Title: Advanced General Relativity - Lecture 1

Date: Jan 09, 2008 10:30 AM

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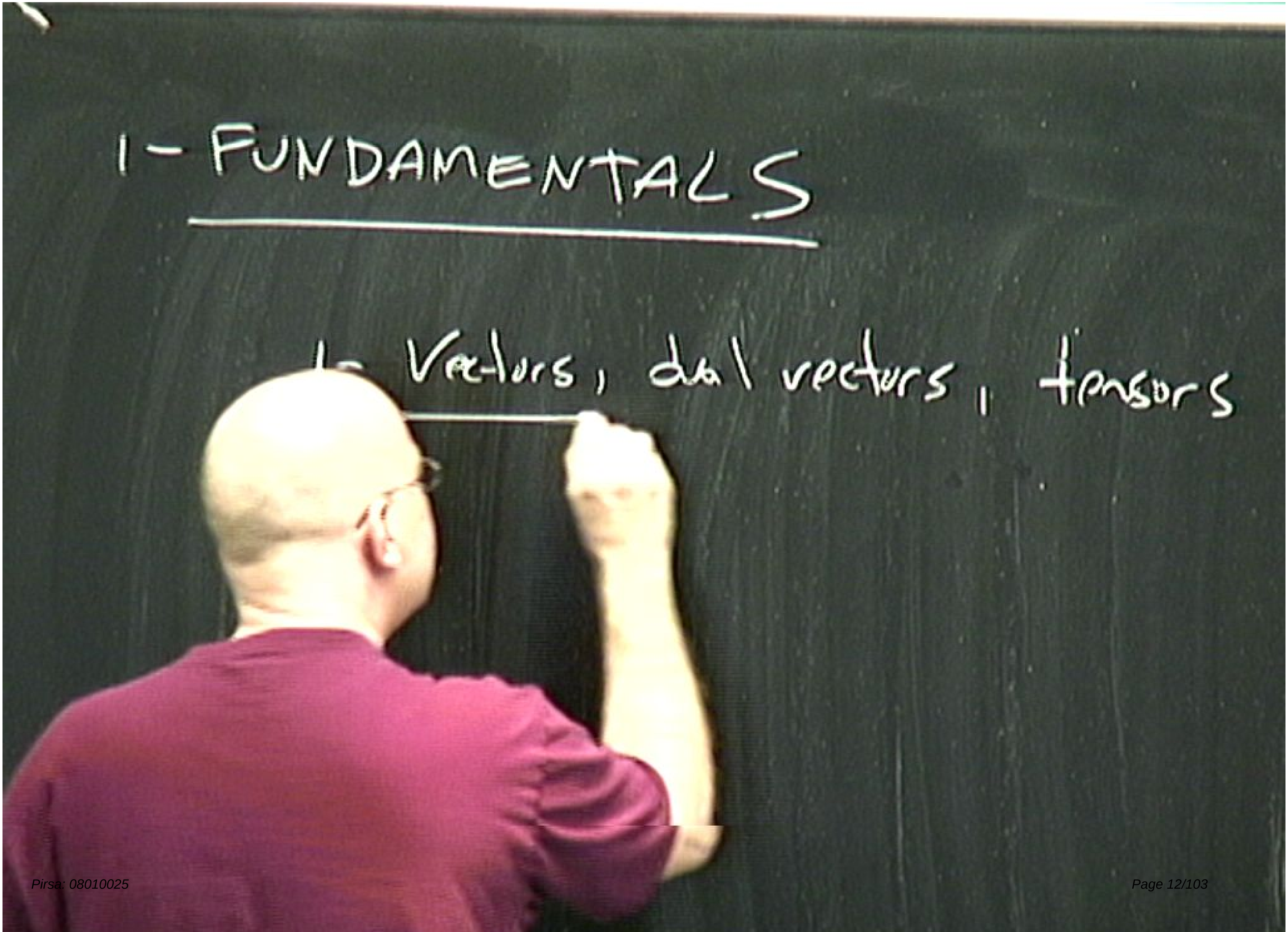
Abstract: Advanced General Relativity

1- FUNDAMENTALS


1- FUNDAMENTALS
1- Vectors, dual vectors, tensors

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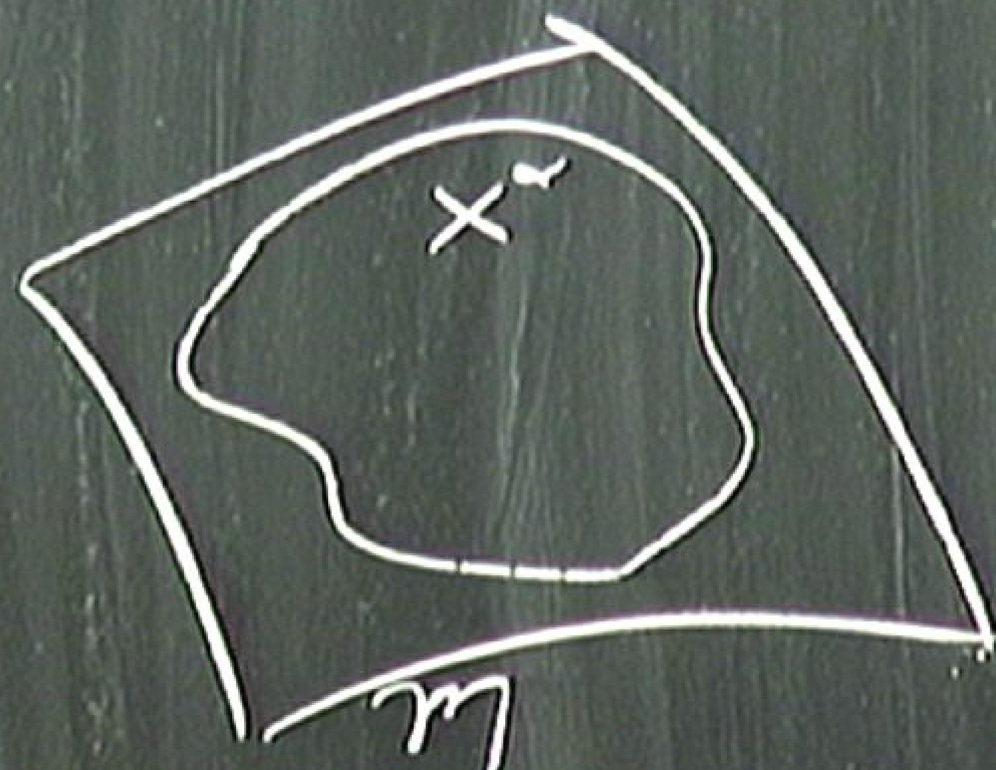
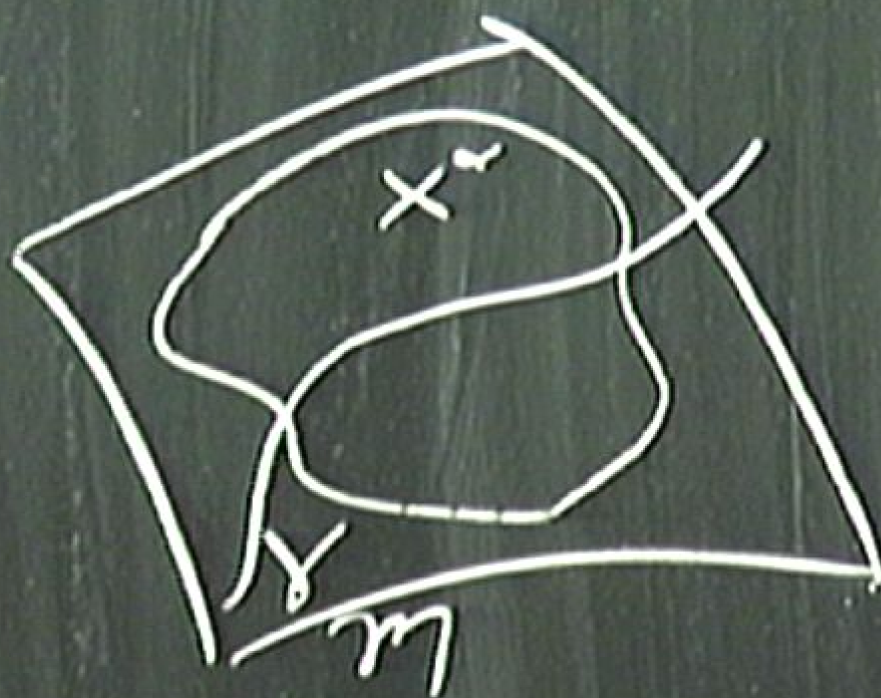
1- Vectors, dual vectors, tensors



1- FUNDAMENTALS

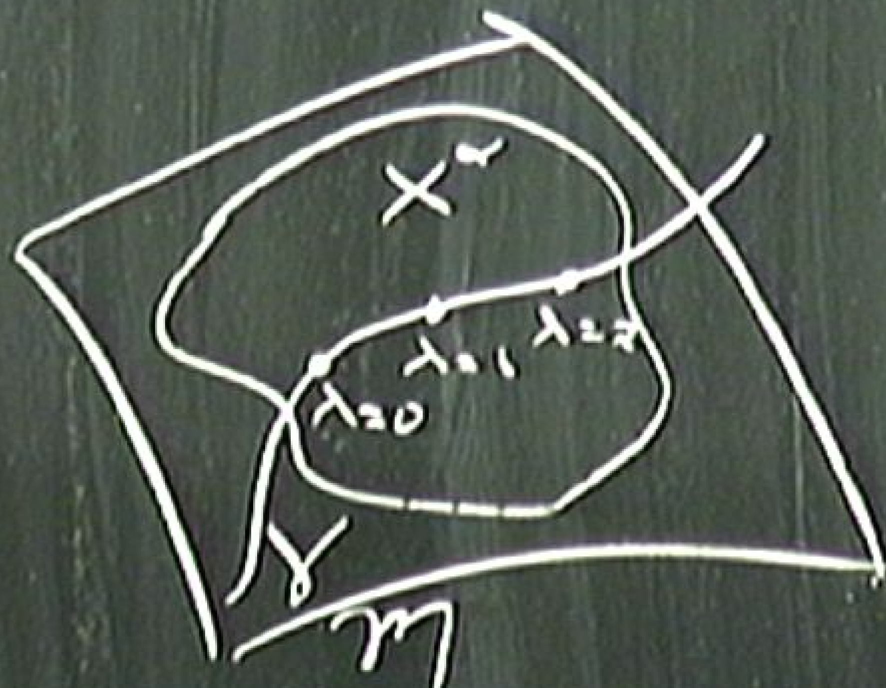
1- Vectors, dual vectors, tensors



$f(x^\alpha)$

1 - FUNDAMENTALS

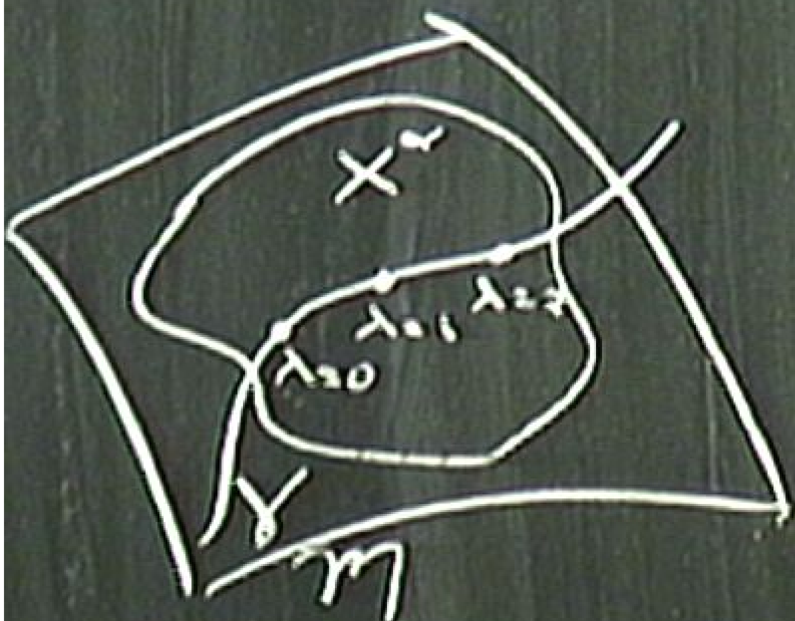
1 - Vectors, dual vectors, tensors



$f(x^\alpha)$

curve : γ :

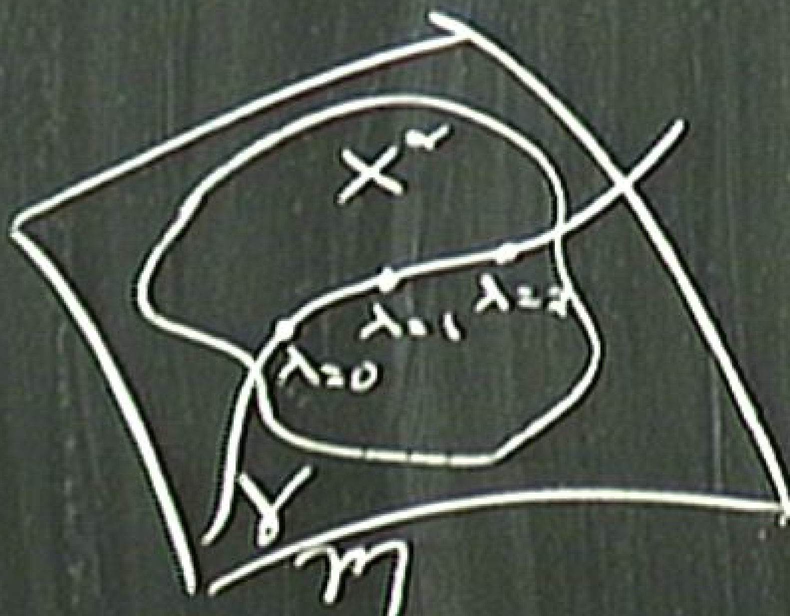
Vectors, dual vectors, tensors



$f(x^\alpha)$

curve : γ : $x^\alpha = z^\alpha(\lambda)$

1- Vectors, dual vectors, tensors

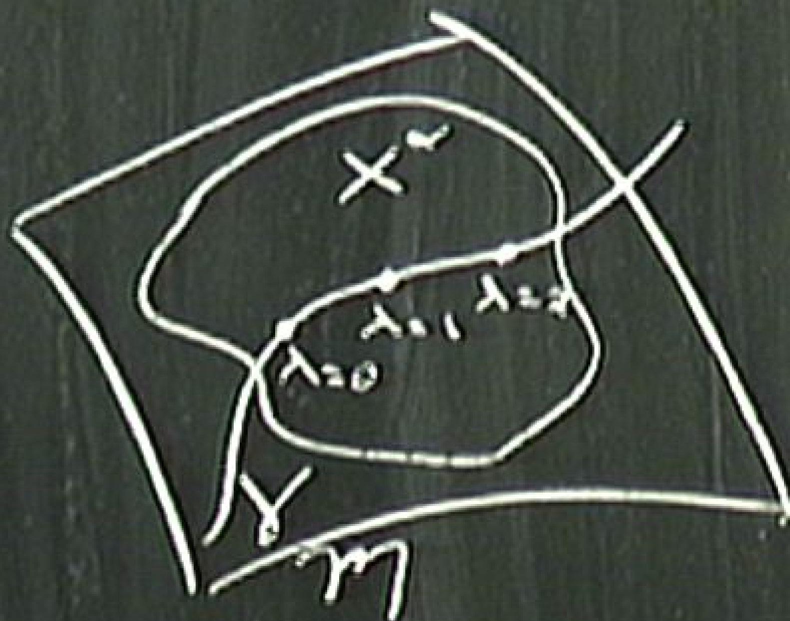


$$f(x^\alpha)$$

curve : $\gamma : \; x^\alpha = z^\alpha(\lambda)$

$$\frac{df}{d\lambda}$$

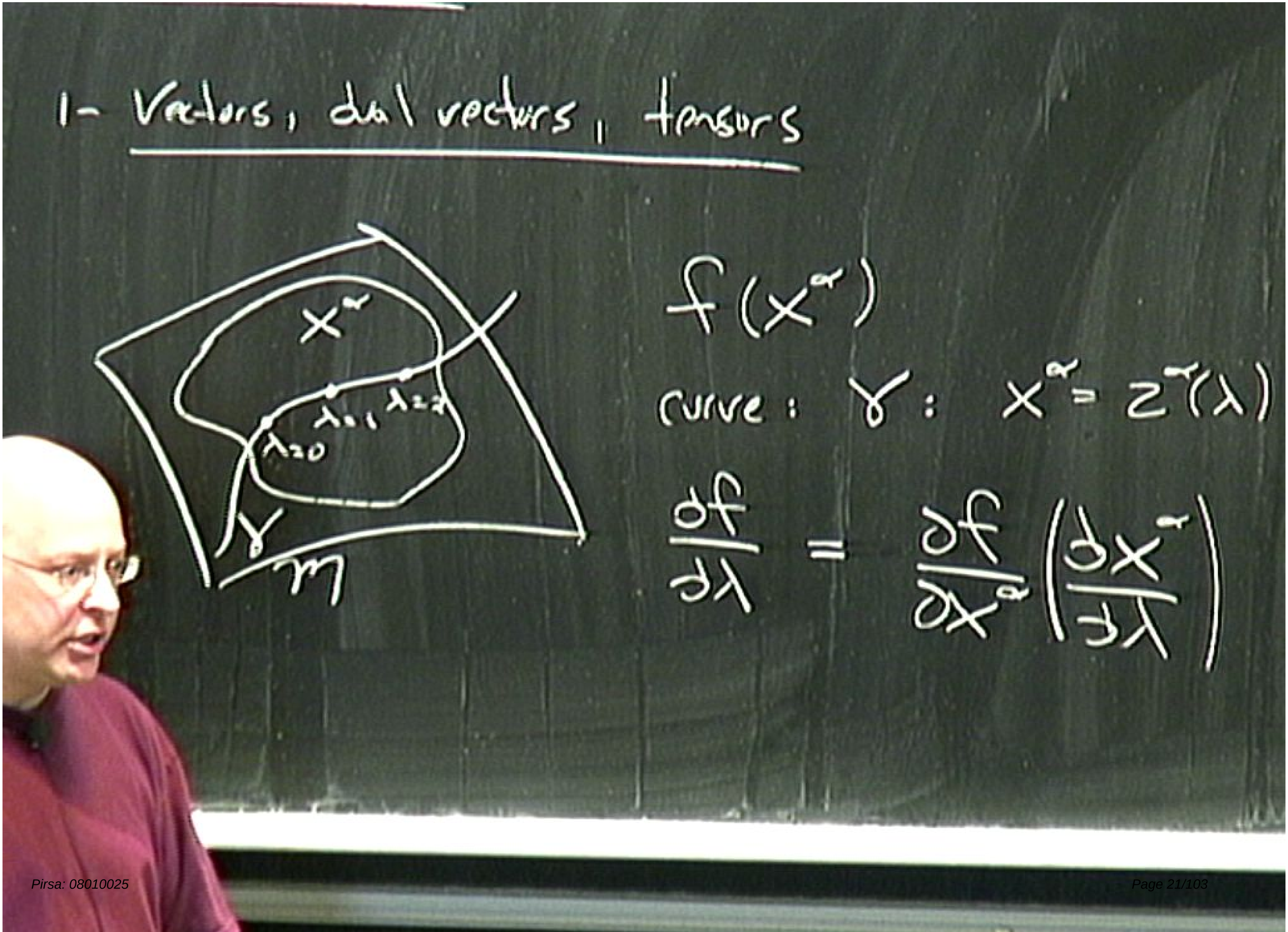
1- Vectors, dual vectors, tensors



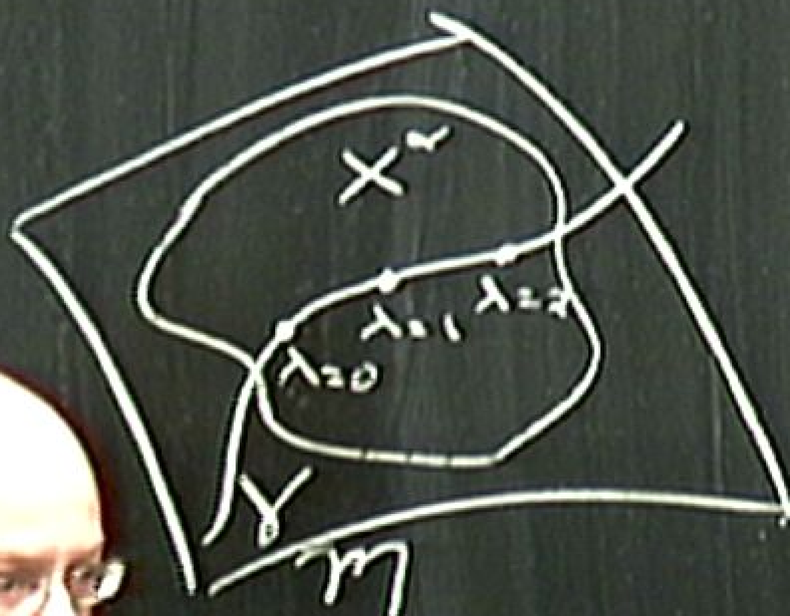
$f(x^\alpha)$

curve : γ : $x^\alpha = z^\alpha(\lambda)$

$$\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \lambda}$$


1- Vectors, dual vectors, tensors
x^α
$\lambda=0$
$\lambda=1$
$\lambda=2$
γ
$\mathcal{M}$
$f(x^\alpha)$
curve : γ : $x^\alpha = z^\alpha(\lambda)$
$\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial \lambda}\right)$

1- Vectors, dual vectors, tensors



$$f(x^\alpha)$$

curve : $\gamma : \; x^\alpha = z^\alpha(\lambda)$

$$\frac{\partial f}{\partial \lambda} = \left(\frac{\partial f}{\partial x^\alpha}\right)\left(\frac{\partial x^\alpha}{\partial \lambda}\right)$$

tangent vector : $U^{\alpha} =$

tangent vector : $u^\alpha = \dfrac{\partial z^\alpha}{\partial \lambda}$ (vector)

gradient dual vector

tangent vector: $u^\alpha = \frac{\partial z^\alpha}{\partial \lambda}$ (vector)

gradient dual vector:

$$f_{,\alpha} = \partial_\alpha f = \nabla_\alpha f = \frac{\partial f}{\partial x^\alpha}$$

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gradient dual vector:

$$f_{,\alpha} = \partial_{\alpha} f = \nabla_{\alpha} f = \frac{\partial f}{\partial x^{\alpha}}$$ (dual vector)

tangent vector: $u^\alpha = \frac{\partial z^\alpha}{\partial \lambda}$ (vector)

gradient dual vector:

$$f_{,\alpha} = \partial_\alpha f = \nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} \quad \text{(dual vector)}$$

scalar: $\frac{\partial f}{\partial \lambda} = f_{,\alpha} u^\alpha$

$f(x^\mu)$

curve: γ: $x^\mu = z^\mu(\lambda)$

$$\frac{df}{d\lambda} = \left(\frac{\partial f}{\partial x^\mu}\right)\left(\frac{dx^\mu}{d\lambda}\right)$$

tangent vector: $U^\mu = \frac{dz^\mu}{d\lambda}$ (vector)

gradient dual vector:

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gradient dual vector:

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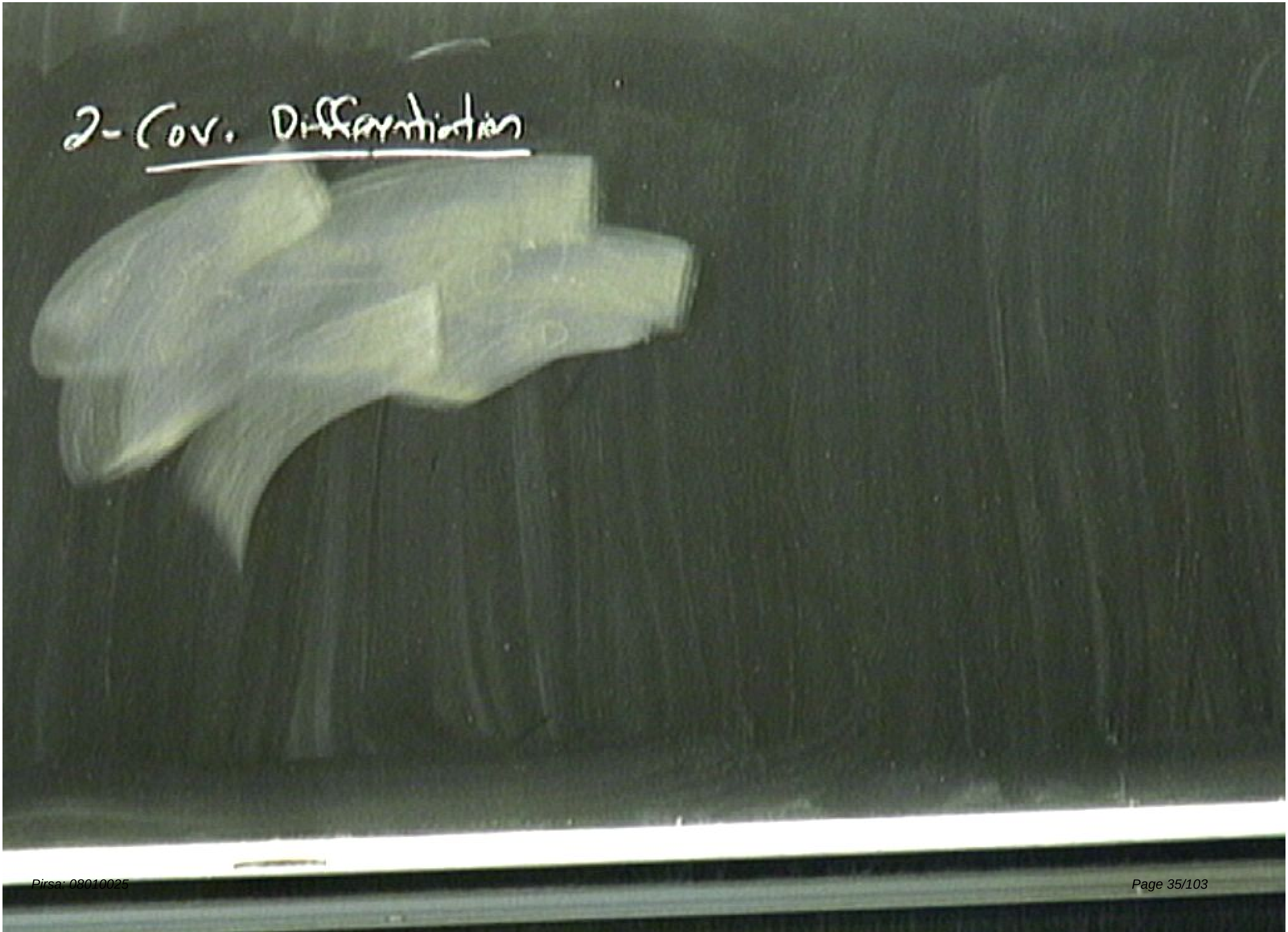
scalar: $\frac{\partial f}{\partial \lambda} = f_{,\alpha} u^{\alpha}$

$$U^{\alpha} \xrightarrow{\text{Wald}} U^{a}$$

$$U^a \longrightarrow U^a$$

4D


2-Cov. Differentiation

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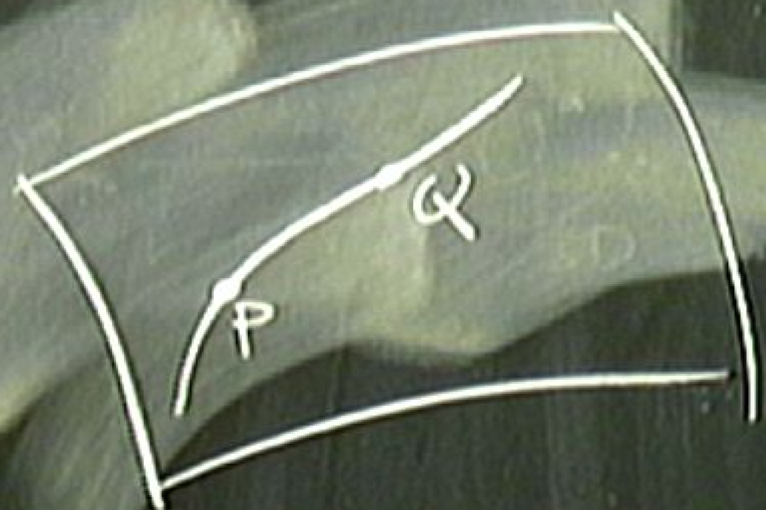
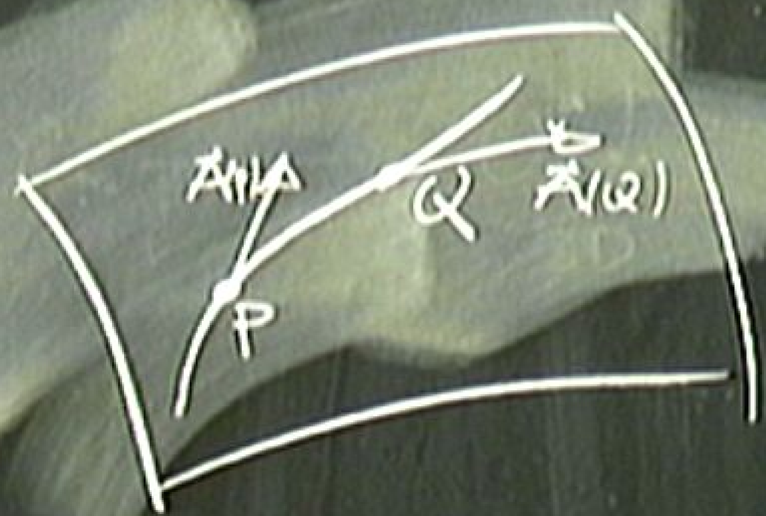
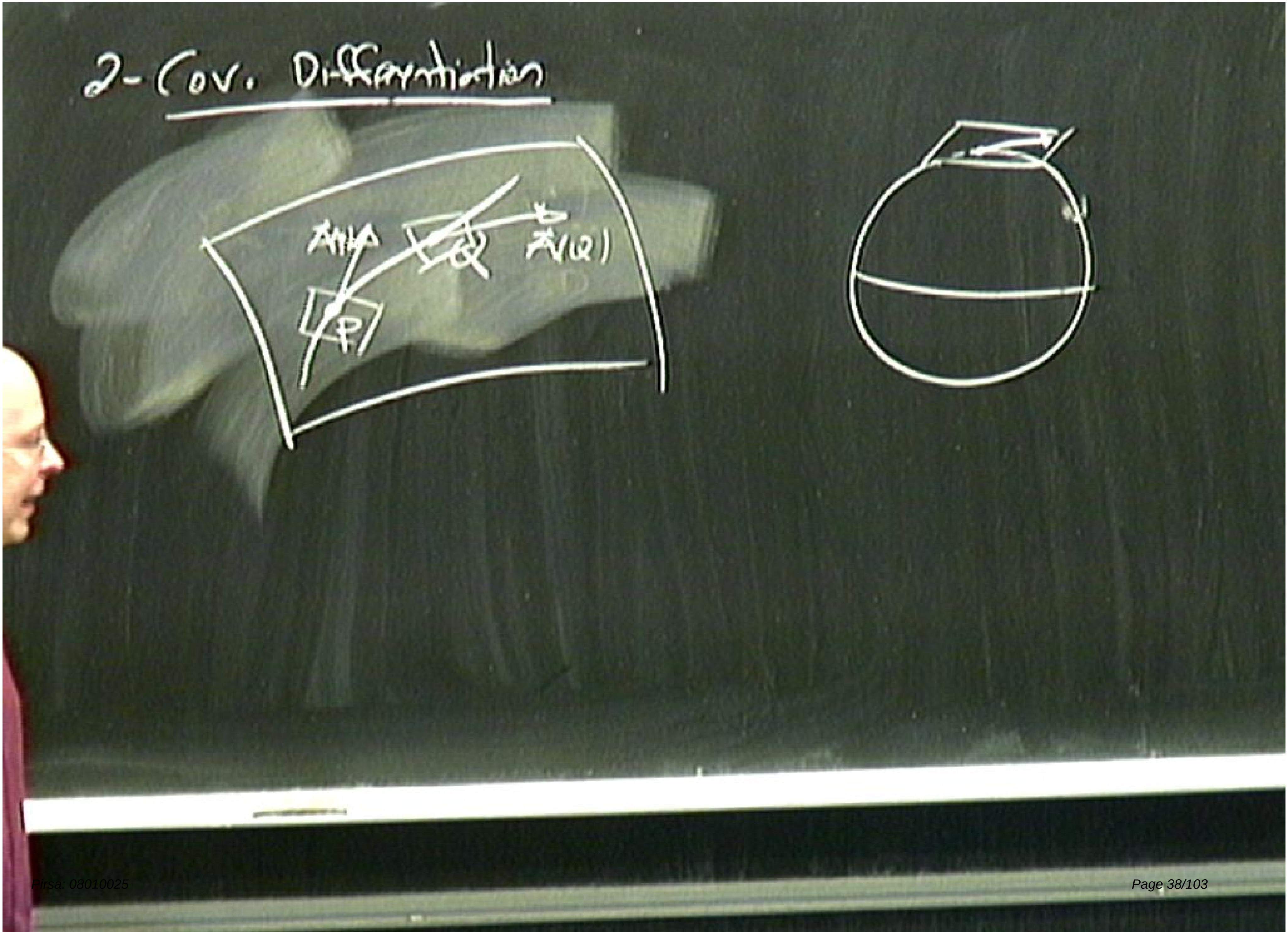


2 - Cov. Differentiation




2- Cov. Differentiation
P
Q
V(Q)

Vectors at P and Q cannot be added or subtracted

→ need a rule to transport

$$A^{\alpha}(Q) \longrightarrow A^{\alpha}_{T}(P)$$

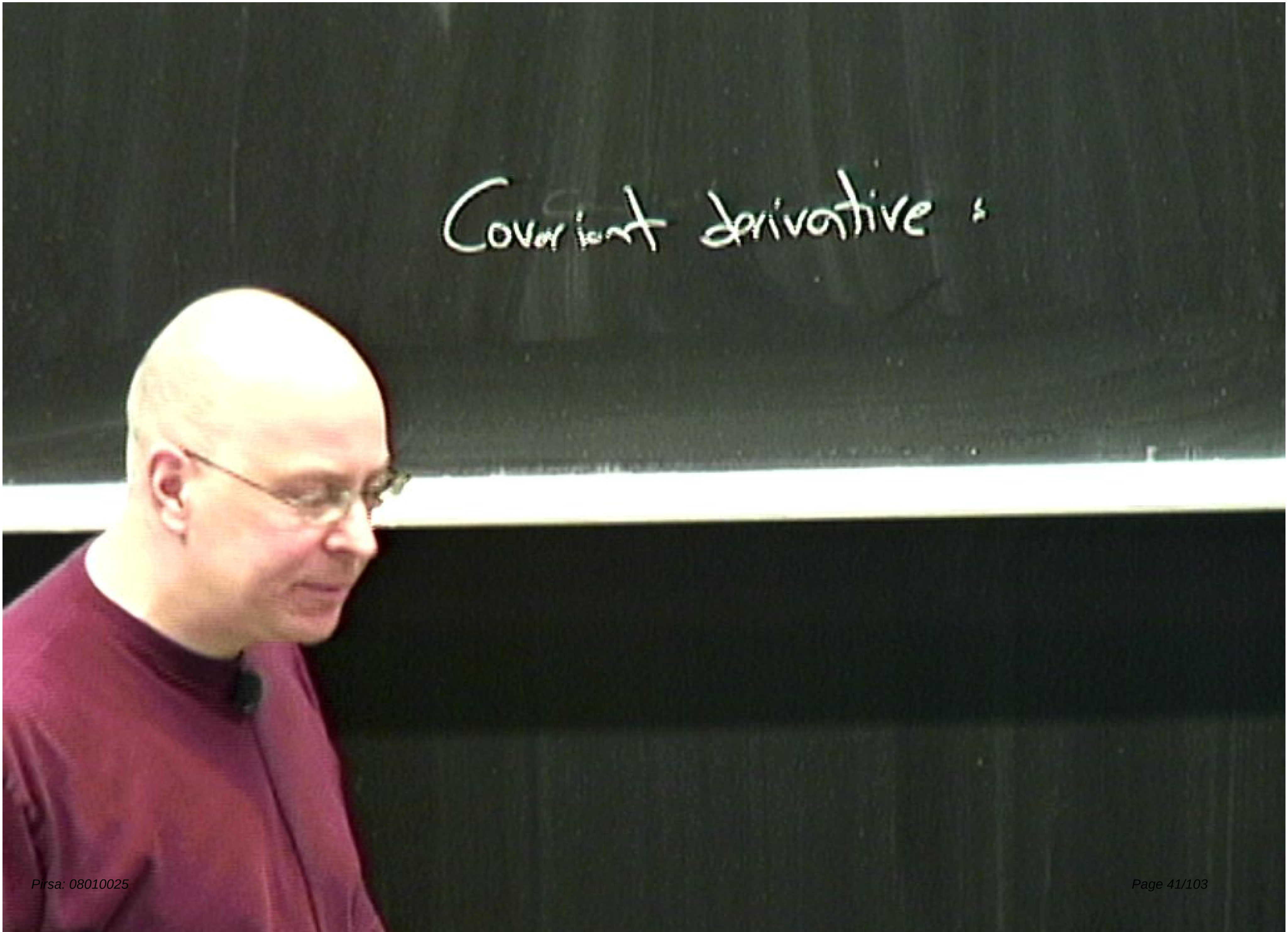
Vectors at P and Q cannot be added or substracted

→ need a <u>rule</u> to transport

$$A^{\alpha}(Q) \longrightarrow A^{\alpha}_{T}(P)$$

→ parallel transport

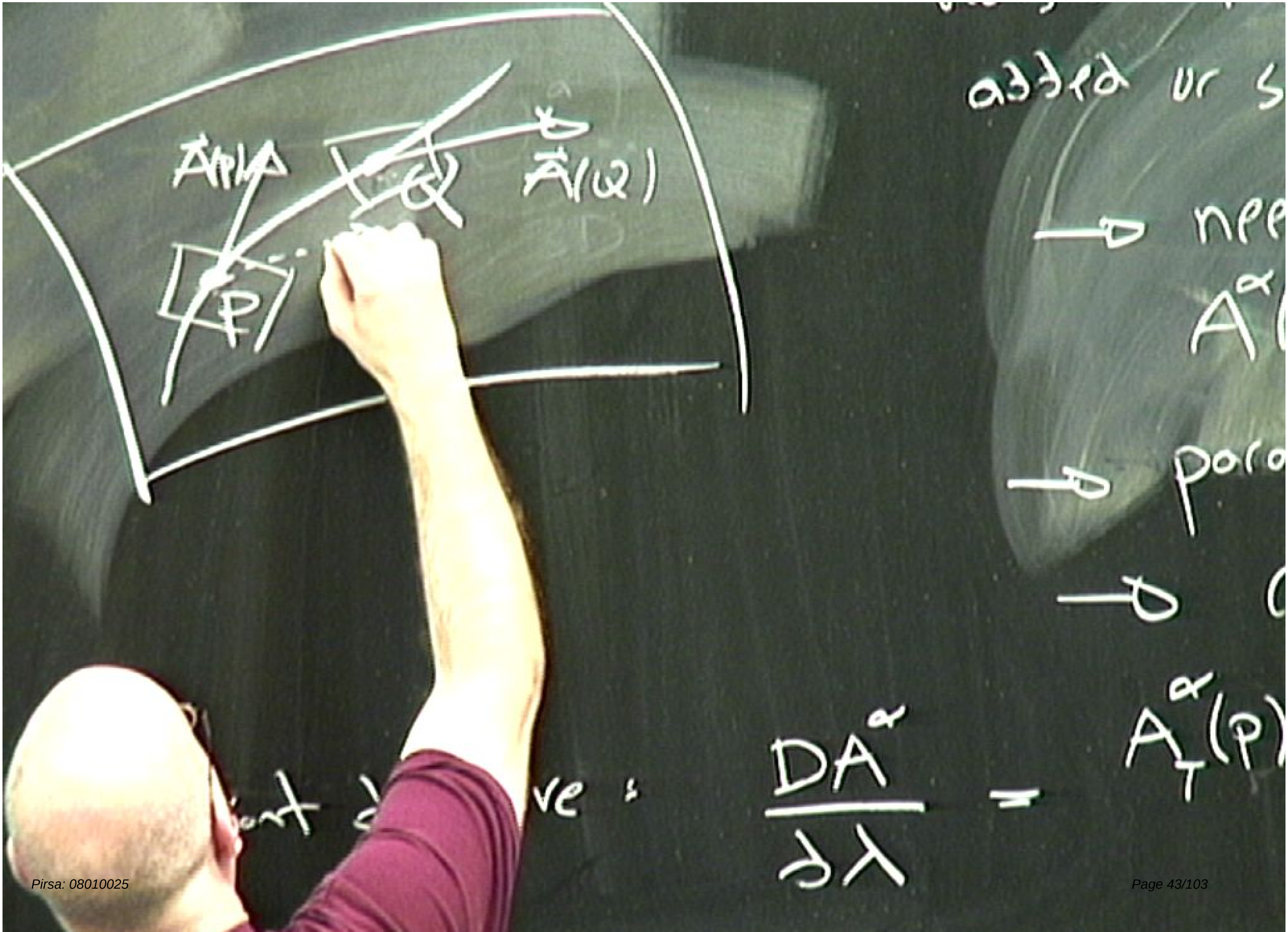
→ connection

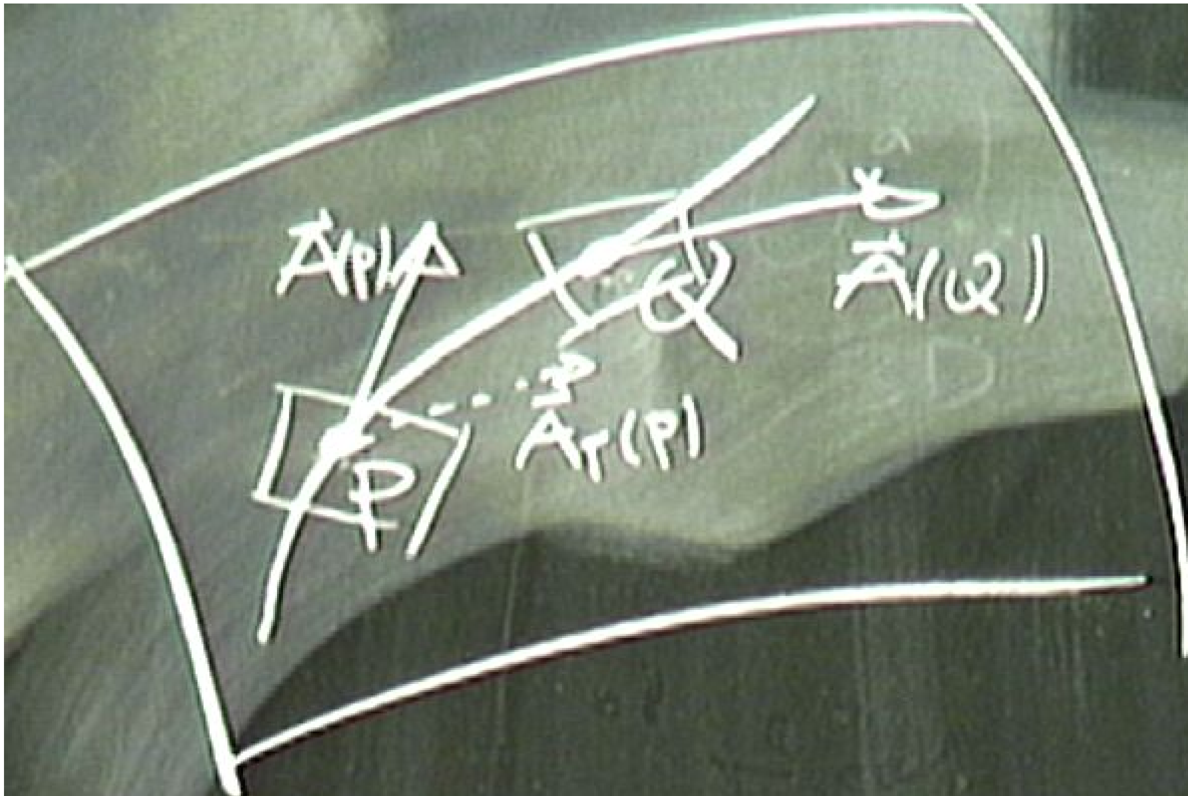

Covariant derivative :

→ paralle

→ con

Covariant derivative : D


$\vec{A}(P)$
$\vec{A}(Q)$
Q
P
added or s
nee
$A^\alpha($
para
ve :
$\frac{DA^\alpha}{d\lambda} =$
$A^\alpha_T(p)$



added or s

→ nee

$A^\alpha($

→ para

→ (

Covariant derivative : $\dfrac{DA^\alpha}{d\lambda} = A_T^\alpha(p)$

$r(p)$

$A^\alpha(Q) \longrightarrow A$

$\longrightarrow$ parallel transport

$\longrightarrow$ connection

derivative :
$$\frac{DA^\alpha}{d\lambda} = \frac{A^\alpha_I(p) - A^\alpha(p)}{d\lambda}$$

$$\frac{DA^{\alpha}}{d\lambda}$$

$$\frac{DA^{\alpha}}{d\lambda} = A^{\alpha}{}_{;\beta} U^{\beta}$$

$$\frac{DA^{\alpha}}{d\lambda} = A^{\alpha}{}_{;\beta}\, U^{\beta} = \left(\nabla_{\beta} A^{\alpha}\right) U^{\beta}$$

$$\frac{DA^\alpha}{d\lambda} = A^\alpha{}_{;\beta}\, U^\beta \equiv \left(\nabla_\beta A^\alpha\right) U^\beta$$

$$= \left(A^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^\gamma\right) U^\beta$$

$$\frac{DA^\alpha}{d\lambda} = A^\alpha{}_{;\beta} u^\beta = (\nabla_\beta A^\alpha) u^\beta$$

$$= \left(A^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^\gamma\right) u^\beta$$

$$\frac{DA^\alpha}{d\lambda} = A^\alpha{}_{;\beta}\, u^\beta = \left(\nabla_\beta A^\alpha\right) u^\beta$$

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connection

$$\frac{DA^{\alpha}}{d\lambda} = A^{\alpha}{}_{;\beta} u^{\beta} = (\nabla_{\beta} A^{\alpha}) u^{\beta}$$

$$= \left(A^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\beta\gamma} A^{\gamma}\right) u^{\beta}$$

... ction

Definition can be extending

to dual vectors and tensors by im... Leib...

$$\frac{DA^\alpha}{d\lambda} = \left(A^\alpha{}_{;\beta}\right) U^\beta = \left(\nabla_\beta A^\alpha\right) U^\beta$$

$$= \left(A^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^\gamma\right) U^\beta$$

connection

Definition can be extended
to dual vectors and tensors by imposing Leibnitz rule.

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$$= \left(A^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^\gamma\right) U^\beta$$

connection

Definition can be extended to dual vectors and tensors by imposing Leibnitz rule.

$$D_\alpha f = \partial_\alpha f$$

$$\frac{DA^{\alpha}}{d\lambda} = \left(A^{\alpha}{}_{;\beta}\right)U^{\beta} = \left(\nabla_{\beta}A^{\alpha}\right)U^{\beta}$$

$$= \left(A^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\beta\gamma}A^{\gamma}\right)U^{\beta}$$

connection

Definition can be extended to dual vectors and tensors by imposing Leibnitz rule.

$$D_{\alpha} f = \partial_{\alpha} f$$

$$D_{\alpha}\left(p_{\beta}A^{\beta}\right)$$

$$\frac{DA^\alpha}{d\lambda} = A^\alpha{}_{;\beta} U^\beta = (\nabla_\beta A^\alpha) U^\beta$$

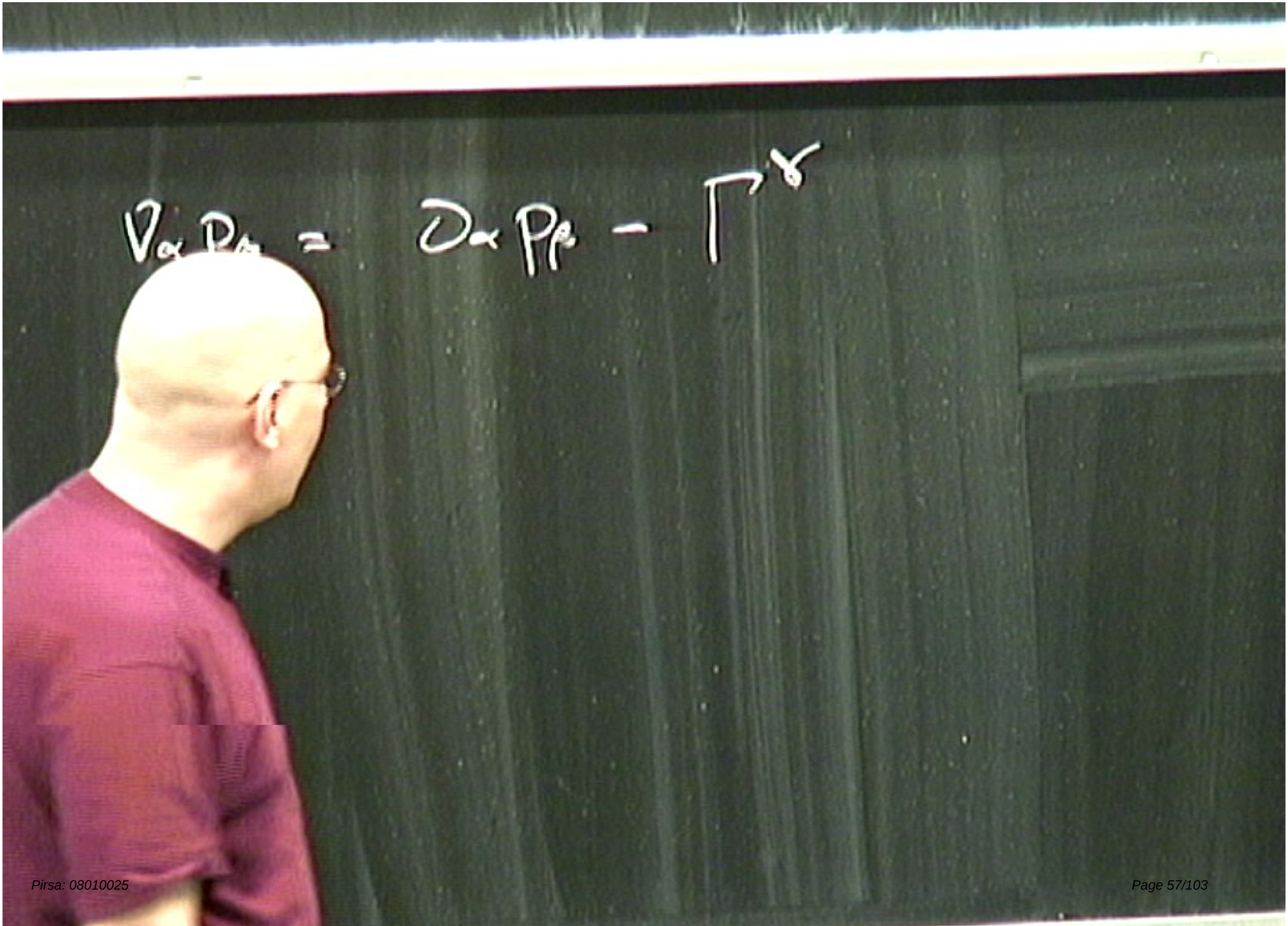
$$= \left(A^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^\gamma\right) U^\beta$$

Definition can be extended ... nection

to dual vectors and tensors by imposing ... bnitz rule.

$$\nabla_\alpha f = \partial_\alpha f$$

$$\nabla_\alpha (p_\beta A^\beta) = (\nabla_\alpha p_\beta) A^\beta + p_\beta (\nabla_\alpha A^\beta)$$


$\nabla_\alpha p_\beta = \partial_\alpha p_\beta - \Gamma^\gamma$

$$\nabla_\alpha p_\beta = \partial_\alpha p_\beta - \Gamma^\gamma_{\alpha\beta}\, p_\gamma$$

$$\nabla_\alpha p_\beta = \partial_\alpha p_\beta - \Gamma^\gamma{}_{\alpha\beta} \, p_\gamma$$


metric

metric

$$g_{\alpha\beta}$$

Metric

$$g_{\alpha\beta} \longrightarrow g^{\alpha\beta} \ : \ g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

$$g_{\alpha\beta} \longrightarrow g^{\alpha\beta} \; : \; g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

$$g = \det[g_{\alpha\beta}]$$

$$g_{\alpha\beta} \to g^{\alpha\beta} \; : \; g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

$$g = \det[g_{\alpha\beta}]$$

map vectors $\leftrightarrow$ dual vectors :

$$A^{\alpha} \to A_{\alpha}$$

$$g_{\alpha\beta} \longrightarrow g^{\alpha\beta} \;:\; g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

$$g = \det[g_{\alpha\beta}]$$

map vectors $\longleftrightarrow$ dual vectors :

$$A^{\alpha} \longrightarrow A_{\alpha} = g_{\alpha\beta} A^{\beta}$$

$$g_{\alpha\beta} \rightarrow g^{\alpha\beta} : g^{\alpha\rho} g_{\rho\gamma} = \delta^\alpha{}_\gamma$$

$$g = \det[g_{\alpha\beta}]$$

map vectors $\leftrightarrow$ dual vectors :

$$A^\alpha \rightarrow A_\alpha = g_{\alpha\beta} A^\beta$$

Christoffel connection: metric compatible.

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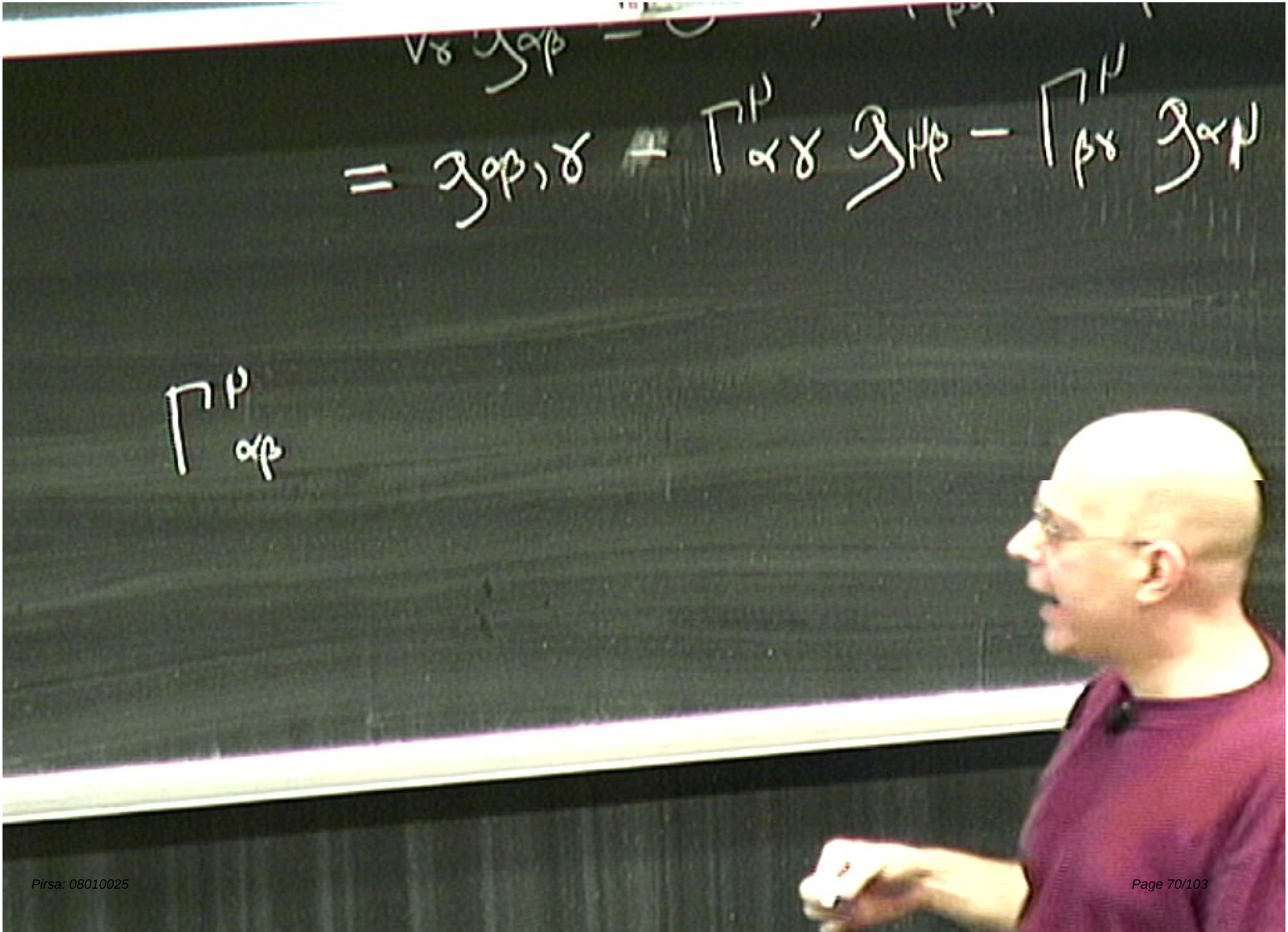
$$\nabla_\gamma g_{\alpha\beta} = 0$$

$$= g_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma}\, g_{\mu\beta} - \Gamma^\mu$$

Christoffel connection: metric compatible.

$$\nabla_\gamma g_{\alpha\beta} = 0$$

$$= g_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma} g_{\mu\beta} - \Gamma^\mu_{\beta\gamma} g_{\alpha\mu}$$


$\nabla_\gamma g_{\alpha\beta} = 0$
$= g_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma} g_{\mu\beta} - \Gamma^\mu_{\beta\gamma} g_{\alpha\mu}$
$\Gamma^\mu_{\alpha\beta}$

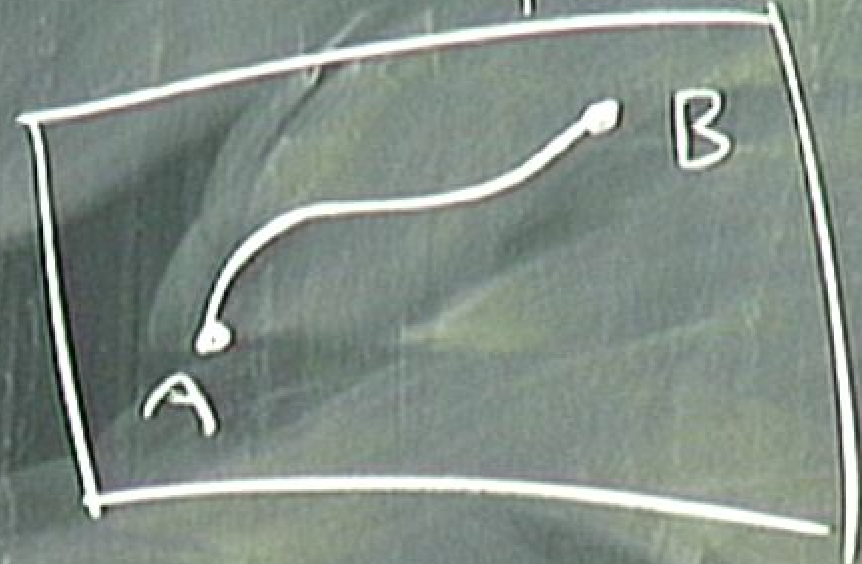
$$= g_{\alpha\beta,\gamma} - \Gamma^{\mu}_{\alpha\gamma}\, g_{\mu\beta} - \Gamma^{\mu}_{\beta\gamma}\, g_{\alpha\mu}$$

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right)$$

3- Geodesics

3- Geodesics

↳ path of extremum length.



Geodesics

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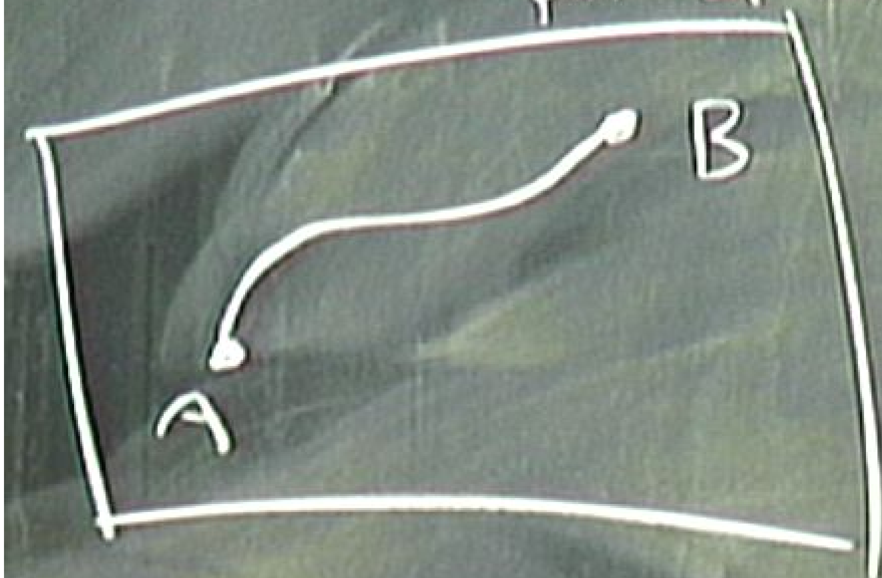
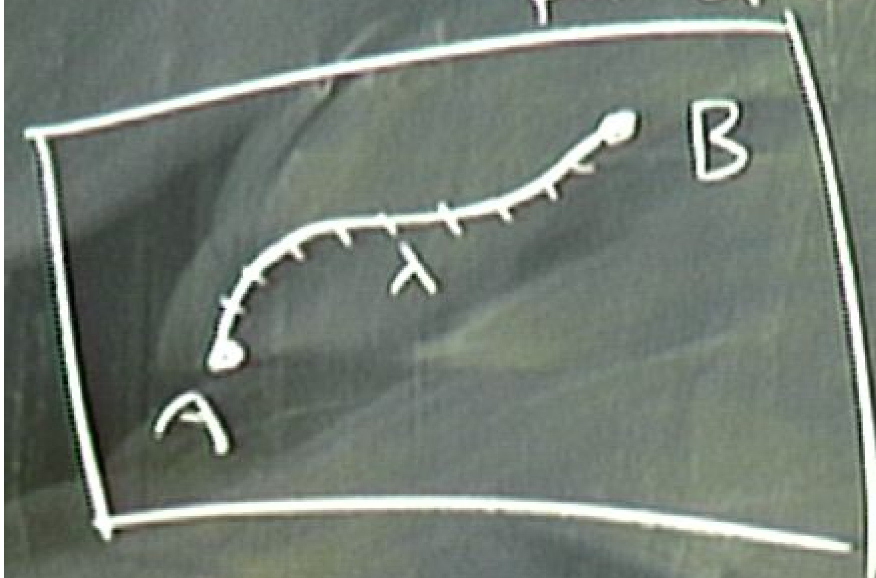


Timelike geodesics

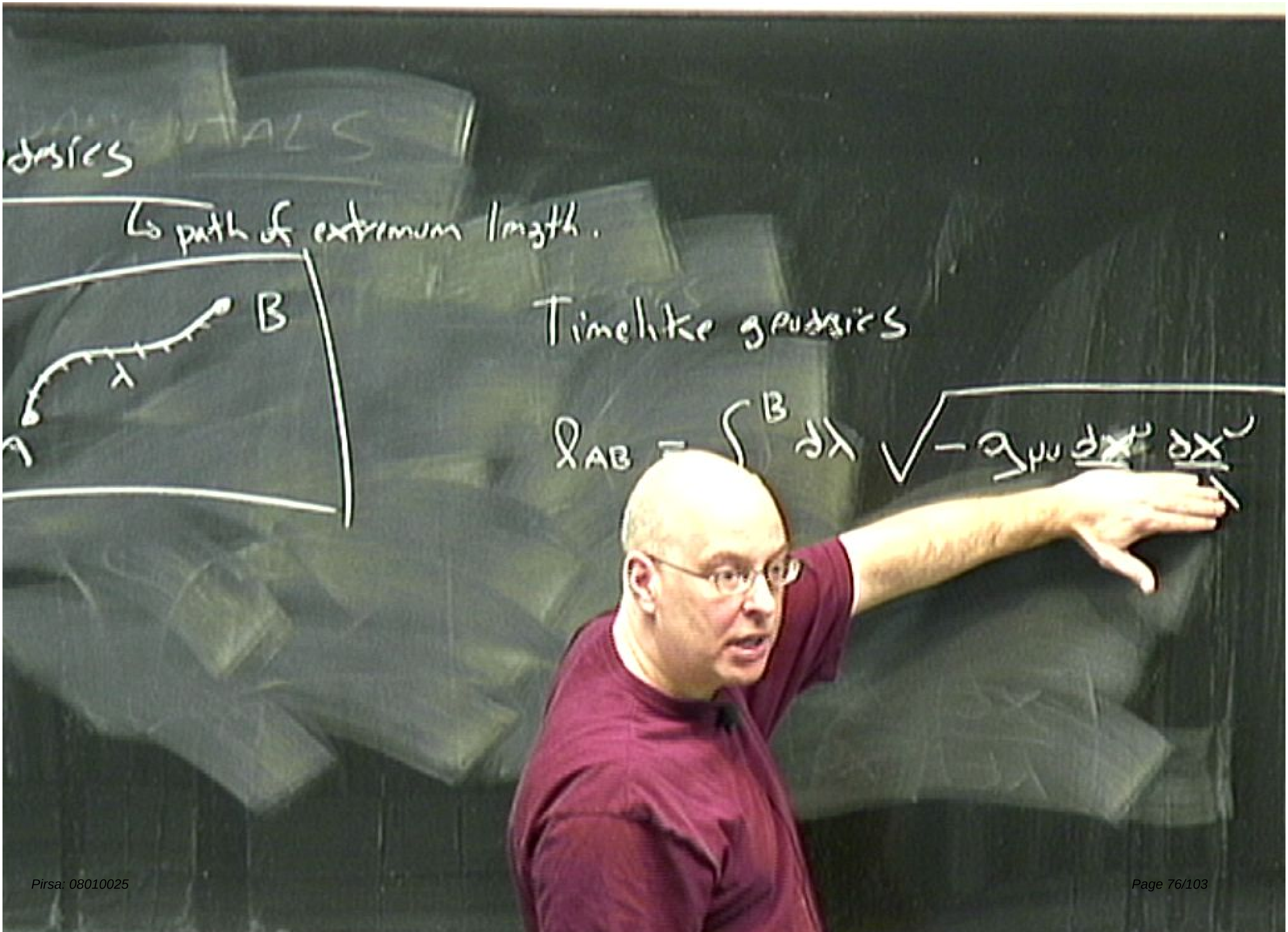
Geodesics

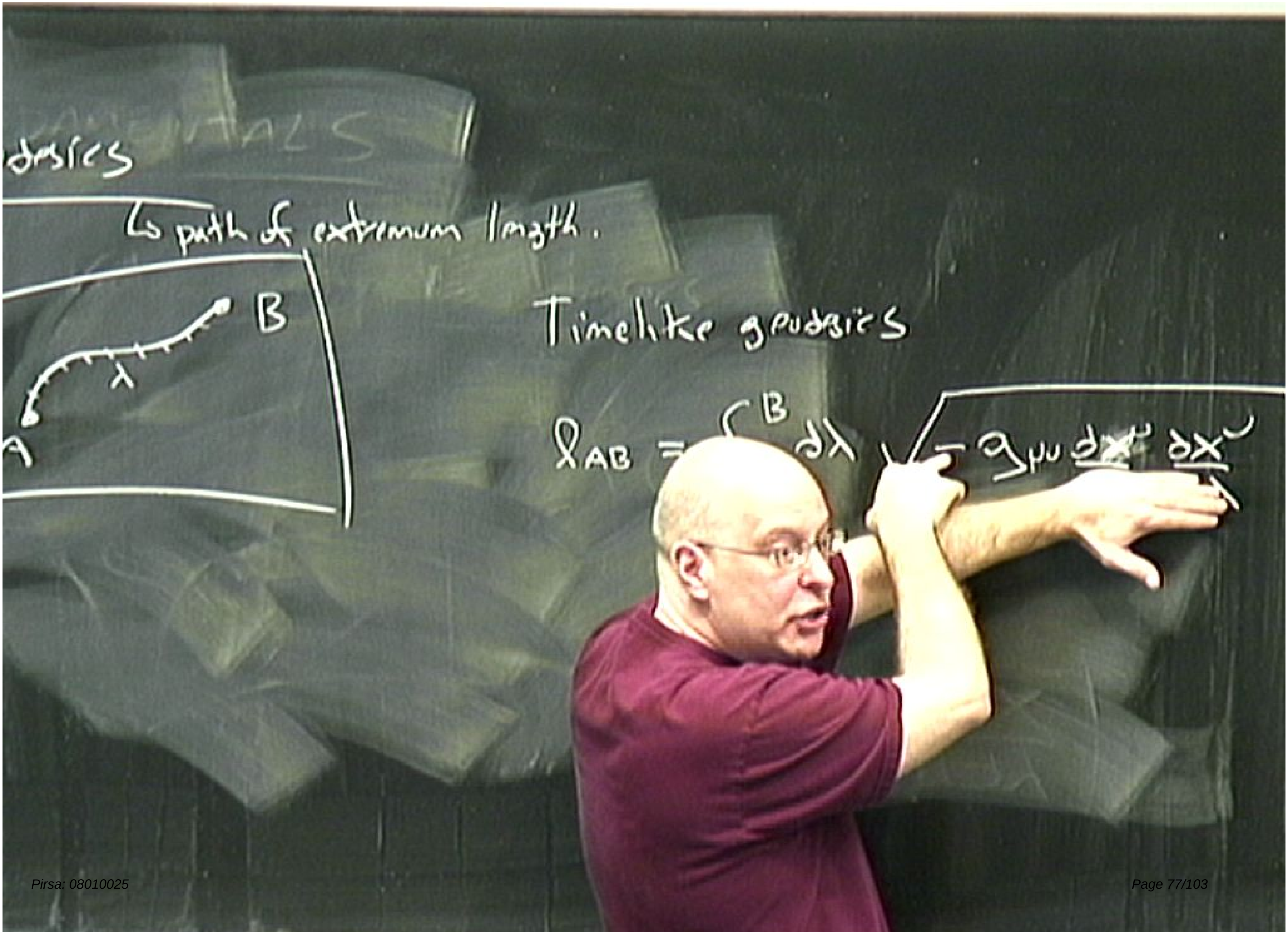
↳ path of extremum length.

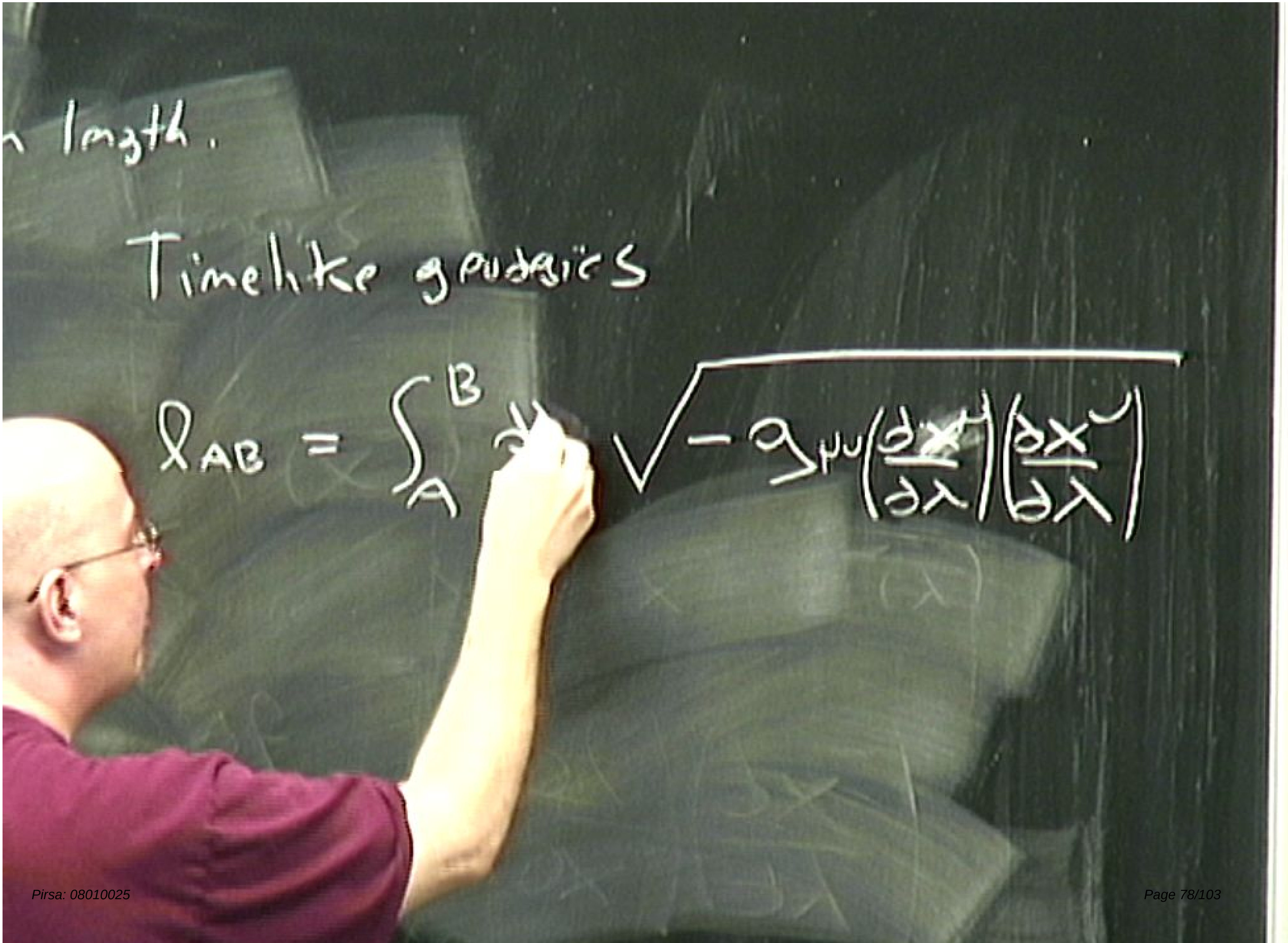


Timelike geodesics

$$\ell_{AB} = \int_A^B d\lambda$$


desics
↳ path of extremum length.
B
λ
Timelike geodesics
$\ell_{AB} = \int^{B} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$


path of extremum length.
B
λ
A
Timelike geodesics
$\ell_{AB} = \int^{B} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$


length.
Timelike geodesics
$\ell_{AB} = \int_A^B d\lambda \sqrt{-g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}$

n length.

Timelike geodesics

$$\ell_{AB} = \int_A^B d\lambda \sqrt{-g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}$$

n length.

Timelike geodesics

$$\ell_{AB} = \int_A^B d\lambda \sqrt{-g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}$$

(invariant under reparameterizations:

$$\lambda \to \lambda'(\lambda)$$

n length.

Timelike geodesics

$$\ell_{AB} = \int_A^B d\lambda \, \underbrace{\sqrt{-g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}}_{L(x^\mu,\, \dot{x}^\mu)}$$

(invariant under reparameterizations:

$$\lambda \to \lambda'(\lambda)\;)$$

Extremization:
$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

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result: $\ddot{x}^\alpha$

Extremization: $\dfrac{\partial}{\partial\lambda}\dfrac{\partial L}{\partial \dot{x}^\alpha} - \dfrac{\partial L}{\partial x^\alpha} = 0$

result: $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta \dot{x}^\gamma$

Extremization: $$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

result: $$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta \dot{x}^\gamma = k\,\dot{x}^\alpha$$

$$k = \frac{1}{L}\frac{dL}{d\lambda}$$

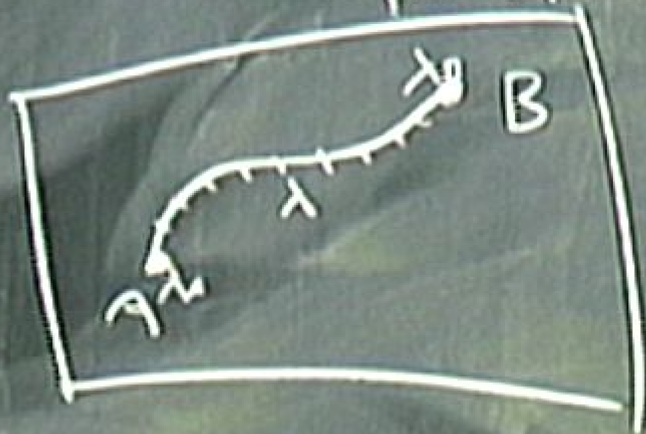
$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

$$\boxed{\begin{aligned} &\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta \dot{x}^\gamma = k\,\dot{x}^\alpha \\ &k = \frac{1}{L}\frac{dL}{d\lambda} \end{aligned}}$$

geodesic equation (general form)

Proper time on curve: $d\tau^2 = -ds^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$

↳ path of extremum length.



Timelike geodesics

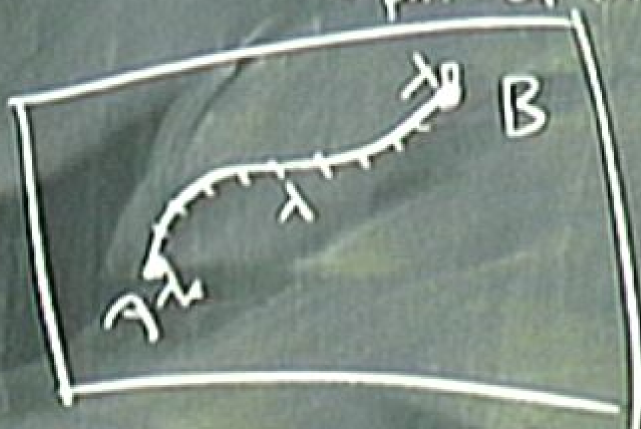
$$l_{AB} = \int_A^B d\lambda \sqrt{-g \cdots}$$

(invariant under reparameter…

$$\lambda \to \lambda'(\lambda)$$

3- Geodesics

↳ path of extremum length.



Timelike geodesics $L(x$

$$\ell_{AB} = \int_A^B d\lambda \sqrt{-g_{\mu\nu}(\frac{d}{d}$$

(invariant under reparameterization

$\lambda \to \lambda'(\lambda)$)

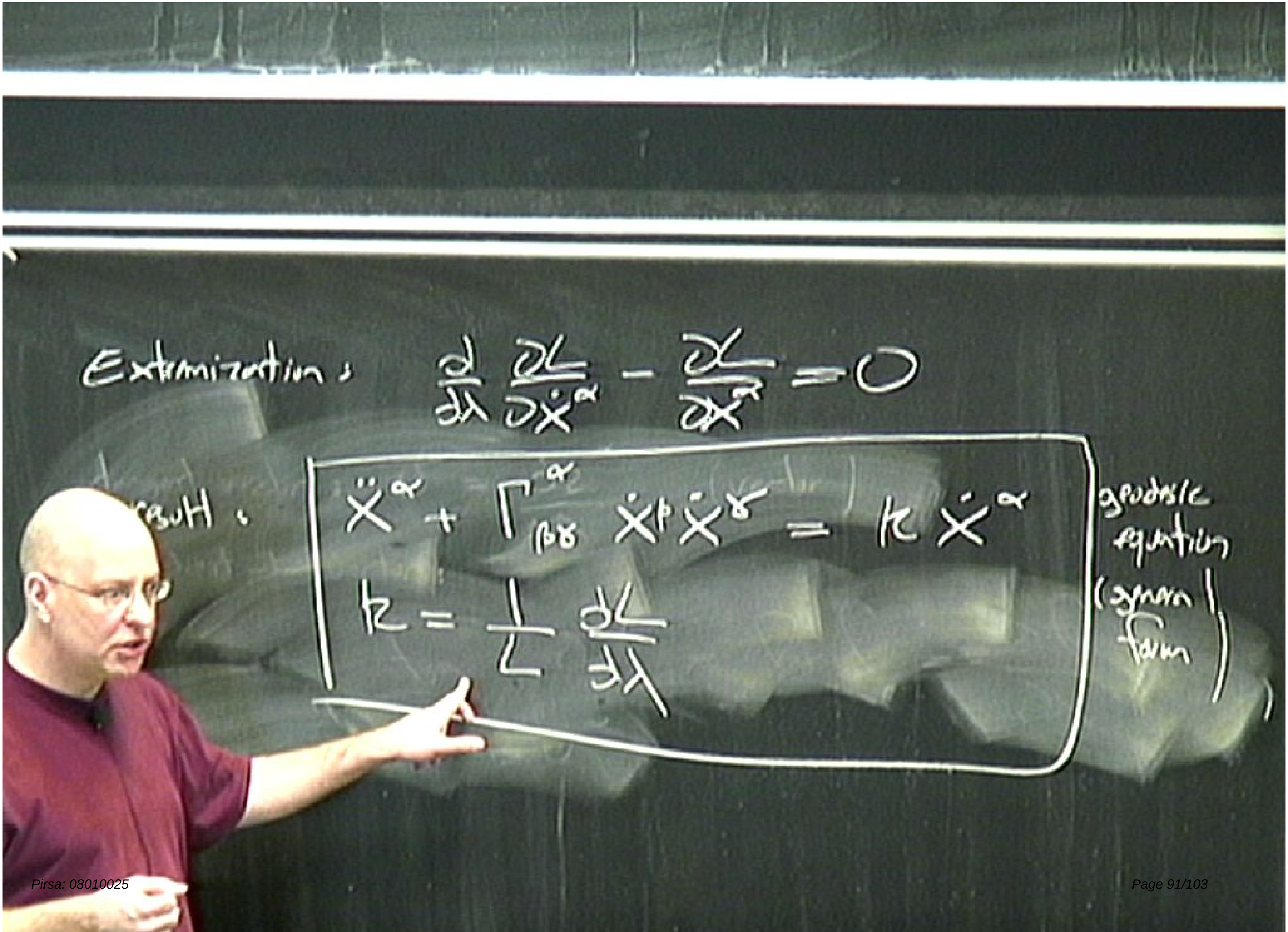
3– Geodesics

↳ path of extremum length.

Timelike geodesics

$$\ell_{AB} = \int_A^B d\lambda \, \overbrace{\sqrt{-g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}}^{L(x^\mu,\,\dot{x}^\mu)}$$

(invariant under reparameterizations: $\lambda \to \lambda'(\lambda)$)


Extremization: $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$
Result: $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = k\,\dot{x}^\alpha$
$k = \frac{1}{L}\frac{dL}{d\lambda}$
geodesic equation (general form)

e on curve: $d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$

If $\lambda \equiv \tau$ after the variation,

then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \rightarrow k = 0$

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then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \Rightarrow k = 0$

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0 \quad \text{geodesic}$$

If $\lambda \equiv \tau$ after the variation,

then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \rightarrow k = 0$

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\,\dot{x}^{\beta}\dot{x}^{\gamma} = 0 \quad \text{geodesic}$$

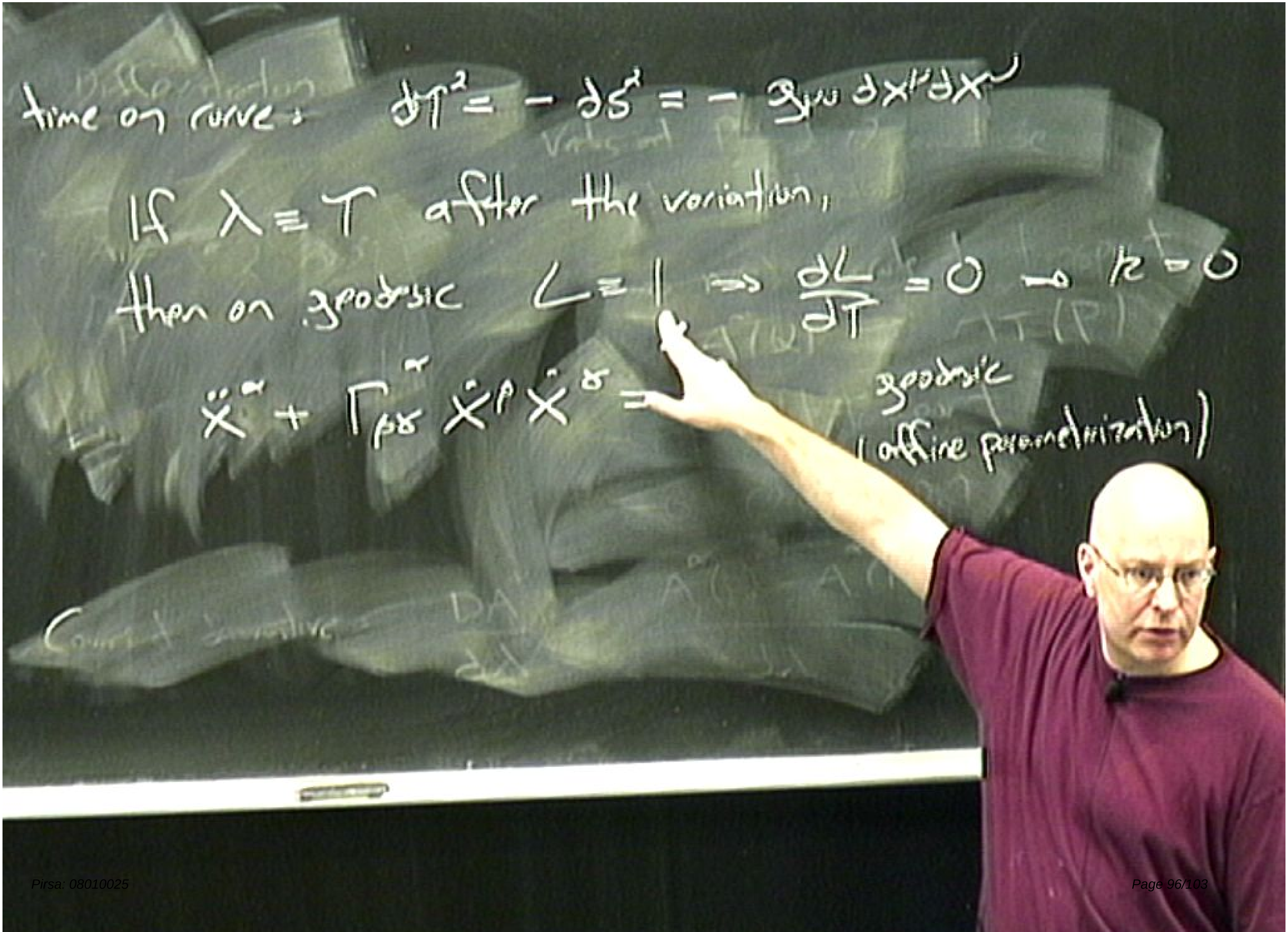
...per time on curve: $d\tau^2 = -ds^2 = -g_{\mu\nu}\,dx^\mu dx^\nu$

If $\lambda \equiv \tau$ after the variation,

then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \Rightarrow k = 0$

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\,\dot{x}^\beta \dot{x}^\gamma = 0$$

geodesic (affine parametrization)


time on curve: $d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$
If $\lambda \equiv \tau$ after the variation,
then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \rightarrow k = 0$
$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma =$
geodesic
(affine parametrization)

If $\lambda \equiv T$ after the variation,

then on geodesic $L \equiv 1 \Rightarrow \frac{dL}{dT} = 0 \rightarrow k = 0$

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0$$ geodesic (affine parametrization)

affine parameter $\lambda^* = aT + b$

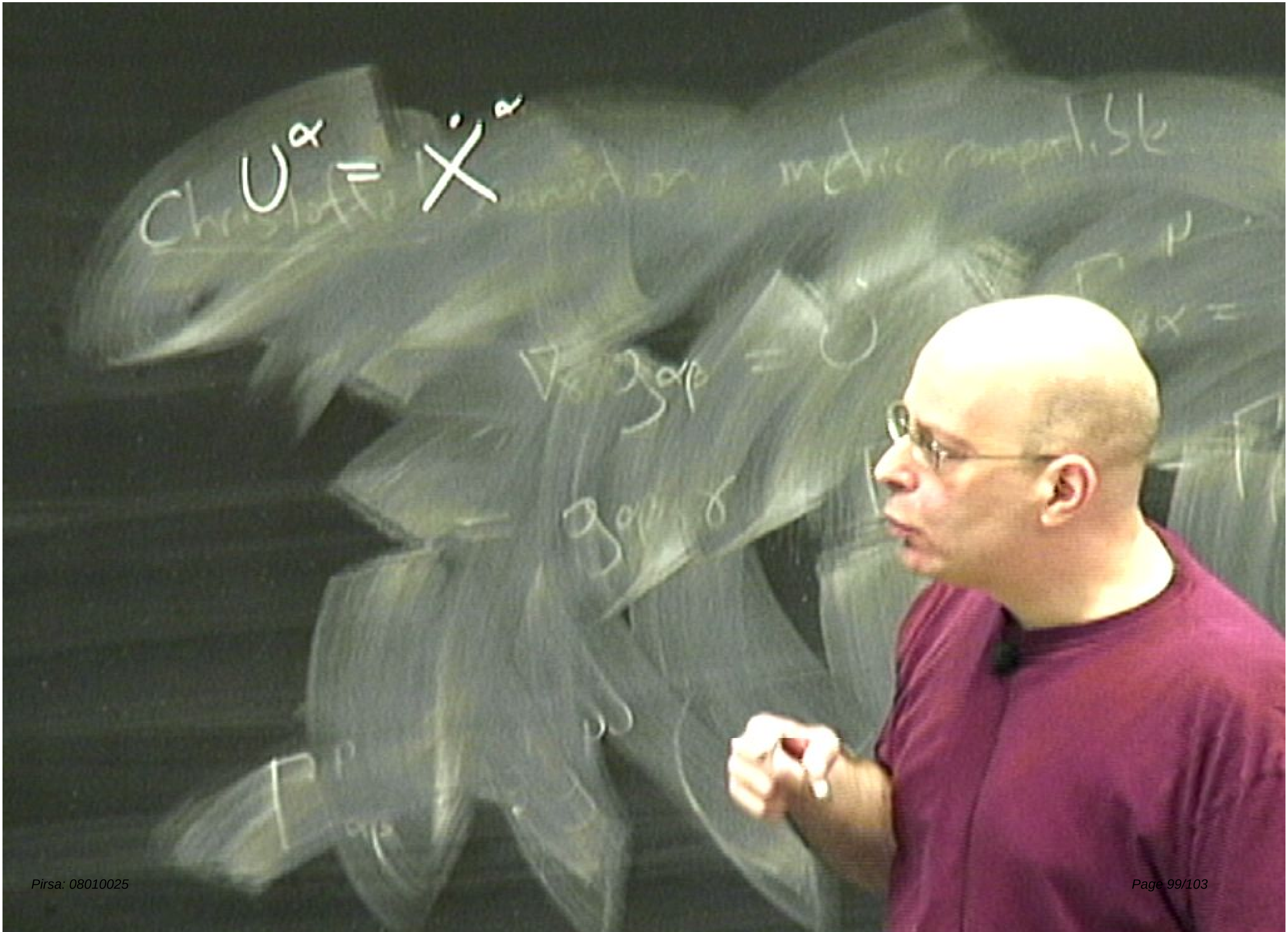
If $\lambda \equiv T$ after the variation,

then on geodesic $L = 1 \Rightarrow \frac{dL}{dT} = 0 \rightarrow k = 0$

$$\boxed{\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0}$$

geodesic (affine parametrization)

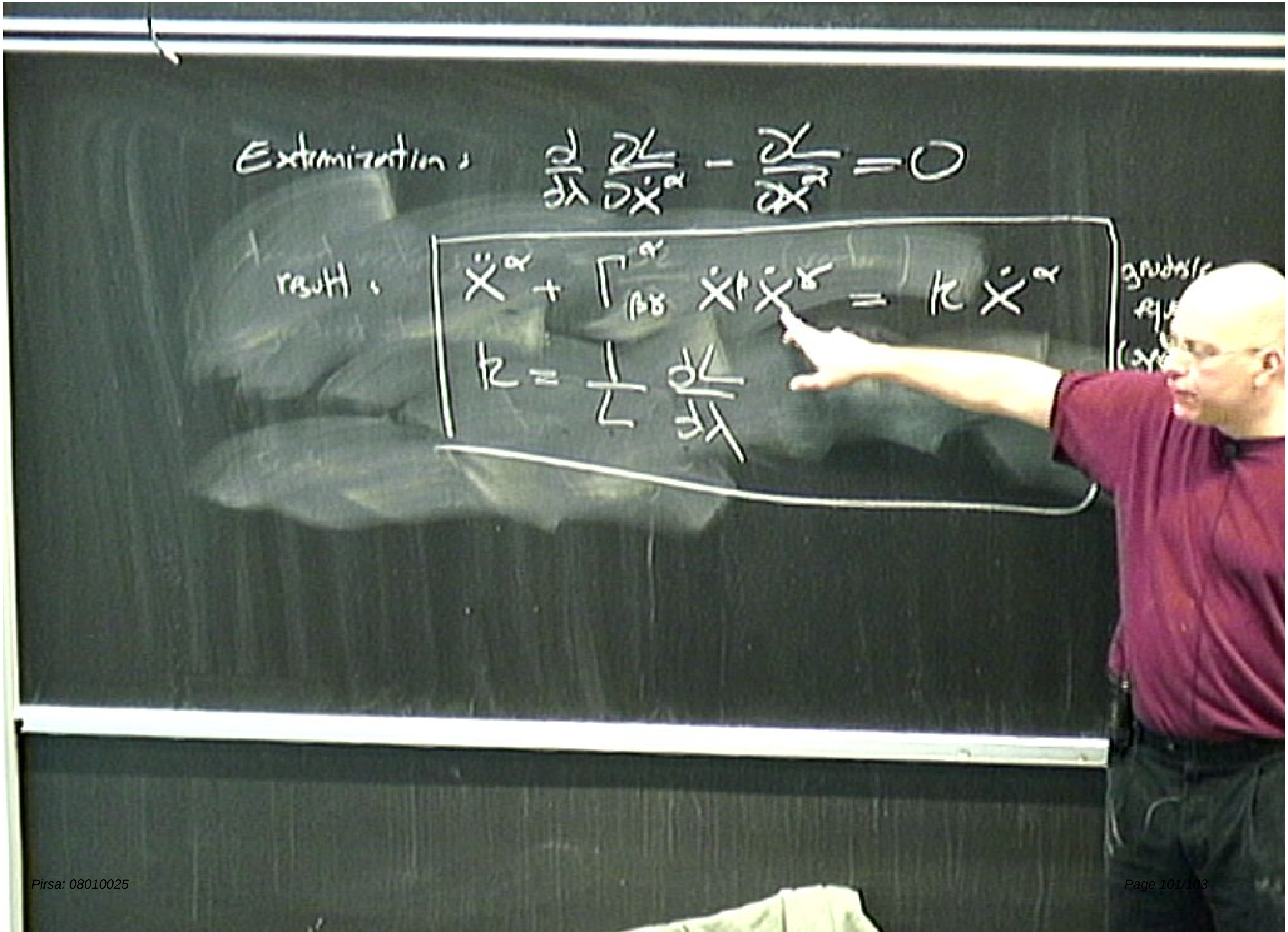
affine parameter $\lambda^{*} = aT + b$


$U^\alpha = \dot{X}^\alpha$

$$U^\alpha = \dot{X}^\alpha$$

$$\dot{U}^\alpha + \Gamma^\alpha_{\beta\gamma} U^\beta U^\gamma = 0$$

$$\frac{DU^\alpha}{d\tau} = 0$$


Extremization: $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$
result: $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = \kappa \dot{x}^\alpha$
$\kappa = \frac{1}{L}\frac{dL}{d\lambda}$

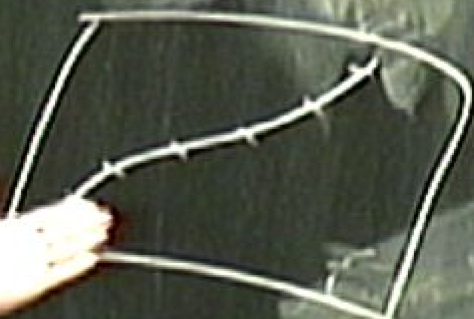
Extremization: $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$

result: $$\boxed{\begin{aligned} &\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = k\dot{x}^\alpha \\ &k = \frac{1}{L}\frac{dL}{d\lambda} \end{aligned}}$$ geodesic equation

Proper time on curve: $d\tau =$ ...

If $\lambda = \tau$ after the variation,

then on geodesic $L = 1 \Rightarrow \frac{dL}{d\tau} = 0 \Rightarrow k =$



$$\boxed{\begin{array}{l} \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0 \\[1em] \text{affine parameter } \lambda^* = a\tau + b \end{array}}$$

geodesic (affine parametrization)