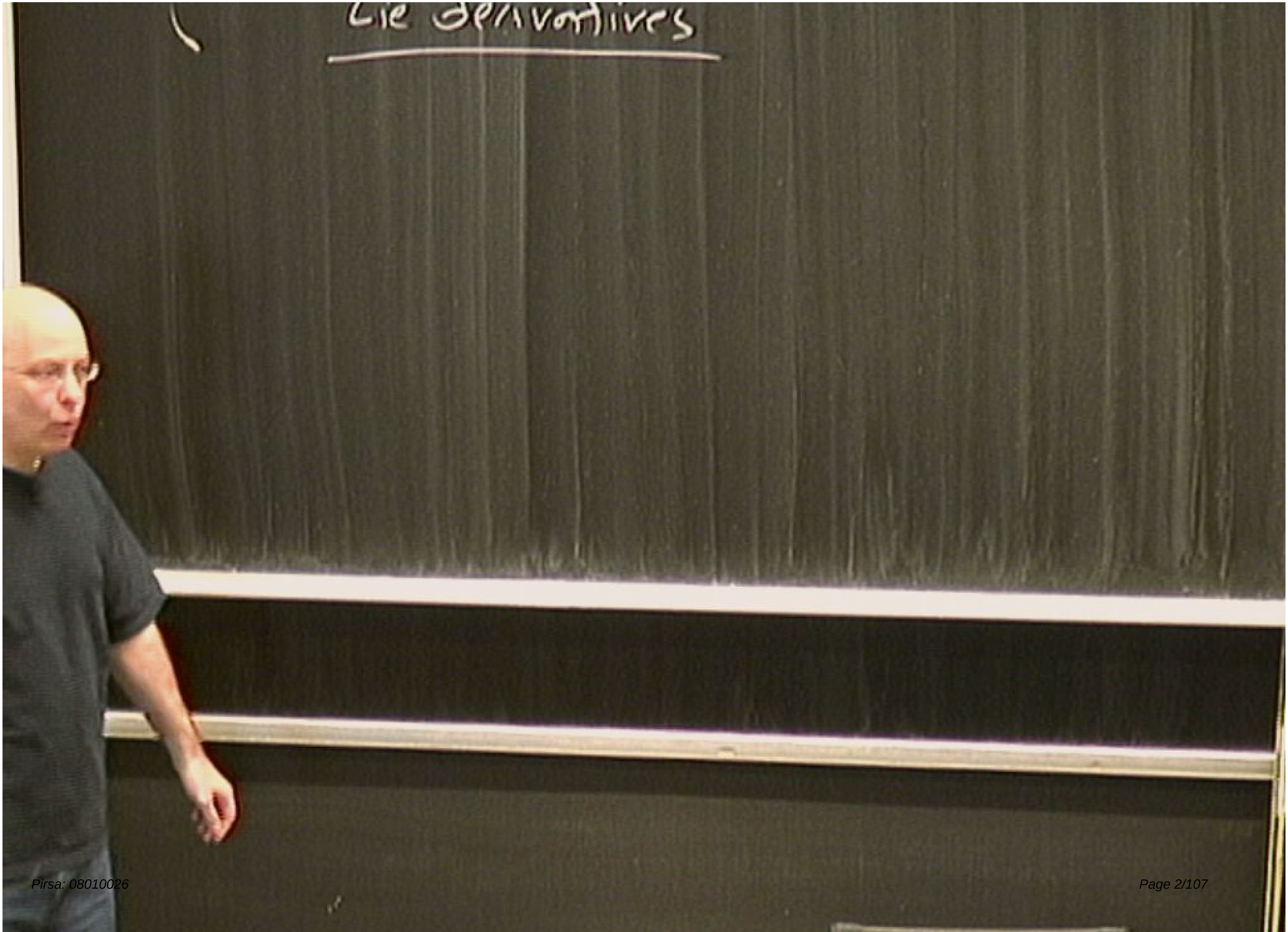
Title: Advanced General Relativity - Lecture 2A

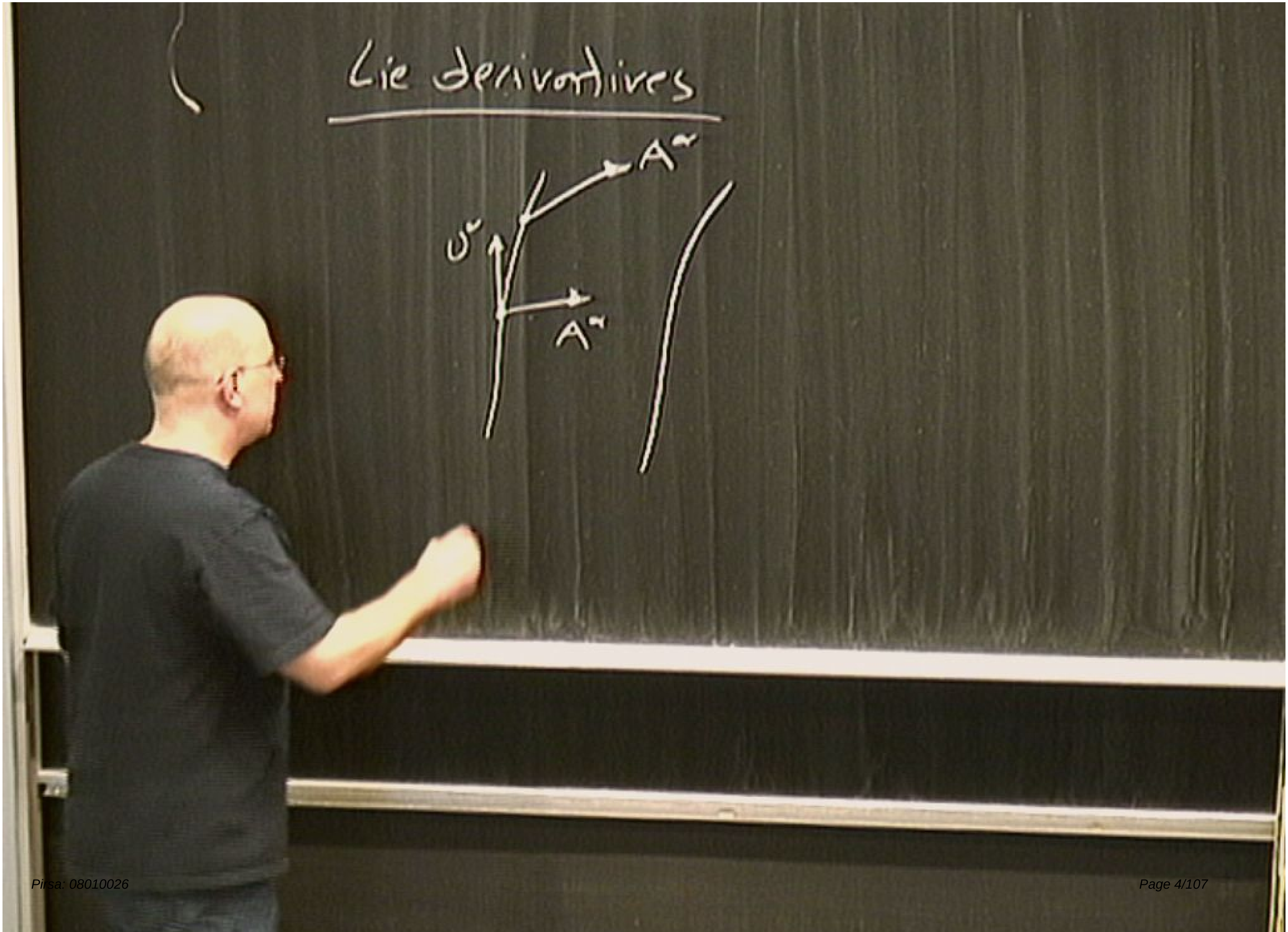
Date: Jan 16, 2008 10:30 AM

URL: http://pirsa.org/08010026

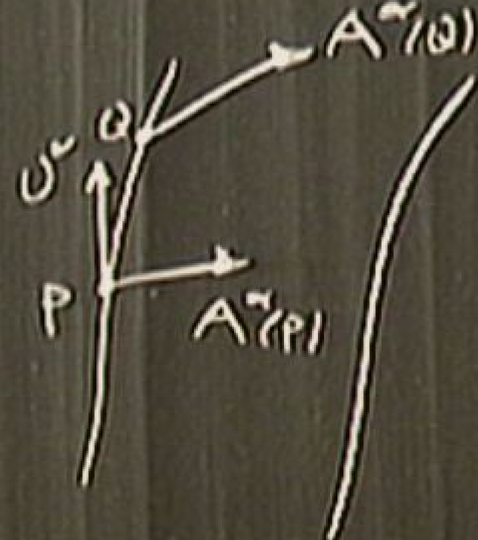
Abstract: Advanced General Relativity

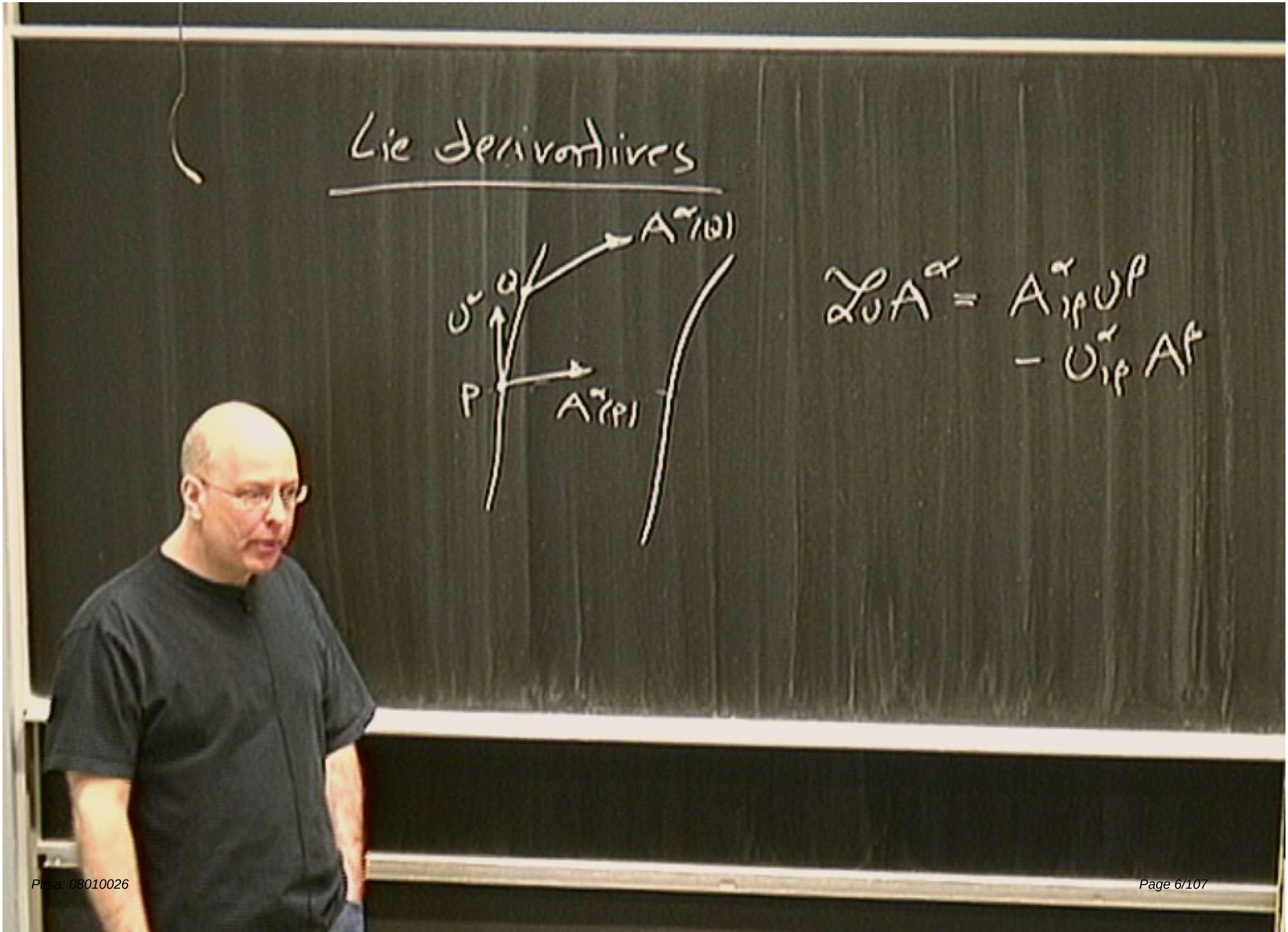

Lie derivatives

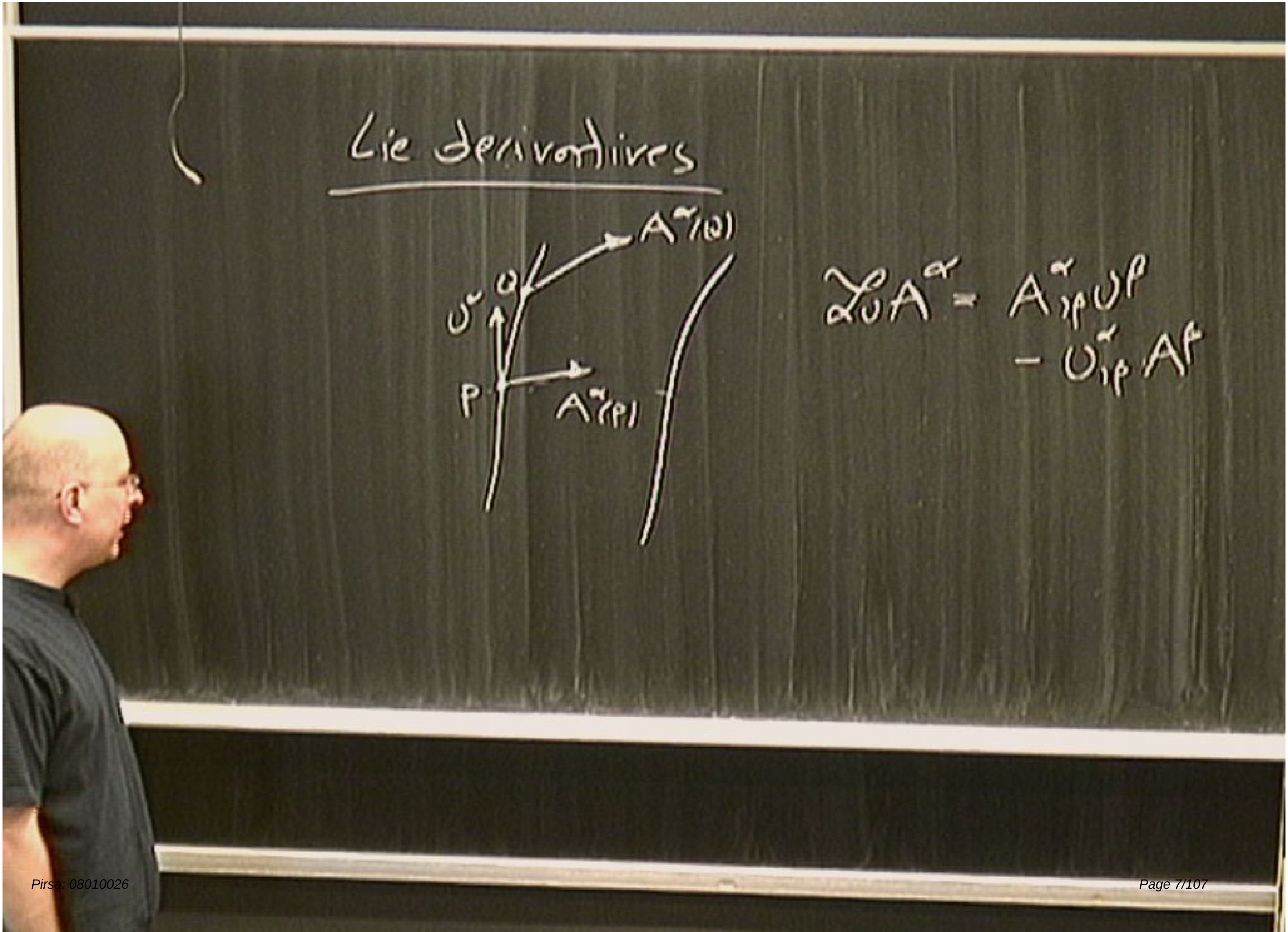
Lie derivatives


Lie derivatives
A^μ
U^μ
A^μ

Lie derivatives




Lie derivatives
$A^\alpha(Q)$
v^α
Q
P
$A^\alpha(P)$
$\mathcal{L}_v A^\alpha = A^\alpha{}_{;\beta} v^\beta - v^\alpha{}_{;\beta} A^\beta$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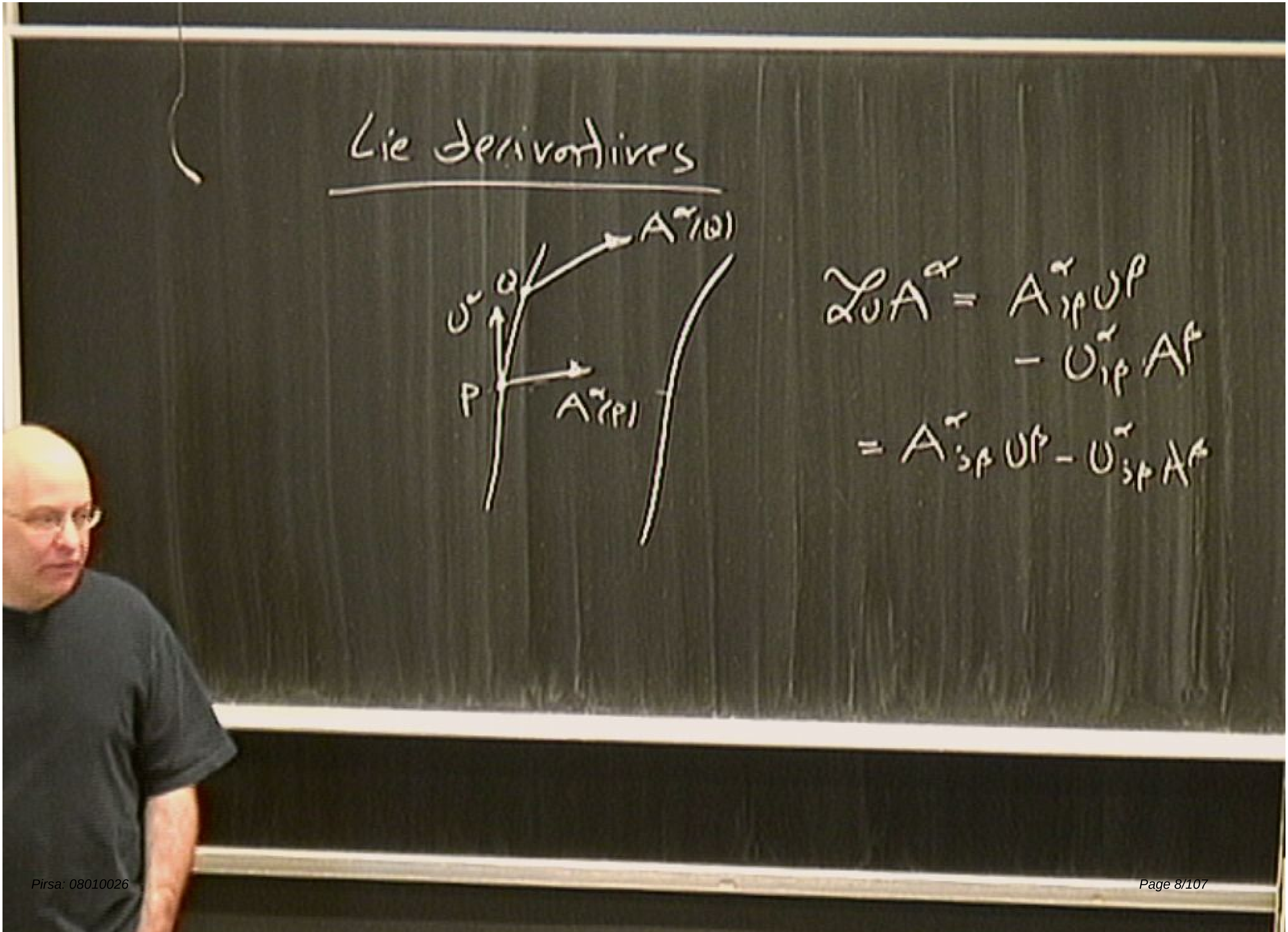
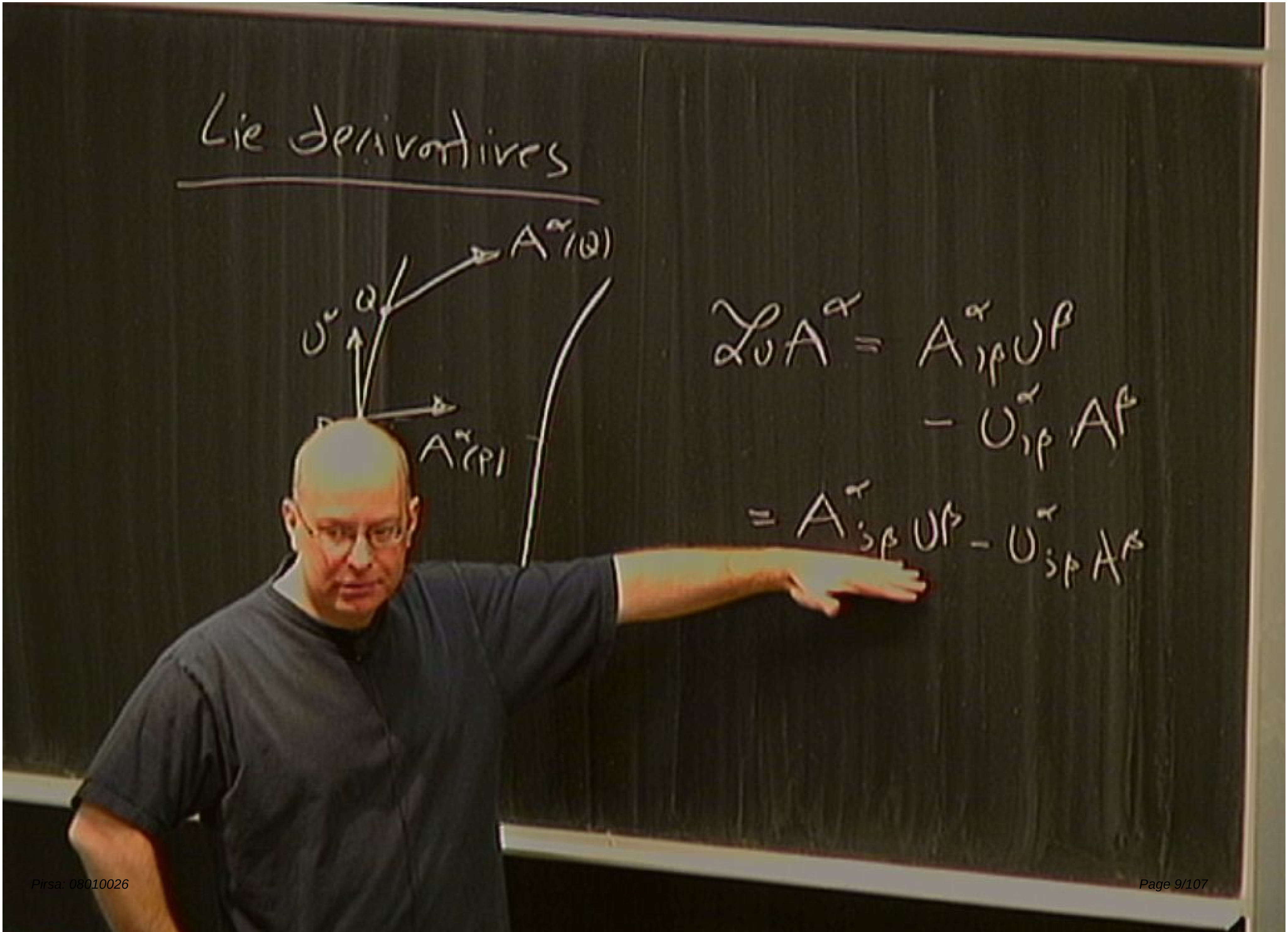
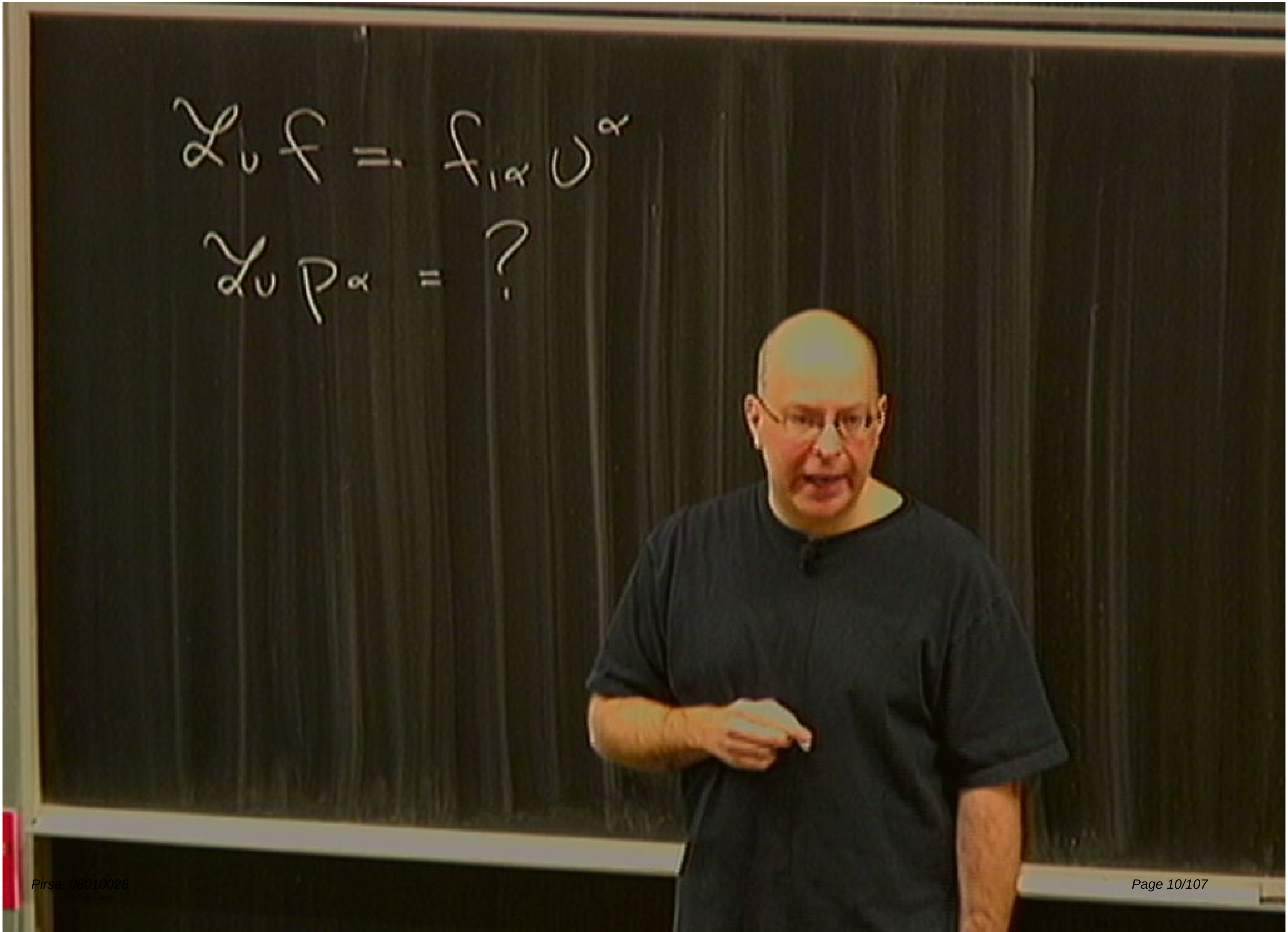

Lie derivatives
$A^\alpha(Q)$
u^α
Q
P
$A^\alpha(P)$
$\mathcal{L}_u A^\alpha = A^\alpha{}_{,\beta} u^\beta - u^\alpha{}_{,\beta} A^\beta$
$= A^\alpha{}_{;\beta} u^\beta - u^\alpha{}_{;\beta} A^\beta$


Lie derivatives
$A^\alpha(Q)$
U^α
Q
$A^\alpha(P)$
$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta$
$- U^\alpha{}_{,\beta} A^\beta$
$= A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta$


$\mathcal{L}_U f = f_{|\alpha} U^{\alpha}$
$\mathcal{L}_U p_\alpha = ?$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$$\mathcal{L}_U p_\alpha = ?$$ $\mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$$\mathcal{L}_U p_\alpha = \; ?$$ $\mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$

$$= p_{\alpha,\beta} A^\alpha U^\beta + \quad A$$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$$\mathcal{L}_U p_\alpha = \,?$$

$\mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$

$$= p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha_{\ ,\beta} U^\beta$$

$$\mathcal{L}_U (p_\alpha A^\alpha) = (\mathcal{L}_U$$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$\mathcal{L}_U p_\alpha = ?$ $\quad \mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$

$$= p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha{}_{,\beta} U^\beta$$

$$\mathcal{L}_U (p_\alpha A^\alpha) = (\mathcal{L}_U p_\alpha) A^\alpha + p_\alpha (\mathcal{L}_U A^\alpha)$$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$\mathcal{L}_U p_\alpha = ?$ $\mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$

$$= p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha_{,\beta} U^\beta$$

$$\mathcal{L}_U (p_\alpha A^\alpha) = (\mathcal{L}_U p_\alpha) A^\alpha + p_\alpha (\mathcal{L}_U A^\alpha)$$

$$(\mathcal{L}_U p_\alpha) A^\alpha =$$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

$\mathcal{L}_U p_\alpha = ?$ $\quad \mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$
$$= p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha_{,\beta} U^\beta$$

$$\mathcal{L}_U (p_\alpha A^\alpha) = (\mathcal{L}_U p_\alpha) A^\alpha + p_\alpha (\mathcal{L}_U A^\alpha)$$

$$(\mathcal{L}_U p_\alpha) A^\alpha = p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha_{,\beta} U^\beta$$
$$- p_\alpha \left(A^\alpha_{,\beta} U^\beta - U^\alpha_{,\beta} A^\beta \right)$$

$$\mathcal{L}_U f = f_{,\alpha} U^\alpha$$

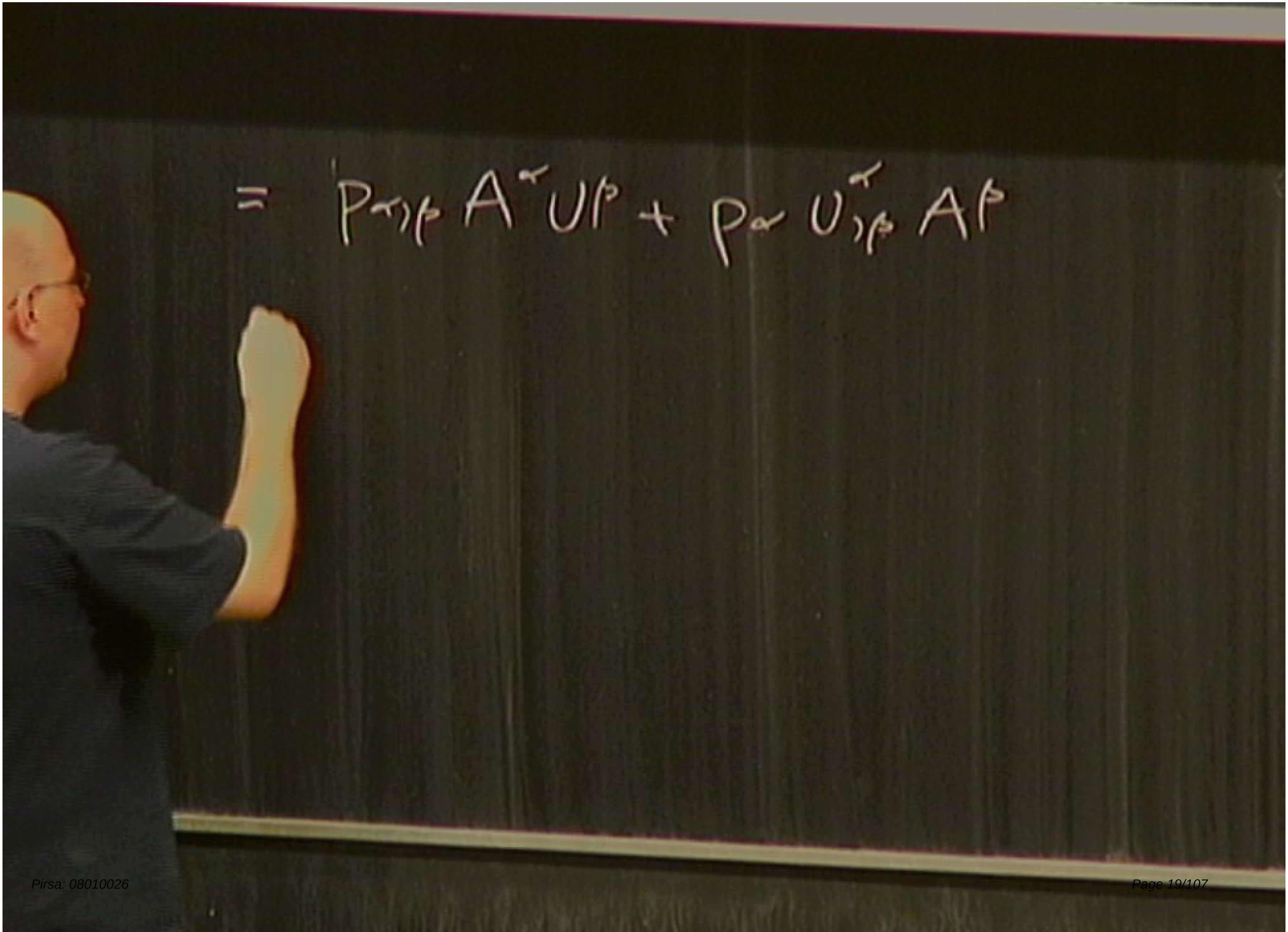
$$\mathcal{L}_U p_\alpha = ?$$ $\mathcal{L}_U$ satisfies product rule

$$\mathcal{L}_U (p_\alpha A^\alpha) = (p_\alpha A^\alpha)_{,\beta} U^\beta$$
$$= p_{\alpha,\beta} A^\alpha U^\beta + p_\alpha A^\alpha{}_{,\beta} U^\beta$$

$$\mathcal{L}_U (p_\alpha A^\alpha) = (\mathcal{L}_U p_\alpha) A^\alpha + p_\alpha (\mathcal{L}_U A^\alpha)$$

$$(\mathcal{L}_U p_\alpha) A^\alpha = p_{\alpha,\beta} A^\alpha U^\beta + \cancel{p_\alpha A^\alpha{}_{,\beta} U^\beta}$$
$$- p_\alpha (\cancel{A^\alpha{}_{,\beta} U^\beta} - U^\alpha{}_{,\beta} A^\beta)$$

$$= \rho_{\alpha;\beta} A^{\alpha} U^{\beta} + \rho_{\alpha} U^{\alpha}{}_{;\beta} A^{\beta}$$


$= \rho_{\alpha;\beta} A^{\alpha} U^{\beta} + \rho_{\alpha} U^{\alpha}{}_{;\beta} A^{\beta}$

$$= p_{\alpha;\beta} A^{\alpha} U^{\beta} + p_{\alpha} U^{\alpha}_{;\beta} A^{\beta}$$

$$= (p_{\alpha;\beta} U^{\beta}$$

$$= \rho_{\alpha;\beta} A^{\alpha} U^{\beta} + \rho_{\alpha} U^{\alpha}{}_{;\beta} A^{\beta}$$

$$= \left(\rho_{\alpha;\beta} U^{\beta} + \rho_{\beta} U^{\beta}{}_{;\alpha} \right) A^{\alpha}$$

$$= p_{\alpha;\beta} A^\alpha U^\beta + p_\alpha U^\alpha_{;\beta} A^\beta$$

$$= \left(p_{\alpha;\beta} U^\beta + p_\beta U^\beta{}_{;\alpha} \right) A^\alpha$$

$$\mathcal{L}_U p_\alpha = p_{\alpha;\beta} U^\beta + p_\beta U^\beta{}_{;\alpha}$$

$$= p_{\alpha;\beta} A^{\alpha} U^{\beta} + p_{\alpha} U^{\alpha}_{;\beta} A^{\beta}$$

$$= \left(p_{\alpha;\beta} U^{\beta} + p_{\beta} U^{\beta}_{;\alpha} \right) A^{\alpha}$$

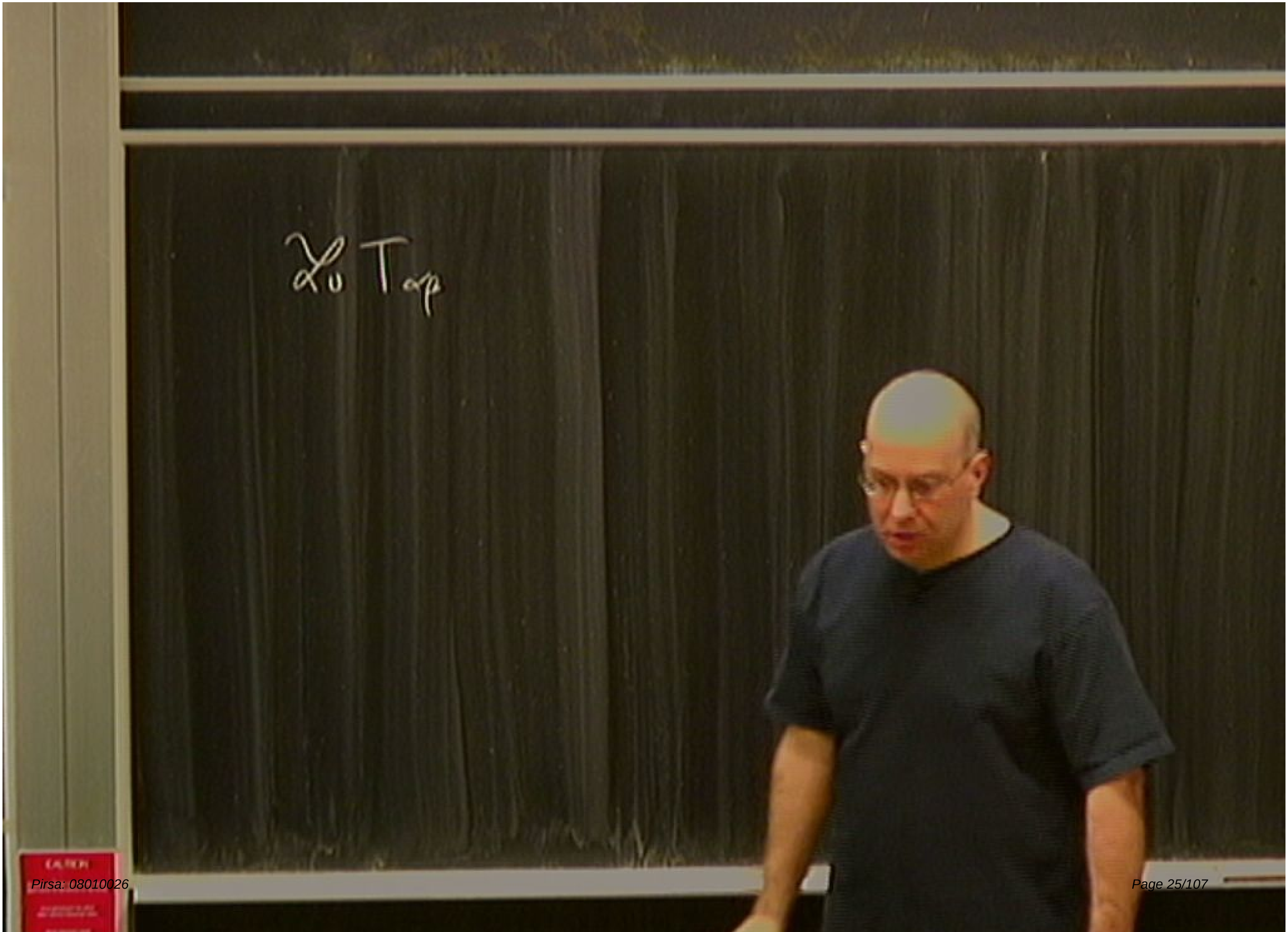
$$\mathcal{L}_U p_{\alpha} = p_{\alpha,\beta} U^{\beta} + p_{\beta} U^{\beta}_{,\alpha}$$

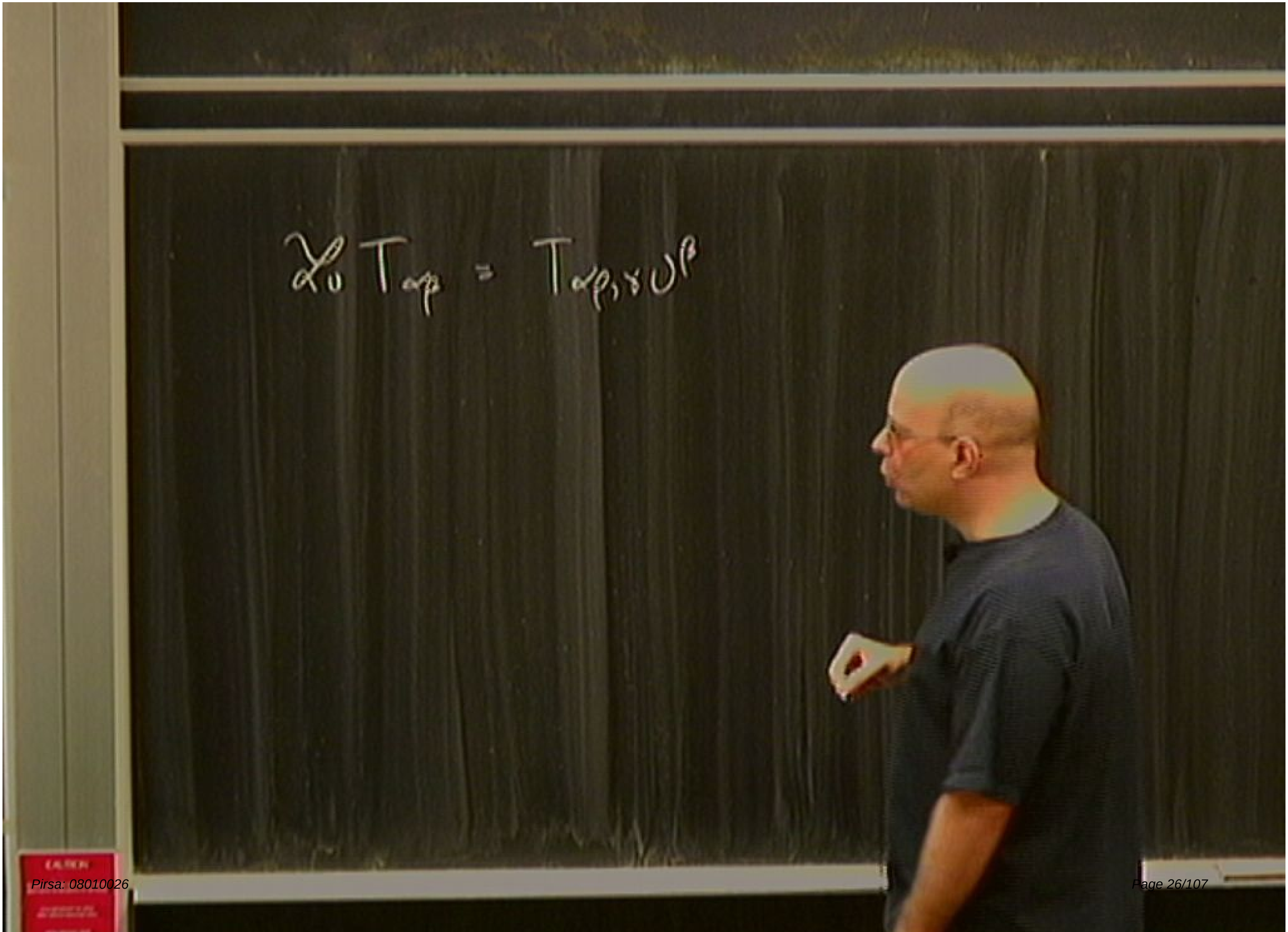
$$= p_{\alpha;\beta} U^{\beta} + p_{\beta} U^{\beta}_{;\alpha}$$

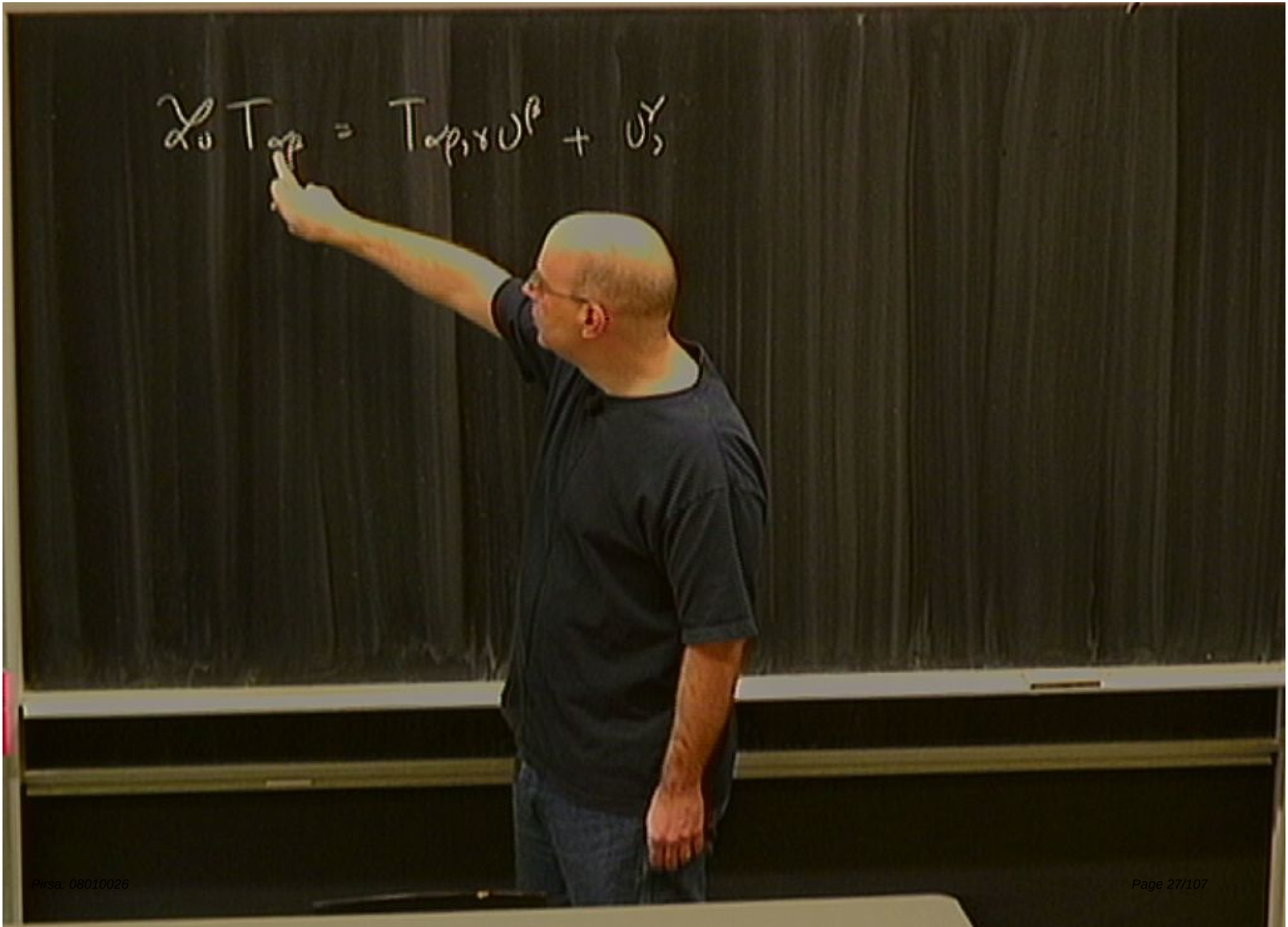
$$= p_{\alpha,\beta} A^{\alpha} U^{\beta} + p_{\alpha} U^{\alpha}{}_{,\beta} A^{\beta}$$

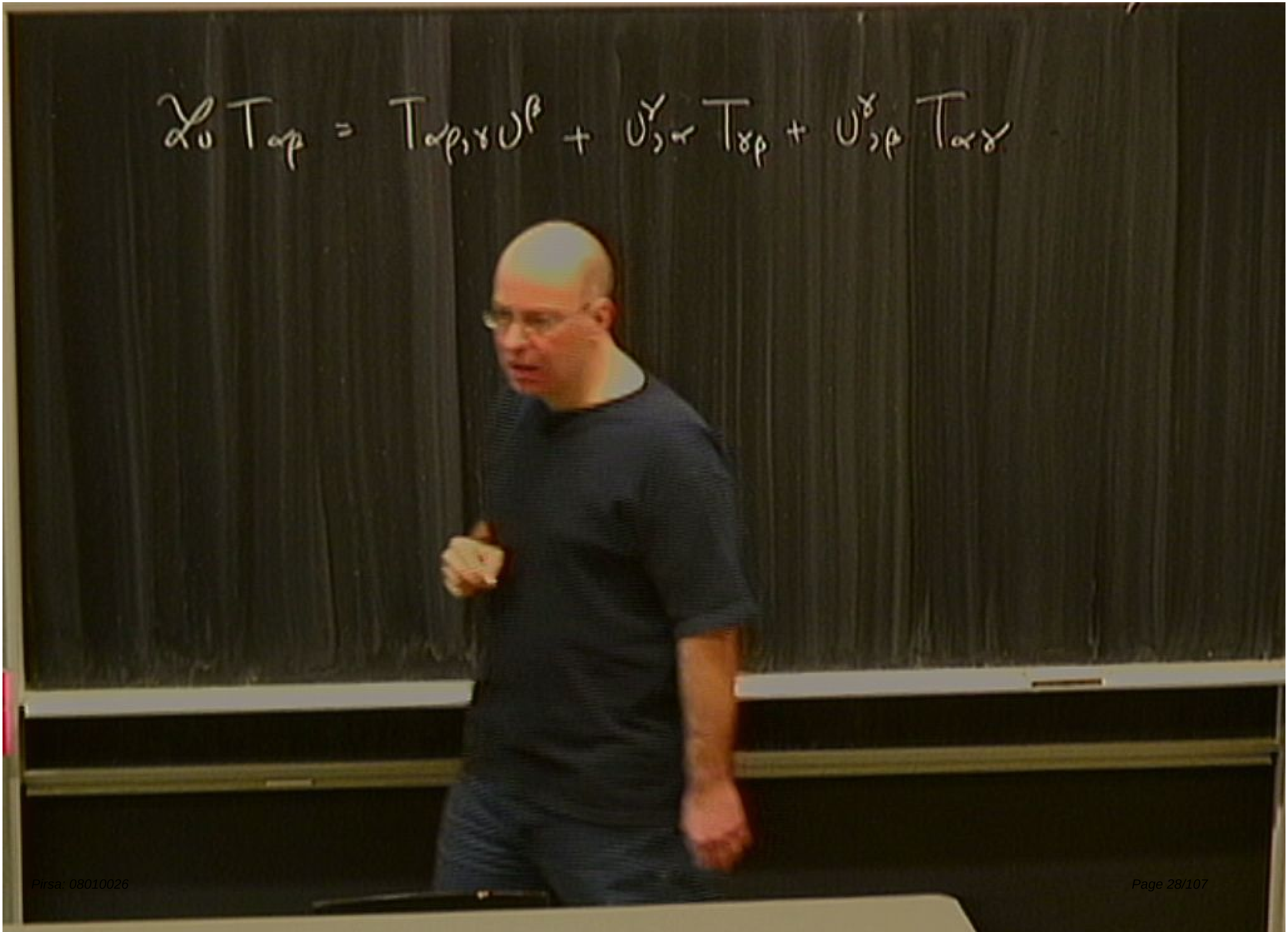
$$= \left(p_{\alpha,\beta} U^{\beta} + p_{\beta} U^{\beta}{}_{,\alpha} \right) A^{\alpha}$$

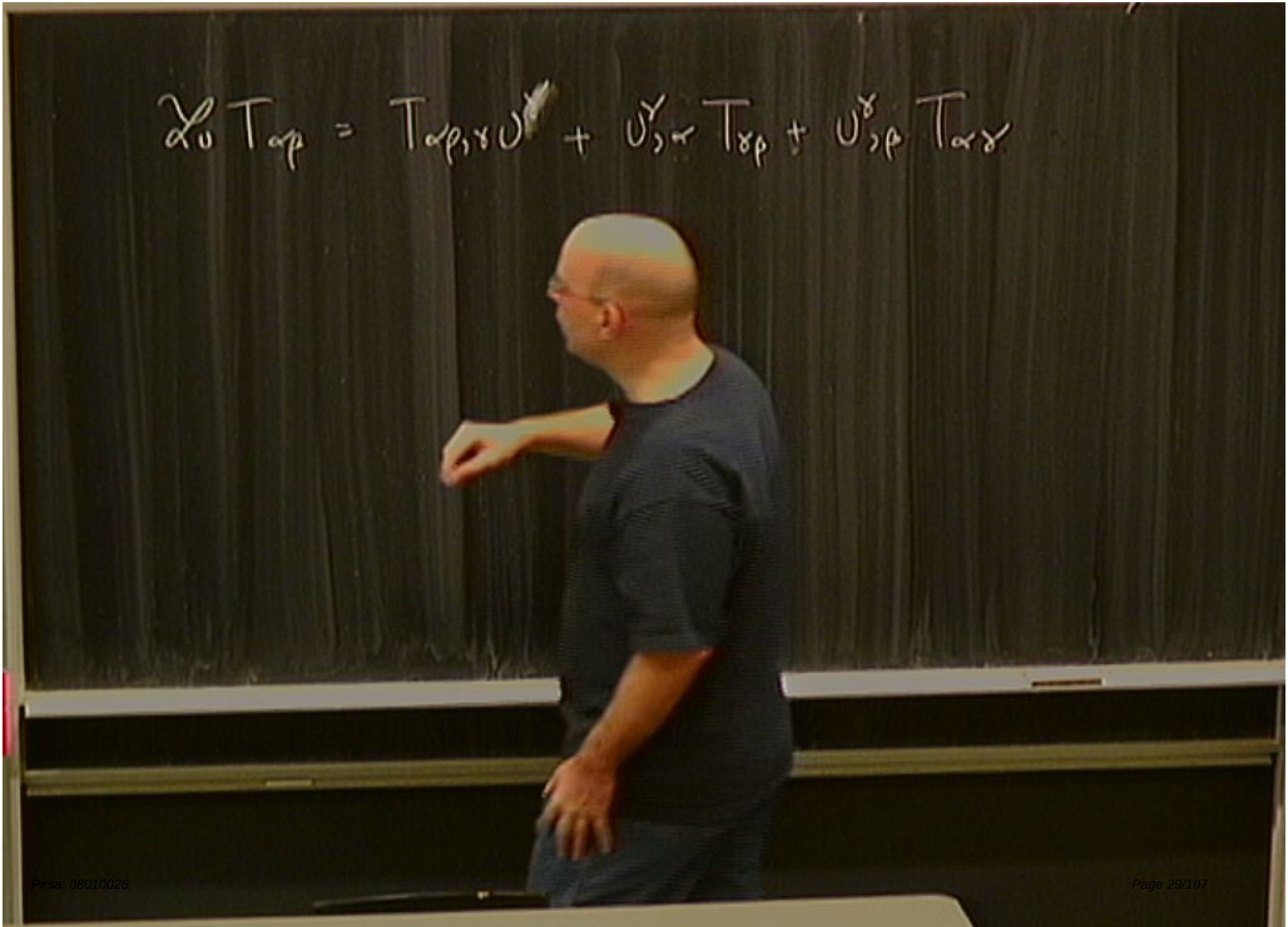
$$\boxed{\begin{aligned} \mathcal{L}_{U} p_{\alpha} &= p_{\alpha,\beta} U^{\beta} + p_{\beta} U^{\beta}{}_{,\alpha} \\ &= p_{\alpha;\beta} U^{\beta} + p_{\beta} U^{\beta}{}_{;\alpha} \end{aligned}}$$

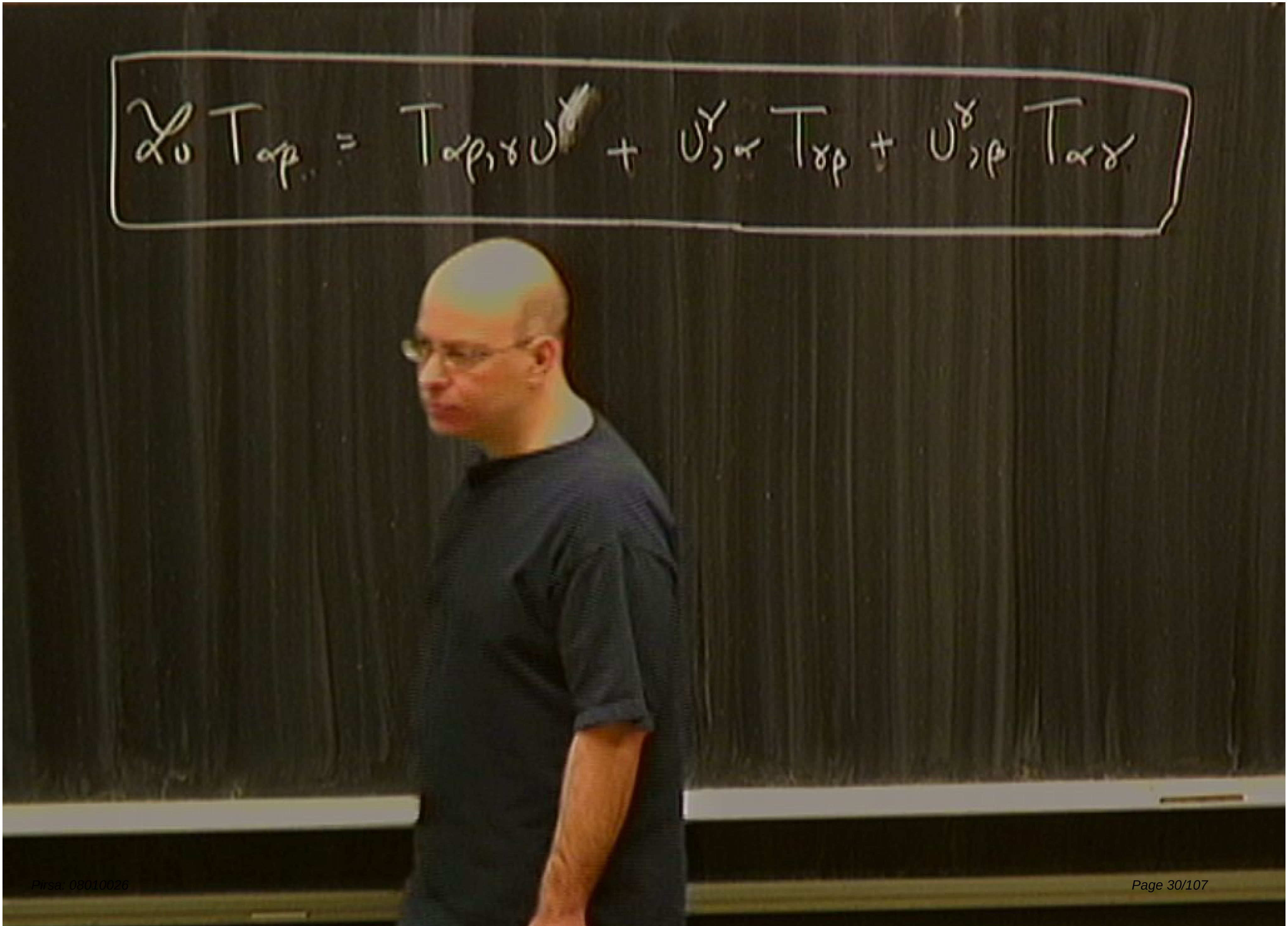

$\mathcal{L}_u T_{ab}$

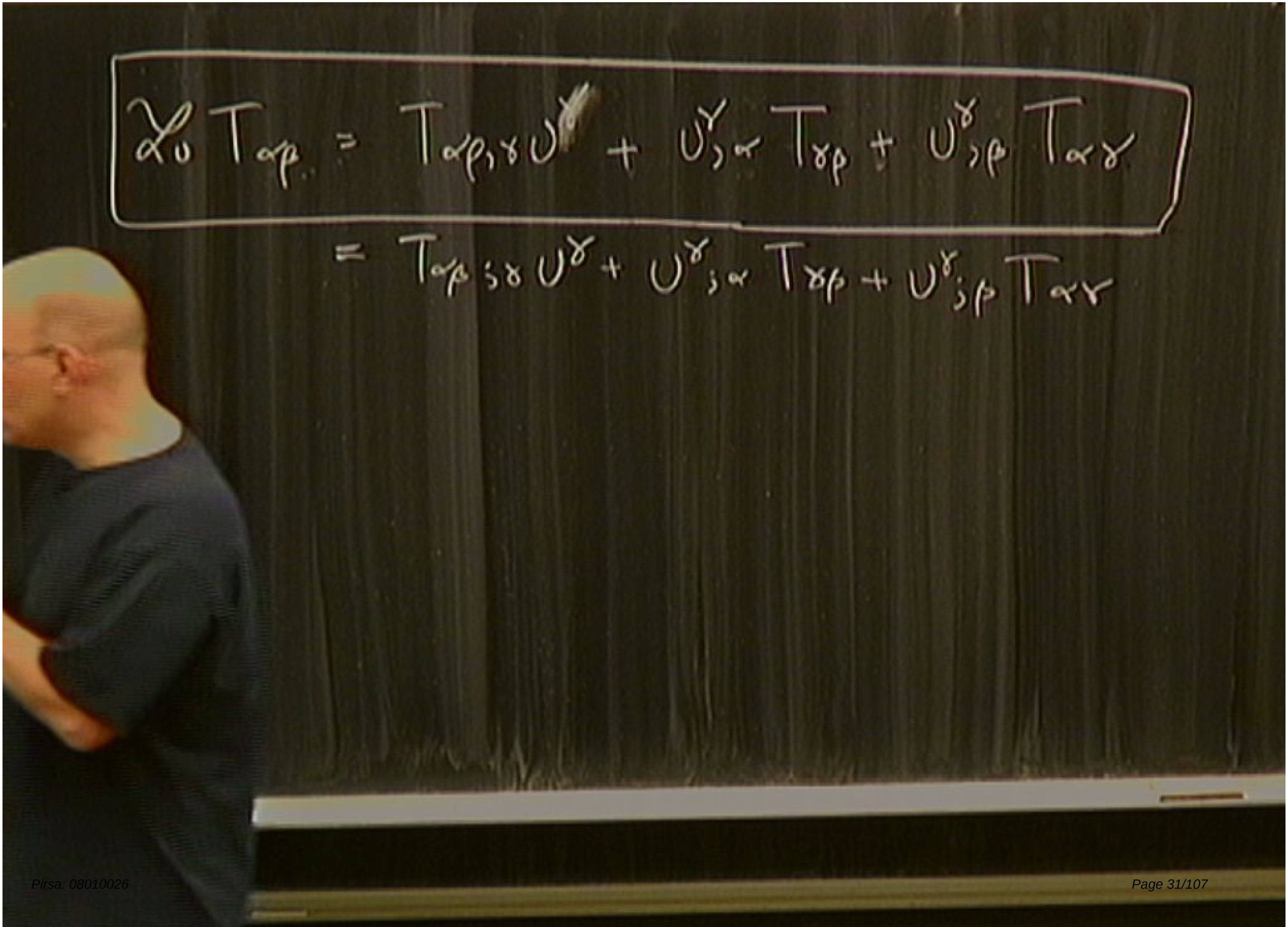

$\mathcal{L}_U T_{\alpha\beta} = T_{\alpha\beta,\gamma} U^{\beta}$


$\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta,\gamma} v^\beta + v^\gamma{}_{,}$


$\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta;\gamma} v^\beta + v^\gamma{}_{;\alpha} T_{\gamma\beta} + v^\gamma{}_{;\beta} T_{\alpha\gamma}$


$\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta;\gamma} v^{\gamma} + v^{\gamma}{}_{;\alpha} T_{\gamma\beta} + v^{\gamma}{}_{;\beta} T_{\alpha\gamma}$

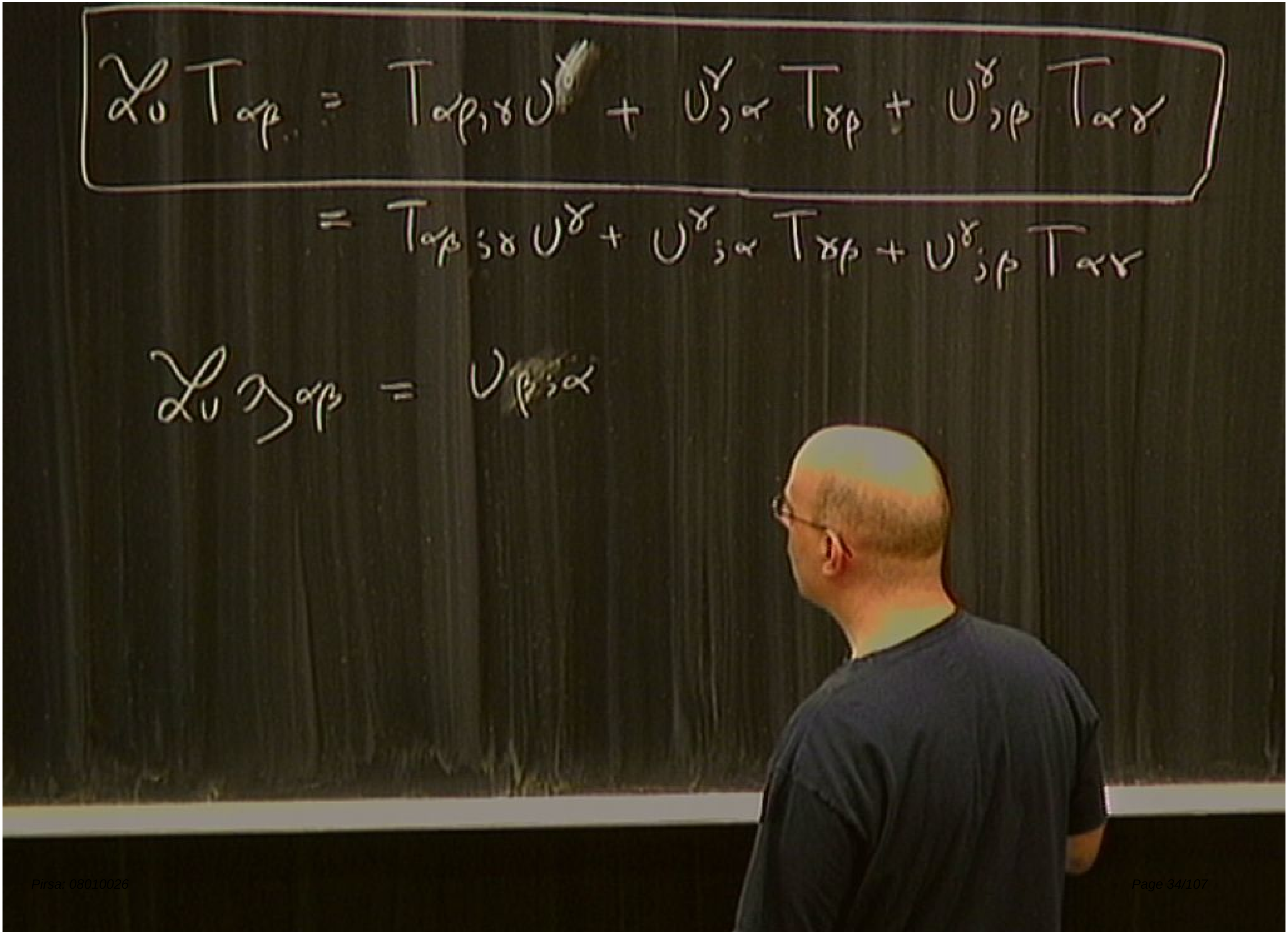

$\boxed{\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta,\gamma} v^{\gamma} + v^{\gamma}{}_{,\alpha} T_{\gamma\beta} + v^{\gamma}{}_{,\beta} T_{\alpha\gamma}}$

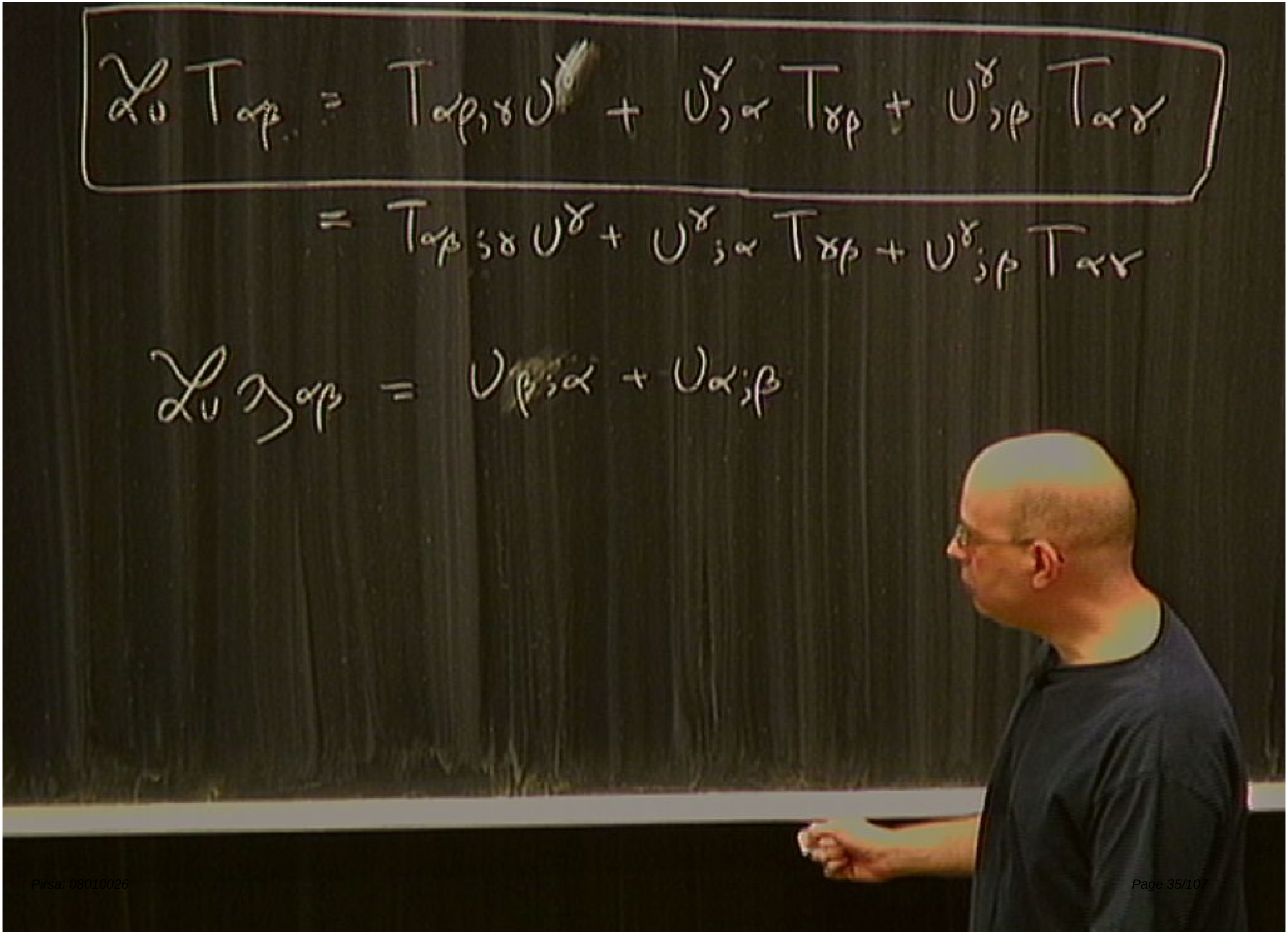

$\boxed{\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta,\gamma} v^{\gamma} + v^{\gamma}{}_{,\alpha} T_{\gamma\beta} + v^{\gamma}{}_{,\beta} T_{\alpha\gamma}}$
$= T_{\alpha\beta;\gamma} v^{\gamma} + v^{\gamma}{}_{;\alpha} T_{\gamma\beta} + v^{\gamma}{}_{;\beta} T_{\alpha\gamma}$

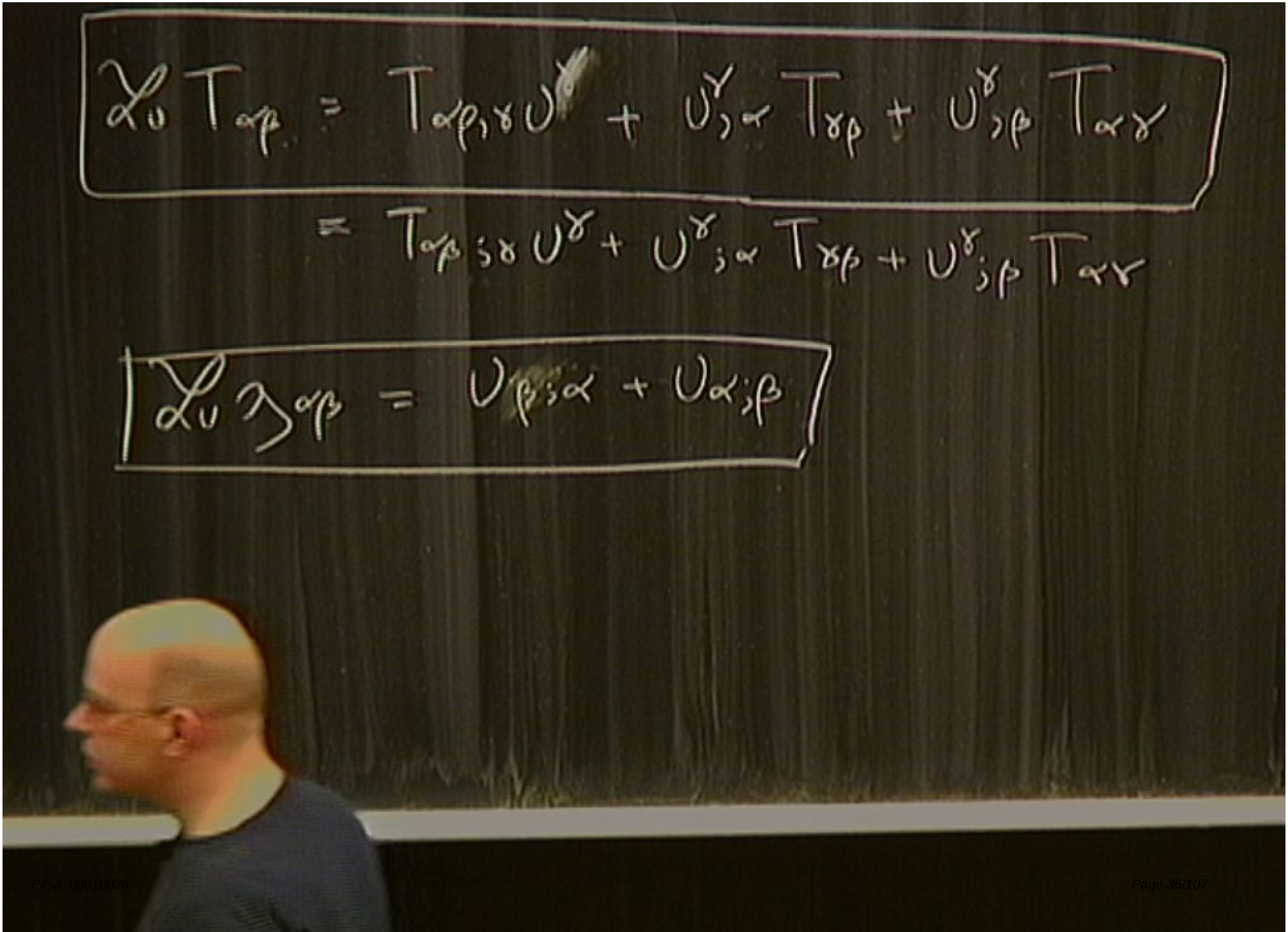
Time-invariant metric (Stationary spacetime)

Time-invariant metric (Stationary spacetime)

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$$


$\boxed{\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta,\gamma} v^\gamma + v^\gamma{}_{,\alpha} T_{\gamma\beta} + v^\gamma{}_{,\beta} T_{\alpha\gamma}}$
$= T_{\alpha\beta;\gamma} v^\gamma + v^\gamma{}_{;\alpha} T_{\gamma\beta} + v^\gamma{}_{;\beta} T_{\alpha\gamma}$
$\mathcal{L}_v g_{\alpha\beta} = v_{\beta;\alpha}$


$\boxed{\mathcal{L}_v T_{\alpha\beta} = T_{\alpha\beta,\gamma} v^\gamma + v^\gamma{}_{,\alpha} T_{\gamma\beta} + v^\gamma{}_{,\beta} T_{\alpha\gamma}}$
$= T_{\alpha\beta;\gamma} v^\gamma + v^\gamma{}_{;\alpha} T_{\gamma\beta} + v^\gamma{}_{;\beta} T_{\alpha\gamma}$
$\mathcal{L}_v g_{\alpha\beta} = v_{\beta;\alpha} + v_{\alpha;\beta}$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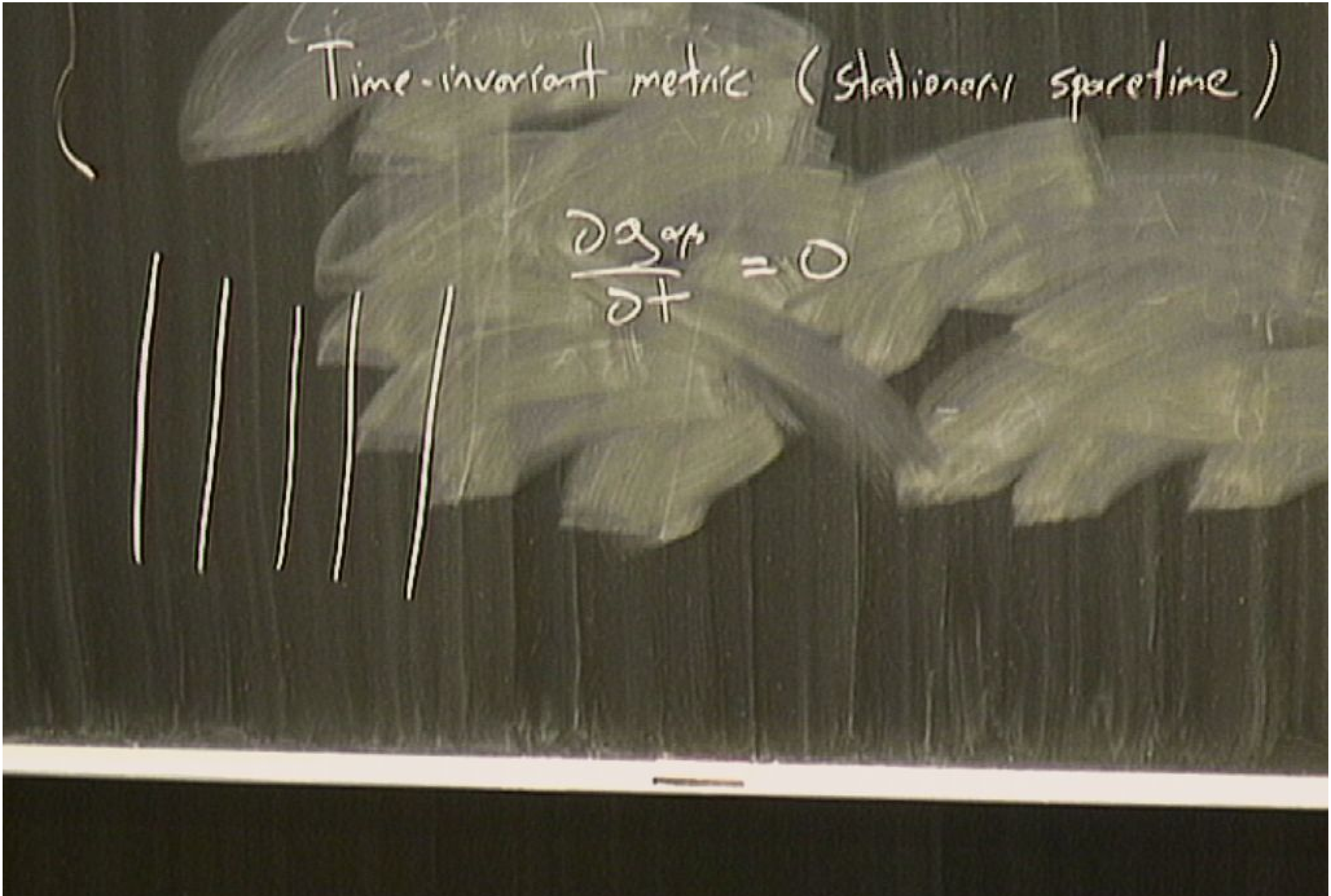
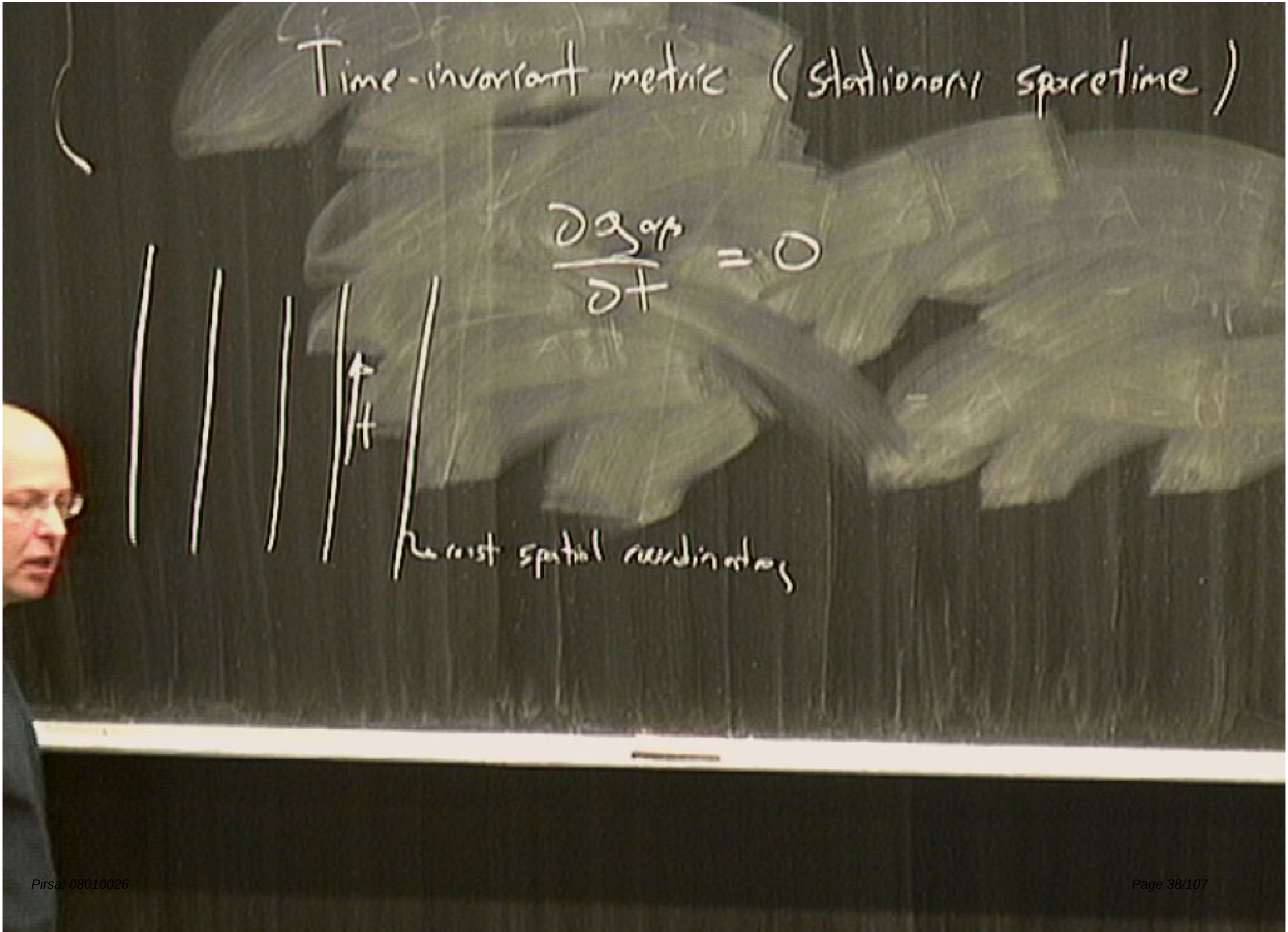
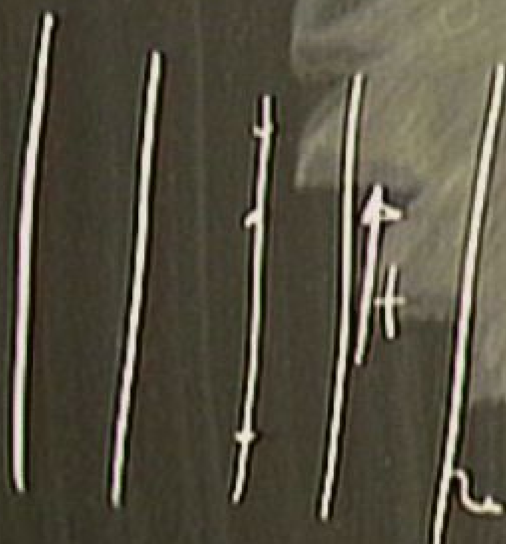
$\boxed{\mathcal{L}_v g_{\alpha\beta} = v_{\beta;\alpha} + v_{\alpha;\beta}}$


Time-invariant metric (stationary spacetime)
$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$


Time-invariant metric (Stationary spacetime)
$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$
t
const spatial coordinates

Time-invariant metric (Stationary spacetime)

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$$



const spatial coordinates

Time-invariant metric (Stationary spacetime)

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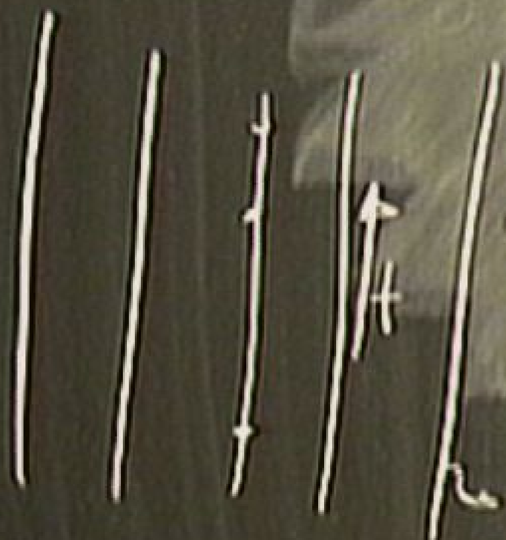

symmetry transformation curves.

const spatial coordinates

Time-invariant metric (Stationary spacetime)

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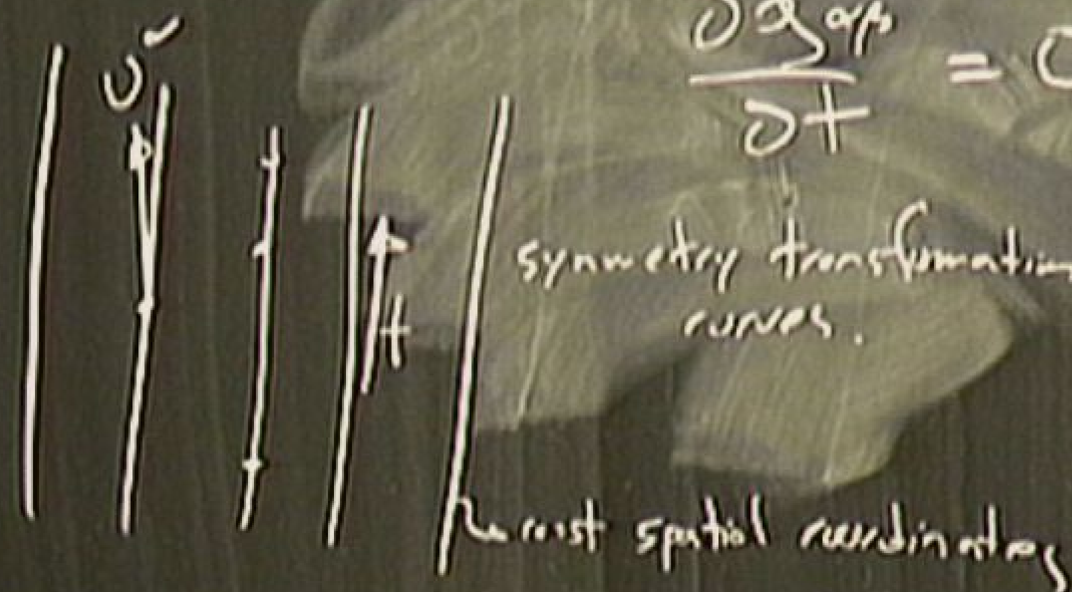
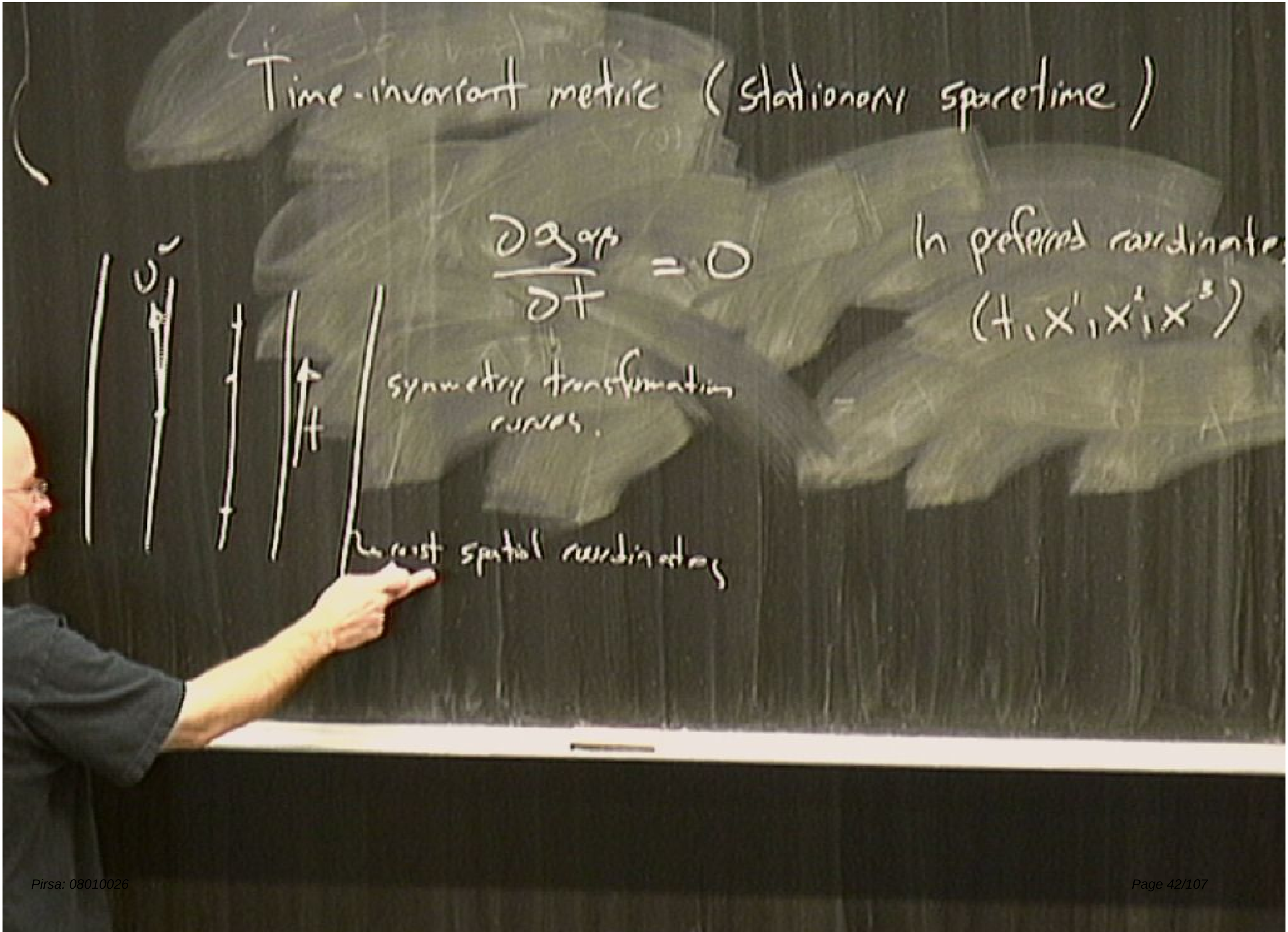
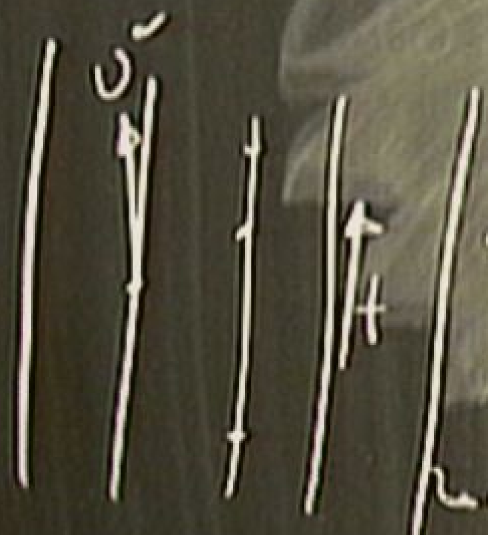



Time-invariant metric (Stationary spacetime)
$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$
In preferred coordinates
(t, x^1, x^2, x^3)
symmetry transformation curves.
t
const spatial coordinates

Time-invariant metric (Stationary spacetime)



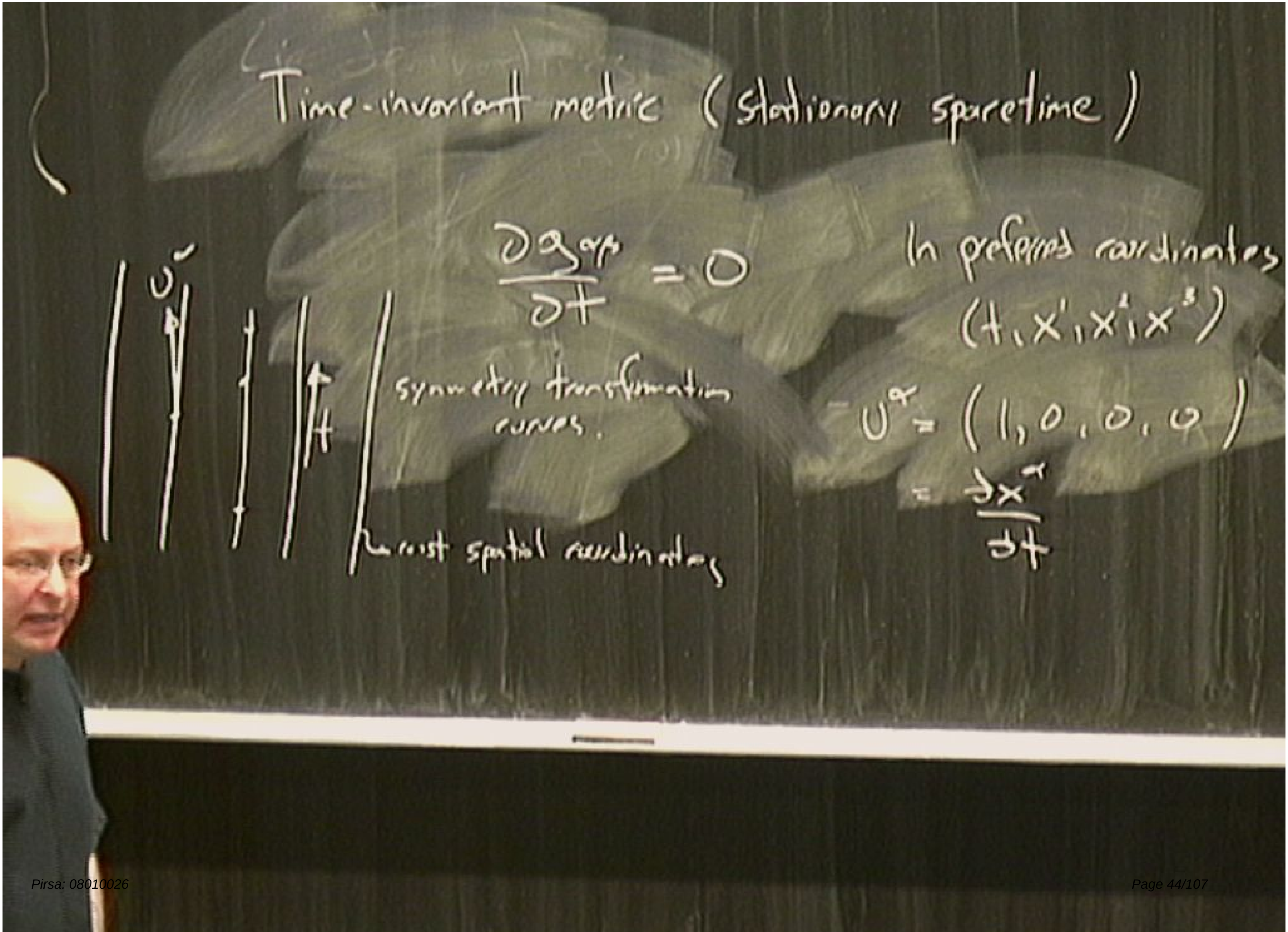
$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

In preferred coordinates (t, x^1, x^2, x^3)

$$u^\alpha = (1, 0, 0, 0)$$

symmetry transformation curves.

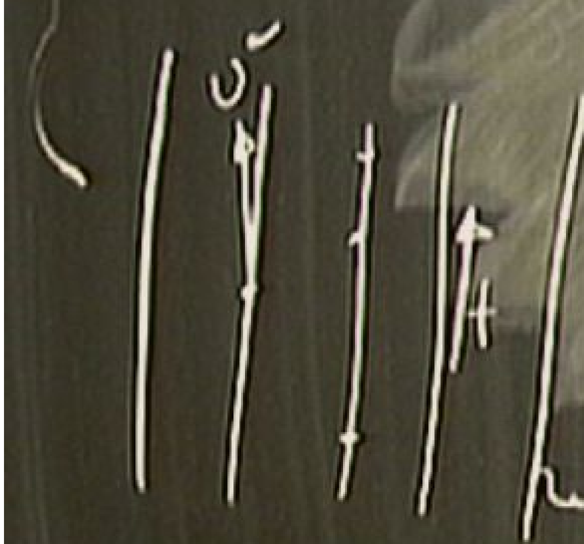
= const spatial coordinates


Time-invariant metric (stationary spacetime)
$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$
In preferred coordinates
(t, x^1, x^2, x^3)
u^α
symmetry transformation curves.
t
$U^\alpha = (1, 0, 0, 0)$
$= \frac{\partial x^\alpha}{\partial t}$
= const spatial coordinates

Time-invariant metric (Stationary spacetime)

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

In preferred coordinates (t, x^1, x^2, x^3)

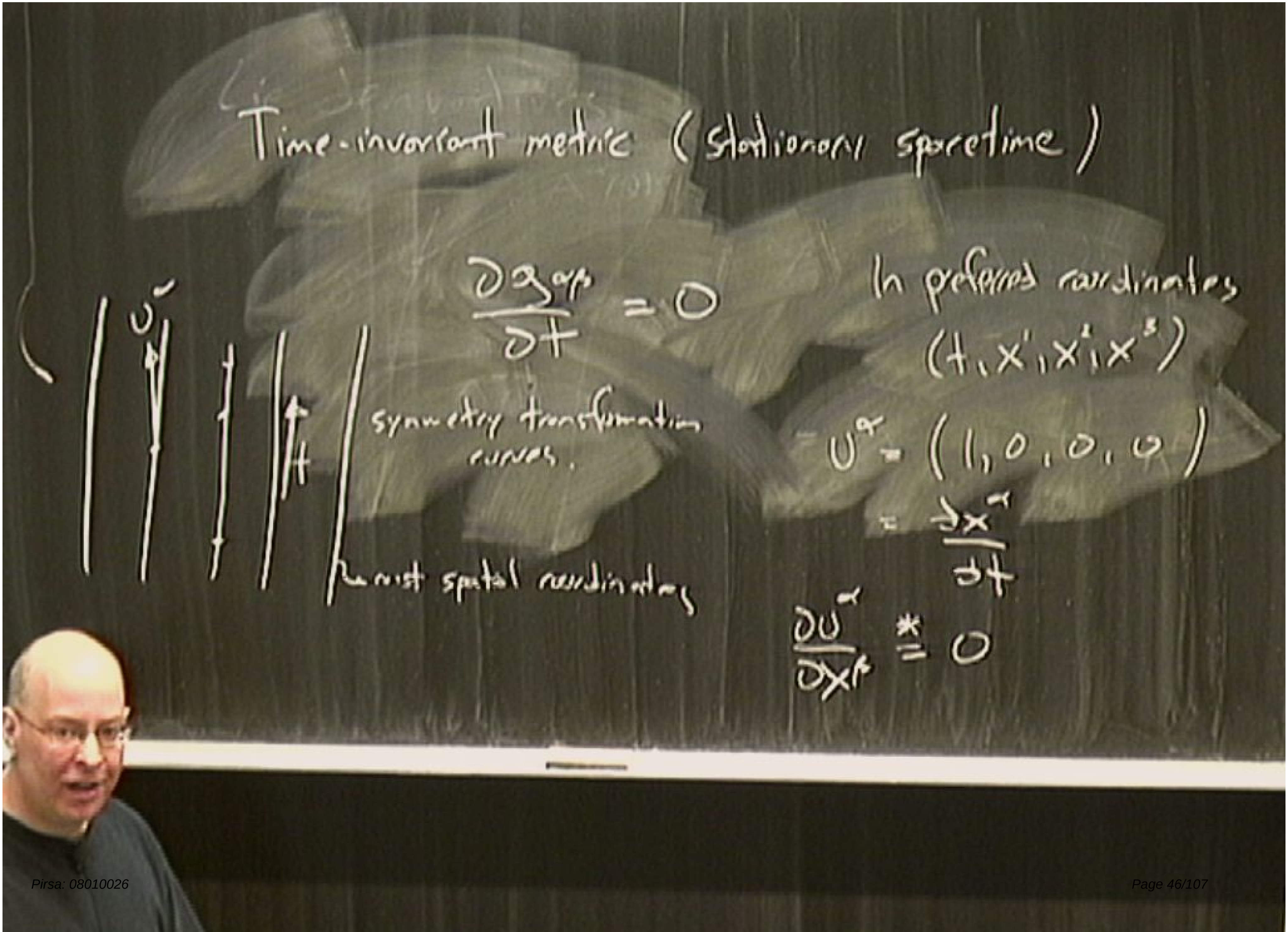


symmetry transformation curves.

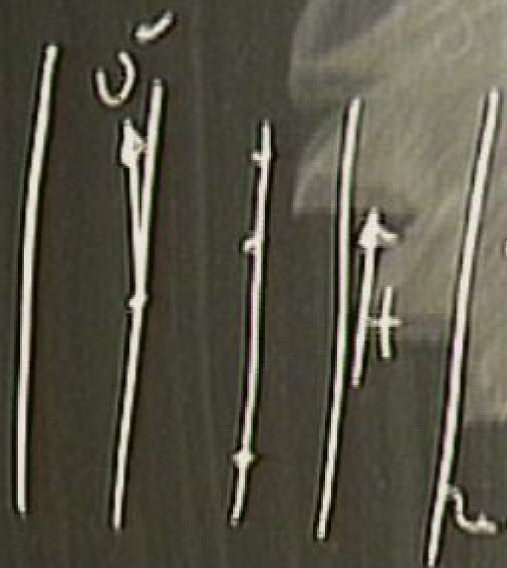
const spatial coordinates

$$U^\alpha = (1, 0, 0, 0) = \frac{\partial x^\alpha}{\partial t}$$

$$\frac{\partial U^\alpha}{\partial x^\beta} = 0$$


Time-invariant metric (Stationary spacetime)
$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$
In preferred coordinates
(t, x^1, x^2, x^3)
u^α
symmetry transformation curves.
constant spatial coordinates
$U^\alpha = (1, 0, 0, 0)$
$= \frac{\partial x^\alpha}{\partial t}$
$\frac{\partial U^\alpha}{\partial x^\beta} \stackrel{*}{=} 0$

Time-invariant metric (Stationary spacetime)



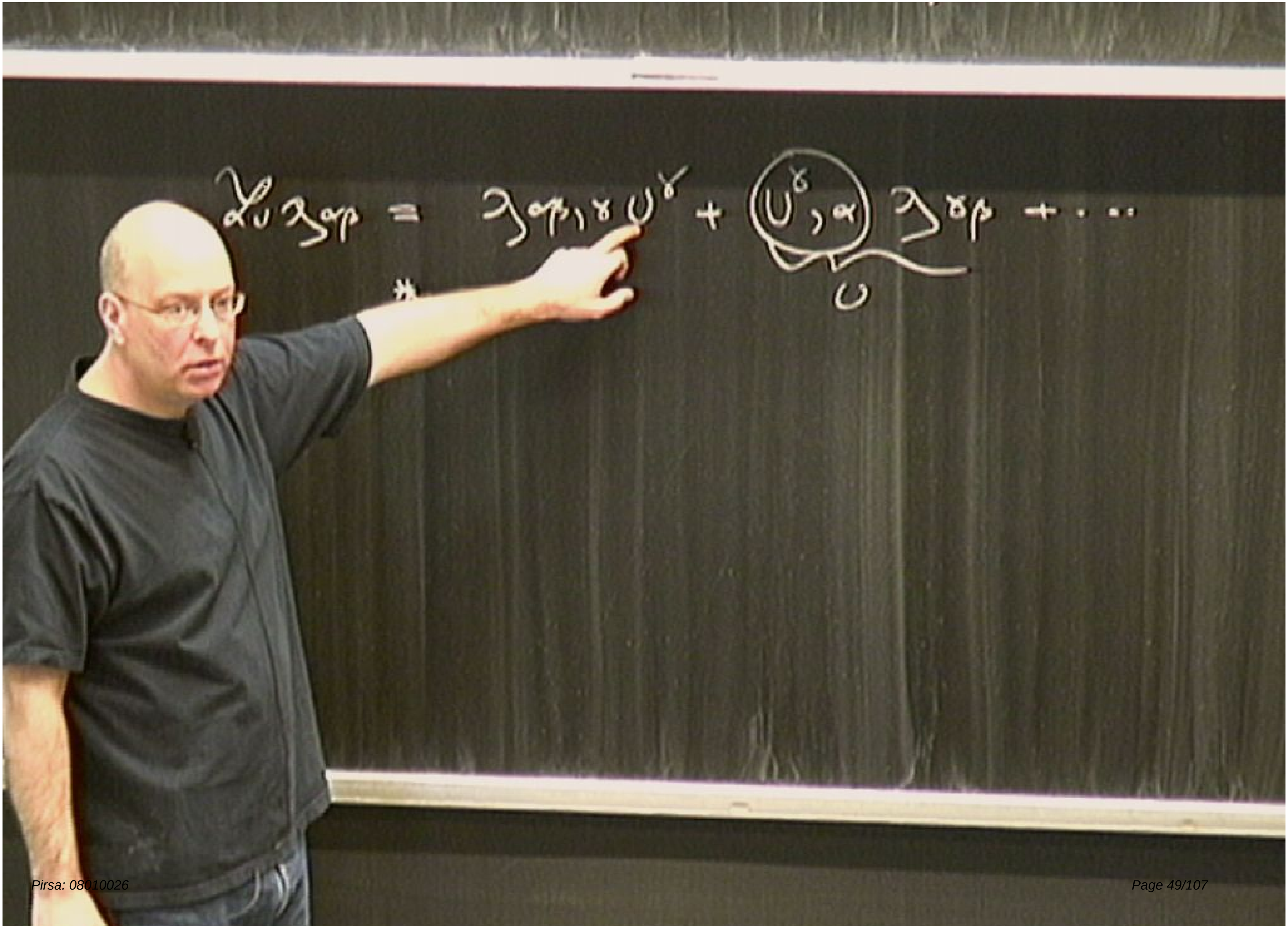
$$\frac{\partial g_{\alpha\beta}}{\partial t} \overset{*}{=} 0$$

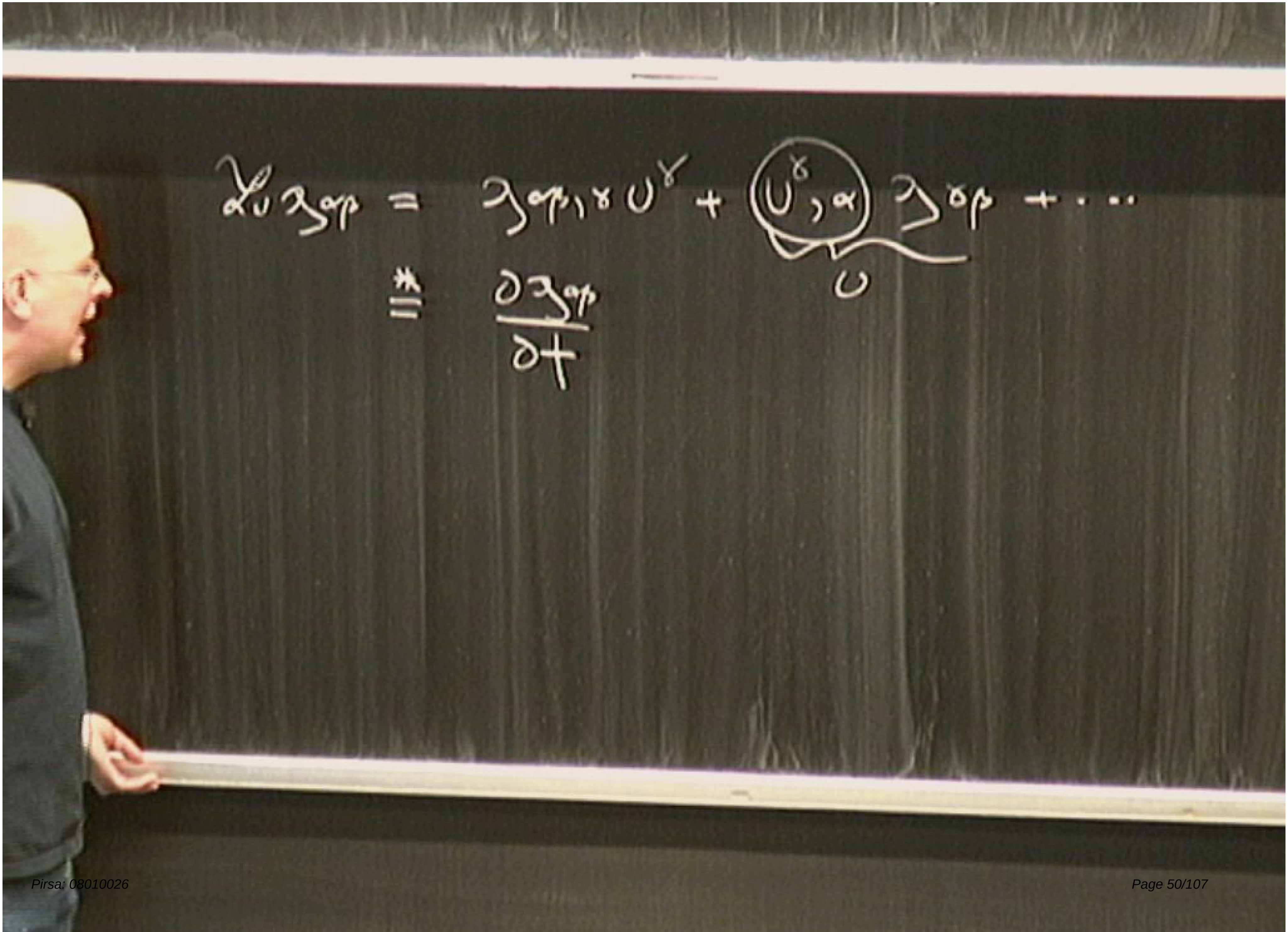
In preferred coordinates (t, x^1, x^2, x^3)

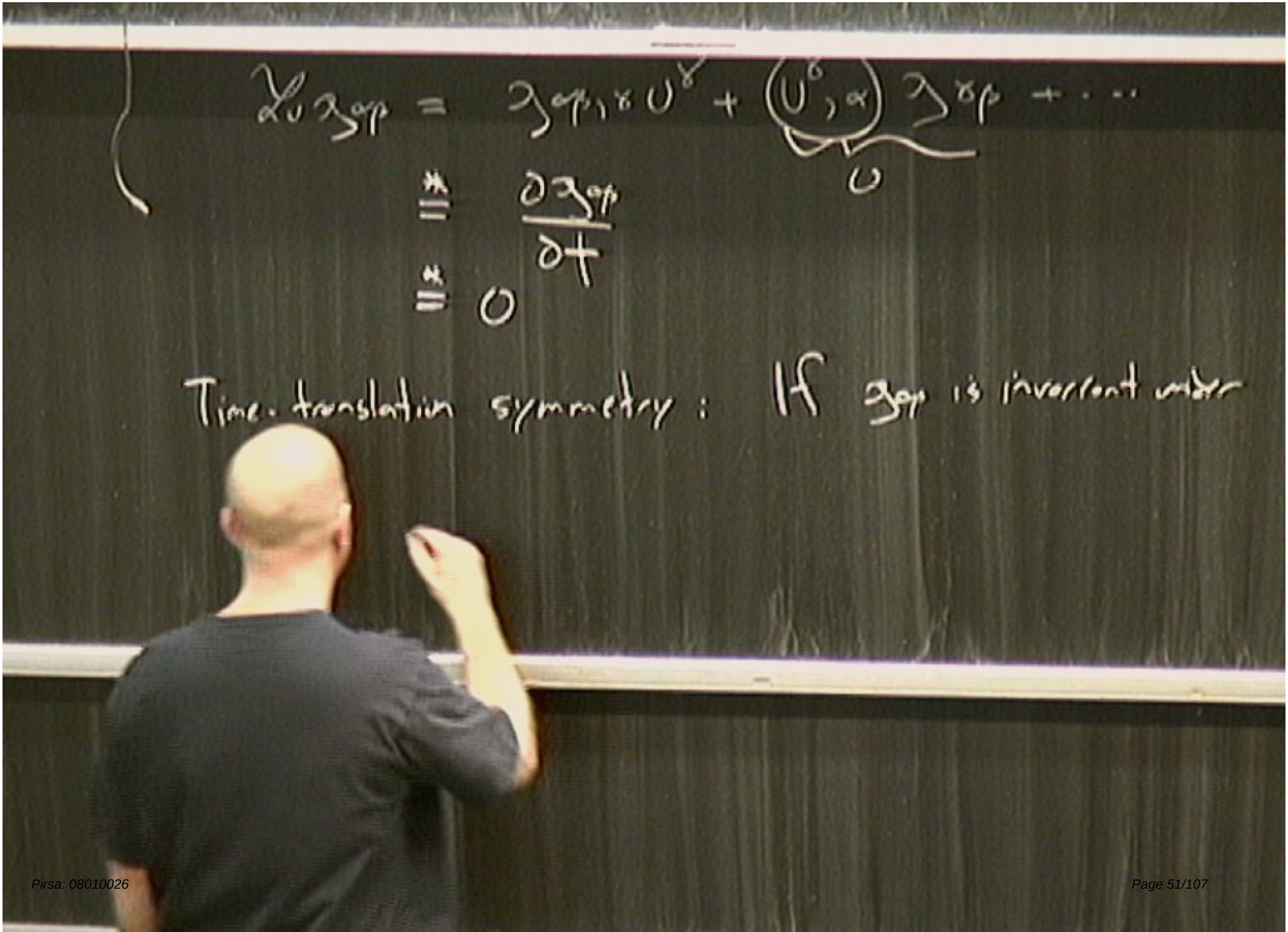
$$u^\alpha \overset{*}{=} (1, 0, 0, 0) = \frac{\partial x^\alpha}{\partial t}$$

$$\frac{\partial u^\alpha}{\partial x^\beta} \overset{*}{=} 0$$

$$\mathcal{L}_u g_{\alpha\beta} = g_{\alpha\beta,\gamma} u^\gamma +$$


$\mathcal{L}_U g_{\alpha\beta} = g_{\alpha\beta,\gamma} U^\gamma + U^\gamma{}_{,\alpha} g_{\gamma\beta} + \dots$


$\mathcal{L}_U g_{\alpha\beta} = g_{\alpha\beta,\gamma}U^\gamma + \underbrace{U^\gamma{}_{,\alpha}\, g_{\gamma\beta}}_{U} + \dots$
$\overset{*}{=} \frac{\partial g_{\alpha\beta}}{\partial t}$


$\mathcal{L}_U g_{\alpha\beta} = g_{\alpha\beta,\gamma} U^\gamma + \underbrace{U^\gamma{}_{,\alpha} g_{\gamma\beta} + \cdots}_{0}$
$\overset{*}{=} \frac{\partial g_{\alpha\beta}}{\partial t}$
$\overset{*}{=} 0$
Time-translation symmetry: If $g_{\alpha\beta}$ is invariant under

$$\mathcal{L}_U g_{\alpha\beta} = g_{\alpha\beta,\gamma} U^\gamma + \underbrace{U^\gamma{}_{,\alpha}}_{0} g_{\gamma\beta} + \cdots$$

$$\stackrel{*}{=} \frac{\partial g_{\alpha\beta}}{\partial t}$$

$$\stackrel{*}{=} 0$$

Time-translation symmetry: If $g_{\alpha\beta}$ is invariant under a translation along curves to which U^α is tangent,

then $\mathcal{L}_U g_{\alpha\beta} = 0$

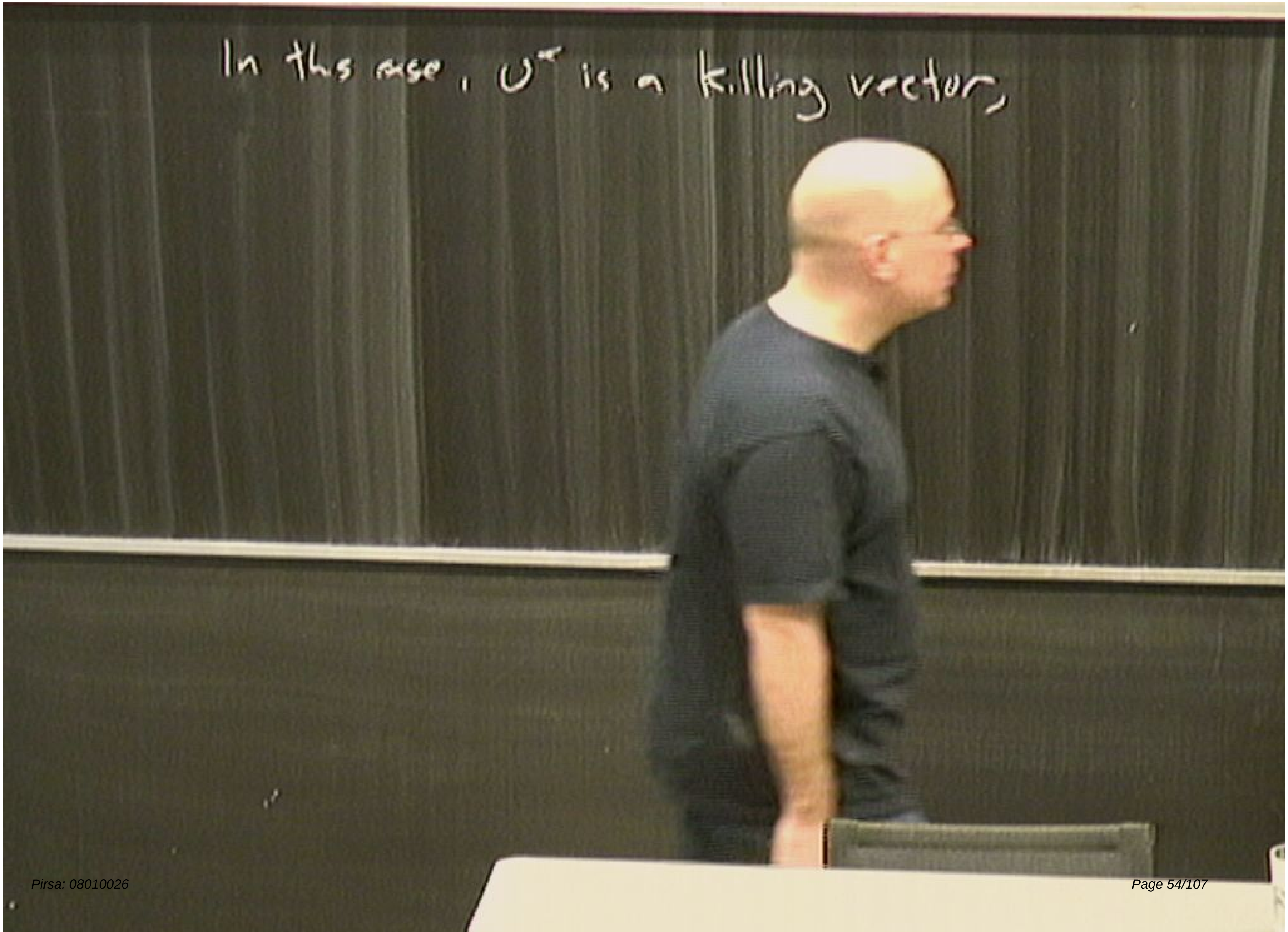
$$\mathcal{L}_U g_{\alpha\beta} = g_{\alpha\beta,\gamma} U^\gamma + \underbrace{U^\gamma{}_{,\alpha}\, g_{\gamma\beta} + \cdots}_{0}$$

$$\overset{*}{=} \frac{\partial g_{\alpha\beta}}{\partial t}$$

$$\overset{*}{=} 0$$

Time-translation symmetry: If $g_{\alpha\beta}$ is invariant under a translation along curves to which U^α ~~is~~ tangent,

then $\boxed{\mathcal{L}_U g_{\alpha\beta} = 0}$


In this case, U^a is a Killing vector,

In this case, U^α is a Killing vector,

$$U_{\alpha;\beta} + U_{\beta;\alpha} = 0$$

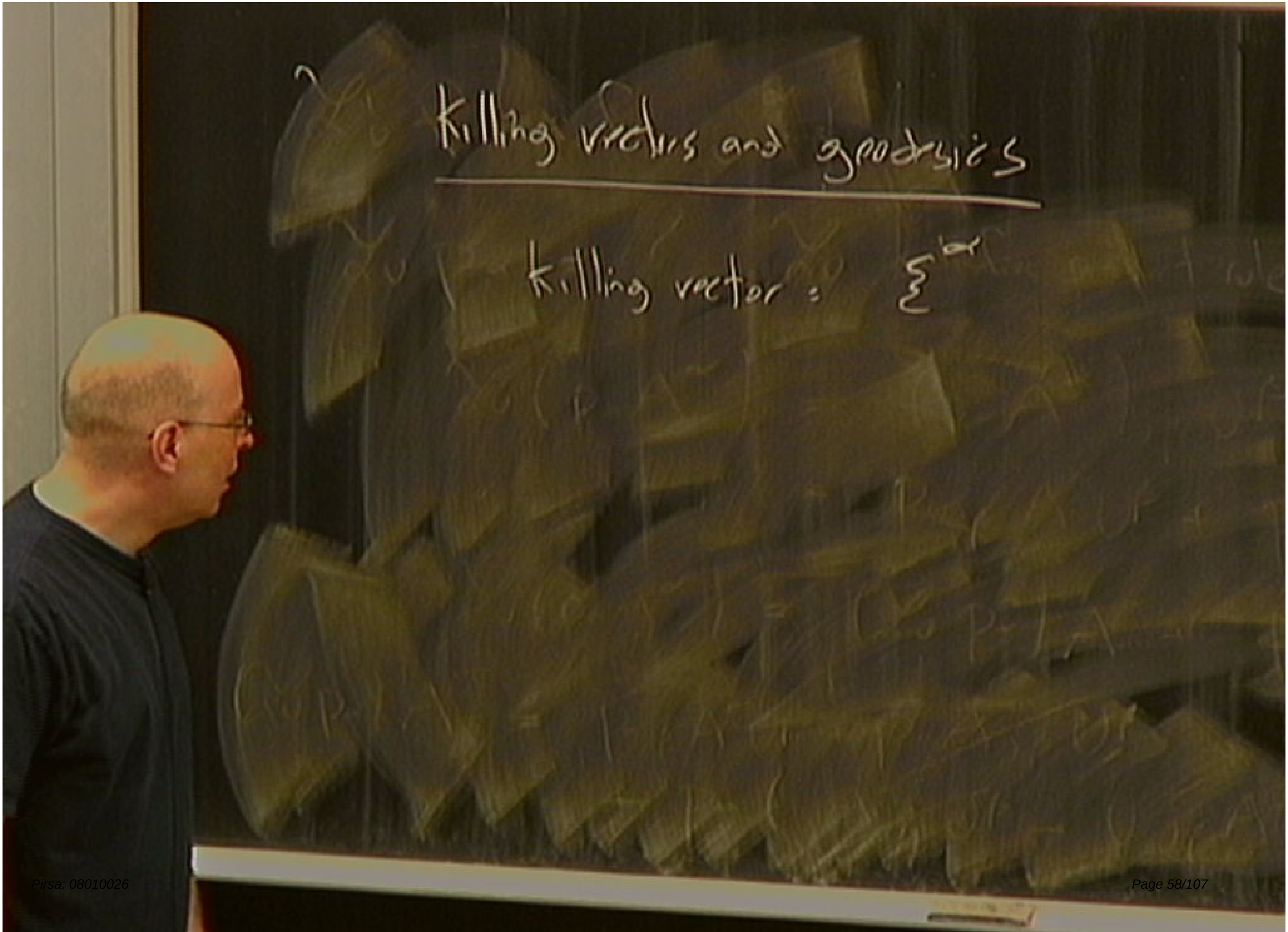
$$\Rightarrow \quad U_{\alpha;\beta}$$

In this case, U^α is a killing vector,

$$U_{\alpha;\beta} + U_{\beta;\alpha} = 0$$

$\Rightarrow$ $U_{\alpha;\beta}$ is antisymmetric

Killing vectors and geodesics


Killing vectors and geodesics
killing vector : ξ^α

Killing vectors and geodesics

Killing vector : ξ^{α} $\qquad \mathcal{L}_{\xi}\, g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

Killing vectors and geodesics

Killing vector: ξ^{α} $\quad \mathcal{L}_{\xi} g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}$

geodesic γ with velocity vector u^{α}

Killing vectors and geodesics

Killing vector: ξ^α $\quad \mathcal{L}_\xi g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

geodesic γ with velocity vector u^α (not Killing)

Killing vectors and geodesics

Killing vector: ξ^{α} $\qquad \mathcal{L}_{\xi}\, g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

geodesic γ with velocity vector u^{α} (not Killing)

↳ affinely parametrized

Killing vectors and geodesics

Killing vector: ξ^α $\qquad \mathcal{L}_\xi g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

geodesic γ with velocity vector u^α (not Killing)

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$$k = u^\alpha \xi_\alpha$$

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Proof: $\frac{dK}{dt} =$

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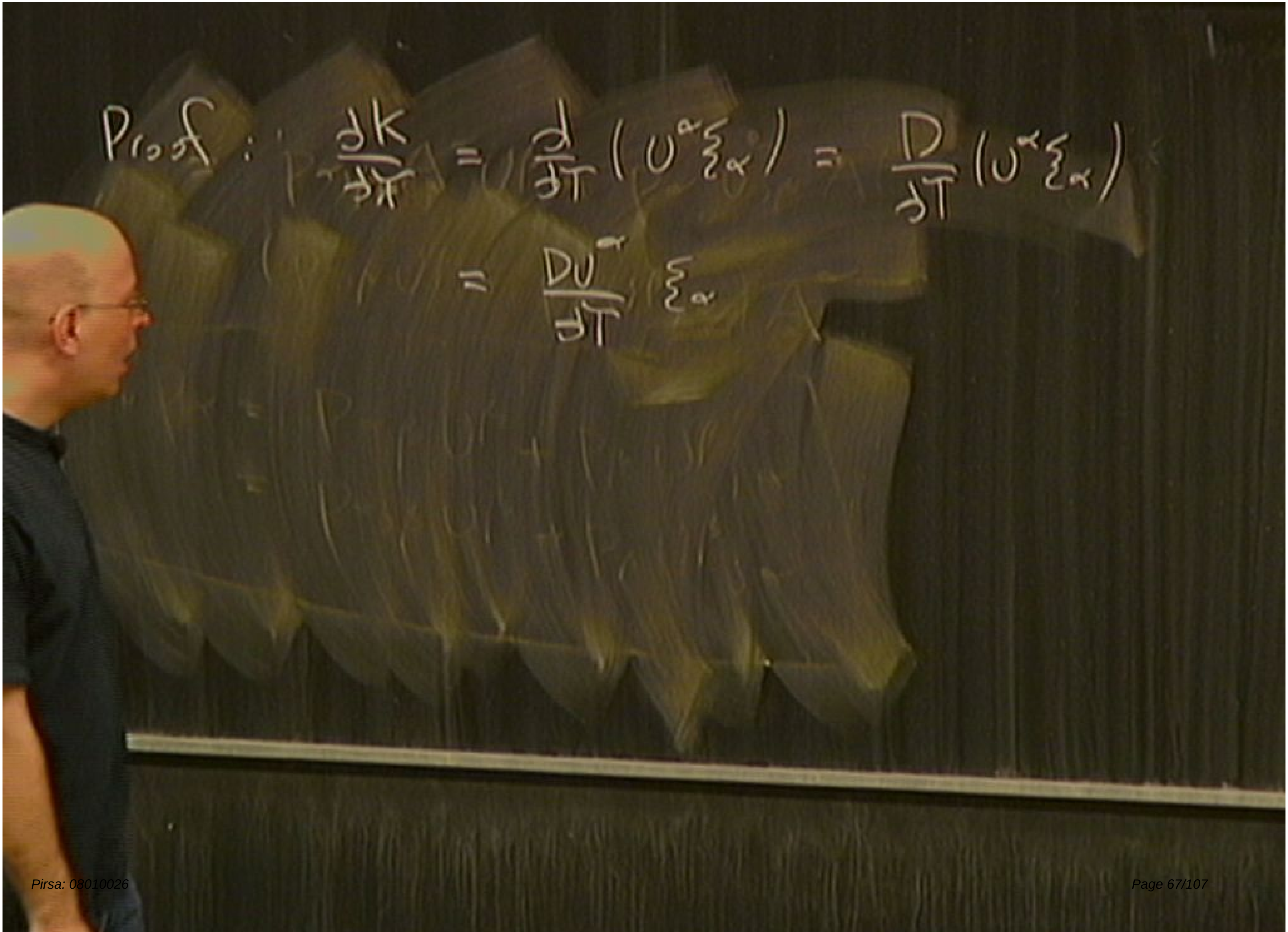
Proof: $\frac{dK}{d\tau} = \frac{d}{d\tau}(u^\alpha \xi_\alpha) = \frac{D}{d\tau}(u^\alpha \xi_\alpha)$
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$$\boxed{\mathcal{L}_u T_{\alpha\beta} = T_{\alpha\beta,\gamma} u^\gamma + u^\gamma{}_{,\alpha} T_{\gamma\beta} + u^\gamma{}_{,\beta} T_{\alpha\gamma}}$$

$$= T_{\alpha\beta;\gamma} u^\gamma + u^\gamma{}_{;\alpha} T_{\gamma\beta} + u^\gamma{}_{;\beta} T_{\alpha\gamma}$$

$$\boxed{\mathcal{L}_u g_{\alpha\beta} = u_{\beta;\alpha} + u_{\alpha;\beta}}$$

$$\xi_{\alpha;\beta} u^\alpha u^\beta = \frac{1}{2} \xi_{\alpha;\beta} u^\alpha u^\beta + \frac{1}{2} \xi$$

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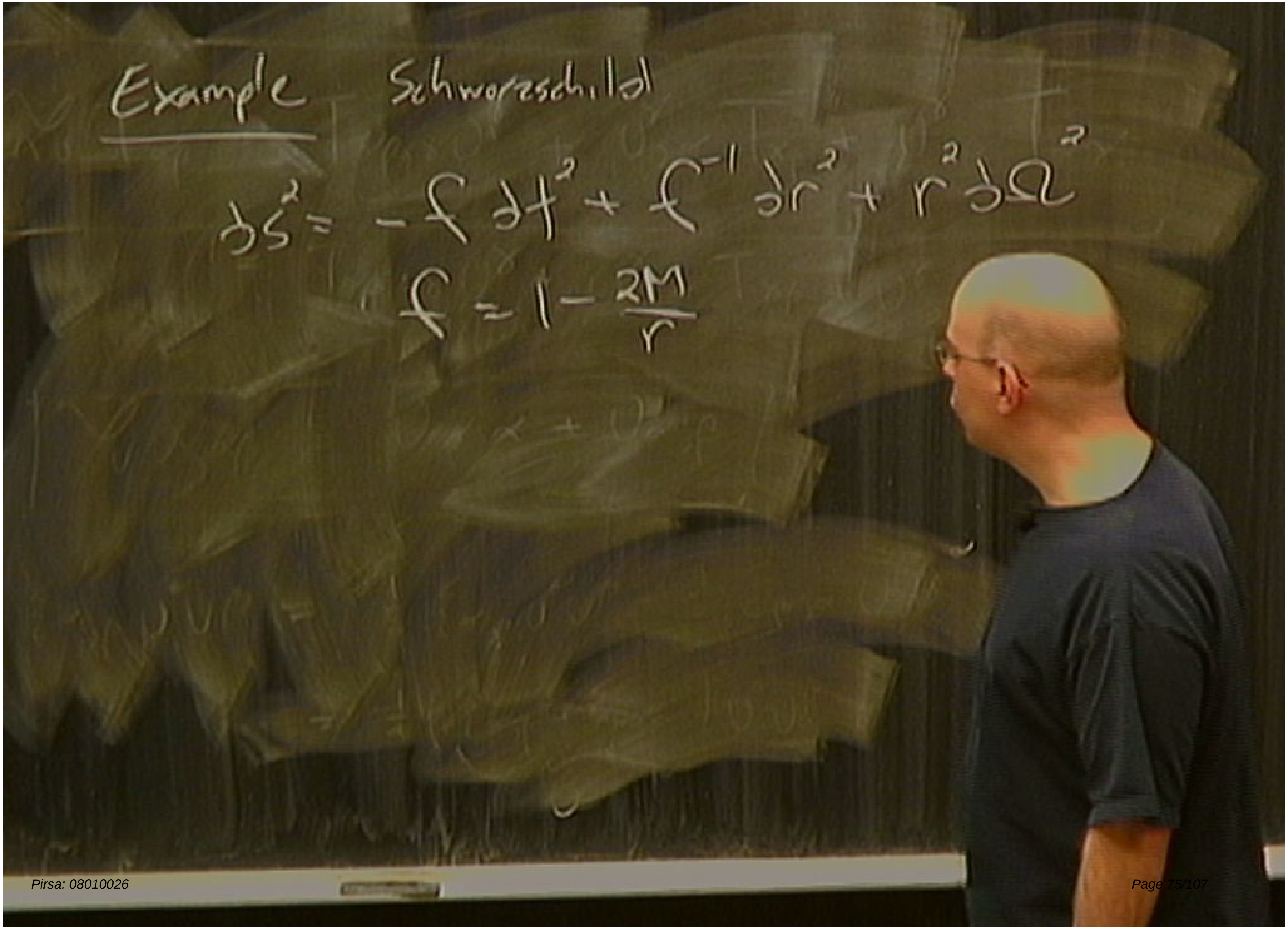
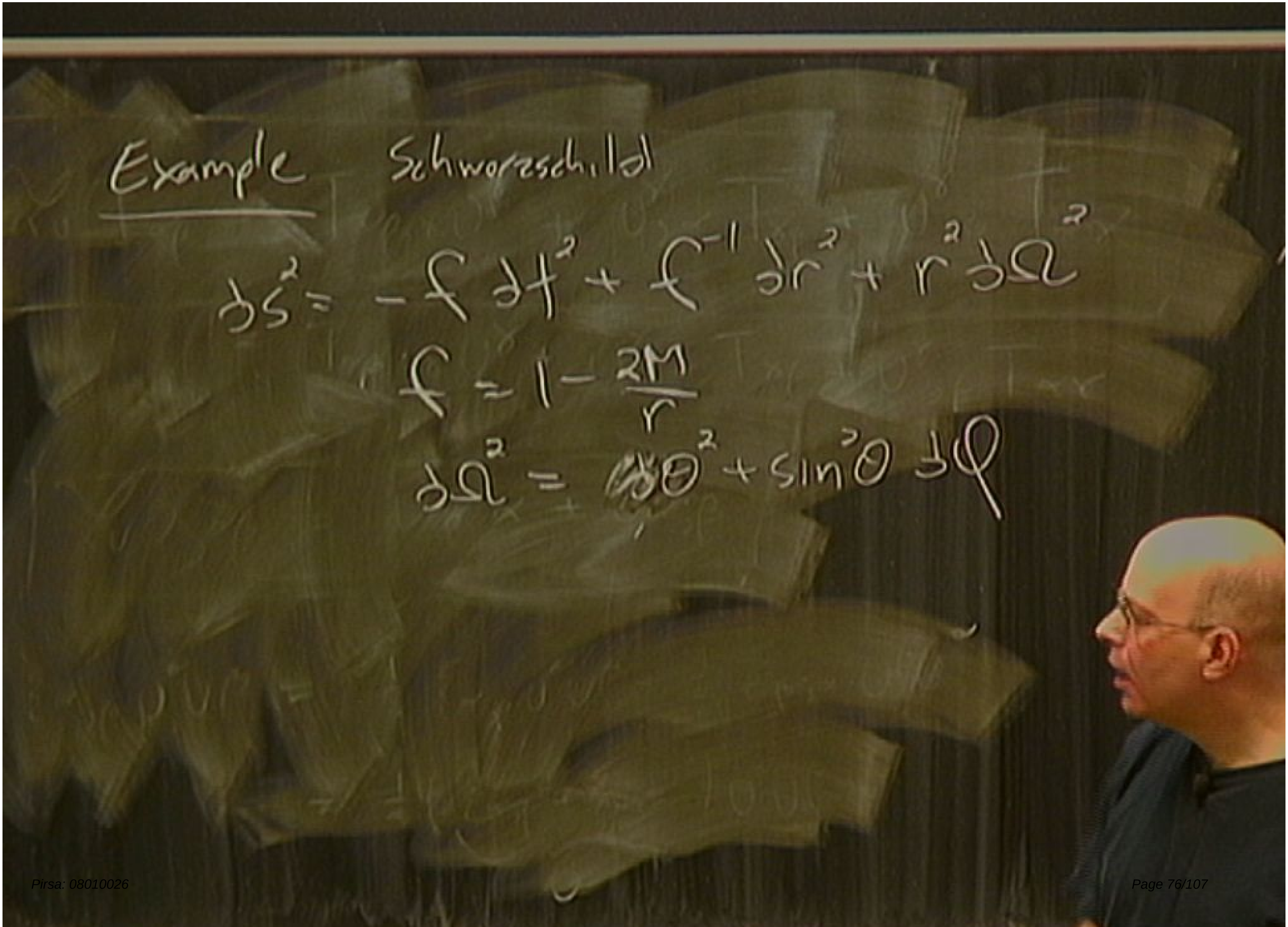
$$= \frac{1}{2} \underbrace{\left(\xi_{\alpha;\beta} + \xi_{\beta;\alpha} \right)}_{0} u^\alpha u^\beta$$


Example Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2$
$f = 1 - \frac{2M}{r}$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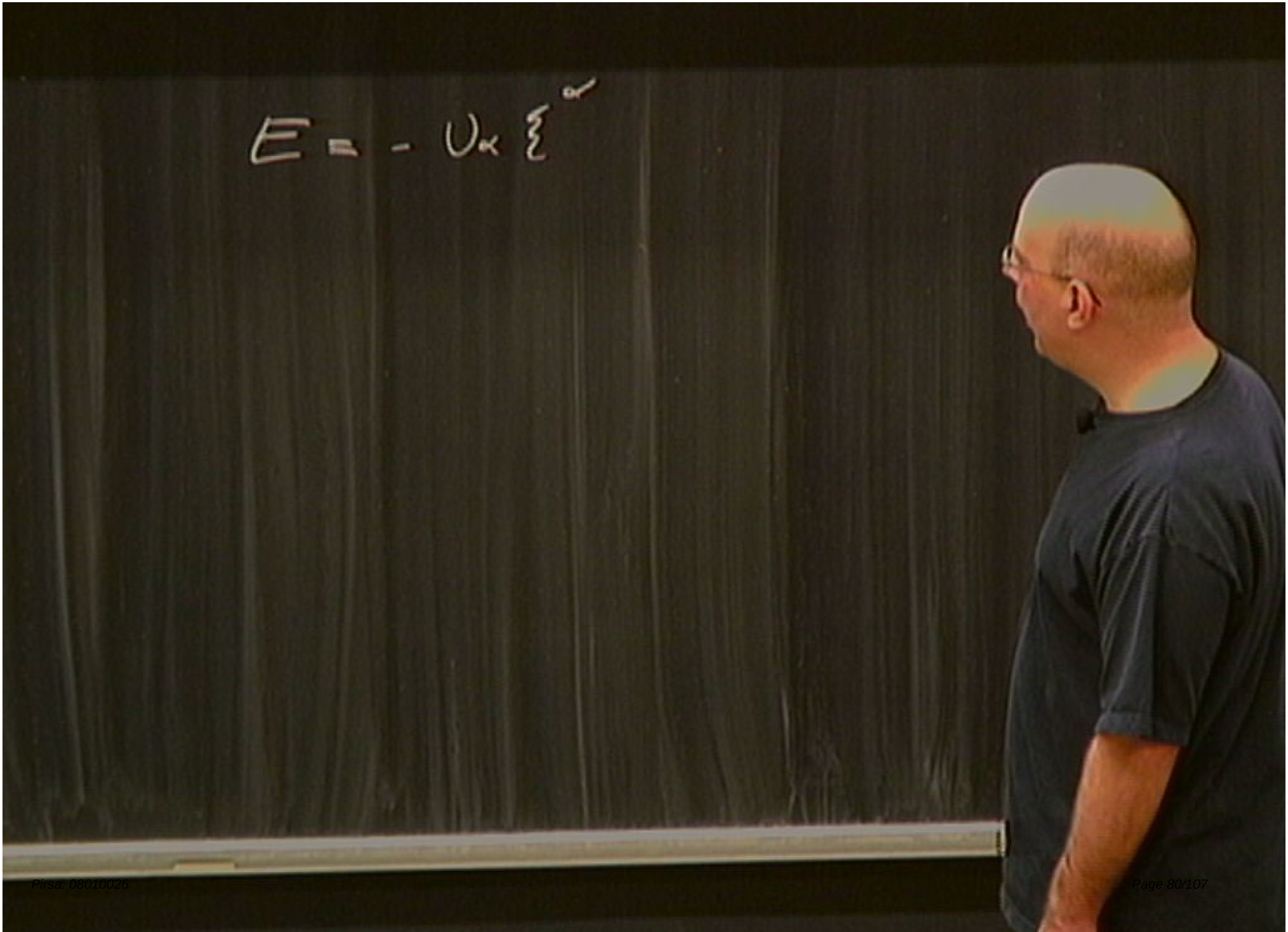
$$u^\alpha = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$$


$E = -U_\alpha \xi^\alpha$

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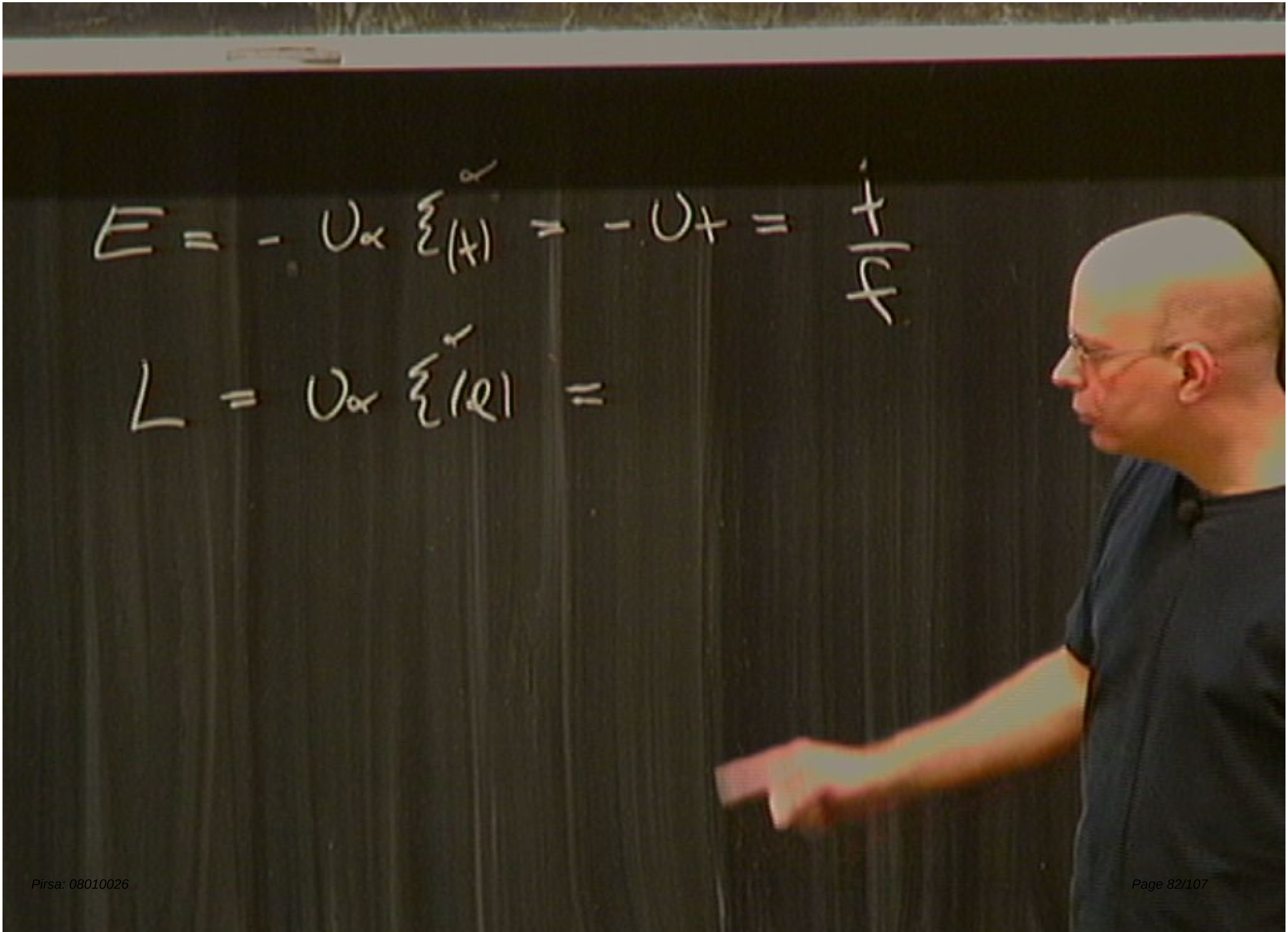
$E = -U_\alpha \xi^\alpha_{(t)} = -U_t = \frac{\dot t}{f}$
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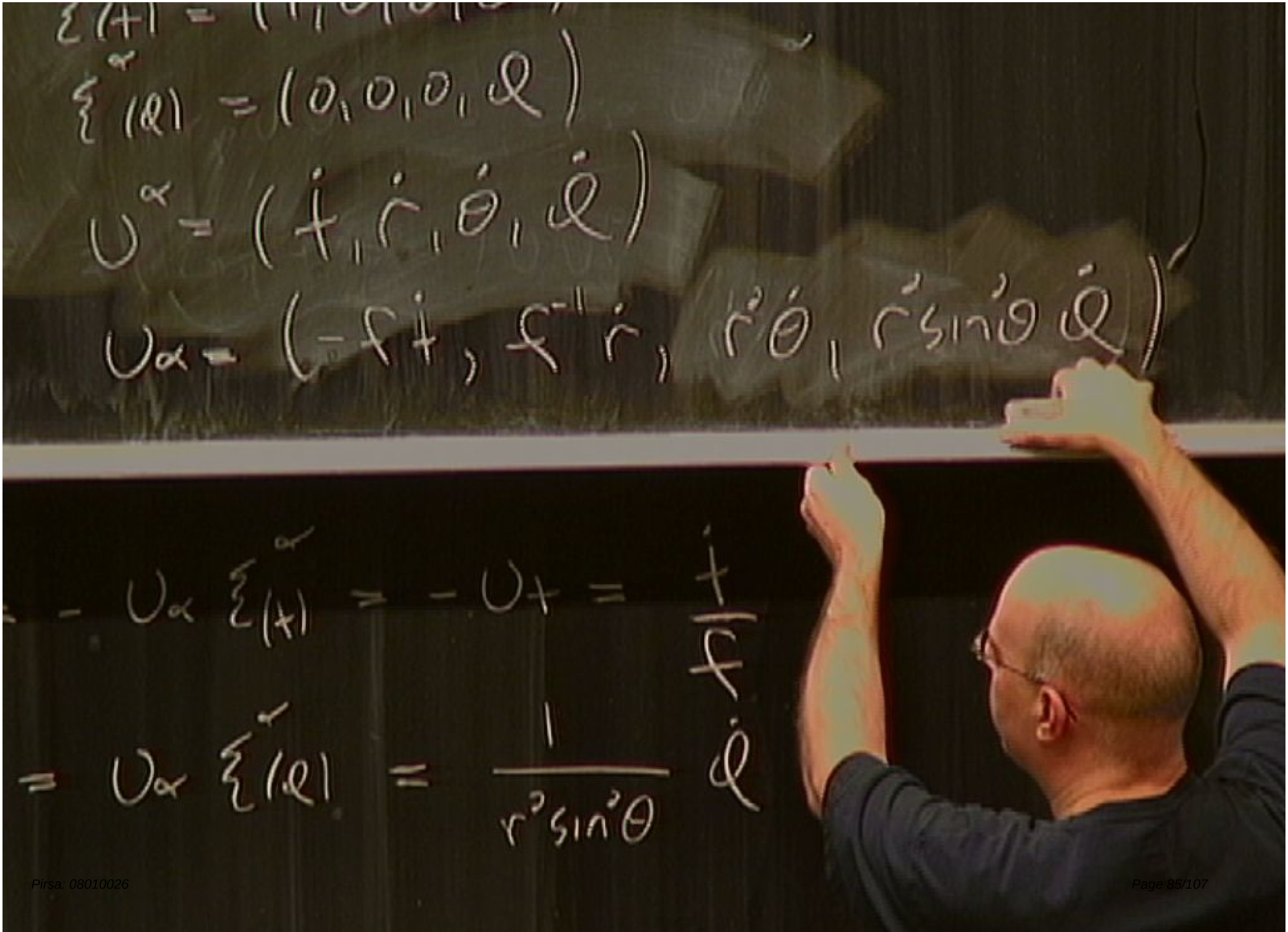

$\xi^\alpha_{(t)} = (1, 0, 0, 0)$
$\xi^\alpha_{(\ell)} = (0, 0, 0, \ell)$
$U^\alpha = (\dot t, \dot r, \dot\theta, \dot\ell)$
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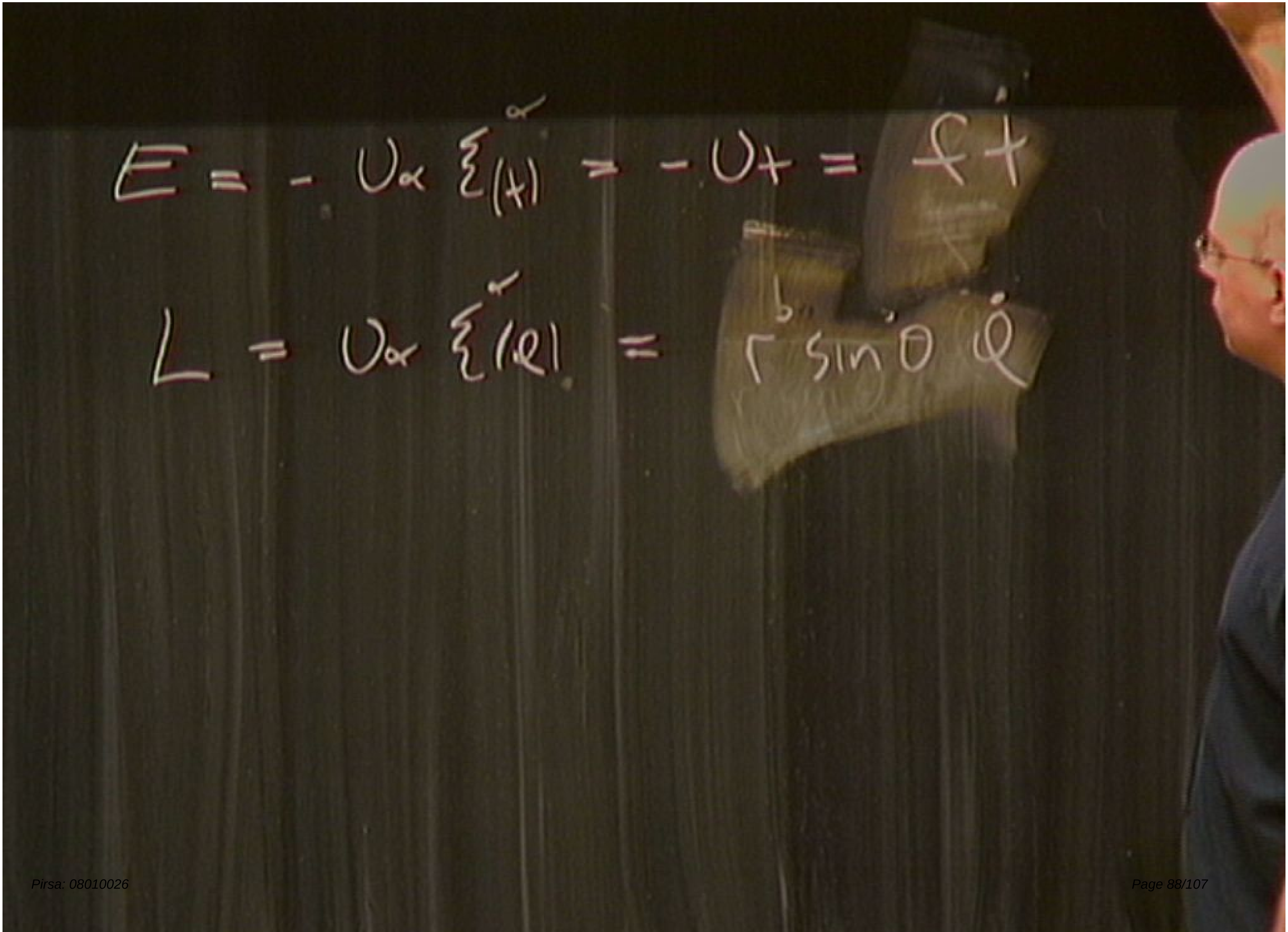
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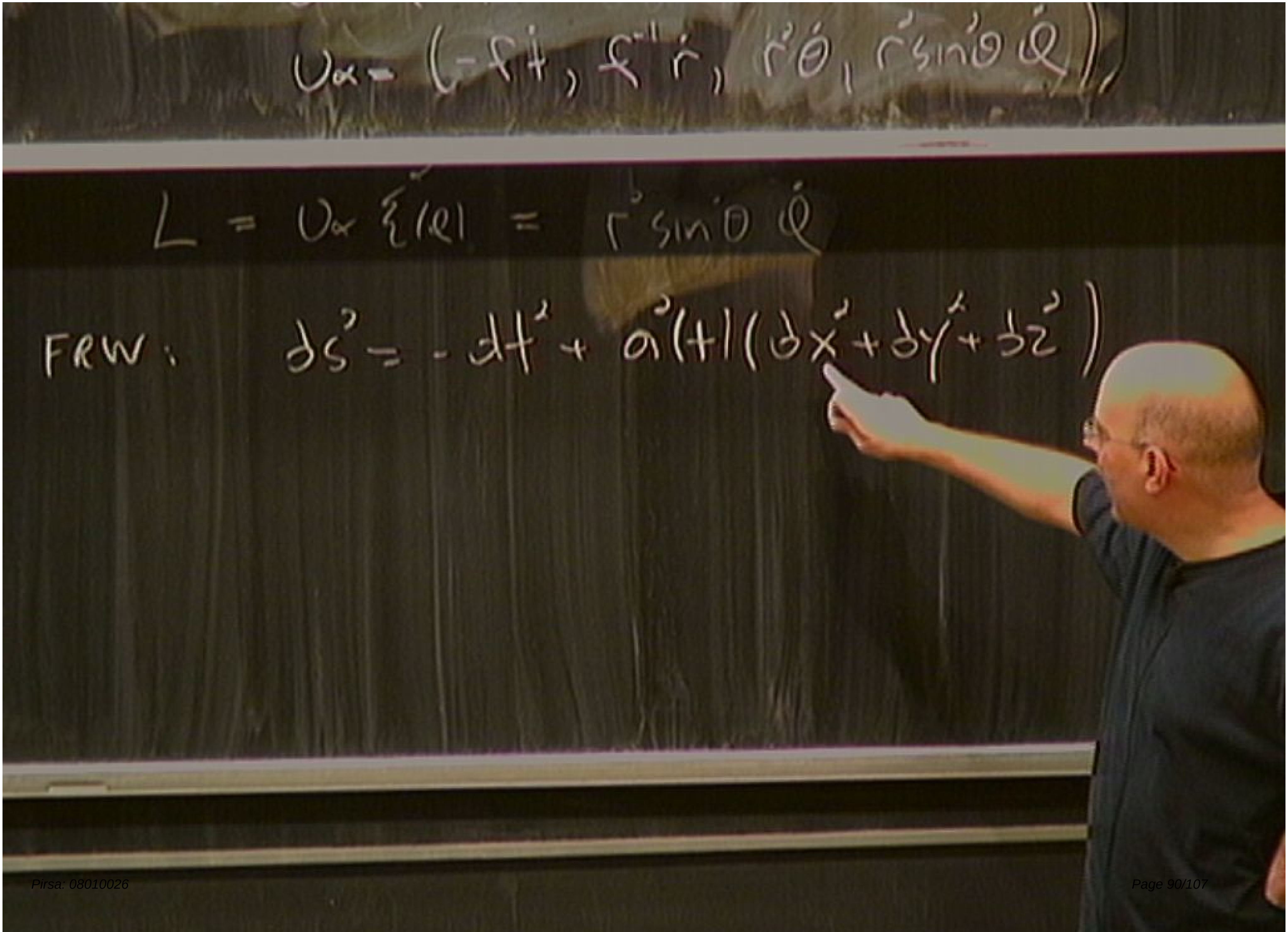
FRW: $$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$


$U_\alpha = (-f\dot{t}, f^{-1}\dot{r}, r^2\dot{\theta}, r^2\sin^2\theta\,\dot{\varphi})$
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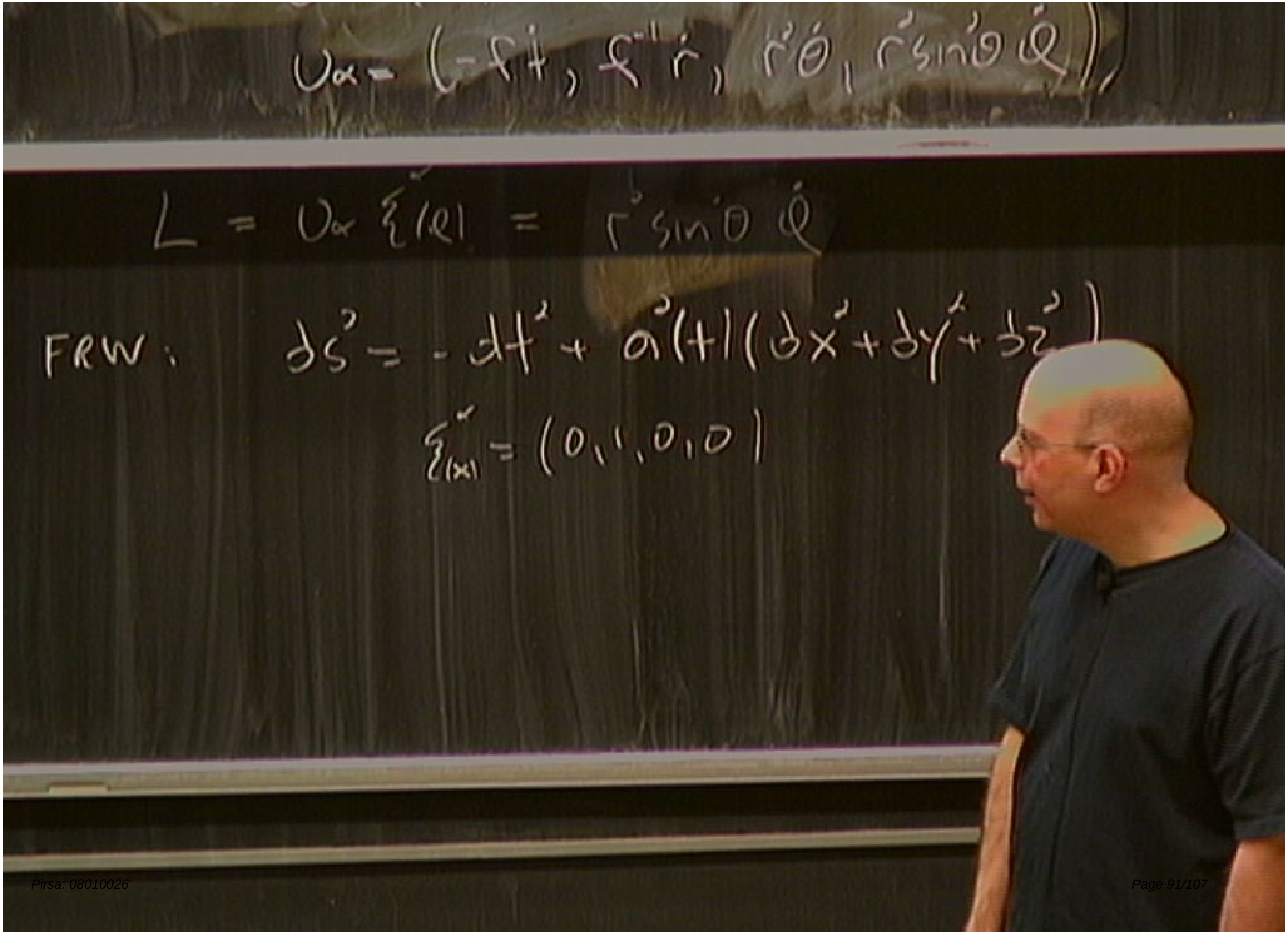
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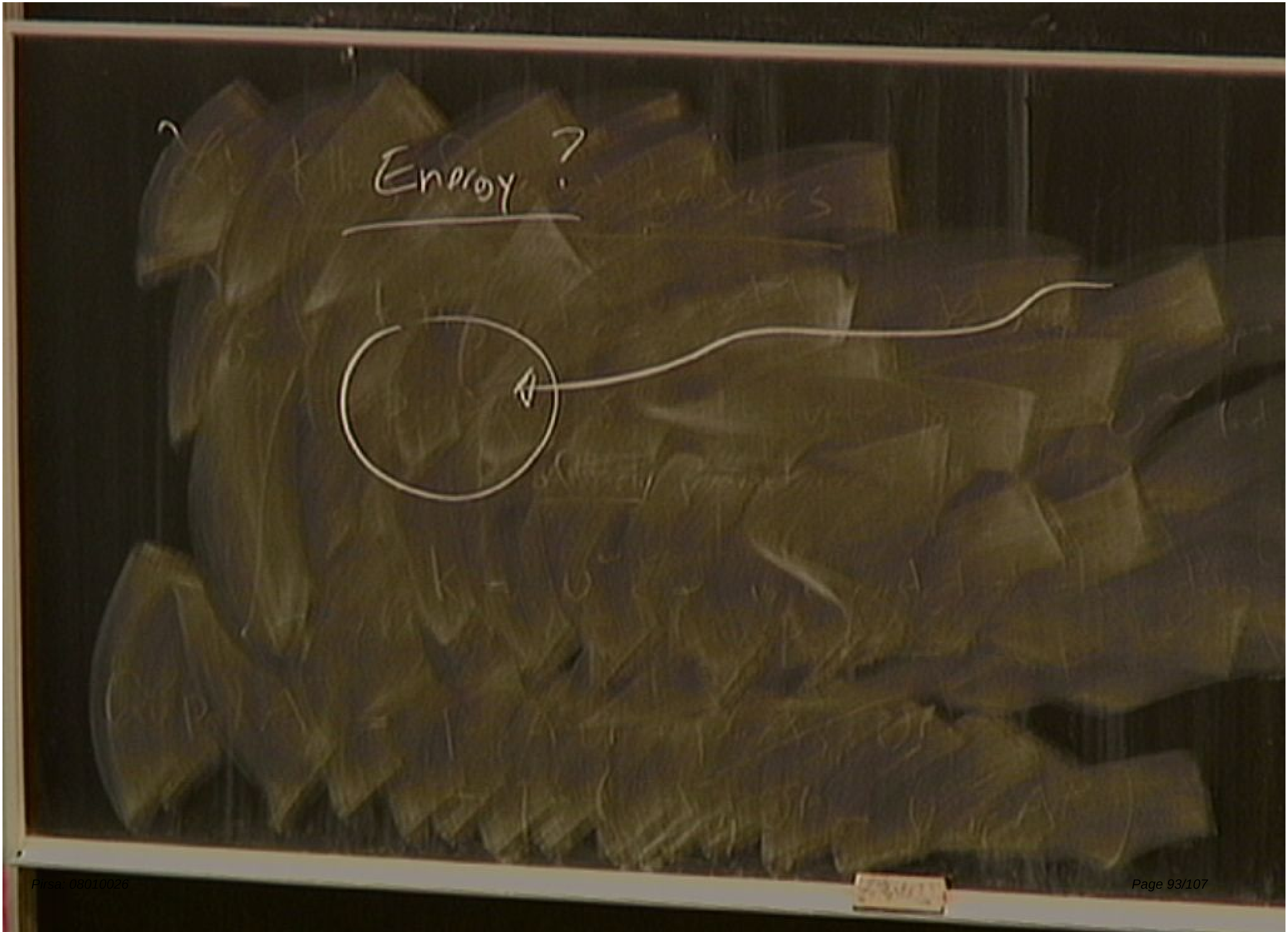
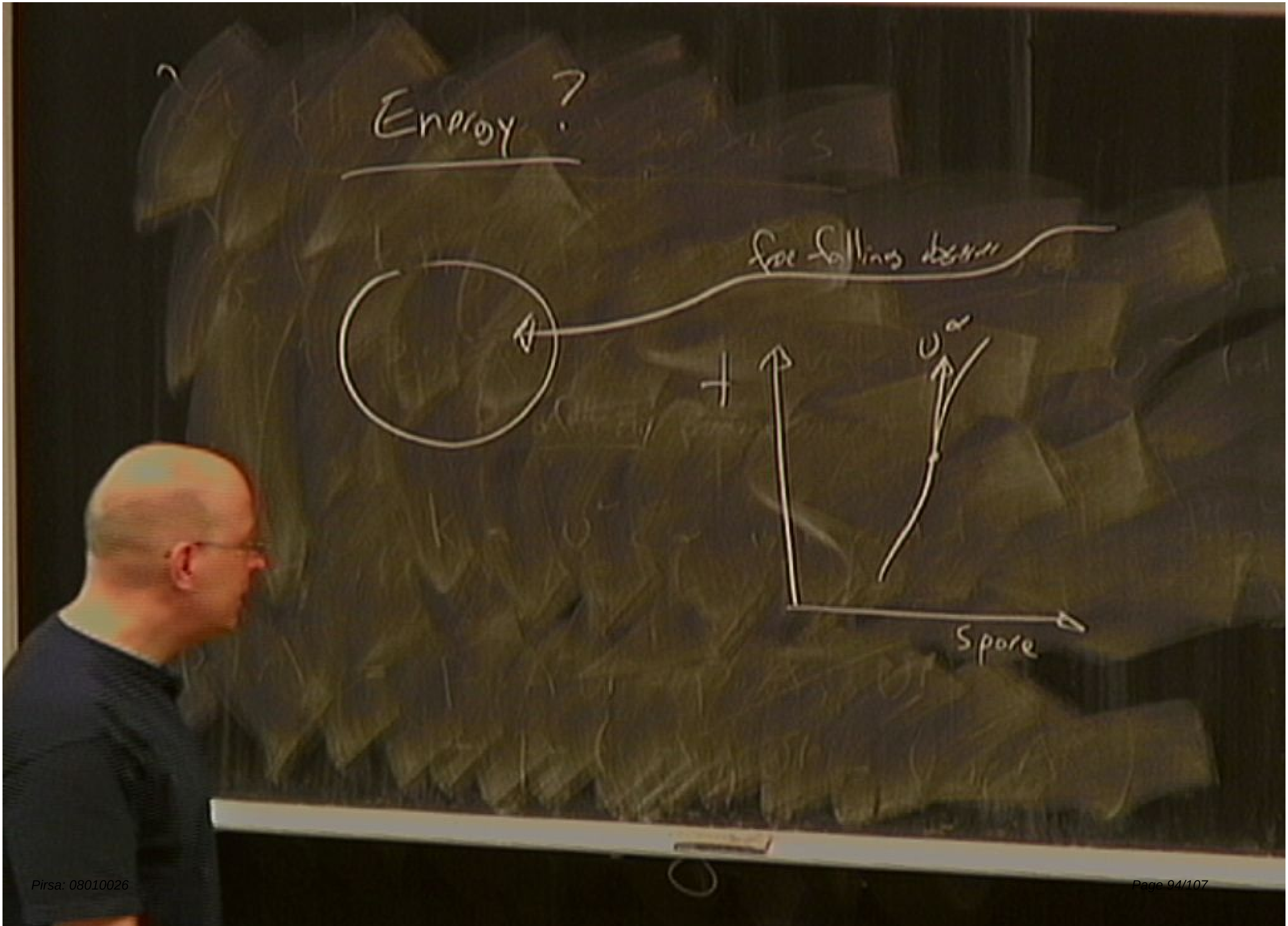
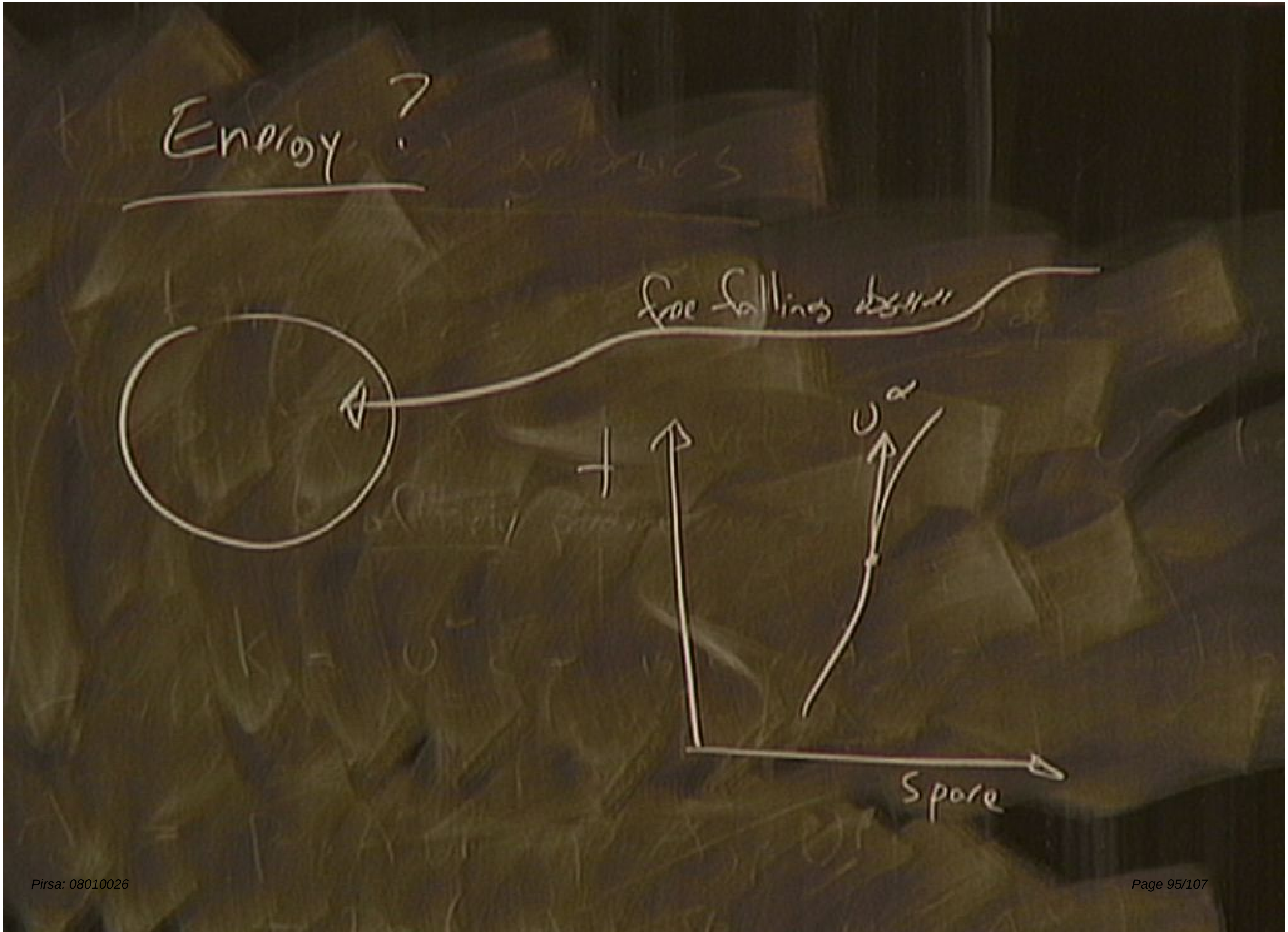
$$\xi^\alpha_{(y)} = (0,0,1,0)$$

$$\vdots$$


Energy ?


Energy ?
free falling observer
t
u^α
Space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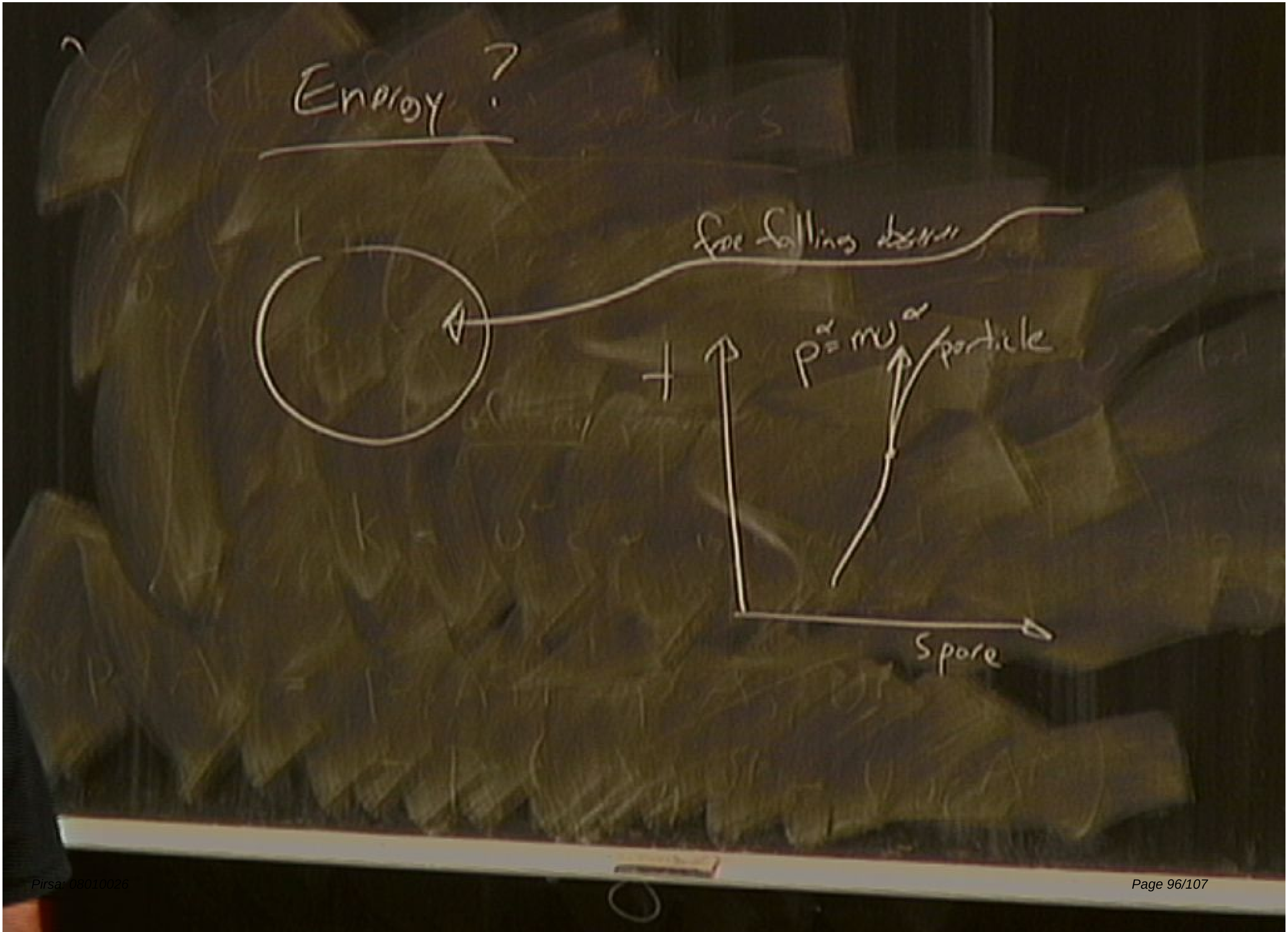
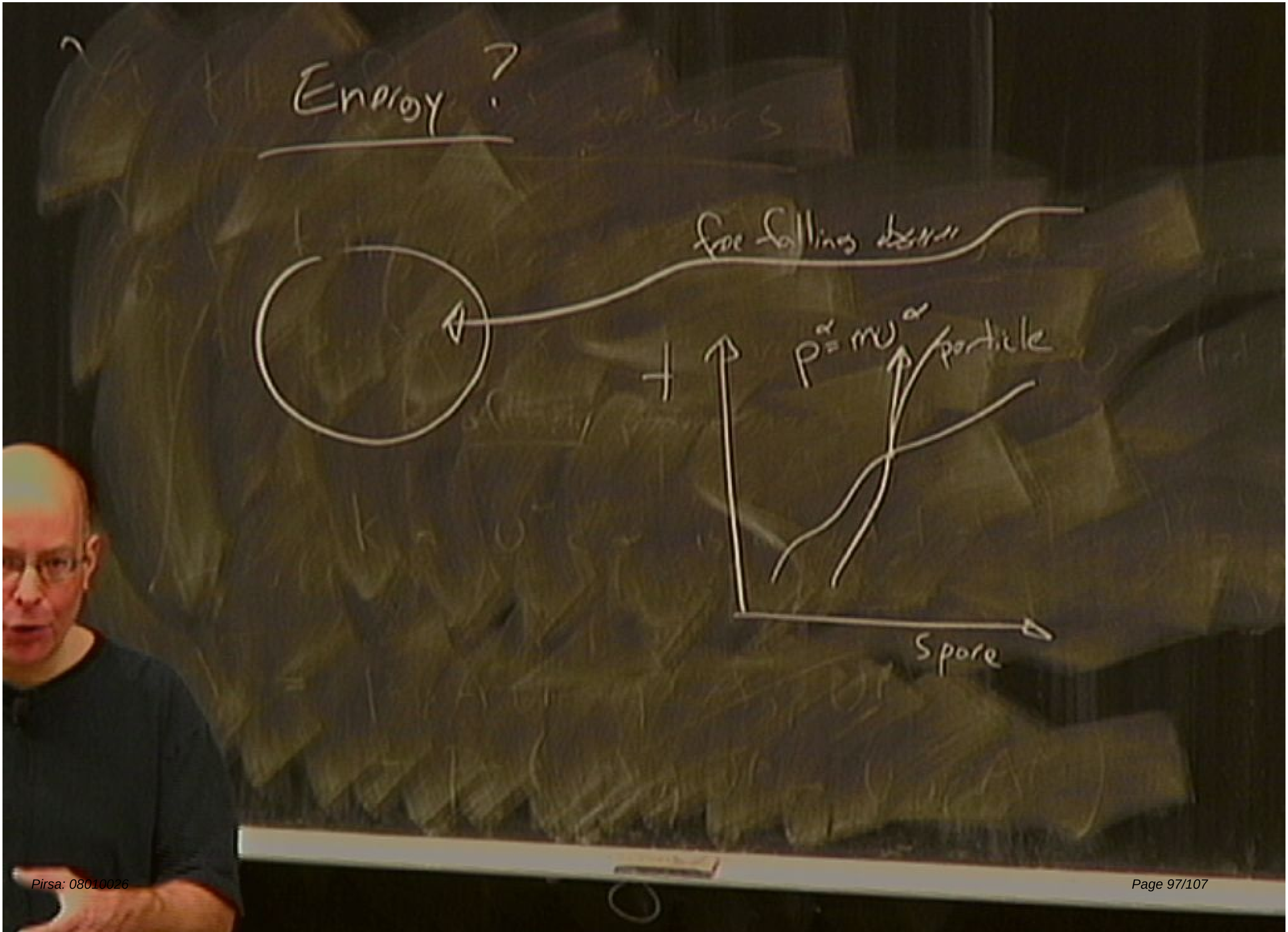

Energy ?
free falling
$p^\alpha = m u^\alpha$
particle
t
Space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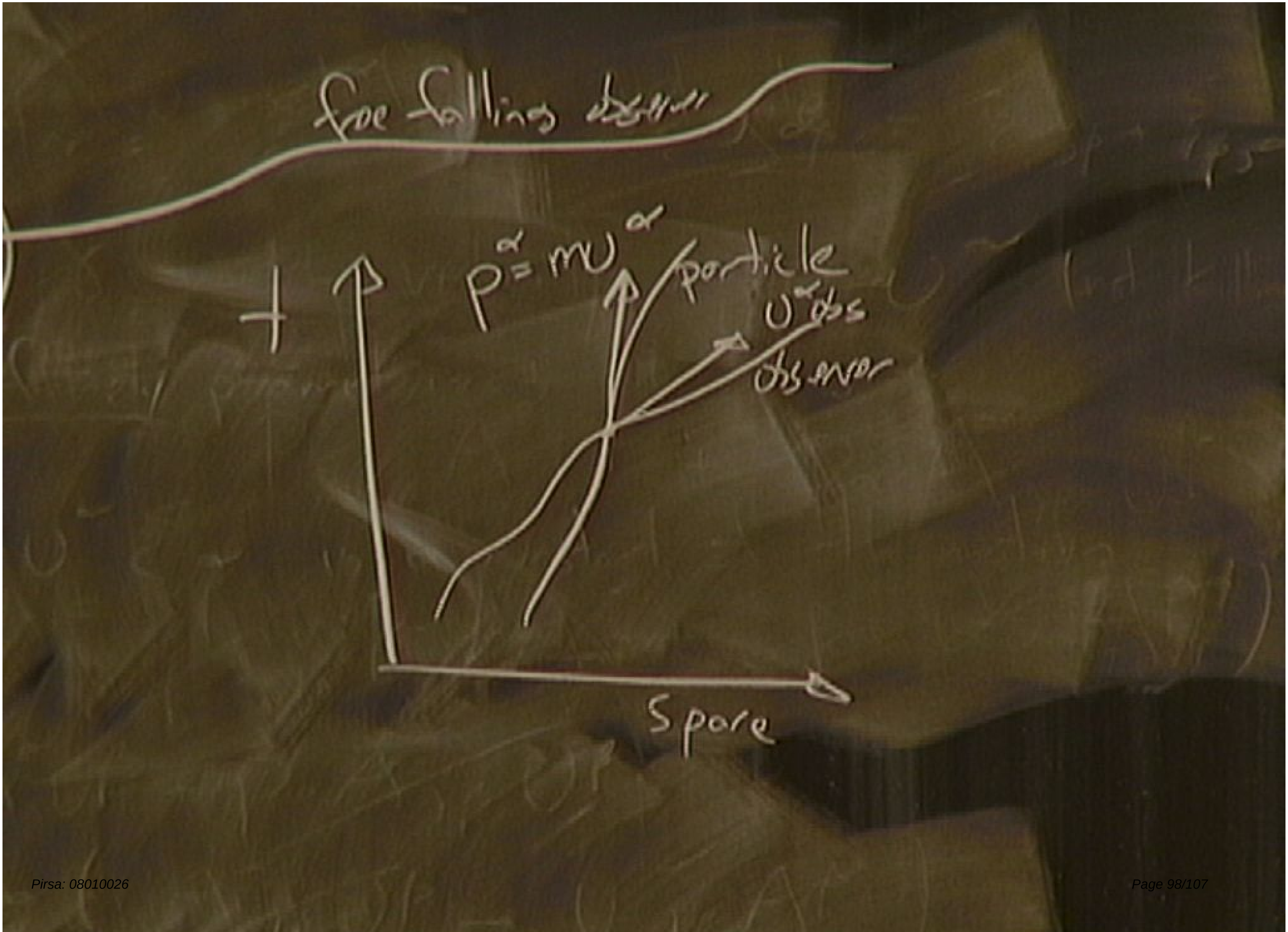

free falling observer
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t
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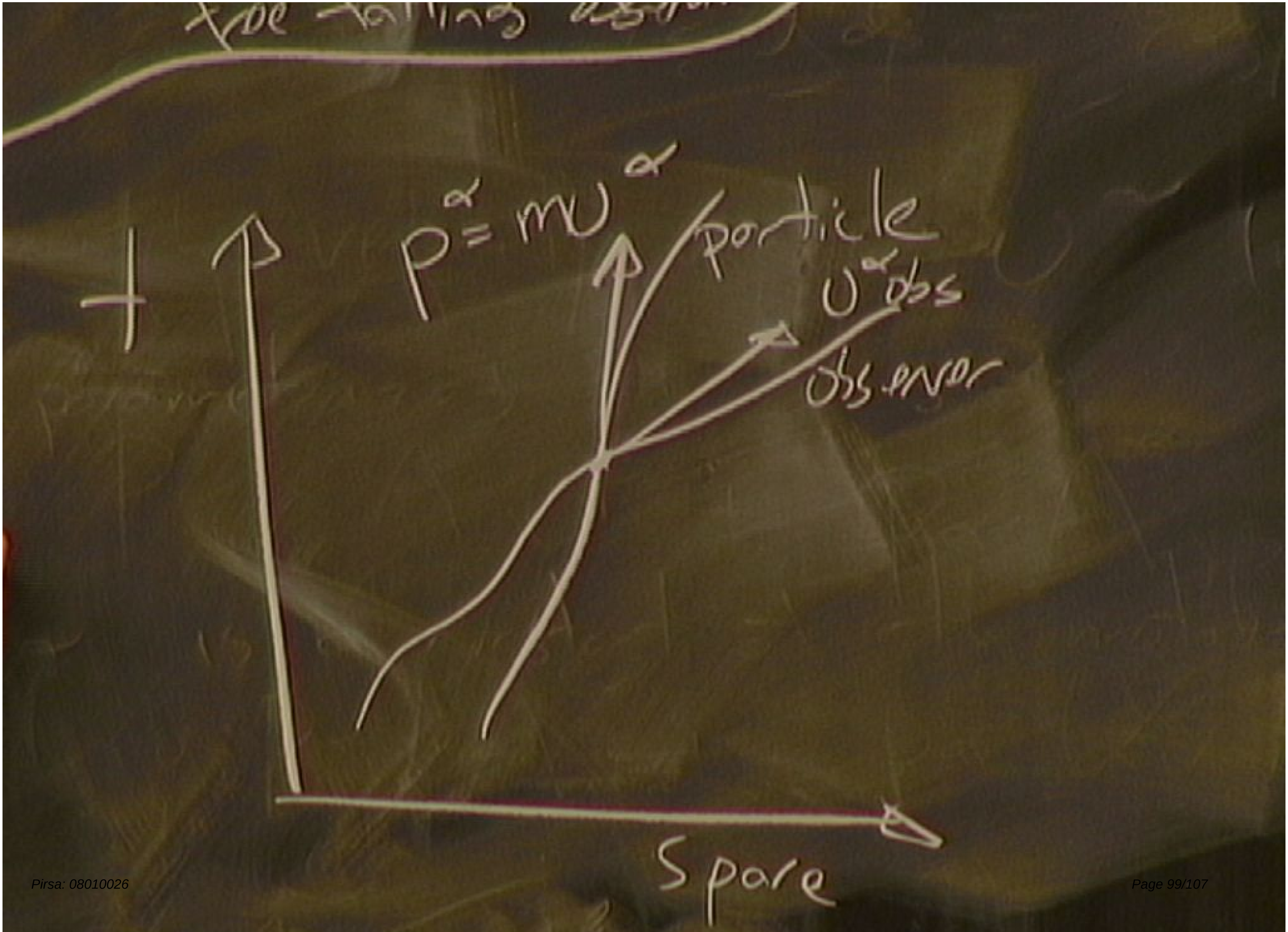
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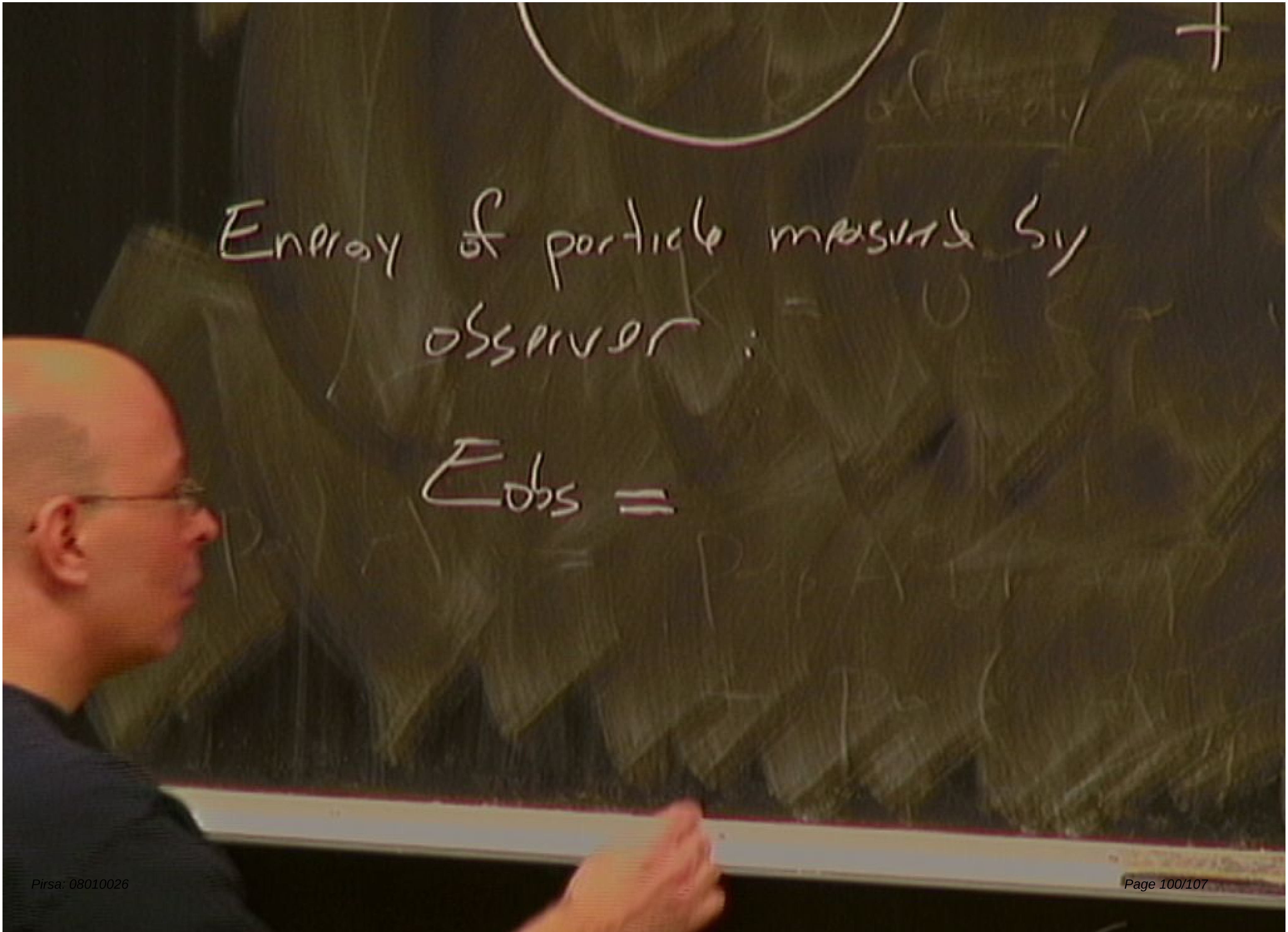
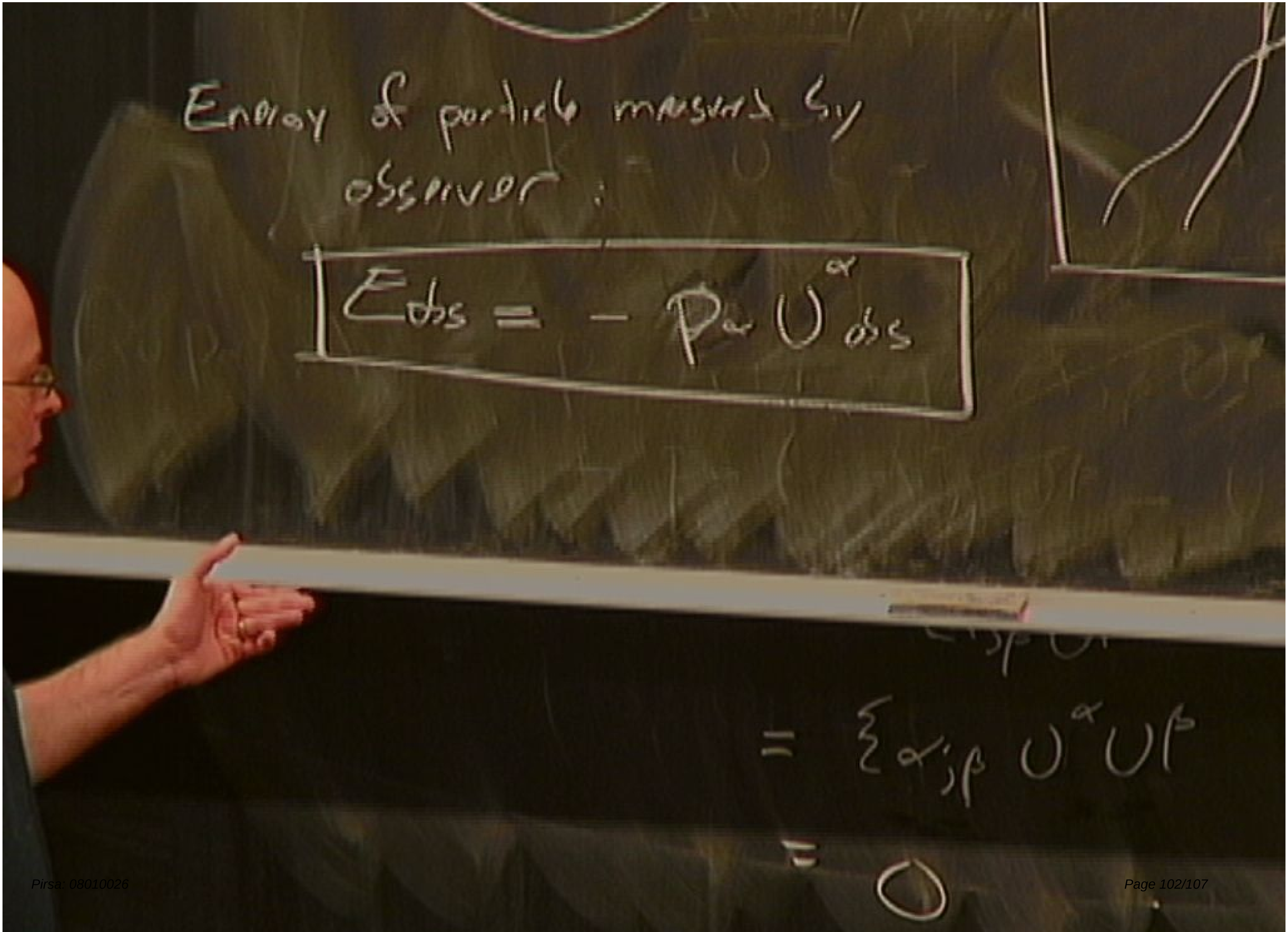

Energy of particle measured by
observer :
$E_{obs} =$

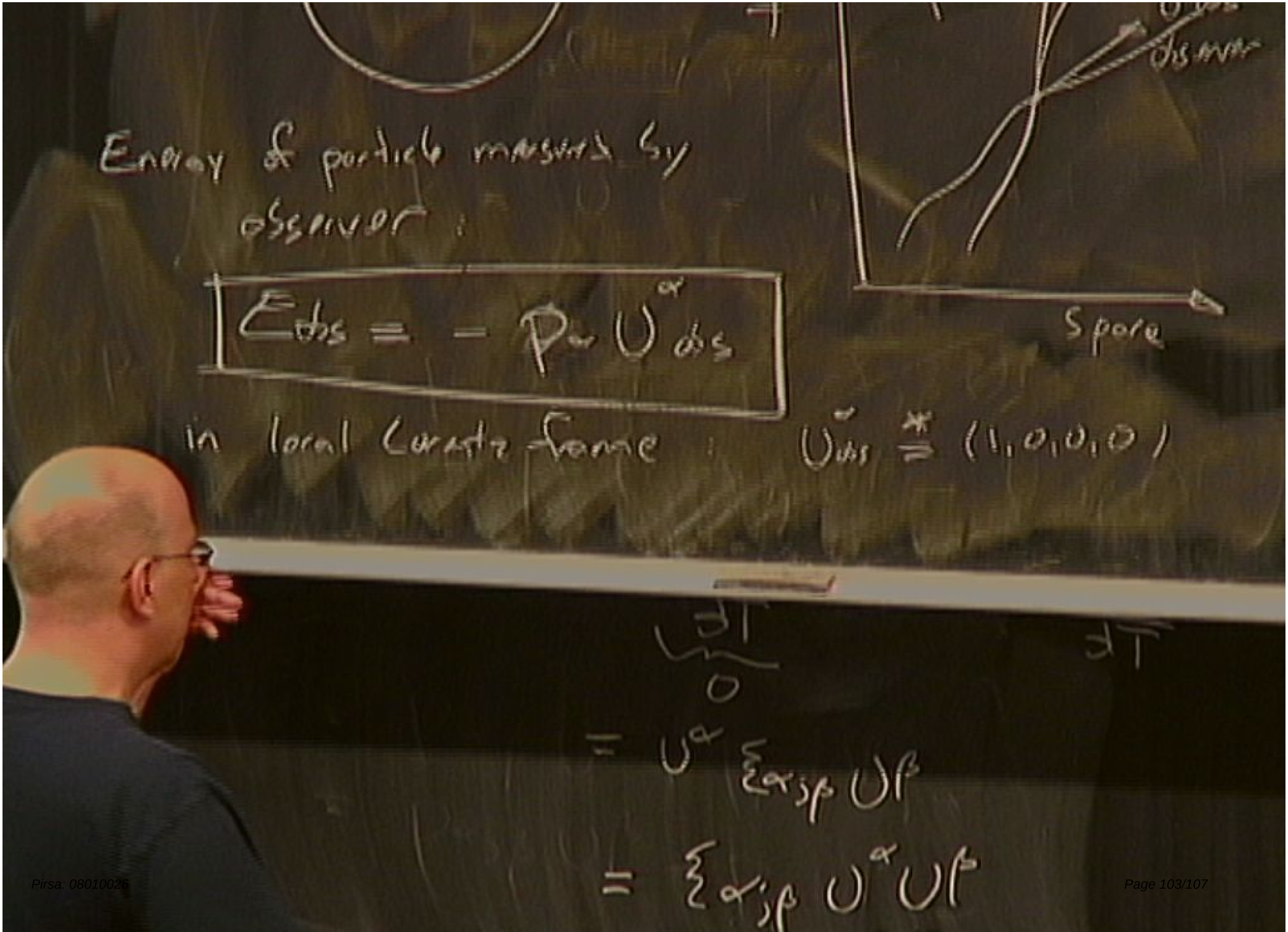
Energy of particle measured by observer:

$$E_{obs} = - p_\alpha U^\alpha_{obs}$$

$$= \xi_{\alpha;\beta} U^\alpha U^\beta$$

$$= 0$$


Energy of particle measured by
observer :
$E_{obs} = - p_\alpha U^\alpha_{obs}$
$= \xi_{\alpha;\beta} U^\alpha U^\beta$
$= 0$


observer
Energy of particle measured by
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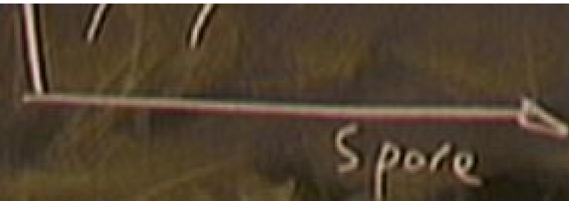
Space
in local Lorentz frame :
$U^\alpha_{obs} \doteq (1,0,0,0)$
$= U^\alpha \, \xi_{\alpha;\beta} U^\beta$
$= \xi_{\alpha;\beta} U^\alpha U^\beta$

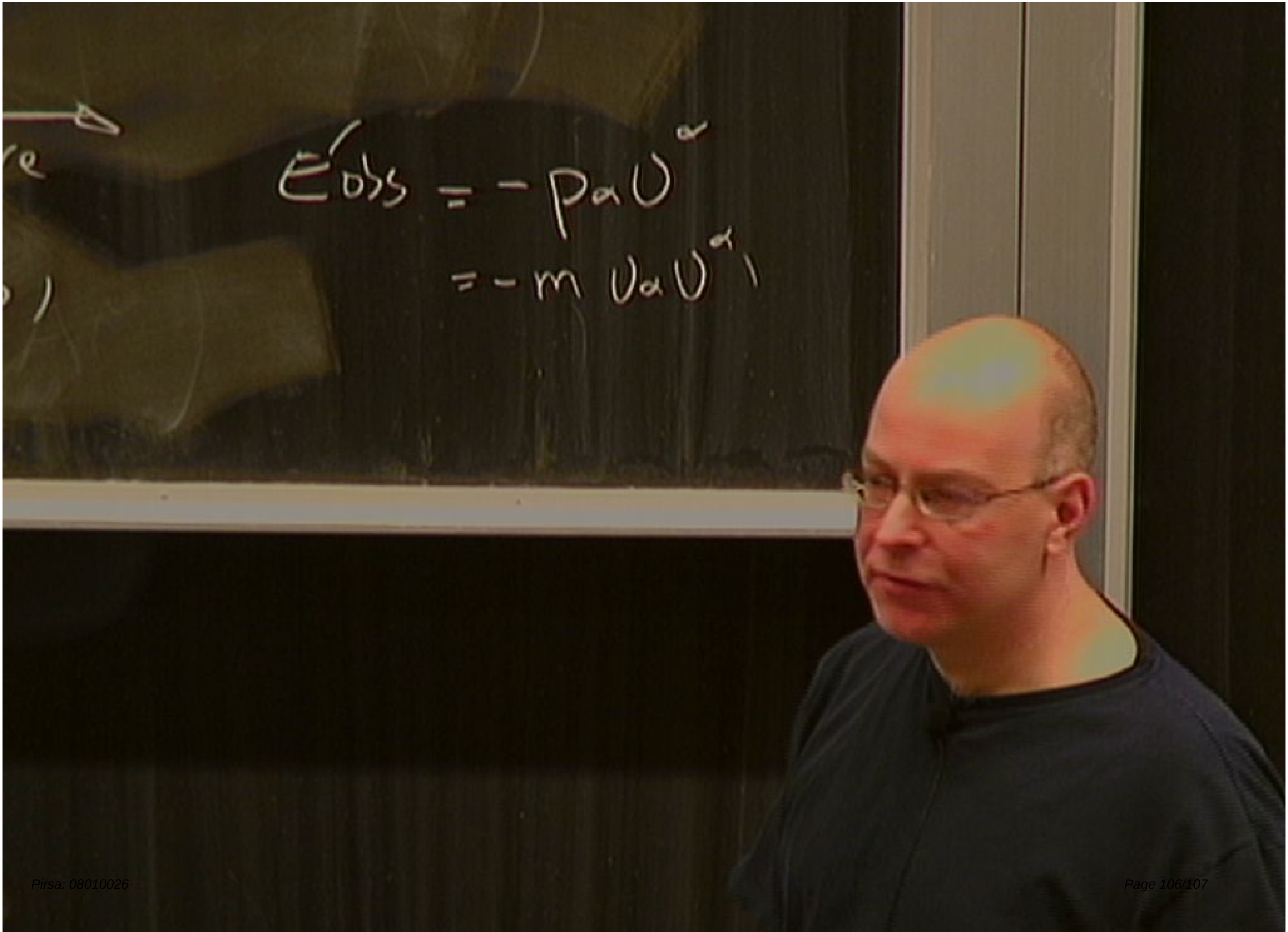
$$\Rightarrow E_{dis} \overset{*}{=} -P_+ \overset{*}{=} P^+$$

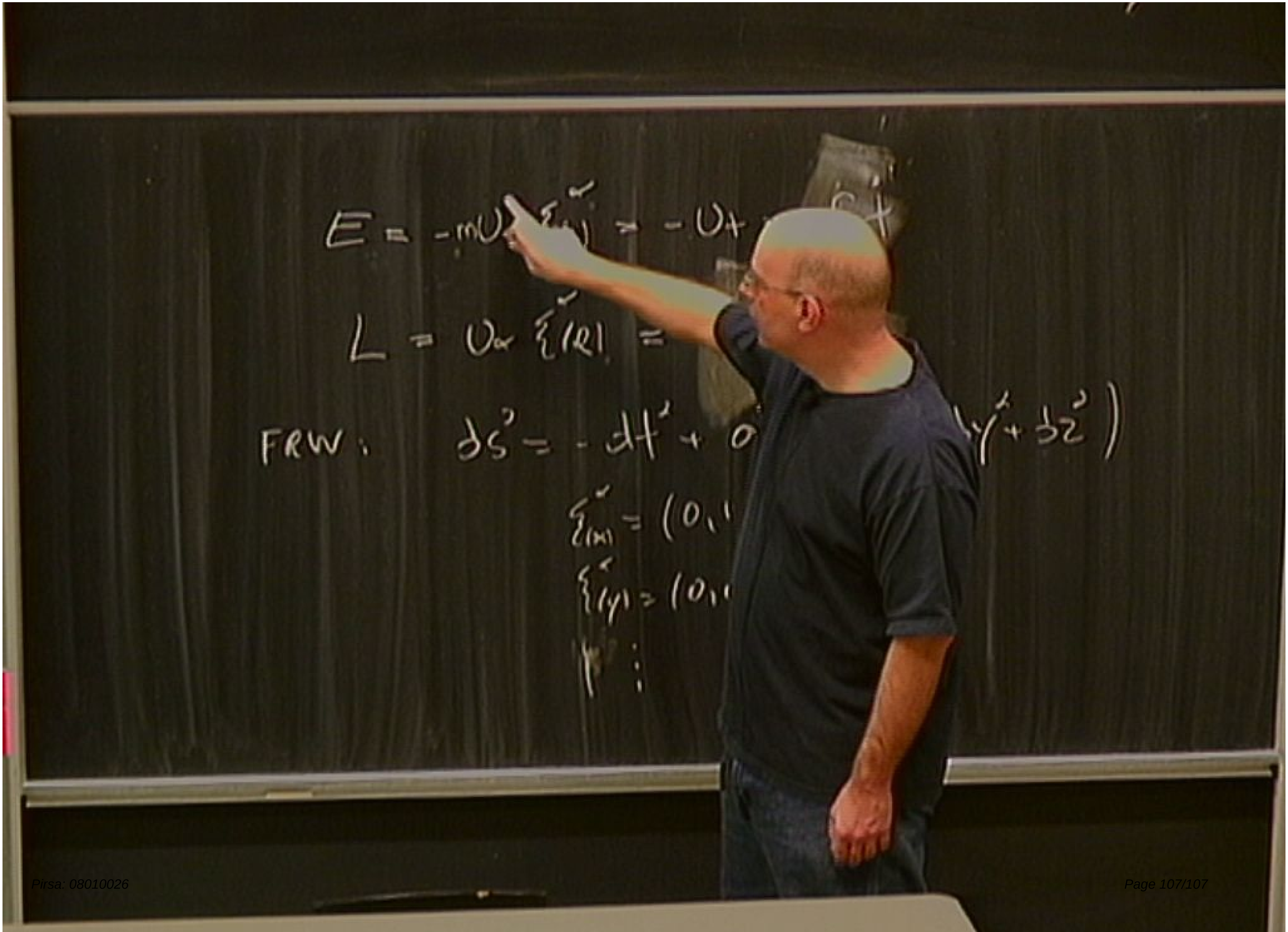
observer:

$$\boxed{E_{obs} = - p_\alpha U^\alpha_{obs}}$$



in local Lorentz frame: $U^\alpha_{obs} \stackrel{*}{=} (1,0,0,0)$


$E_{obs} = -p_a U^a$
$= -m U_a U^a$,


FRW :