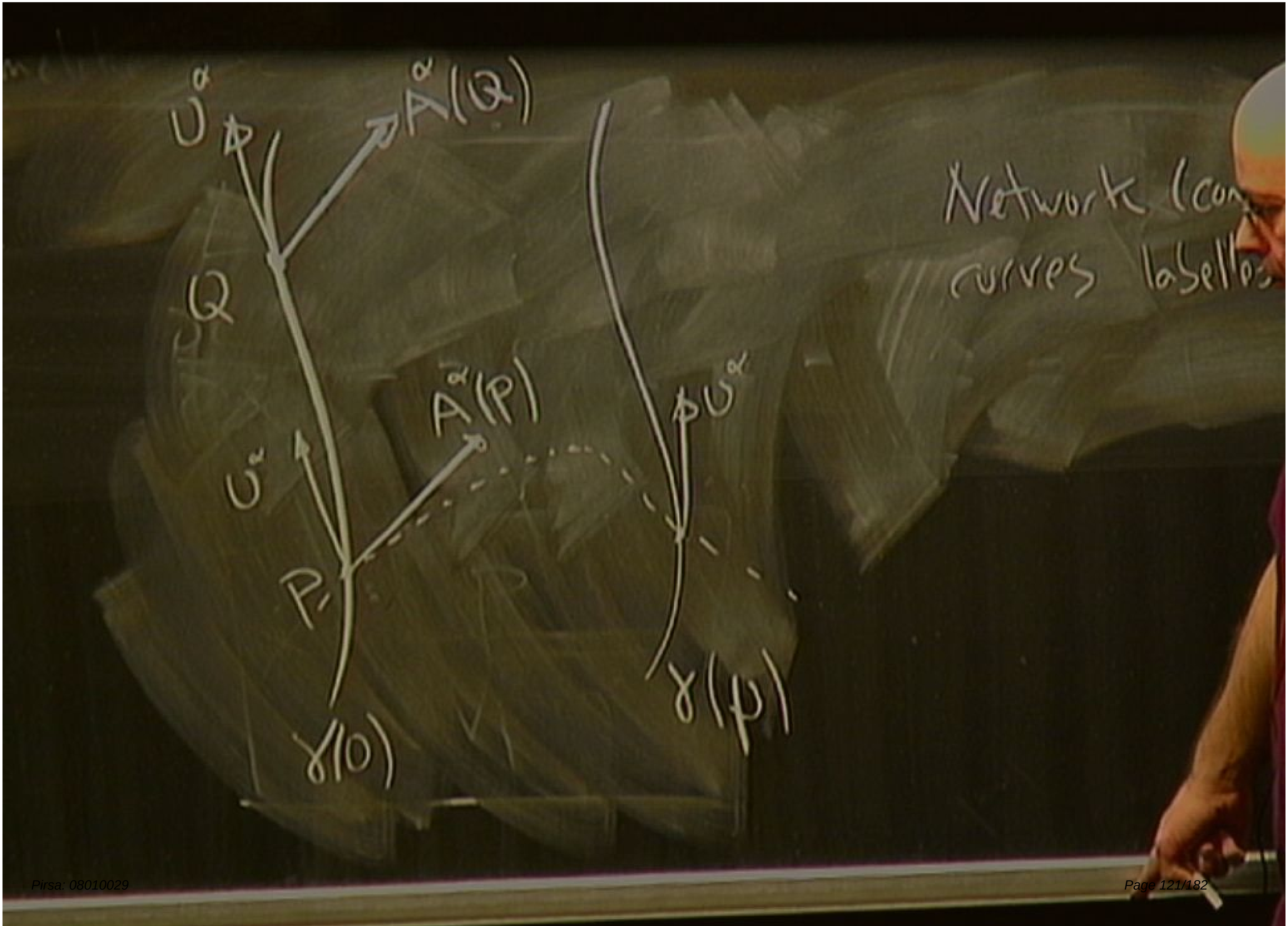

U^α
$A^\alpha(Q)$
Q
U^α
$A(P)$
P
U^α
$\gamma(p)$


U^α
$A^\alpha(Q)$
Q
$A^\alpha(P)$
U^α
P
$\gamma(0)$
μU^α
$\gamma(\mu)$
Network

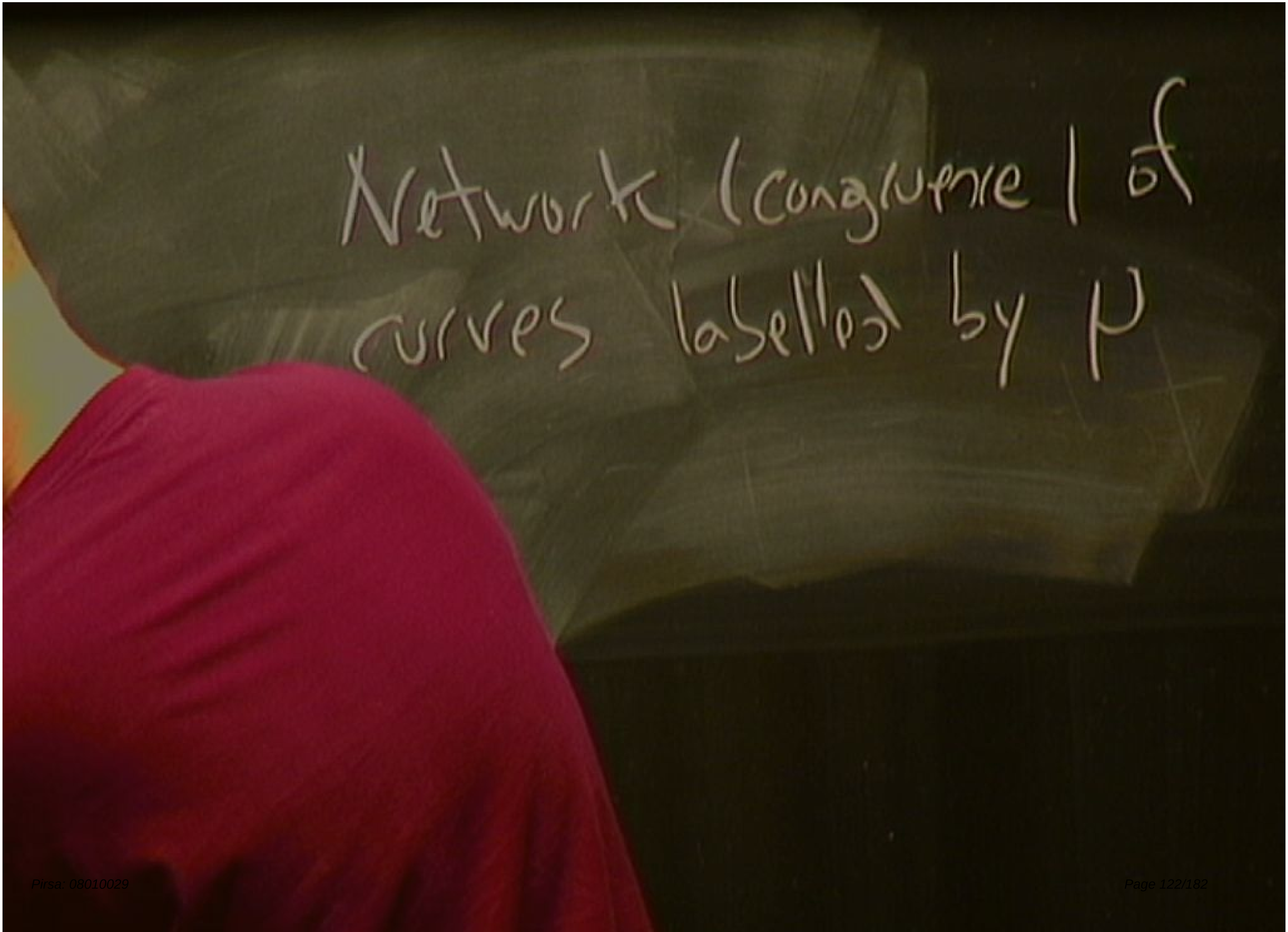


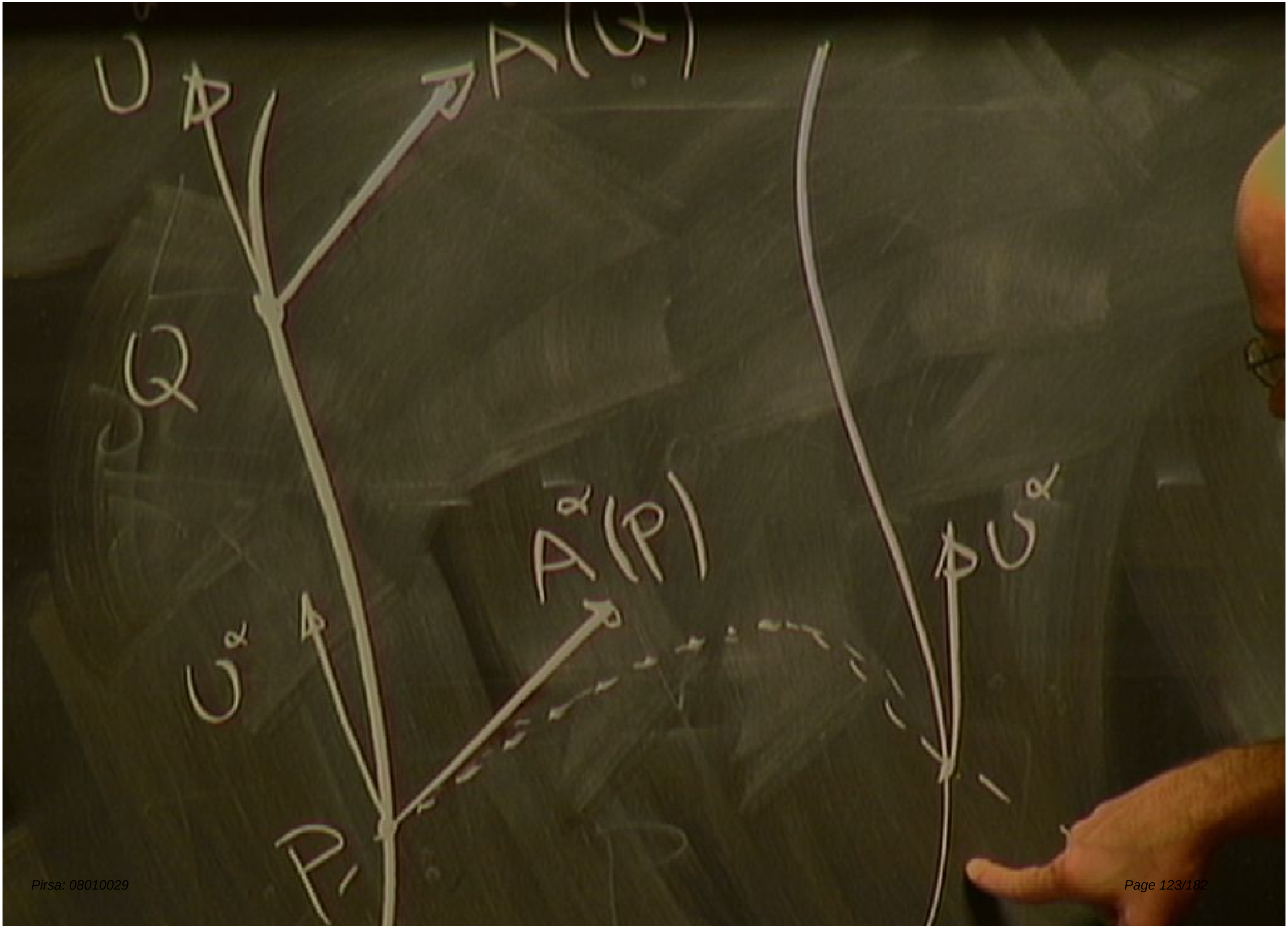
Q
U^α
$A^\alpha(p)$
pU^α
P
$\gamma(0)$
$\gamma(p)$


Network
curves
Q
P
U^α
$A^\alpha(P)$
$\gamma(0)$
$\gamma(\lambda)$

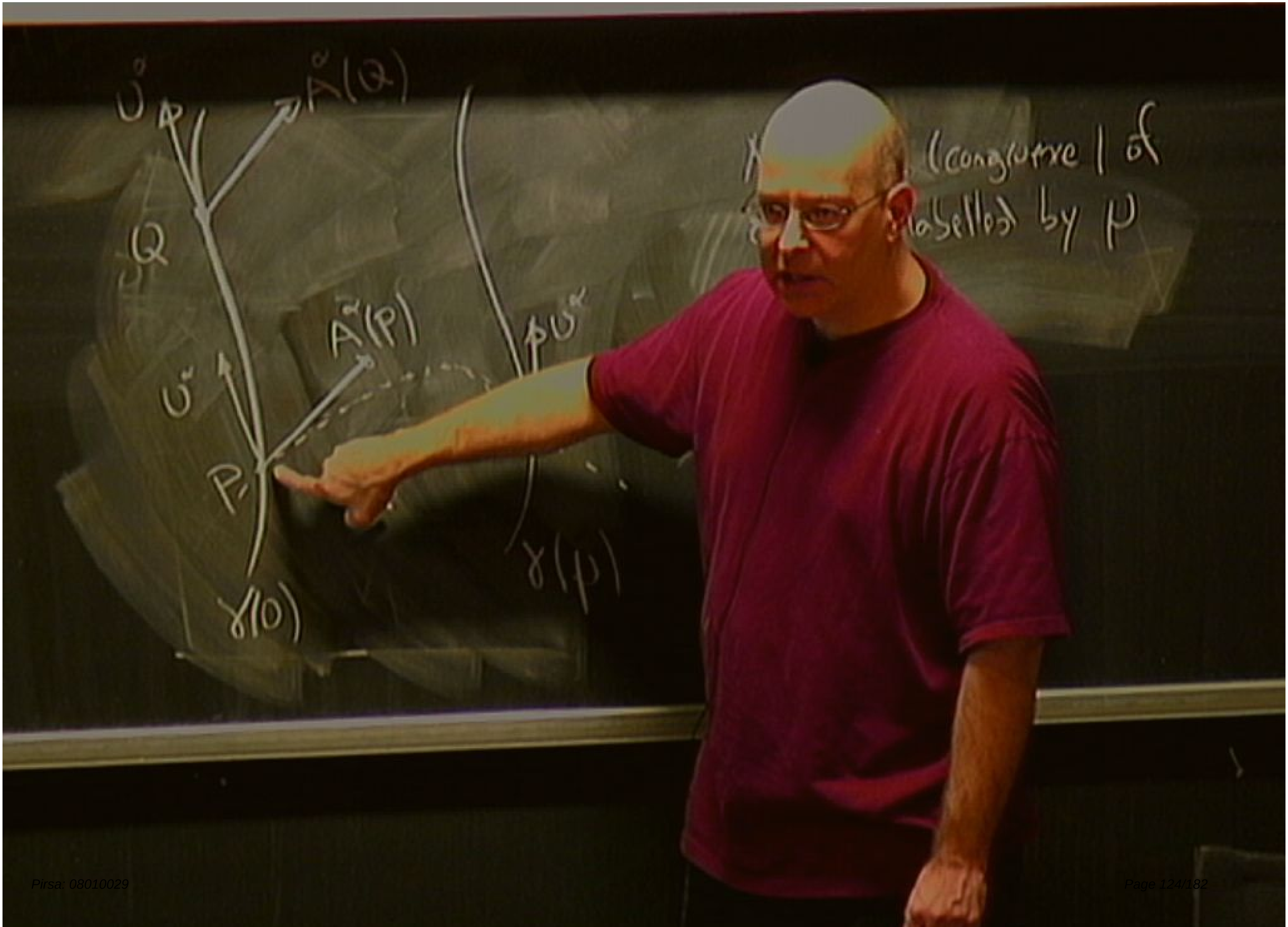


U^α
$A^\alpha(Q)$
Q
$A^\alpha(P)$
U^α
P
U^α
$\gamma(0)$
$\gamma(p)$
Network (con
curves labelle

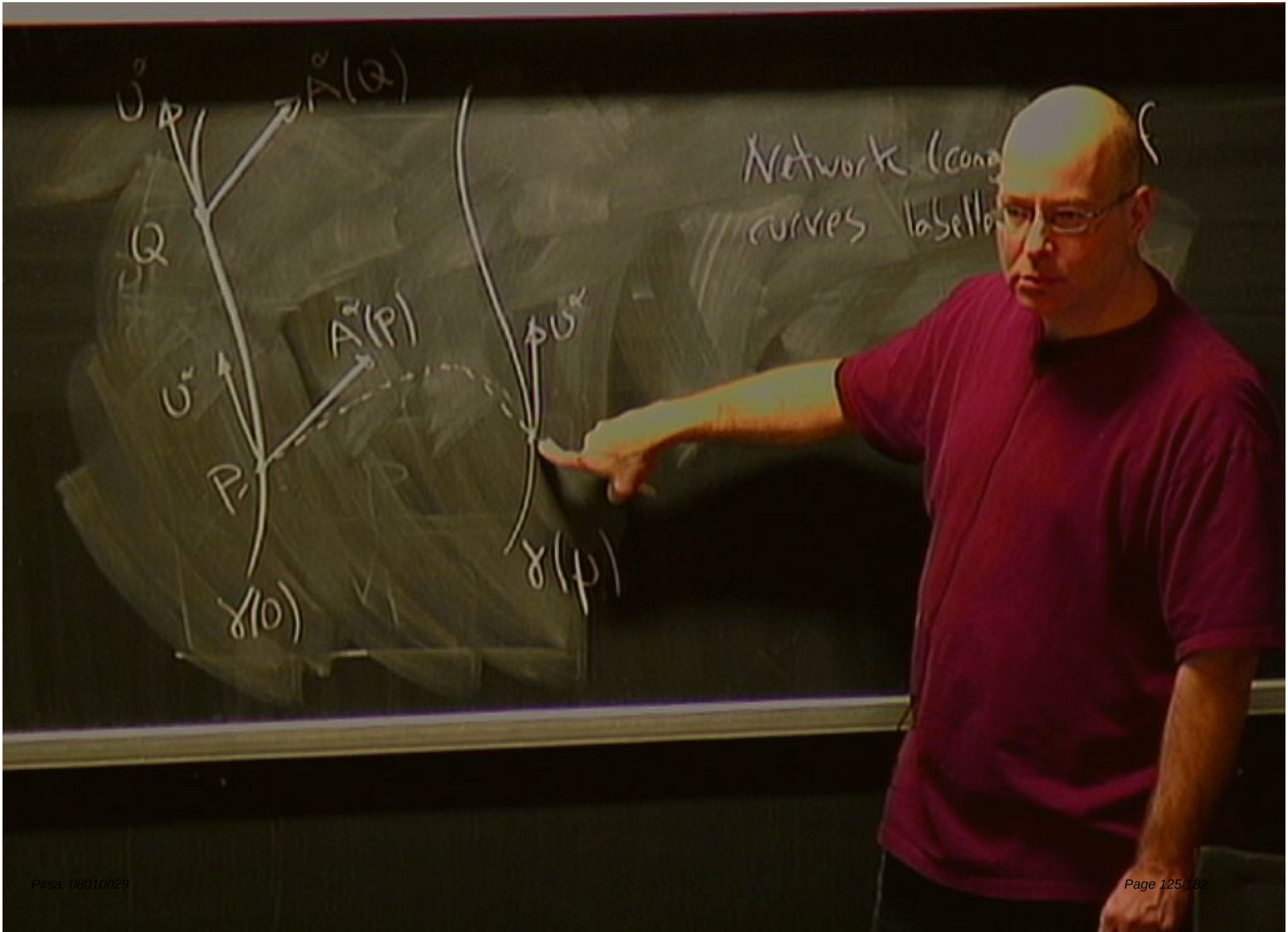

Network (congruence) of
curves labelled by μ



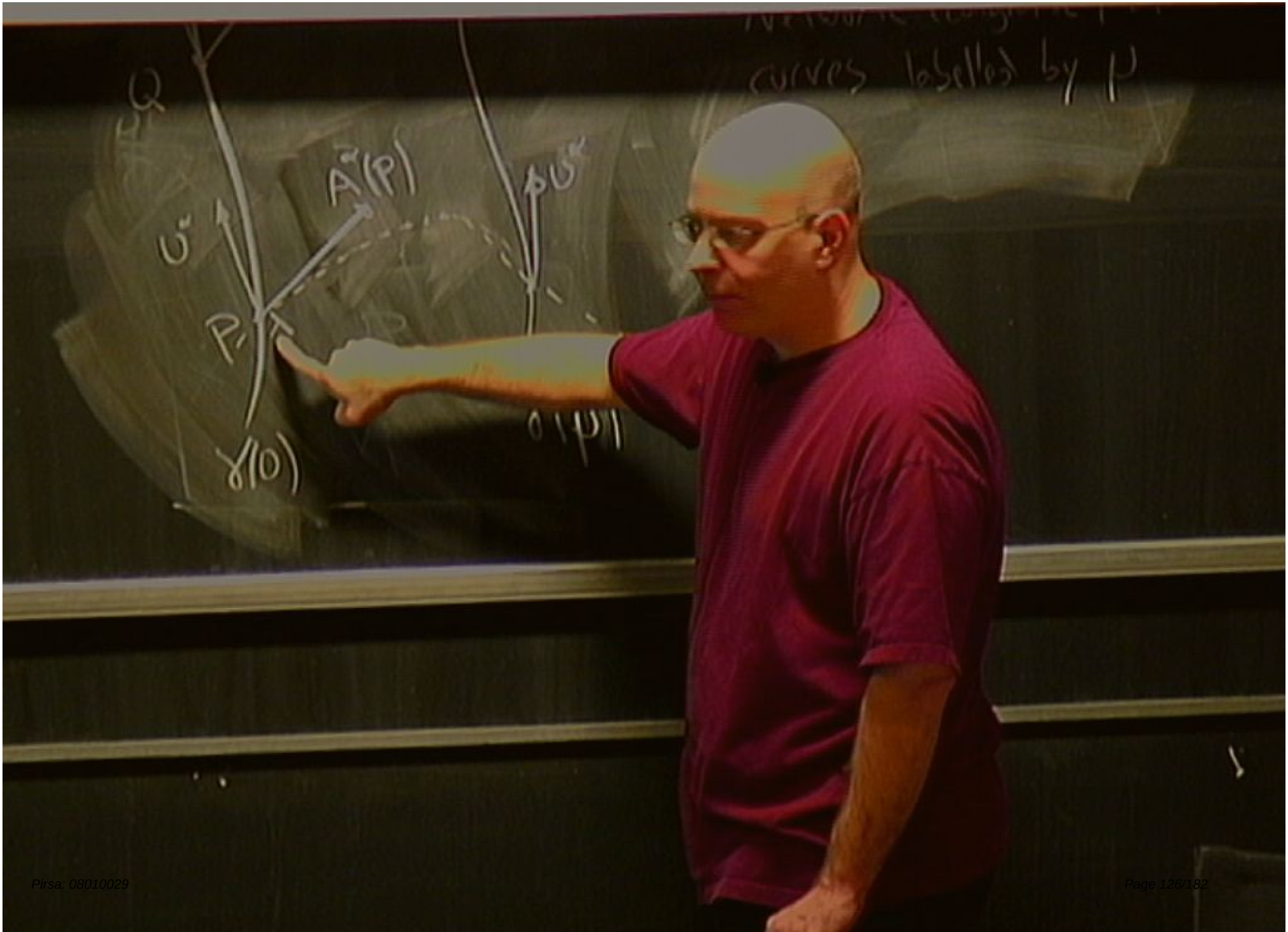
U
A(Q)
Q
A^α(P)
U^α
P
U^α

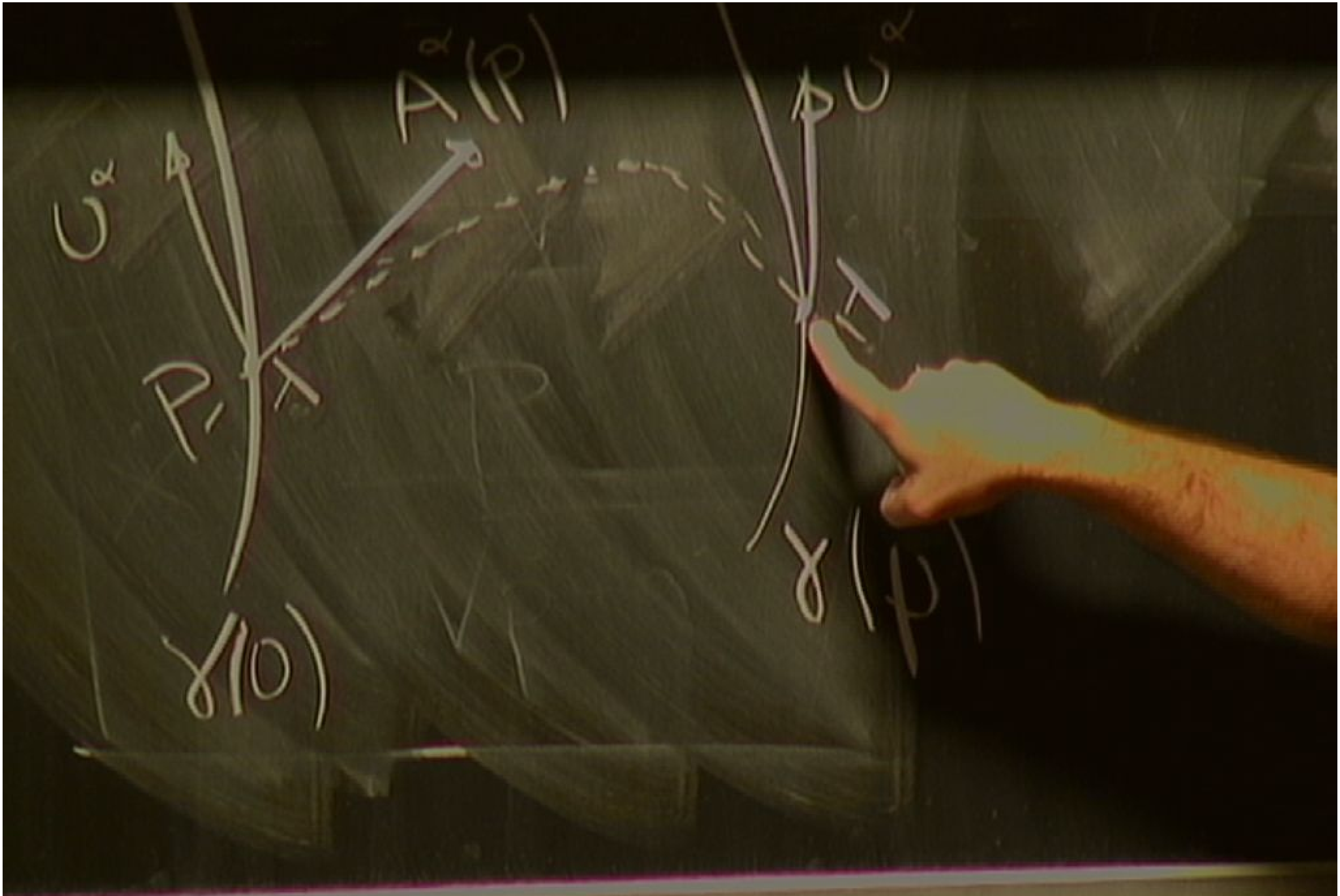


U^α
$A^\alpha(Q)$
Q
$A^\alpha(P)$
U^α
P
$\gamma(0)$
U^α
$\gamma(\mu)$
(congruence) of
labelled by μ

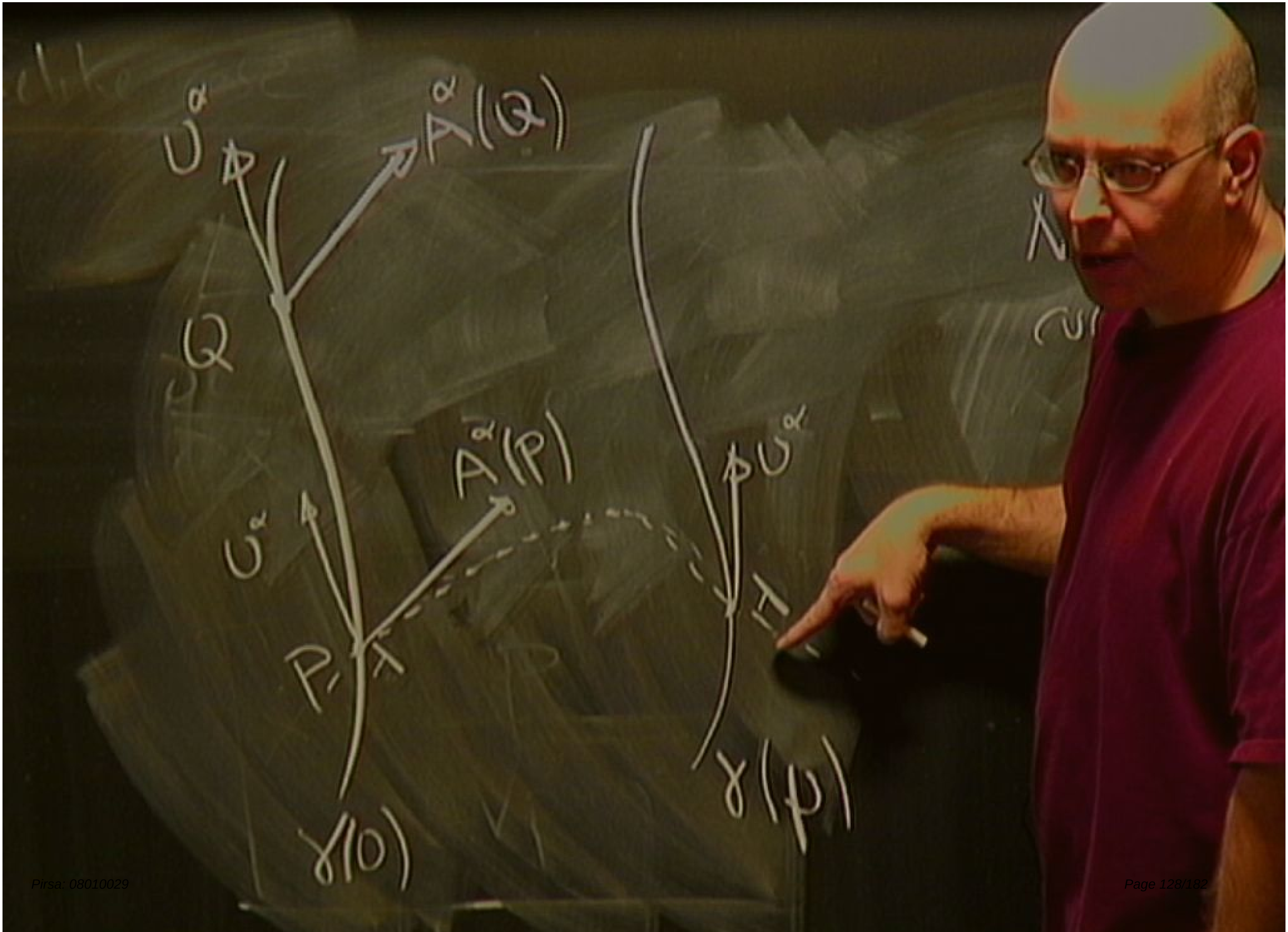


U^α
$A^\alpha(Q)$
Q
$A^\alpha(P)$
U^α
U^α
P
$\gamma(0)$
$\gamma(p)$
Network (cong
curves labell

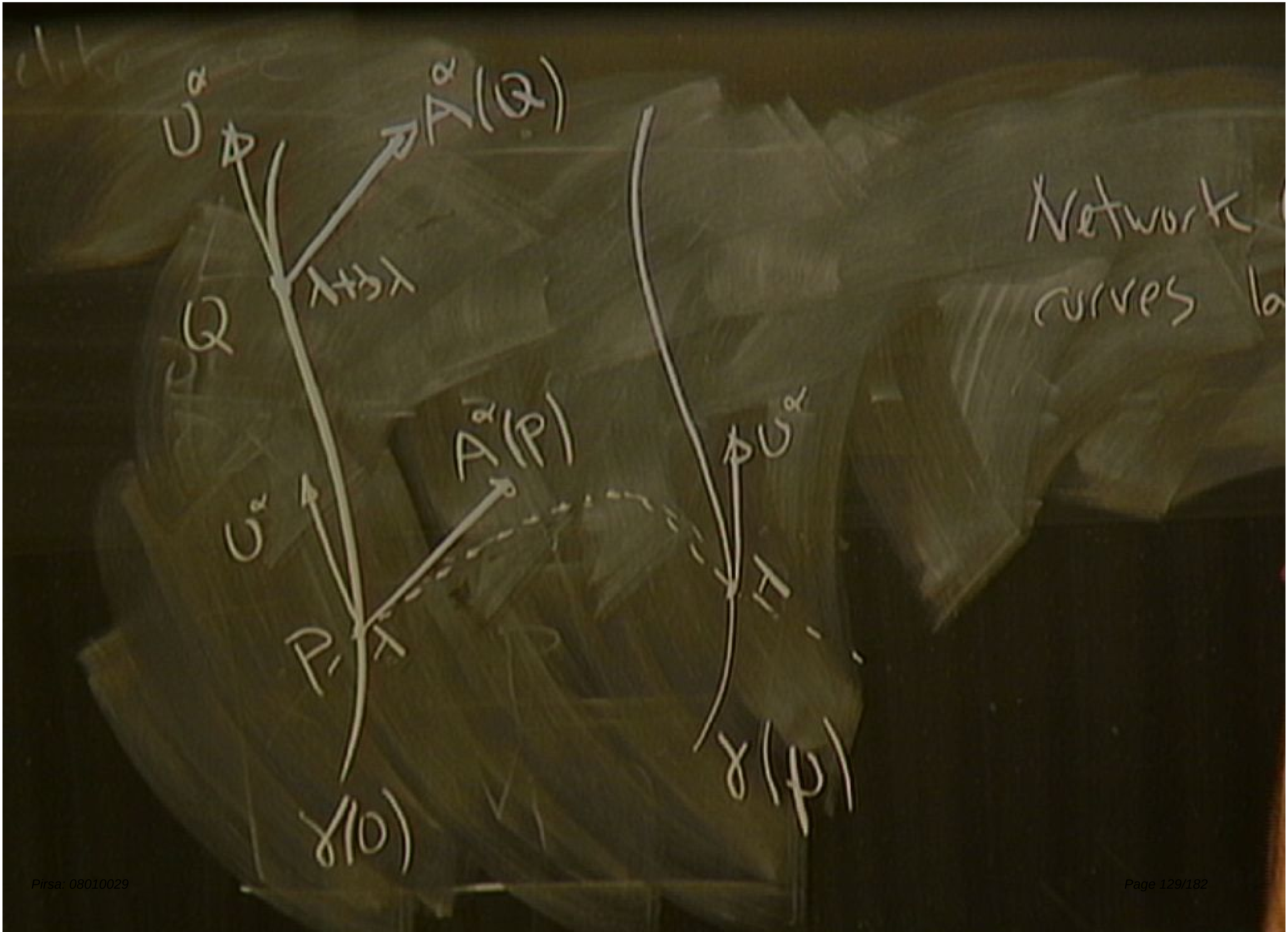

curves labelled by μ
Q
A^α(p)
U^α
P
γ(0)
σ(μ)

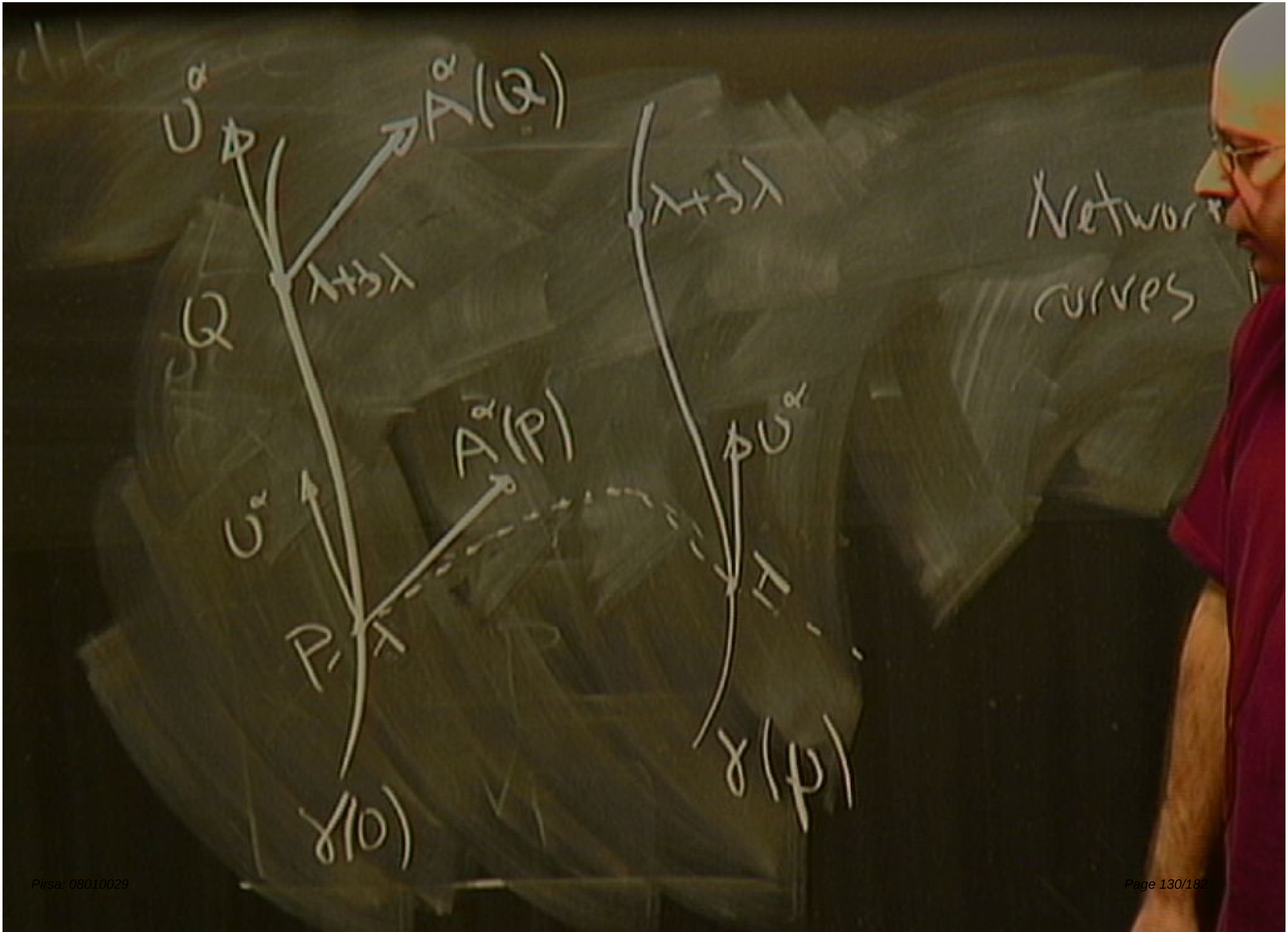


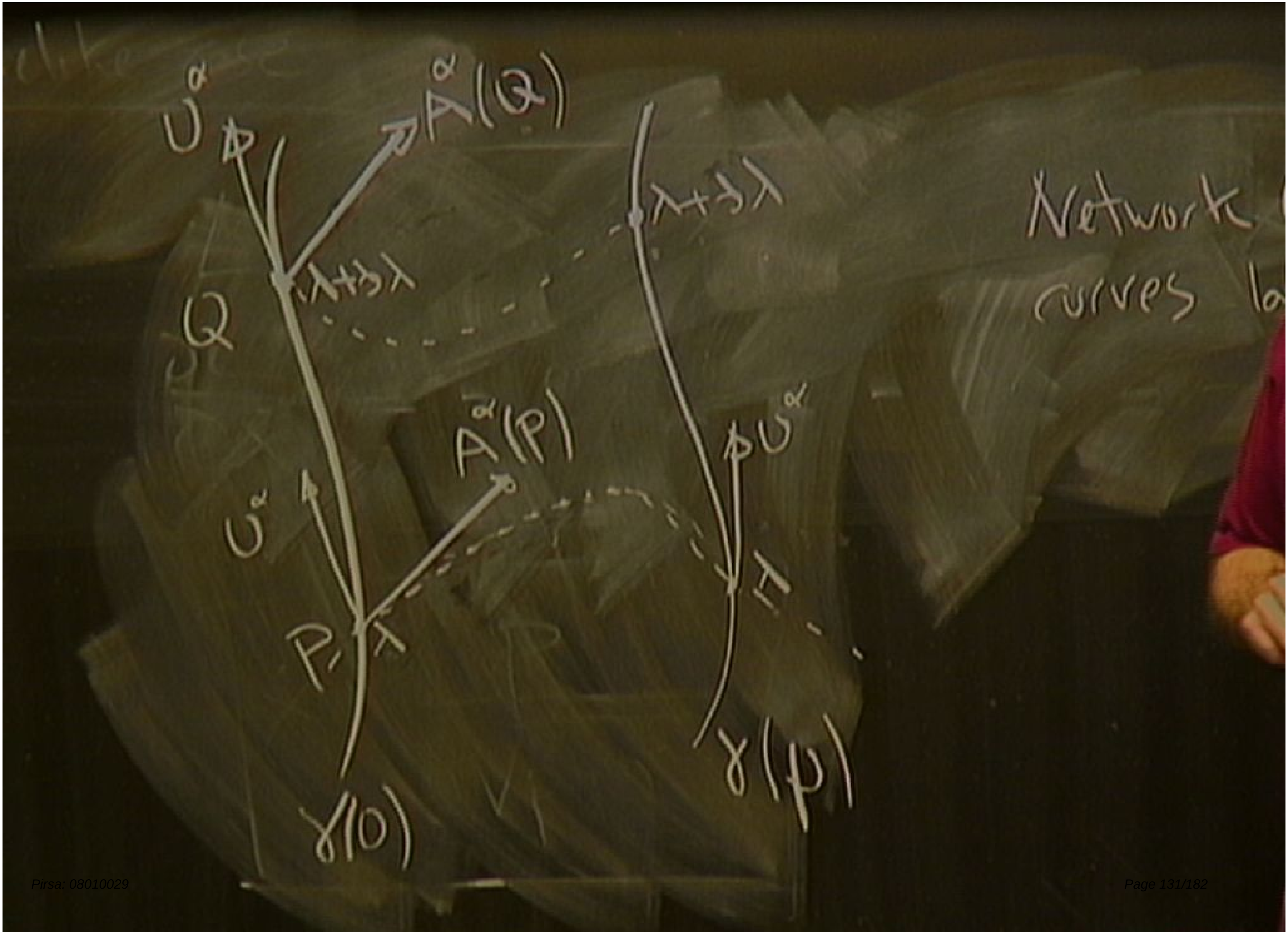
U^α
$A^\alpha(P)$
P
$\gamma(0)$
$\gamma(\mu)$

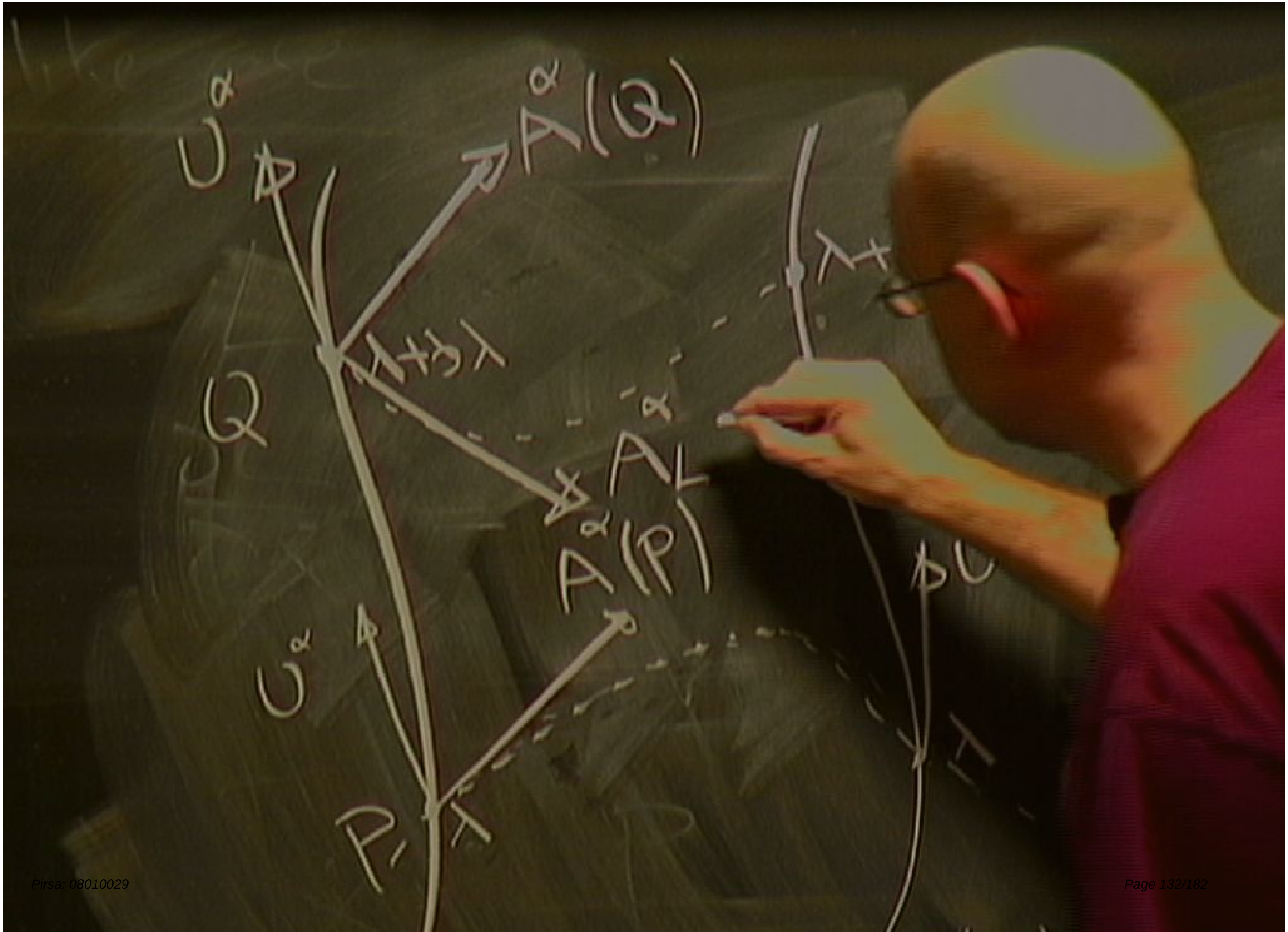


U^α
$A^\alpha(Q)$
Q
U^α
$A^\alpha(P)$
P
U^α
$\gamma(0)$
$\gamma(\lambda)$

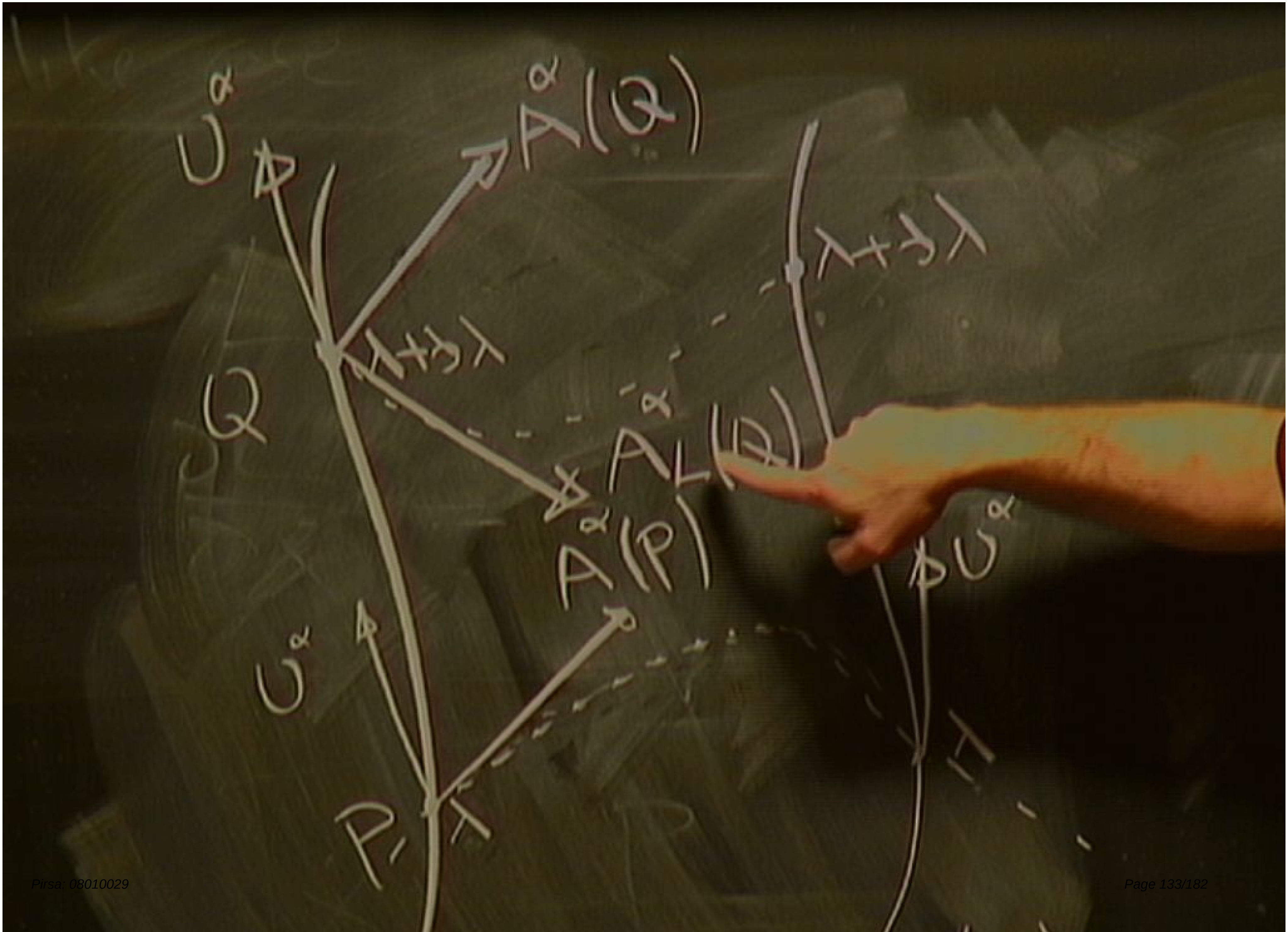

U^α
$A^\alpha(Q)$
$\lambda+\delta\lambda$
Q
Network
curves
$A^\alpha(P)$
U^α
U^α
P
λ
$\gamma(0)$
$\gamma(p)$


U^α
A^α(Q)
λ+δλ
Q
λ+δλ
A^α(P)
U^α
U^α
P
λ
λ
γ(0)
γ(μ)
Network
curves

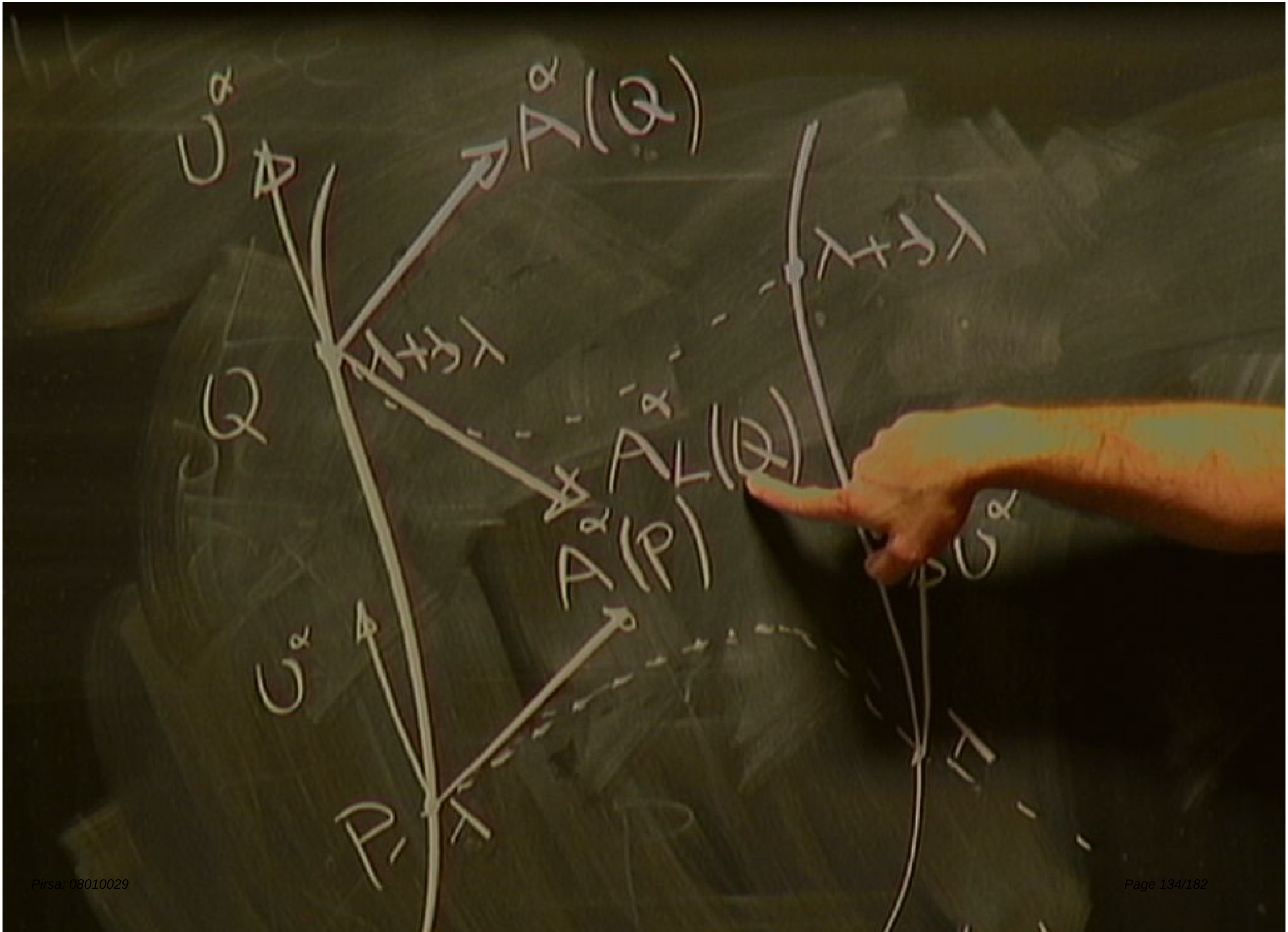

U^α
$A^\alpha(Q)$
$\lambda+\partial\lambda$
$\lambda+\partial\lambda$
Network
curves
Q
$A^\alpha(P)$
U^α
U^α
P
λ
λ
$\gamma(0)$
$\gamma(p)$



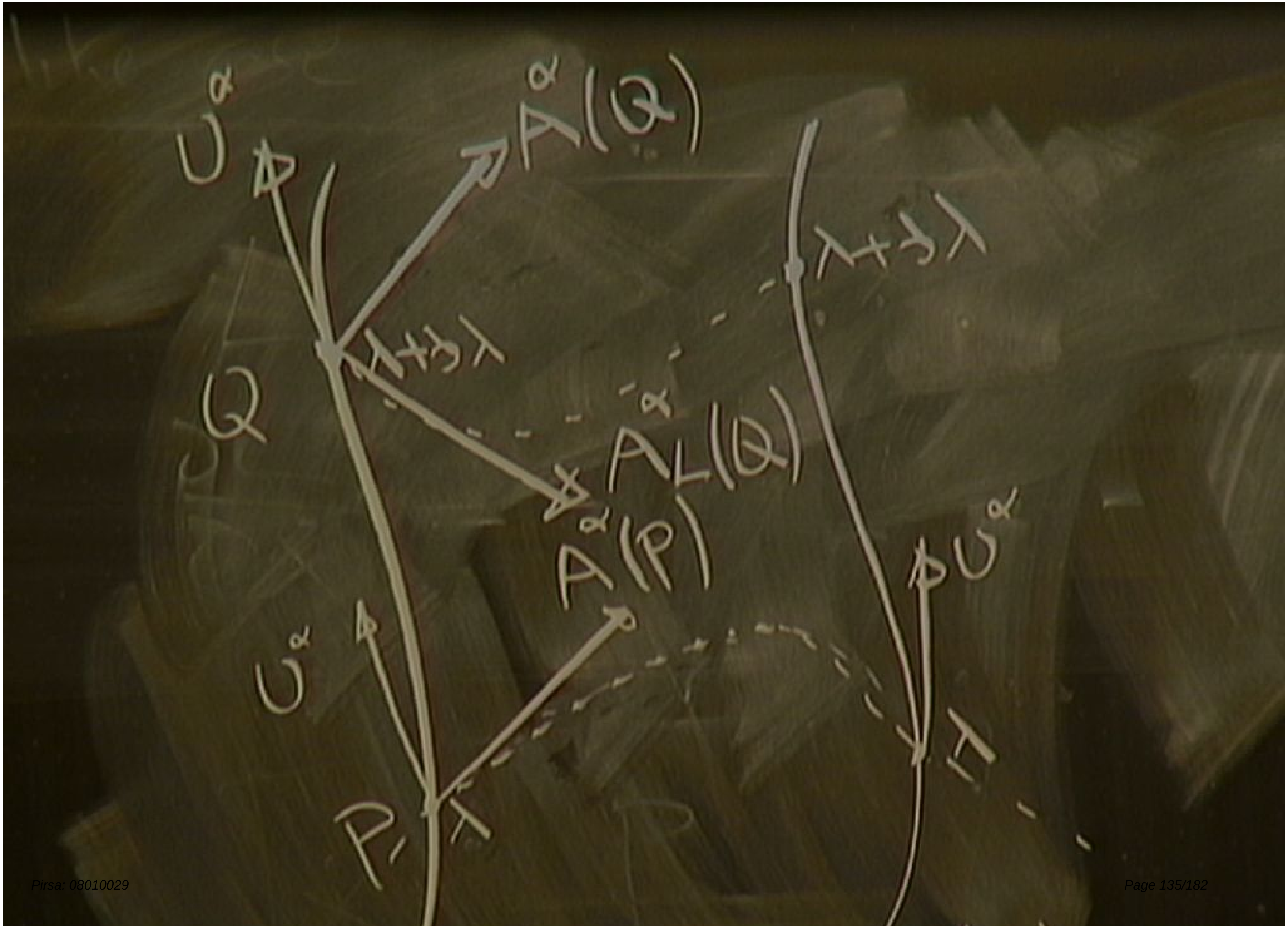
U^α
$A^\alpha(Q)$
Q
A_L^α
$A^\alpha(P)$
U^α
P



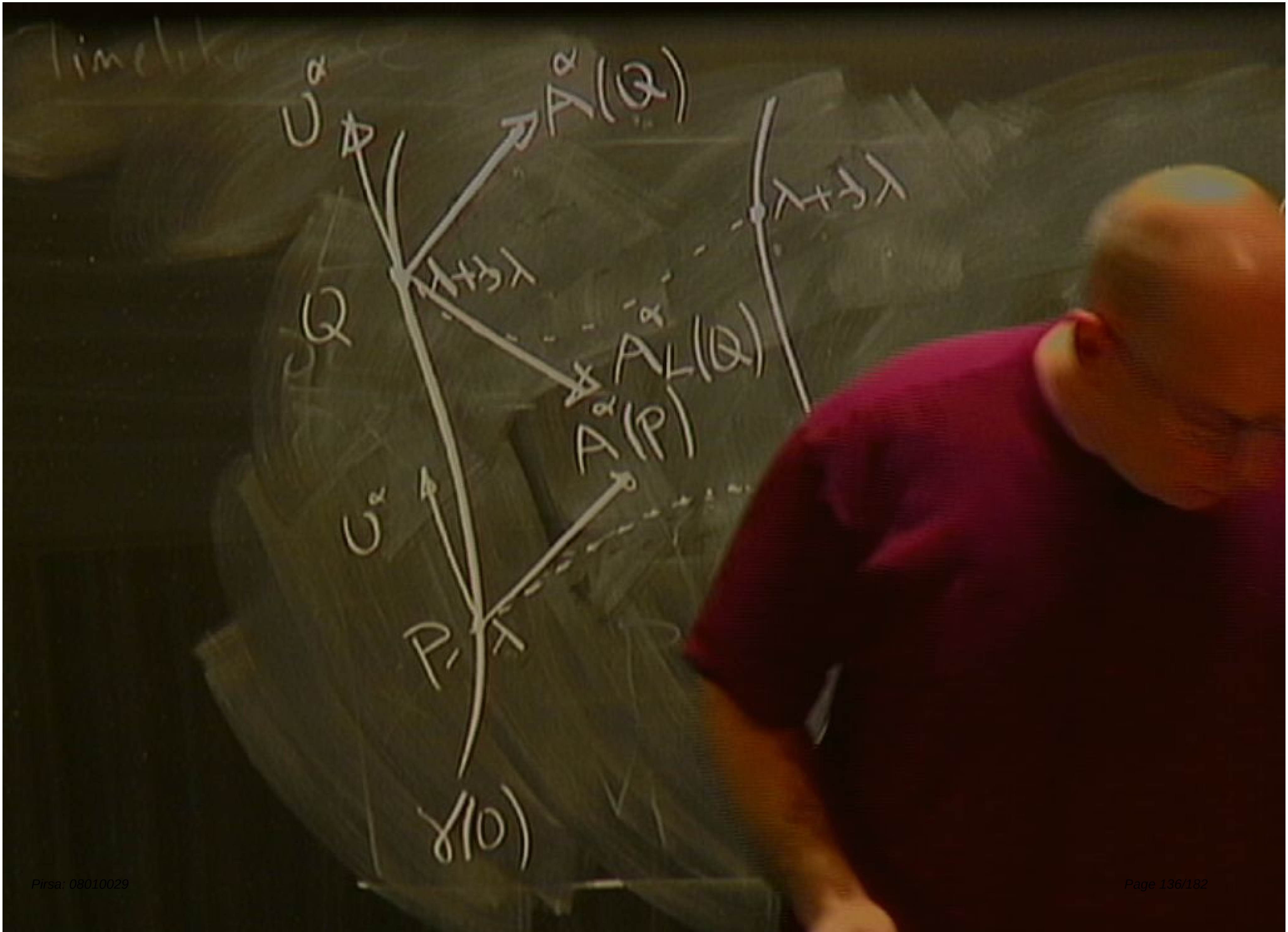
U^α
A^α(Q)
λ+δλ
Q
λ+δλ
A^α_L(Q)
A^α(P)
U^α
U^α
P, λ
λ

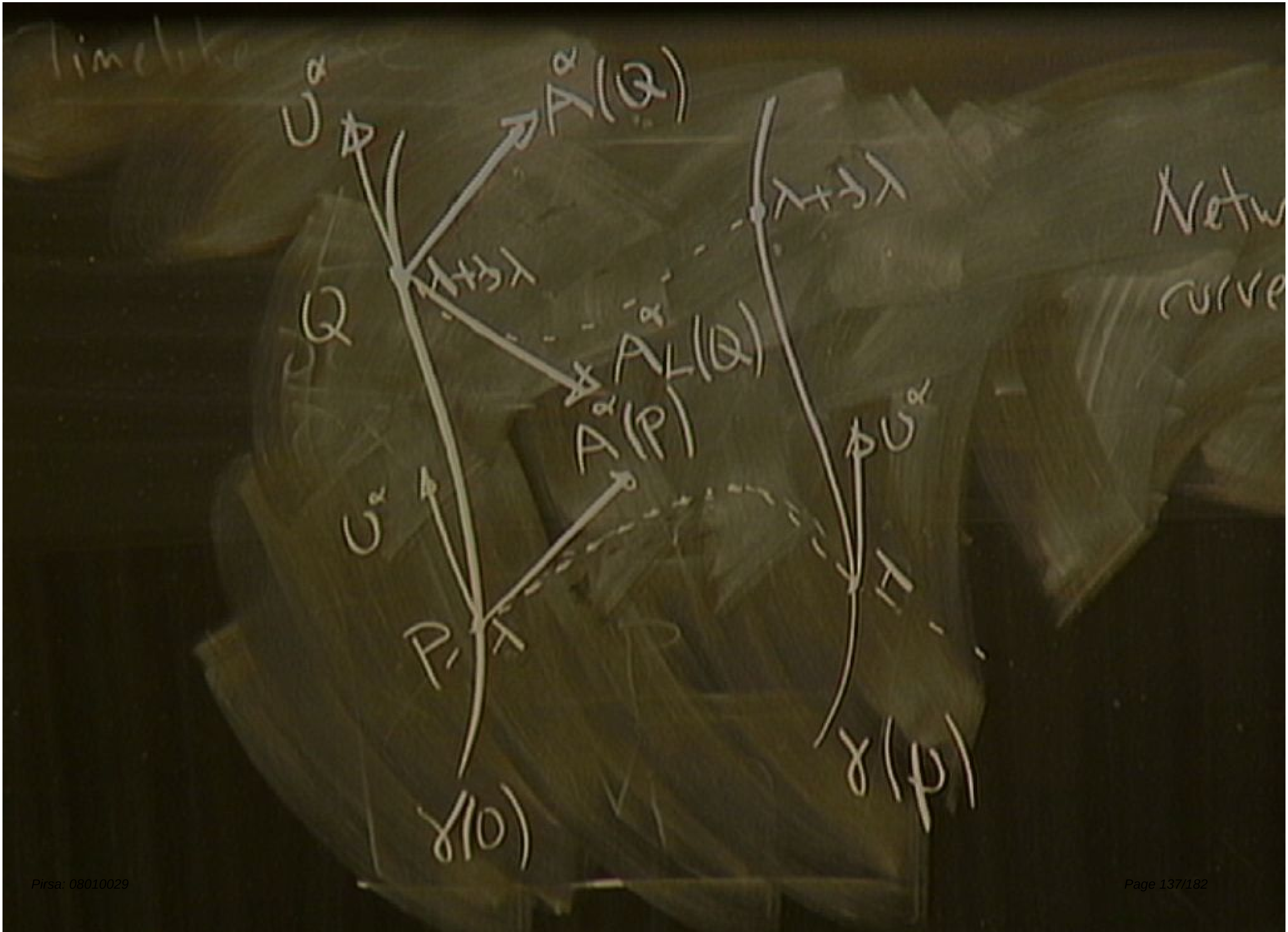


U^α
$A^\alpha(Q)$
$\lambda+\delta\lambda$
$\lambda+\delta\lambda$
Q
$A_L^\alpha(Q)$
$A^\alpha(P)$
U^α
U^α
P, λ
λ



U^α
$A^\alpha(Q)$
$\lambda+\delta\lambda$
Q
$\lambda+\delta\lambda$
$A_L^\alpha(Q)$
$A^\alpha(P)$
U^α
U^α
P
λ
λ


Timelike case
U^α
A^α(Q)
λ+∂λ
Q
A_⊥^α(Q)
A^α(P)
U^α
P
λ
γ(0)



U^α
$A^\alpha(Q)$
$\lambda+\delta\lambda$
$\lambda+\delta\lambda$
Netw
curve
Q
$A_L^\alpha(Q)$
$A^\alpha(P)$
U^α
U^α
P
λ
λ
$\gamma(0)$
$\gamma(p)$

Def

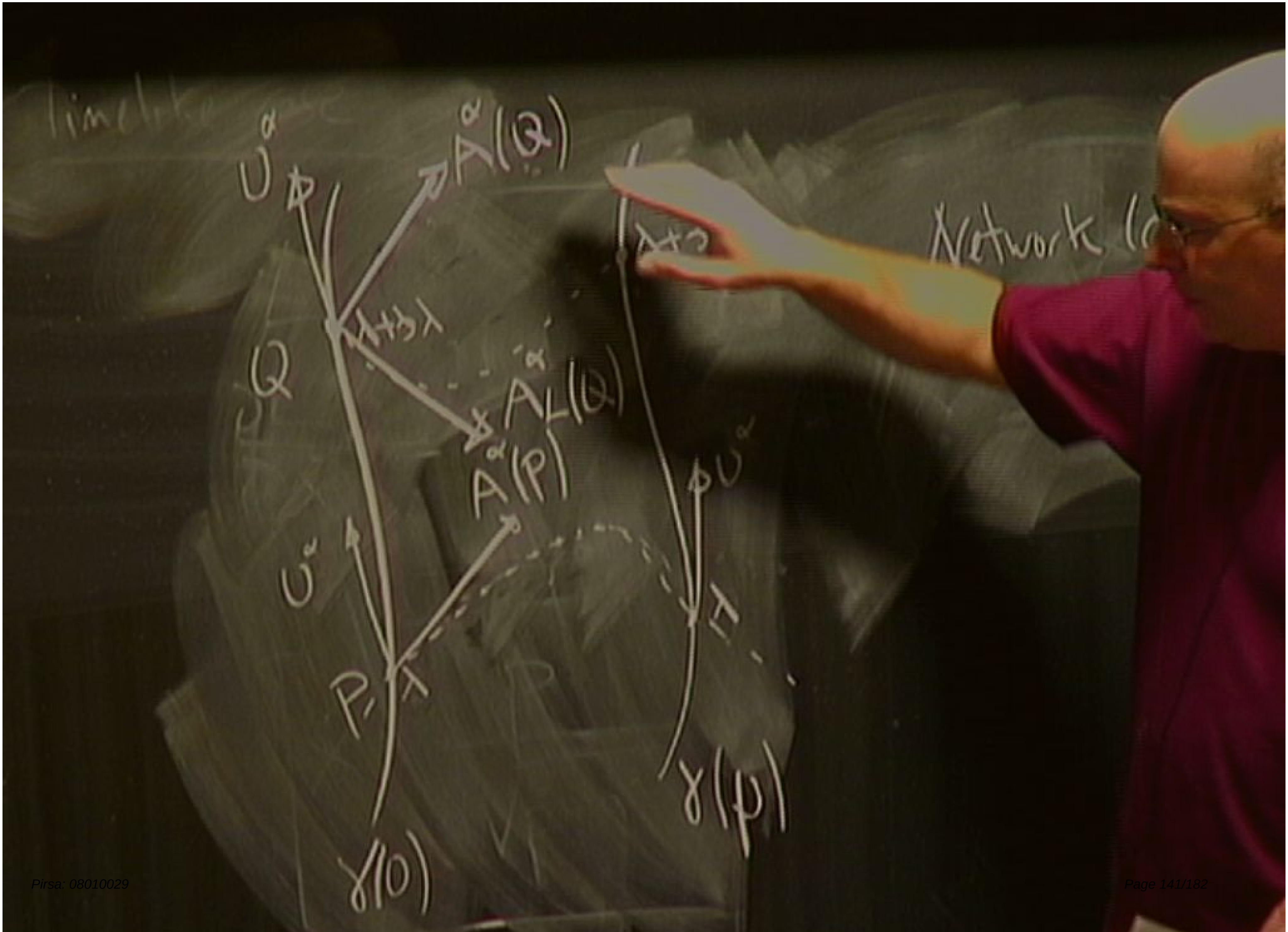
$$\mathcal{L}_U A^\alpha \big|_p = A^\alpha(Q) -$$

Def

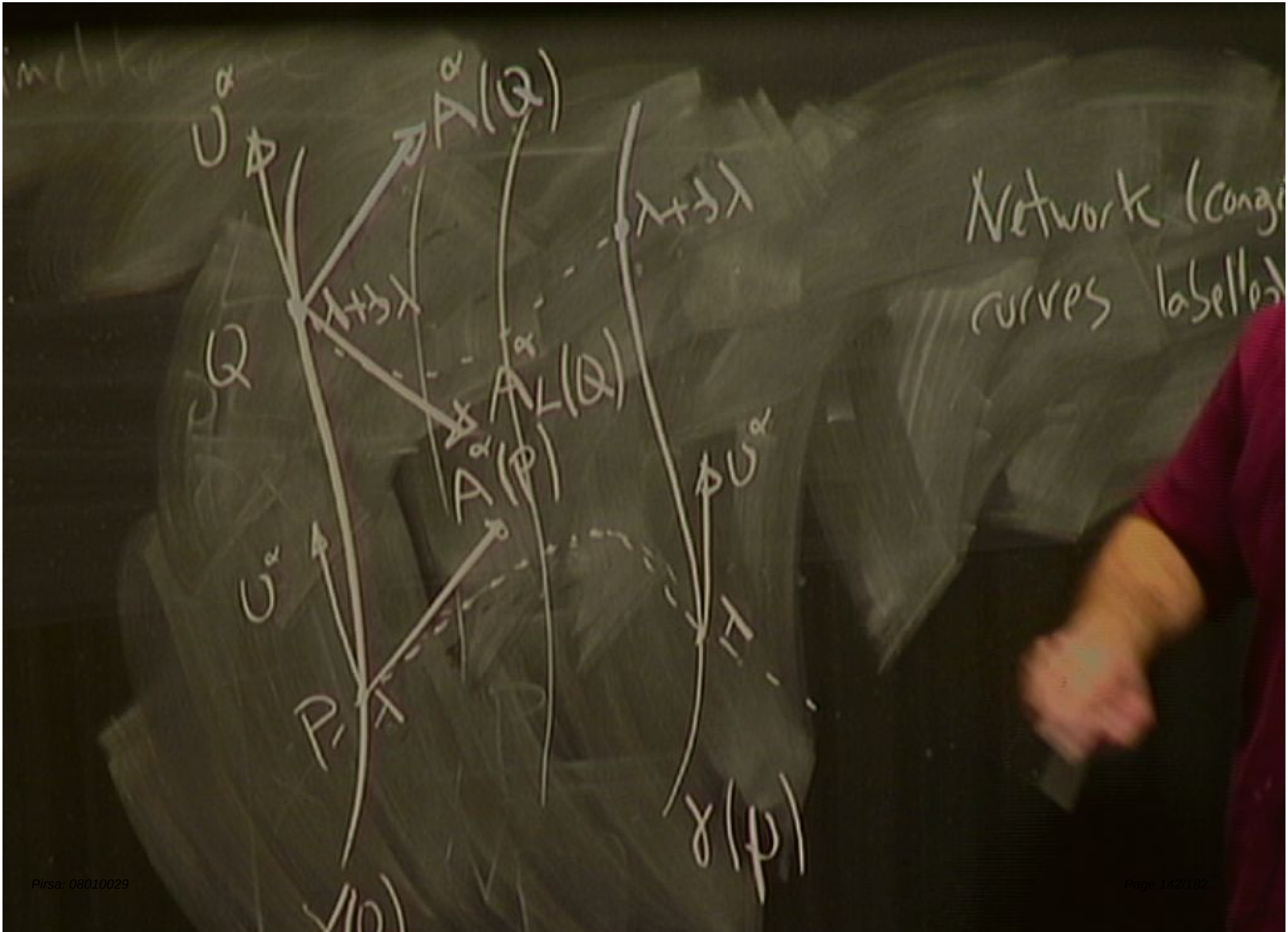
$$\mathcal{L}_u A^\alpha \Big|_P = \frac{A^\alpha(Q) - A^\alpha_{\angle}(Q)}{\partial \lambda}$$

Def

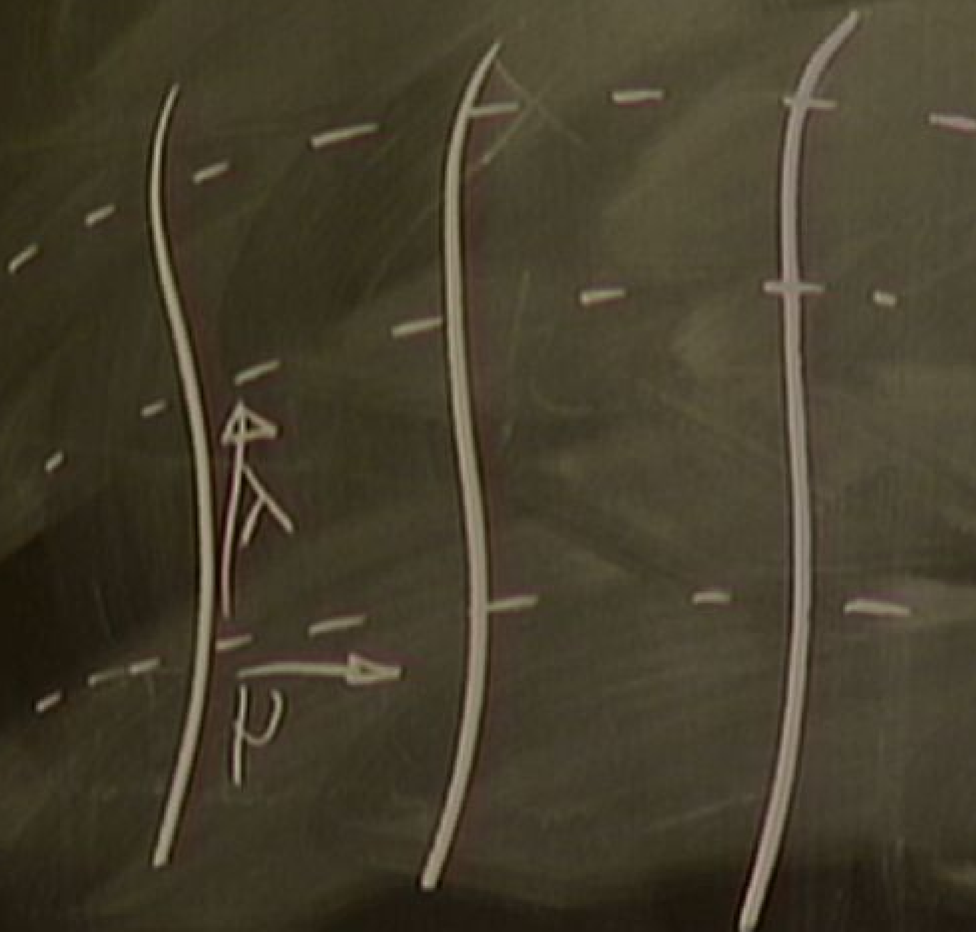
$$\mathcal{L}_U A^\alpha \Big|_P = \frac{A^\alpha(Q) - A^\alpha}{\partial\lambda}$$

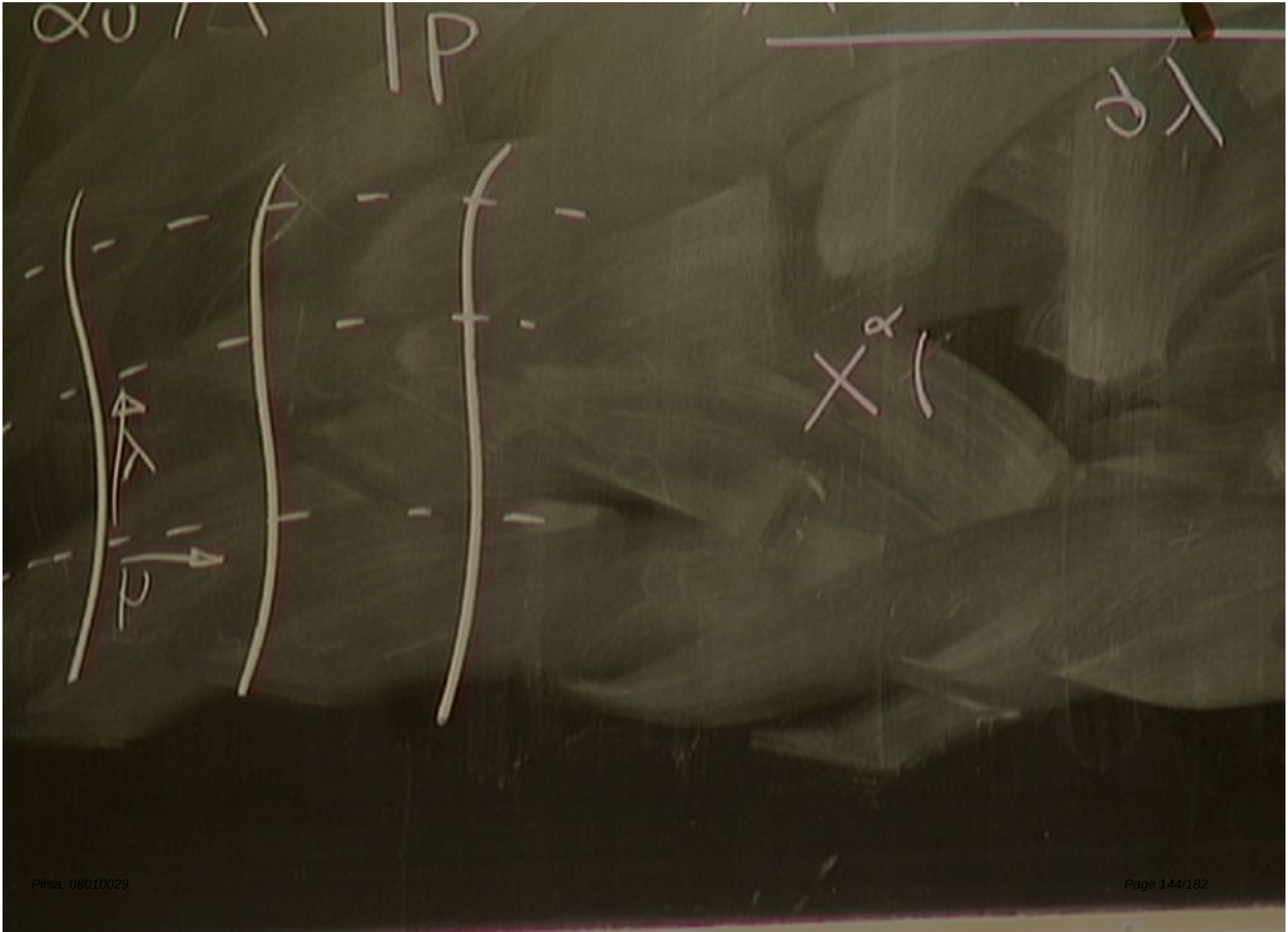


Network
U^α
$A^\alpha(Q)$
Q
$A_\perp^\alpha(Q)$
$A^\alpha(p)$
U^α
P
$\gamma(0)$
$\gamma(p)$


Q
P
γ(p)

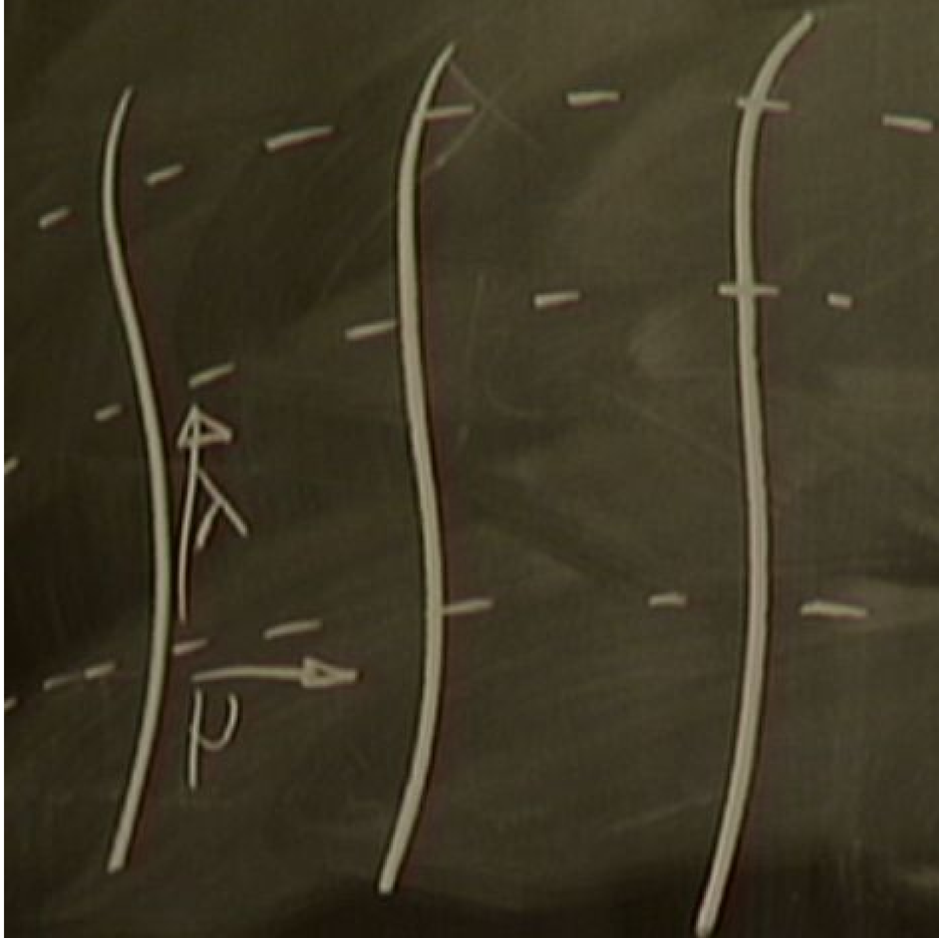
$$\mathcal{L}_u A^\alpha \Big|_p = \underline{A^\alpha(Q)}$$



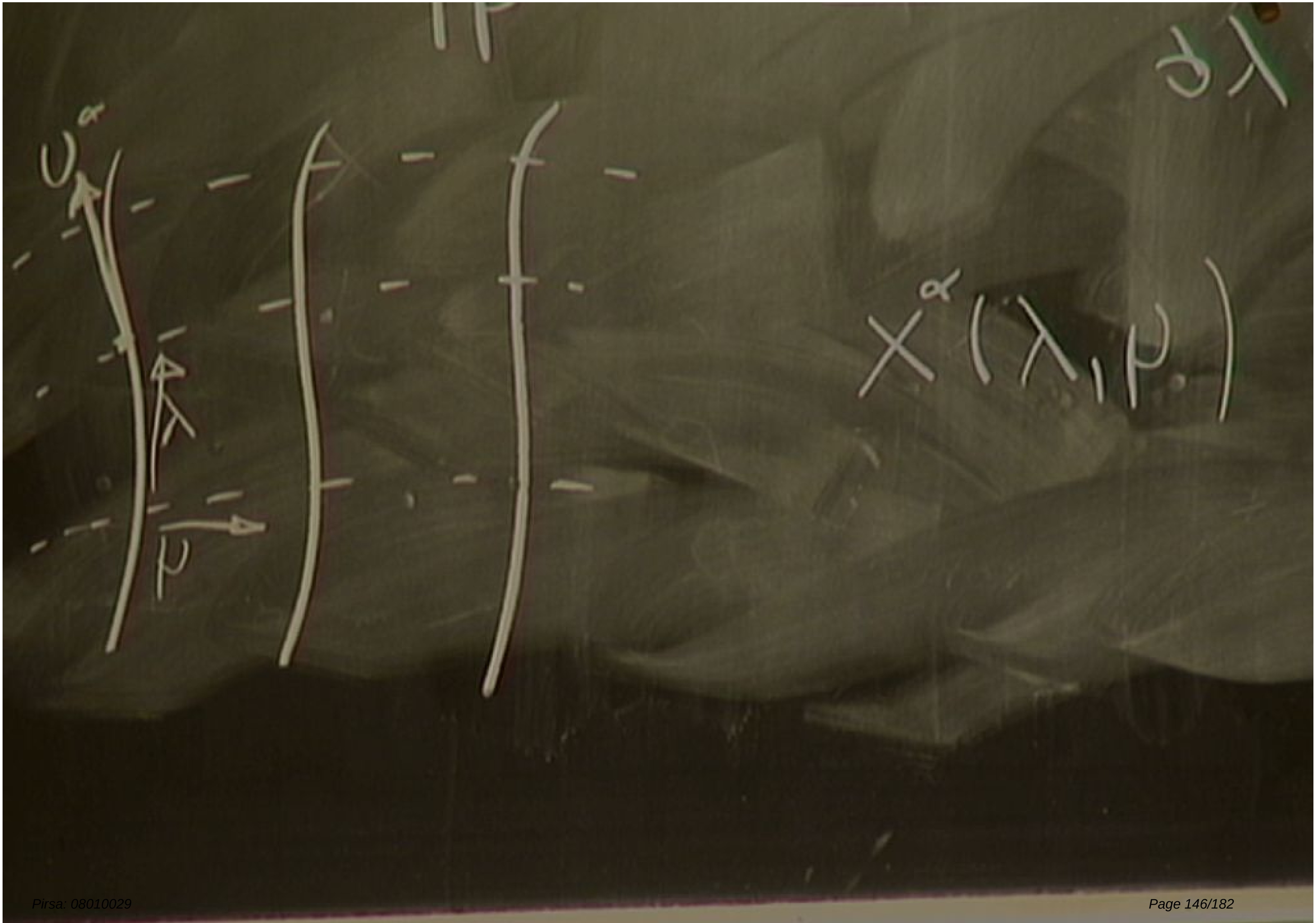


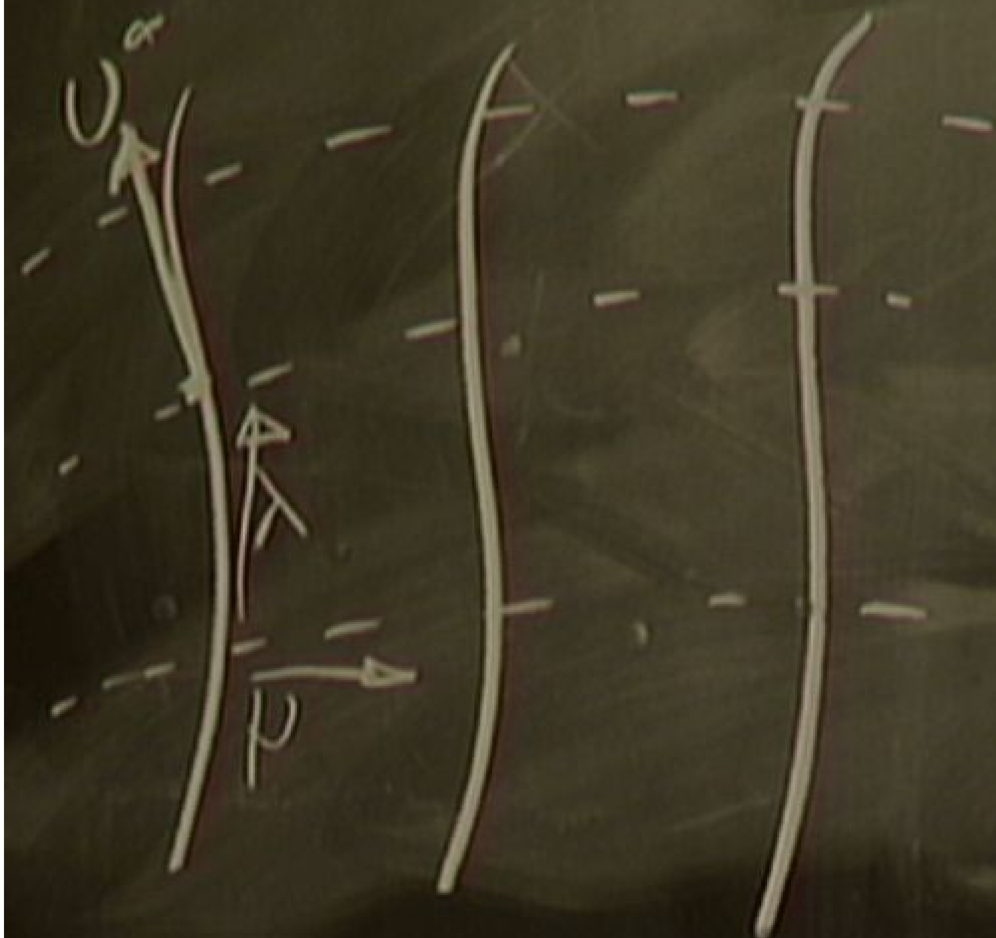
$\alpha \sigma \lambda \quad | p$

$\partial \lambda$



$X^\alpha(\lambda, \mu)$


u^α
λ
μ
$x^\alpha(\lambda, \mu)$

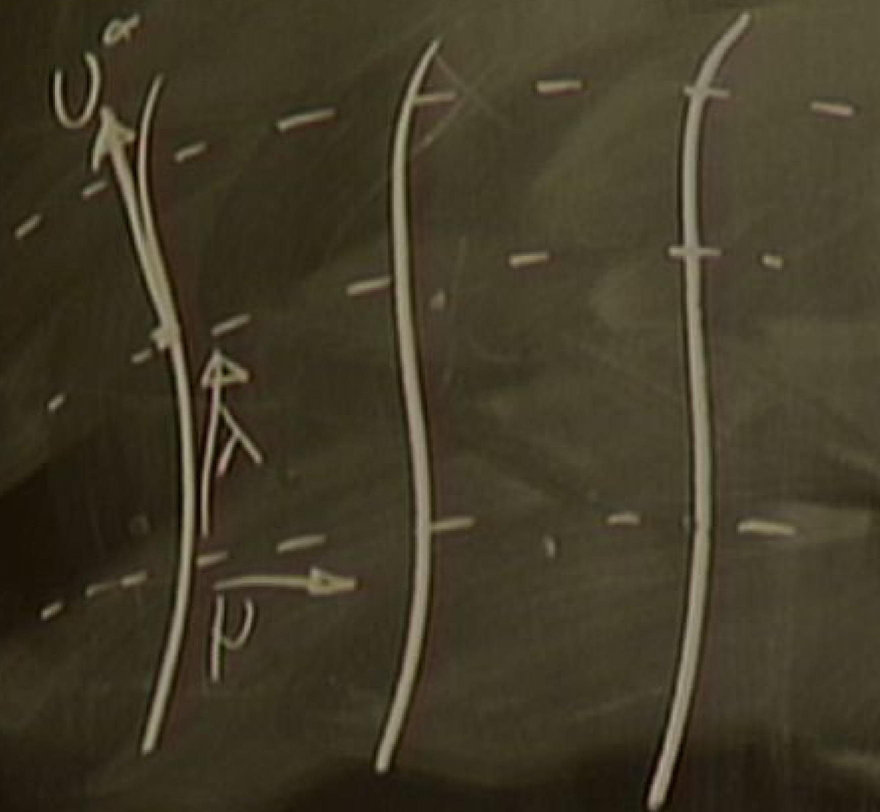


$$x^\alpha(\lambda, \mu)$$

$$U^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu$$

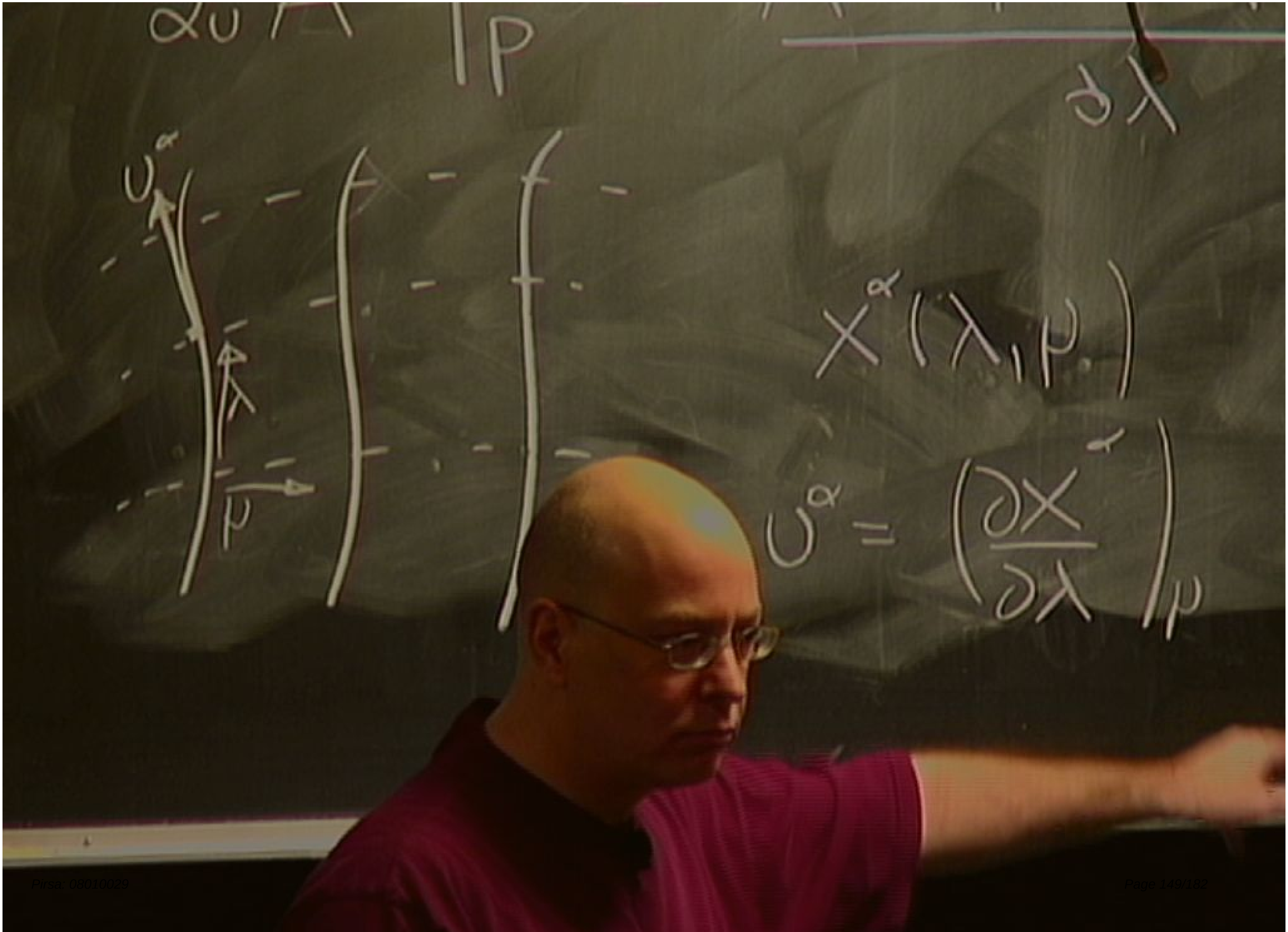
$\alpha u A \mid p$

$\partial\lambda$



$$x^\alpha(\lambda, \mu)$$

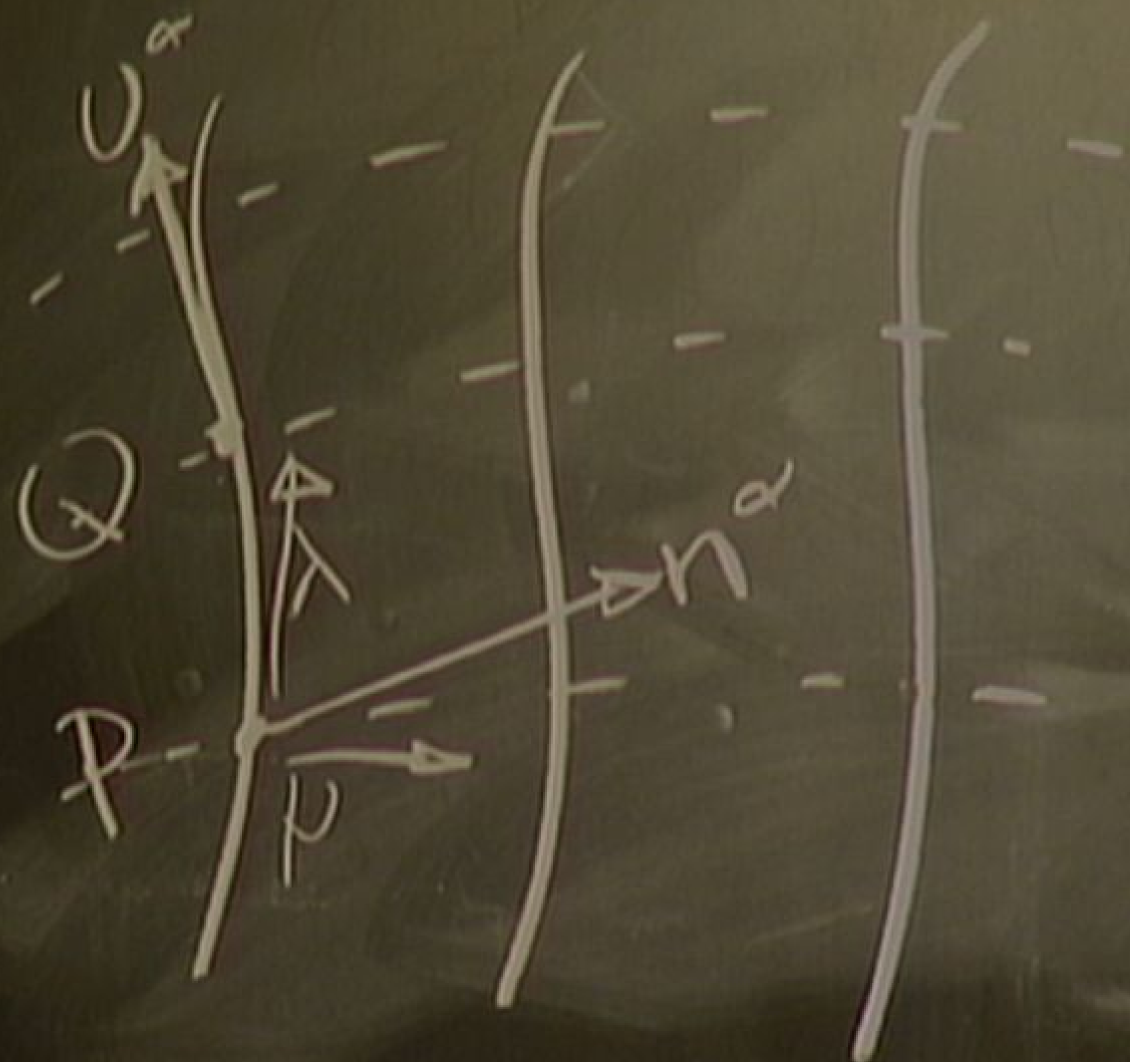
$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu$$

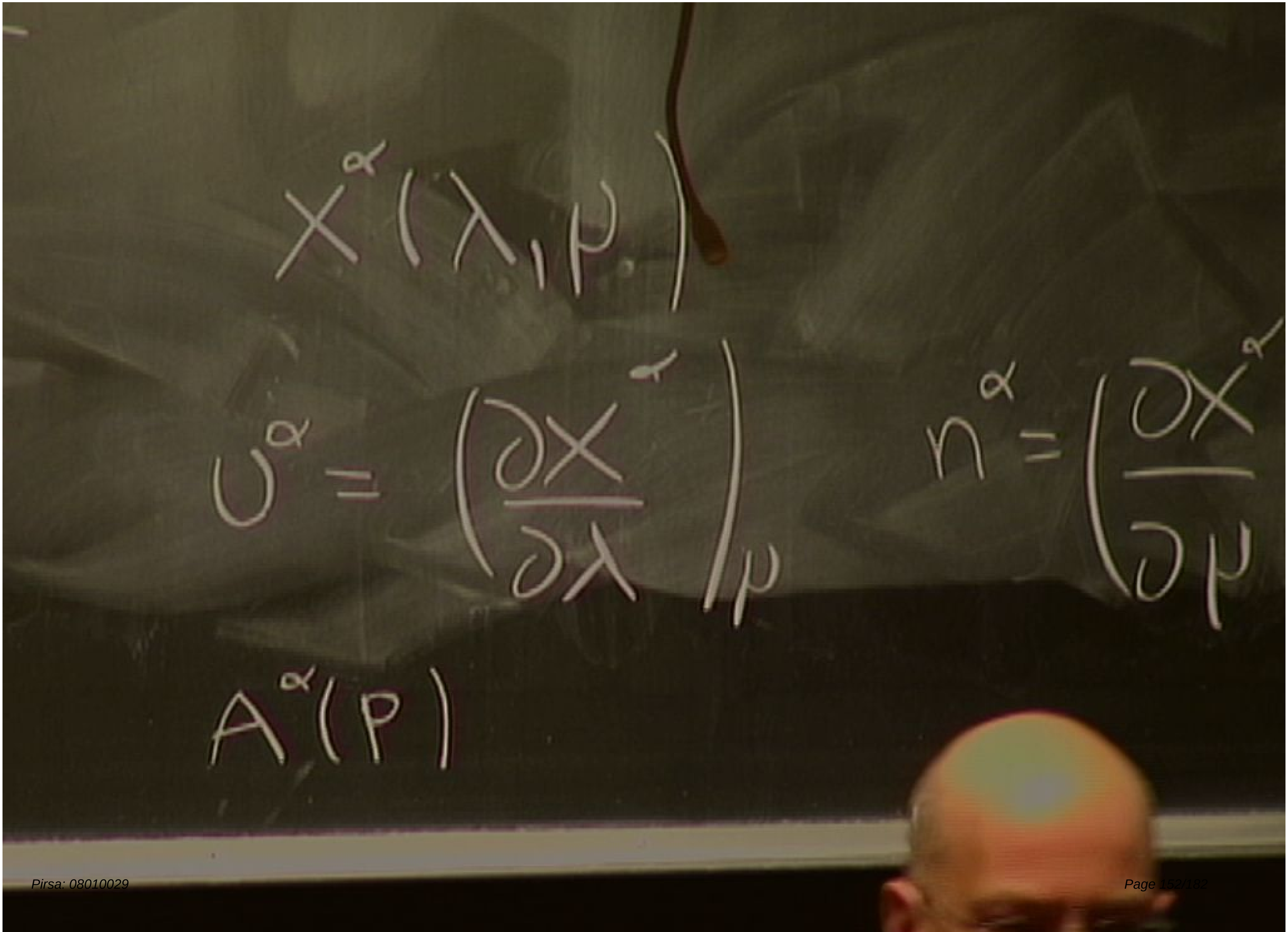

$x^\alpha(\lambda,\mu)$
$U^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu$

$$x^\alpha(\lambda,\mu)$$

$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu \qquad n^\alpha = \left(\frac{\partial x^\alpha}{\partial \mu}\right)_\lambda$$

$$\mathcal{L}_u A \big|_P =$$




$x^\alpha(\lambda,\mu)$
$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu$
$n^\alpha = \left(\frac{\partial x^\alpha}{\partial \mu}\right.$
$A^\alpha(P)$

$$x^\alpha(\lambda,\mu)$$

$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu \qquad n^\alpha = \left(\frac{\partial x^\alpha}{\partial \mu}\right)_\lambda$$

$$A^\alpha(P) = n$$

$$x^\alpha(\lambda,\mu)$$

$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu \qquad n^\alpha = \left(\frac{\partial x^\alpha}{\partial \mu}\right)_\lambda$$

$$A^\alpha(P) = n^\alpha(P)$$

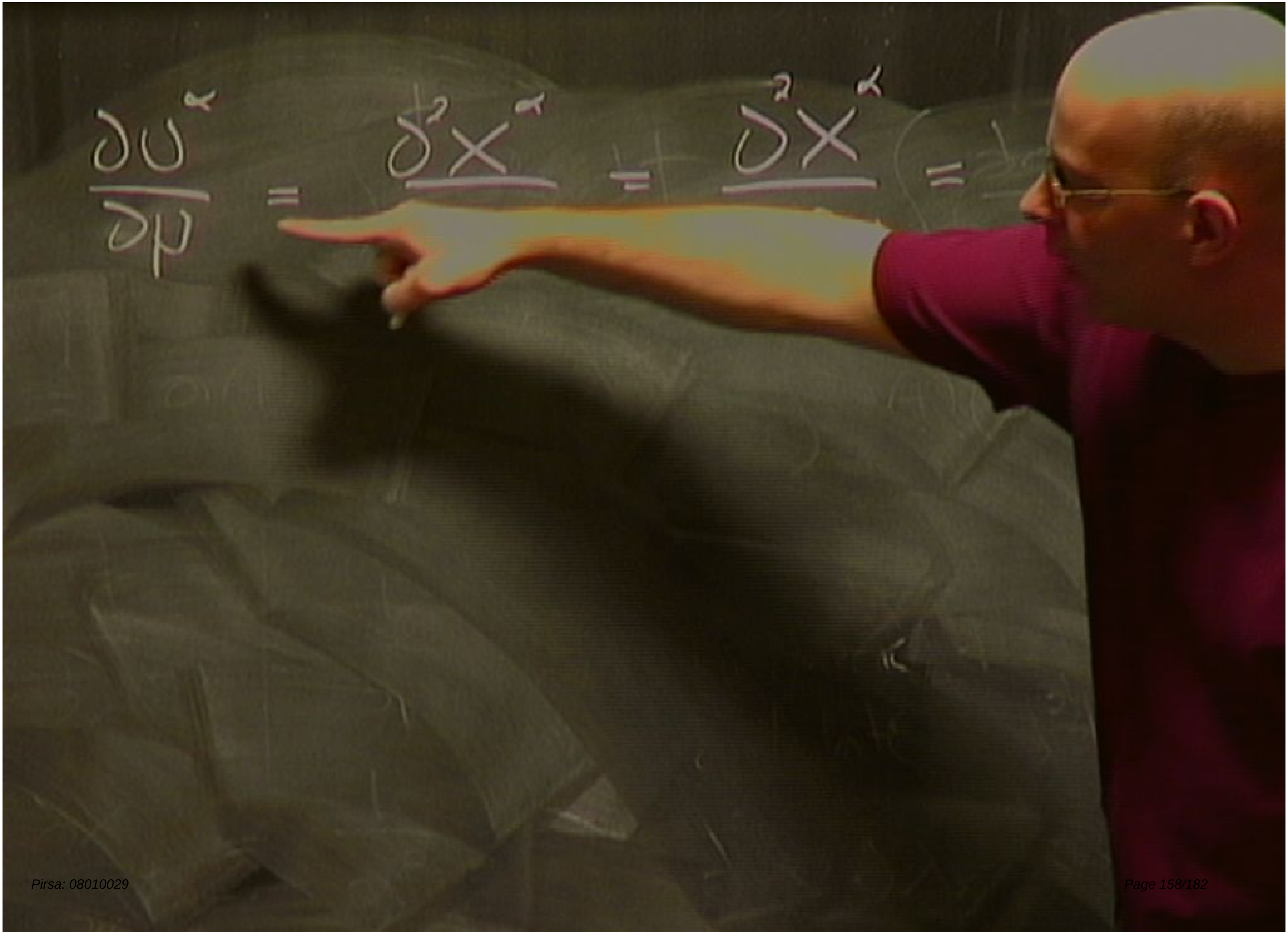
$$x^\alpha(\lambda,\mu)$$

$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_\mu \qquad n^\alpha = \left(\frac{\partial x^\alpha}{\partial \mu}\right)_\lambda$$

$$A^\alpha(P) = n^\alpha(P) \; ; \; A^\alpha_L(Q) = n^\alpha(Q)$$

$$\frac{\partial u^{\alpha}}{\partial \mu} =$$

$$\frac{\partial u^{\alpha}}{\partial \mu} = \frac{\partial^2 x^{\alpha}}{\partial \mu \, \partial \lambda}$$



$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} =$$

$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

u^α

$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

$$u^\alpha{}_{,\beta} n^\beta$$

$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

$$u^\alpha{}_{,\beta}\, n^\beta$$

$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

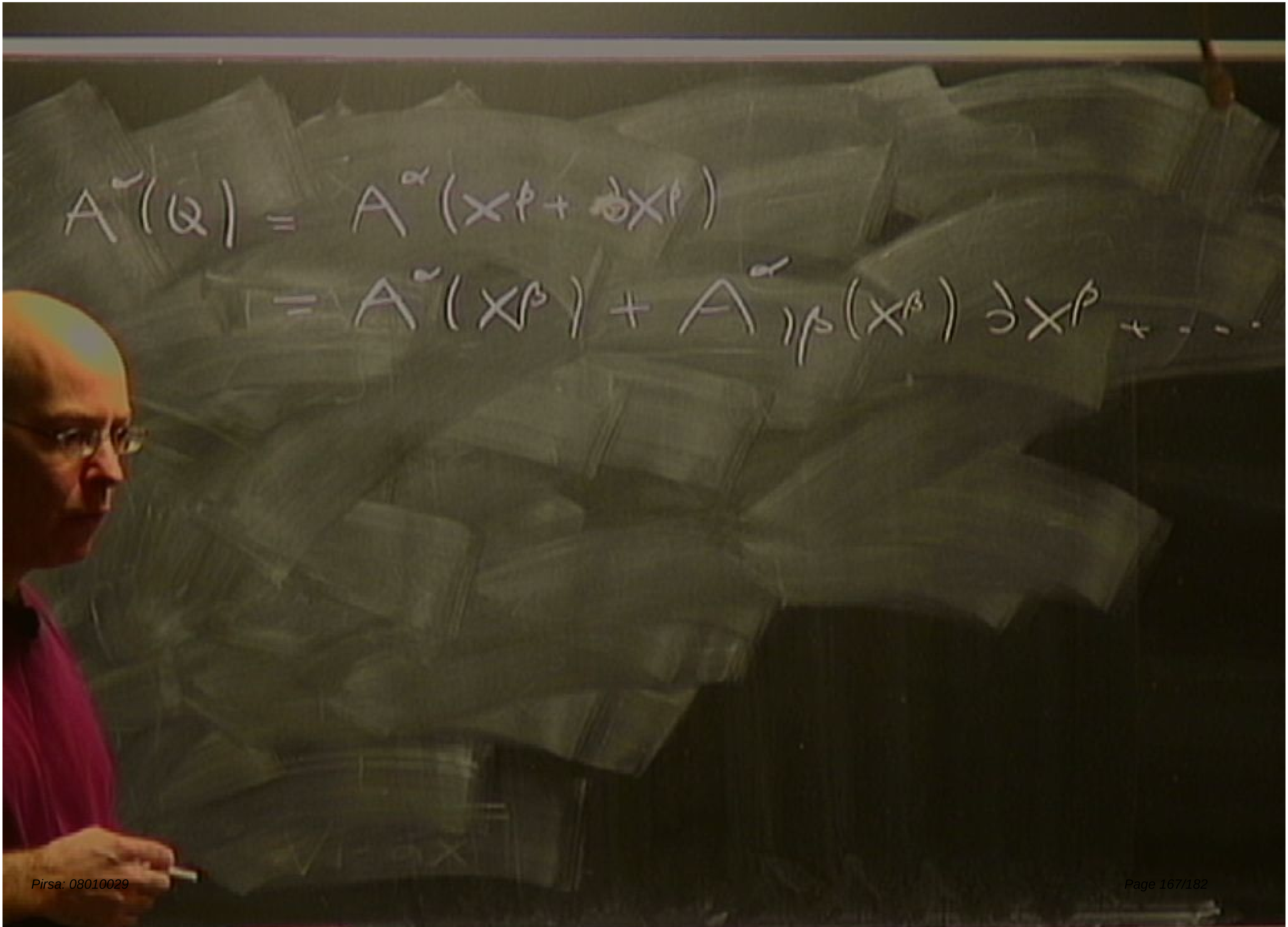
$$u^\alpha{}_{,\beta} n^\beta = n^\alpha{}_{,\beta} u^\beta$$

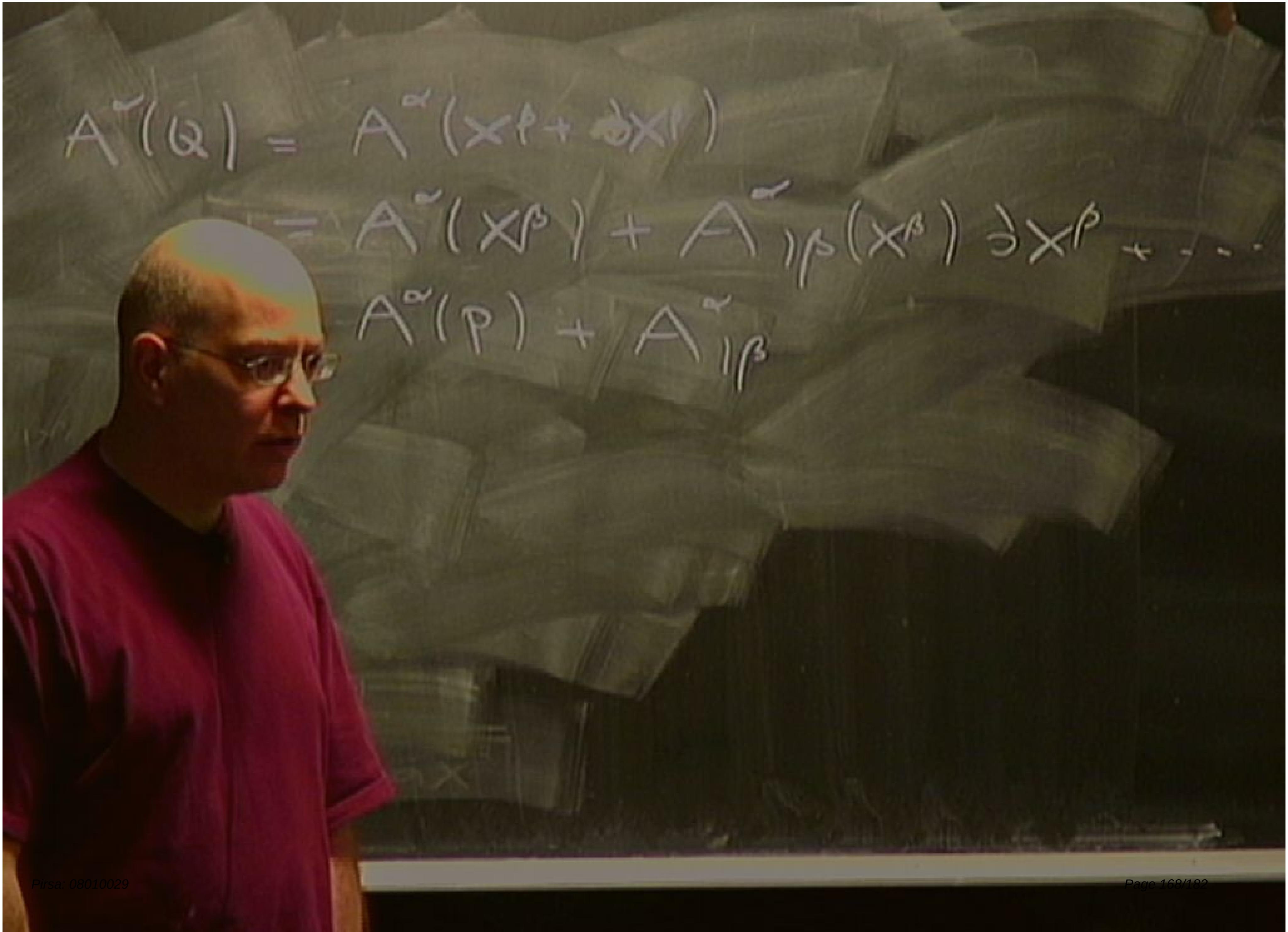
$$\frac{\partial u^\alpha}{\partial \mu} = \frac{\partial^2 x^\alpha}{\partial \mu \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda \partial \mu} = \frac{\partial n^\alpha}{\partial \lambda}$$

$$\frac{\partial u^\alpha}{\partial \mu} = u^\alpha{}_{,\beta} n^\beta \qquad \frac{\partial n^\alpha}{\partial \lambda} = n^\alpha{}_{,\beta} u^\beta$$

$$\boxed{u^\alpha{}_{,\beta} n^\beta = n^\alpha{}_{,\beta} u^\beta}$$

$$A^{\alpha}(Q) = A^{\alpha}($$


$A^{\alpha}(Q) = A^{\alpha}(x^{\beta} + \partial x^{\beta})$
$= A^{\alpha}(x^{\beta}) + A^{\alpha}{}_{,\beta}(x^{\beta})\,\partial x^{\beta} + \ldots$


$A^\alpha(Q) = A^\alpha(x^\beta + \partial x^\beta)$
$= A^\alpha(x^\beta) + A^\alpha{}_{,\beta}(x^\beta)\,\partial x^\beta + \ldots$
$A^\alpha(p) + A^\alpha{}_{,\beta}$

$$A^\alpha(Q) = A^\alpha(x^\beta + \partial x^\beta)$$

$$= A^\alpha(x^\beta) + A^\alpha{}_{,\beta}(x^\beta)\,\partial x^\beta + \dots$$

$$= A^\alpha(P) + A^\alpha{}_{,\beta}\,U^\beta\,\partial\lambda$$

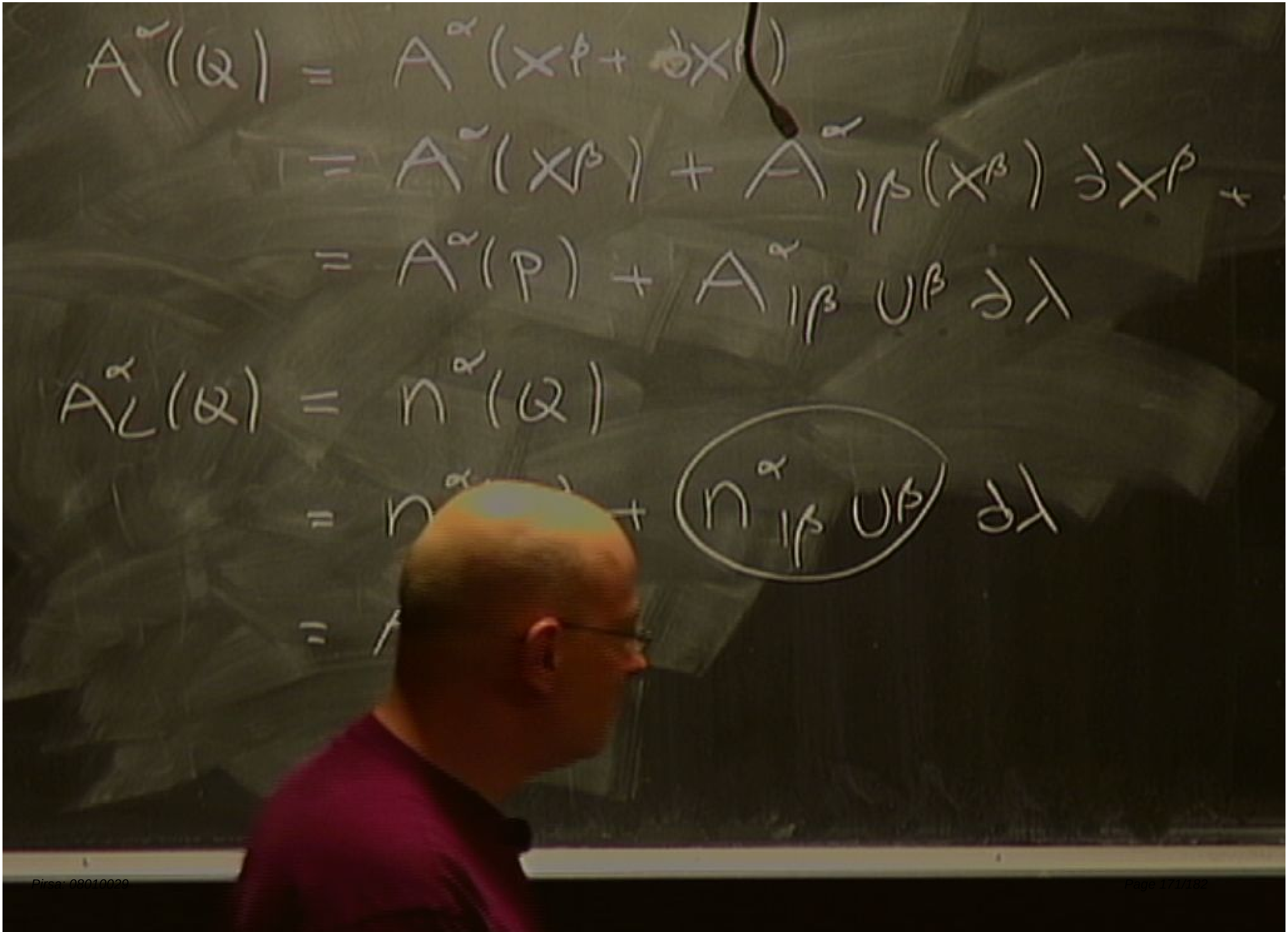
$$A^\alpha(Q) = A^\alpha(x^\beta + \partial x^\beta)$$

$$= A^\alpha(x^\beta) + A^\alpha{}_{|\beta}(x^\beta)\,\partial x^\beta + \ldots$$

$$= A^\alpha(P) + A^\alpha{}_{|\beta}\, U^\beta\, \partial\lambda$$

$$A_L^\alpha(Q) = n^\alpha(Q)$$

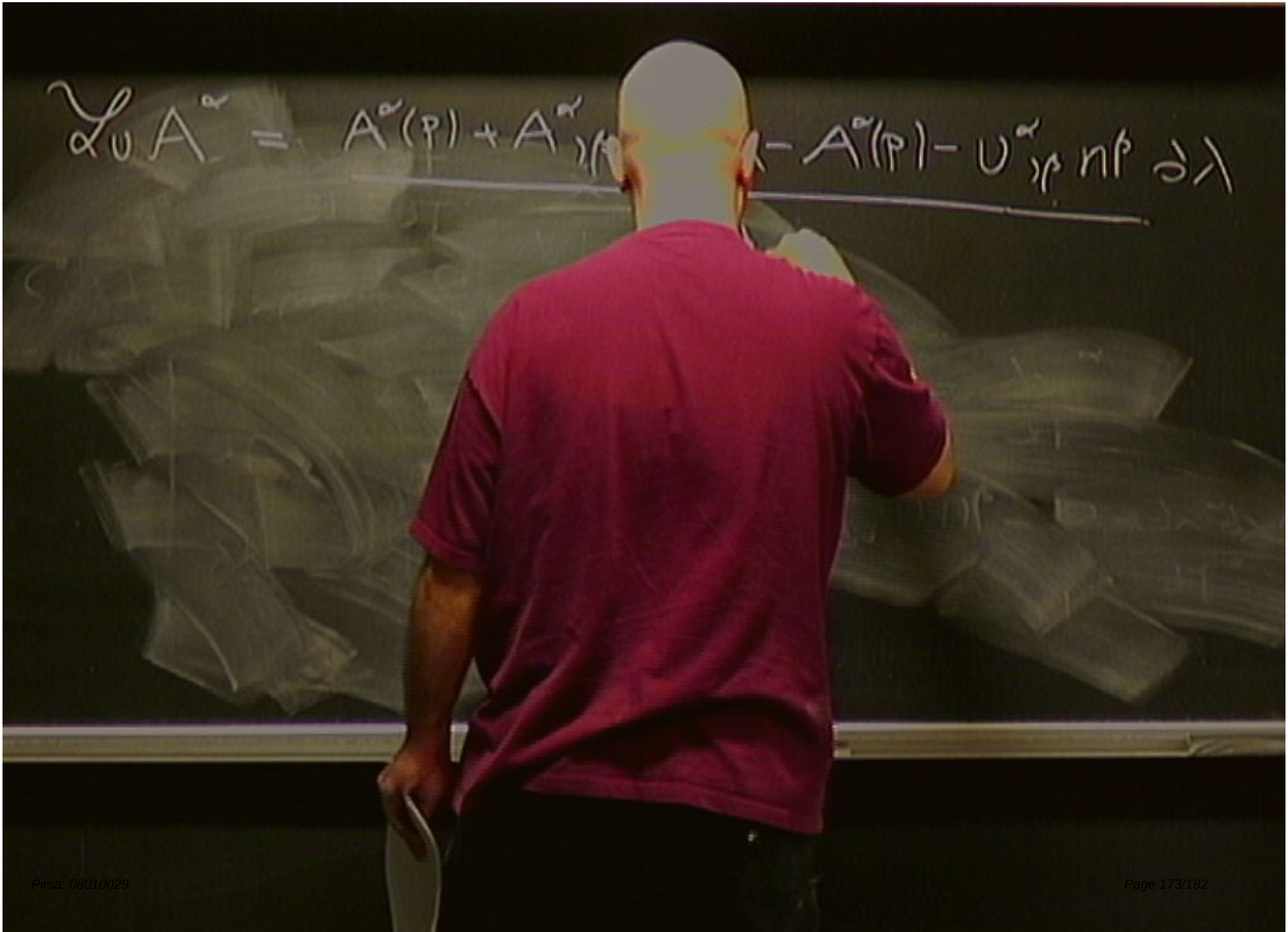
$$= n^\alpha(P) + n^\alpha{}_{|\beta}\, U^\beta\, \partial\lambda$$

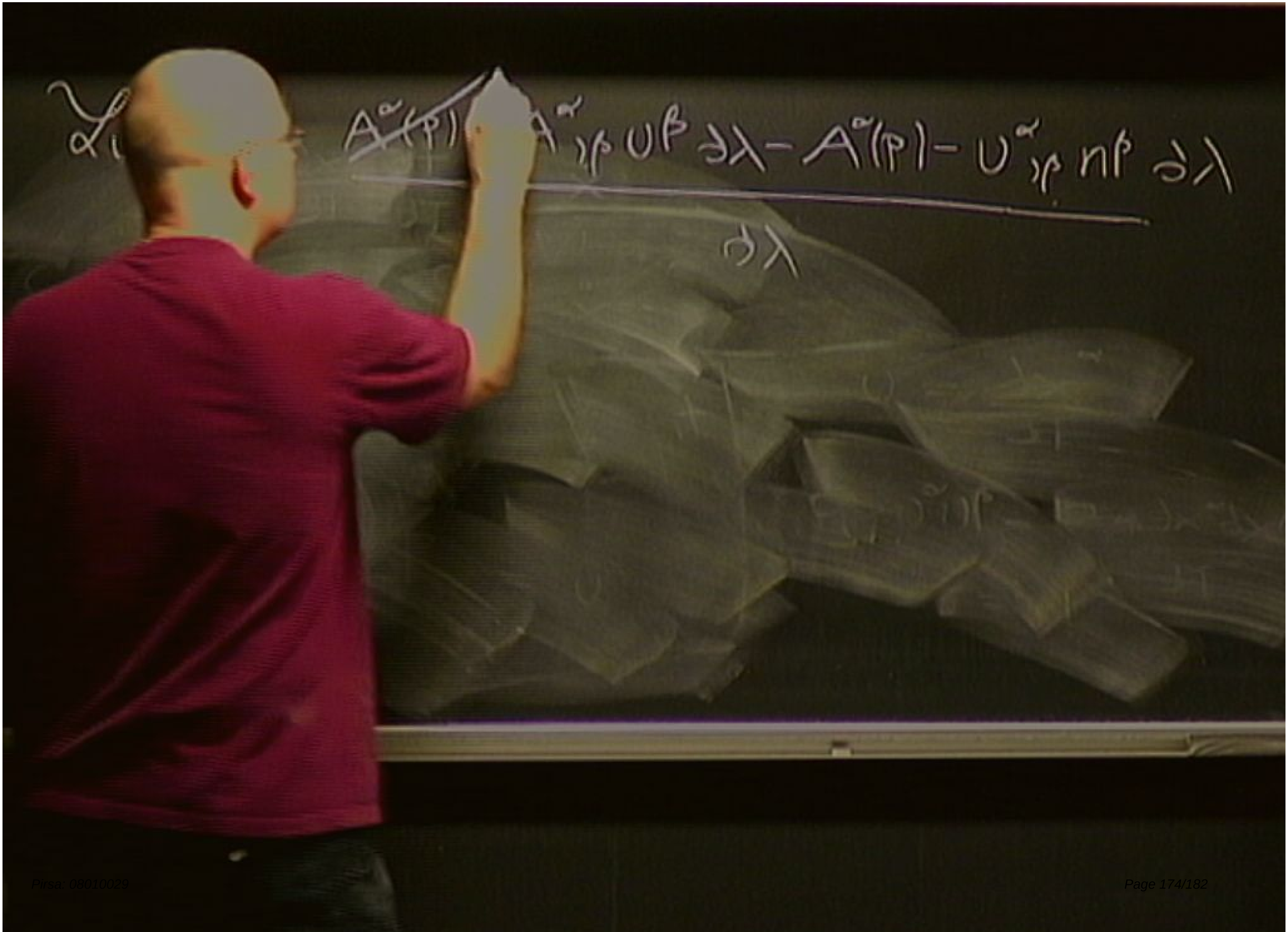

$A^\alpha(Q) = A^\alpha(x^\beta + \partial x^\beta)$
$= A^\alpha(x^\beta) + A^\alpha{}_{|\beta}(x^\beta)\,\partial x^\beta +$
$= A^\alpha(P) + A^\alpha{}_{|\beta}\,U^\beta\,\partial\lambda$
$A^\alpha_L(Q) = n^\alpha(Q)$
$= n^\alpha \quad + n^\alpha{}_{|\beta}\,U^\beta\,\partial\lambda$
$=$

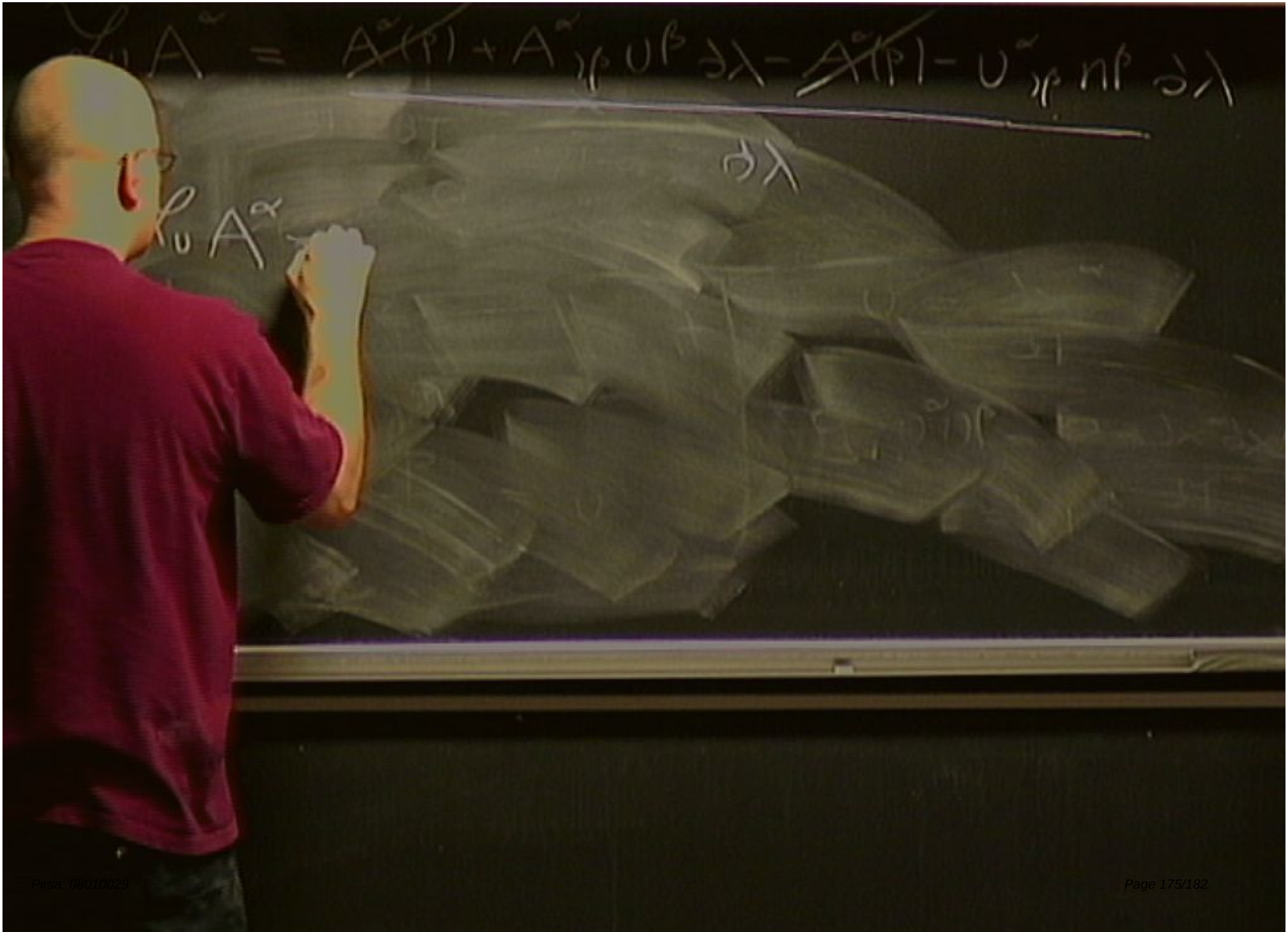
$$A^\alpha_L(Q) = n^\alpha(Q)$$

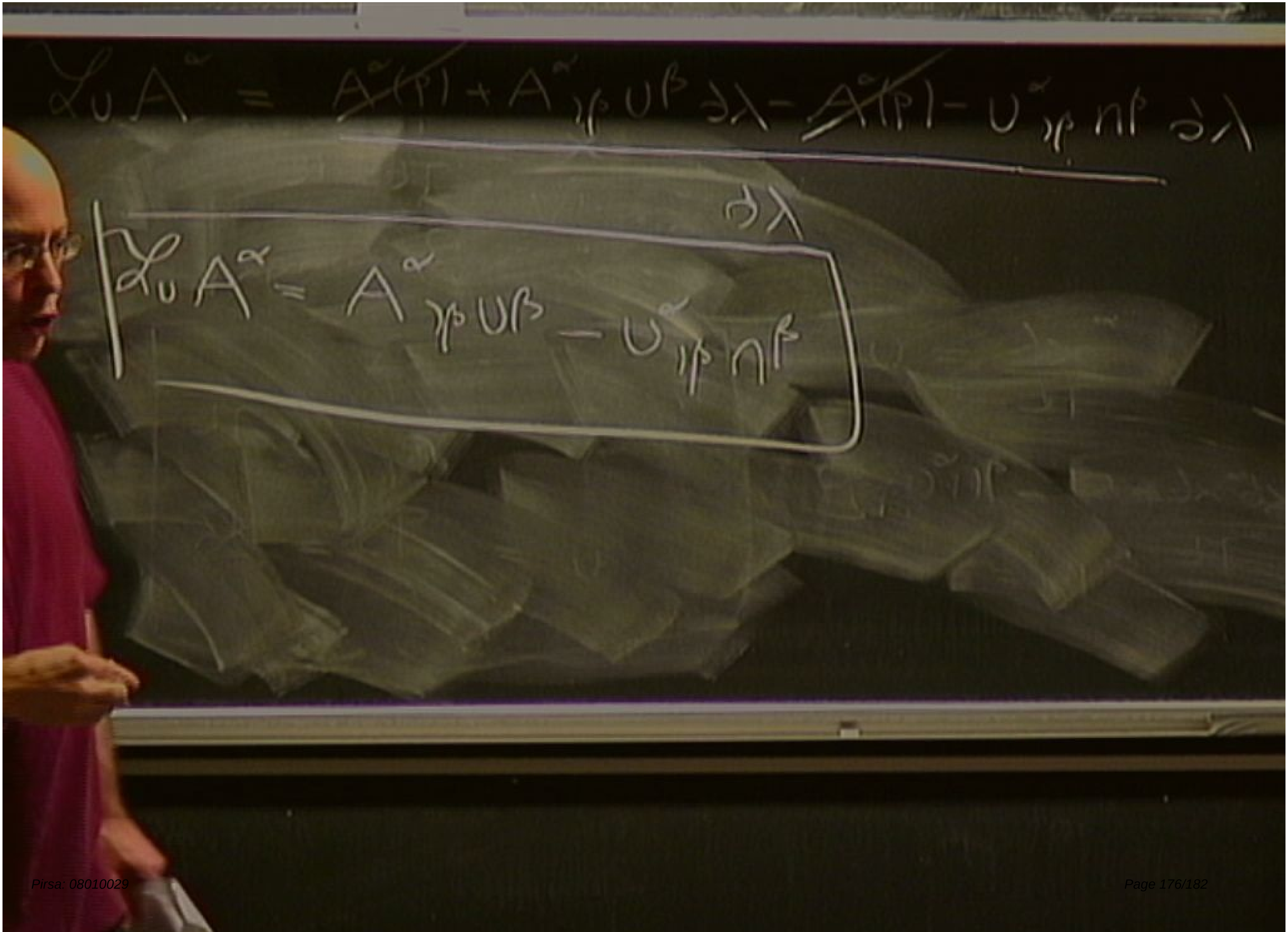
$$= n^\alpha(P) + n^\alpha{}_{;\beta}\, u^\beta \, d\lambda$$

$$= A^\alpha(P) + u^\alpha{}_{;\beta}\, n^\beta \, d\lambda$$




A^α;β U^β ∂λ − A^α(p) − U^α;β n^β ∂λ
∂λ

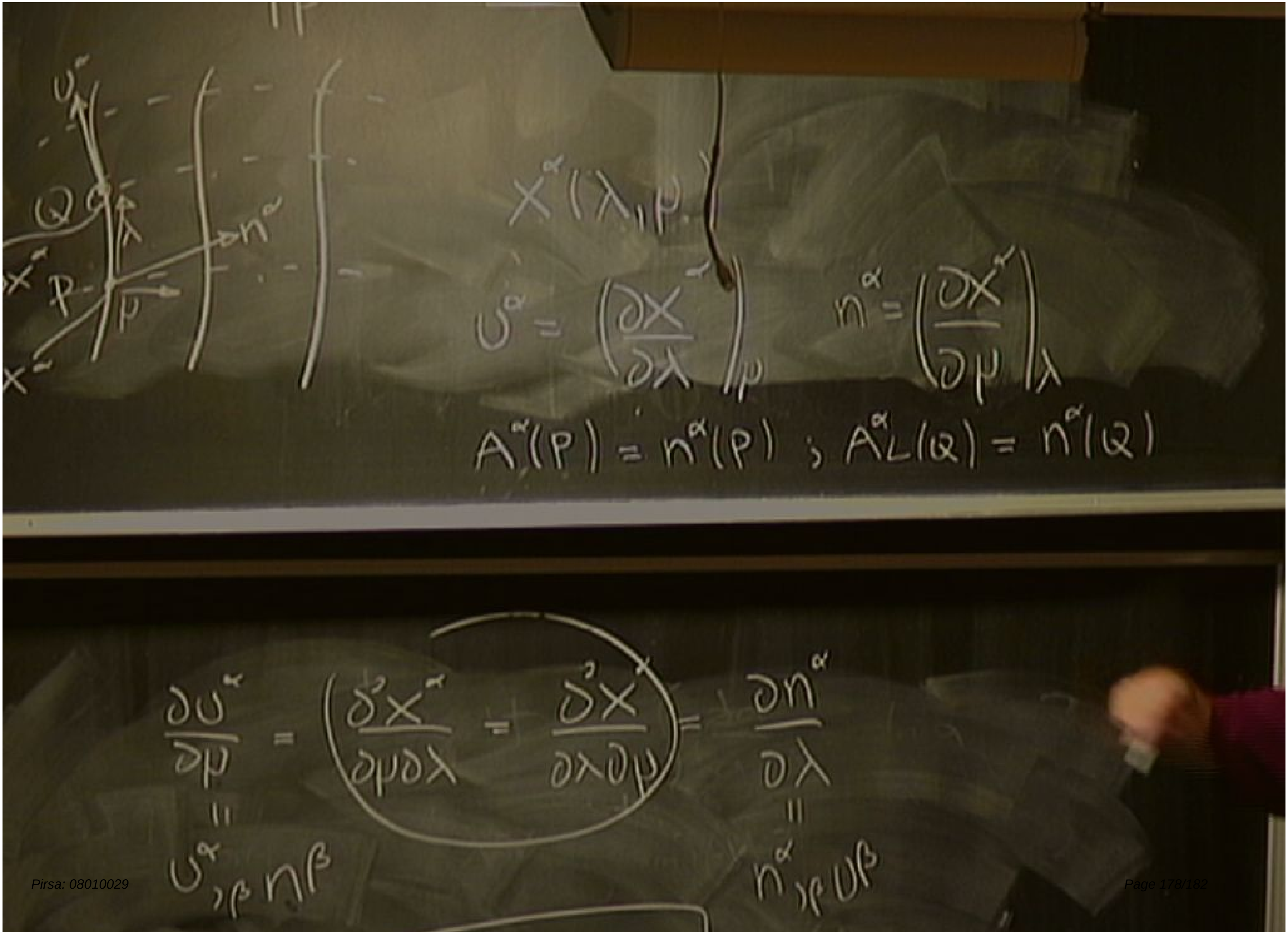

$\mathcal{L}_U A^\alpha = \cancel{A^\alpha(p)} + A^\alpha{}_{,\beta} U^\beta \partial\lambda - \cancel{A^\alpha(p)} - U^\alpha{}_{,\beta} A^\beta \partial\lambda$
$\partial\lambda$
$\mathcal{L}_U A^\alpha$


$\mathcal{L}_u A^\alpha = \cancel{A^\alpha(p)} + A^\alpha{}_{,\beta} u^\beta \partial\lambda - \cancel{A^\alpha(p)} - u^\alpha{}_{,\beta} n^\beta \partial\lambda$
$\partial\lambda$
$\boxed{\mathcal{L}_u A^\alpha = A^\alpha{}_{,\beta} u^\beta - u^\alpha{}_{,\beta} n^\beta}$

$$\mathcal{L}_u A^\alpha = \cancel{A^\alpha(p)} + A^\alpha{}_{,\beta} u^\beta \partial\lambda - \cancel{A^\alpha(p)} - u^\alpha{}_{,\beta} A^\beta \partial\lambda$$

$\partial\lambda$

$$\boxed{\mathcal{L}_u A^\alpha = A^\alpha{}_{,\beta} u^\beta - u^\alpha{}_{,\beta} A^\beta}$$


X^α(λ, μ)
u^α = (∂x^α/∂λ)_μ
n^α = (∂x^α/∂μ)_λ
A^α(P) = n^α(P) ; A^α_L(Q) = n^α(Q)
∂u^α/∂μ = ∂²x^α/∂μ∂λ = ∂²x^α/∂λ∂μ = ∂n^α/∂λ
u^α_{;β} n^β
n^α_{;β} u^β

$$\mathcal{L}_u A = \cancel{A(p)} + A_{;\beta} u^\beta \partial\lambda - \cancel{A(p)}$$

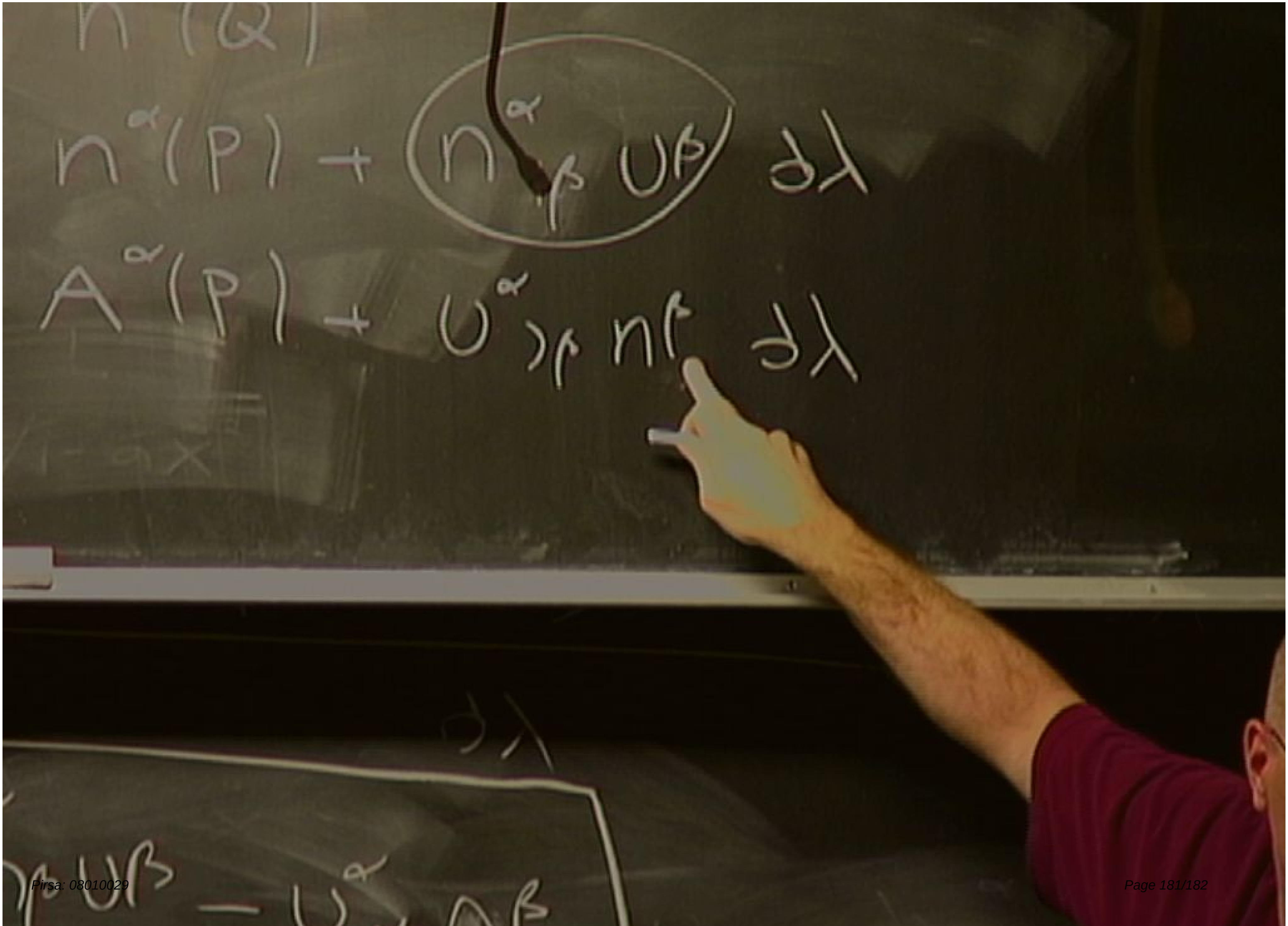
$$\partial\lambda$$

$$\boxed{\mathcal{L}_u A^\alpha = A^\alpha{}_{;\beta} u^\beta - u^\alpha{}_{;\beta} A^\beta}$$

$$\mathcal{L}_u A = \cancel{A(p)} + A_{;\beta} U^\beta \partial\lambda - \cancel{A(p)}$$

$$\partial\lambda$$

$$\boxed{\mathcal{L}_u A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta}$$


$n^\alpha(P) + n^\alpha{}_{\gamma\beta} U^\beta d\lambda$
$A^\alpha(P) + U^\alpha{}_{\gamma\beta} n^\beta d\lambda$

$$n(p) + n_{;\beta} U^\beta \, \partial\lambda$$

$$A^\alpha(p) + U^\alpha_{;\beta} A^\beta \, \partial\lambda$$

$$\frac{) + A^\alpha_{;\beta} U^\beta \, \partial\lambda - \cancel{A^\alpha(p)} - U^\alpha_{;\beta} n^\beta \, \partial\lambda}{\partial\lambda}$$

$$\boxed{\; n U^\beta - U^\alpha_{;\beta} n^\beta \;}$$