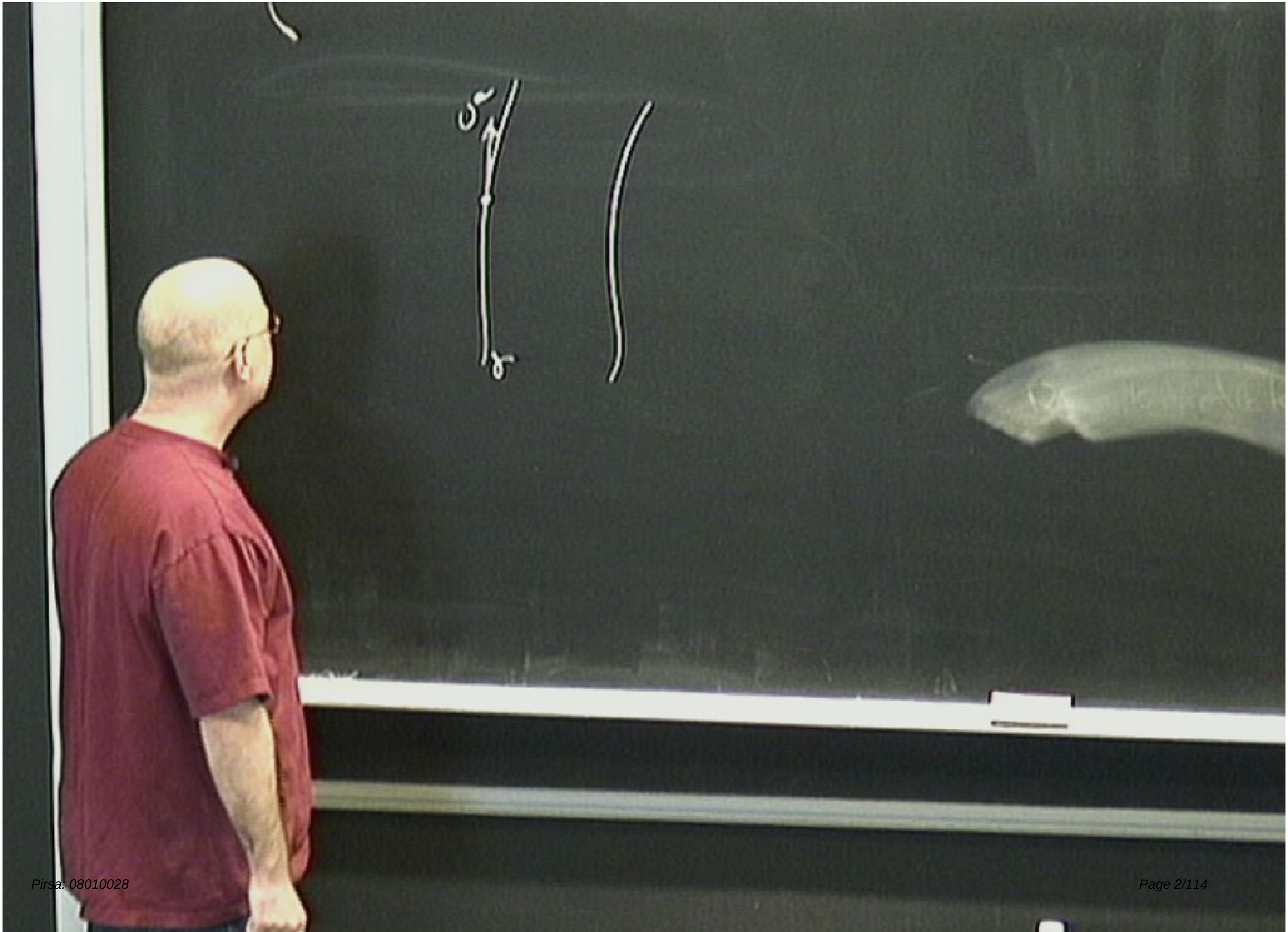
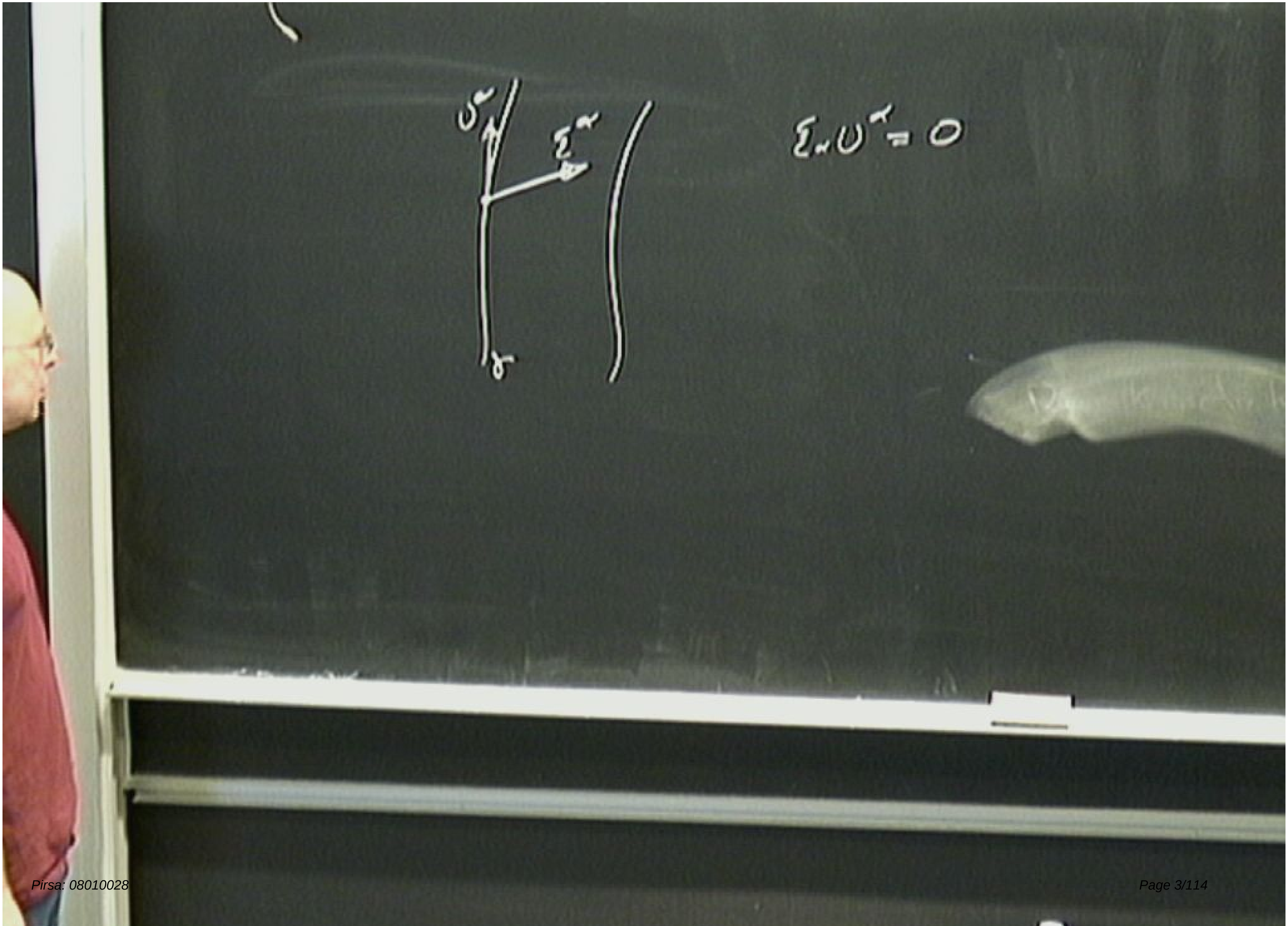
Title: Advanced General Relativity - Lecture 4A

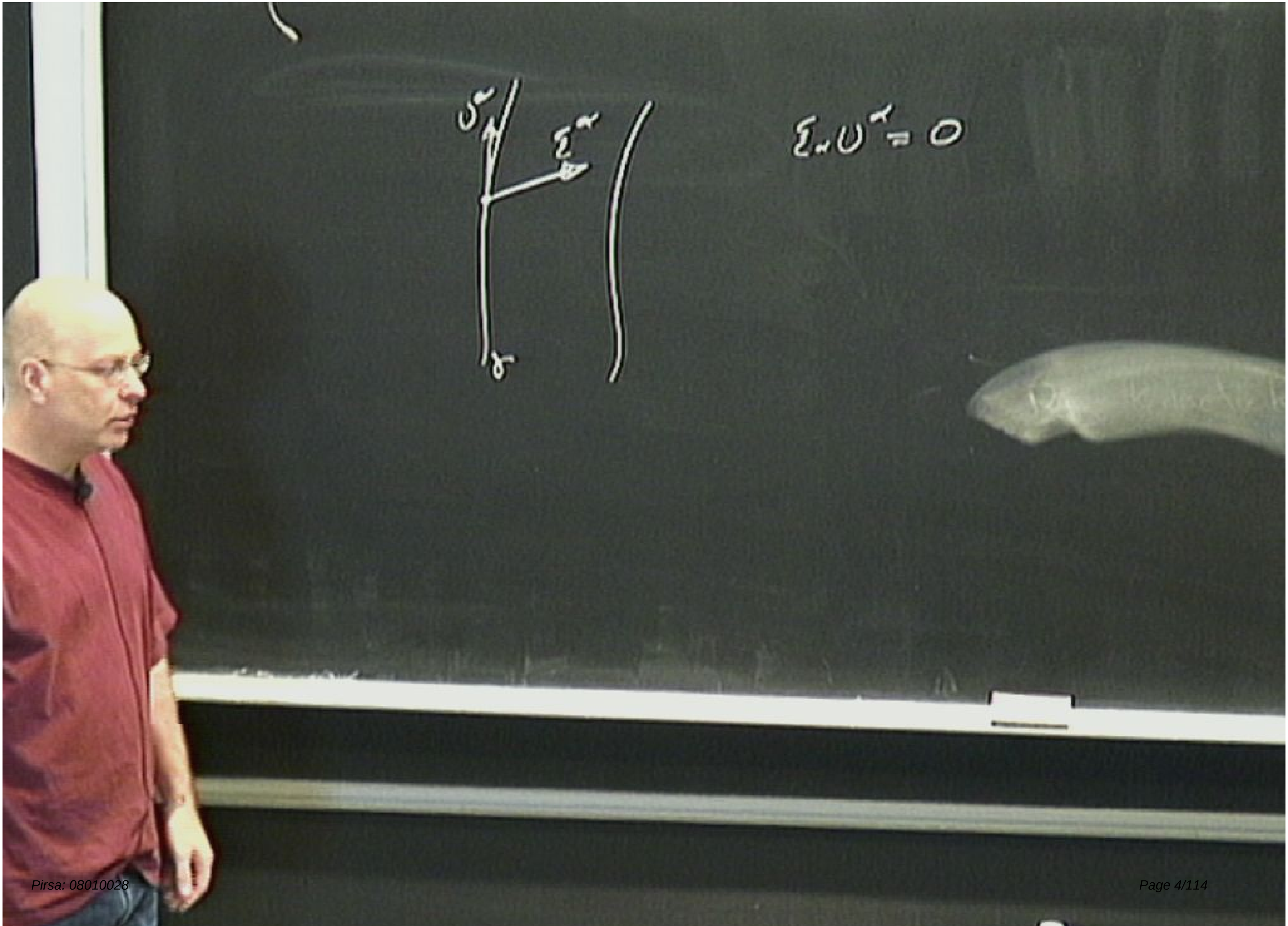
Date: Jan 30, 2008 10:30 AM

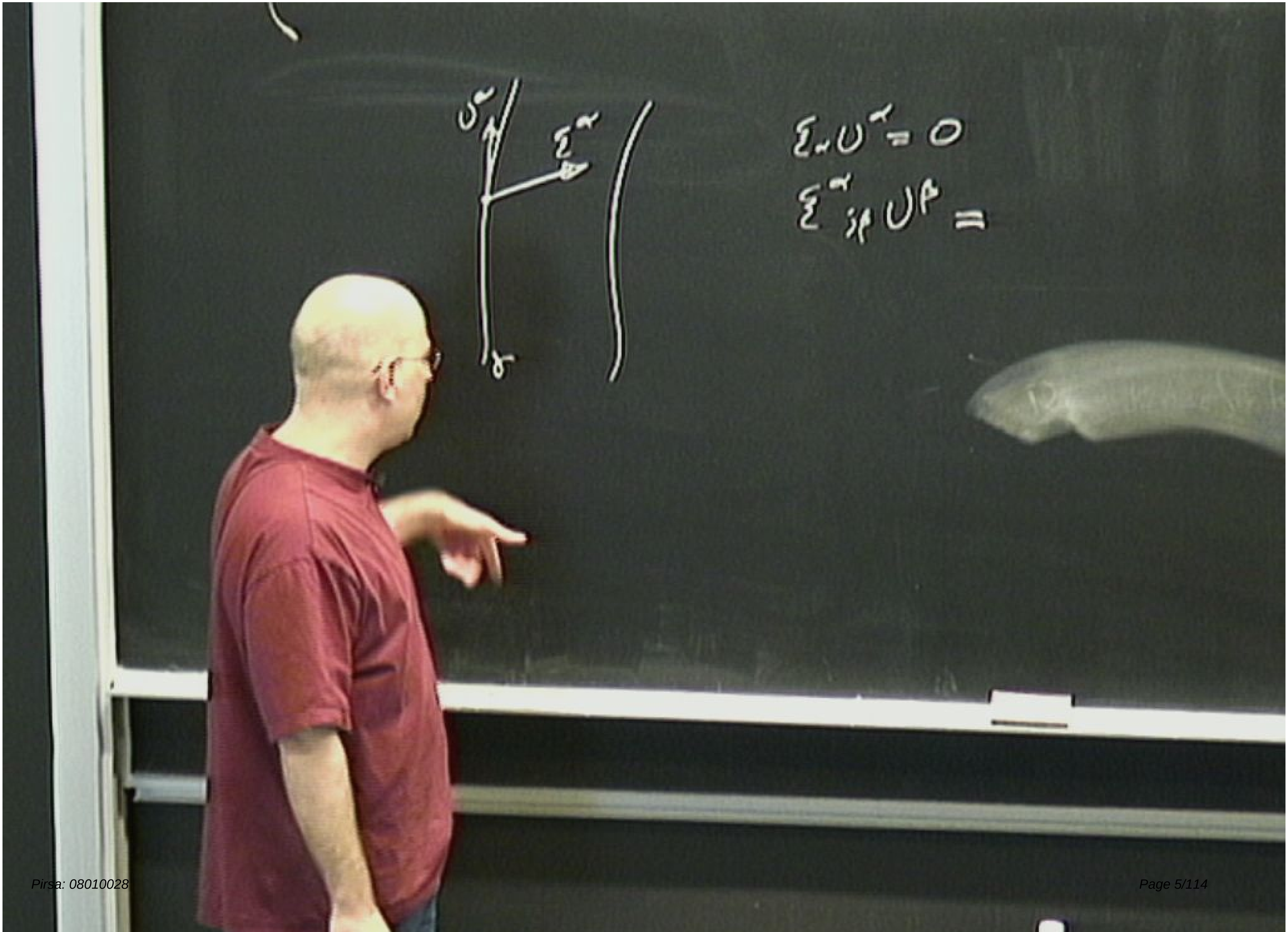
URL: http://pirsa.org/08010028

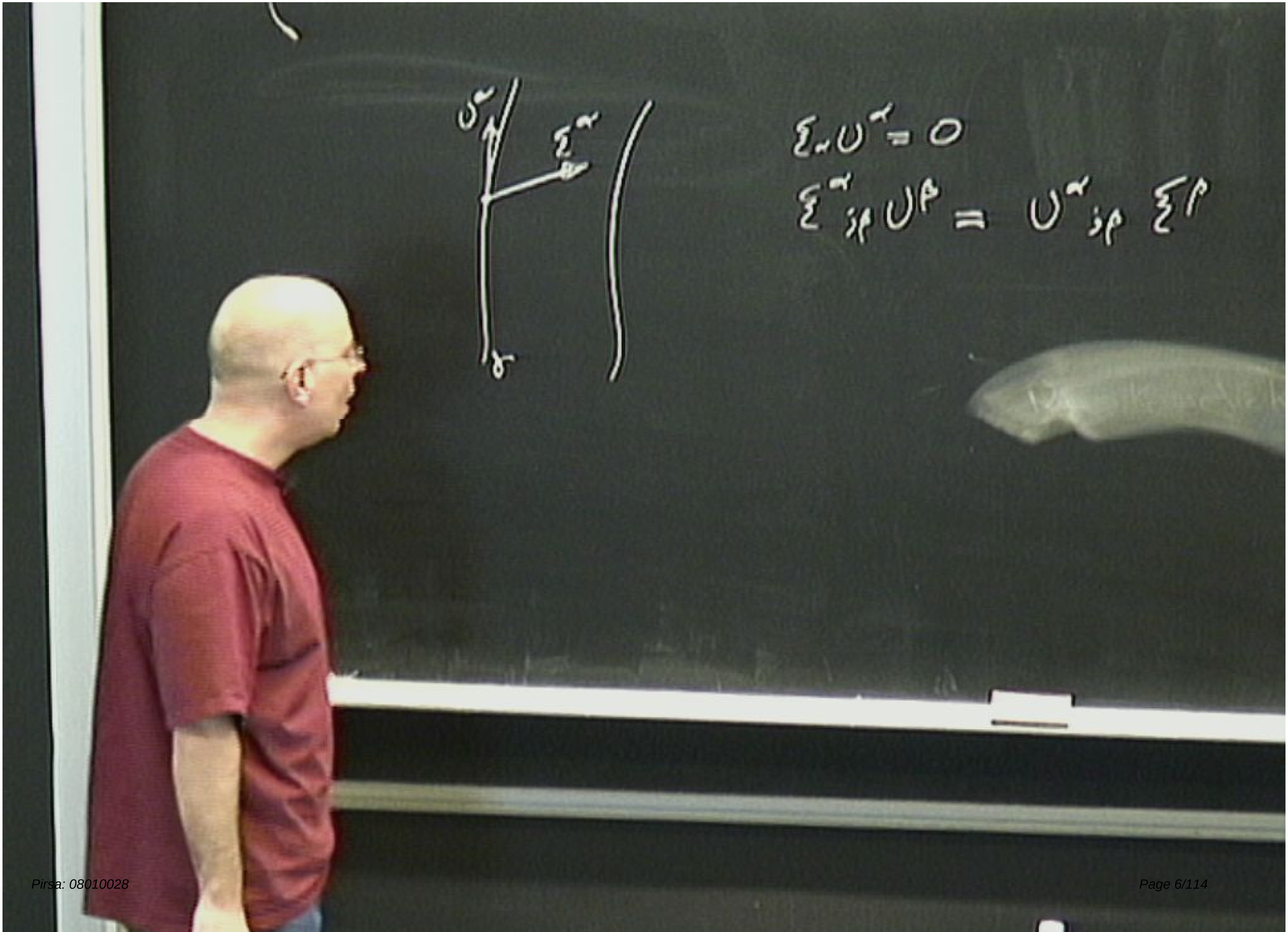
Abstract: Advanced General Relativity

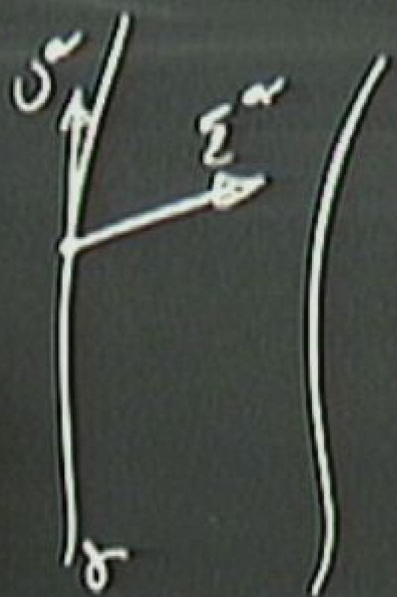



U^α
ξ^α
γ
$\xi_\alpha U^\alpha = 0$


$\xi_\mu U^\mu = 0$

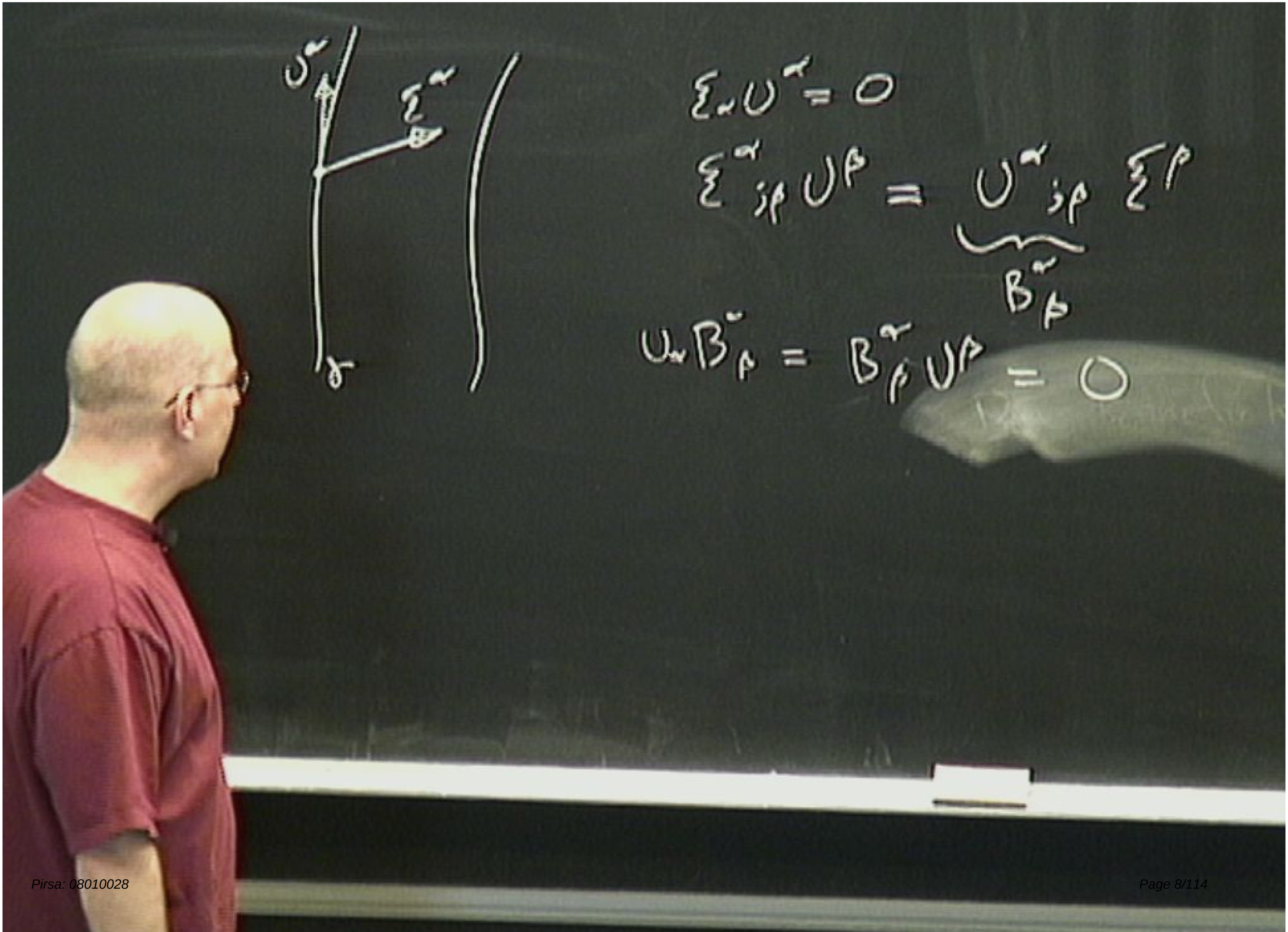

$\xi_\alpha U^\alpha = 0$
$\xi^\alpha{}_{;\beta} U^\beta =$


ξ_α U^α = 0
ξ^α_;β U^β = U^α_;β ξ^β



$$\xi_\alpha U^\alpha = 0$$

$$\xi^\alpha{}_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta}}_{B^\alpha{}_\beta} \xi^\beta$$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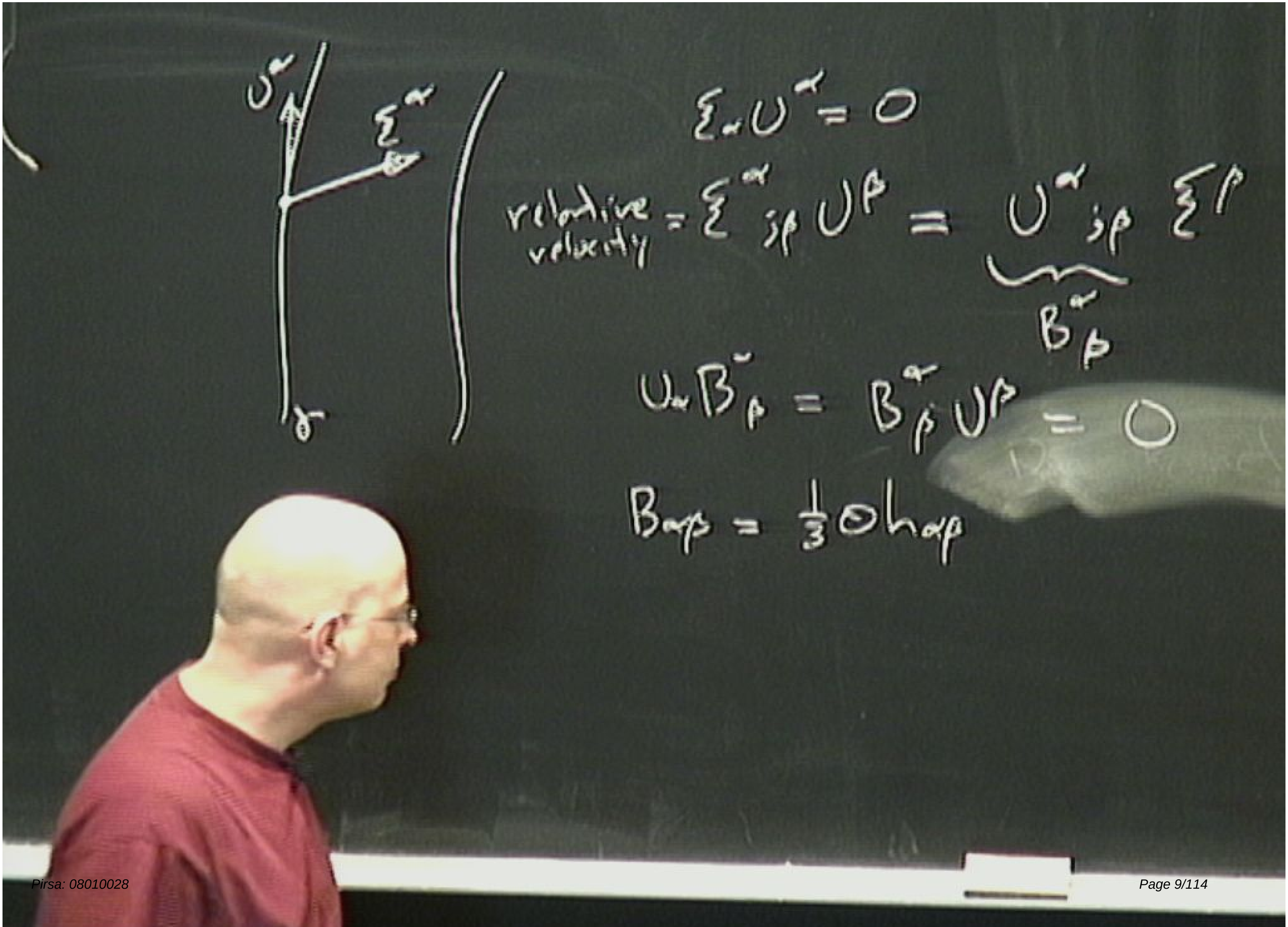
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γ
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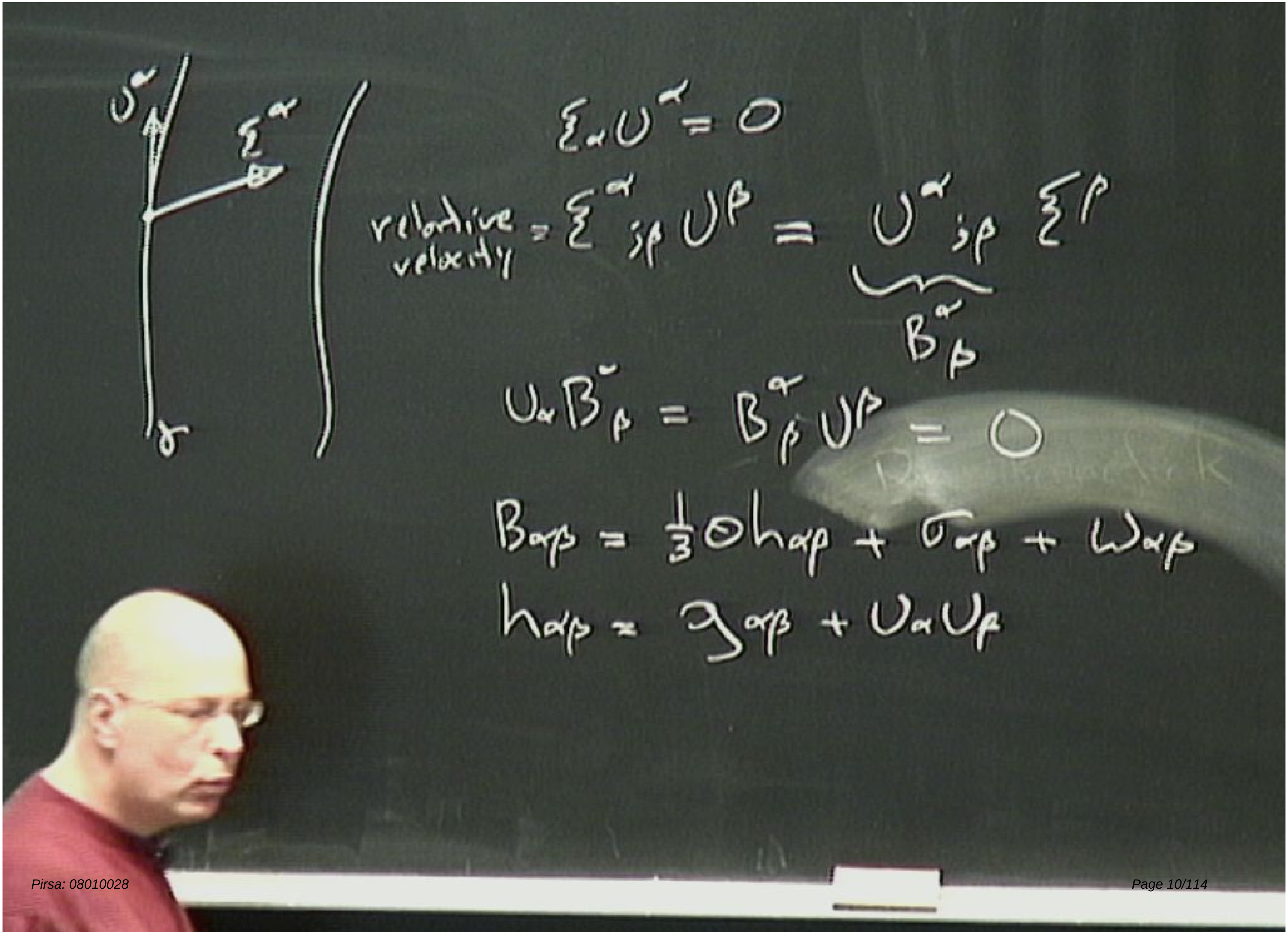
relative velocity $= \xi^\alpha{}_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta}}_{B^\alpha{}_\beta} \xi^\beta$
$U_\alpha B^\alpha{}_\beta = B^\alpha{}_\beta U^\beta = 0$
$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta}$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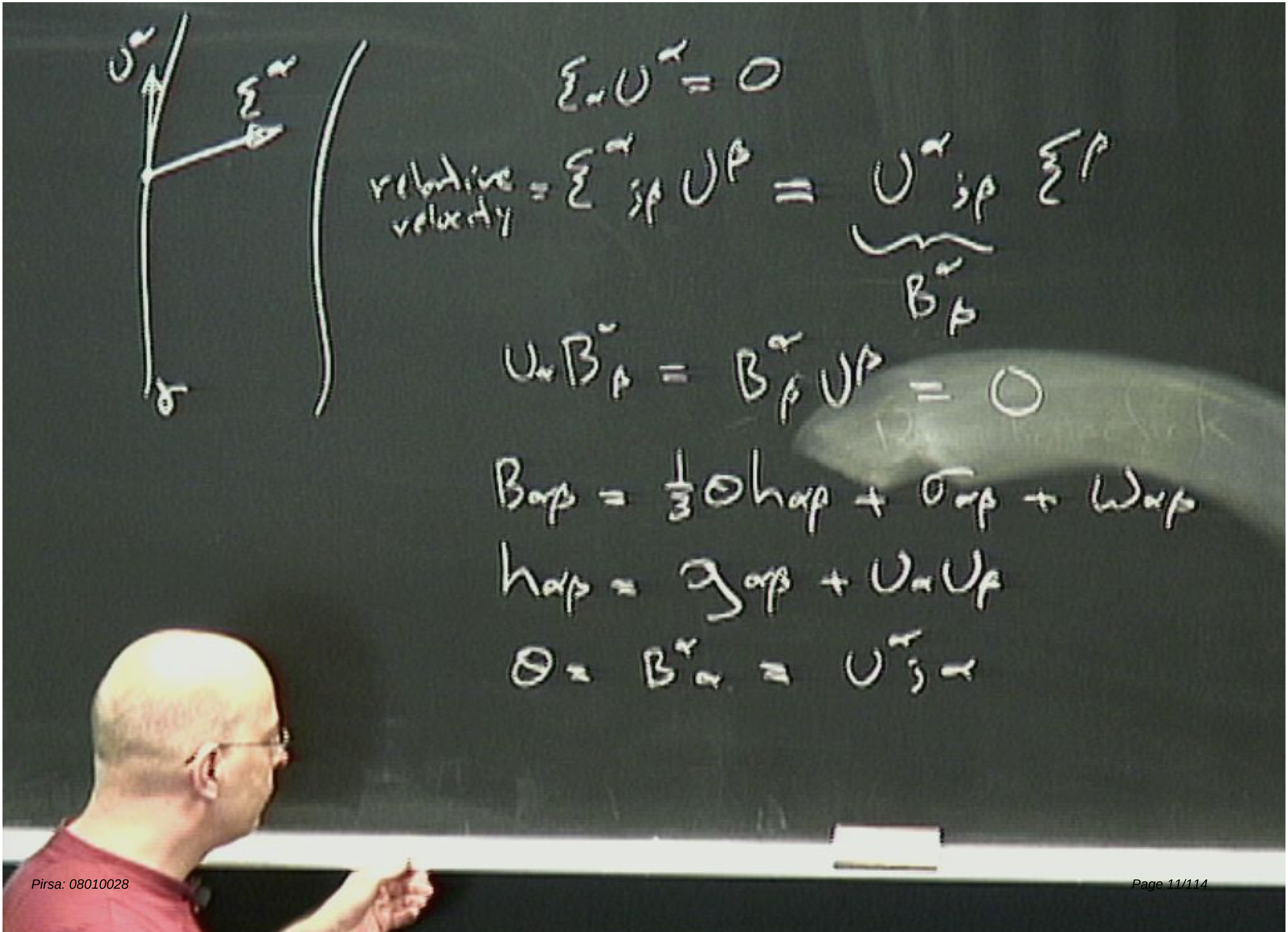

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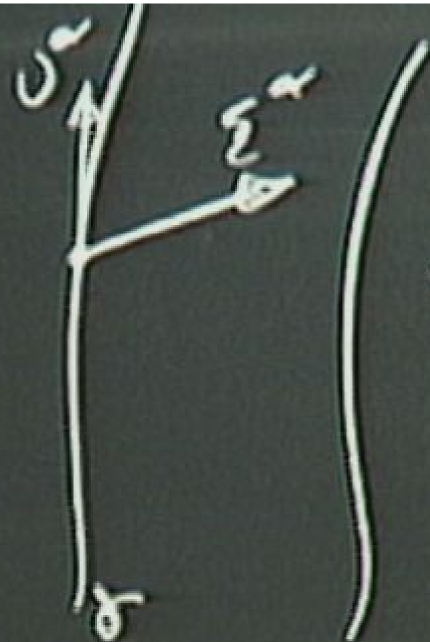

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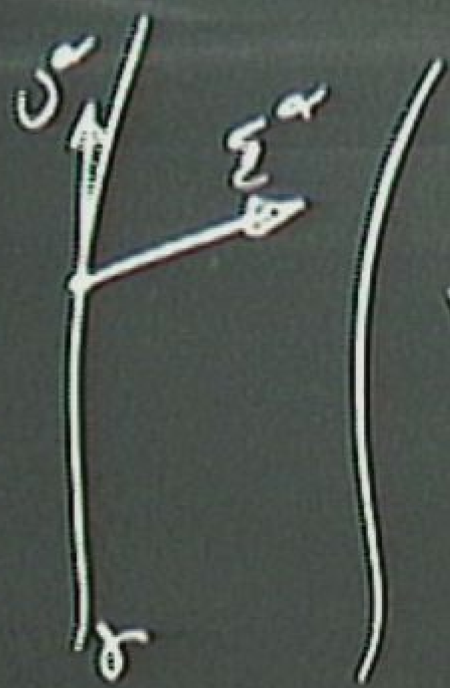
$$\sigma_{\alpha\beta} = B_{(\alpha\beta)}$$



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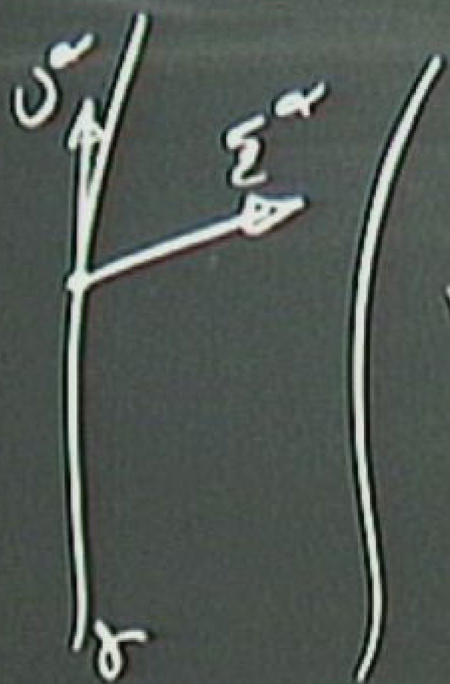

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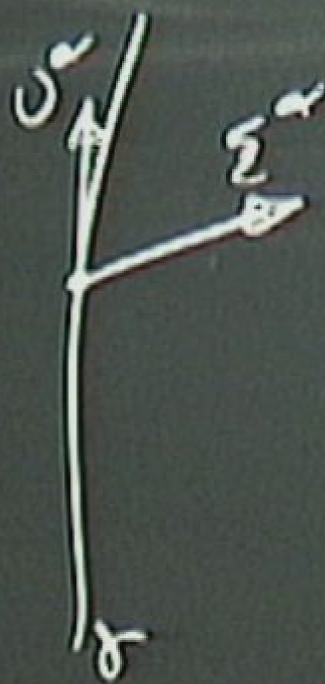

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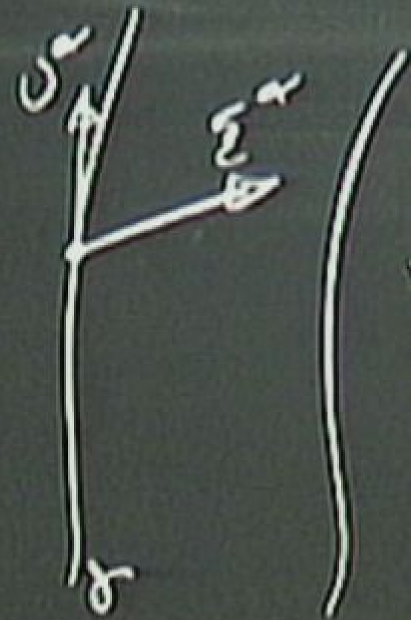

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$$\left\{\begin{array}{l} B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha{}_\alpha = U^\alpha{}_{;\alpha} \quad \text{expansion} \\ \sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B_{[\alpha\beta]} \end{array}\right.$$

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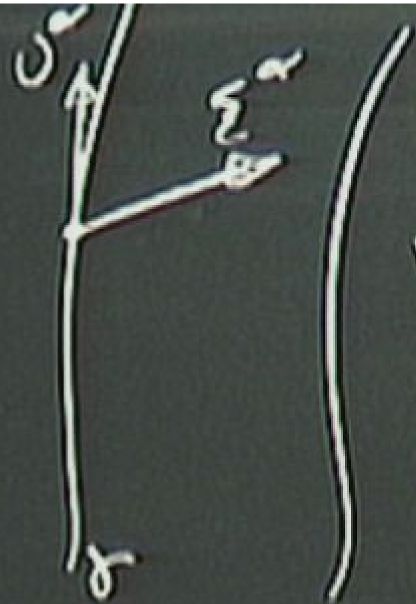

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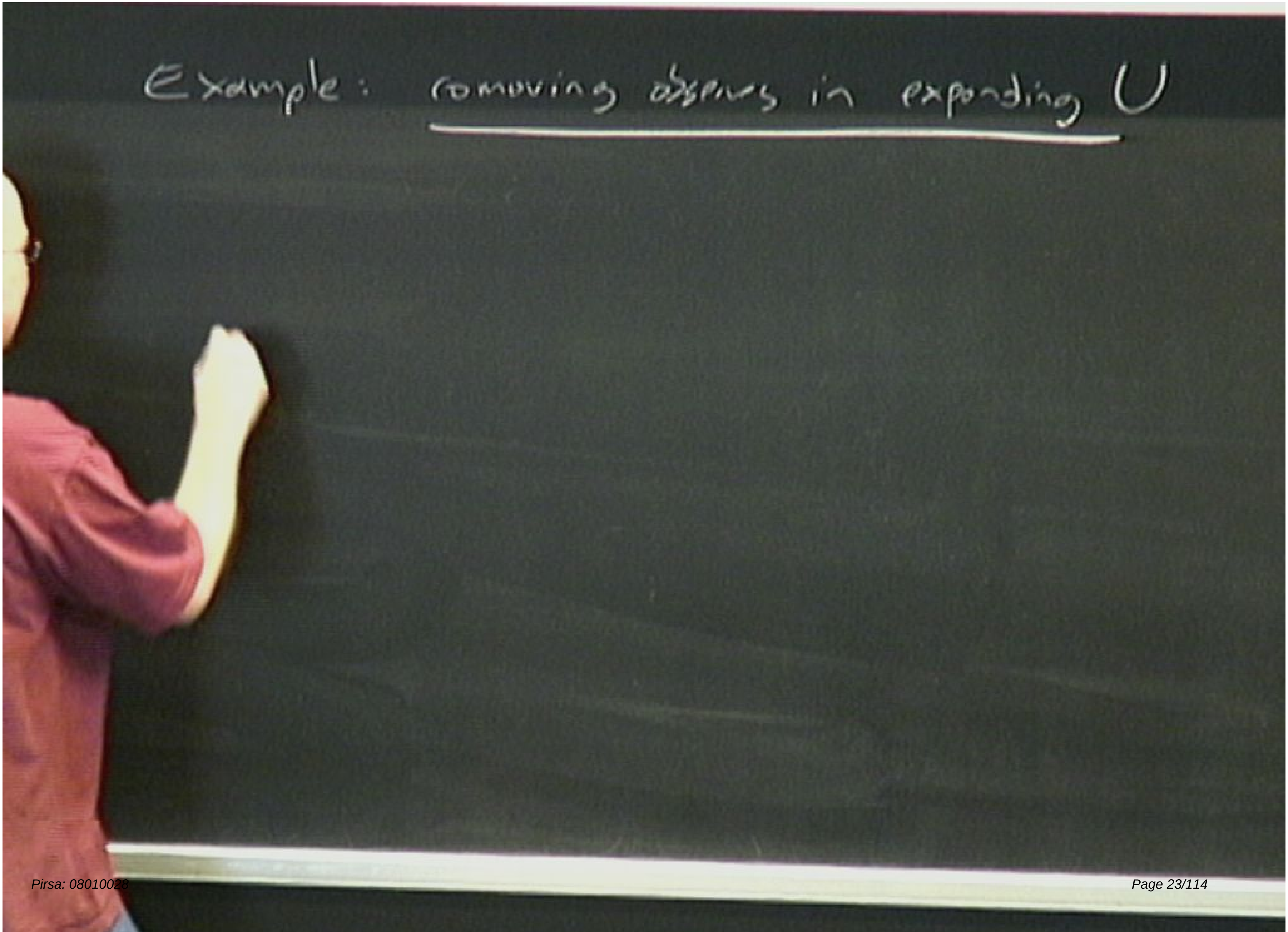
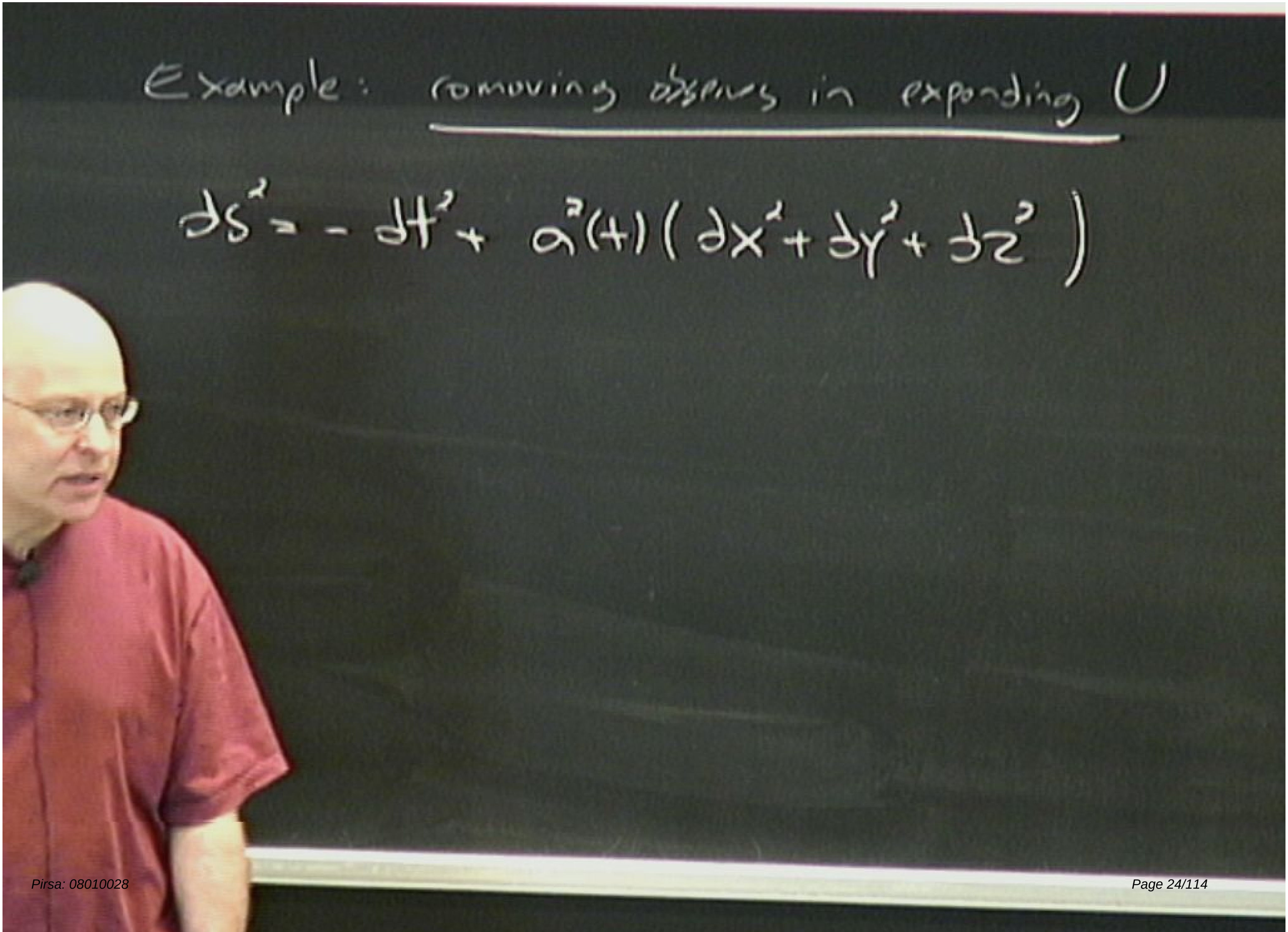
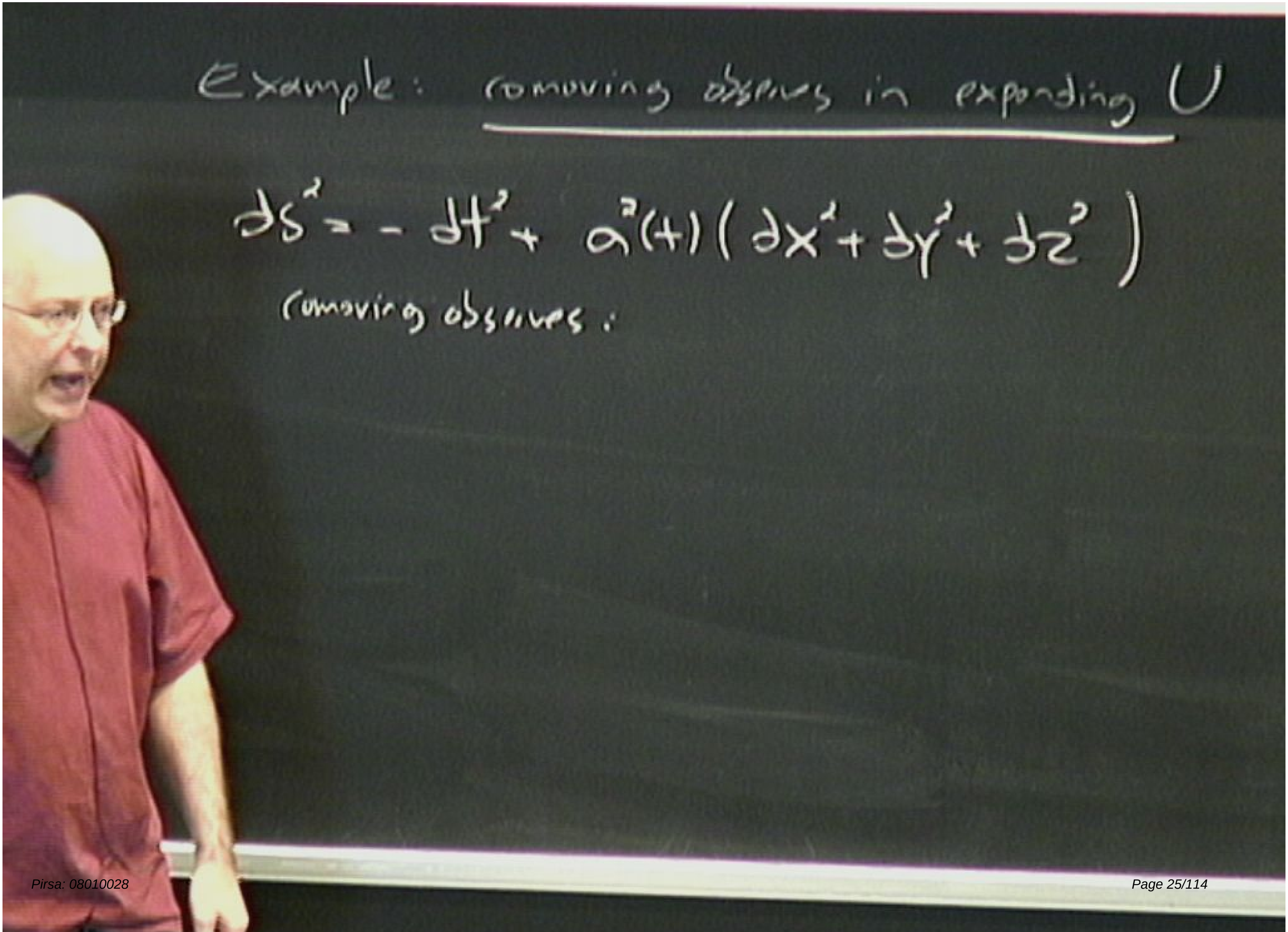

Example: comoving observers in expanding U


Example: comoving observers in expanding U
$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2)$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comoving observers:

Example: comoving observers in expanding U

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comoving observers: $U^{\alpha} = (1, 0, 0, 0)$

Example: Comoving observers in expanding U

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$$U_\alpha = (-1, 0, 0, 0)$$

Example: comoving observers in expanding U

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$$= -\frac{\partial t}{\partial x^\alpha}$$

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$$= -\partial_\alpha t$$

Example: comoving observes in expanding U

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(hypersurface orthogon

Example: comoving observers in expanding U

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comoving observers: $U^\alpha = (1,0,0,0)$

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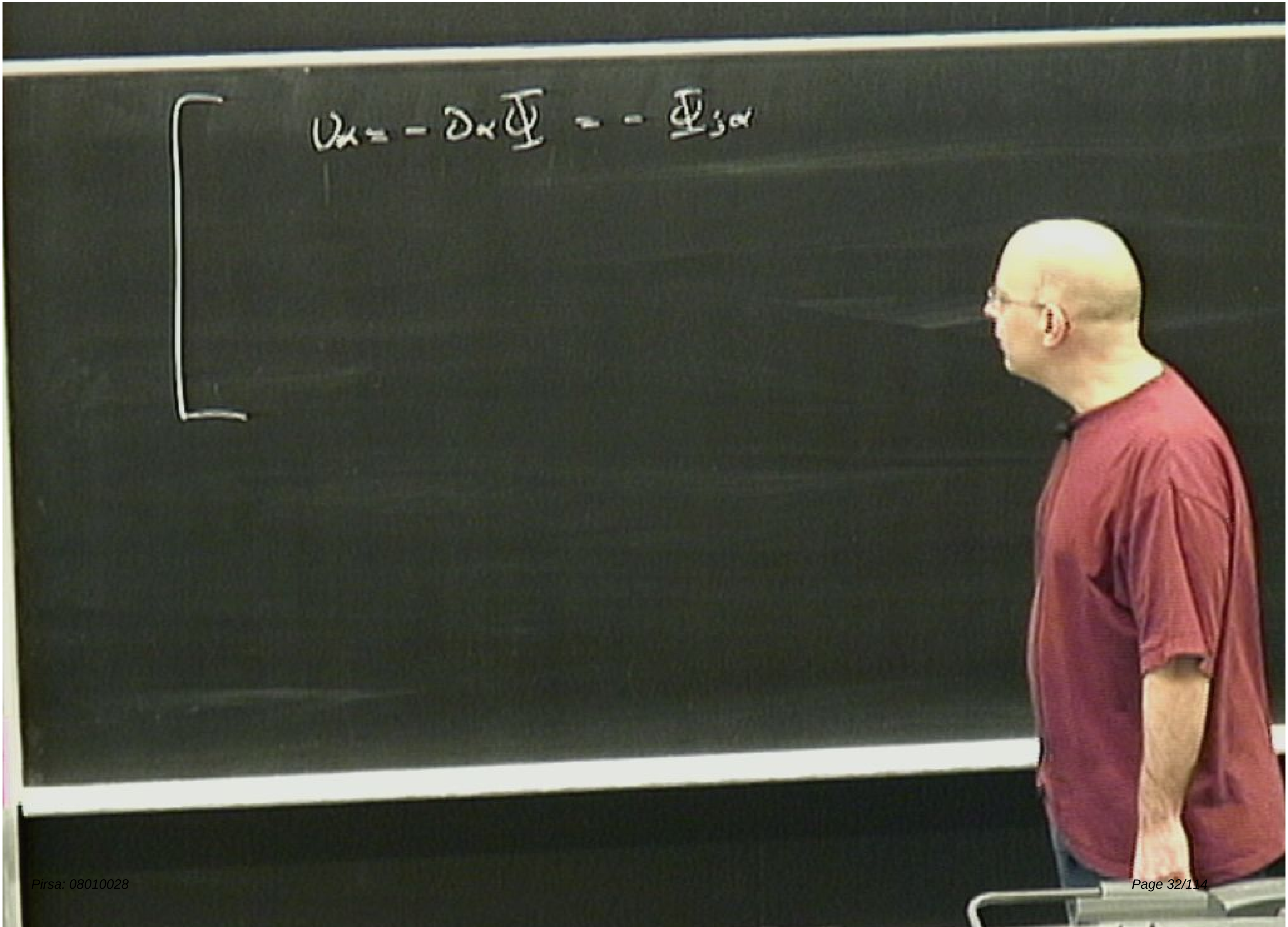
$$= -\frac{\partial t}{\partial x^\alpha}$$

$$= -\partial_\alpha t \quad \text{(hypersurface orthogonality)}$$


$v_\alpha = -D_\alpha \Phi = -\Phi_{;\alpha}$

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$$v_{\alpha;\beta} = -\Phi_{;\alpha\beta}$$

$$U_\alpha = - D_\alpha \Phi = - \Phi_{;\alpha}$$

$$U_{\alpha;\beta} = - \Phi_{;\alpha\beta} = - \Phi_{;\beta\alpha}$$

$$v_\alpha = -D_\alpha \Phi = -\Phi_{;\alpha}$$

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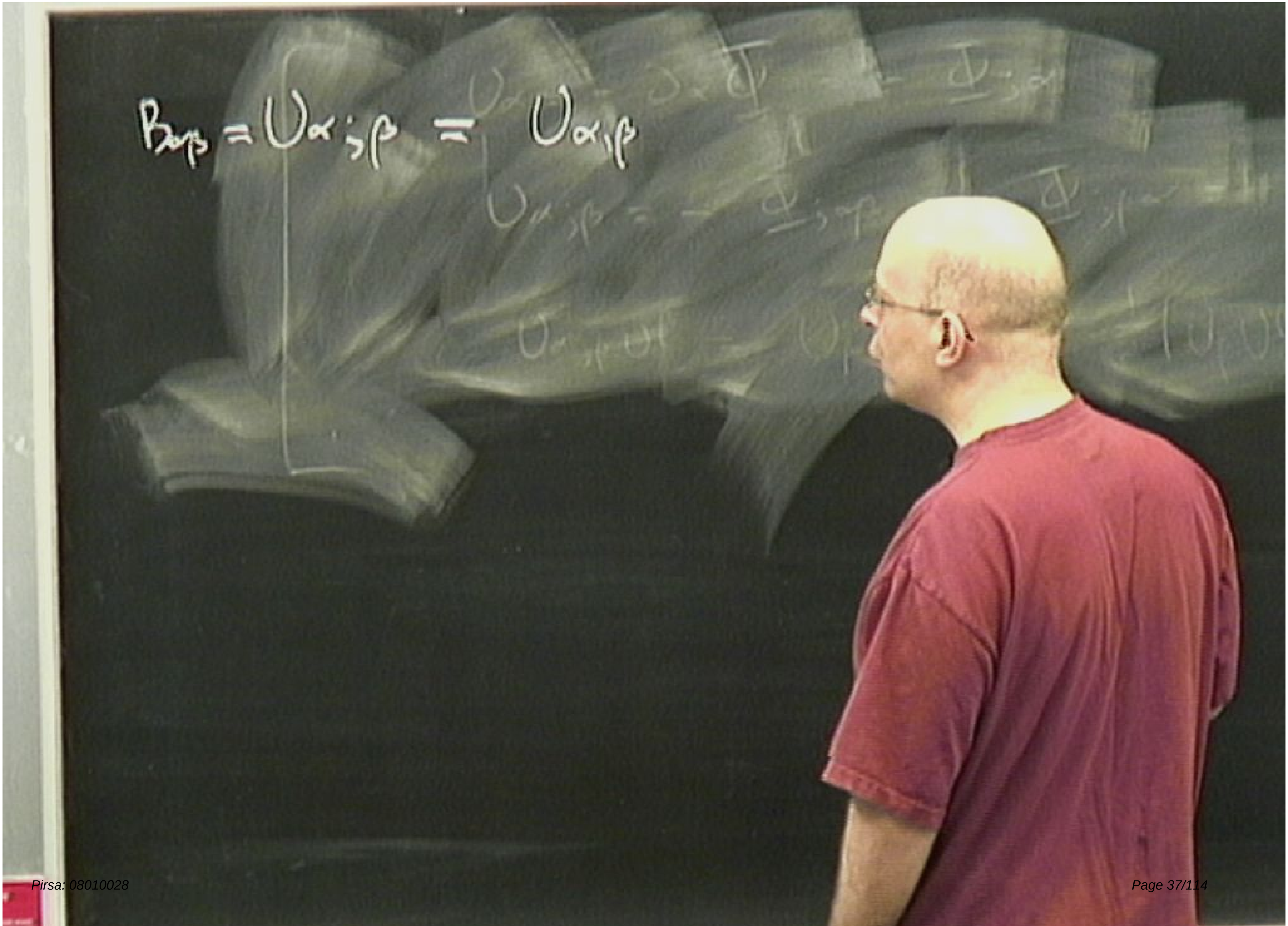
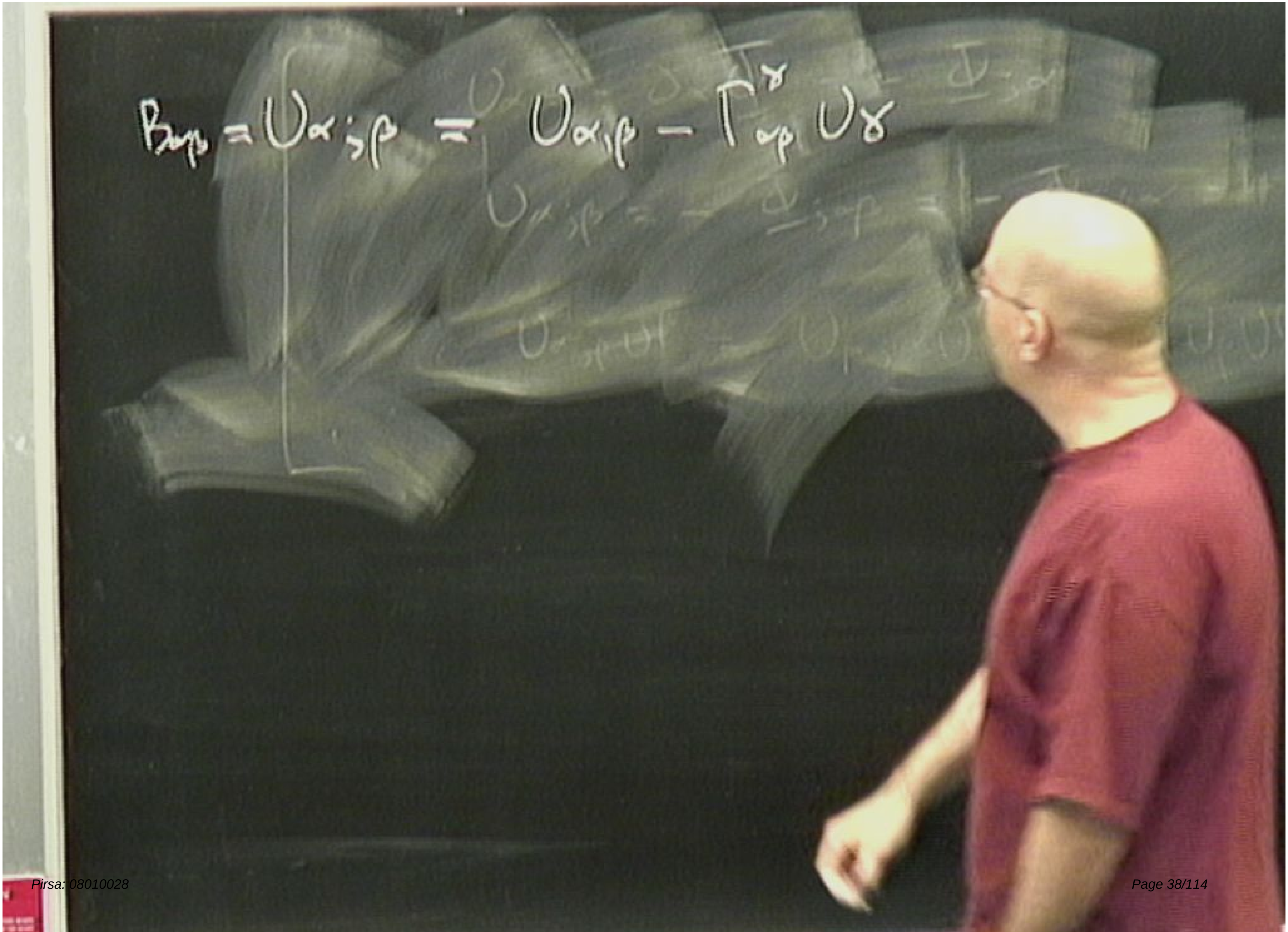
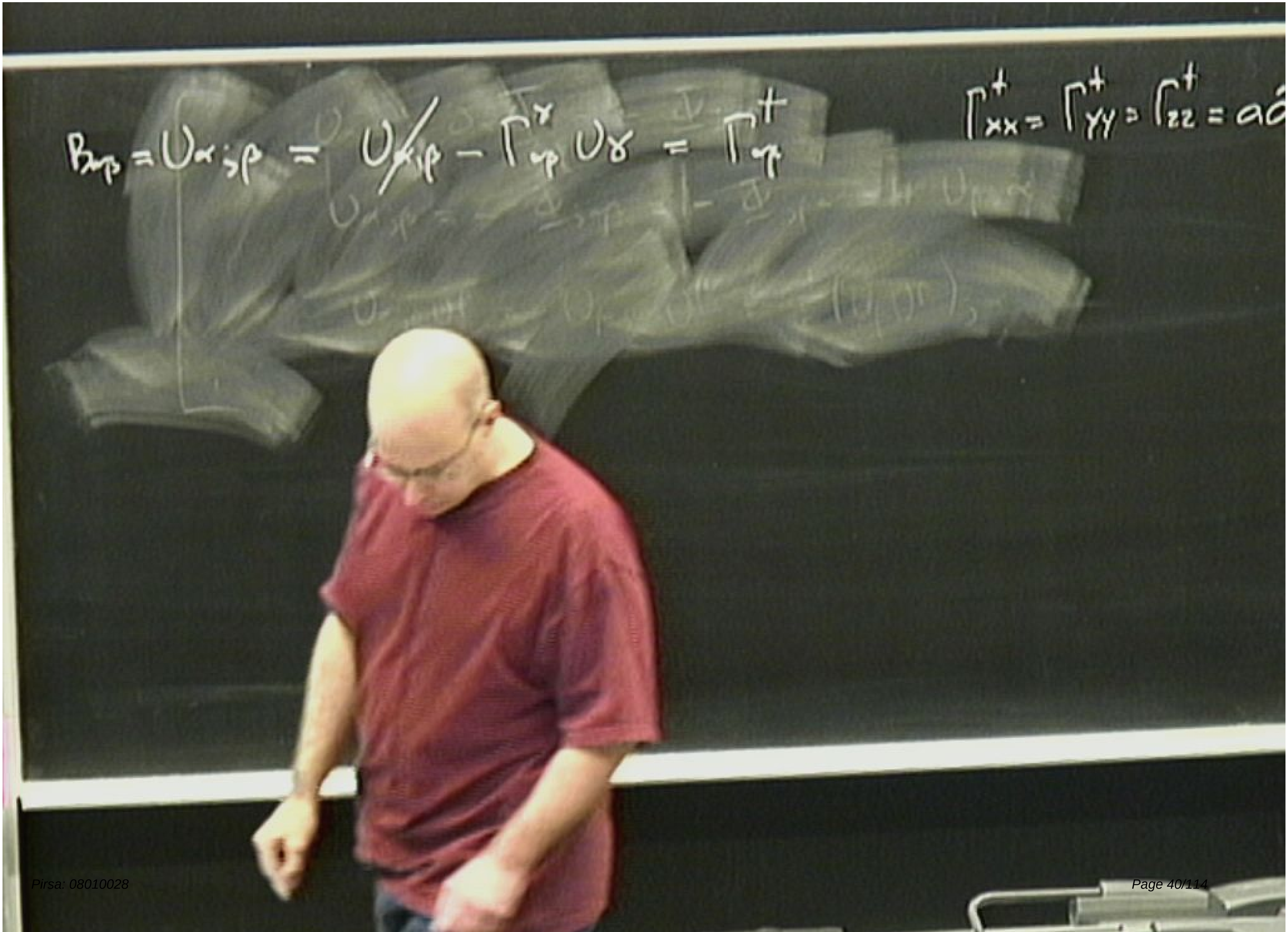
$$v_{\alpha;\beta} = -\Phi_{;\alpha\beta} = -\Phi_{;\beta\alpha} = v_{\beta;\alpha}$$

$$v_{\alpha;\beta} v^\beta = v_{\beta;\alpha} v^\beta = \frac{1}{2}(v_\beta v^\beta)_{;\alpha}$$


$B_{\alpha\beta} = U_{\alpha;\beta} = U_{\alpha,\beta}$


$B_{\alpha\beta} = U_{\alpha;\beta} = U_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma}$

$$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma} = \Gamma^{t}_{\alpha\beta}$$


$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma} = \Gamma^{t}_{\alpha\beta}$
$\Gamma^{t}_{xx} = \Gamma^{t}_{yy} = \Gamma^{t}_{zz} = a\dot{a}$

$$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_\gamma = \Gamma^{t}_{\alpha\beta}$$

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$$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma} = \Gamma^{t}_{\alpha\beta}$$

$$= \mathrm{diag}\left(0, a\dot{a}, a\dot{a}, a\dot{a}\right)$$

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$$h_{\alpha\beta} =$$

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$$= \mathrm{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$$

$$\Gamma^{t}_{xx} = \Gamma^{t}_{yy} = \Gamma^{t}_{zz} = a\dot{a}$$

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$$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

$$\Gamma^{t}_{xx} =$$

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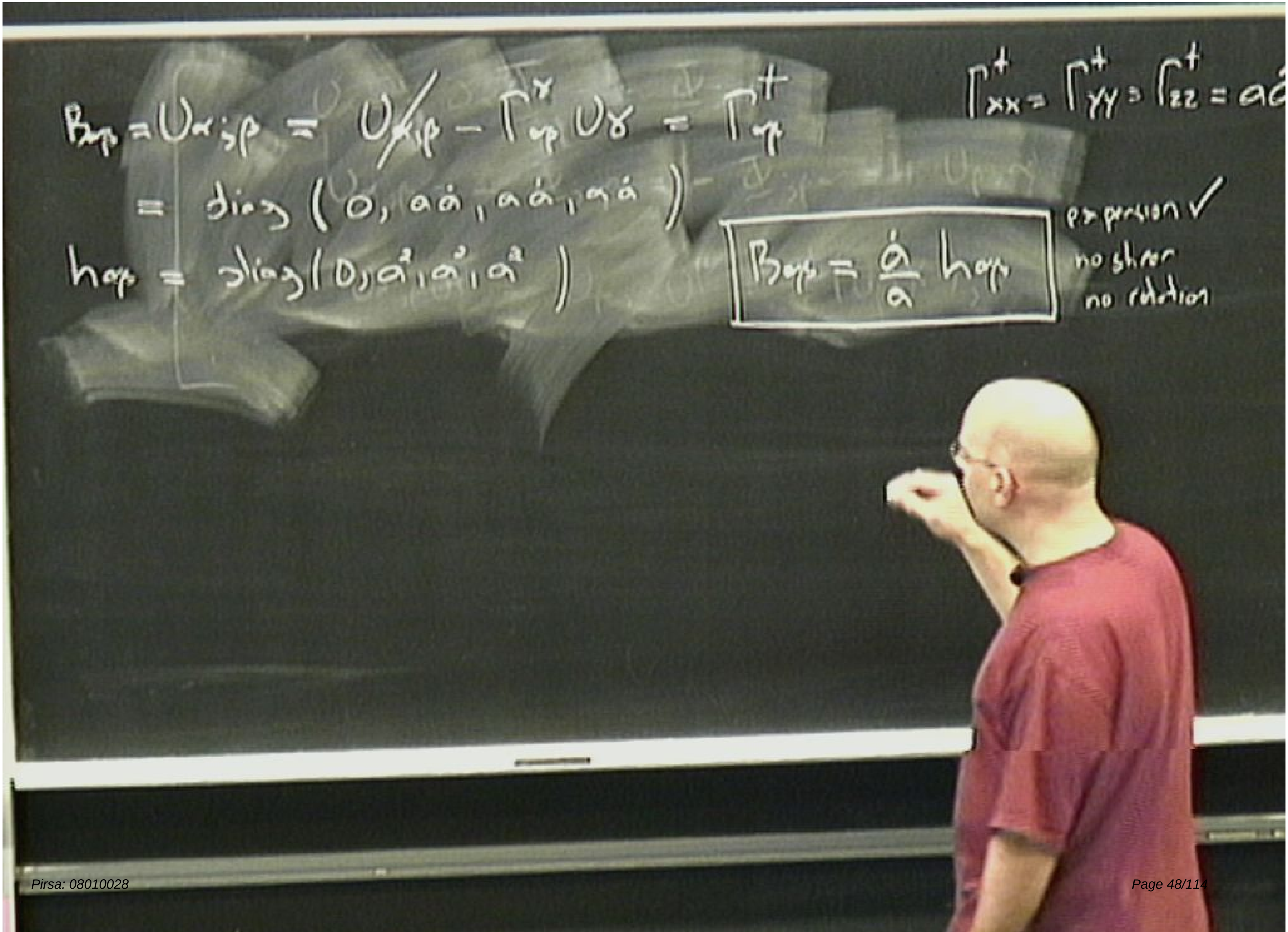
$$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$$

$$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$$



$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_\gamma = \Gamma^{t}_{\alpha\beta}$
$= \mathrm{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$
$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$
$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$
$\Gamma^{t}_{xx} = \Gamma^{t}_{yy} = \Gamma^{t}_{zz} = a\dot{a}$
expansion ✓
no shear
no rotation

$$\boxed{B_{\alpha\beta} = \frac{\dot a}{a} h_{\alpha\beta}}$$

expansion ✓
no shear
no rotation

$$\Theta = \frac{3\dot a}{a}$$

$$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_\gamma = \Gamma^{t}_{\alpha\beta}$$

$$= \mathrm{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$$

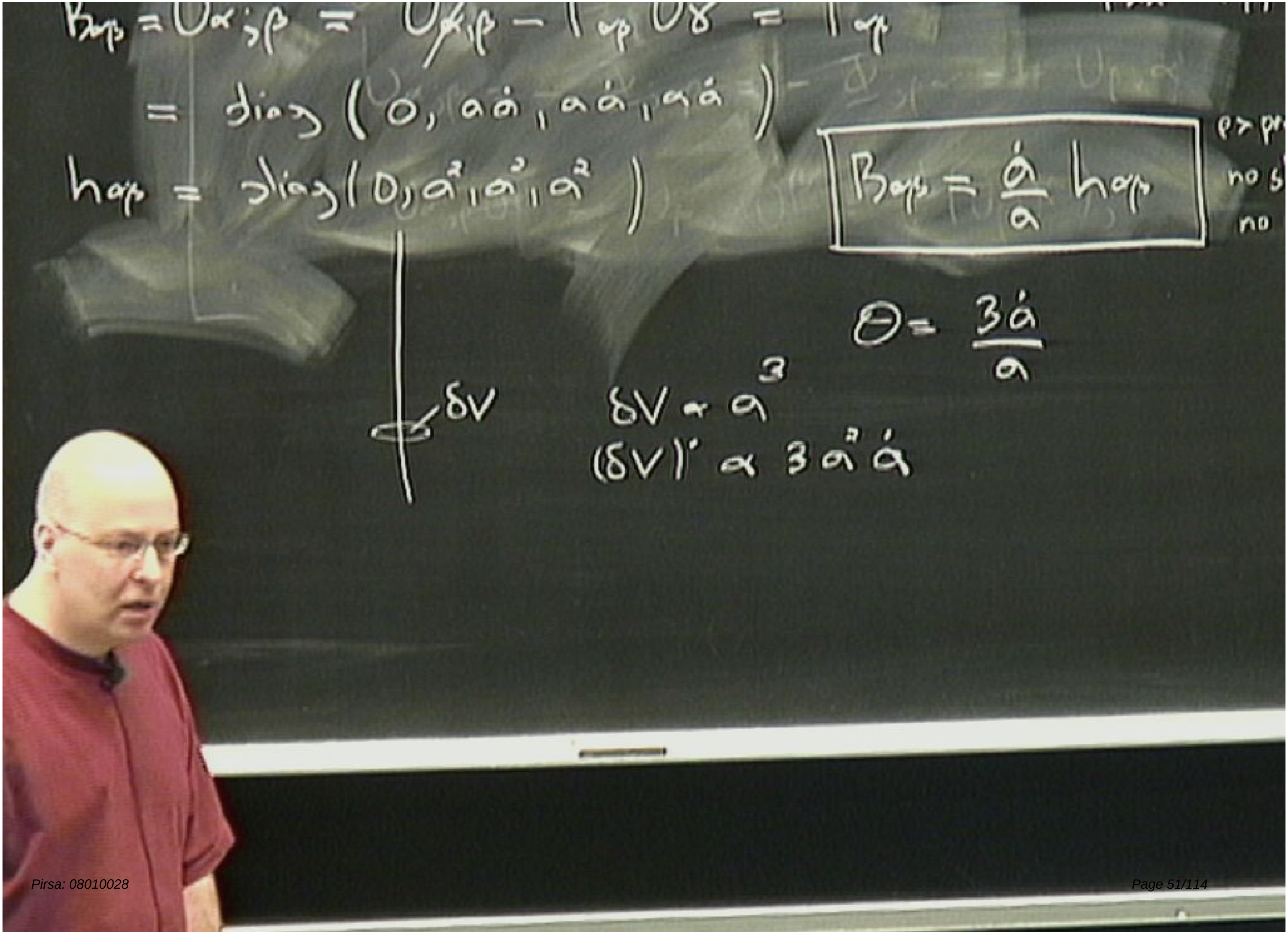
$$\Gamma^{t}_{xx} = \Gamma^{t}_{yy} = \Gamma_{z\ldots}$$

$$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$$

expansion
no shear
no rotation

$$\theta = \frac{3\dot{a}}{a}$$

δV $\qquad \delta V \propto a^3$


$B_{\alpha\beta} = U_{\alpha;\beta} = U_{\alpha,\beta} - \Gamma_{\alpha\beta} U_\gamma = \Gamma_{\alpha\beta}$
$= \mathrm{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$
$h_{\alpha\beta} = \mathrm{diag}(0, a^2, a^2, a^2)$
$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$
$\theta = \frac{3\dot{a}}{a}$
δV
$\delta V \propto a^3$
$(\delta V)^{\cdot} \propto 3a^2\dot{a}$

$$B_{\alpha\beta} = U_{\alpha;\beta} = \cancel{U_{\alpha,\beta}} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma} = \Gamma_{\alpha\beta}$$

$$= \mathrm{diag}\left(0, a\dot{a}, a\dot{a}, a\dot{a}\right)$$

$$h_{\alpha\beta} = \mathrm{diag}\left(0, a^2, a^2, a^2\right)$$

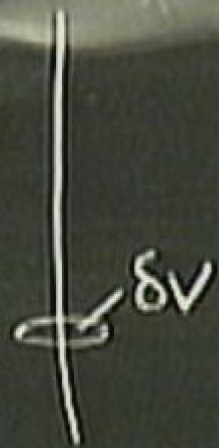
$$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$$

expr

no s

no

$$\theta = \frac{3\dot{a}}{a}$$



$$\delta V \propto a^3$$

$$(\delta V)^{\cdot} \propto 3a^2\dot{a}$$

$$\frac{1}{\delta V}(\delta V)^{\cdot} = \frac{3\dot{a}}{a}$$

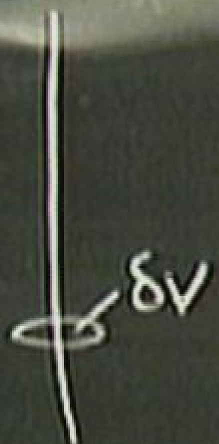
$$\beta = \partial_\alpha u_\beta - \Gamma_{\alpha\beta}\,\partial\gamma = \Gamma_{\alpha\beta}$$

$$(0, a\dot a, a\dot a, a\dot a)$$

$$\mathrm{diag}(0, a^2, a^2, a^2)$$

$$\boxed{B_{\alpha\beta} = \frac{\dot a}{a} h_{\alpha\beta}}$$

expansion ✓
no shear
no rotation



$$\theta = \frac{3\dot a}{a} = \frac{1}{\delta V}(\delta V)'$$

$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2\dot a$$

$$\frac{1}{\delta V}(\delta V)' = \frac{3\dot a}{a}$$

$$\ldots_{\beta} = U_{\alpha;\beta} - \Gamma^{\gamma}_{\alpha\beta} U_{\gamma} = \Gamma_{\alpha\beta}$$

$$\ldots\mathrm{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$\mathrm{diag}(0, a^2, a^2, a^2)$$

$$\boxed{B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}}$$

expansion ✓
no shear
no rotation

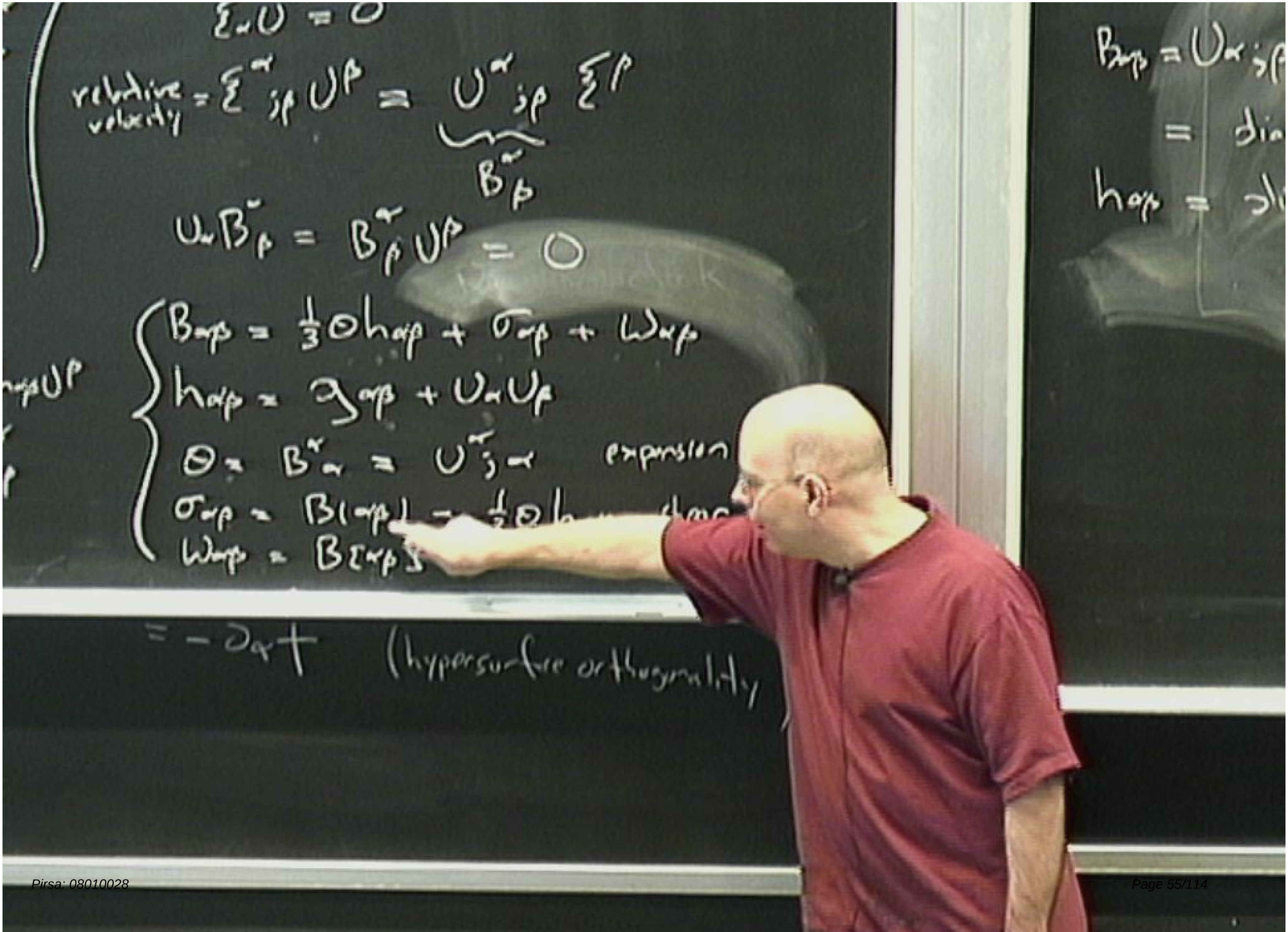
$$\boxed{\Theta = \frac{3\dot{a}}{a} = \frac{1}{\delta V}(\delta V)'}$$

δV

$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2\dot{a}$$

$$\frac{1}{\delta V}(\delta V)' = \frac{3\dot{a}}{a}$$



$\xi_\alpha U = 0$
relative velocity $= \xi^\alpha{}_{;\beta} U^\beta = \underbrace{U^\alpha{}_{;\beta}}_{B^\alpha{}_\beta} \xi^\beta$
$U_\alpha B^\alpha{}_\beta = B^\alpha{}_\beta U^\beta = 0$
$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$
$\Theta = B^\alpha{}_\alpha = U^\alpha{}_{;\alpha}$ expansion
$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\Theta h$
$\omega_{\alpha\beta} = B_{[\alpha\beta]}$
$= -\partial_\alpha t$ (hypersurface orthogonality
$B_{\alpha\beta} = U_{\alpha;\beta}$
$h_{\alpha\beta}$ =

Dynamics: evolution equations for Θ, σ_{ab}, ω_{ab}

Dynamics: evolution equations for Θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$

$$B_{\alpha\beta\zeta\mu} U^{\mu} =$$

Dynamics : evolution equations for Θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$

$$B_{\alpha\beta;\mu} U^\mu = U_{\alpha;\beta\mu} U^\mu$$

Dynamics: evolution equations for $\Theta, \sigma_{\alpha\beta}, \omega_{\alpha\beta}$

$$B_{\alpha\beta;\mu} U^{\mu} = U_{\alpha;\beta\mu} U^{\mu}$$

$$= \left(U_{\alpha;\mu\beta} - R_{\alpha\nu\beta\mu} U^{\nu} \right) U^{\mu}$$

$$B_{\alpha\beta;\mu} U^\mu = U_{\alpha;\beta\mu} U^\mu$$

$$= \left(U_{\alpha;\mu\beta} - R_{\alpha\nu\beta\mu} U^\nu \right) U^\mu$$

$$= U_{\alpha;\mu\beta} U^\mu - R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

$$B_{\alpha\beta;\mu} U^{\mu} = U_{\alpha;\beta\mu} U^{\mu}$$

$$= \left(U_{\alpha;\mu\beta} - R_{\alpha\nu\beta\mu} U^{\nu} \right) U^{\mu}$$

$$= U_{\alpha;\mu\beta} U^{\mu} - R_{\alpha\mu\beta\nu} U^{\mu} U^{\nu}$$

$$= \left(U_{\alpha;\mu} U^{\mu} \right)_{;\beta} - U_{\alpha;\mu} U^{\mu}{}_{;\beta} - R_{\alpha\mu\beta\nu} U^{\mu} U^{\nu}$$

$$B_{\alpha\beta;\mu}U^\mu = U_{\alpha;\beta\mu}U^\mu$$

$$= \left(U_{\alpha;\mu\beta} - R_{\alpha\nu\mu\beta}U^\nu\right)U^\mu$$

$$= U_{\alpha;\mu\beta}U^\mu - R_{\alpha\mu\beta\nu}U^\mu U^\nu$$

$$= \cancel{\left(U_{\alpha;\mu}U^\mu\right)_{;\beta}} - U_{\alpha;\mu}U^\mu{}_{;\beta} - R_{\alpha\mu\beta\nu}U^\mu U^\nu$$

$$B_{\alpha\beta;\mu}U^\mu = - \ \ldots \ - R_{\alpha\mu\beta\nu}U^\mu U^\nu$$

$$B_{\alpha\beta;\rho}U^\rho = U_{\alpha;\beta\rho}U^\rho$$

$$= \left(U_{\alpha;\rho\beta} - R_{\alpha\nu\rho\beta}U^\nu\right)U^\rho$$

$$= U_{\alpha;\rho\beta}U^\rho - R_{\alpha\rho\beta\nu}U^\rho U^\nu$$

$$= \left(\cancel{U_{\alpha;\rho}U^\rho}\right)_{;\beta} - U_{\alpha;\rho}U^\rho{}_{;\beta} - R_{\alpha\rho\beta\nu}U^\rho U^\nu$$

$$B_{\alpha\beta;\rho}U^\rho = -B_{\alpha\rho}B^\rho{}_\beta - R_{\alpha\rho\beta\nu}U^\rho U^\nu$$

$$= U_{\alpha;\beta\mu}U^\mu - R_{\alpha\mu\beta\nu}U^\mu U^\nu$$

$$= (\cancel{U_{\alpha;\mu}U^\mu})_{;\beta} - U_{\alpha;\mu}U^\mu{}_{;\beta} - R_{\alpha\mu\beta\nu}U^\mu$$

$$\boxed{B_{\alpha\beta;\mu}U^\mu = - B_{\alpha\mu}B^\mu{}_\beta - R_{\alpha\mu\beta\nu}U^\mu U^\nu}$$

Take the trace :

$$= U_{\alpha;\mu\beta} U^\mu - R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

$$= \cancel{(U_{\alpha;\mu} U^\mu)_{;\beta}} - U_{\alpha;\mu} U^\mu{}_{;\beta} - R_{\alpha\mu\beta\nu} U^\mu$$

$$\boxed{B_{\alpha\beta;\mu} U^\mu = - B_{\alpha\mu} B^\mu{}_\beta - R_{\alpha\mu\beta\nu} U^\mu U^\nu}$$

Take the trace : $\frac{d\theta}{d\tau}$

$$= U_{\alpha;\mu;\beta} U^\mu - R_{\alpha\mu\beta\nu} U^\mu U^\nu$$

$$= \cancel{(U_{\alpha;\mu} U^\mu)_{;\beta}} - U_{\alpha;\mu} U^\mu{}_{;\beta} - R_{\alpha\mu\beta\nu} U^\mu$$

$$\boxed{B_{\alpha\beta;\mu} U^\mu = - B_{\alpha\mu} B^\mu{}_\beta - R_{\alpha\mu\beta\nu} U^\mu U^\nu}$$

Take the trace : $\frac{d\theta}{d\tau} = - B_{\alpha\mu} B^{\mu\alpha}$

$$= \cancel{(U_{\alpha;\mu} U^{\mu})_{;\beta}} - U_{\alpha;\mu} U^{\mu}{}_{;\beta} - R_{\alpha\mu\beta\nu}$$

$$\boxed{B_{\alpha\beta;\mu} U^{\mu} = - B_{\alpha\mu} B^{\mu}{}_{\beta} - R_{\alpha\mu\beta\nu} U^{\mu} U^{\nu}}$$

Take the trace: $$\frac{d\theta}{d\tau} = - B_{\alpha\mu} B^{\mu\alpha} - R_{\mu\nu} U^{\mu} U^{\nu}$$

$$B_{\alpha\beta;\mu}U^{\mu} = -B_{\alpha\mu}B^{\mu}{}_{\beta} - R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu}$$

re : $$\frac{d\theta}{d\tau} = -B_{\alpha\mu}B^{\mu\alpha} - R_{\mu\nu}U^{\mu}U^{\nu}$$

$$B_{\alpha\mu}B^{\mu\alpha} = \left(\tfrac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu}\right)\left(\tfrac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha}\right)$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left(\tfrac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right)\left(\tfrac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \tfrac{1}{9}\theta^2$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left(\frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right)\left(\frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9}\theta^2 (3)$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left(\frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right)\left(\frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu}\sigma$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left(\frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left(\frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu}\sigma^{\alpha\mu} - \omega_{\alpha\mu}\omega^{\alpha\mu}$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left(\tfrac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left(\tfrac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \tfrac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu}\sigma^{\alpha\mu} - \omega_{\alpha\mu}\omega^{\alpha\mu}$$

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}U^{\alpha}U^{\beta}}$$

Raychaudhuri's equation.

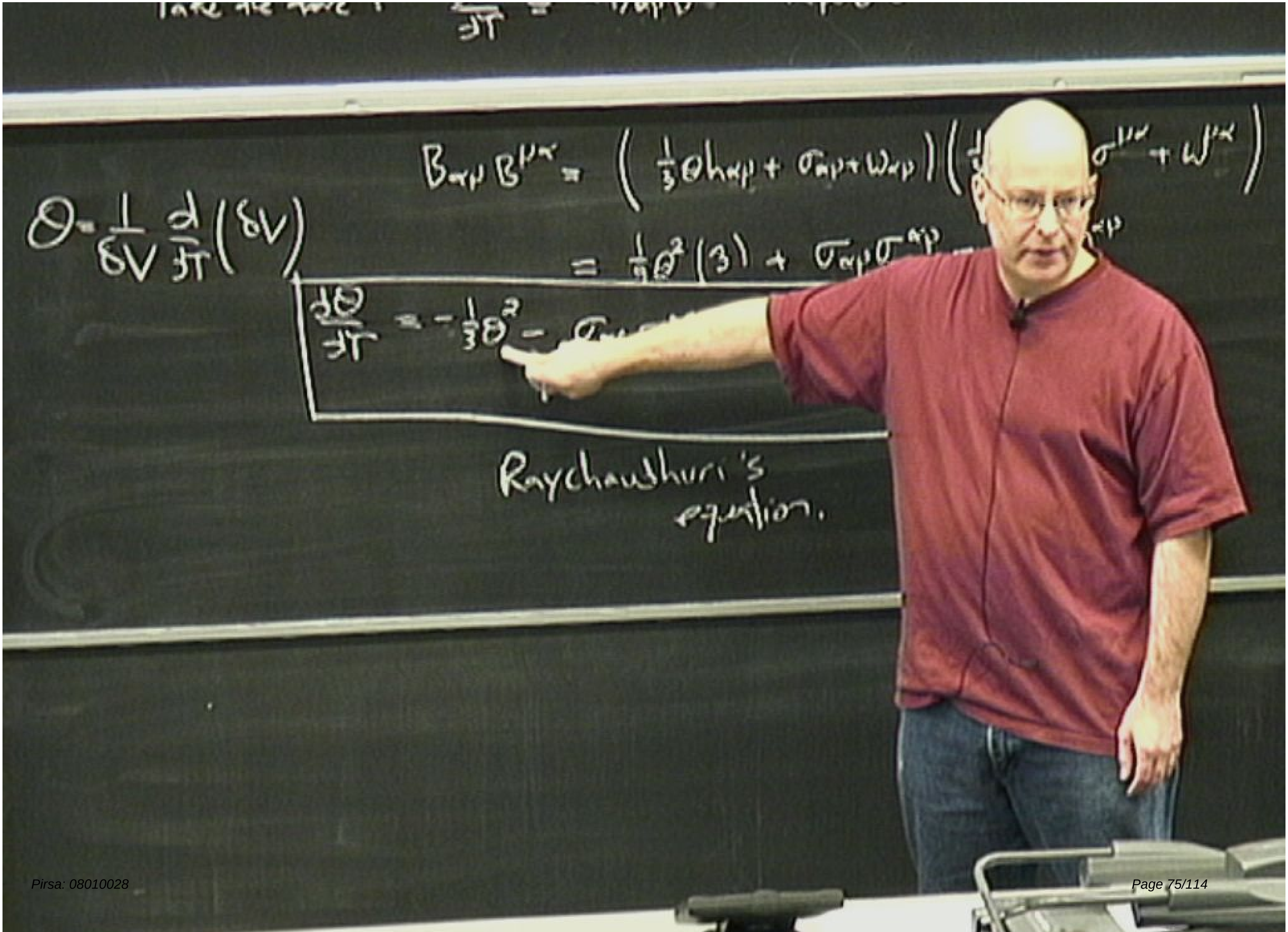
$$B_{\alpha\mu}B^{\mu\alpha} = \left(\tfrac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu}\right)\left(\tfrac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha}\right)$$

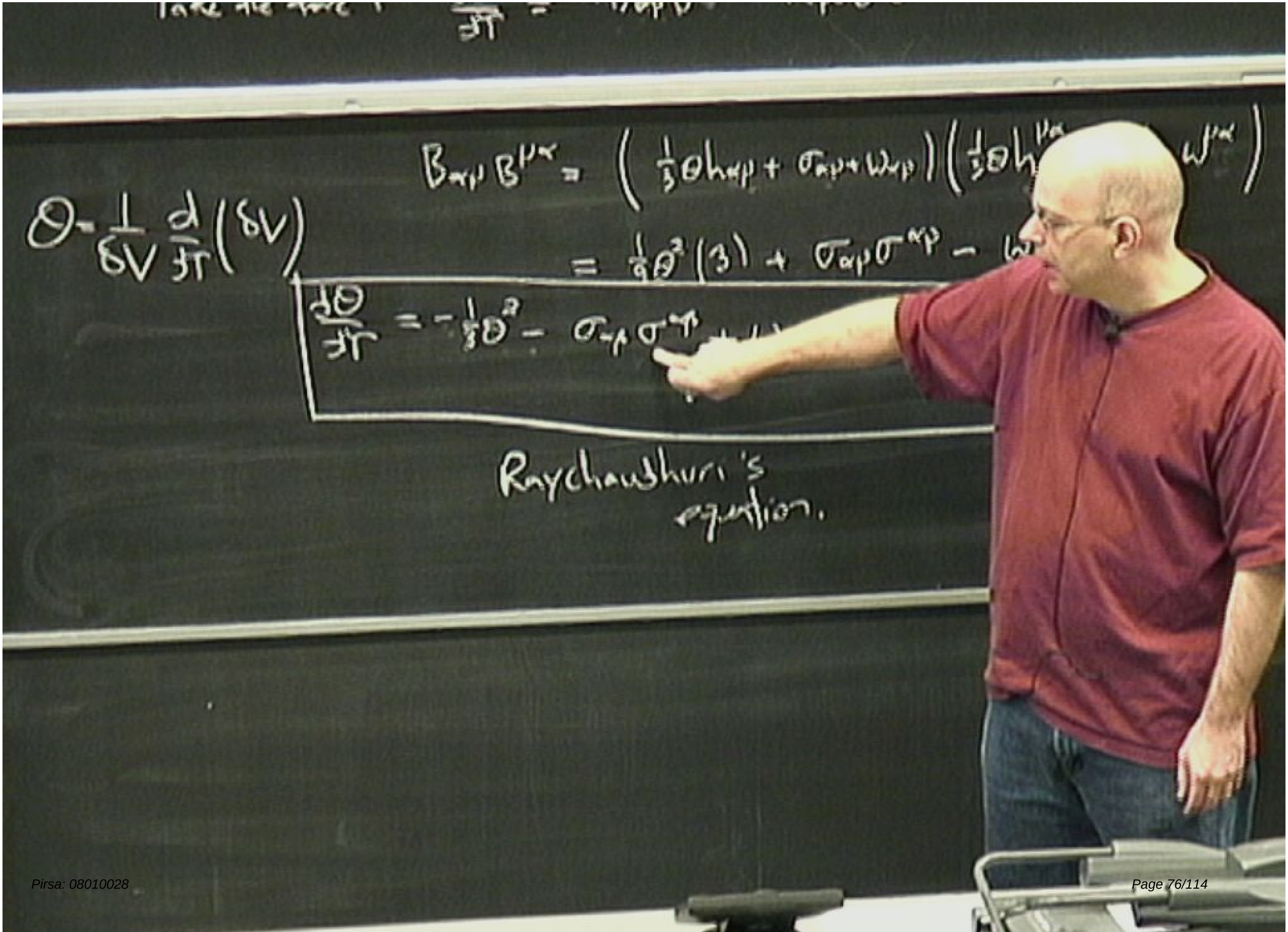
$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}(\delta V)$$

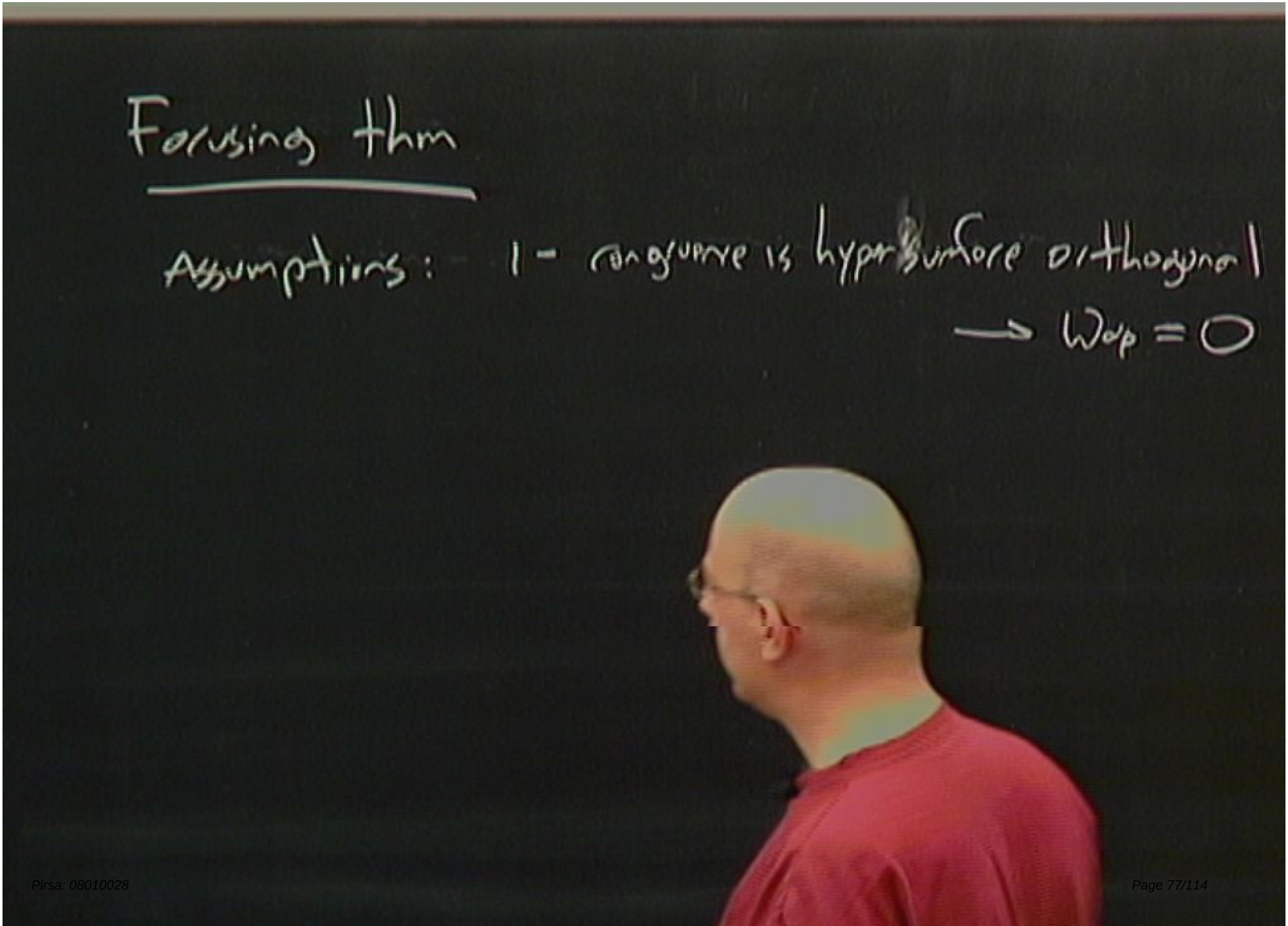
$$= \tfrac{1}{9}\theta^2(3) + \sigma_{\alpha\mu}\sigma^{\alpha\mu} - \omega_{\alpha\mu}\omega^{\alpha\mu}$$

$$\boxed{\frac{d\theta}{d\tau} = -\tfrac{1}{3}\theta^2 - \sigma_{\alpha\mu}\sigma^{\alpha\mu} + \omega_{\alpha\mu}\omega^{\alpha\mu} - R_{\alpha\mu}U^{\alpha}U^{\mu}}$$

Raychaudhuri's equation.


θ = 1/δV d/dτ (δV)
Raychawdhuri's
equation.


$\theta = \frac{1}{\delta V} \frac{d}{d\tau} (\delta V)$
$B_{\alpha\beta} B^{\beta\alpha} = \left(\frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \right)\left(\frac{1}{3}\theta h^{\beta\alpha}$
$\omega^{\beta\alpha})$
$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\beta}\sigma^{\alpha\beta} -$
$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta}$
Raychaudhuri's
equation.


Focusing thm
Assumptions: 1 - congruence is hypersurface orthogonal
→ $\omega_{ab} = 0$

Focusing thm

Assumptions: 1- congruence is hypersurface orthogonal

$$\longrightarrow \omega_{\alpha\beta} = 0$$

2- $R_{\alpha\beta} u^\alpha u^\beta \geq 0$

Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$\longrightarrow \omega_{\alpha\beta} = 0$

2 - $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) U^\alpha U^\beta \geq 0$$

Strong-energy condition

Focusing thm

Assumptions: 1- congruence is hypersurface orthogonal

$$\longrightarrow \omega_{\alpha\beta} = 0$$

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$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) u^\alpha u^\beta \geq 0$$

strong-energy condition

$$T_{\alpha\beta} u^\alpha u^\beta + \frac{1}{2} T \geq 0$$

Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

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Strong-energy condition

$$T_{\alpha\beta} U^\alpha U^\beta + \tfrac{1}{2} T \geq 0$$

Vacuum energy: $T_{\alpha\beta} = \Lambda g_{\alpha\beta}$

Focusing thm

Assumptions: 1- congruence is hypersurface orthogonal

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Strong-energy condition

$$T_{\alpha\beta} u^\alpha u^\beta + \tfrac{1}{2} T \geq 0$$

Vacuum energy: $T_{\alpha\beta} = \Lambda g_{\alpha\beta}$

$$-\Lambda + \tfrac{1}{2}(4)\Lambda = 3\Lambda \geq 0$$

Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$\longrightarrow \omega_{\alpha\beta} = 0$

2 - $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) U^\alpha U^\beta \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T \geq 0$$

$$-3\Lambda \geq 0$$

Vacuum energy: $T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$

$$+\Lambda \cancel{+} \frac{1}{2}(4)\Lambda =$$

Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$\longrightarrow W_{\alpha\beta} = 0$

2 - $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) U^\alpha U^\beta \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T \geq 0$$

$-3\Lambda \geq 0$

Vacuum energy: $T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$

$+\Lambda - \frac{1}{2}(4)\Lambda =$

Focusing thm

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$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) U^\alpha U^\beta \geq 0$$

Strong-energy condition

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Vacuum energy: $T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$

$$+\Lambda \cancel{+} \frac{1}{2}(4)\Lambda =$$

Focusing thm

Assumptions: 1- congruence is hypersurface orthogonal

$\longrightarrow \omega_{\alpha\beta} = 0$

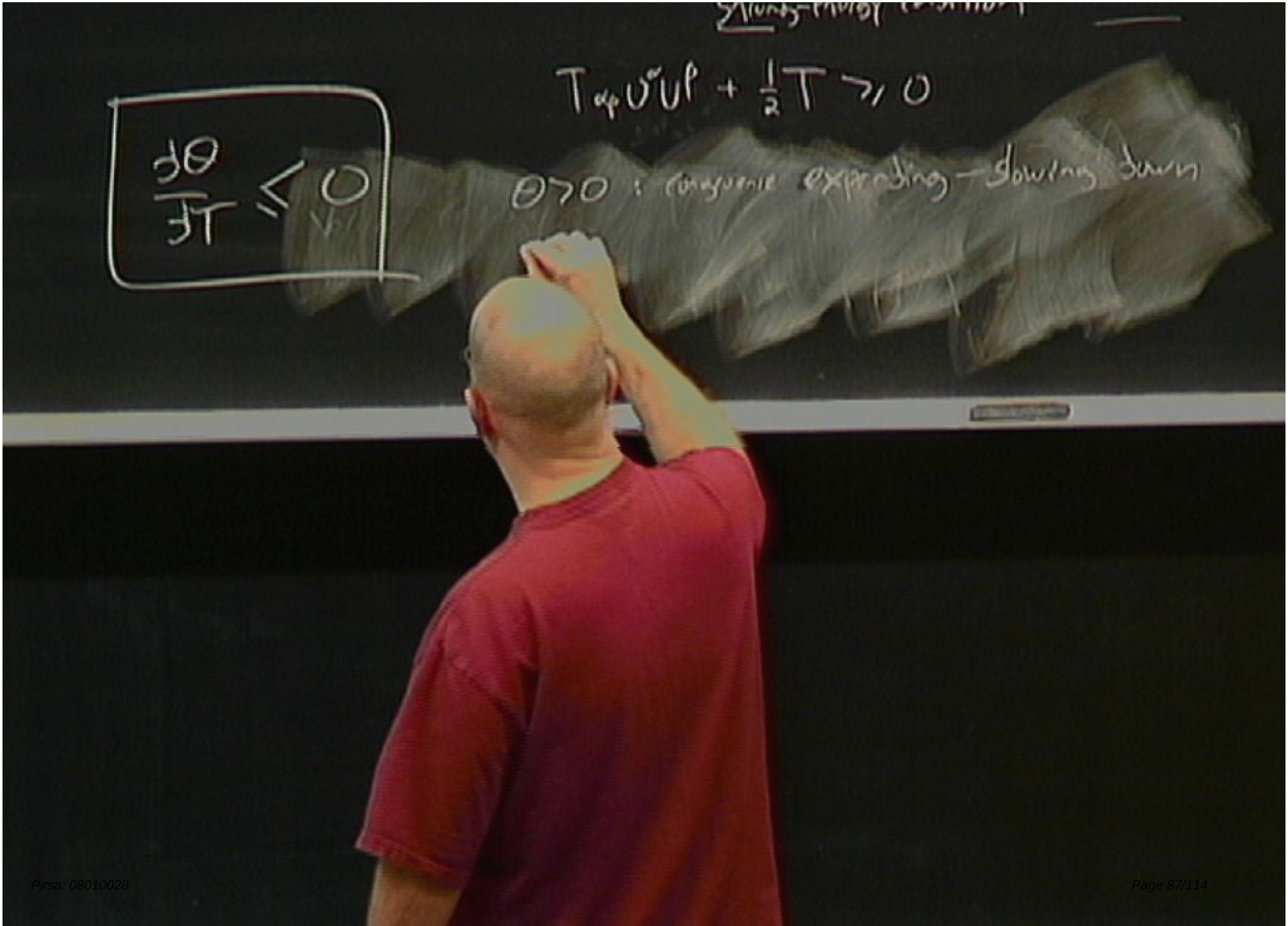
2- $R_{\alpha\beta} u^\alpha u^\beta \geq 0$

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) u^\alpha u^\beta \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} u^\alpha u^\beta + \frac{1}{2} T \geq 0$$

$$\boxed{\frac{d\theta}{d\tau} \leq 0}$$


$T_{\alpha\beta}U^\alpha U^\beta + \frac{1}{2}T \geq 0$
$\frac{d\theta}{d\tau} \leq 0$
$\theta > 0$: congruence expanding – slowing down

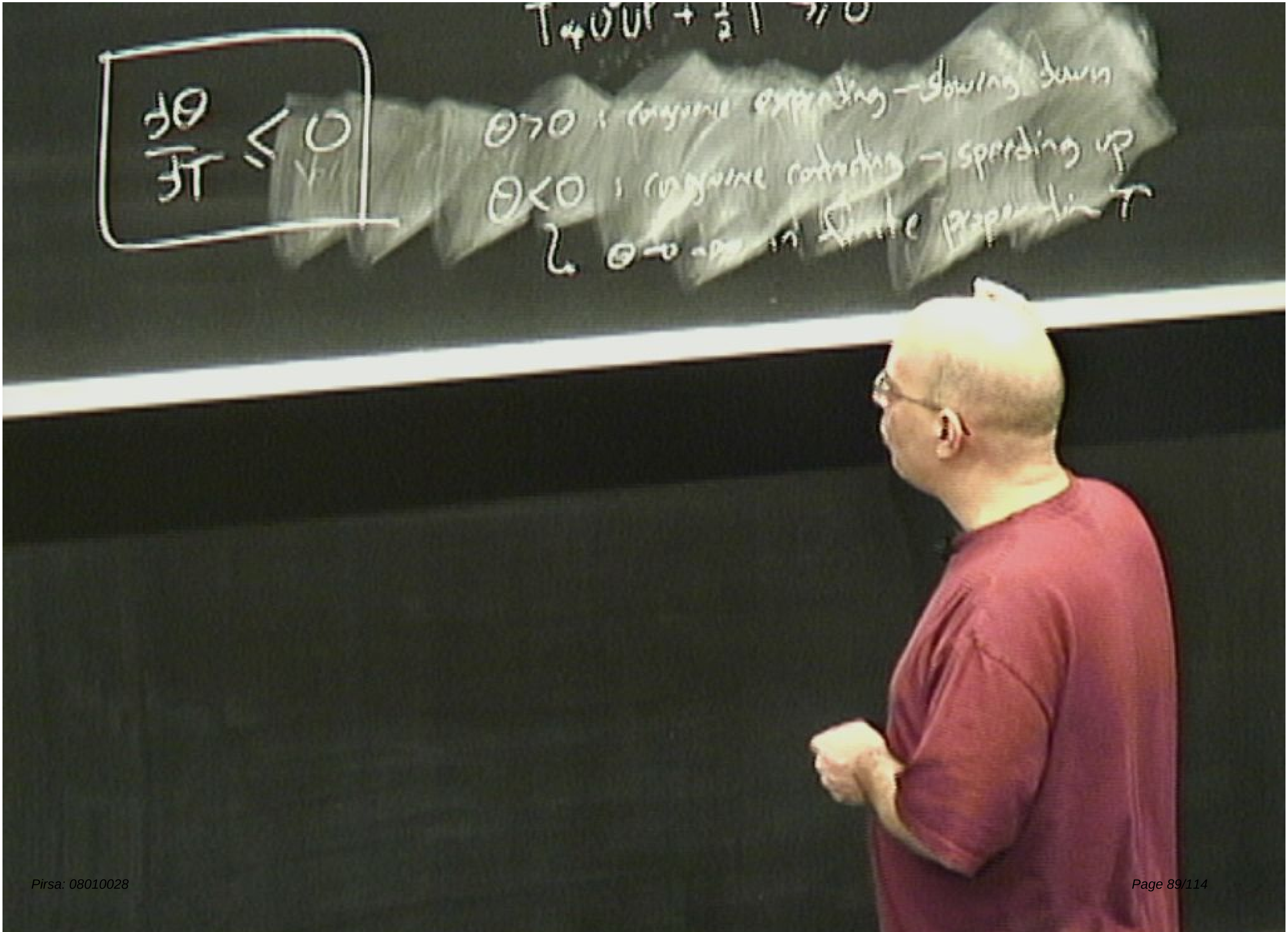
Strong-energy condition

$$T_{\alpha\beta}U^{\alpha}U^{\beta} + \frac{1}{2}T \geq 0$$

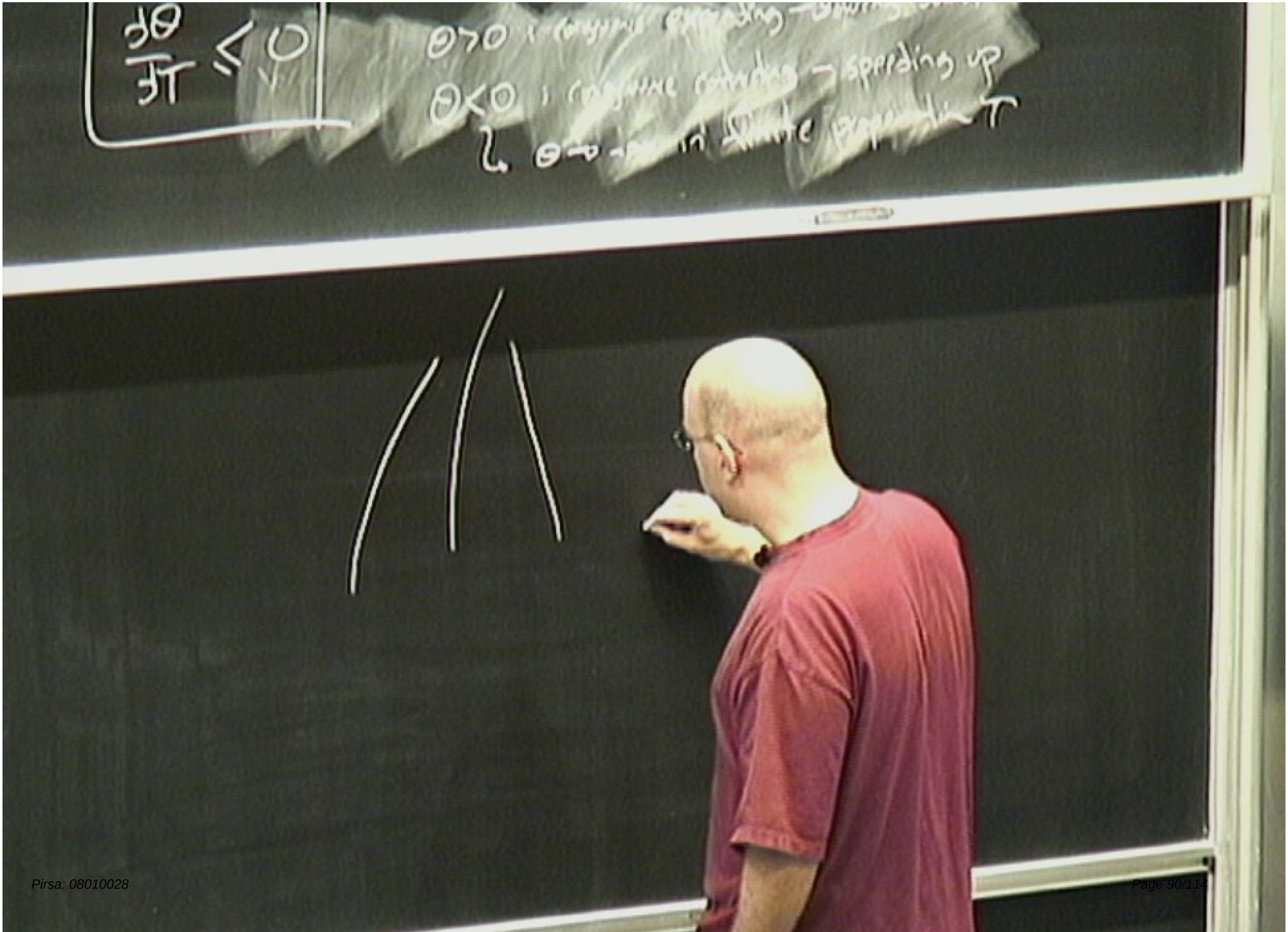
$$\boxed{\frac{d\theta}{d\tau} \leq 0}$$

$\theta > 0$: congruence expanding — slowing down

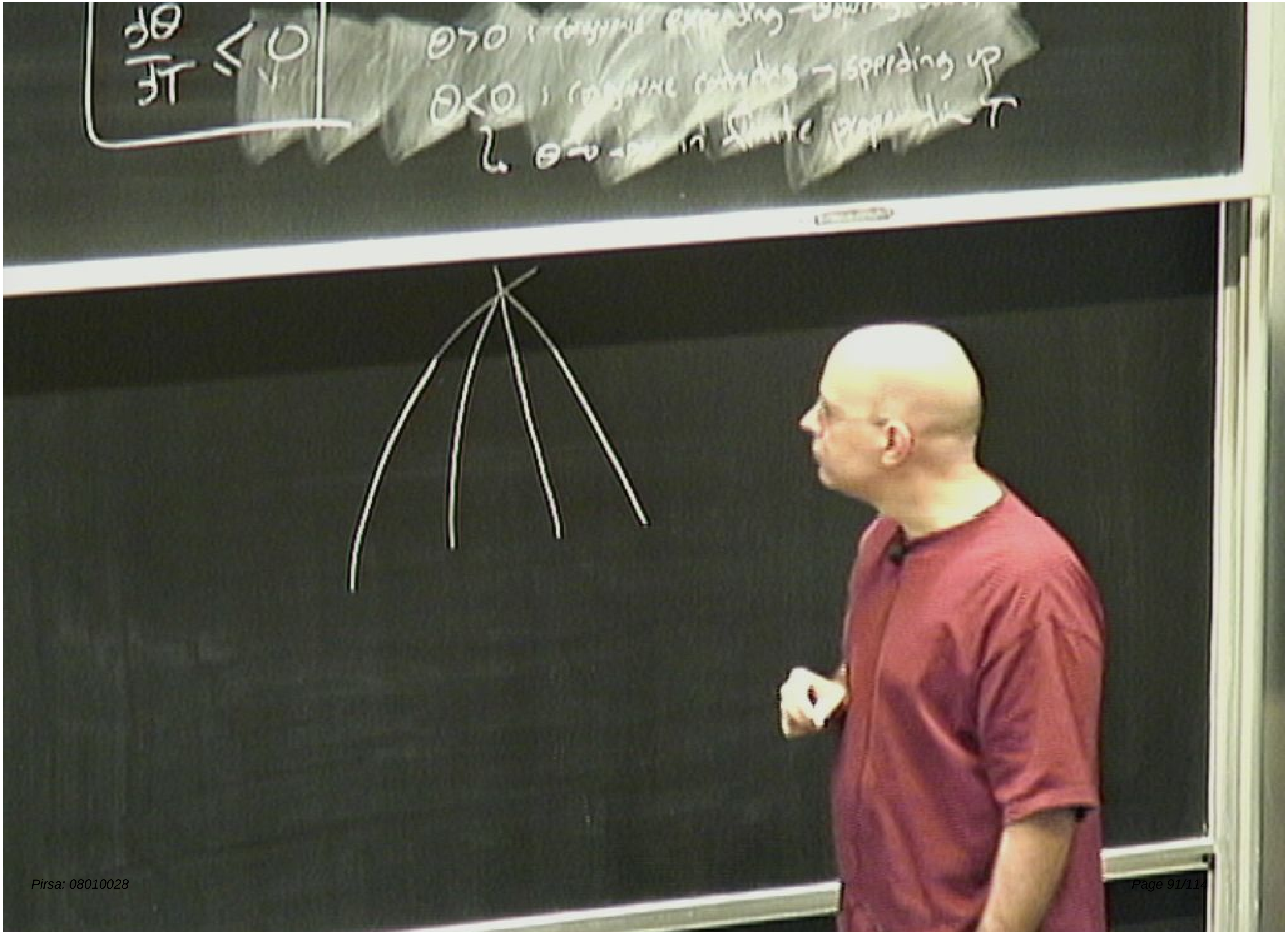
$\theta < 0$: congruence contracting — speeding up

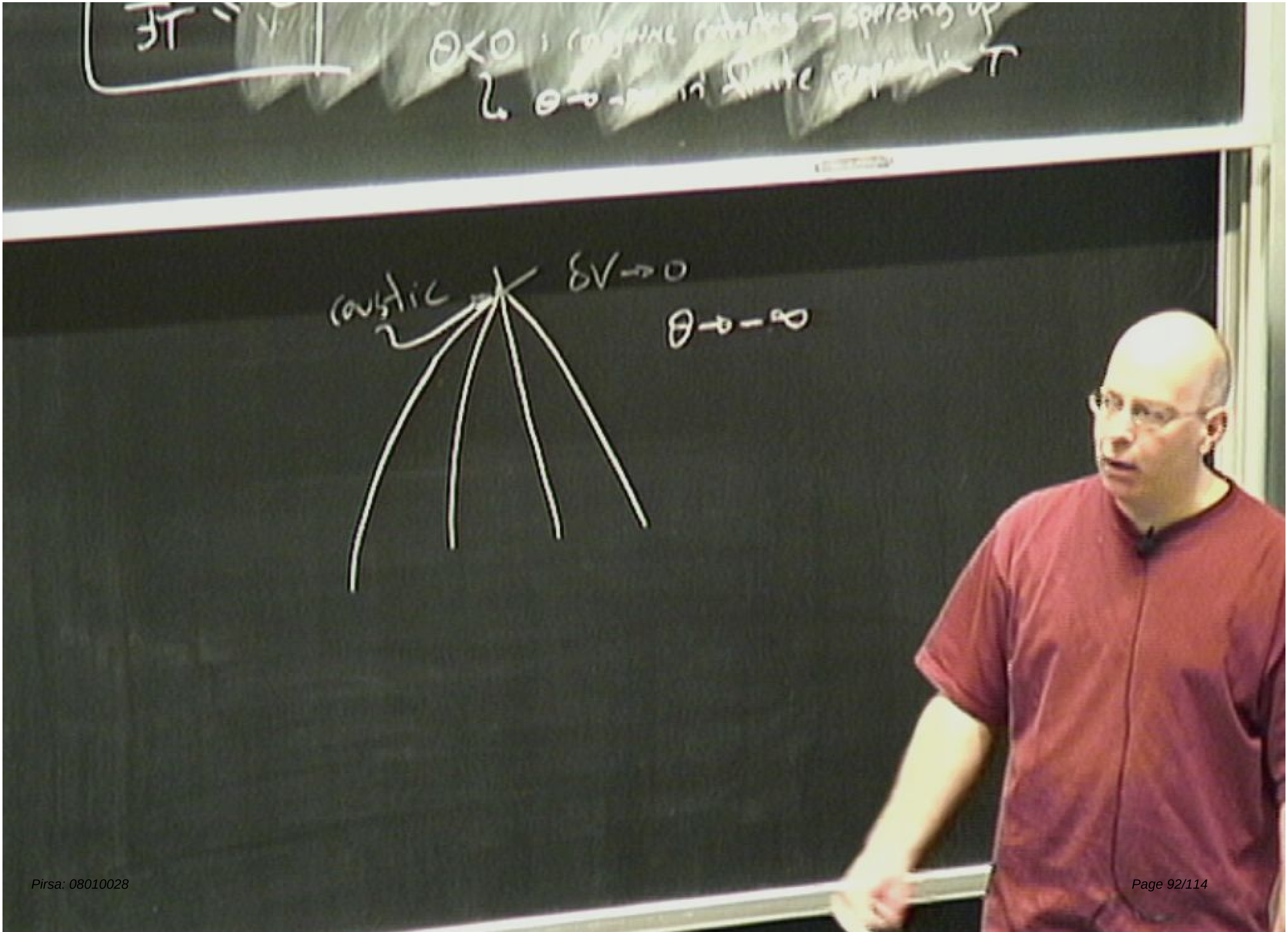


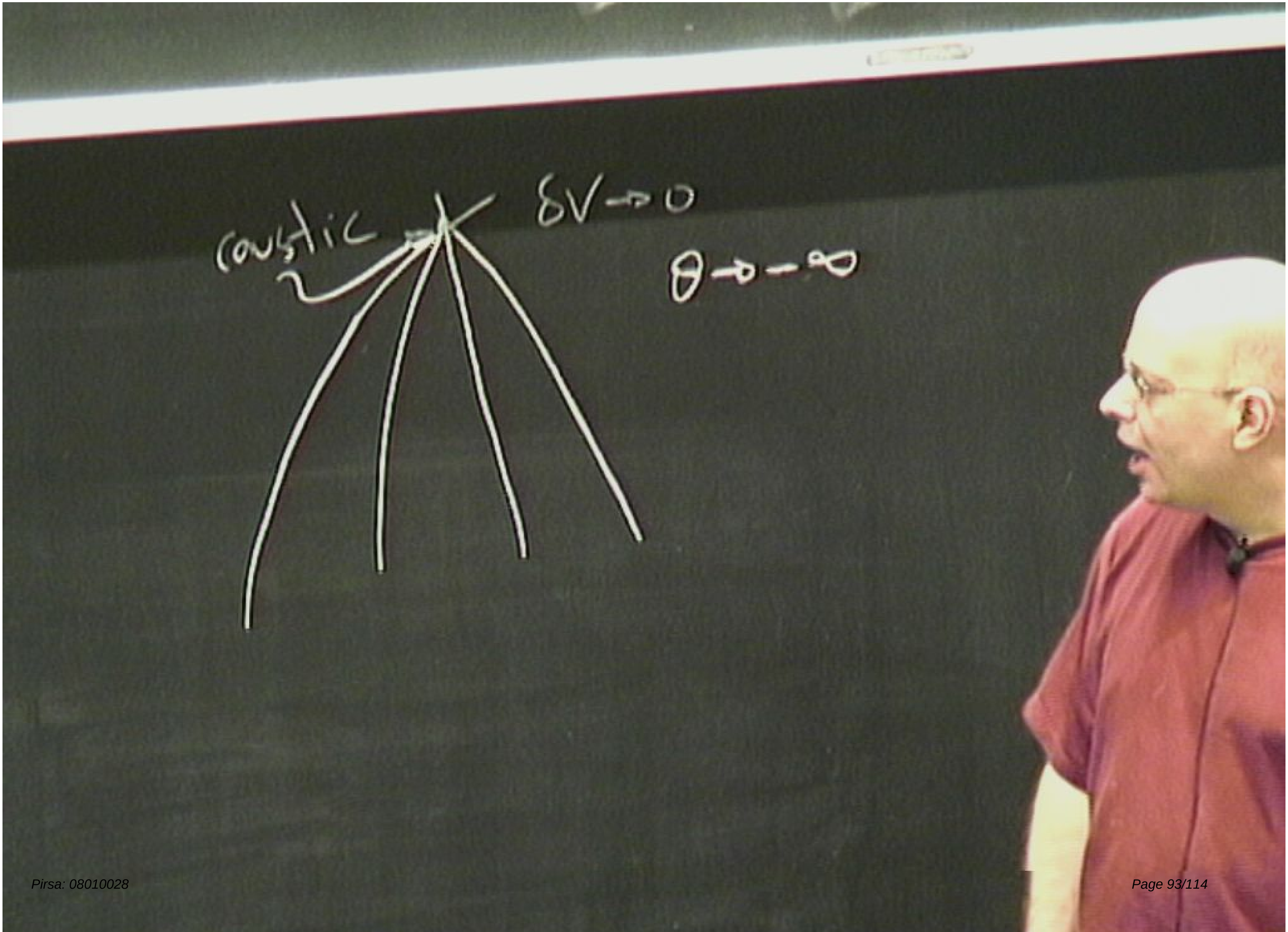
dθ/dτ ≤ 0
θ>0 : congruence expanding → slowing down
θ<0 : congruence contracting → speeding up



$\frac{\partial\theta}{\partial\tau} \leq 0$
$\theta > 0$: congruence expanding
$\theta < 0$: congruence contracting → speeding up


θ>0 : congruence expanding
θ<0 : congruence contracting → speeding up


caustic
δV → 0
θ → −∞


caustic
δV → 0
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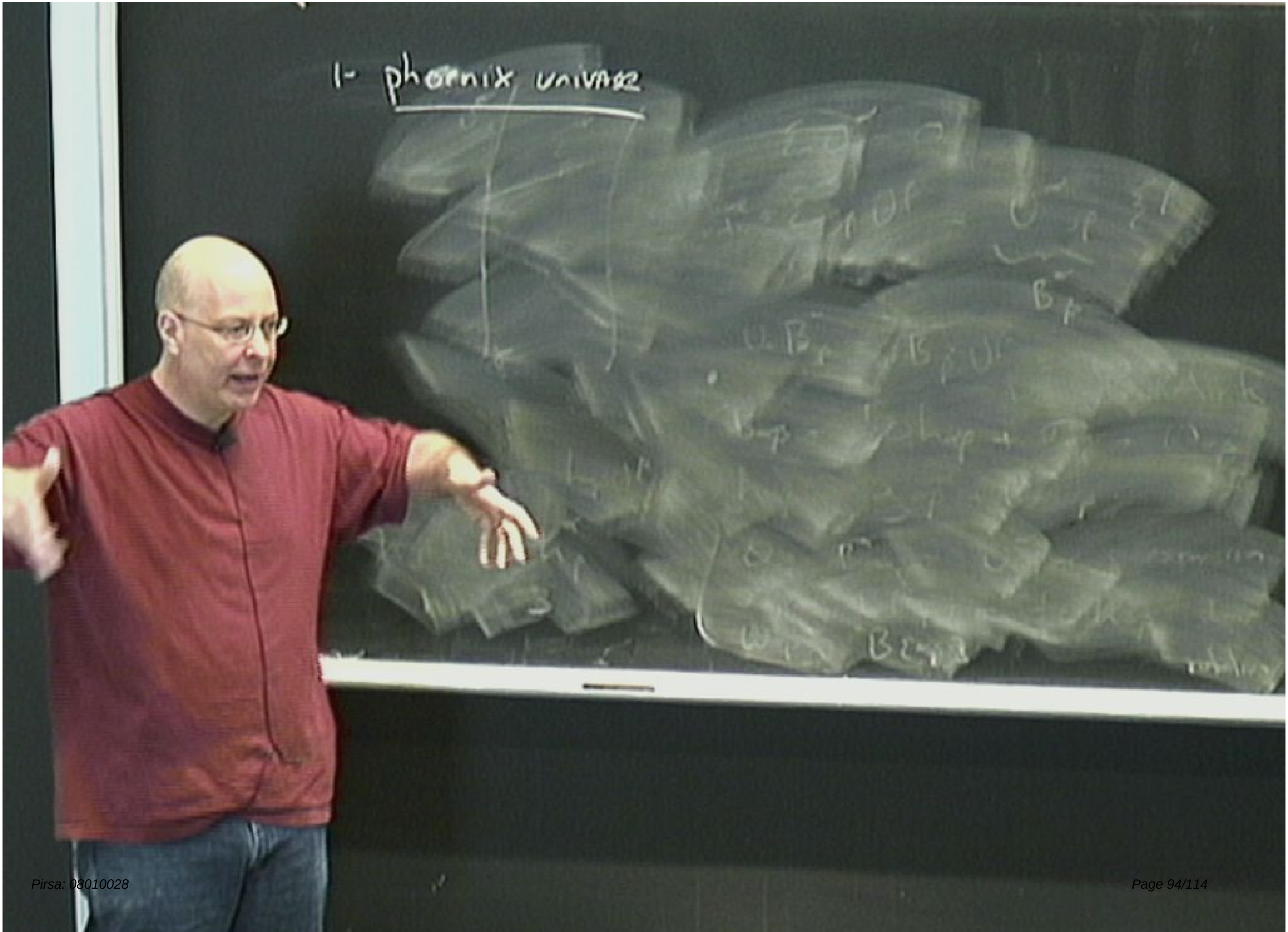
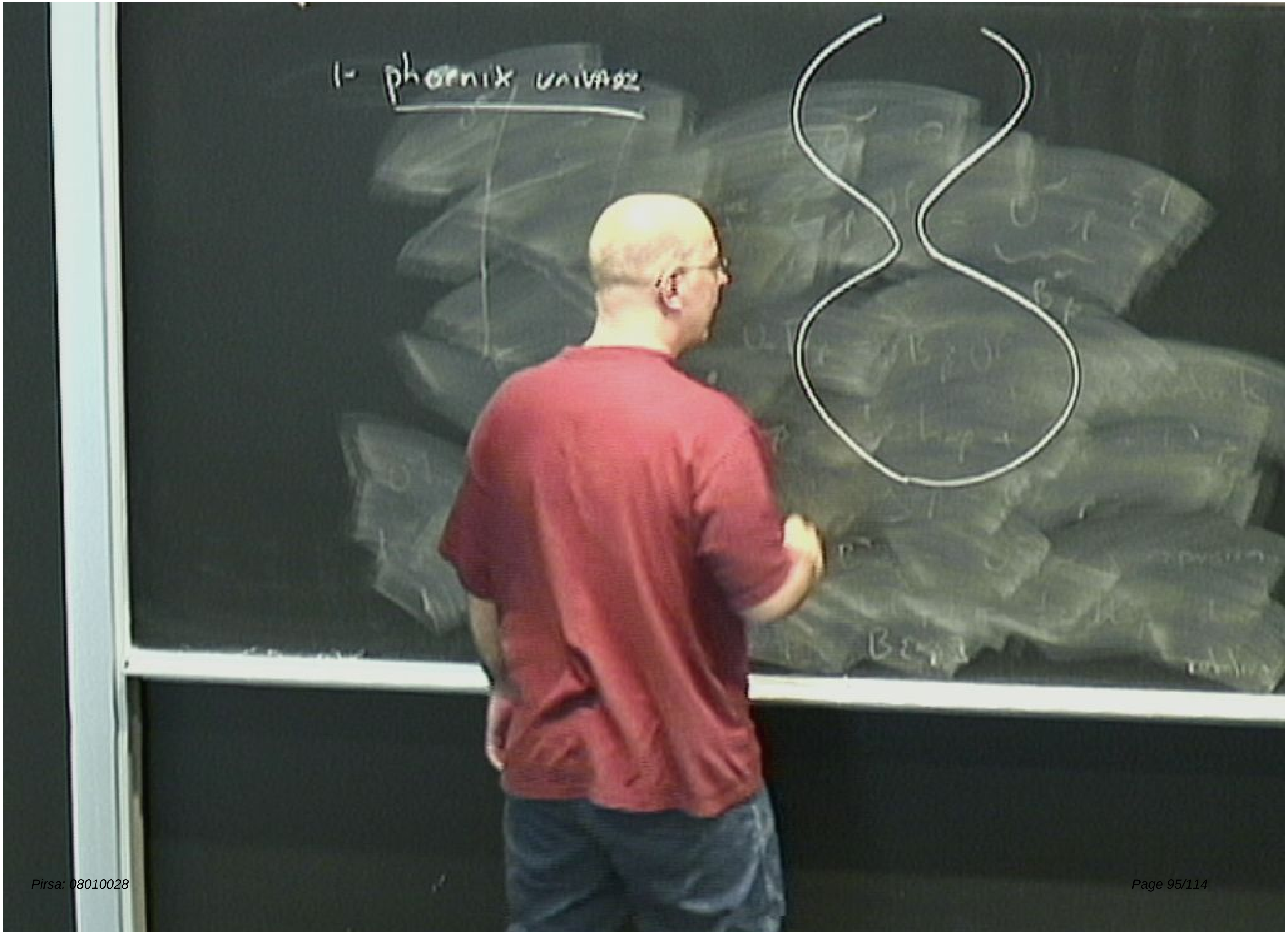
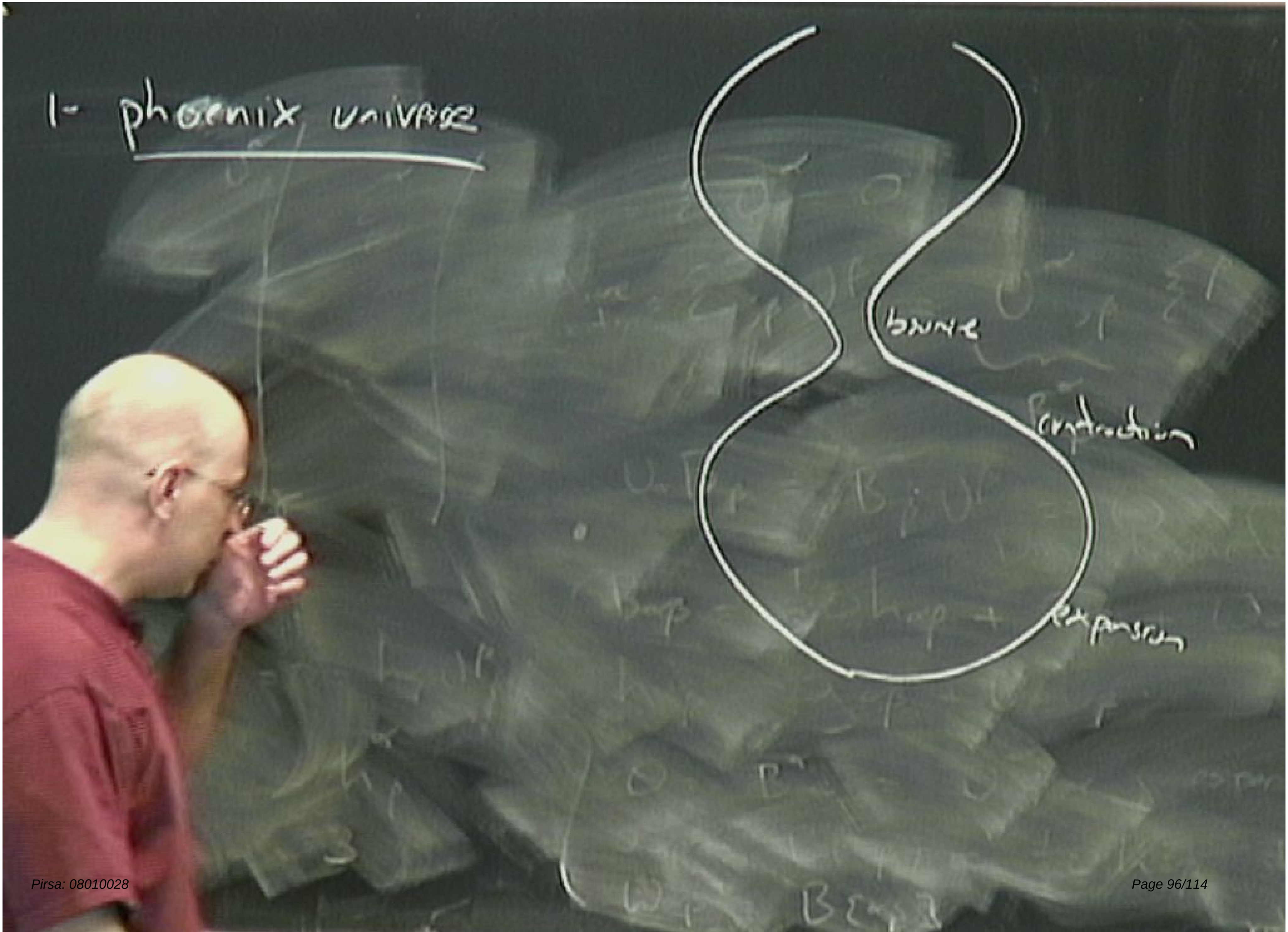
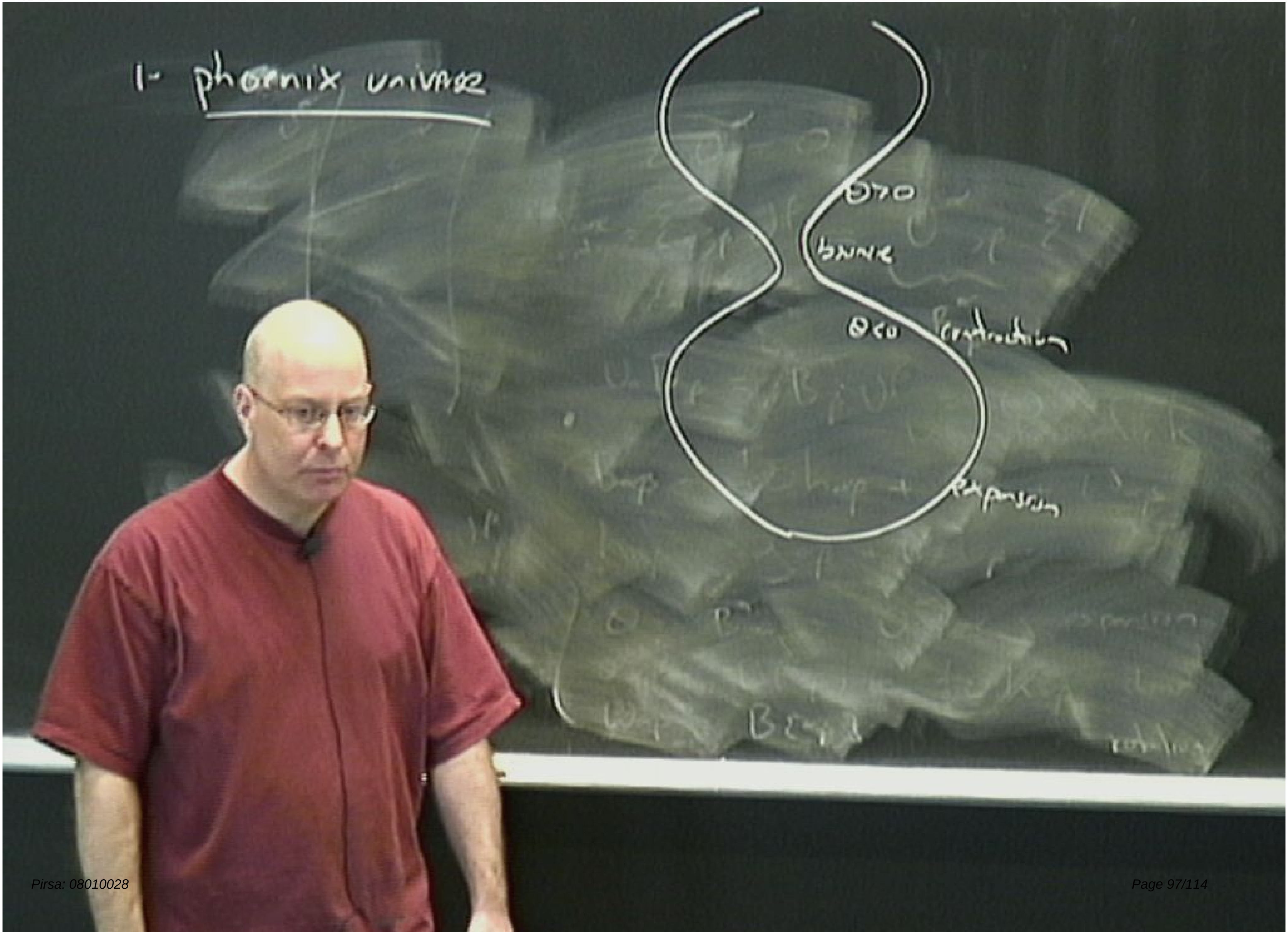
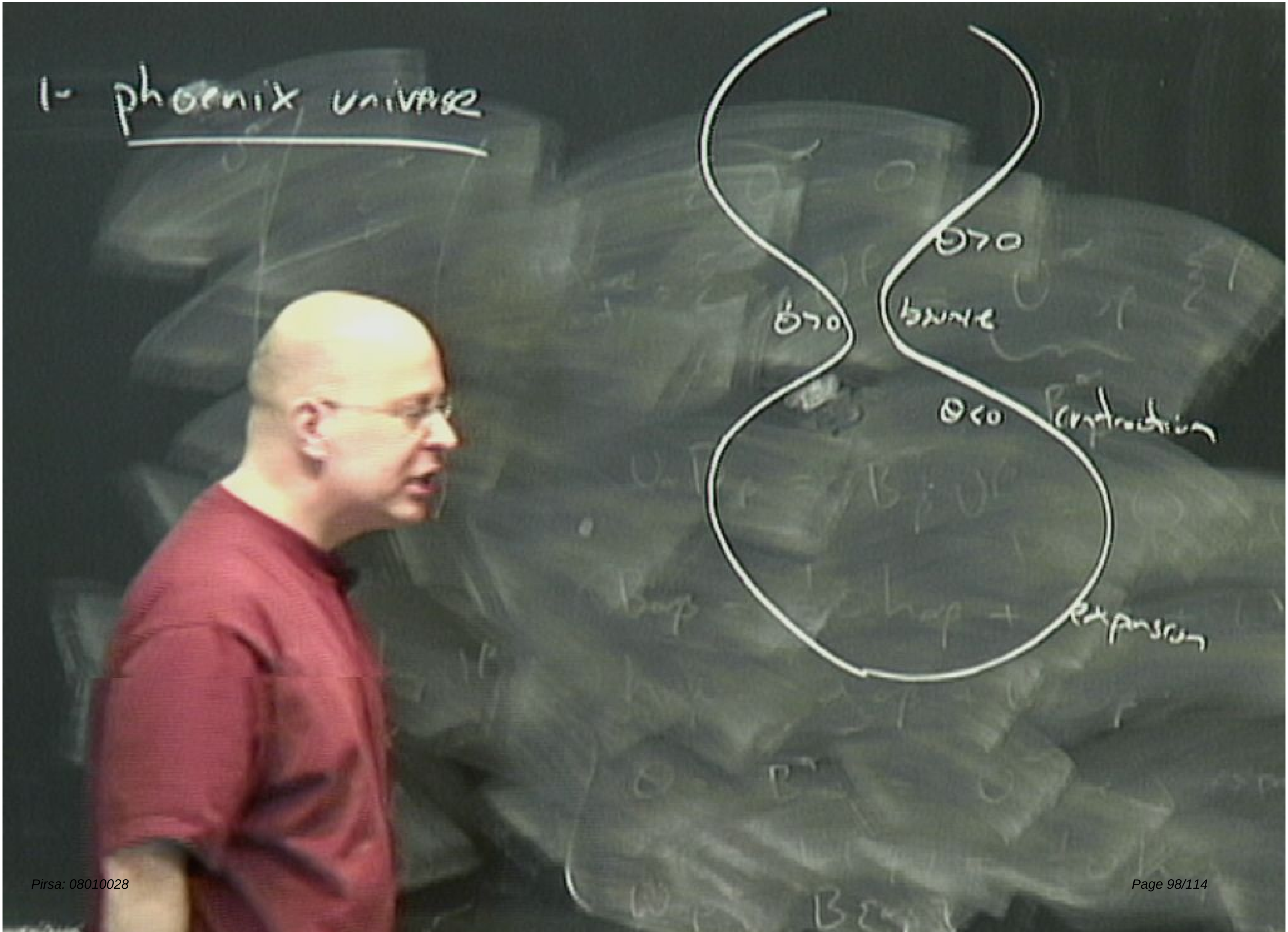

1- phoenix universe

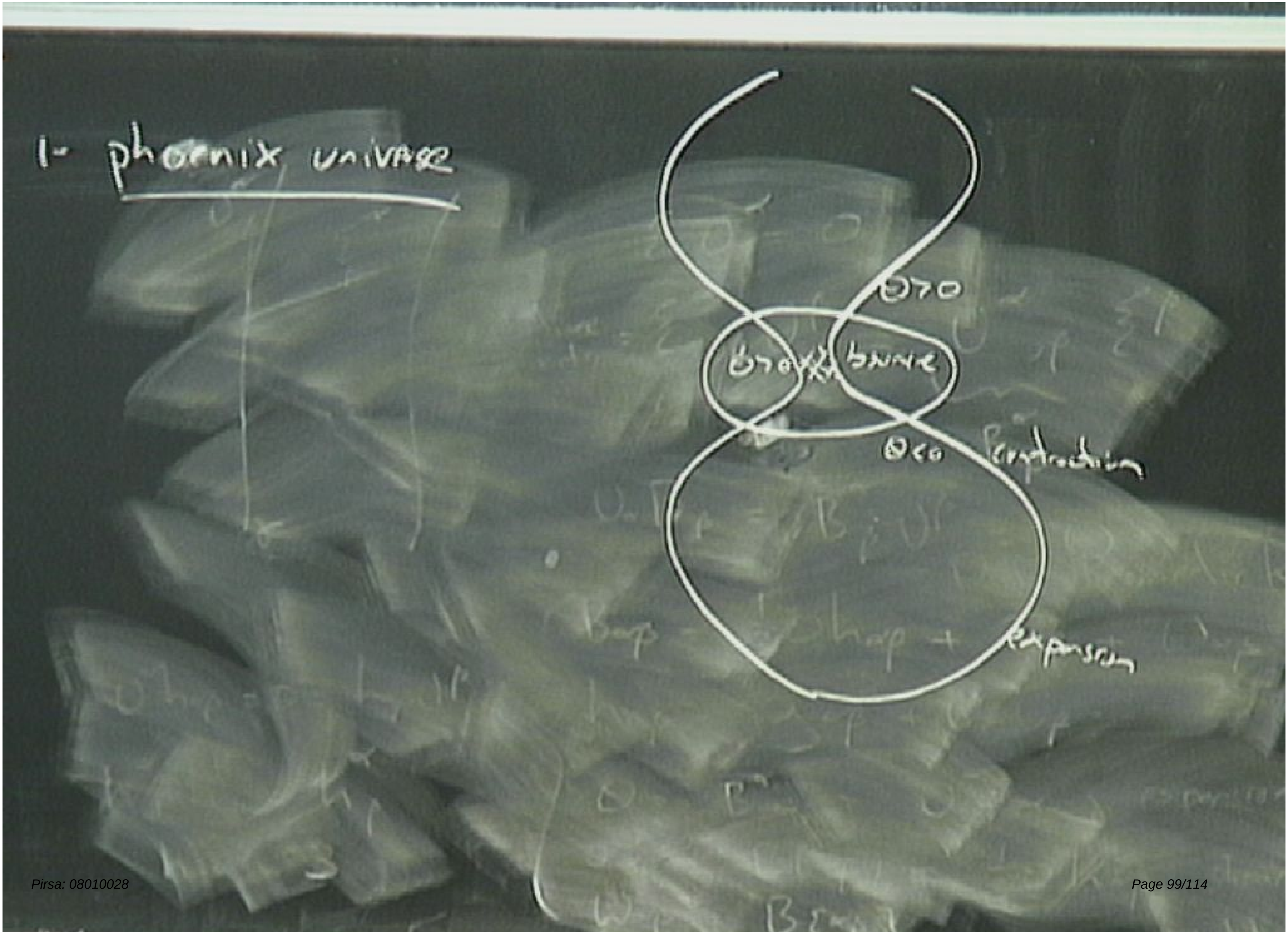

1- phoenix universe

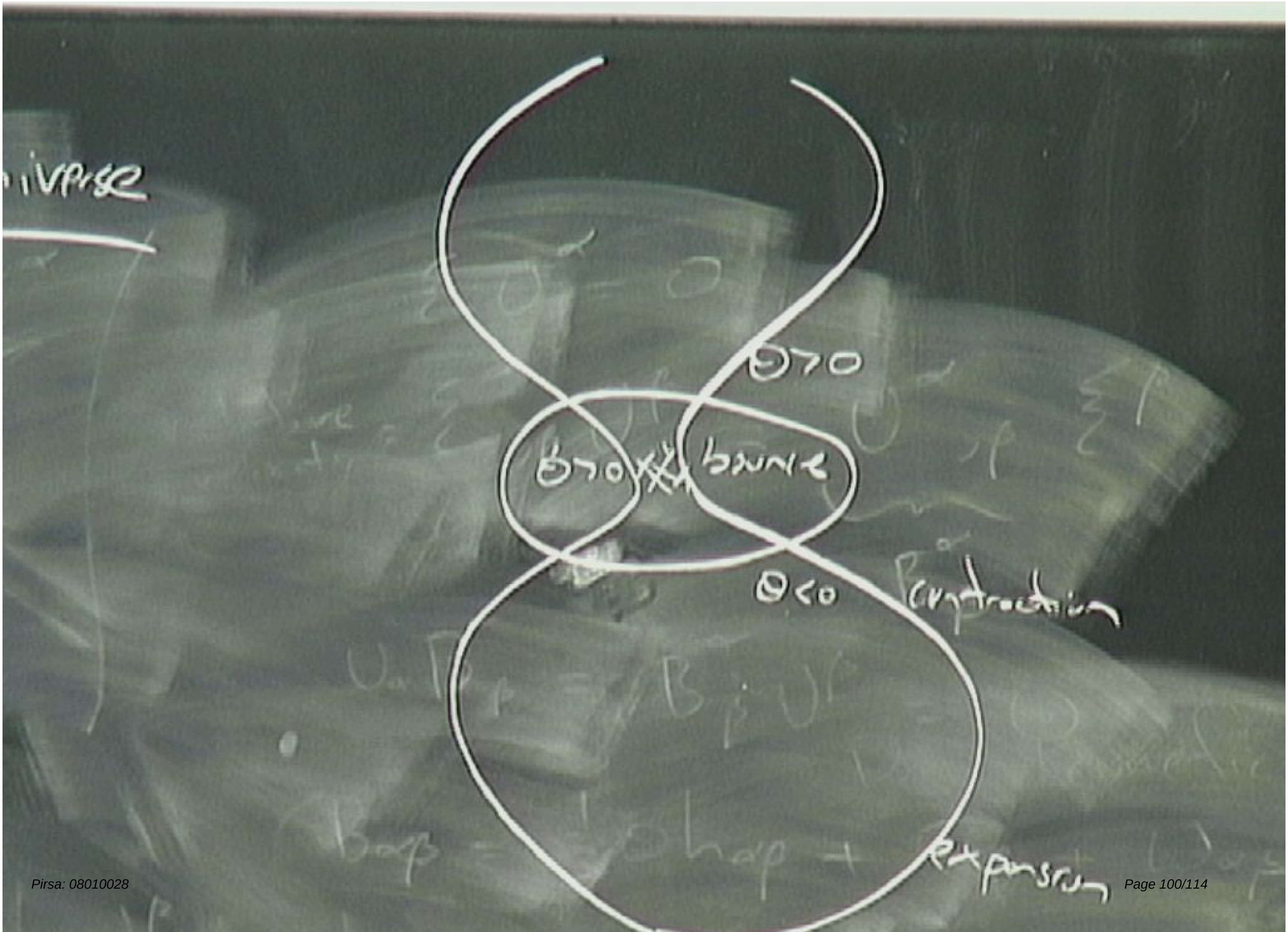


1- phoenix universe
brane
contraction
expansion

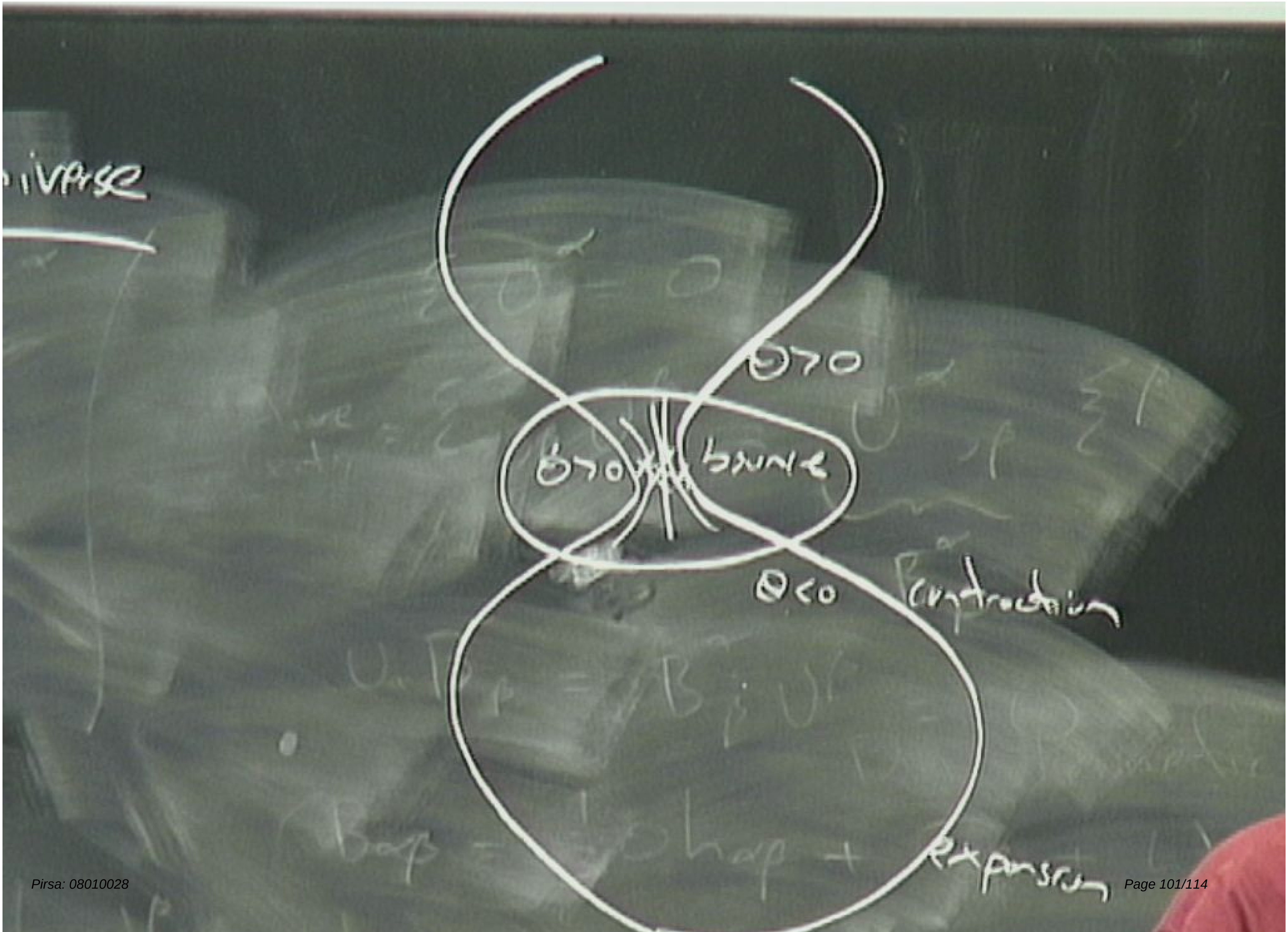

1- phoenix universe
Θ>0
Θ<0 contraction
expansion

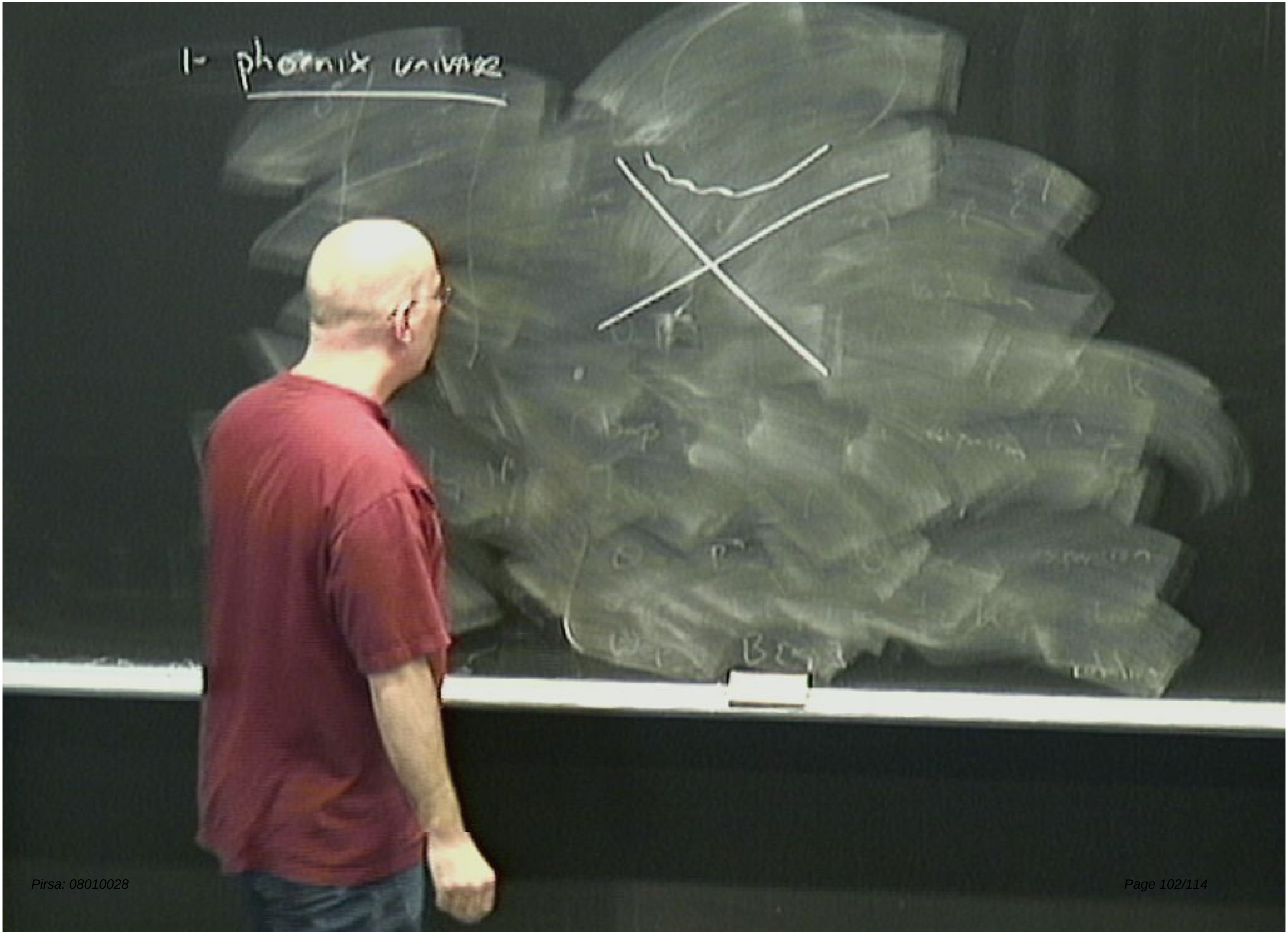

1- phoenix universe
$\theta > 0$
$\theta > 0$
bounce
$\theta < 0$
contraction
expansion

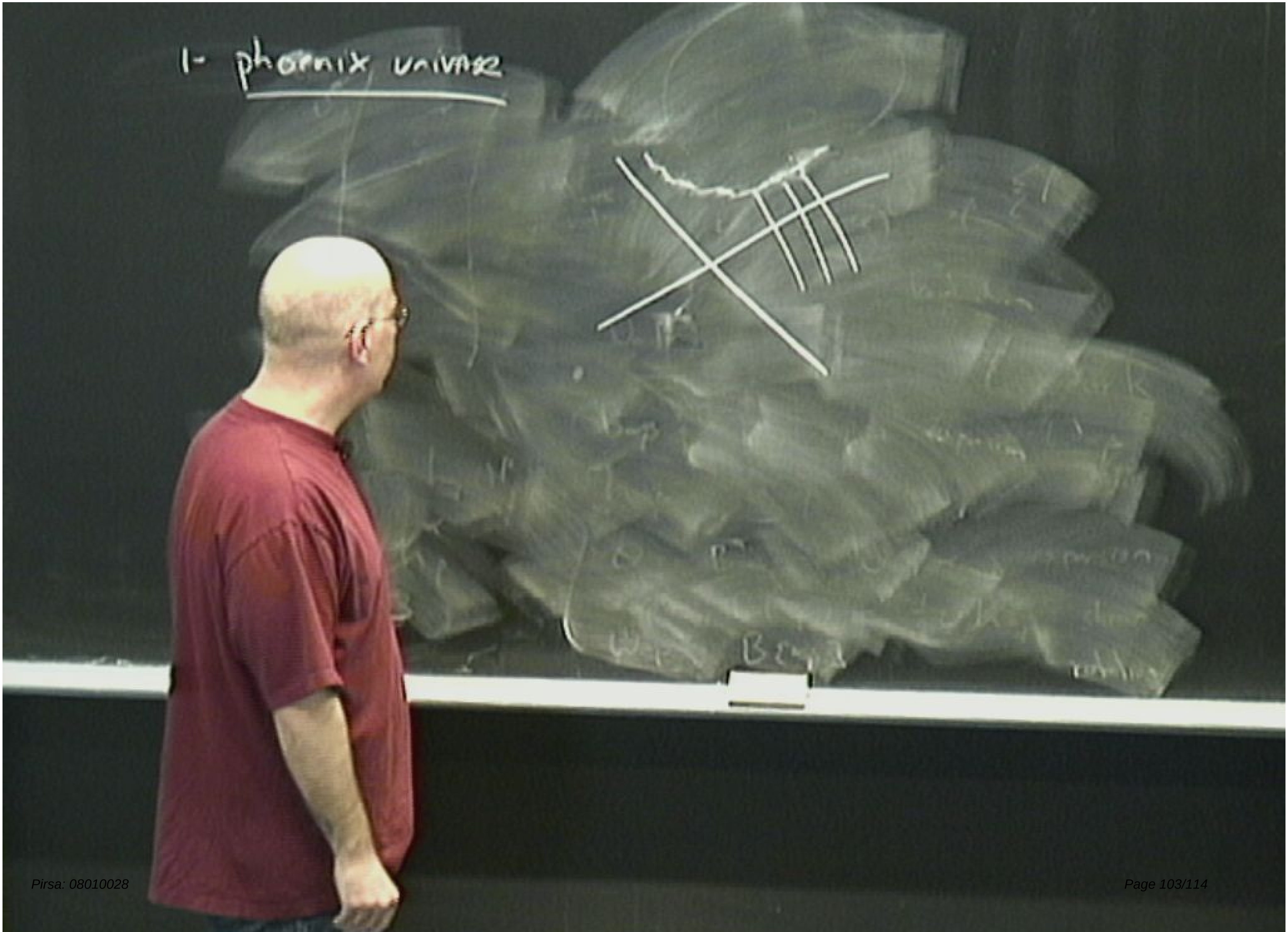

1- phoenix universe
Ω>0
brane
Ω<0
contraction
expansion

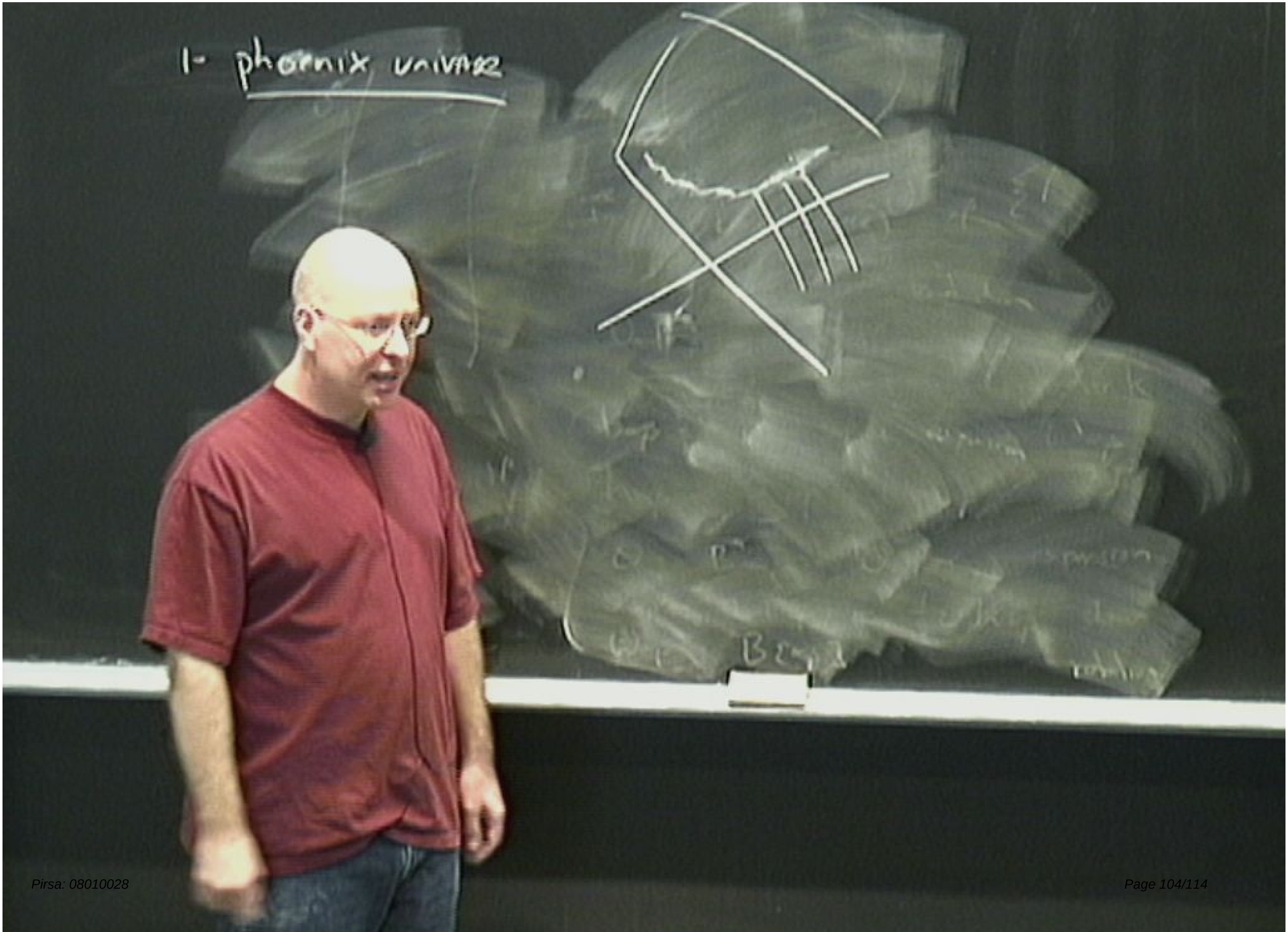


niverse
θ>0
θ>0 bounce
θ<0
contraction
expansion


Θ>0
Θ>0 brane
Θ<0
contraction
expansion


1- phoenix universe


1- phoenix universe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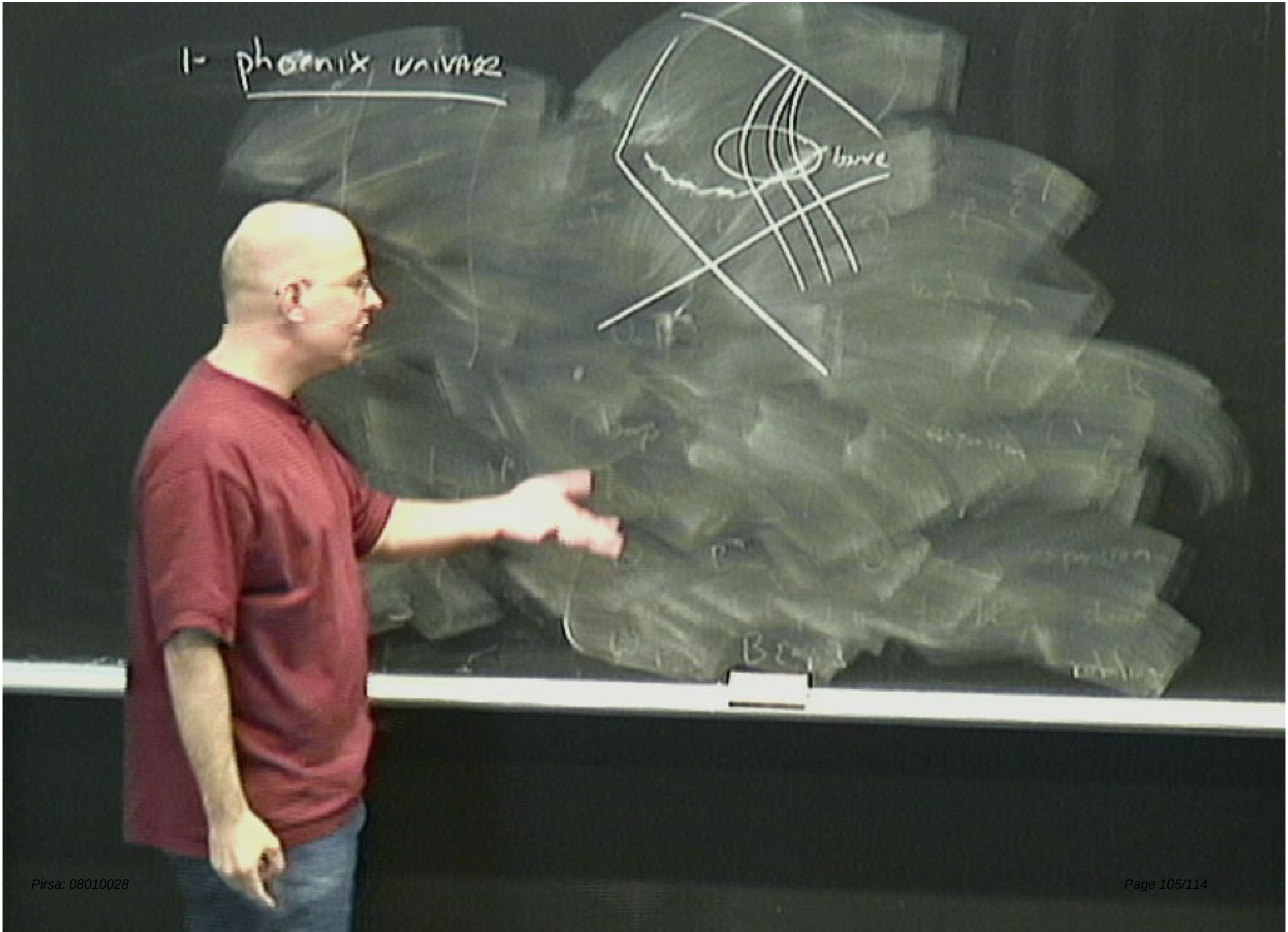
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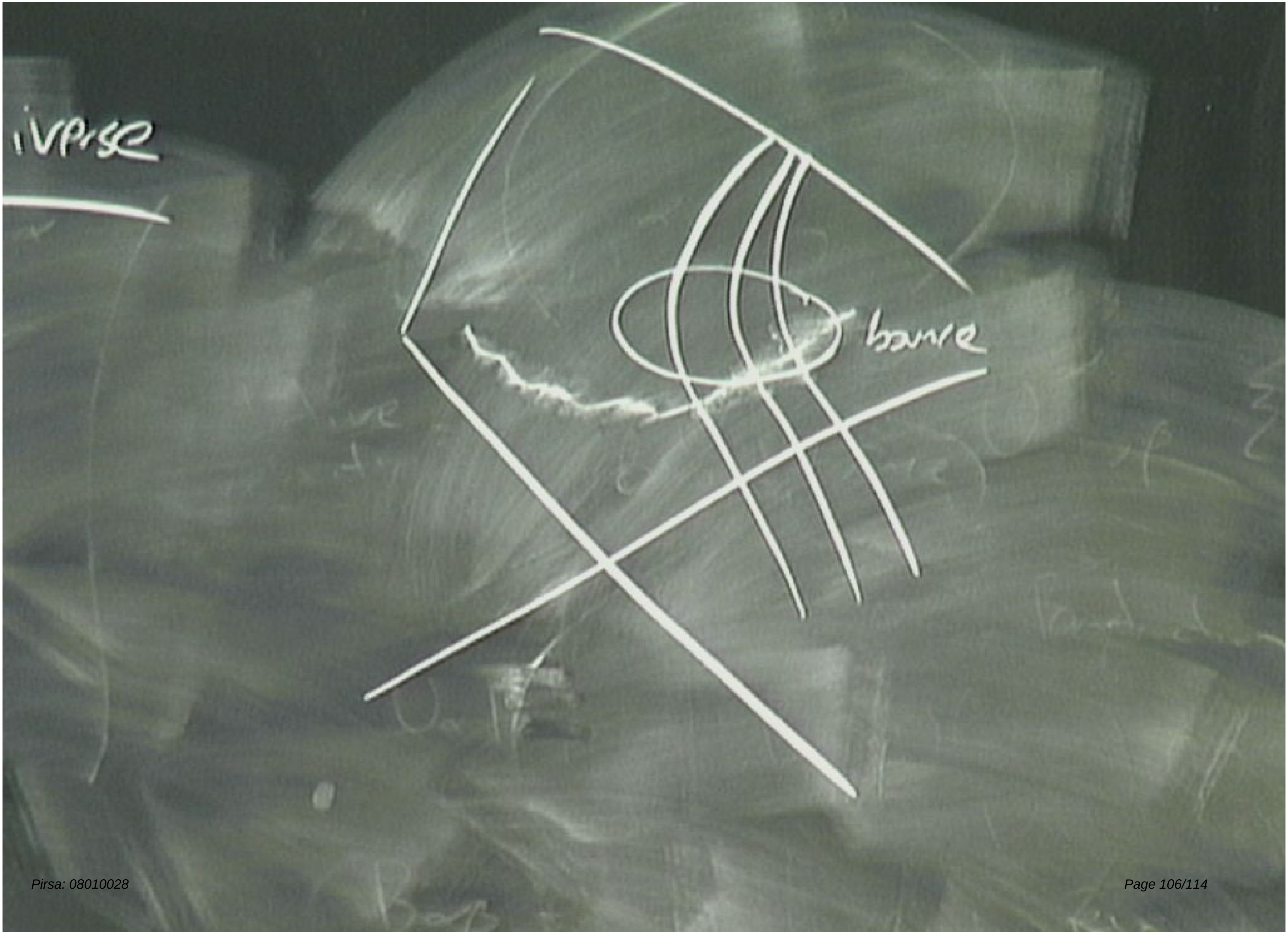
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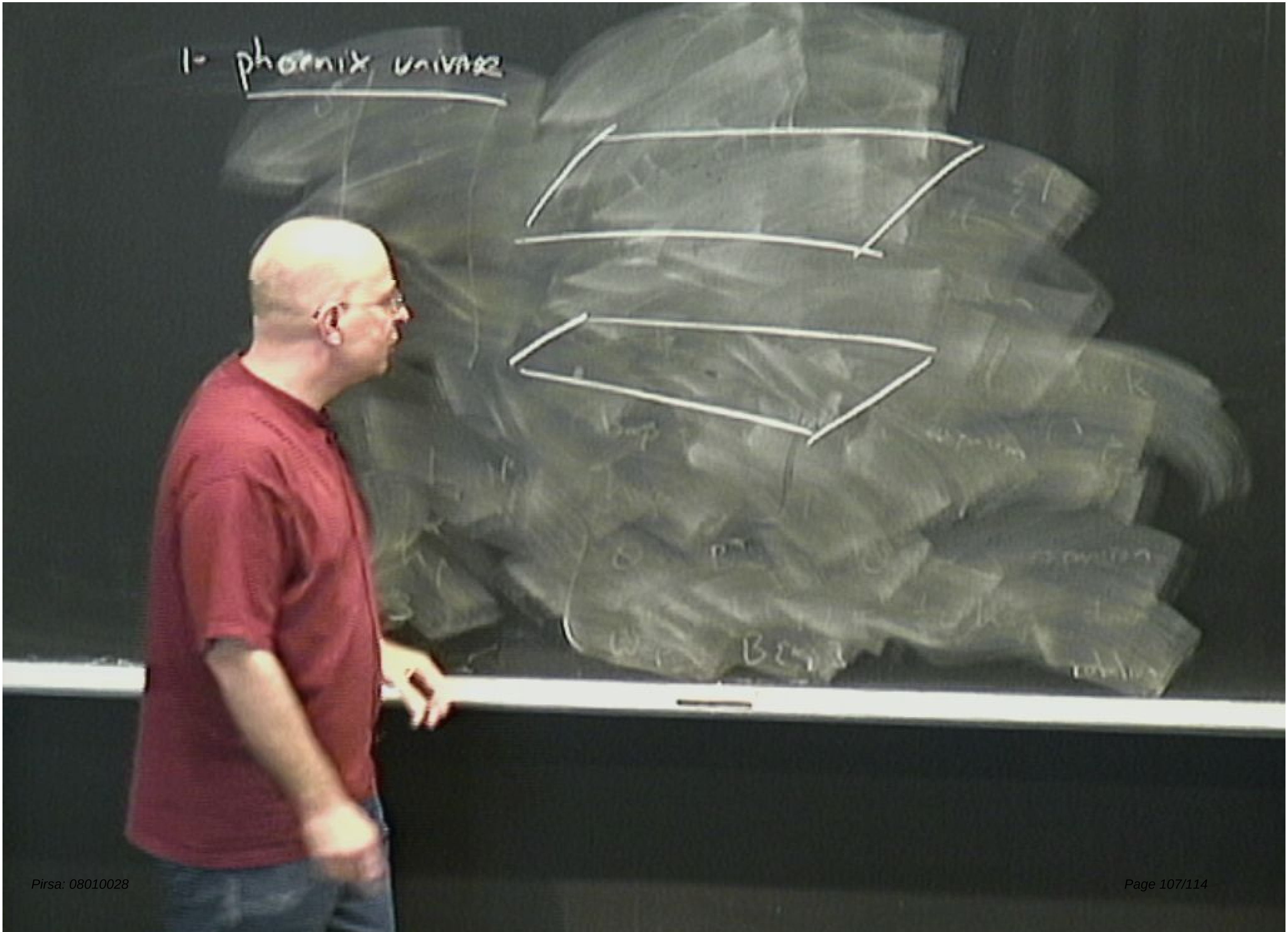
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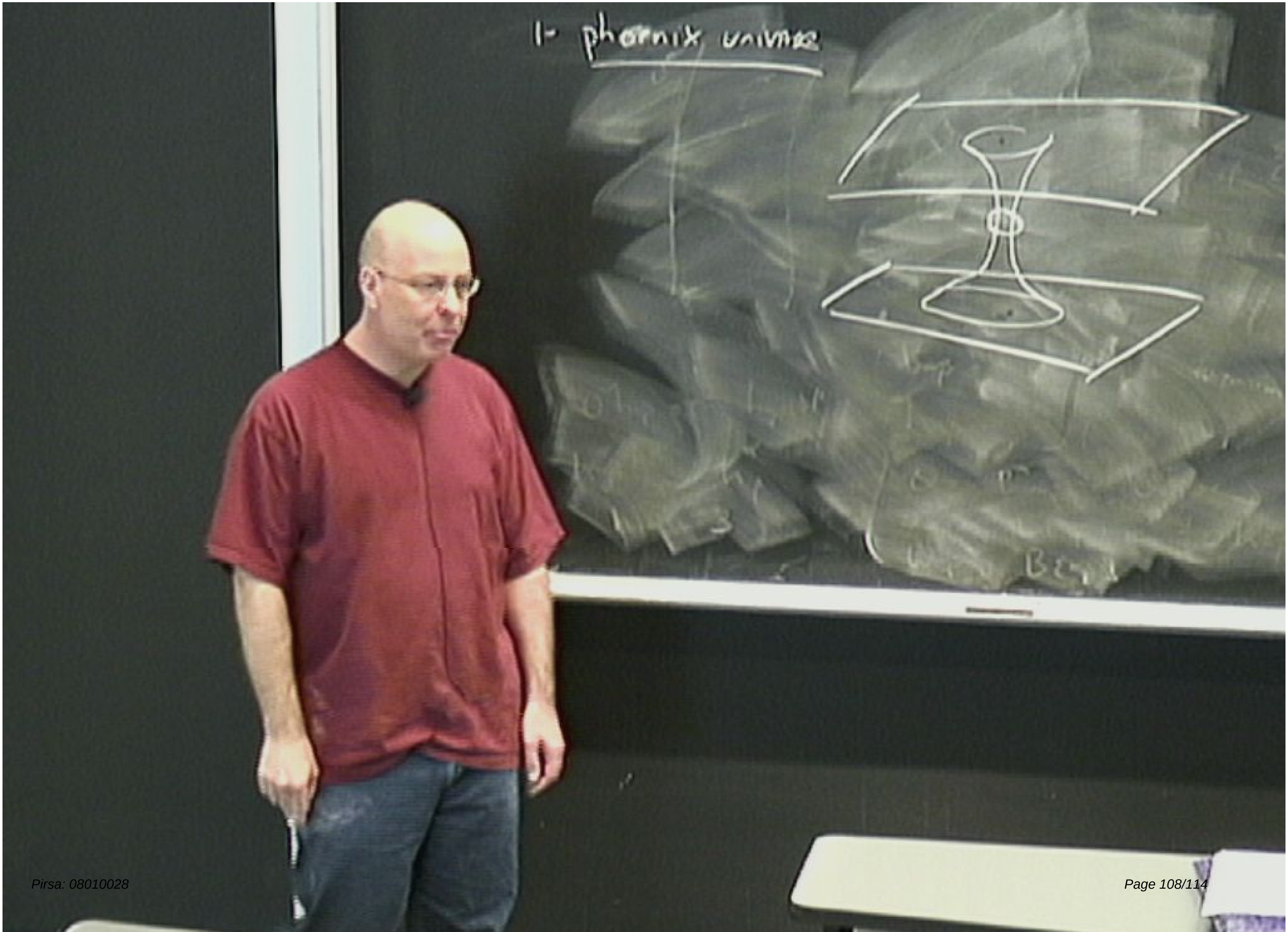
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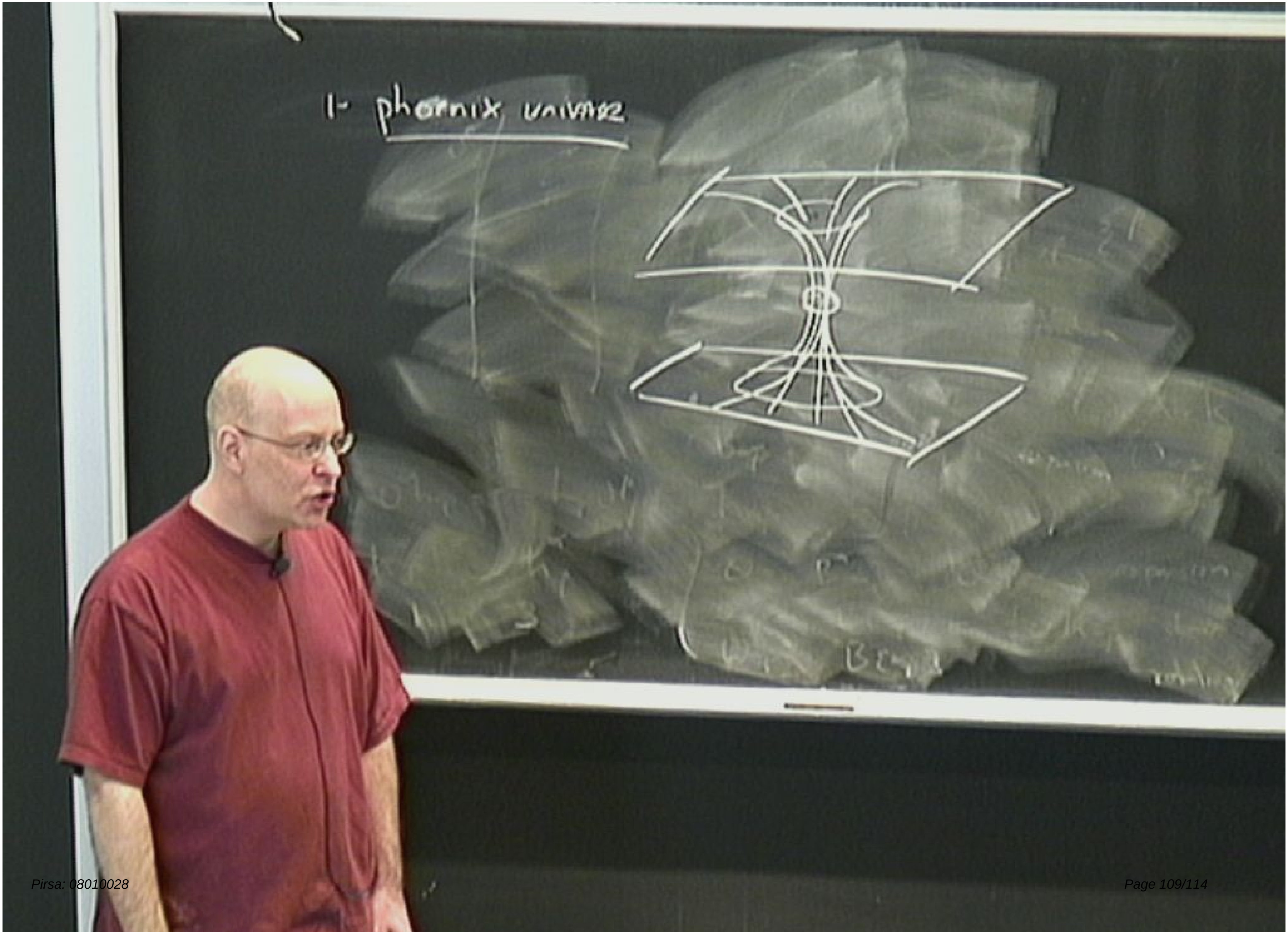
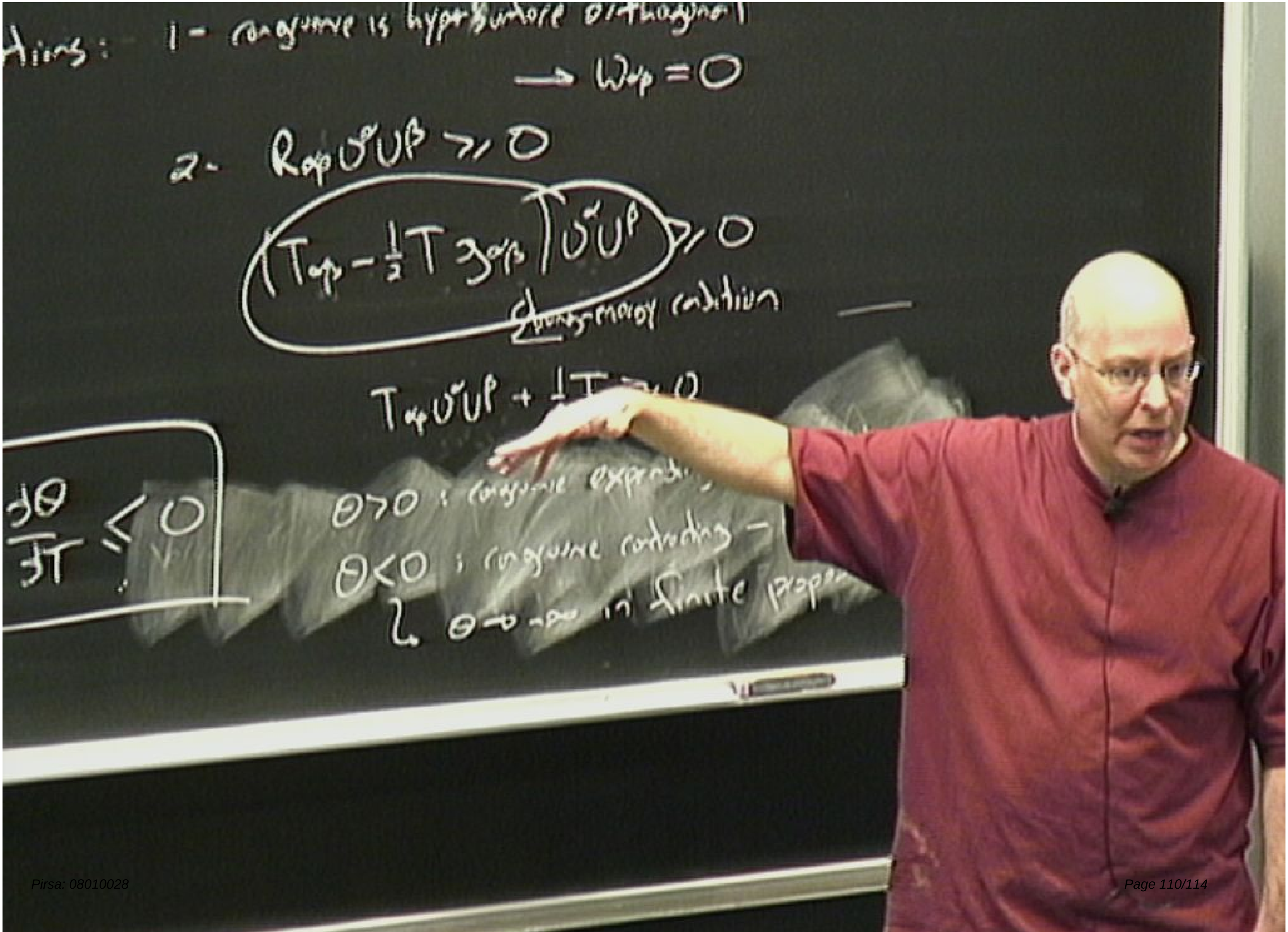
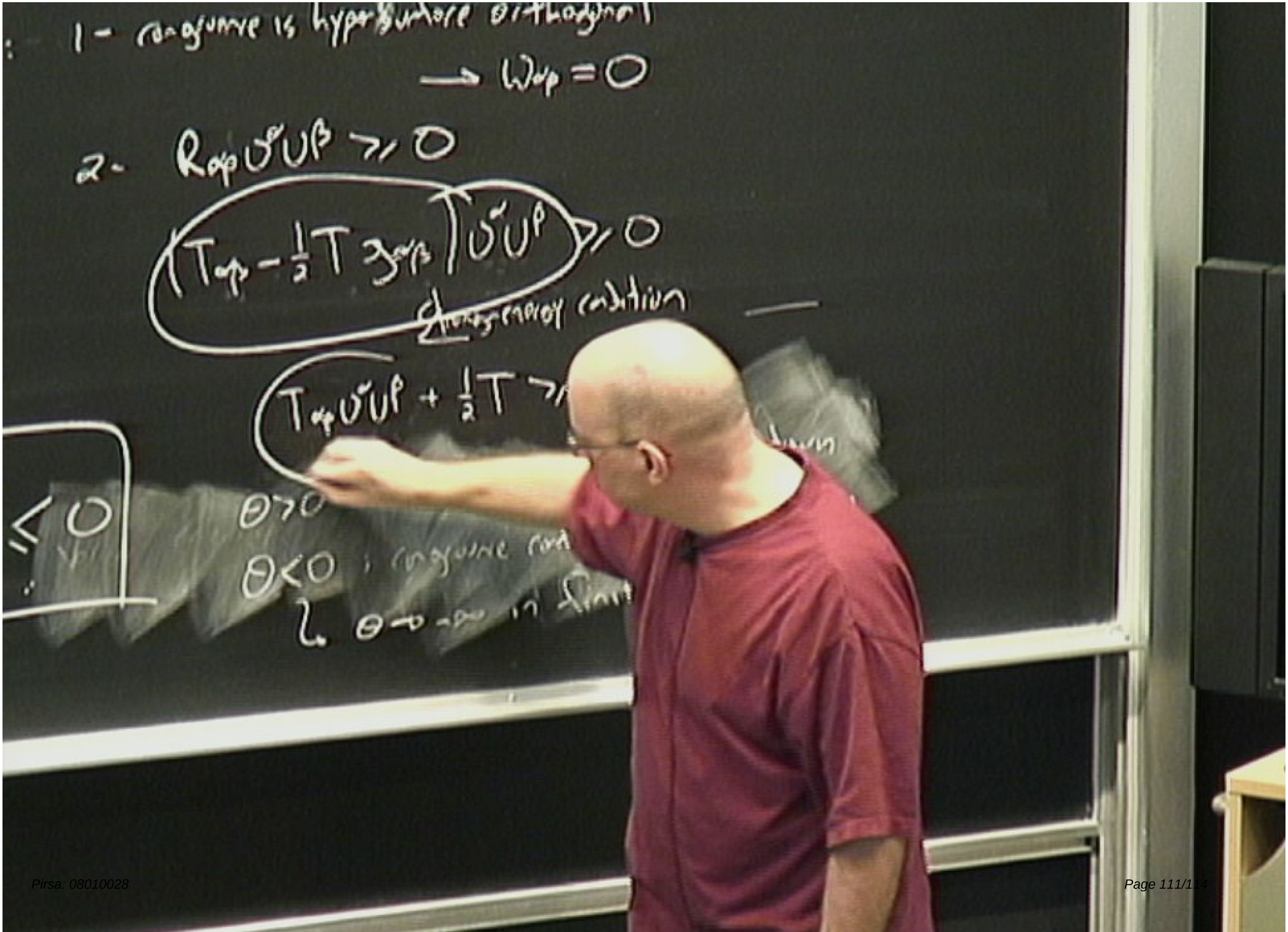
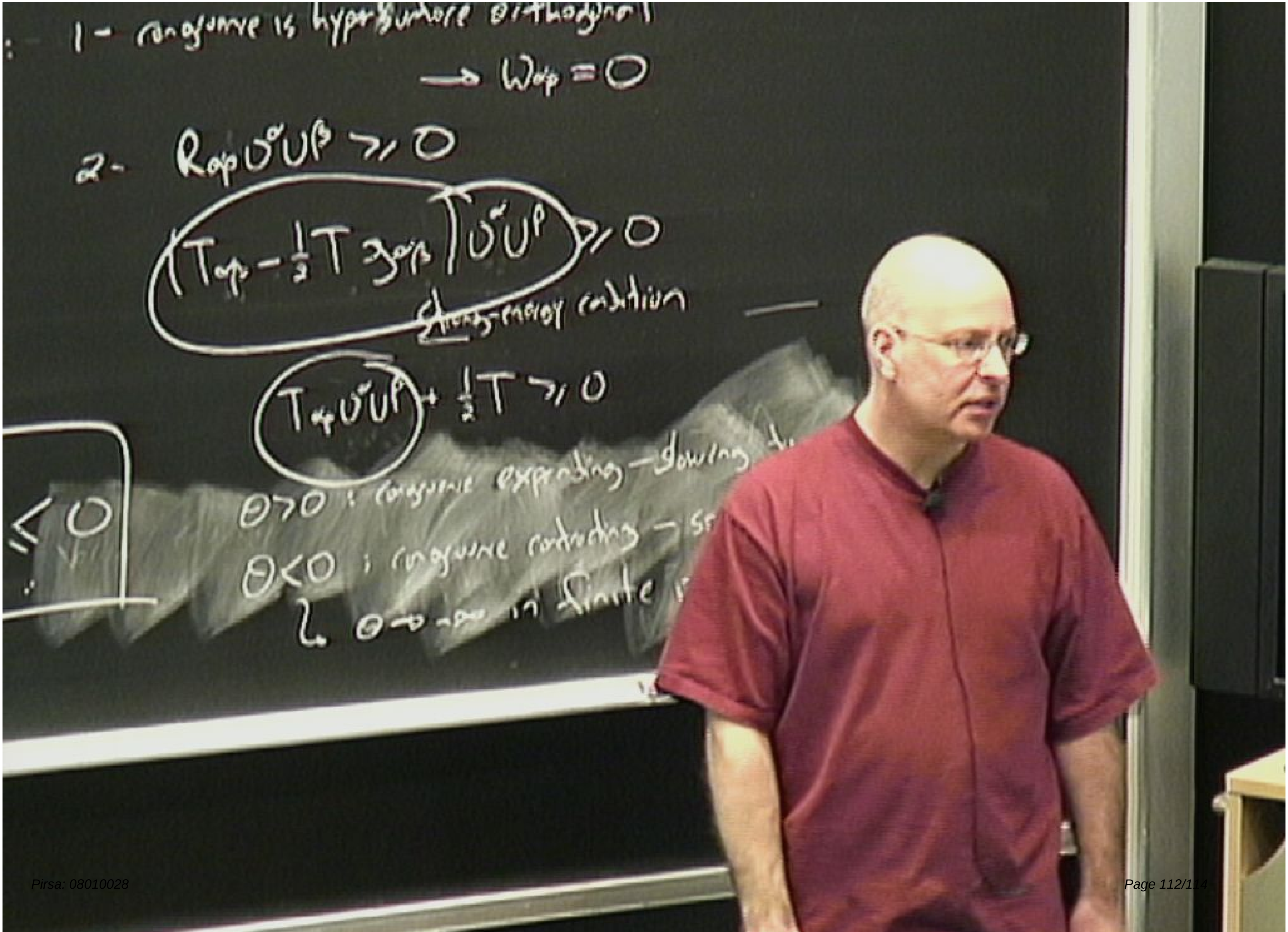
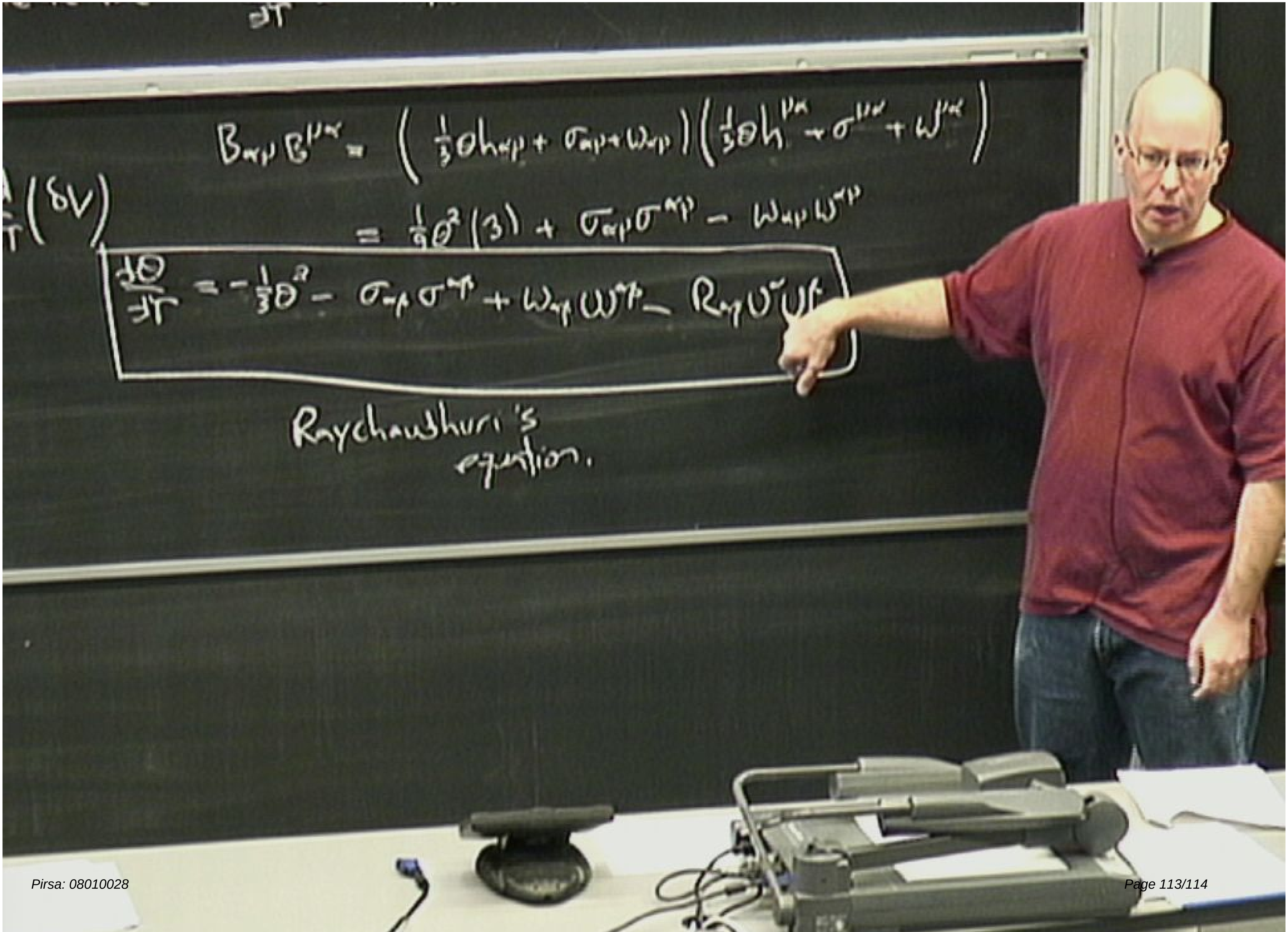

1- phoenix universe


1 - congruence is hypersurface orthogonal
→ ωαβ = 0
2 - Rαβ UαUβ ≥ 0
(Tαβ − ½ T gαβ) UαUβ ≥ 0
energy condition
Tαβ UαUβ + ½ T ≥ 0
dθ/dτ ≤ 0
θ>0 : congruence expanding
θ<0 : congruence contracting
finite proper


1 - congruence is hypersurface orthogonal
→ $\omega_{\alpha\beta} = 0$
2 - $R_{\alpha\beta}u^\alpha u^\beta \geq 0$
$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) u^\alpha u^\beta \geq 0$
strong energy condition
$T_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2} T \geq$
$\theta > 0$
$\theta < 0$: congruence


1 - congruence is hypersurface orthogonal
→ $W_{\alpha\beta} = 0$
2 - $R_{\alpha\beta}U^\alpha U^\beta \geq 0$
$(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta})U^\alpha U^\beta \geq 0$
strong-energy condition
$T_{\alpha\beta}U^\alpha U^\beta + \frac{1}{2}T \geq 0$
$\theta > 0$: congruence expanding
$\theta < 0$: congruence contracting


$B_{\alpha\mu} B^{\mu\alpha} = \left(\frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right)\left(\frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$
$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu}\sigma^{\alpha\mu} - \omega_{\alpha\mu}\omega^{\alpha\mu}$
$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\mu}\sigma^{\alpha\mu} + \omega_{\alpha\mu}\omega^{\alpha\mu} - R_{\mu\nu}U^\mu U^\nu}$
Raychaudhuri's equation.

Take the trace : $\frac{d\theta}{d\tau} = -B_{\alpha\mu}B^{\mu\alpha} - R_{\mu\nu}U^{\mu}U^{\nu}$

$$B_{\alpha\mu}B^{\mu\alpha} = \left(\tfrac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu}\right)\left(\tfrac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha}\right)$$

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}(\delta V)$$

$$= \tfrac{1}{9}\theta^2(3) + \sigma_{\alpha\mu}\sigma^{\alpha\mu} - \omega_{\alpha\mu}\omega^{\alpha\mu}$$

$$\boxed{\frac{d\theta}{d\tau} = -\tfrac{1}{3}\theta^2 - \sigma_{\alpha\mu}\sigma^{\alpha\mu} + \cancel{\omega_{\alpha\mu}\omega^{\alpha\mu}} - R_{\alpha\mu}U^{\alpha}U^{\mu}}$$

Raychaudhuri's equation.