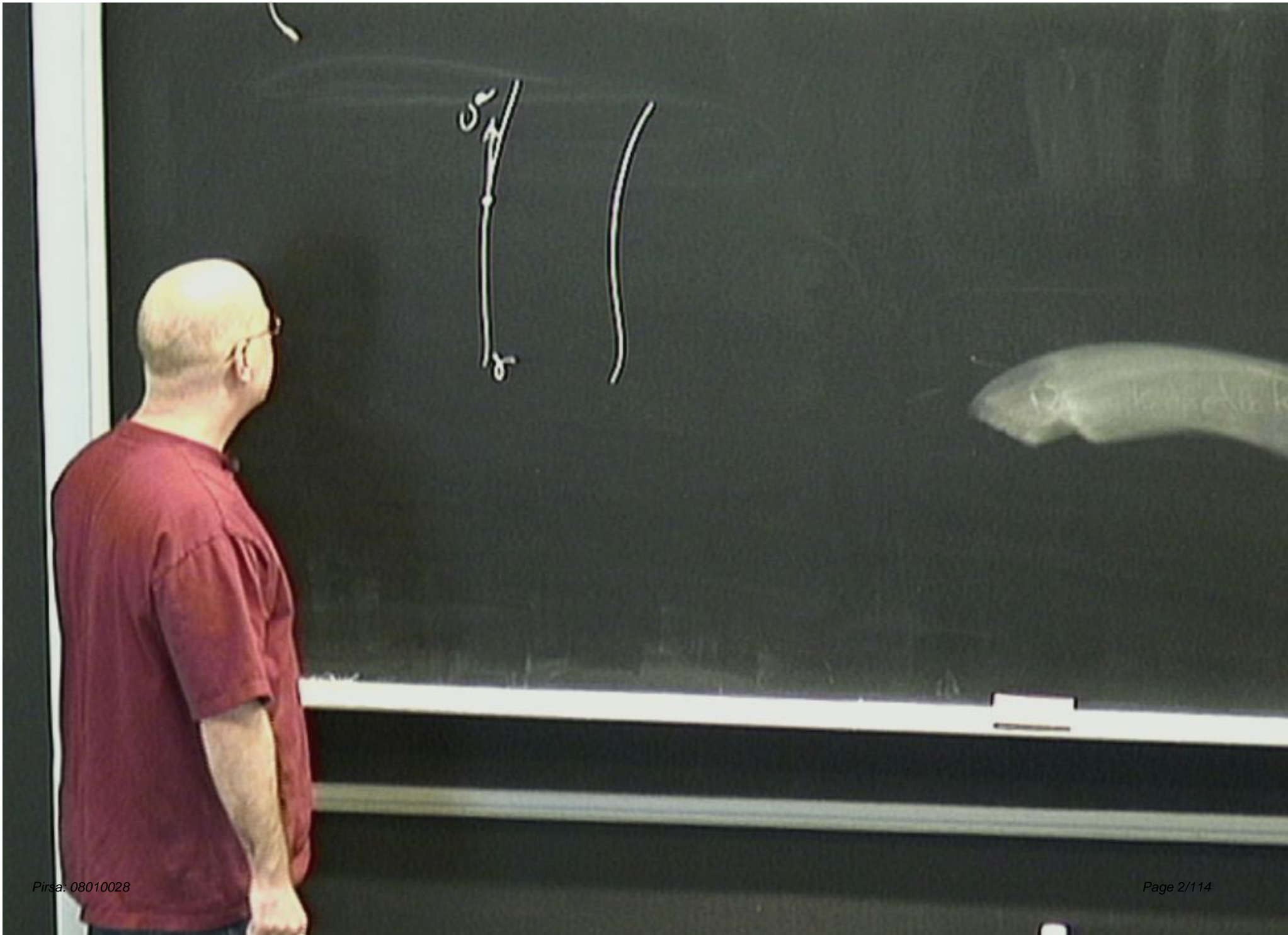


Title: Advanced General Relativity - Lecture 4A

Date: Jan 30, 2008 10:30 AM

URL: <http://pirsa.org/08010028>

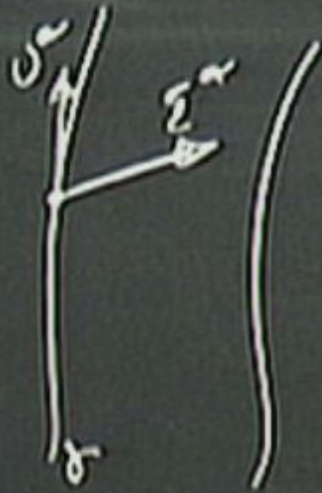
Abstract: Advanced General Relativity





$$\sum \sigma_x = 0$$



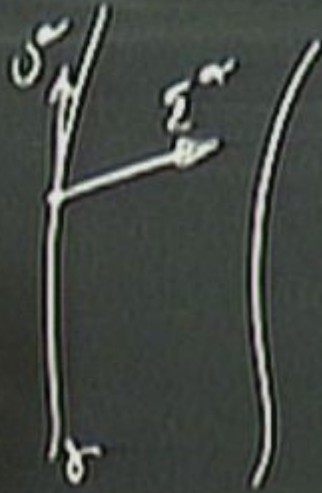


$$\sum u^T = 0$$



$$\sum_{\mu} U^{\mu} = 0$$

$$\sum_{\mu} z_{\mu} U^{\mu} =$$



$$\sum_{\alpha} U^{\alpha} = 0$$

$$\sum_{\alpha} \xi^{\alpha} U^{\alpha} = U^{\alpha} \rho_{\alpha} \xi^{\alpha}$$



$$\Sigma_{\alpha} U^{\alpha} = 0$$

$$\Sigma^{\alpha}{}_{\beta} U^{\beta} = \underbrace{U^{\alpha}{}_{\beta}}_{B^{\alpha}{}_{\beta}} \Sigma^{\beta}$$





$$\xi_{\alpha} U^{\alpha} = 0$$

$$\xi^{\alpha}{}_{;\beta} U^{\beta} = \underbrace{U^{\alpha}{}_{;\beta}}_{B^{\alpha}{}_{\beta}} \xi^{\beta}$$

$$U_{\alpha} B^{\alpha}{}_{\beta} = B^{\alpha}{}_{\beta} U^{\beta} = 0$$



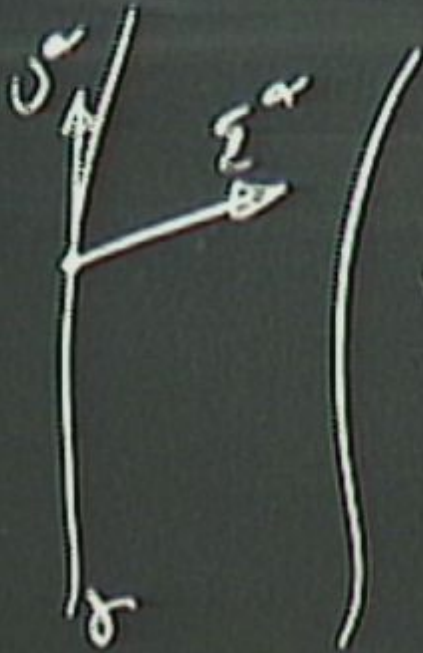


$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{;\beta} U^\beta = \underbrace{U^\alpha_{;\beta}}_{B^\alpha_\beta} \Sigma^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta}$$



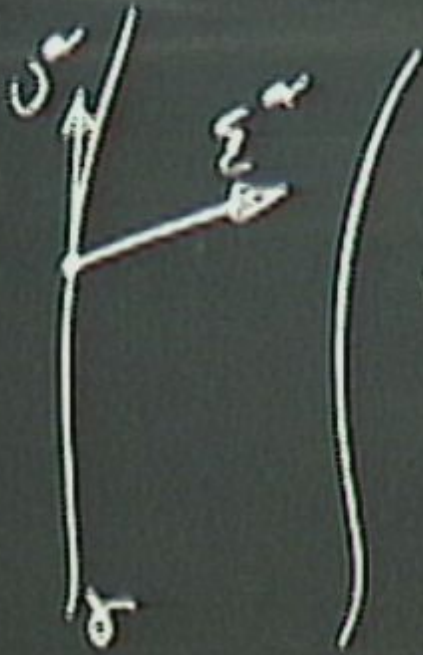
$$\xi_{\alpha} U^{\alpha} = 0$$

relative velocity =  $\xi^{\alpha} \xi_{\beta} U^{\beta} = \underbrace{U^{\alpha} \xi_{\beta}}_{B^{\alpha}_{\beta}} \xi^{\beta}$

$$U_{\alpha} B^{\alpha}_{\beta} = B^{\alpha}_{\beta} U^{\beta} = 0$$

$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$h_{\alpha\beta} = \gamma_{\alpha\beta} + U_{\alpha} U_{\beta}$$



$$\Sigma_\alpha U^\alpha = 0$$

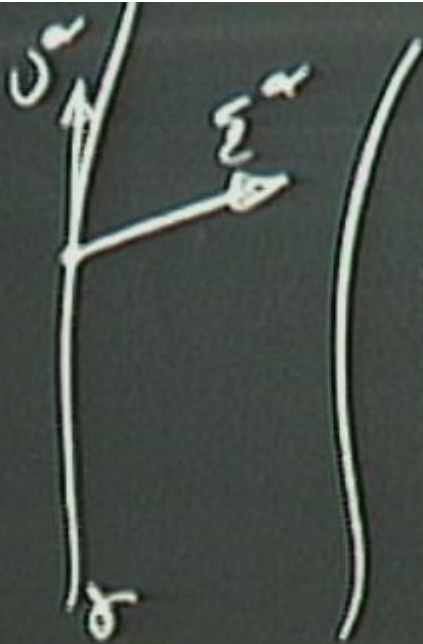
relative velocity =  $\Sigma^\alpha \text{ ; } \rho U^\rho = \underbrace{U^\alpha \text{ ; } \rho}_{B^\alpha_\rho} \Sigma^\rho$

$$U_\alpha B^\alpha_\rho = B^\alpha_\rho U^\rho = 0$$

$$B_{\alpha\rho} = \frac{1}{3} \Theta h_{\alpha\rho} + \sigma_{\alpha\rho} + \omega_{\alpha\rho}$$

$$h_{\alpha\rho} = g_{\alpha\rho} + U_\alpha U_\rho$$

$$\Theta = B^\alpha_\alpha = U^\alpha \text{ ; } \alpha$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{; \beta} U^\beta = U^\alpha_{; \beta} \Sigma^\beta$

$B^\alpha_\beta$

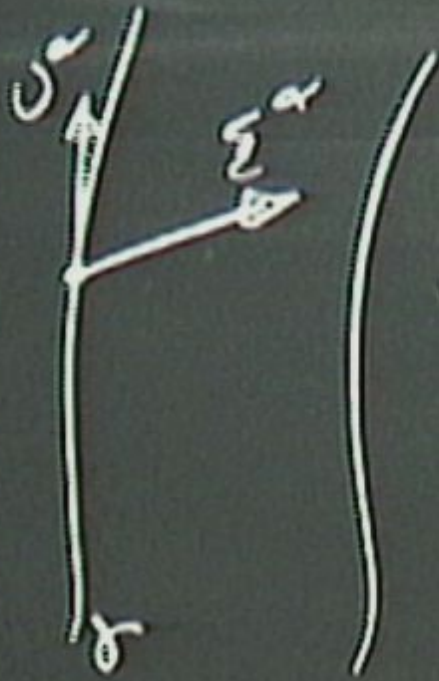
$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

$$\Theta = B^\alpha_\alpha = U^\alpha_{; \alpha}$$

$$\sigma_{\alpha\beta} = B(\alpha\beta)$$



$$\xi_\alpha U^\alpha = 0$$

relative velocity =  $\xi^\alpha_{;\beta} U^\beta = \underbrace{U^\alpha_{;\beta}}_{B^\alpha_\beta} \xi^\beta$

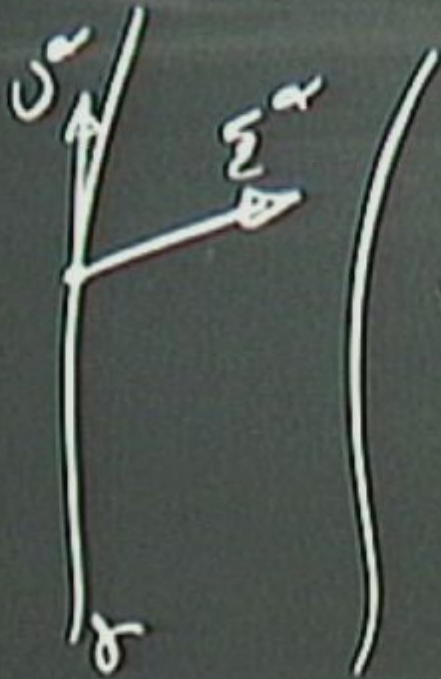
$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + U_\alpha U_\beta$$

$$\Theta = B^\alpha_\alpha = U^\alpha_{;\alpha}$$

$$\sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta}$$



$$\xi_\alpha U^\alpha = 0$$

relative velocity =  $\xi^\alpha_{;\beta} U^\beta = \underbrace{U^\alpha_{;\beta}}_{B^\alpha_\beta} \xi^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

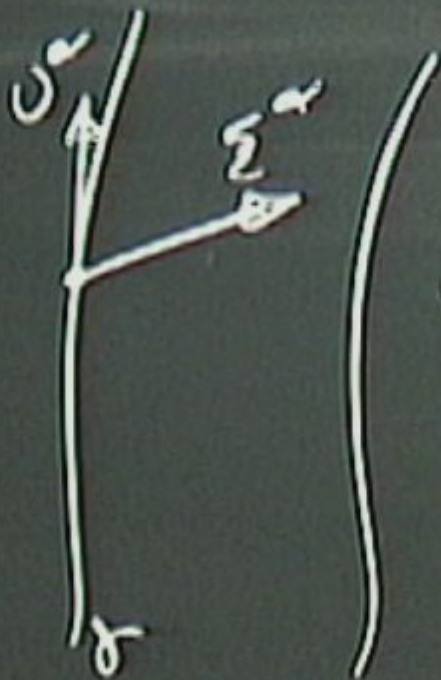
$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

$$\Theta = B^\alpha_\alpha = U^\alpha_{;\alpha}$$

$$\sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta}$$

$$\omega_{\alpha\beta} = B[\alpha\beta]$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{;\beta} U^\beta = \underbrace{U^\alpha_{;\beta}}_{B^\alpha_\beta} \Sigma^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha_{;\alpha} \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B[\alpha\beta] \end{array} \right.$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{; \beta} U^\beta = \underbrace{U^\alpha_{; \beta}}_{B^\alpha_\beta} \Sigma^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$U^\alpha_{; \beta} U^\beta = 0 = h_{\alpha\beta} U^\beta$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha_{; \alpha} \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B[\alpha\beta] \end{array} \right.$$





$$\xi_\alpha U^\alpha = 0$$

relative velocity =  $\xi^\alpha \xi_\beta U^\beta = \underbrace{U^\alpha \xi_\beta}_{B^\alpha_\beta} \xi^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\alpha = 0$$

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

$$h^\alpha_\mu h^\mu_\beta =$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha \xi_\alpha \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B[\alpha\beta] \end{array} \right.$$



$$\sum_{\alpha} U^{\alpha} = 0$$

relative velocity =  $\sum^{\alpha} z^{\alpha} U^{\beta} = \underbrace{U^{\alpha} z^{\beta}}_{B^{\alpha}_{\beta}} \xi^{\beta}$

$$U_{\alpha} B^{\alpha}_{\beta} = B^{\alpha}_{\beta} U^{\alpha} = 0$$

$$U^{\alpha} h_{\alpha\beta} = 0 = h_{\alpha\beta} U^{\alpha}$$

$$h^{\alpha}_{\mu} h^{\mu}_{\beta} = h^{\alpha}_{\beta}$$

$$\left\{ \begin{aligned} B_{\alpha\beta} &= \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} &= \mathcal{I}_{\alpha\beta} + U_{\alpha} U_{\beta} \\ \Theta &= B^{\alpha}_{\alpha} = U^{\alpha} z_{\alpha} \\ \sigma_{\alpha\beta} &= B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} &= B[\alpha\beta] \end{aligned} \right.$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{\beta} U^\beta = \underbrace{U^\alpha_{\beta}}_{B^\alpha_\beta} \Sigma^\beta$

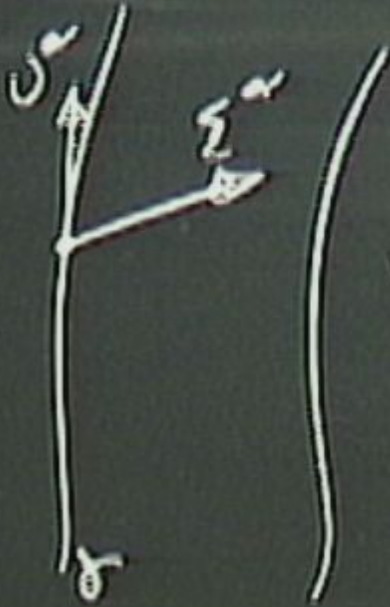
$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

$$h^\alpha_\rho h^\rho_\beta = h^\alpha_\beta$$

$$h^\alpha_\alpha = 3$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha_{;\alpha} \\ \sigma_{\alpha\beta} = B(\omega_{\alpha\beta}) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B[\Sigma_{\alpha\beta}] \end{array} \right.$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha \zeta_\beta U^\beta = \underbrace{U^\alpha \zeta_\beta}_{B^\alpha_\beta} \Sigma^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

$$h^\alpha_\mu h^\mu_\beta = h^\alpha_\beta$$

$$h^\alpha_\alpha = 3$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = \gamma_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha \zeta_\alpha \quad \text{expansion} \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \\ \omega_{\alpha\beta} = B[\alpha\beta] \end{array} \right.$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{;\beta} U^\beta = U^\alpha_{;\beta} \Sigma^\beta$

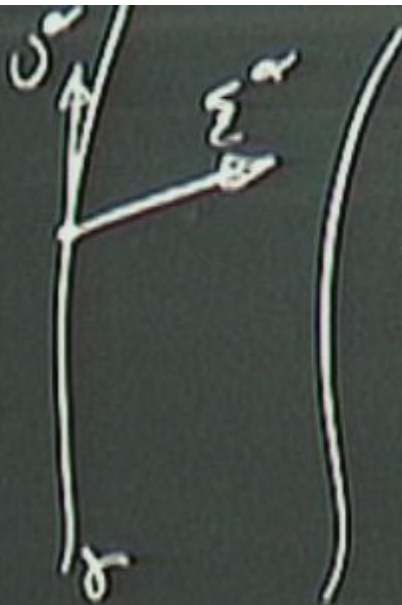
$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

$$h^\alpha_\mu h^\mu_\nu = h^\alpha_\nu$$

$$h^\alpha_\beta h^\beta_\gamma = h^\alpha_\gamma$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha_{;\alpha} \quad \text{expansion} \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \quad \text{shear} \\ \omega_{\alpha\beta} = B[\alpha\beta] \end{array} \right.$$



$$\Sigma_\alpha U^\alpha = 0$$

relative velocity =  $\Sigma^\alpha_{j\beta} U^\beta = \underbrace{U^\alpha_{j\beta}}_{B^\alpha_\beta} \Sigma^\beta$

$$U_\alpha B^\alpha_\beta = B^\alpha_\beta U^\beta = 0$$

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

$$h^\alpha_\mu h^\mu_\beta = h^\alpha_\beta$$

$$h^\alpha_\alpha = 3$$

$$\left\{ \begin{array}{l} B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ h_{\alpha\beta} = \mathcal{I}_{\alpha\beta} + U_\alpha U_\beta \\ \Theta = B^\alpha_\alpha = U^\alpha_{j\alpha} \quad \text{expansion} \\ \sigma_{\alpha\beta} = B(\alpha\beta) - \frac{1}{3}\Theta h_{\alpha\beta} \quad \text{shear} \\ \omega_{\alpha\beta} = B[\alpha\beta] \quad \text{rotation} \end{array} \right.$$

Example: removing objects in expanding U

Example: comoving observers in expanding U

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$



Example: comoving observers in expanding U

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

comoving observers:

Example: comoving observers in expanding U

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

comoving observers:  $U^\alpha = (1, 0, 0, 0)$

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$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

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$$U_\alpha = (-1, 0, 0, 0)$$

← Example: comoving observers in expanding

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

comoving observers:  $U^\alpha = (1, 0, 0, 0)$

$$U_\alpha = (-1, 0, 0, 0)$$

$$= -\frac{\partial}{\partial t}$$

Example: comoving observers in expanding

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

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$$U_\alpha = (-1, 0, 0, 0)$$

$$= -\frac{\partial}{\partial t}$$

$$= -\partial_\alpha t$$

Example: comoving observers in expanding

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

comoving observers:  $U^\alpha = (1, 0, 0, 0)$

$$U_\alpha = (-1, 0, 0, 0)$$

$$= -\frac{\partial}{\partial t}$$

$$= -\partial_\alpha t$$

(hypersurface orthogonal)

Example: removing observers in expanding

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2)$$

removing observers:  $U^\alpha = (1, 0, 0, 0)$

$$U_\alpha = (-1, 0, 0, 0)$$

$$= -\frac{\partial t}{\partial x^\alpha}$$

$$= -\partial_\alpha t \quad (\text{hypersurface orthogonality})$$

$$U_\alpha = -\partial_\alpha \Phi - - \Phi_{,\alpha}$$





$$U_\alpha = -\partial_\alpha \Phi = -\Phi_{,\alpha}$$

$$U_{\alpha\beta} = -\Phi_{,\alpha\beta}$$

$$U_\alpha = -\partial_\alpha \Phi = -\Phi_{;\alpha}$$

$$U_{\alpha;\beta} = -\Phi_{;\beta\alpha} = -\Phi_{;\alpha\beta}$$

$$U_\alpha = -\partial_\alpha \Phi = -\Phi_{;\alpha}$$

$$U_{\alpha;\beta} = -\Phi_{;\alpha\beta} = -\Phi_{;\beta\alpha} = U_{\beta;\alpha}$$

$$U_\alpha = -\partial_\alpha \Phi = -\Phi_{;\alpha}$$

$$U_{\alpha;\beta} = -\Phi_{;\alpha\beta} = -\Phi_{;\beta\alpha} = U_{\beta;\alpha}$$

$$U_{\alpha;\beta} U^\beta = U_{\beta;\alpha} U^\beta = \frac{1}{2} (U_\mu U^\mu)_{;\alpha}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\alpha,\beta}$$

$$U_{\alpha;\beta} = -\frac{\Phi_{;\beta}}{\Phi}$$

$$U_{\alpha;\beta} = \frac{\Phi_{;\beta}}{\Phi}$$



$$R_{\alpha\beta} = U_{\alpha;\beta} = U_{\alpha,\beta} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$U_{\alpha;\beta} = \frac{\partial U_{\alpha}}{\partial x^{\beta}} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$U_{\alpha;\beta} = \frac{\partial U_{\alpha}}{\partial x^{\beta}} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$R_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\delta$$





$$R_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\dot{a}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\dot{a}$$

$$R_{\alpha\beta} = U_{\alpha;\rho} U^{\rho}{}_{\beta} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\dot{a}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} =$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\dot{a}$$

$$\begin{aligned}
 P_{pp} &= U_{\alpha}^T \rho = U_{\alpha}^T \rho - \Gamma_{\alpha}^T U_{\alpha} = \Gamma_{\alpha}^T \\
 &= \text{diag}(0, a^2, a^2, a^2) \\
 h_{pp} &= \text{diag}(0, a^2, a^2, a^2)
 \end{aligned}$$

$$\Gamma_{xx}^T = \Gamma_{yy}^T = \Gamma_{zz}^T = a^2$$

$$B_{\alpha\beta} = U\alpha; \beta = U \cancel{\alpha; \beta} - \Gamma_{\alpha\beta}^{\gamma} U\gamma = \Gamma_{\alpha\beta}^{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \text{diag}(0, a^2, a^2, a^2)$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \text{diag}(0, a^2, a^2, a^2)$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

$$\Gamma_{\alpha\beta}^{\gamma}$$

$$\begin{aligned}
 B_{\alpha\beta} &= U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma} \\
 &= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a}) \\
 h_{\alpha\beta} &= \text{diag}(0, a^2, a^2, a^2)
 \end{aligned}$$

$$\Gamma_{xx}^+ = \Gamma_{yy}^+ = \Gamma_{zz}^+ = a\dot{a}$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

expression ✓  
 no shear  
 no rotation





$$U_{\alpha} = T_{\alpha\beta}$$

$$\dot{a}, \alpha \dot{a}$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

expansion ✓  
no shear  
no rotation

$$\Theta = \frac{3\dot{a}}{a}$$

$$P_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \text{diag}(0, a^2, a^2, a^2)$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

expression  
no shear  
no rotation

$$\delta V$$

$$\delta V + a^3$$

$$\Theta = \frac{3\dot{a}}{a}$$

$$h_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \text{diag}(0, a^2, a^2, a^2)$$

$$\beta_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

$$\delta V$$

$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2 \dot{a}$$

$$\Theta = \frac{3\dot{a}}{a}$$

$$K_{\alpha\beta} = U_{\alpha;\beta} = U_{\beta;\alpha} - \Gamma_{\alpha\beta}^{\gamma} U_{\gamma} = \Gamma_{\alpha\beta}^{\gamma} U_{\gamma}$$

$$= \text{diag}(0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$h_{\alpha\beta} = \text{diag}(0, a^2, a^2, a^2)$$

$$B_{\alpha\beta} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

$$\delta V$$

$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2 \dot{a}$$

$$\frac{1}{\delta V} (\delta V)' = \frac{3\dot{a}}{a}$$

$$\Theta = \frac{3\dot{a}}{a}$$

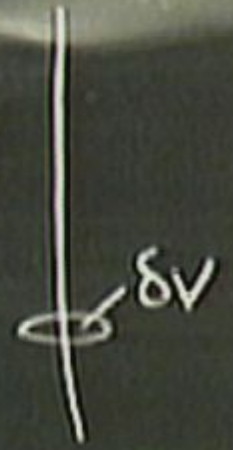
$$p = U_{k,p} - U_p U_k = U_p$$

$$\rightarrow (0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$\rightarrow (0, a^2, a^2, a^2)$$

$$B_{\text{exp}} = \frac{\dot{a}}{a} h_{\text{exp}}$$

expression ✓  
no shear  
no rotation



$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2 \dot{a}$$

$$\Theta = \frac{3\dot{a}}{a} = \frac{1}{\delta V} (\delta V)'$$

$$\frac{1}{\delta V} (\delta V)' = \frac{3\dot{a}}{a}$$

$$p = U_{\alpha\beta} - T_{\alpha\beta} U_{\gamma} = T_{\alpha\beta}$$

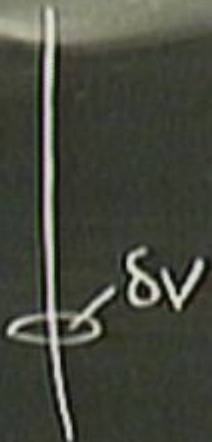
$$\rightarrow (0, a\dot{a}, a\dot{a}, a\dot{a})$$

$$\rightarrow (0, a^2, a^2, a^2)$$

$$B_{\text{exp}} = \frac{\dot{a}}{a} h_{\alpha\beta}$$

expression ✓  
no shear  
no rotation

$$\Theta = \frac{3\dot{a}}{a} = \frac{1}{\delta V} (\delta V)'$$



$$\delta V \propto a^3$$

$$(\delta V)' \propto 3a^2 \dot{a}$$

$$\frac{1}{\delta V} (\delta V)' = \frac{3\dot{a}}{a}$$

$$\sum_{\alpha} U^{\alpha} = 0$$

relative velocity =  $\sum^{\alpha}_{; \beta} U^{\beta} = \underbrace{U^{\alpha}_{; \beta}}_{B^{\alpha}_{\beta}} \sum^{\beta}$

$$U_{\alpha} B^{\alpha}_{\beta} = B^{\alpha}_{\beta} U^{\alpha} = 0$$

$$B_{\alpha\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$h_{\alpha\beta} = \gamma_{\alpha\beta} + U_{\alpha} U_{\beta}$$

$$\Theta = B^{\alpha}_{\alpha} = U^{\alpha}_{; \alpha} \quad \text{expansion}$$

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{2} \Theta h_{\alpha\beta}$$

$$\omega_{\alpha\beta} = B_{[\alpha\beta]}$$

$$= -\partial_{\alpha} t \quad (\text{hypersurface orthogonality})$$

$$P_{\alpha\beta} = U_{\alpha} U_{\beta}$$

$$= \text{diag}$$

$$h_{\alpha\beta} = \gamma_{\alpha\beta}$$

Dynamics: evolution equations for  $\Theta$ ,  $T_{op}$ ,  $W_{op}$



Dynamics: evolution equations for  $\Theta$ , Top,  $U_{\text{top}}$

$$B_{\alpha\beta\gamma\mu} U^{\mu} =$$

Dynamics: evolution equations for  $\Theta$ ,  $T_{\alpha\beta}$ ,  $U_{\alpha\beta}$

$$B_{\alpha\beta\gamma\mu} U^{\mu} = U_{\alpha\beta\gamma\mu} U^{\mu}$$

Dynamics: evolution equations for  $\Theta$ ,  $T_{\alpha\beta}$ ,  $U_{\alpha\beta}$

$$\begin{aligned} B_{\alpha\beta\gamma\mu} U^\mu &= U_{\alpha\beta\gamma\mu} U^\mu \\ &= (U_{\alpha\beta\gamma\mu} - R_{\alpha\beta\gamma\mu} U^\nu) U^\mu \end{aligned}$$

$$\begin{aligned}
 B_{\alpha\beta} U^{\mu} &= U_{\alpha\beta\rho} U^{\mu} \\
 &= (U_{\alpha\beta\rho} - R_{\alpha\beta\rho\sigma} U^{\sigma}) U^{\mu} \\
 &= U_{\alpha\beta\rho} U^{\mu} - R_{\alpha\beta\rho\sigma} U^{\mu} U^{\sigma}
 \end{aligned}$$

$$\begin{aligned}
B_{\alpha\beta\gamma\mu} U^\mu &= U_{\alpha\gamma\beta\mu} U^\mu \\
&= (U_{\alpha\gamma\mu\beta} - R_{\alpha\gamma\beta\mu} U^\mu) U^\mu \\
&= U_{\alpha\gamma\mu\beta} U^\mu - R_{\alpha\gamma\beta\mu} U^\mu U^\mu \\
&= (U_{\alpha\gamma\mu\beta} U^\mu)_{;\beta} - U_{\alpha\gamma\mu} U^\mu_{;\beta} - R_{\alpha\gamma\beta\mu} U^\mu U^\mu
\end{aligned}$$

$$U \alpha \beta \beta U^T = U \alpha \beta \beta U^T$$

$$= (U \alpha \beta \beta - R \alpha \beta \beta U^T) U^T$$

$$= U \alpha \beta \beta U^T - R \alpha \beta \beta U^T U^T$$

$$= (U \alpha \beta \beta U^T)_{\beta \beta} - U \alpha \beta \beta U^T_{\beta \beta} - R \alpha \beta \beta U^T_{\beta \beta}$$

$$B_{\alpha \beta \beta} U^T = - R \alpha \beta \beta U^T_{\beta \beta}$$

$$U_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} = U_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta}$$

$$= (U_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta}) U^{\alpha\beta\gamma\delta}$$

$$= U_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta}$$

$$= (\cancel{U_{\alpha\beta\gamma\delta}})_{\alpha\beta\gamma\delta} - U_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta}$$

$$B_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} = - B_{\alpha\beta\gamma\delta} B^{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta} U^{\alpha\beta\gamma\delta}$$

$$\begin{aligned}
 &= U_{\alpha\beta\gamma\delta} U^{\mu\nu} - R_{\alpha\beta\gamma\delta} U^{\mu\nu} \\
 &= (\cancel{U_{\alpha\beta\gamma\delta}})_{\beta\gamma} - U_{\alpha\beta\gamma\delta} U^{\mu\nu} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}
 \end{aligned}$$

$$\boxed{B_{\alpha\beta\gamma\delta} U^{\mu\nu} = -B_{\alpha\beta\gamma\delta} U^{\mu\nu} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}}$$

Take the trace :



$$\begin{aligned}
 &= U_{\alpha\beta\gamma\delta} U^{\mu\nu} - R_{\alpha\beta\gamma\delta} U^{\mu\nu} \\
 &= (U_{\alpha\beta\gamma\delta})_{;\beta} - U_{\alpha\beta\gamma} U^{\mu}{}_{;\beta} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}
 \end{aligned}$$

$$\boxed{B_{\alpha\beta\gamma\delta} U^{\mu\nu} = -B_{\alpha\beta\gamma} B^{\mu}{}_{\delta} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}}$$

Take the trace :  $\frac{\partial \theta}{\partial T}$

$$\begin{aligned}
 &= U_{\alpha\beta\gamma\delta} U^{\mu\nu} - R_{\alpha\beta\gamma\delta} U^{\mu\nu} \\
 &= (U_{\alpha\beta\gamma\delta})_{;\beta} - U_{\alpha\beta\gamma} U^{\mu}_{;\beta} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}
 \end{aligned}$$

$$\boxed{B_{\alpha\beta\gamma\delta} U^{\mu\nu} = -B_{\alpha\beta\gamma} B^{\mu}_{\delta} - R_{\alpha\beta\gamma\delta} U^{\mu\nu}}$$

Take the trace :

$$\frac{\partial \theta}{\partial T} = -B_{\alpha\beta} B^{\alpha\beta}$$

$$= (\cancel{U_{\alpha\beta\gamma\delta}})_{;\beta} - U_{\alpha\beta\gamma\delta} U^{\mu}_{;\beta} - R_{\alpha\mu\beta\gamma}$$

$$\boxed{B_{\alpha\beta\gamma\delta} U^{\mu} = -B_{\alpha\mu\beta\gamma} - R_{\alpha\mu\beta\gamma} U^{\mu}}$$

Take the trace :  $\frac{d\theta}{dT} = -B_{\alpha\mu\beta\gamma} - R_{\mu\nu} U^{\mu} U^{\nu}$

$$\partial_{\alpha\beta} \partial^{\mu\nu} U^{\rho} = -B_{\alpha\mu} B^{\nu\rho} - R_{\alpha\mu\rho\nu} U^{\mu\nu}$$

$$\kappa = \frac{\partial \Theta}{\partial T} = -B_{\alpha\mu} B^{\mu\alpha} - R_{\mu\nu} U^{\mu\nu}$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \right) \left( \frac{1}{3} \Theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$
$$= \frac{1}{9}\theta^2$$

$$\begin{aligned} B_{\alpha\mu} B^{\mu\alpha} &= \left( \frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right) \\ &= \frac{1}{9}\theta^2 (3) \end{aligned}$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$
$$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu}\sigma^{\mu\alpha}$$

$$\begin{aligned} B_{\alpha\mu} B^{\mu\alpha} &= \left( \frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right) \\ &= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu} \sigma^{\alpha\mu} - \omega_{\alpha\mu} \omega^{\alpha\mu} \end{aligned}$$



$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9}\theta^2 (3) + \sigma_{\alpha\mu} \sigma^{\alpha\mu} - \omega_{\alpha\mu} \omega^{\alpha\mu}$$

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\mu} \sigma^{\alpha\mu} + \omega_{\alpha\mu} \omega^{\alpha\mu} - R_{\alpha\mu} U^{\alpha} U^{\mu}$$

Raychaudhuri's  
equation.

$$\Theta = \frac{1}{\delta V} \frac{d}{dT} (\delta V)$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3} \Theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3} \Theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9} \Theta^2 (3) + \sigma_{\alpha\mu} \sigma^{\mu\alpha} - \omega_{\alpha\mu} \omega^{\mu\alpha}$$

$$\frac{d\Theta}{dT} = -\frac{1}{3} \Theta^2 - \sigma_{\alpha\mu} \sigma^{\mu\alpha} + \omega_{\alpha\mu} \omega^{\mu\alpha} - R_{\alpha\mu} U^{\alpha} U^{\mu}$$

Raychaudhuri's equation.

$$\Theta = \frac{1}{\delta V} \frac{d(\delta V)}{dT}$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3} \Theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3} \Theta \delta^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9} \Theta^2 (3) + \sigma_{\alpha\mu} \sigma^{\mu\alpha}$$

$$\frac{d\Theta}{dT} = -\frac{1}{3} \Theta^2 - \sigma_{\alpha\mu} \sigma^{\mu\alpha}$$

Raychaudhuri's equation.

$$\Theta = \frac{1}{\delta V} \frac{d}{dT} (\delta V)$$

$$B_{\alpha\beta} B^{\mu\nu} = \left( \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \right) \left( \frac{1}{3} \Theta h^{\mu\nu} + \sigma^{\mu\nu} + \omega^{\mu\nu} \right)$$

$$= \frac{1}{9} \Theta^2 (3) + \sigma_{\alpha\beta} \sigma^{\alpha\beta} - \omega_{\alpha\beta} \omega^{\alpha\beta}$$

$$\frac{d\Theta}{dT} = -\frac{1}{3} \Theta^2 - \sigma_{\alpha\beta} \sigma^{\alpha\beta} + \omega_{\alpha\beta} \omega^{\alpha\beta}$$

Raychaudhuri's equation.

# Focusing thm

Assumptions: 1 - language is hyperbolic orthogonal  
 $\rightarrow W_{op} = 0$

## Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal  
 $\rightarrow W_{ab} = 0$

2 -  $R_{ab}U^aU^b > 0$

# Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$$\rightarrow \omega_{\alpha\beta} = 0$$

2.  $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) U^\alpha U^\beta \geq 0$$

Strong-energy condition

# Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$$\rightarrow \omega_{\alpha\beta} = 0$$

$$2 - R_{\alpha\beta} \tilde{u}^{\alpha} \tilde{u}^{\beta} \geq 0$$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) \tilde{u}^{\alpha} \tilde{u}^{\beta} \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} \tilde{u}^{\alpha} \tilde{u}^{\beta} + \frac{1}{2} T \geq 0$$



## Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$$\rightarrow \omega_{[a} \omega_{b]} = 0$$

2.  $R_{\alpha\beta} \tilde{U}^\alpha \tilde{U}^\beta \gg 0$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) \tilde{U}^\alpha \tilde{U}^\beta \gg 0$$

Strong energy condition

$$T_{\alpha\beta} \tilde{U}^\alpha \tilde{U}^\beta + \frac{1}{2} T \gg 0$$

vacuum energy:  $T_{\alpha\beta} = \Lambda g_{\alpha\beta}$

## Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$$\rightarrow \omega_{\alpha\beta} = 0$$

2.  $R_{\alpha\beta} \tilde{U}^{\alpha} \tilde{U}^{\beta} \geq 0$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) \tilde{U}^{\alpha} \tilde{U}^{\beta} \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} \tilde{U}^{\alpha} \tilde{U}^{\beta} + \frac{1}{2} T \geq 0$$

vacuum energy:  $T_{\alpha\beta} = \Lambda g_{\alpha\beta}$

$$-\Lambda + \frac{1}{2} (4) \Lambda = 3\Lambda \geq 0$$

# Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal  
 $\rightarrow W_{\alpha\beta} = 0$

$$2 - R_{\alpha\beta} \tilde{U}^{\alpha} \tilde{U}^{\beta} \geq 0$$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) \tilde{U}^{\alpha} \tilde{U}^{\beta} \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} \tilde{U}^{\alpha} \tilde{U}^{\beta} + \frac{1}{2} T \geq 0$$

$$-3\Lambda \geq 0$$

vacuum energy:  $T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$

$$+\Lambda \neq \frac{1}{2} (4)\Lambda = \Lambda$$

# Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal  
 $\rightarrow W_{\alpha\beta} = 0$

$$2 - R_{\alpha\beta} U^{\alpha} U^{\beta} \geq 0$$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) U^{\alpha} U^{\beta} \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} U^{\alpha} U^{\beta} + \frac{1}{2} T \geq 0$$

vacuum energy:

$$T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \Lambda \frac{1}{2} (4) \Lambda =$$

$$-3\Lambda/2 < 0$$

# Focusing thm

Assumptions: 1 - congruence is hypersurface orthogonal

$$\rightarrow W_{\alpha\beta} = 0$$

2 -  $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) U^\alpha U^\beta \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T \geq 0$$

$$-3\Lambda \geq 0$$

vacuum energy:  $T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$

$$+\Lambda + \frac{1}{2} (4)\Lambda = \Lambda$$

# Focusing thm

Assumptions: 1- congruence is hypersurface orthogonal

$$\rightarrow W_{\alpha\beta} = 0$$

2-  $R_{\alpha\beta} \dot{U}^\alpha \dot{U}^\beta \geq 0$

$$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) \dot{U}^\alpha \dot{U}^\beta \geq 0$$

Strong-energy condition

$$T_{\alpha\beta} \dot{U}^\alpha \dot{U}^\beta + \frac{1}{2} T \geq 0$$

$$\frac{d\theta}{dt} \leq 0$$

Stoichiometry condition

$$T_{\text{up}} \frac{dU}{dP} + \frac{1}{2} T \gg 0$$

$$\frac{d\theta}{dT} \leq 0$$

$\theta > 0$  : consequence expanding - slowing down

Steady-state condition

$$T_{up} v_{up} + \frac{1}{2} T > 0$$

$$\frac{d\theta}{dT} \leq 0$$

$\theta > 0$  : congruence expanding - slowing down

$\theta < 0$  : congruence contracting - speeding up



$$\frac{d\theta}{dT} \leq 0$$

$$T \propto \sqrt{L} + \frac{1}{2} l \propto \sqrt{L}$$

$\theta > 0$  : cone expanding - slowing down

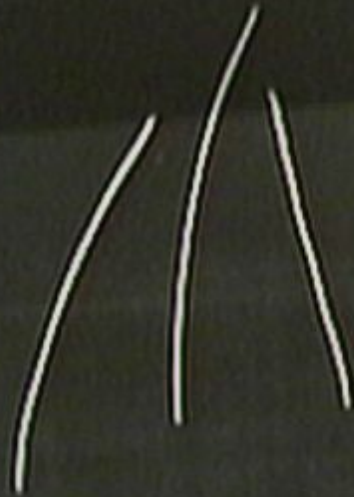
$\theta < 0$  : cone contracting - speeding up

$\theta \rightarrow 0$  as  $L$  approaches  $\infty$



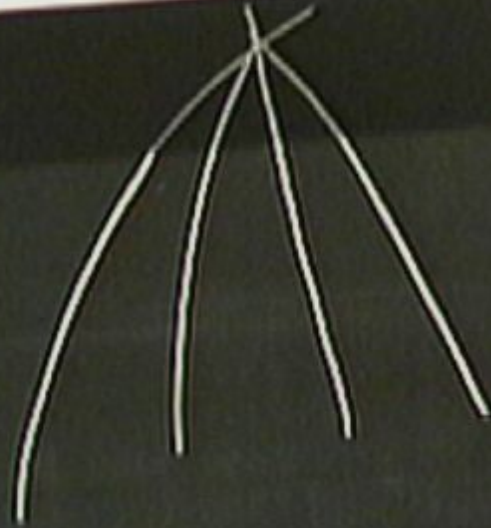
$$\frac{d\theta}{dT} \leq 0$$

$\theta > 0$  : resource expanding - spreading out  
 $\theta < 0$  : resource contracting - spreading up  
 $\hookrightarrow \theta \rightarrow 0$  in finite population  $N$



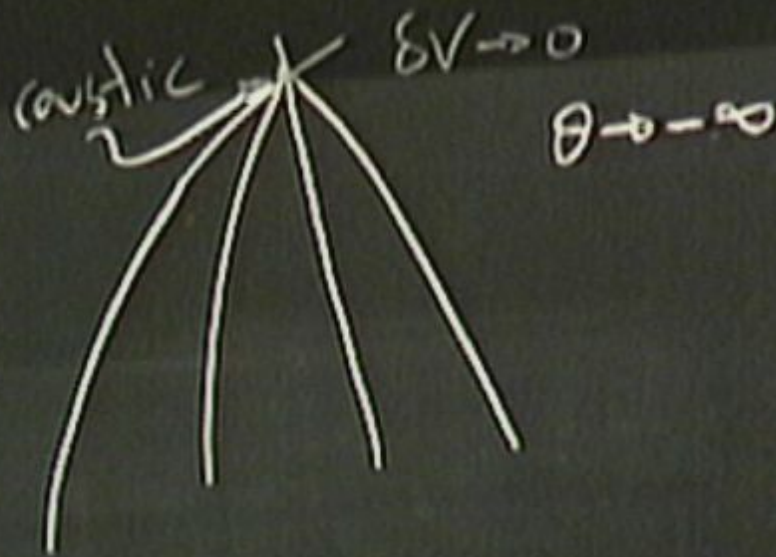
$$\frac{d\theta}{dT} \leq 0$$

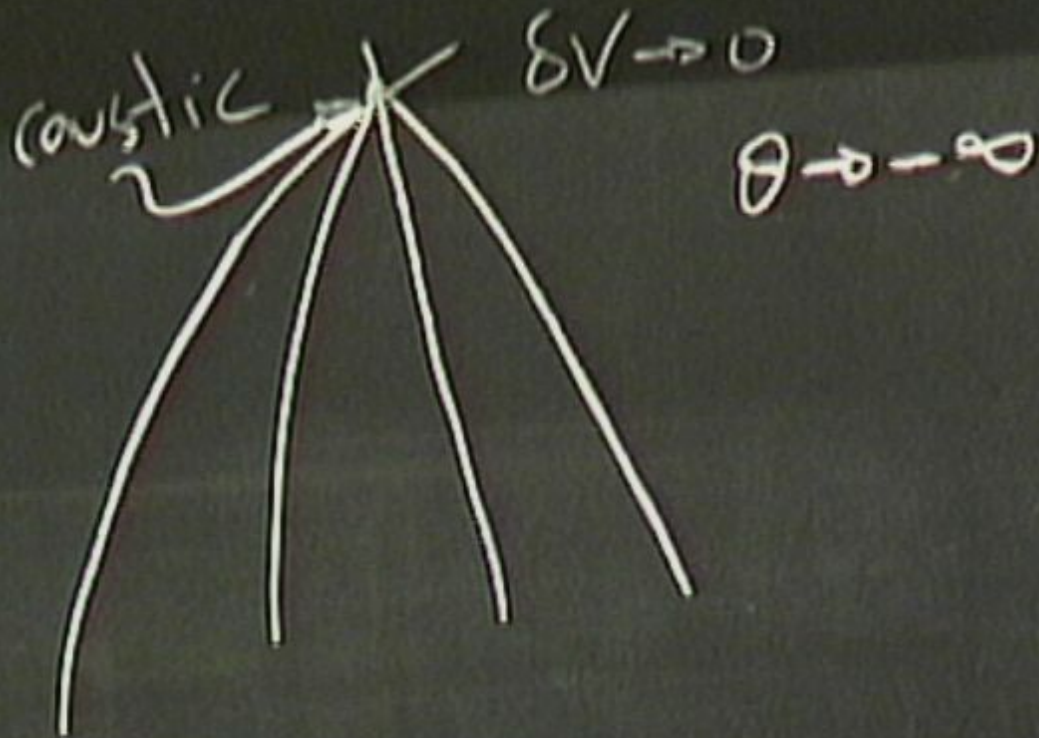
$\theta > 0$  : longwave spreading - spreading down  
 $\theta < 0$  : longwave spreading - spreading up  
 $\theta \rightarrow 0$  : in static equilibrium  $\uparrow$



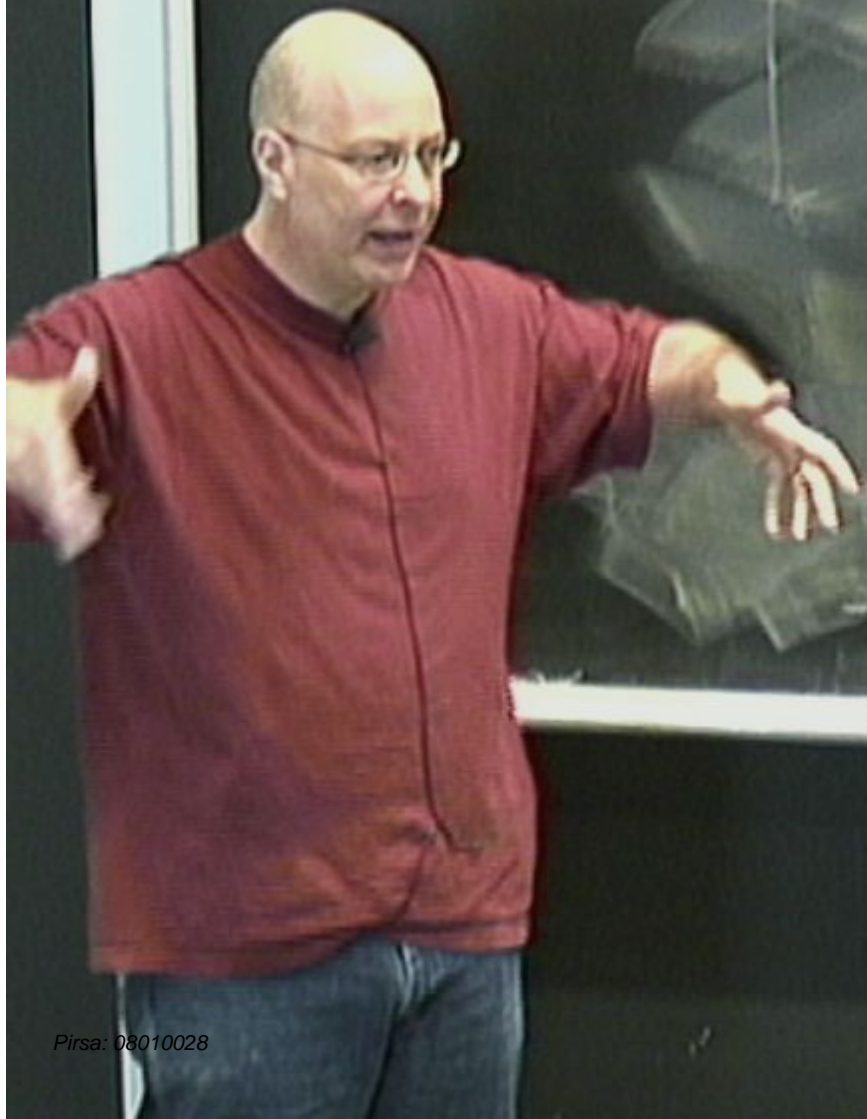
JT

$\theta < 0$  : negative energies - spreading up  
 $\theta \rightarrow -\infty$  in finite potential  $V$





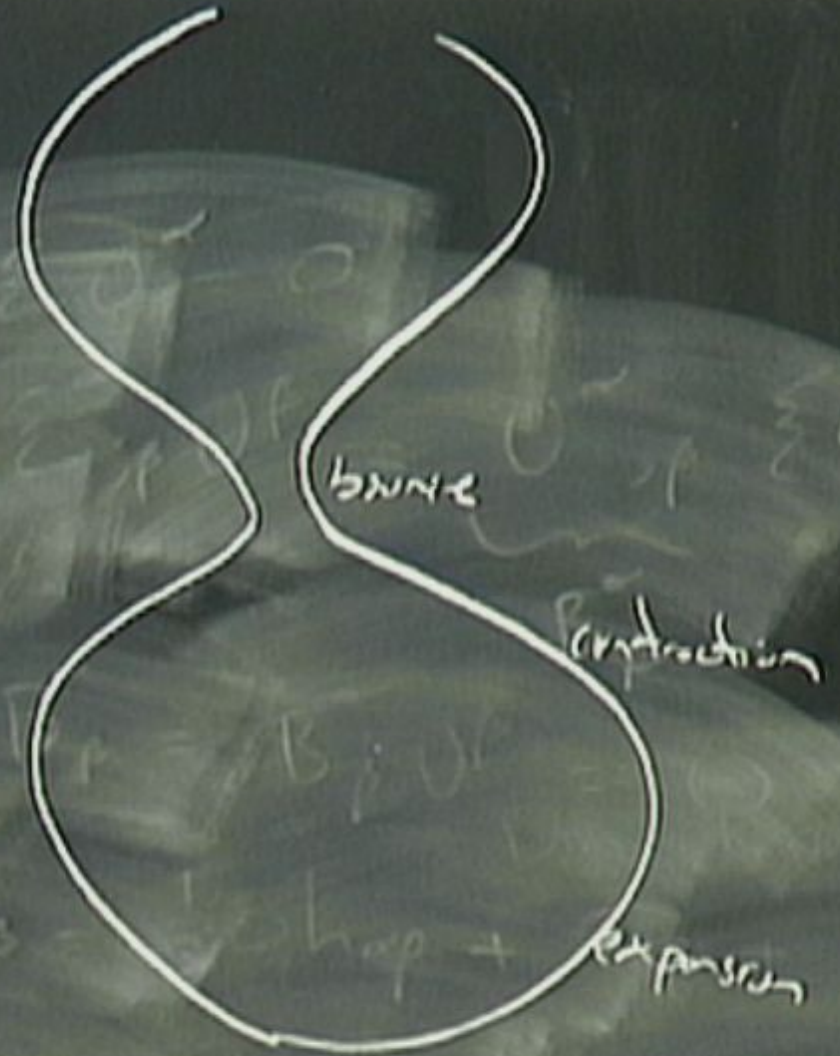
1- phoenix universe



1- phoenix univaz



1- phoenix universe

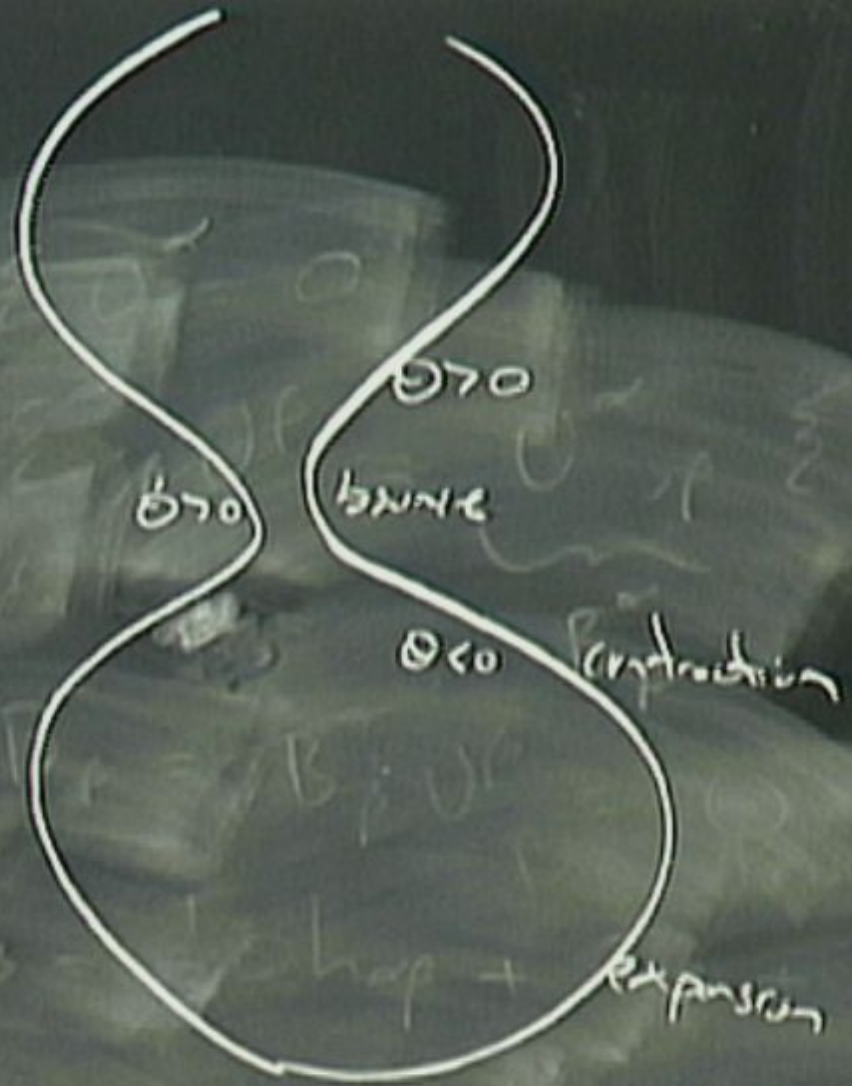




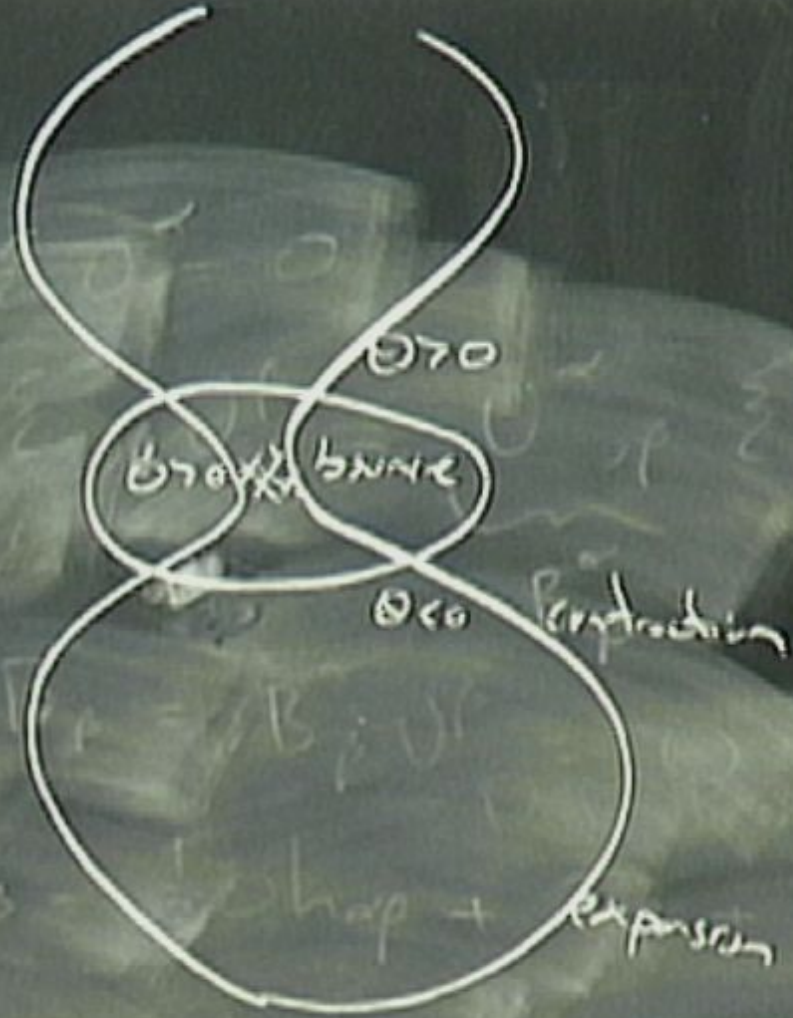
1- phoenix universe



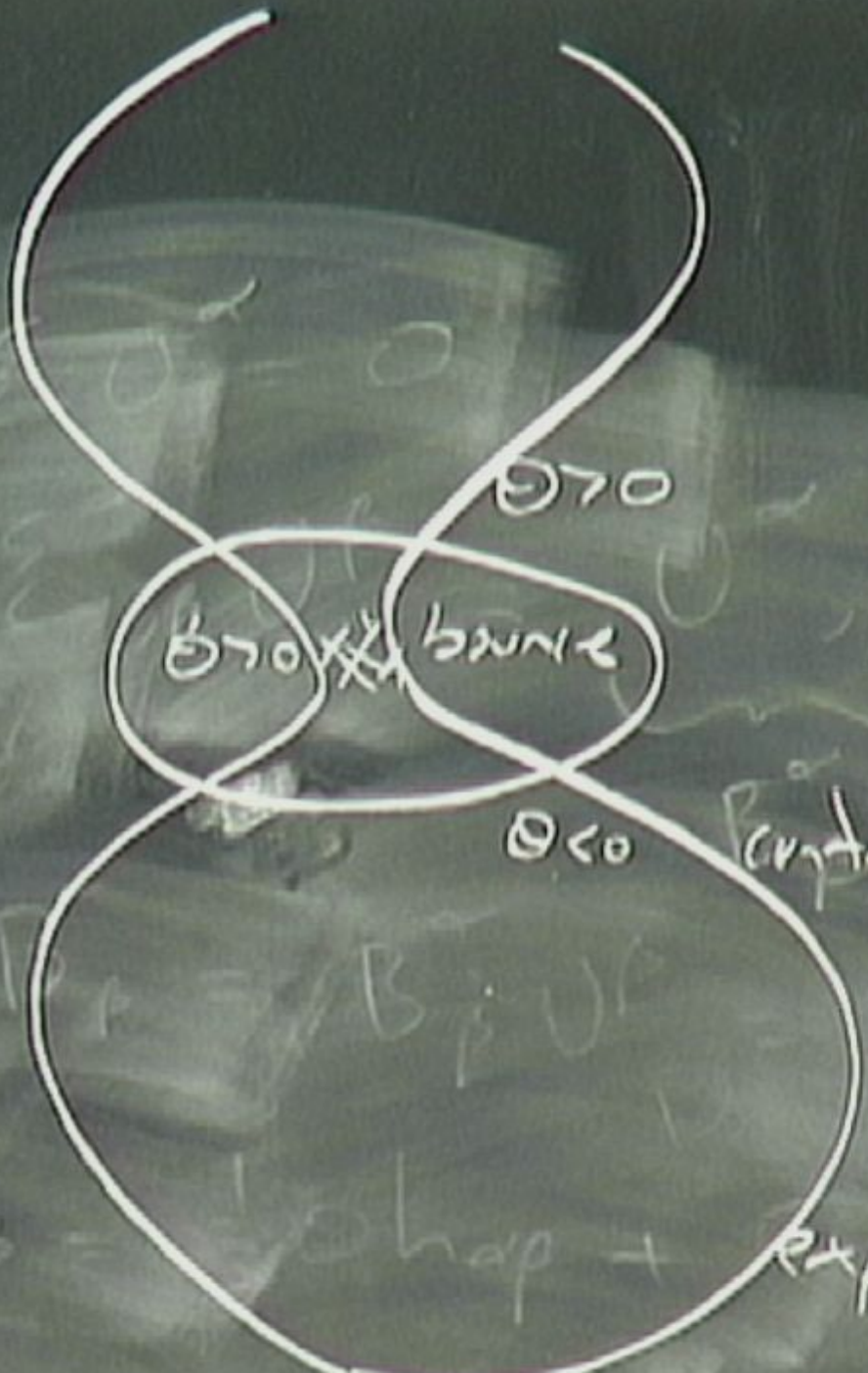
1- phoenix universe



# 1- phoenix universe



diverge



$\theta 70$

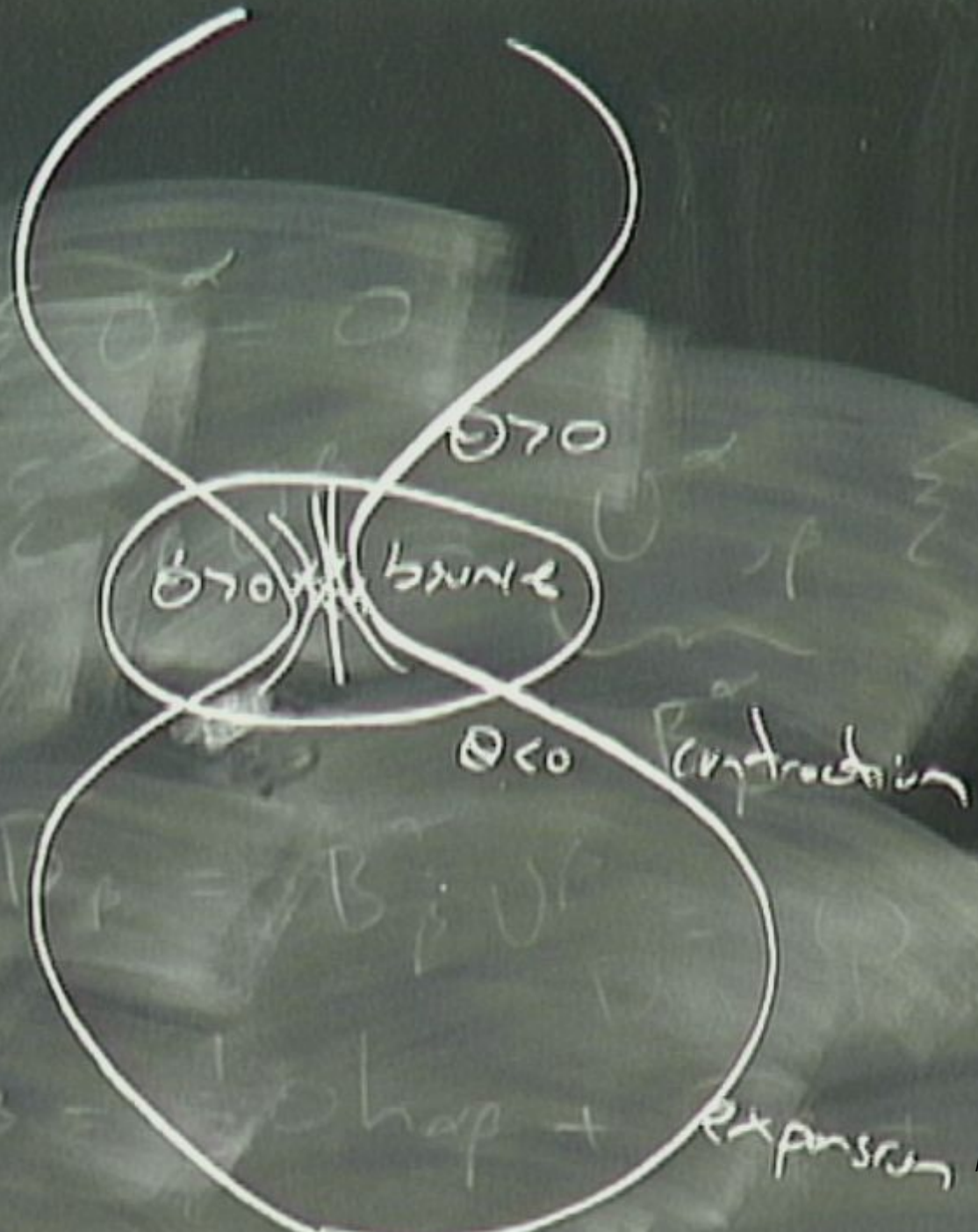
$\theta 20$  ~~XXXX~~ bunkle

$\theta 20$

contraction

expansion

diverge



1- phoenix univ



1- phoenix universe



1- phoenix univer

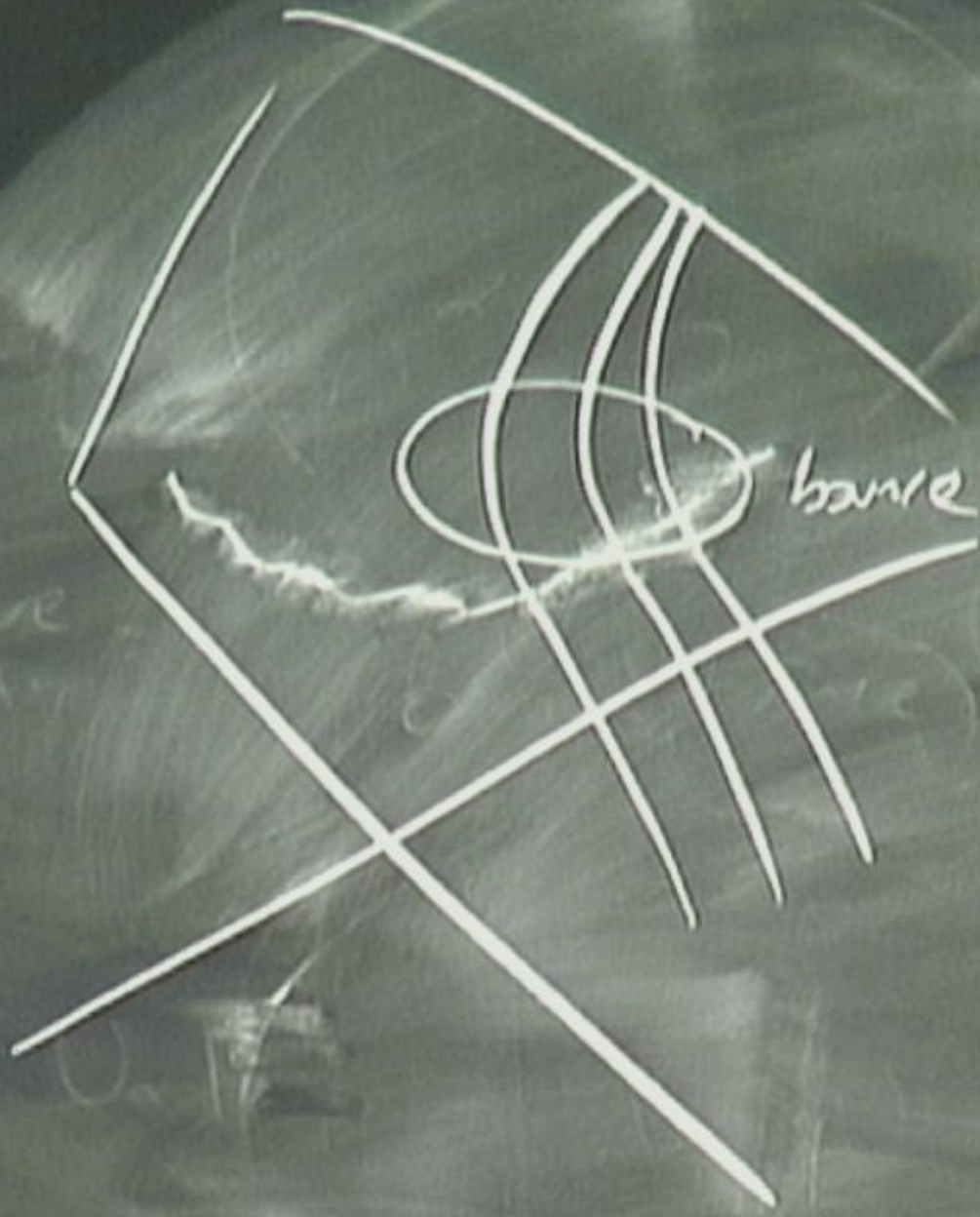




1- phoenix universe



ivrise

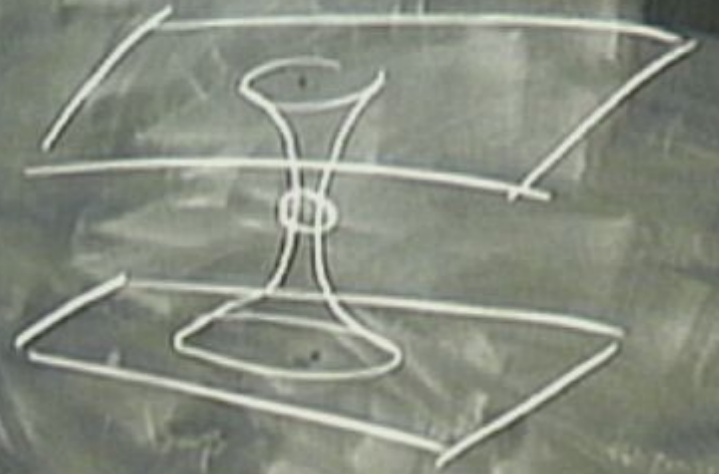


banne

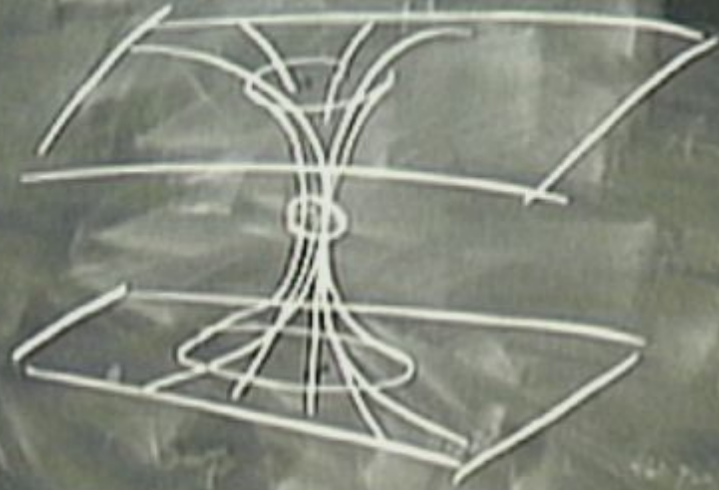
1- phoenix universe



1- phoenix units



1- phoenix universe



Hints: 1 - congruence is hyperbolic orthogonal  
 $\rightarrow W_{up} = 0$

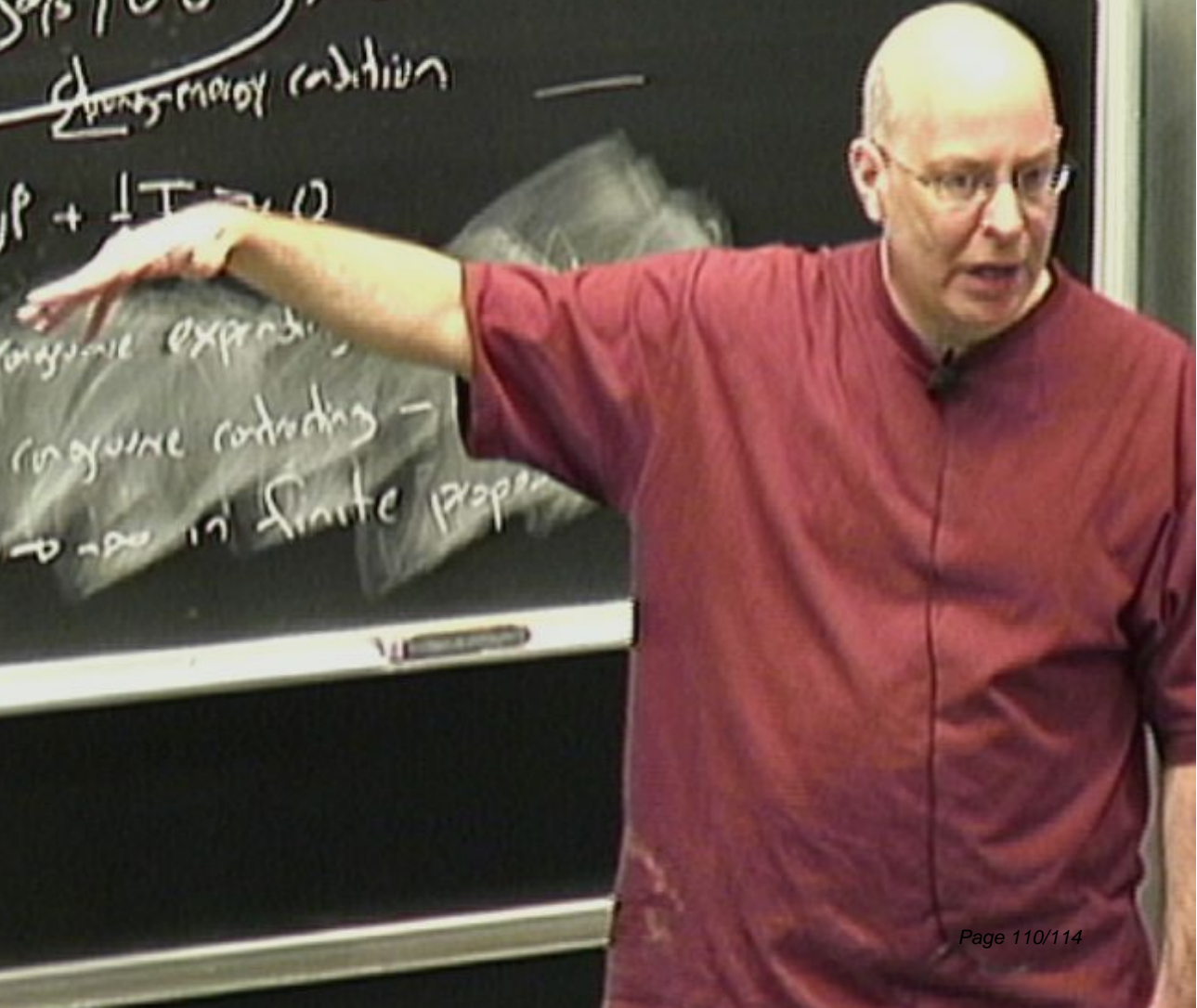
2.  $R_{up} \sigma_{up} \succ 0$

$(T_{up} - \frac{1}{2} T_{30\beta}) \sigma_{up} \succ 0$   
symmetry condition

$T_{up} \sigma_{up} + \frac{1}{2} T_{30\beta} = 0$

$\theta > 0$  : congruence expanding  
 $\theta < 0$  : congruence contracting -  
 $\hookrightarrow \theta \rightarrow -\infty$  in finite proper

$\frac{d\theta}{dT} \leq 0$



1 - conjugate is hypersurface orthogonal

$$\rightarrow W_{\text{eff}} = 0$$

2.  $R_{\alpha\beta} U^\alpha U^\beta \geq 0$

$(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) U^\alpha U^\beta \geq 0$   
strong energy condition

$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T \geq 0$



$\theta > 0$

$\theta < 0$  : conjugate condition

$\hookrightarrow \theta \rightarrow -\infty$  in limit

1 - congruence is hyperbolic & orthogonal

$$\rightarrow W_{\text{top}} = 0$$

2.  $R_{\alpha\beta\gamma\delta} U^\alpha U^\beta U^\gamma U^\delta > 0$

$(T_{\text{top}} - \frac{1}{2} T_{\text{top}}) U^\alpha U^\beta U^\gamma U^\delta > 0$   
Strong-energy condition

$T_{\alpha\beta} U^\alpha U^\beta + \frac{1}{2} T > 0$

$\theta > 0$  : congruence expanding - slowing down

$\theta < 0$  : congruence contracting - speeding up

$\theta \rightarrow -\infty$  in finite time





$$B_{\alpha\beta} B^{\mu\nu} = \left( \frac{1}{3} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \right) \left( \frac{1}{3} \theta h^{\mu\nu} + \sigma^{\mu\nu} + \omega^{\mu\nu} \right)$$

$$= \frac{1}{9} \theta^2 (3) + \sigma_{\alpha\beta} \sigma^{\alpha\beta} - \omega_{\alpha\beta} \omega^{\alpha\beta}$$

$$\frac{d\theta}{dt} = -\frac{1}{3} \theta^2 - \sigma_{\alpha\beta} \sigma^{\alpha\beta} + \omega_{\alpha\beta} \omega^{\alpha\beta} - R_{\alpha\beta} U^{\alpha} U^{\beta}$$

Raychaudhuri's equation.

Take the trace:  $\frac{\partial \Theta}{\partial T} = -B_{\alpha\mu} B^{\mu\alpha} - R_{\mu\nu} U^\nu U^\mu$

$$\Theta = \frac{1}{\delta V} \frac{d}{dT} (\delta V)$$

$$B_{\alpha\mu} B^{\mu\alpha} = \left( \frac{1}{3} \Theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu} \right) \left( \frac{1}{3} \Theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha} \right)$$

$$= \frac{1}{9} \Theta^2 (3) + \sigma_{\alpha\mu} \sigma^{\mu\alpha} - \omega_{\alpha\mu} \omega^{\mu\alpha}$$

$$\frac{\partial \Theta}{\partial T} = -\frac{1}{3} \Theta^2 - \underbrace{(\sigma_{\alpha\mu} \sigma^{\mu\alpha})}_{\text{circled}} + \cancel{\omega_{\alpha\mu} \omega^{\mu\alpha}} - R_{\alpha\mu} U^\mu U^\alpha$$

Raychaudhuri's equation.