Title: Hamiltonian Implementation/Simulation & Phase Measurement

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Abstract: This is a talk in two parts. The first part is on evolution of a system under a Hamiltonian. First, a general method for implementing evolution under a Hamiltonian using entanglement and classical communication is presented. This method improves on previous methods by requiring less entanglement and communication, as well as allowing more general Hamiltonians to be implemented. Next, a method for simulating evolution under a sparse Hamiltonian using a quantum computer is presented. When H acts on n qubits, and has at most a constant number of nonzero entries in each row/column, we may select any positive integer k such that the simulation requires O((log*n)t^(1+1/2k)) accesses to matrix entries of H. The second part of the talk is on adaptive measurements of optical phase. Standard measurement schemes, using each resource independently, lead to a phase uncertainty that scales as 1/sqrt(N). It has long been conjectured that it should be possible to achieve a precision limited only by the Heisenberg uncertainty principle, dramatically improving the scaling to 1/N. I present a Heisenberg-limited phase estimation procedure which has been demonstrated experimentally. We use multiple applications of the phase shift on unentangled single-photon states, and generalize Kitaev\\\'s phase estimation algorithm using adaptive measurement theory to achieve a standard deviation scaling at the Heisenberg limit.

# Areas of Research

- Hamiltonian implementation
- Hamiltonian simulation
- Phase measurements
  - Optimal adaptive dyne measurements
  - Adaptive measurements for interferometry
  - Continuous adaptive measurements
  - Heisenberg limited interferometry with multiple passes
- Capacities
  - Relations between different capacities for unitary gates
  - Capacities of qubit channels
- Processing of single photon sources
- Bell inequalities with loss

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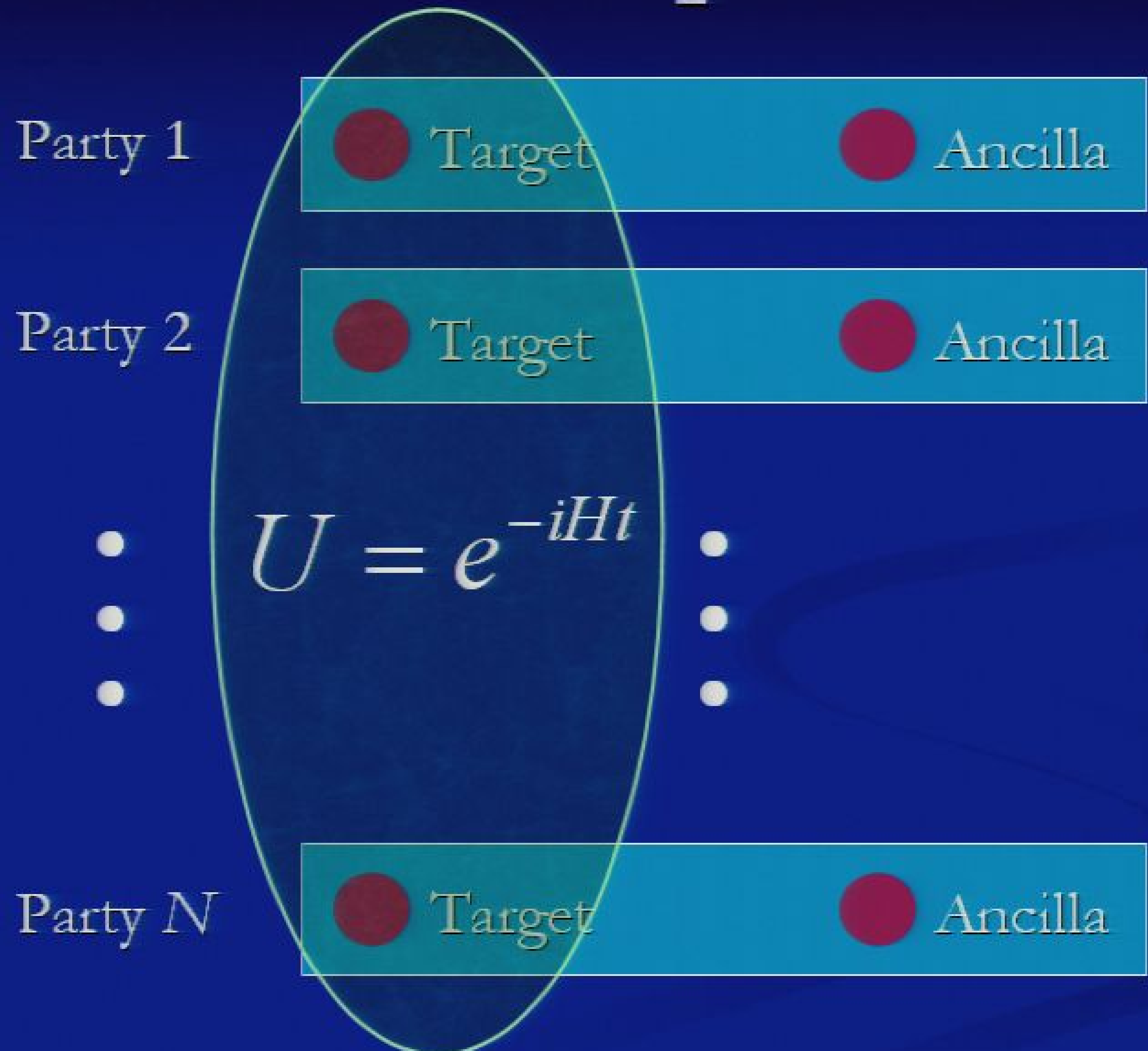
# Hamiltonian Implementation

| | | |
|---|---|---|
| Party 1 | Target | Ancilla |
| Party 2 | Target | Ancilla |
| ⋮ | ⋮ | |
| Party $N$ | Target | Ancilla |



D. W. Berry, Phys. Rev. A **75**, 032349 (2007).

# Hamiltonian Implementation

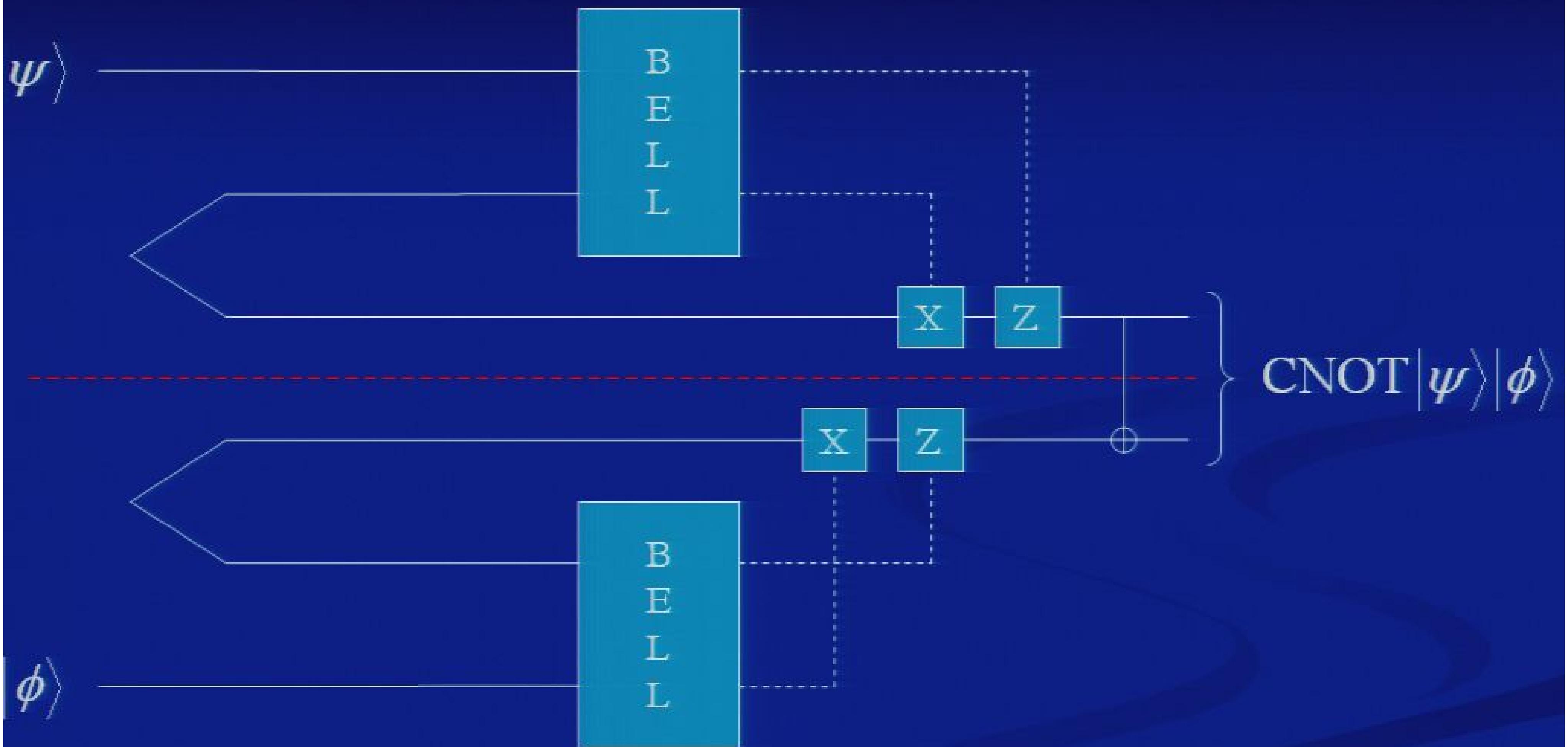






D. W. Berry, Phys. Rev. A **75**, 032349 (2007).

# Teleportation scheme for CNOT




D. Gottesman and I. L. Chuang, Nature **402**, 390 (1999).

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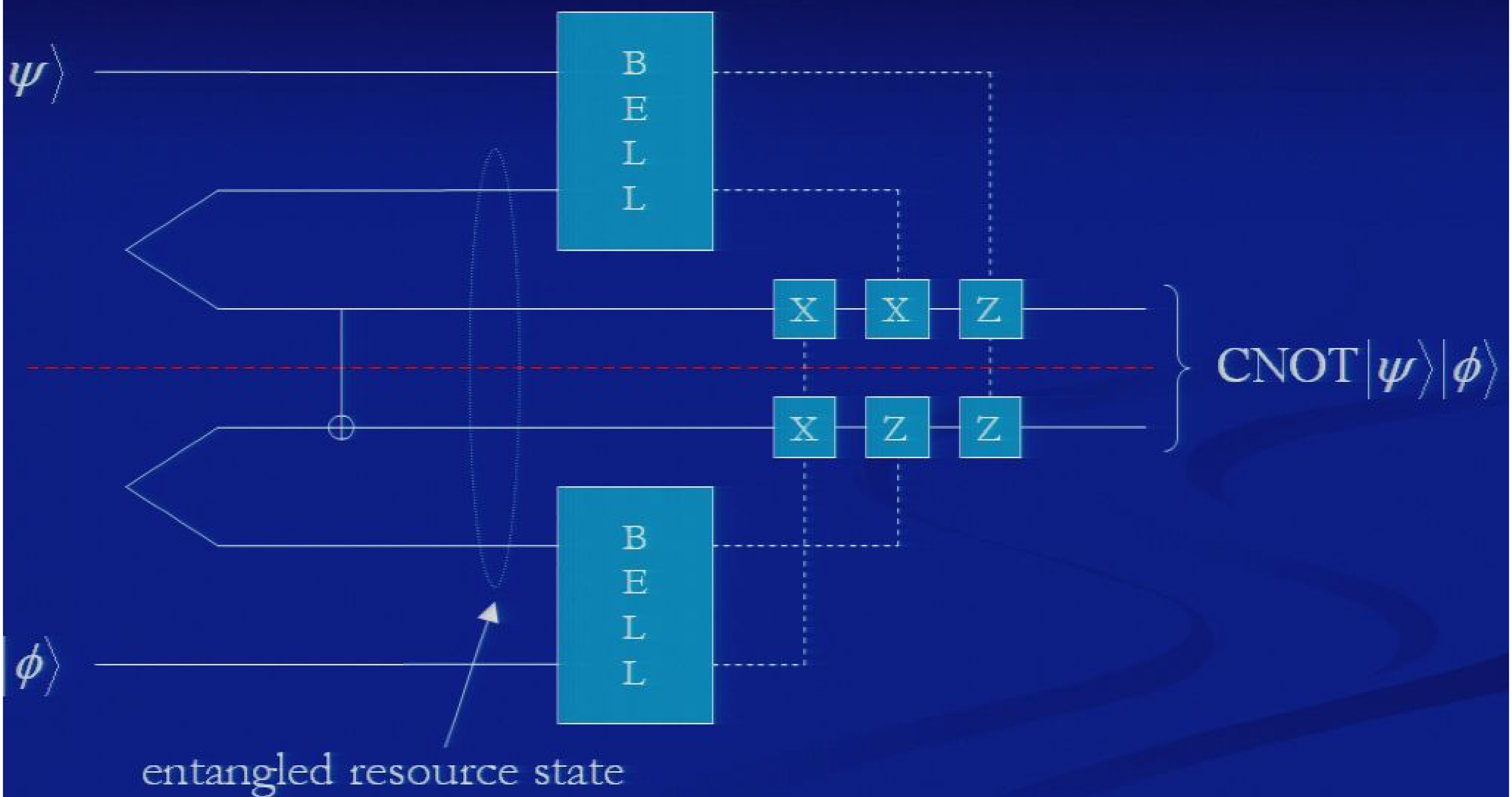




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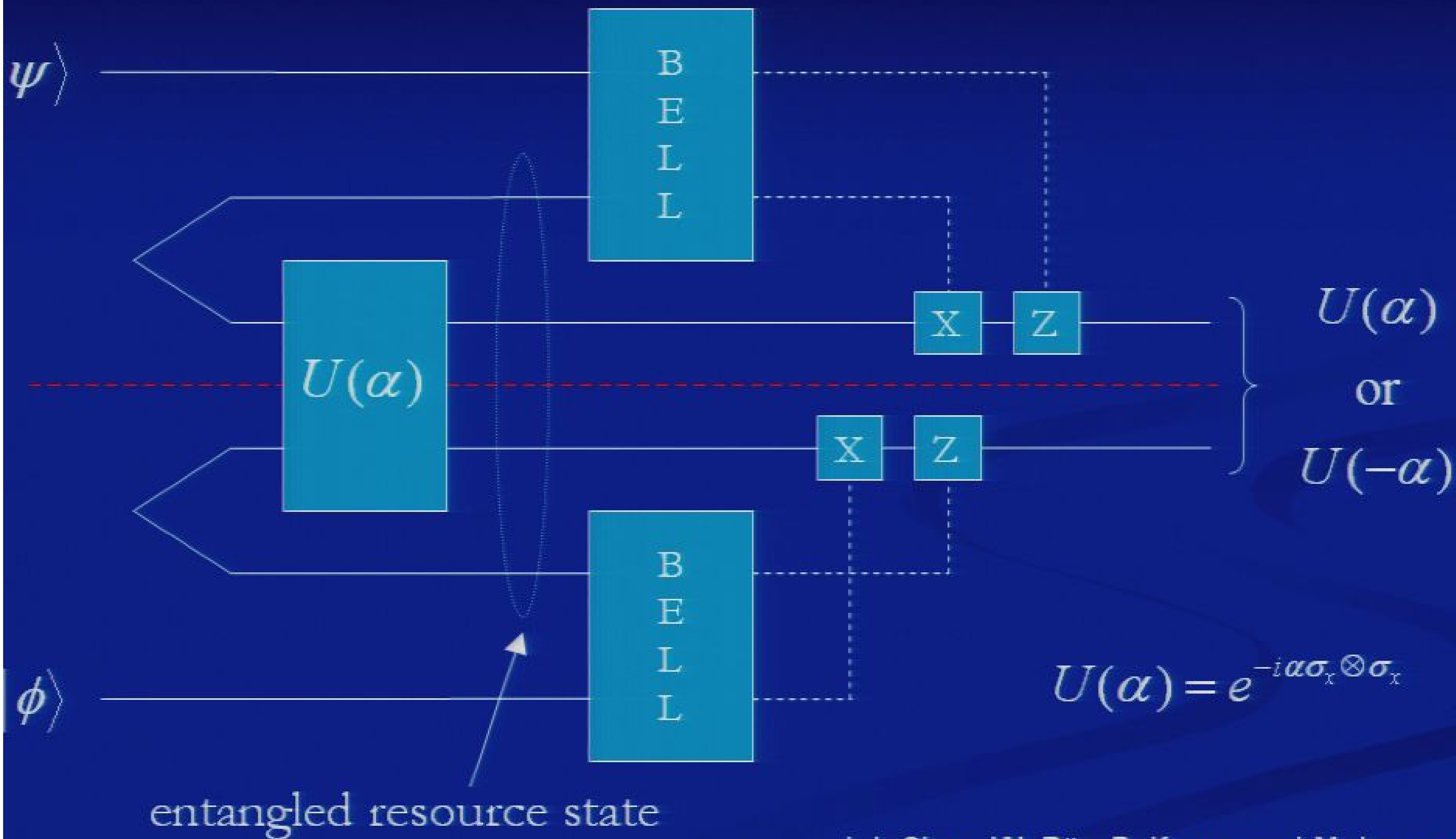
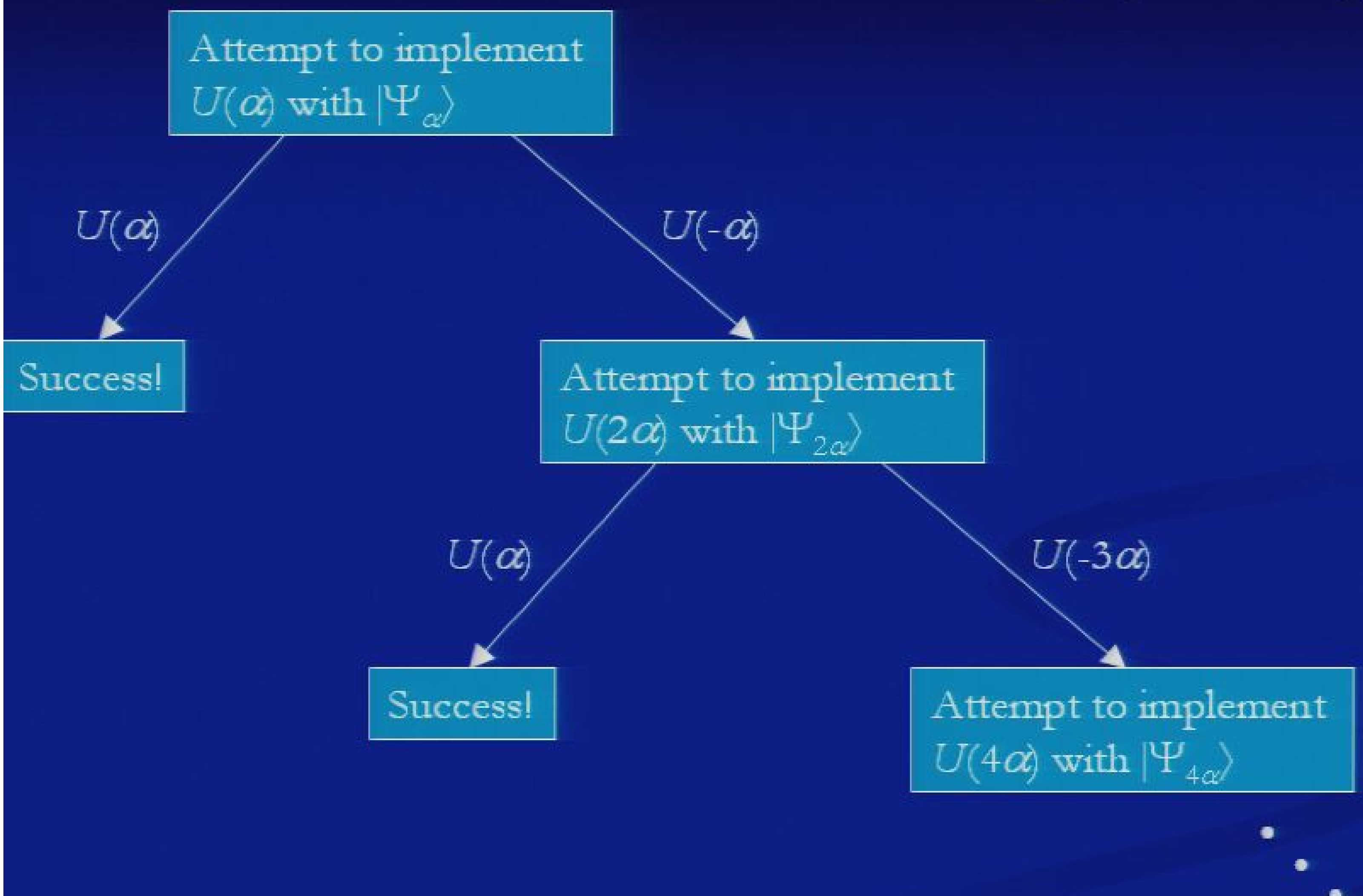
# Cirac scheme for $\exp(-i\alpha\sigma_x \otimes \sigma_x)$




$$U(\alpha) = e^{-i\alpha\sigma_x \otimes \sigma_x}$$

J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).

# Cirac scheme for $\exp(-i\alpha\sigma_x \otimes \sigma_x)$




For $\alpha = \pi/2^N$, eventually the correction is $U(2^{N-1}\alpha) = U(\pi/2)$, which is local.

# Entanglement consumption for Cirac scheme

- The entanglement used in the first step is $E(\alpha)$.

$$E(\alpha) = -\sin\alpha \log\sin\alpha - \cos\alpha \log\cos\alpha$$

- The entanglement used in the second step is $E(2\alpha)$ if there was failure, and 0 otherwise. Since the probability of failure is 50%, the average entanglement used is $E(2\alpha)/2$.

- The entanglement used in the third step is $E(4\alpha)/4$, and so forth.

- For $\alpha_N = \pi/2^N$, there are $N-2$ steps which consume entanglement, with the expected entanglement consumption being

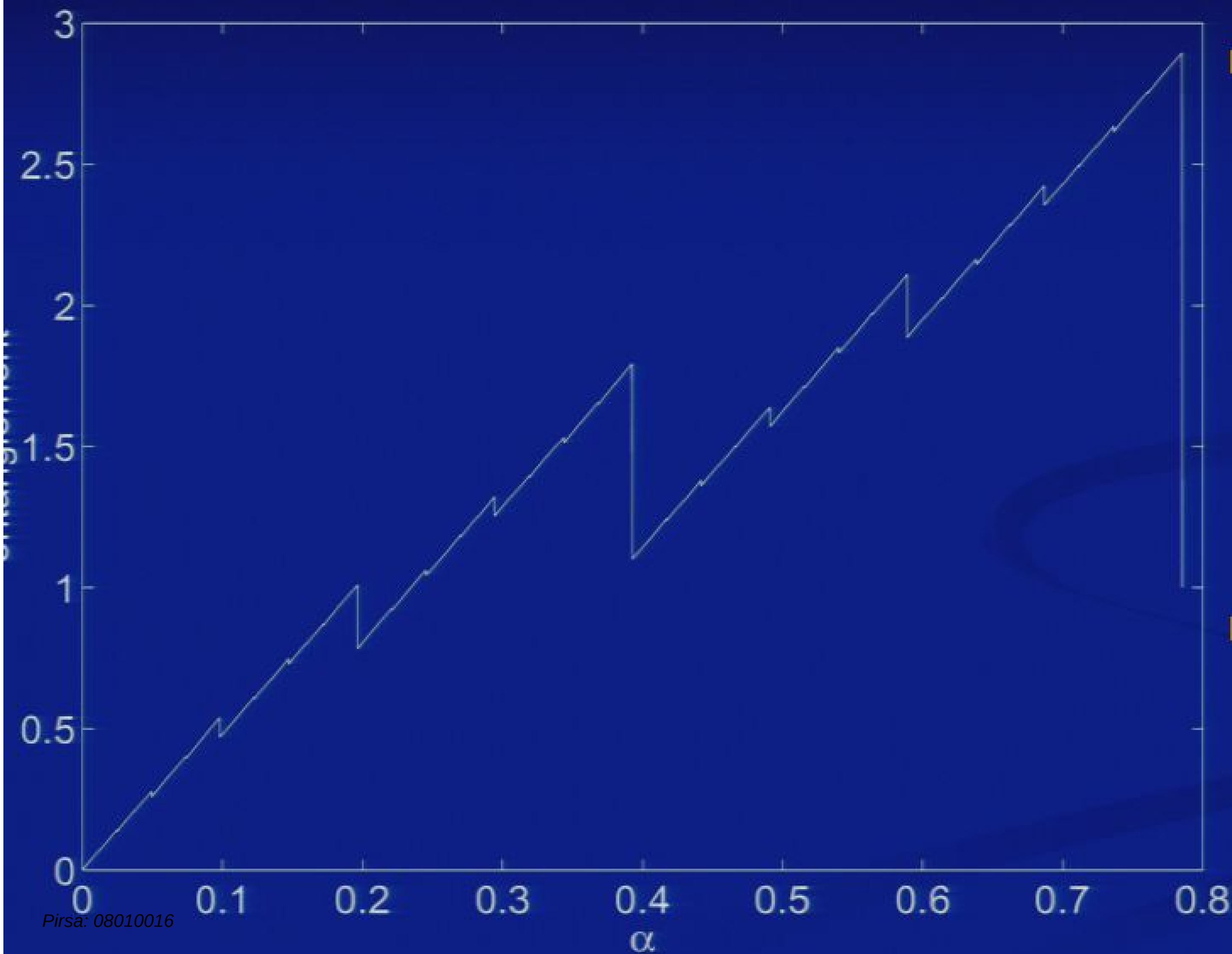
$$\bar{E}(\alpha_N) = \sum_{n=0}^{N-1} E(2^n \alpha_N)/2^n = \sum_{n=0}^{N-1} E(2^{n-N}\pi)/2^n$$

$$= \frac{\alpha_N}{\pi} \sum_{k=1}^{N} 2^k E(\alpha_k) < \alpha_N f_\infty$$

where

$$f_\infty = \frac{1}{\pi} \sum_{k=1}^{\infty} 2^k E(\alpha_k) = 5.9793\ldots$$

# General α




- In the Cirac scheme, for general $\alpha$, one would expand it as

$$\alpha = \sum_{N=0}^{\infty} b_N \alpha_N$$

where the $b_N$ are bits.

- The entanglement consumption is

$$\bar{E}(\alpha) = \sum_{N=0}^{\infty} b_N \bar{E}(\alpha_N)$$

$$< \alpha f_{\infty}$$

# Part 1: Simplifying the ancilla

- We use the entangled resource[1]

$$|\Psi\rangle = \cos(\alpha)|00\rangle + i\sin(\alpha)|11\rangle$$

- We want to convert this to the operation

$$U(\alpha) = \cos(\alpha) I \otimes I + i\sin(\alpha)\sigma_z \otimes \sigma_z$$

- We start by applying controlled phase operations between the ancilla qubits and the target qubits; this results in the *stator*[2]

$$\cos(\alpha)|00\rangle \otimes I \otimes I + i\sin(\alpha)|11\rangle \otimes \sigma_z \otimes \sigma_z$$

- We perform a Hadamard on the first ancilla qubit, then measure in the computational basis. This gives (with sign depending on the result)

$$\cos(\alpha)|0\rangle \otimes I \otimes I \pm i\sin(\alpha)|1\rangle \otimes \sigma_z \otimes \sigma_z$$



[1] L. Chen and Y.-X. Chen, Phys. Rev. A **71**, 054302 (2005).



[2] B. Reznik, Y. Aharonov, and B. Groisman, Phys. Rev. A **65**, 032312 (2002).

# Part 1: Simplifying the ancilla

- We can then perform a phase correction using a $\sigma_z$ operation on the other ancilla.
- We perform a Hadamard and computational basis measurement on the remaining ancilla, giving the operation

$$U(\pm\alpha) = \cos(\alpha) I \otimes I \pm i \sin(\alpha) \sigma_z \otimes \sigma_z$$

- In this case we *can not* correct the sign, because there are no remaining ancilla qubits.
- In the case that the sign is wrong, we can simply count this as a failure, and attempt to apply $U(2\alpha)$. We proceed in the same way as for Cirac's scheme, and the expected entanglement consumption is identical.

# Part 2: Reducing the entanglement

- We use the more general entangled resource
$$|\Psi(\beta)\rangle = \cos(\beta)|00\rangle + i\sin(\beta)|11\rangle$$
where $\alpha \neq \beta$, and usually $\alpha < \beta$.
- We apply the first part of the procedure as before, giving the stator
$$\cos(\beta)|0\rangle \otimes I \otimes I + i\sin(\beta)|1\rangle \otimes \sigma_z \otimes \sigma_z$$
- We now apply a projective measurement on the remaining qubit in the basis
$$\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle, \qquad \sin(\gamma)|0\rangle - \cos(\gamma)|1\rangle$$
- Given a result corresponding to projection onto the first state, we obtain
$$\cos(\beta)\cos(\gamma) I \otimes I + i\sin(\beta)\sin(\gamma)\sigma_z \otimes \sigma_z$$
- We select $\gamma$ such that
$$\tan(\gamma)\tan(\beta) = \tan(\alpha)$$
- Provided the correct measurement result is obtained we obtain the desired unitary.

# Part 2: Reducing the entanglement

- For the other measurement result the operation performed is proportional to

$$\cos(\beta)\sin(\gamma)I\otimes I - i\sin(\beta)\cos(\gamma)\sigma_z\otimes\sigma_z$$

- This is again of the class $U(\alpha)$, so we can correct for it using the same approach as before.

- We again use a sequence of corrections for the case of failure.

# Part 2: Reducing the entanglement

Examples:

1. $\beta = \alpha$ – this is just the scheme given before.

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$$\sigma_z \otimes \sigma_z \quad \downarrow$$

$$\sin(\alpha)\sigma_z \otimes \sigma_z - i\cos(\alpha) I \otimes I$$

$$\downarrow \quad i$$

$$\cos(\alpha) I \otimes I + i\sin(\alpha)\sigma_z \otimes \sigma_z$$

# Part 2: Reducing the entanglement

Examples:

3. $\beta = \mathbf{arctan}\sqrt{\mathbf{tan}\,\alpha}$ – we have $\gamma=\beta$, and the probability for failure is

$$\frac{1}{2}\sin^2 2\beta \approx 2\alpha$$

where the approximation holds for small $\alpha$.

$\Rightarrow$ For small $\alpha$ the probability for success is close to 1.

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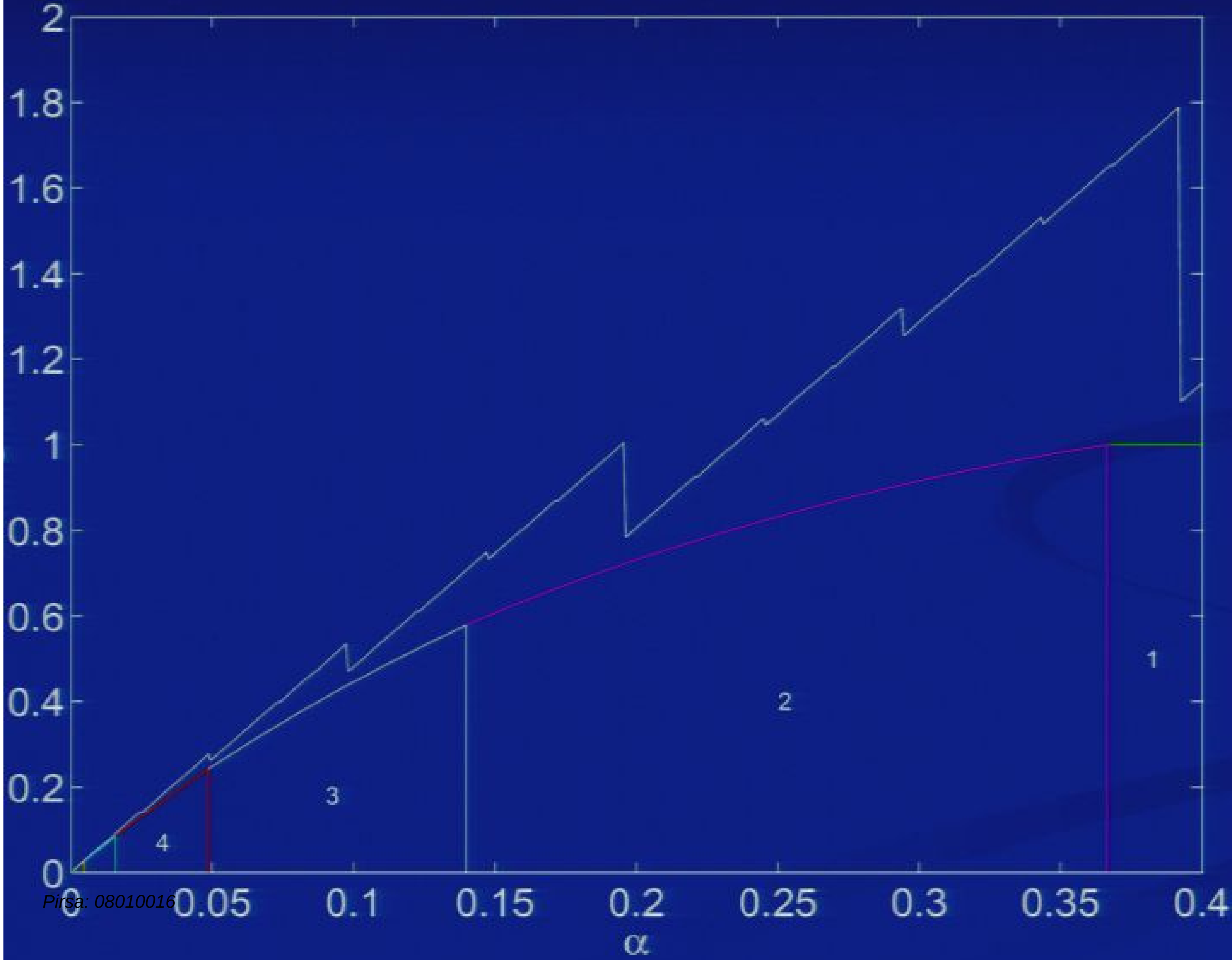
$\Rightarrow$ For small $\alpha$ the probability for success is close to 1.

4. Numerically optimise the value of $\beta$ at each step to minimise the entanglement consumption. Given a certain number of failures, the correction is performed deterministically using one ebit.

$\Rightarrow$ The asymptotic entanglement consumption is approximately $5.6418\alpha$, as opposed to $5.9793\alpha$.

# Part 2: Reducing the entanglement




- We can directly perform the scheme regardless of $\alpha$, rather than needing to perform a sum.
- The entanglement consumed is reduced, particularly for larger values of $\alpha$.

# Part 3: $H_1 \otimes \ldots \otimes H_N$ Hamiltonian

- Consider the multipartite Hamiltonian

$$U(\alpha) = \cos(\alpha) I^{\otimes N} + i \sin(\alpha) \sigma_z^{\otimes N}$$

- We use the entangled resource state

$$|\Psi(\beta)\rangle = \cos(\beta)|0\rangle^{\otimes N} + i \sin(\beta)|1\rangle^{\otimes N}$$

- All parties apply controlled phase between the ancilla qubit and the target qubit.

$$\cos(\beta)|0\rangle^{\otimes N} \otimes I^{\otimes N} \pm i \sin(\beta)|1\rangle^{\otimes N} \otimes \sigma_z^{\otimes N}$$

- Parties 1 to $N-1$ apply a Hadamard followed by a computational basis measurement.

$$\cos(\beta)|0\rangle \otimes I^{\otimes N} \pm i \sin(\beta)|1\rangle \otimes \sigma_z^{\otimes N}$$

- The sign depends on the measurement results.
- The rest of the scheme proceeds as before!

W. Dür and J. I. Cirac, Phys. Rev. A 64, 012317 (2001).

# Part 3: $H_1 \otimes \ldots \otimes H_N$ Hamiltonian

- Diagonalise the $H_j$ via local unitaries to give

$$H_{\text{diag}} = \Delta \bigotimes_j \text{diag}(a_{1,j}, a_{2,j}, a_{3,j}, \ldots, a_{d_j-1,j}, a_{d_j,j})$$

where $\Delta = \|H\|$, $a_{l,j} \in [-1,1]$, $d_j$ is dimension $H_j$ acts on.

- We perform chain of simulations

$$\sigma_z^{\otimes N} \to H^2_{A_1B_1} \otimes \sigma_z^{\otimes N-1} \to H^2_{A_1B_1} \otimes H^2_{A_2B_2} \otimes \sigma_z^{\otimes N-2}$$

$$\ldots \to \bigotimes_j H^2_{A_jB_j} \to H_{\text{diag}} / \Delta$$

where

$$H^2_{A_jB_j} = \text{diag}(a_{1,j}, -a_{1,j}, \ldots, a_{d_j,j}, -a_{d_j,j})$$

# Part 3: $H_1 \otimes \ldots \otimes H_N$ Hamiltonian

- We wish to perform the general simulation

$$\bigotimes_{n=1}^{j-1} H^2_{A_n B_n} \otimes \sigma_z^{\otimes N-j+1} \to \bigotimes_{n=1}^{j} H^2_{A_n B_n} \otimes \sigma_z^{\otimes N-j}$$

- First append $d_j$ dimensional ancilla $B_j$ so the $j$'th term in tensor product is diag$(1,-1,\ldots,1,-1)$.
- Local unitaries $U_l$ exchange $(2l-1)$'th and $(2l)$'th basis vectors of $A_jB_j$.
- For small time $\delta t$, apply $U_l$ at time $p_l\delta t$, and again at time $\delta t$ (for $p_l = (a_{l,j}+1)/2$).

This is based the method in
A. M. Childs, D. W. Leung, and G. Vidal, IEEE Trans. Inf. Theory **50**, 1189 (2004).

# Part 3: $H_1 \otimes \ldots \otimes H_N$ Hamiltonian

- The total entanglement consumption is only

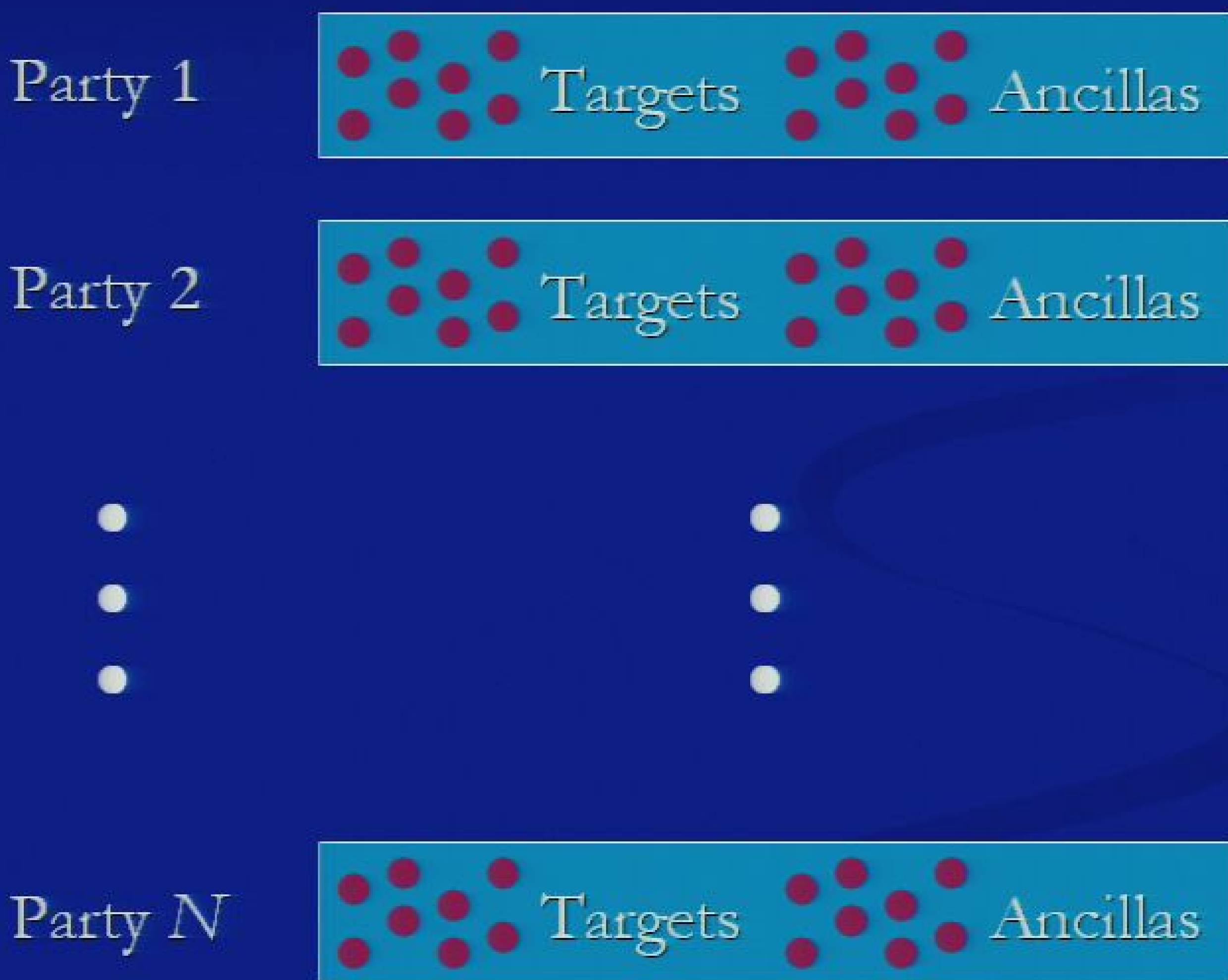
$$5.6418\|H\|t$$

- For more general Hamiltonians $H = \sum_k H_k$, where each $H_k$ is a tensor product Hamiltonian.

- These can be simulated using a Taylor expansion, with average entanglement consumption

$$5.6418t\sum_k \|H_k\|.$$

# Part 4: Reduced communication



# Part 4: Reduced communication

- We wish to implement $U(\alpha)^{\otimes M}$.
- At stage $l$ (after $l-1$ failures) we need to perform the correction on (on average) $M_l = Mp(l)$ times.
- The resource state required is

$$|\Psi_l\rangle = [\cos(\beta_l)|0\rangle^{\otimes N} + i\sin(\beta_l)|1\rangle^{\otimes N}]^{\otimes M_l}$$

- We can express the resource state as

$$|\Psi_l\rangle = \sum_{\mathbf{i}} \mu_{\mathbf{i}} |\mathbf{i}\rangle$$

where

$$\mathbf{i} = (i_1, \ldots, i_{M_l}), \qquad |\mathbf{i}\rangle = |i_1\rangle \otimes \ldots \otimes |i_{M_l}\rangle, \qquad \mu_{\mathbf{i}} = \prod_m \sin^{i_m}(\beta_l)\cos^{1-i_m}(\beta_l)$$

- We now retain only typical sequences of the $i$. Denoting the set of typical $\mathbf{i}$ by $S_l$, we approximate the resource state by

$$|\tilde{\Psi}_l\rangle \propto \sum_{\mathbf{i}\in S_l} \mu_{\mathbf{i}} |\mathbf{i}\rangle$$

- The number of elements in the set $S_l$ is approximately

$$2^{M_l E(\beta_l)}$$

# Part 4: Reduced communication

- Previously each party performed a Hadamard and a computational basis measurement.
- Now we perform a measurement in the Fourier transform basis in the space of typical sequences.
- The measurement result may be represented by a $M_l E(\beta_l)$ bit string.
- The communication from each party to the final party is, on average, $p(l)E(\beta_l)$ per implementation.
- The total communication to the final party is the same as the entanglement consumed!

# Summary

This work improves on previous methods for implementing unitaries in the following ways:

1. We have simplified the ancilla.
2. The entanglement required is reduced.
3. We may implement multipartite unitaries.
4. The communication required may be reduced to the entanglement consumed except for the communication from the final party.

# Hamiltonian Simulation

Dominic W. Berry
Centre for Quantum Computer Technology, Macquarie University
with G Ahokas (Calgary), B C Sanders (Calgary), R Cleve (Waterloo), P Hoyer (Calgary), N Wiebe (Calgary)

MITACS

25 years cifar
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NSERC CRSNG

D. W. Berry, G. Ahokas, R. Cleve, and B. C. Sanders, Comm. Math. Phys. **270**, 359 (2007).

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INFORMATICS iCORE CIRCLE OF RESEARCH EXCELLENCE

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Thanks to Richard Cleve, Nathan Wiebe and Barry Sanders for slides in this section!

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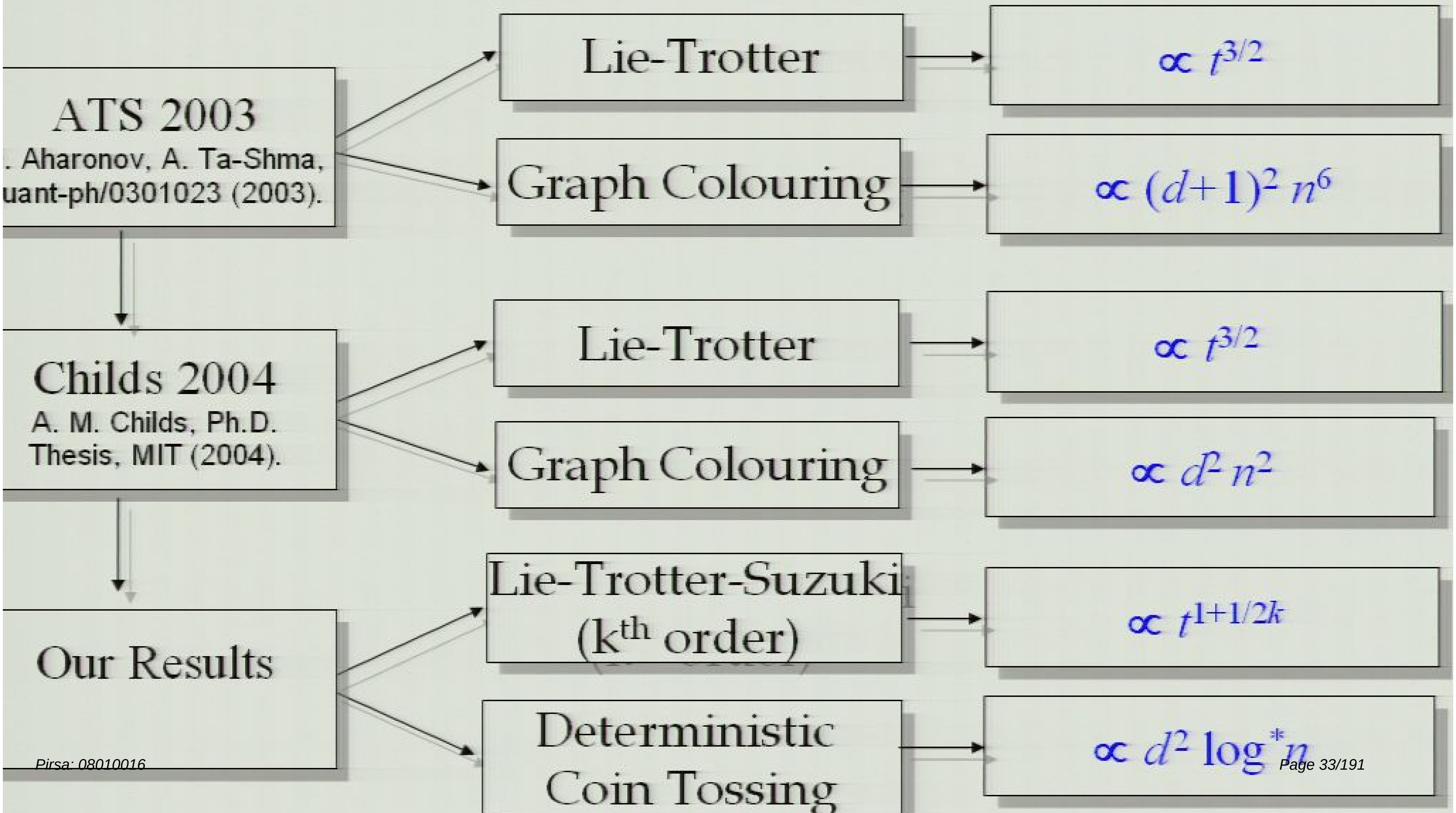
# Background

Feynman 1982: Quantum Computer would efficiently simulate dynamics of quantum systems.

Lloyd 1996: Formalized conjecture, assumed tensor product structure, showed efficient algorithm.



# Optimal in *t*; nearly constant in *n*

$\log^* n$ is the height of the smallest tower of powers of 2 that exceeds $n$:

$$2^2 = 4$$
$$2^{2^2} = 2^{(2^2)} = 16$$
$$2^{2^{2^2}} = 2^{\left(2^{(2^2)}\right)} = 2^{16} = 65536$$

so $\log^*(65536) = 4.$

Our Results

- Lie-Trotter-Suzuki ($k^{th}$ order) → $\propto t^{1+1/2k}$
- Deterministic Coin Tossing → $\propto d^2 \log^* n$

# Lie-Trotter-Suzuki integrators

- $H$: sum of local Hamiltonians $H = \sum_{i=1}^{m} H_i$
- Trotter $(m=2)$: $e^{iHt} \approx (e^{iH_1 t/2r} e^{iH_2 t/r} e^{iH_1 t/2r})^r$, $H = H_1 + H_2$.
- Number of steps for quantum computer $N \propto t^{3/2}$.
- Suzuki generalization of Trotter formula:

$$S_2(\lambda) = \prod_{j=1}^{m} e^{H_j \lambda/2} \prod_{j'=m}^{1} e^{H_{j'} \lambda/2}, \quad p_k = \left(4 - 4^{1/(2k-1)}\right)^{-1}$$

$$S_{2k}(\lambda) = [S_{2k-2}(p_k\lambda)]^2 S_{2k-2}((1-4p_k)\lambda)[S_{2k-2}(p_k\lambda)]^2$$

5 terms

- Suzuki proves for small $\lambda$:

$$\left\| \exp\left(\sum_{j=1}^{m} H_j \lambda\right) - S_{2k}(\lambda) \right\| \in O(|\lambda|^{2k+1})$$



M. Suzuki, Phys. Lett. A **146**, 319 (1990); J. Math. Phys. **32**, 400 (1991).

# Lemma: Strict bound for Lie-Trotter-Suzuki

$$\left\| \exp\left(-it\sum_{i=1}^{m} H_i\right) - \left[S_{2k}\left(-i\frac{t}{r}\right)\right]^r \right\| \le 2\frac{\left(2m5^{k-1}q_k\tau\right)^{2k+1}}{(2k+1)!\,r^{2k}}$$

$$\tau = t \times \max_j \left\| H_j \right\|$$

$$q_k = \prod_{k'=2}^{k} \left(1 - 4p_{k'}\right)$$

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$$\tau = t \times \max_j \left\|H_j\right\|$$

$$q_k = \prod_{k'=2}^{k}\left(1-4p_{k'}\right)$$

$$12m5^{k-1}q_k\tau / r \le 1,$$

$$\frac{3}{2}\frac{\left(2m5^{k-1}q_k\tau\right)^{2k+1}}{(2k+1)!\,r^{2k}} \le 1.$$

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$$r = \left\lceil \left(2m5^{k-1}q_k\tau\right)^{1+1/2k}\left[\frac{2}{(2k+1)!\epsilon}\right]^{1/2k} \right\rceil$$

# Theorem: Simulation cost nearly linear in time

Theorem: $$N \le \frac{m 5^{2k} \left( m q_k \tau \right)^{1+1/2k}}{\left[ (2k+1)! \varepsilon \right]^{1/2k}}$$

for $(2k+1)!\varepsilon \le 1 \le 2m5^{k-1} q_k \tau$

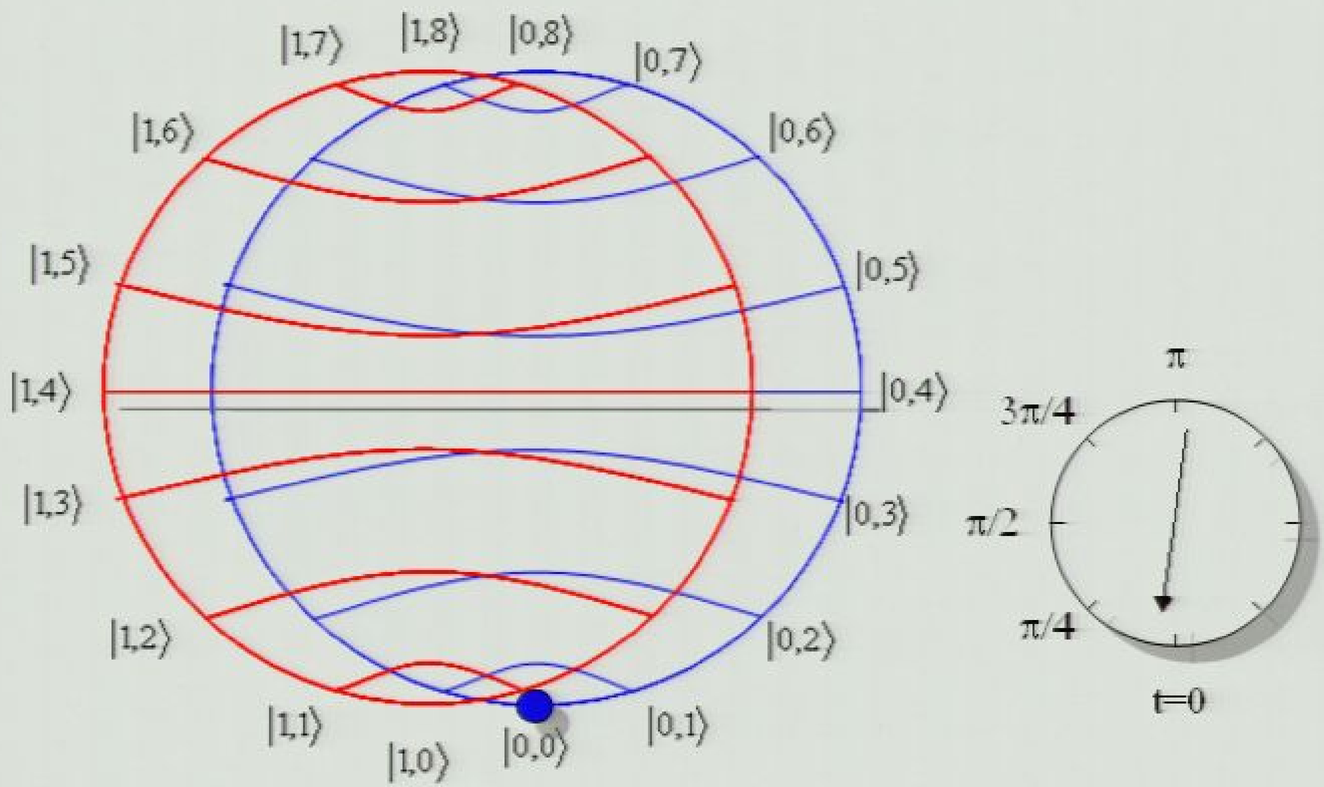
Optimal choice of $k$: $k \approx \frac{1}{2}\sqrt{\log_5\left(\frac{m\tau}{\varepsilon}\right)}$

Then $N \le 2m^2\tau \exp\left[ 2\sqrt{\ln 5 \times \ln\left(m\tau/\varepsilon\right)} \right]$

# Simulation time cannot be sublinear in *t*

**Theorem:** For all positive integers $N$ there exists a row-computable 2-sparse Hamiltonian $H$ such that simulating the evolution of $H$ for scaled time $\tau = \pi N/2$ within precision $1/4$ requires at least $\tau/2\pi$ queries to $H$.

$X_j =$ 0 1 1 0 1 0 0 1



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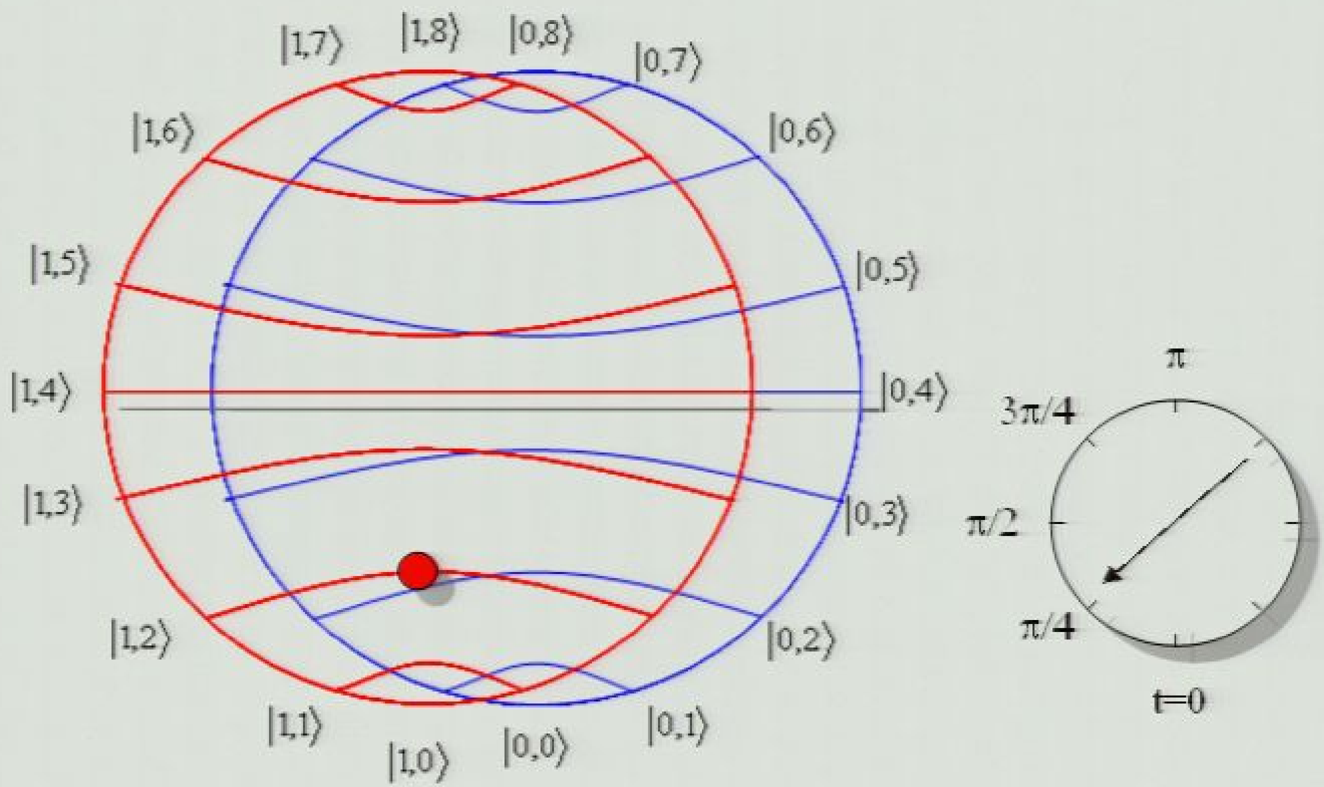

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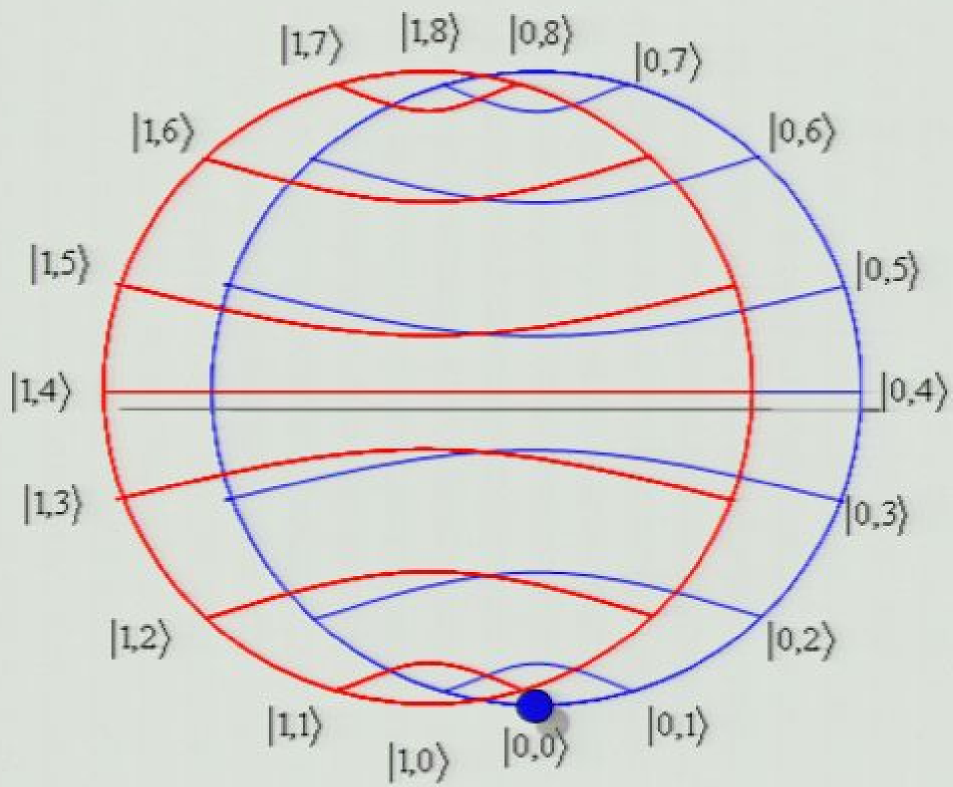
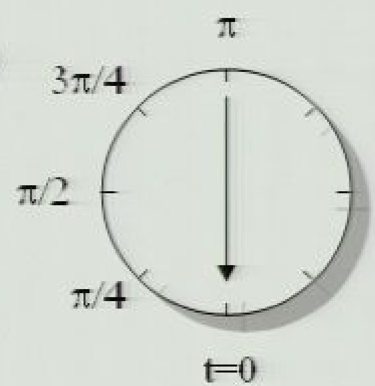
**Theorem:** For all positive integers $N$ there exists a row-computable 2-sparse Hamiltonian $H$ such that simulating the evolution of $H$ for scaled time $\tau = \pi N/2$ within precision $1/4$ requires at least $\tau/2\pi$ queries to $H$.

$X_j =$ 0 1 1 0 1 0 0 1



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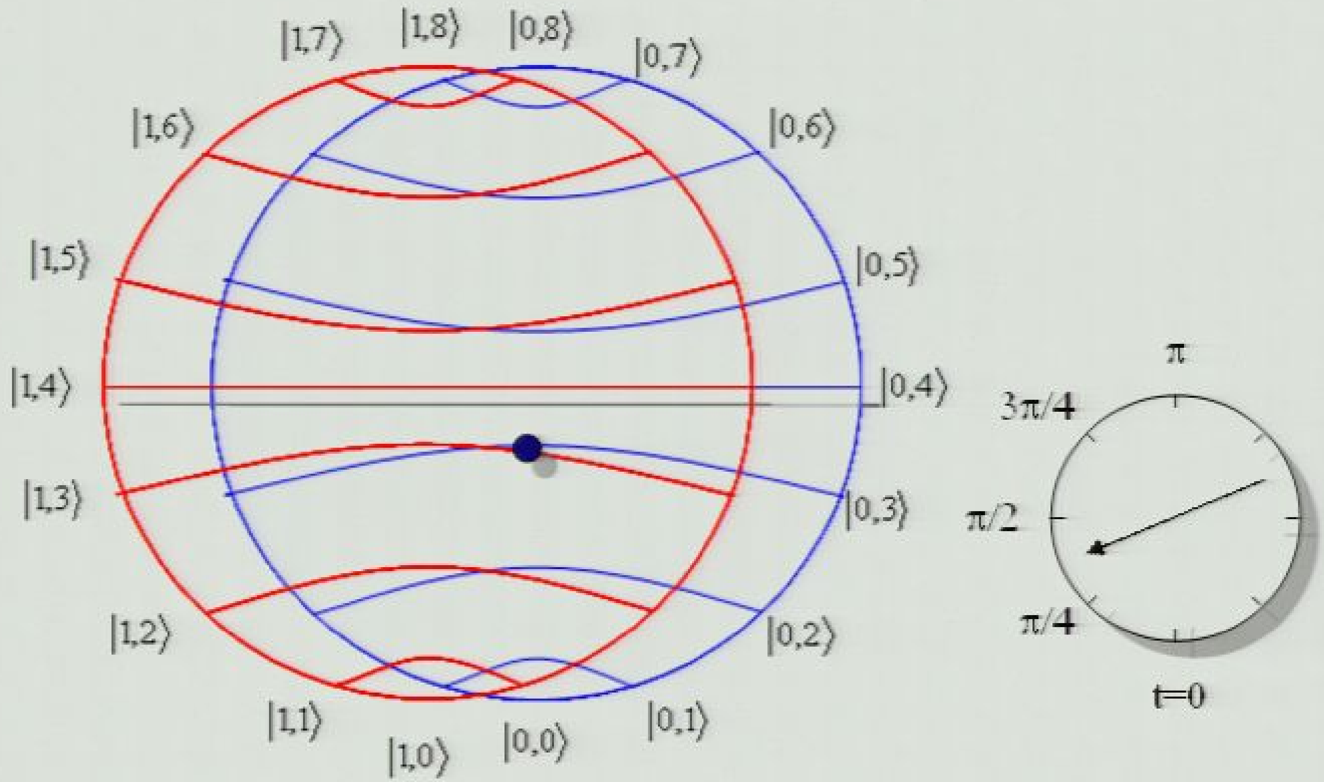

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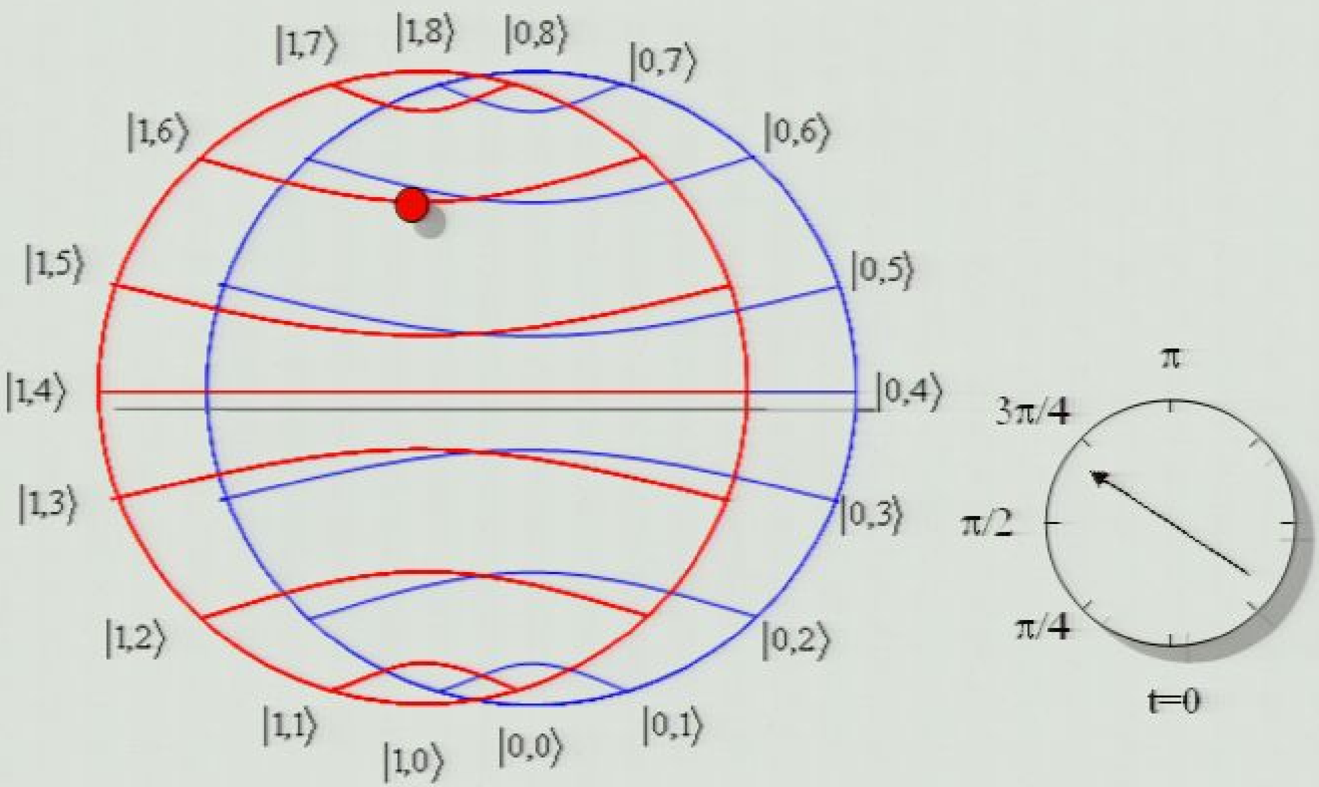

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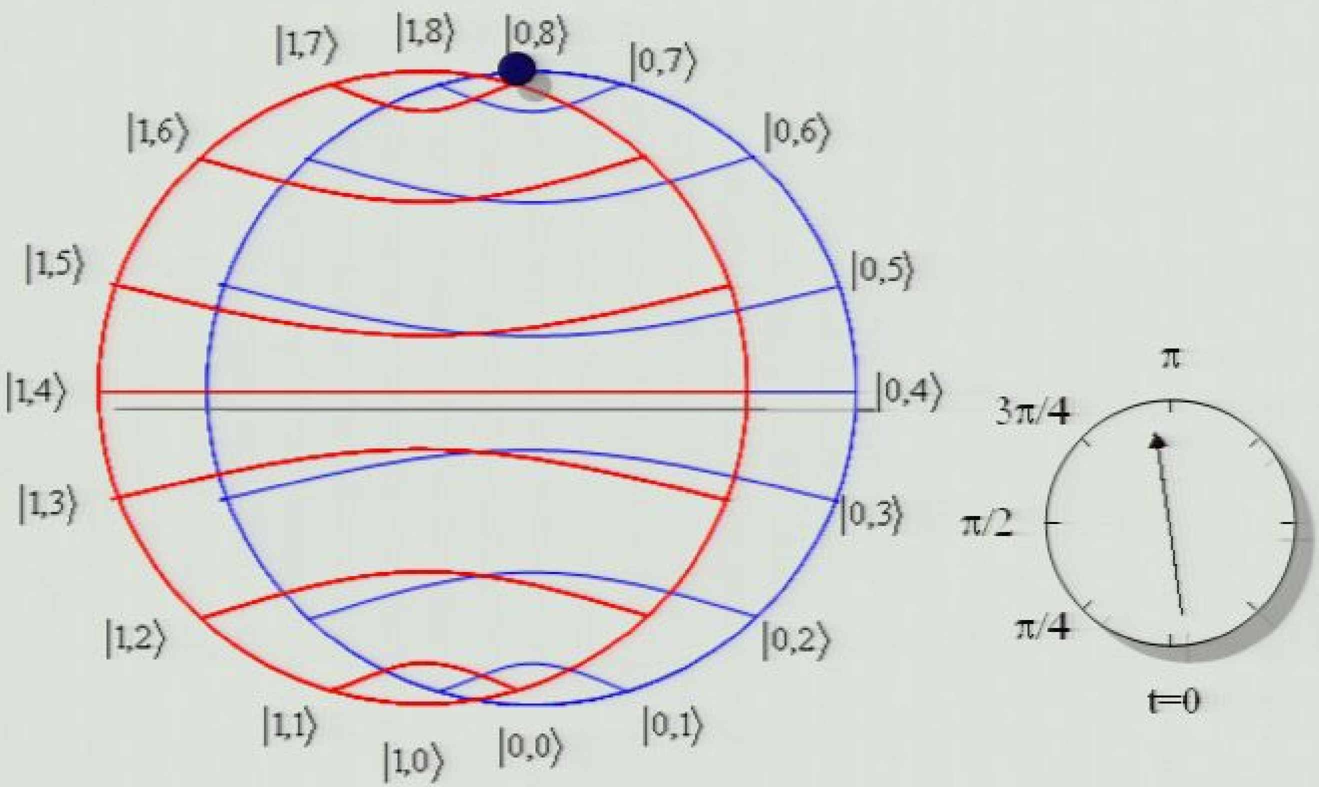


# Lemma (decomposition of *H* unknown)

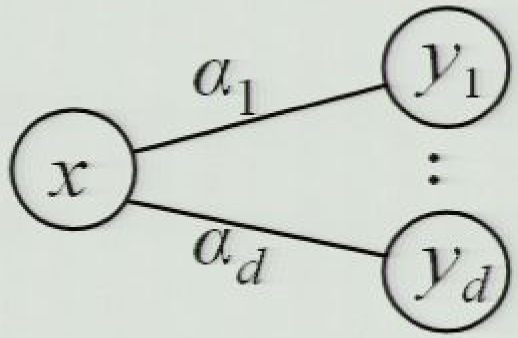
$\exists$ decomposition $H = \sum_{j=1}^{m} H_j$, with each $H_j$ 1-sparse,

such that $m = 6d^2$, and each query to any $H_j$ can be

simulated by $O(\log^* n)$ queries to $H$.

# Graph associated with $H$

Connect $x$ to $y_k(x)$ with an edge of weight $\alpha_k(x)$



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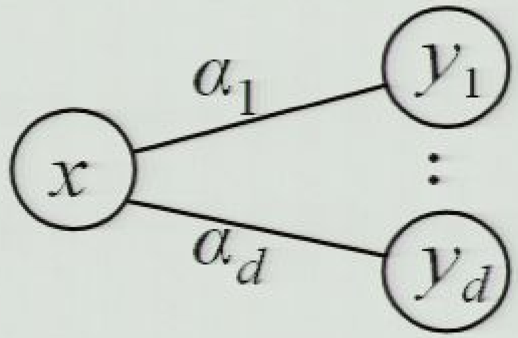


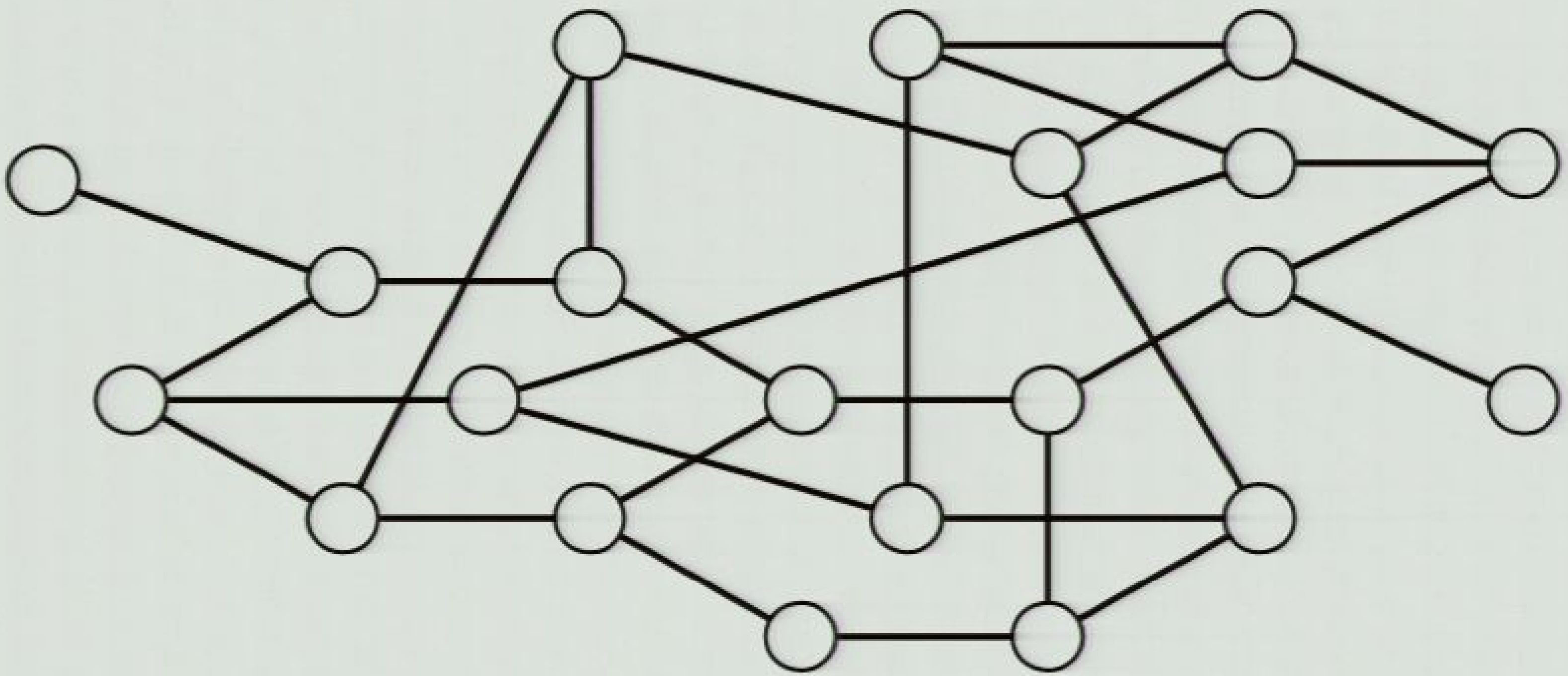

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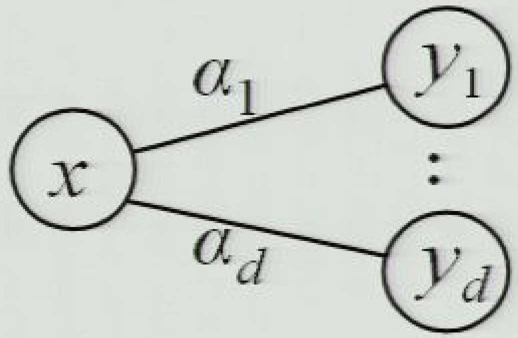
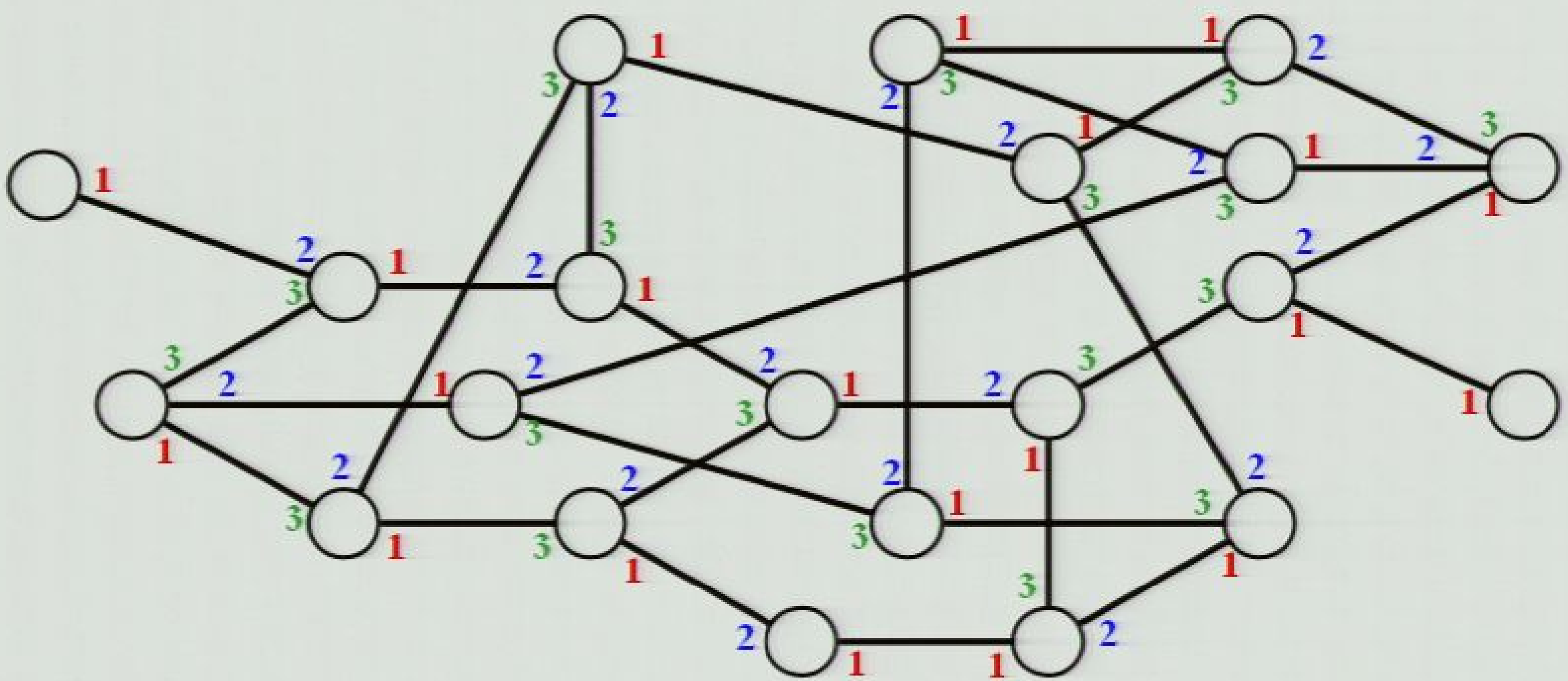
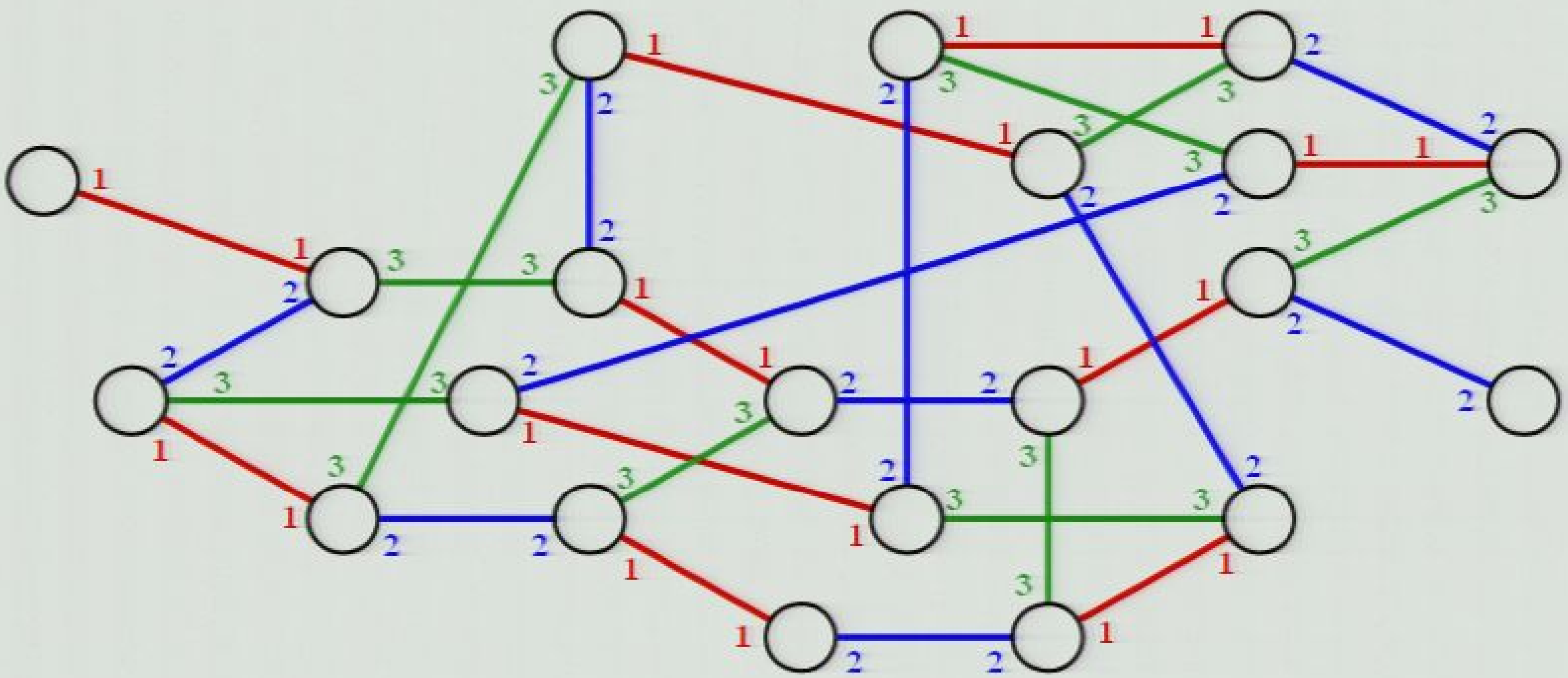


# *Symmetrically* labeled graphs



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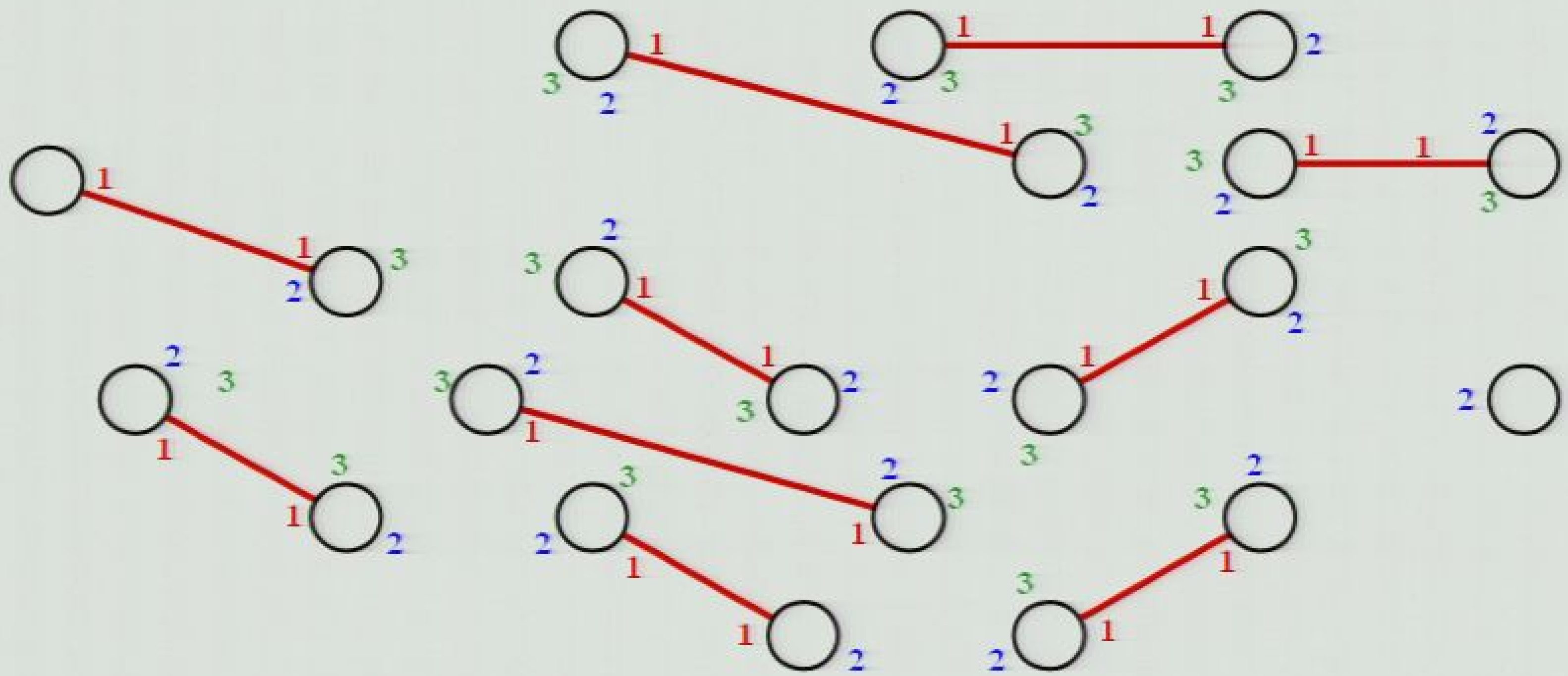

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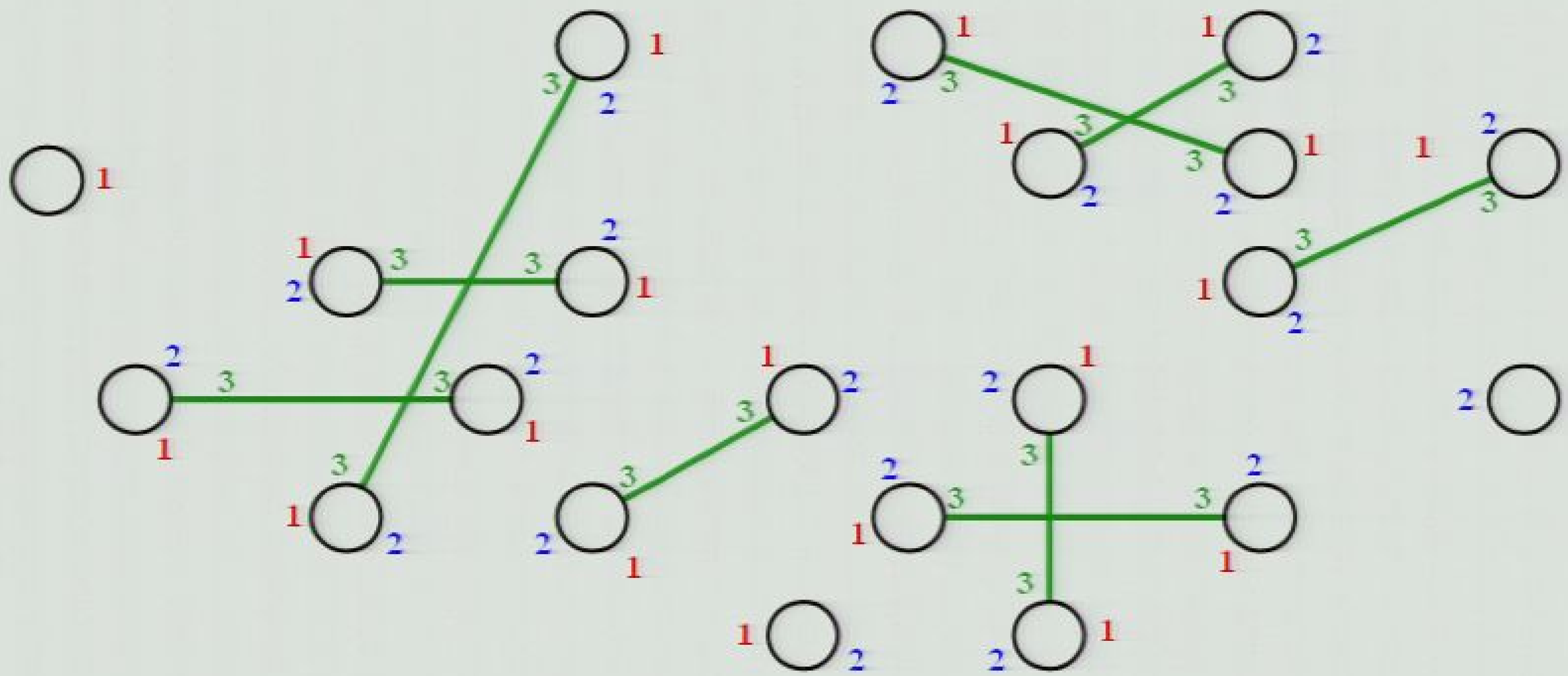

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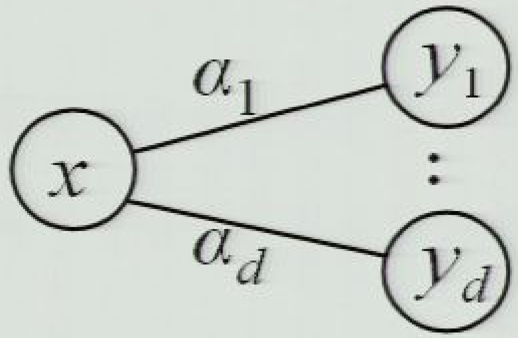


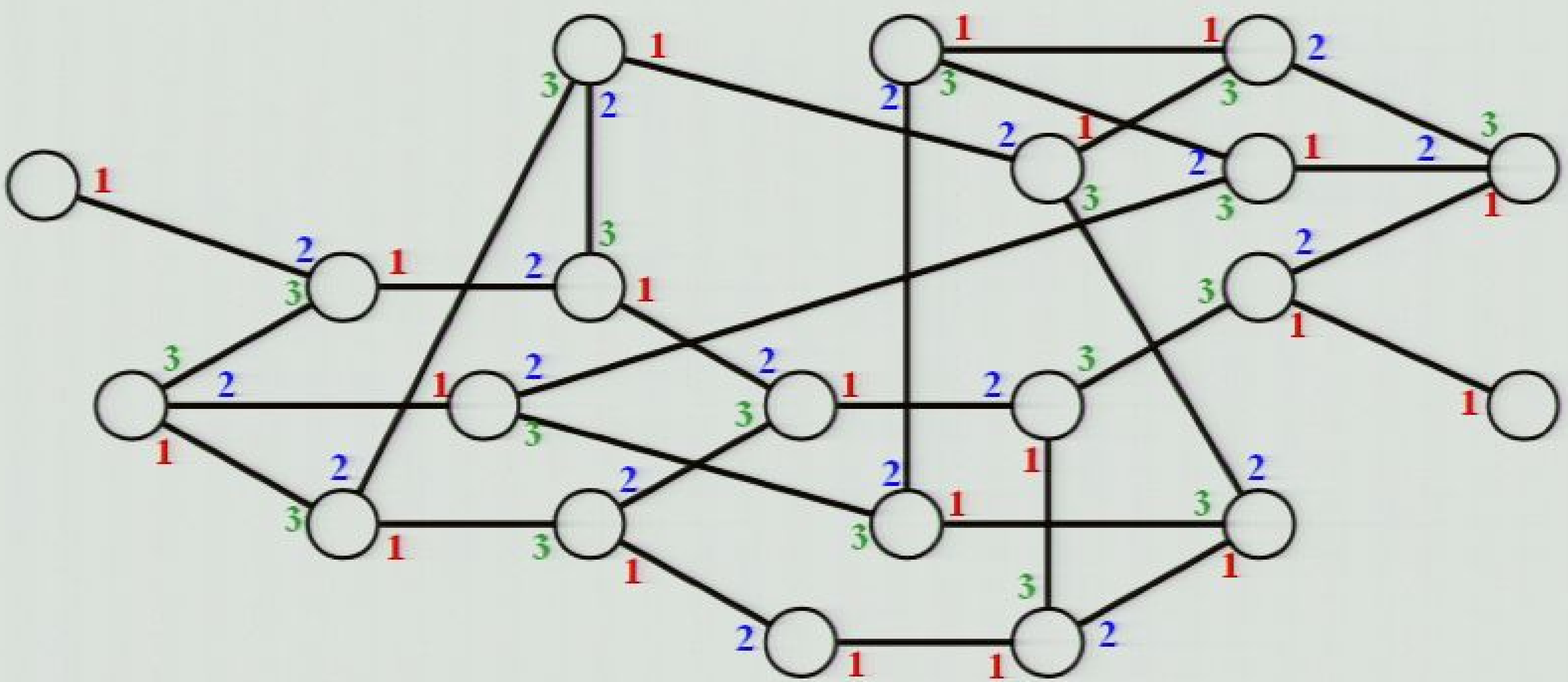

# Non-symmetric case

Modify labeling to be symmetric (with an overhead cost)

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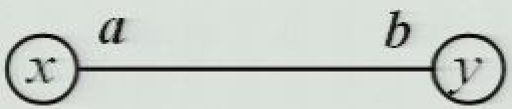


with $x < y$

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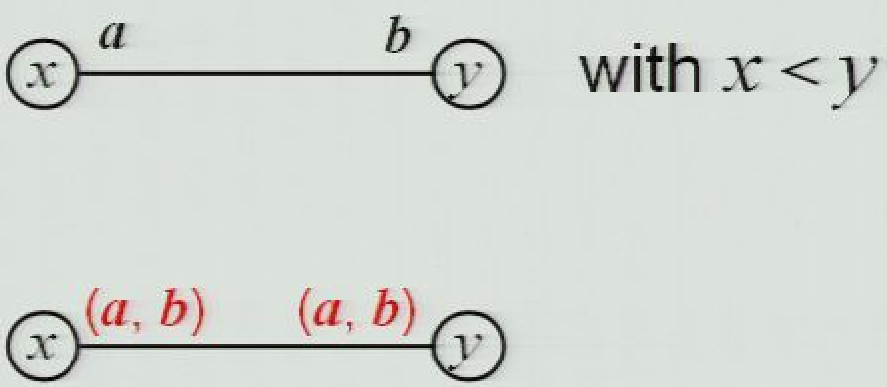

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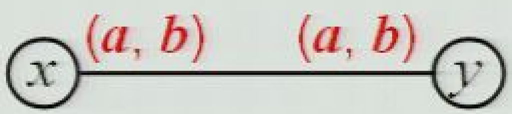
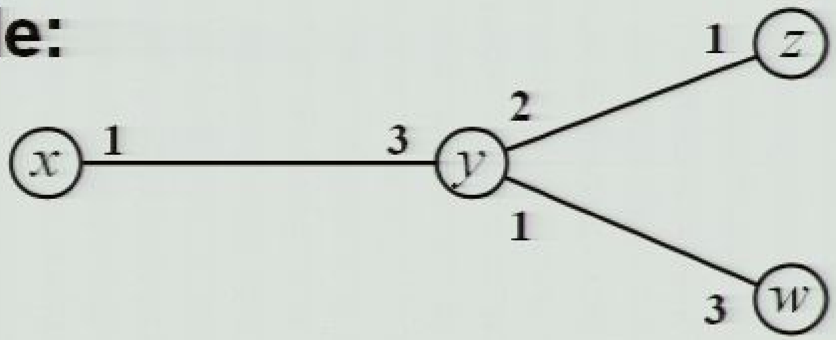



We now have $d^2$ labels instead of $d$ labels, but a ***symmetric*** labeling

**Example:**




with $z < y$

with $y < w$

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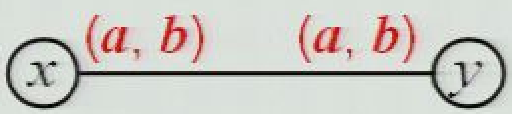


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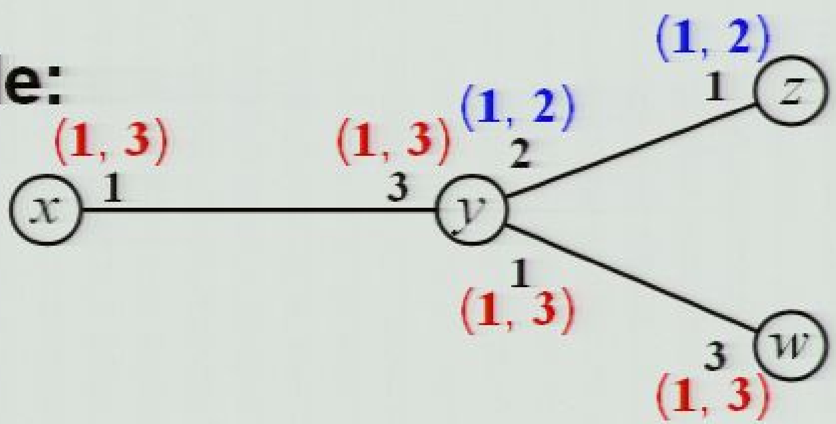


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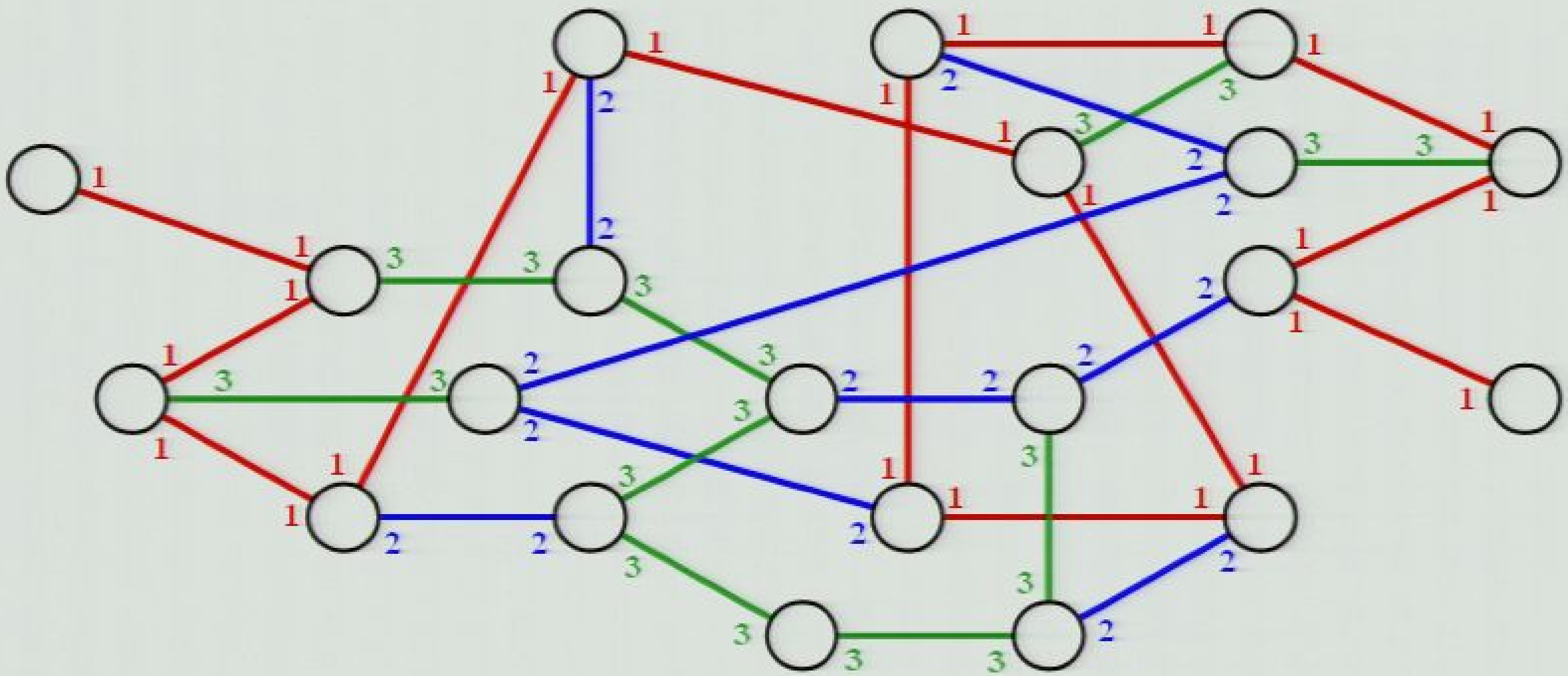
# Graph with monochromatic paths



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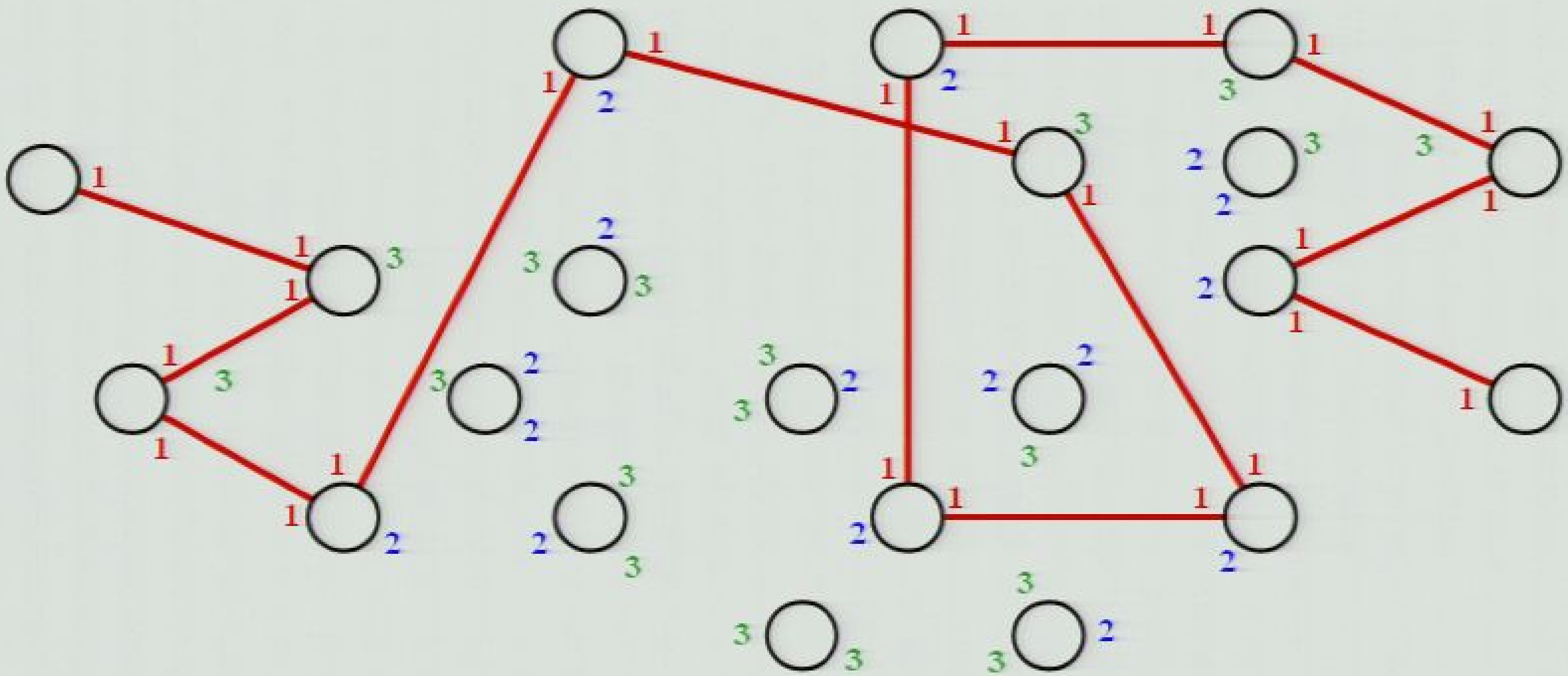

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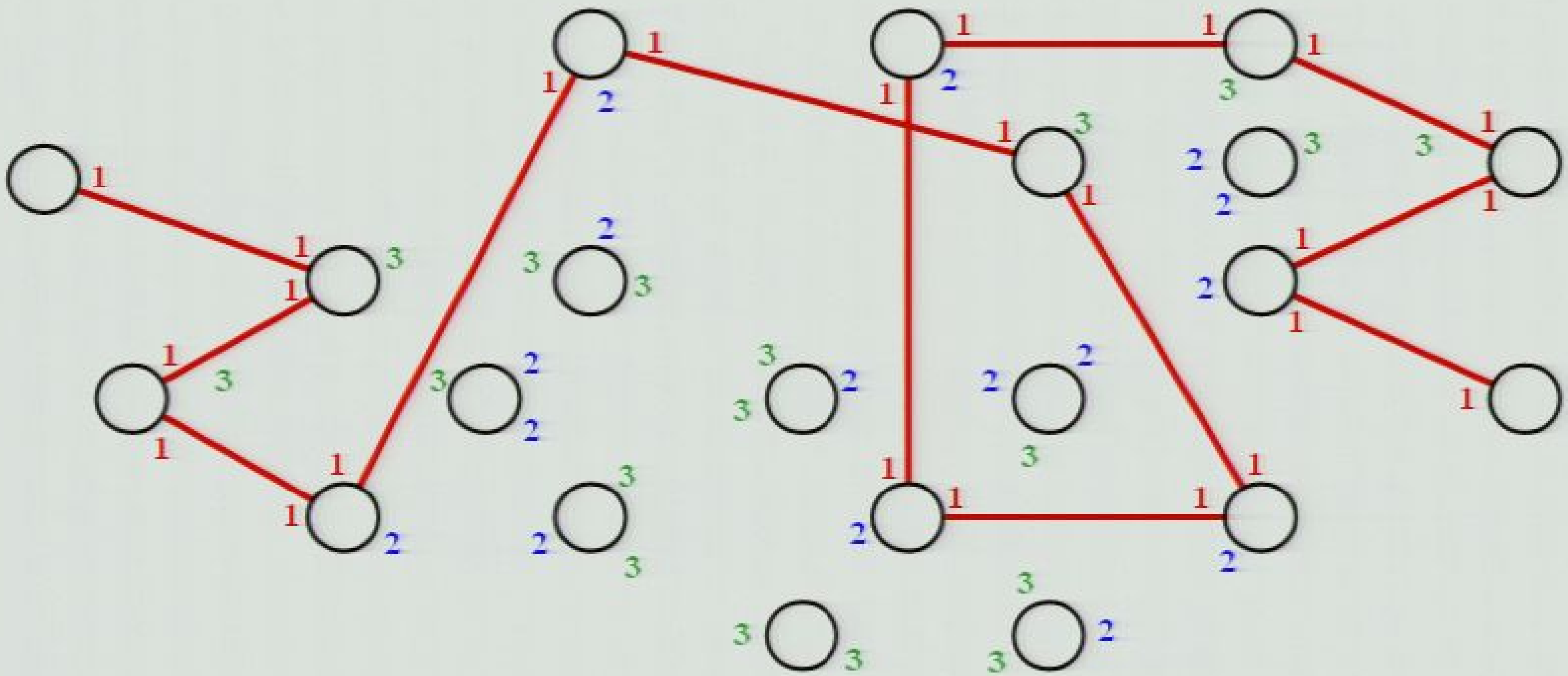
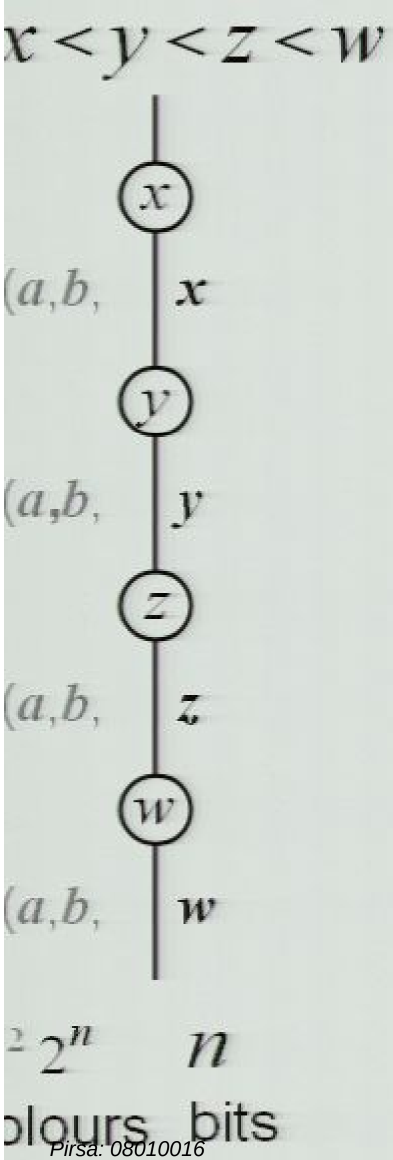


To break up the paths, we increase the number of colours




# "Deterministic coin-tossing"

R. Cole, U. Vishkin, Inform. and Control **70**, 32 (1986).

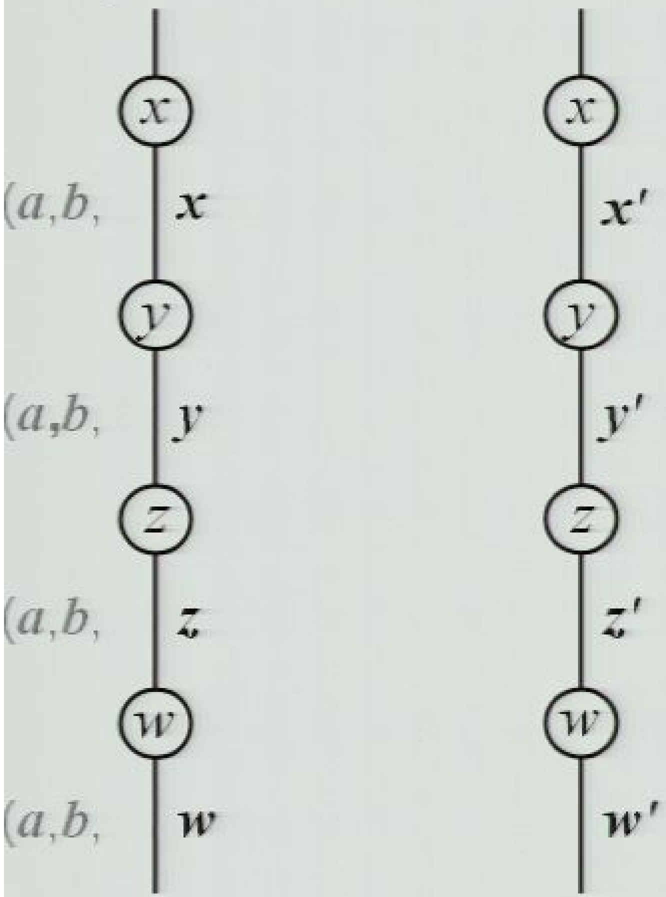
010 ↓

**Example:** $y = 01\textcolor{red}{0}00101$

$z = 01\textcolor{red}{1}01101$

Then $y' = (010,0)$

$x < y < z < w$




$2^n$ $n$

olours bits

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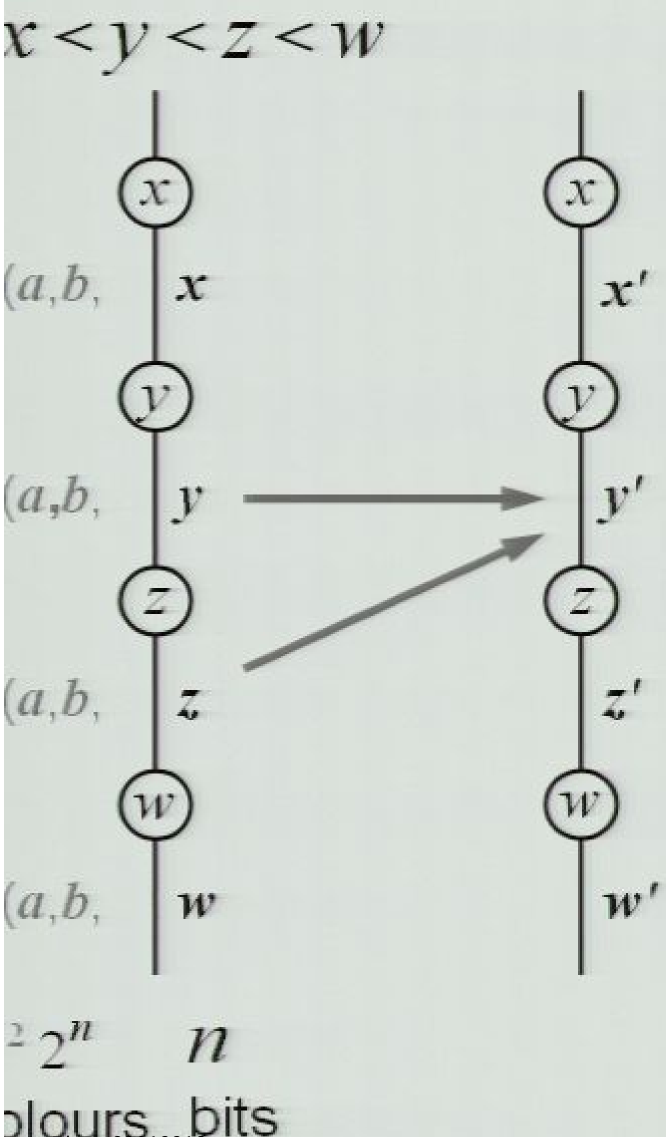


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**Example:** $y = 01\underset{\uparrow 010}{0}00101$

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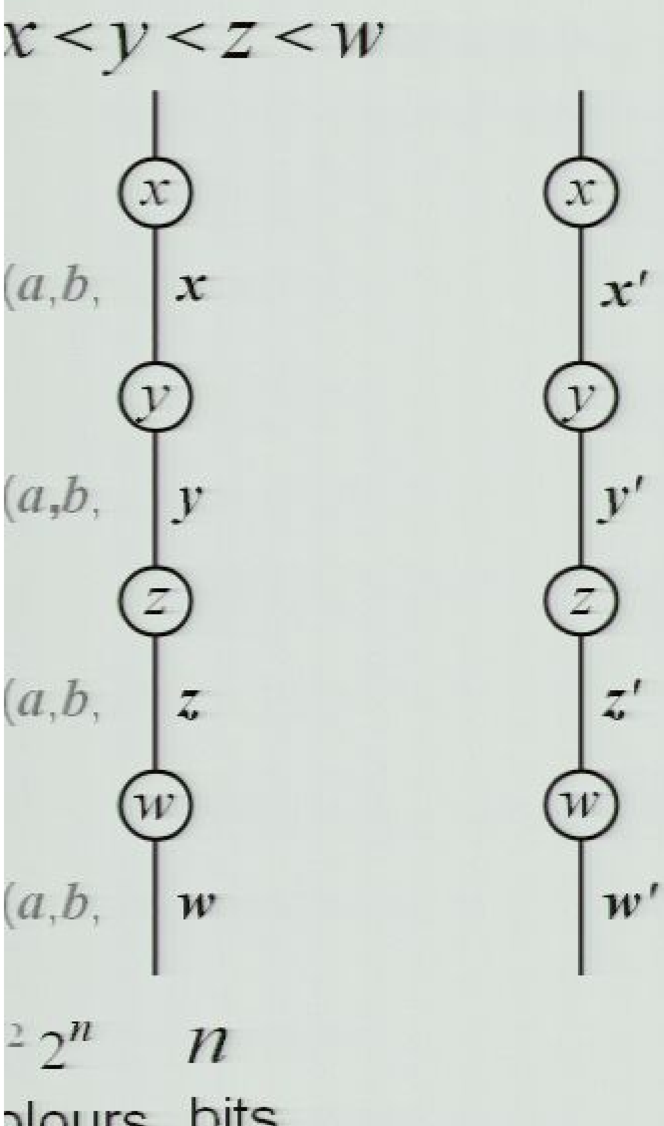


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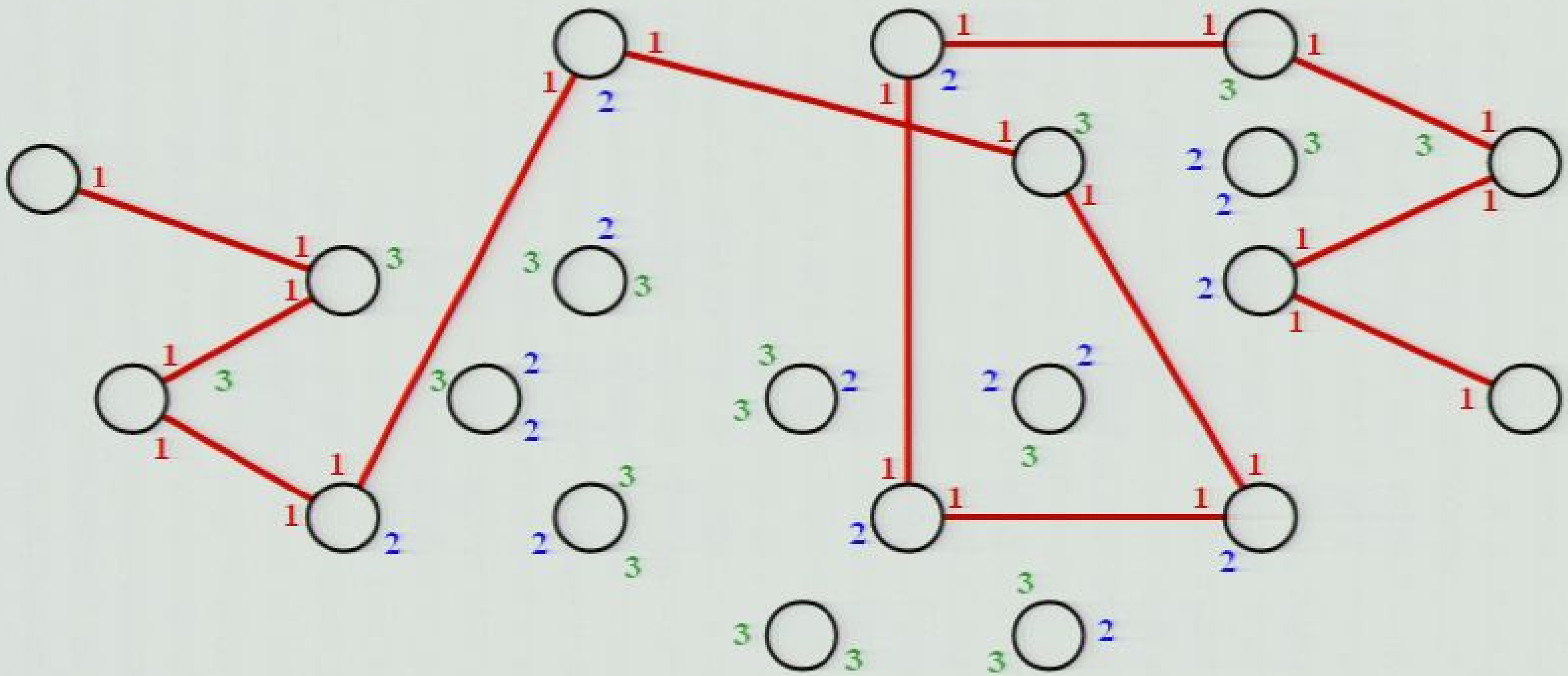

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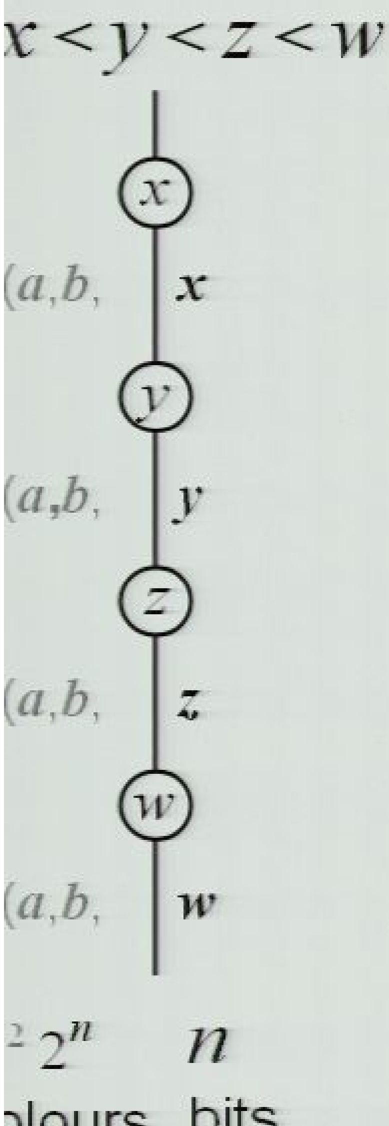


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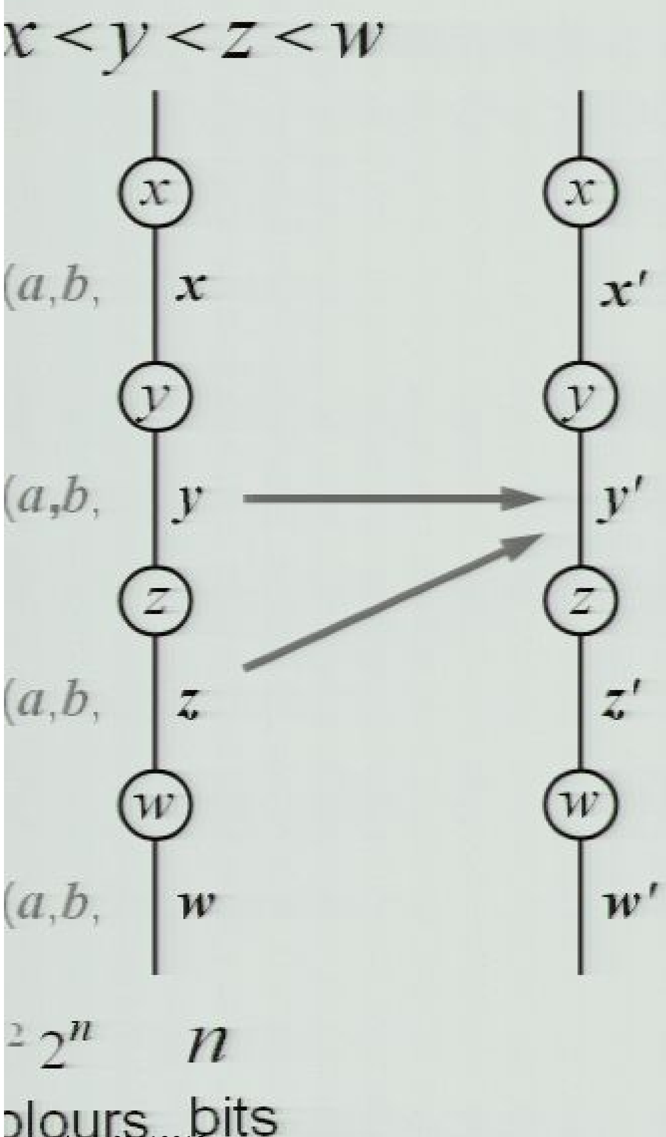


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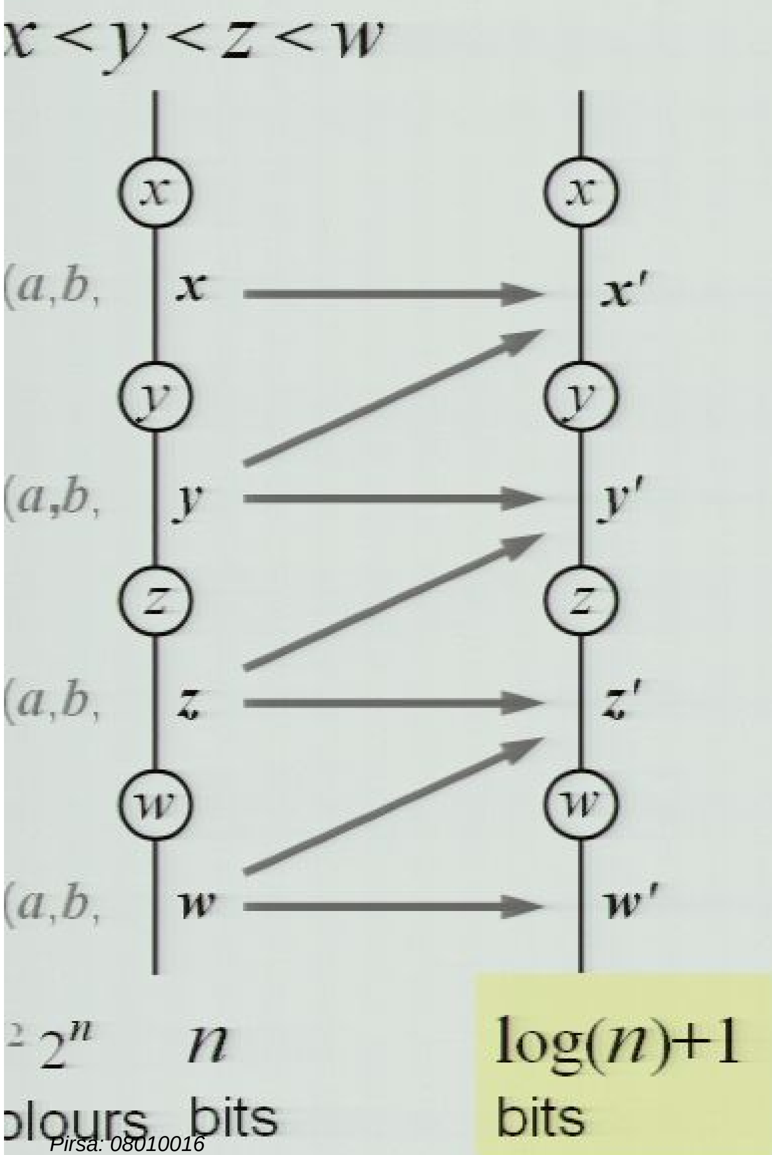


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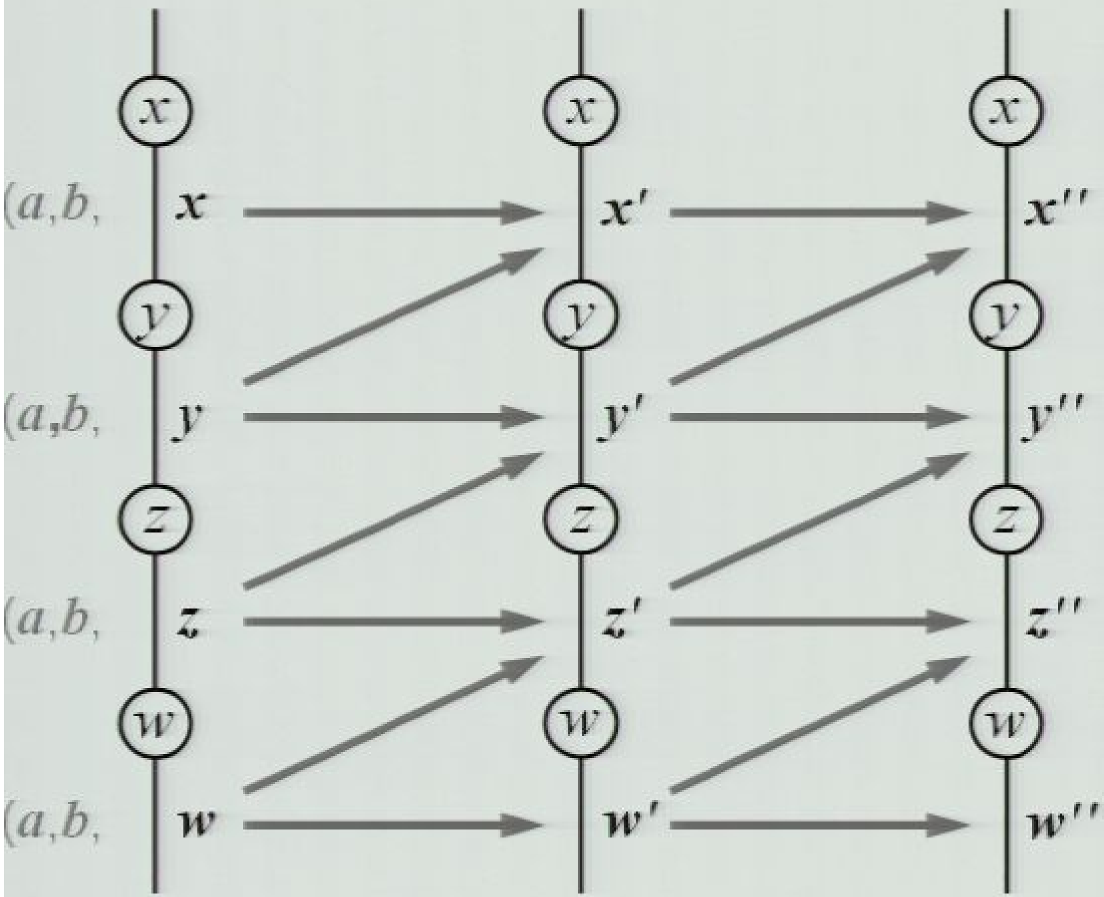


$^2\ 2^n$ olours $n$ bits

$\log(n)+1$ bits

$\log(\log(n)+1)+1$ bits

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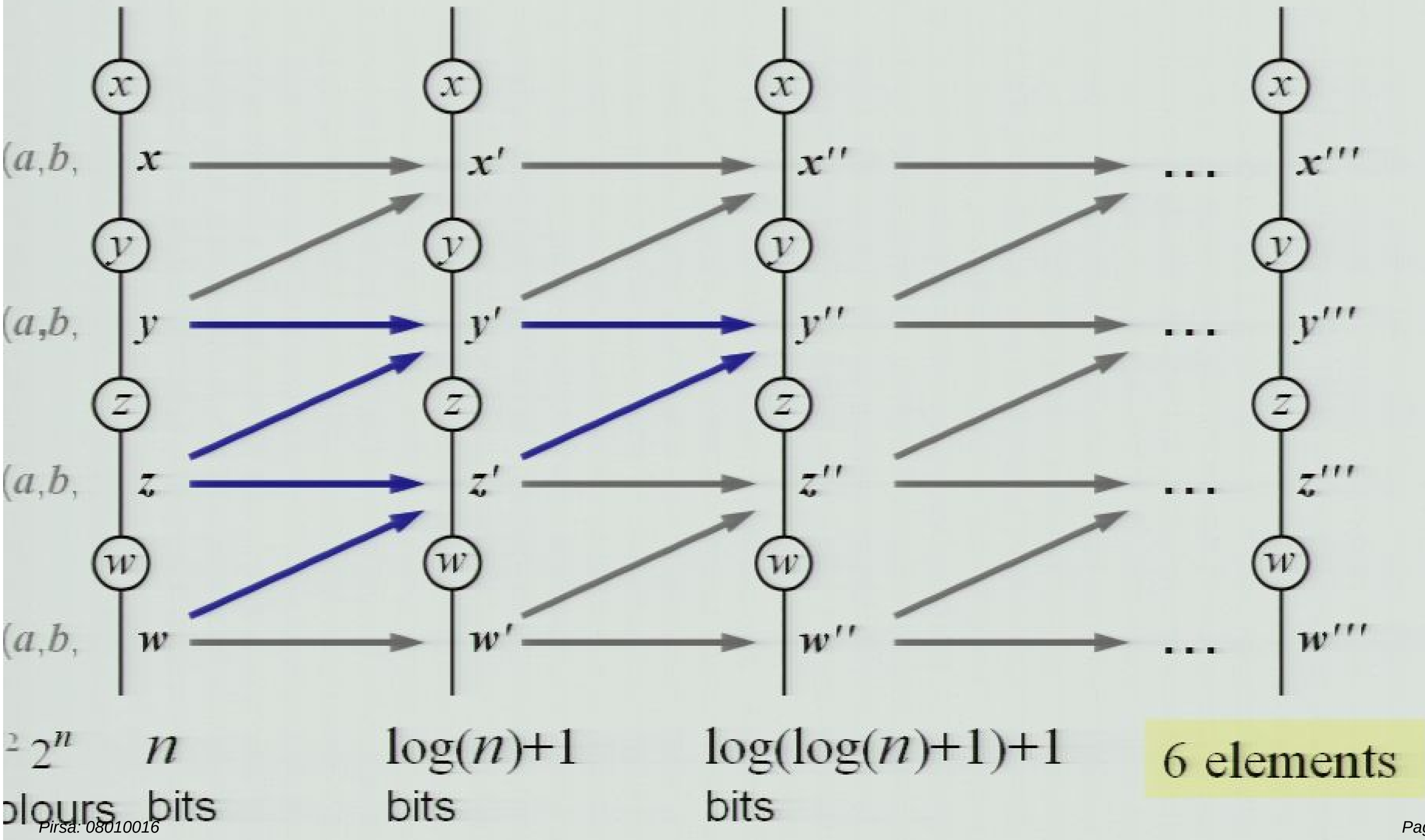

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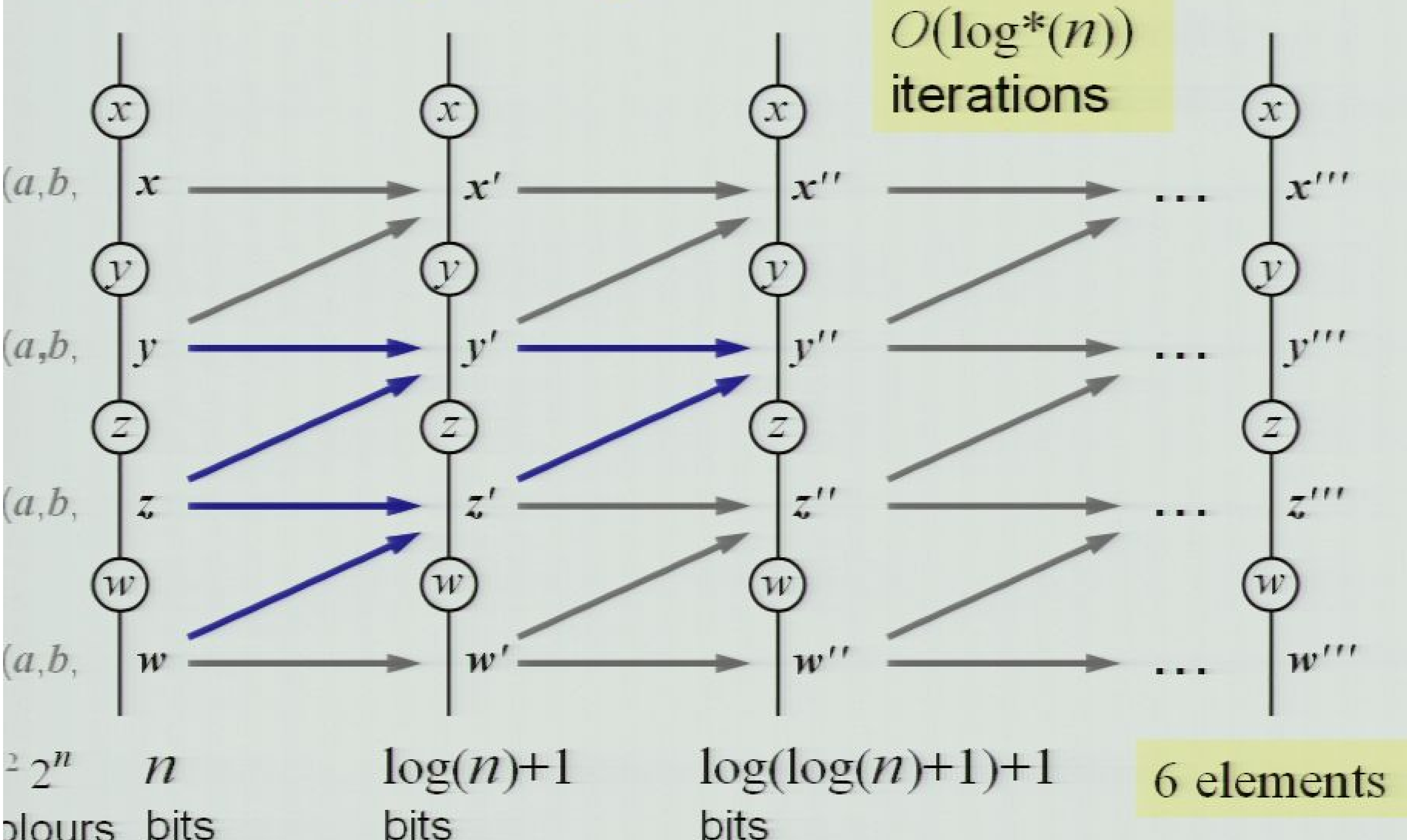


**Theorem:** The number of black-box calls for given $k$ is

$$N_{\mathrm{bb}} \in O\left((\log^* n) d^2 5^{2k} (d^2 q_k \tau)^{1+1/2k} / [(2k+1)!\epsilon]^{1/2k}\right)$$

with $\log^* n \equiv \min\{r | \log_2^{(r)} n < 2\}$ (the $^{(r)}$ indicating the iterated logarithm).

## Sketch of Proof:

# of $H_j$'s is $m = 6d^2$. Need to call the black-box $O(\log^* n)$ times for each $H_j$.

Substituting into theorem for upper bound on $N$ gives result.

# Summary

- We use the Lie-Trotter-Suzuki integrators for scaling close to linear in the evolution time, $t$.
- Simulations sublinear in $t$ are impossible.
- We place an upper bound on the number of steps needed, rather than just giving the scaling. This allows us to estimate the optimal order, $k$.
- For sparse Hamiltonians we show how to decompose them with scaling close to constant in the number of qubits, $n$.

# Adaptive Phase Estimation

Dominic Berry & Howard Wiseman

+

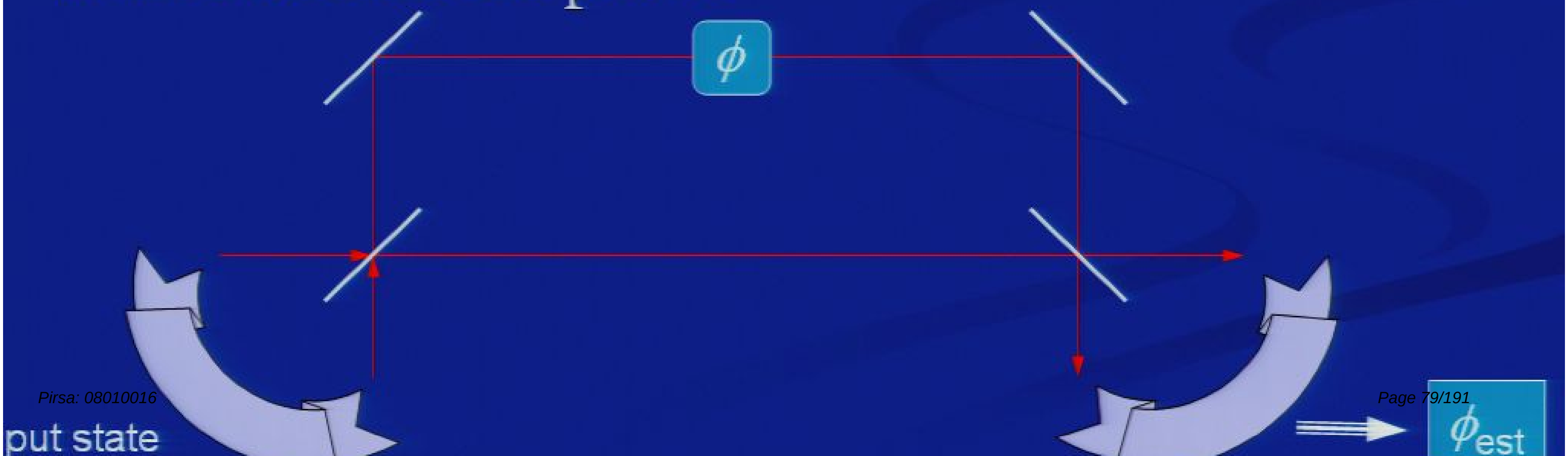
Elanor Huntington, Brendon Higgins,
Steve Bartlett, Geoff Pryde

CQCT, School of Information Technology and Electrical Engineering, UNSW, Canberra
Centre for Quantum Dynamics, School of Science, Griffith University, Brisbane
CQCT, Quantum Information Science & Cryptography, Macquarie University, Sydney
Quantum Information Theory group, School of Physics, University of Sydney
CQCT, Centre for Quantum Dynamics, School of Science, Griffith University, Brisbane

# Types of phase

- Single-mode phase


- Interferometric phase



# Single-mode phase


- The canonical phase measurement is gives[1]

$$P(\phi) = \mathrm{Tr}[\rho F(\phi)]$$

where

$$F(\phi) = \frac{1}{2\pi} |\phi\rangle\langle\phi|, \qquad |\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle$$

- If there is an upper limit $N$ on the photon number, the optimal state has phase uncertainty[2]

$$\Delta\phi = \tan\left(\frac{\pi}{N+2}\right) \sim \frac{\pi}{N}$$



[1] U. Leonhardt *et al*, Phys. Rev. A **51**, 84 (1995).

[2] G. S. Summy and D. T. Pegg, Opt. Comm. **77**, 75 (1990).

# Single-mode phase

Input state $\Longrightarrow$ $\phi_{\text{est}}$

Here Δϕ is the square root of the Holevo phase variance

$$\Delta\phi^2 = \left|\left\langle e^{i\phi}\right\rangle\right|^{-2} - 1$$

This is equivalent to the usual standard deviation for narrowly peaked distributions.

It also satisfies the uncertainty principle

$$\Delta n \Delta\phi \geq 1/2$$

- The canoni

  where

  $F(\phi)$

- If there is an ... number, the optimal state has ... ase uncertainty[2]

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# The Heisenberg limit

- The Heisenberg uncertainty principle gives

$$\Delta n \Delta \phi \geq 1/2$$

where

$$\Delta \phi^2 = \left| \left\langle e^{i\phi} \right\rangle \right|^{-2} - 1$$

- This implies the lower bound for scaling of the phase uncertainty

$$\Delta \phi \geq \frac{1}{2\Delta n}$$

- For an upper limit on the photon number, $N$,

$$N \geq \Delta n$$

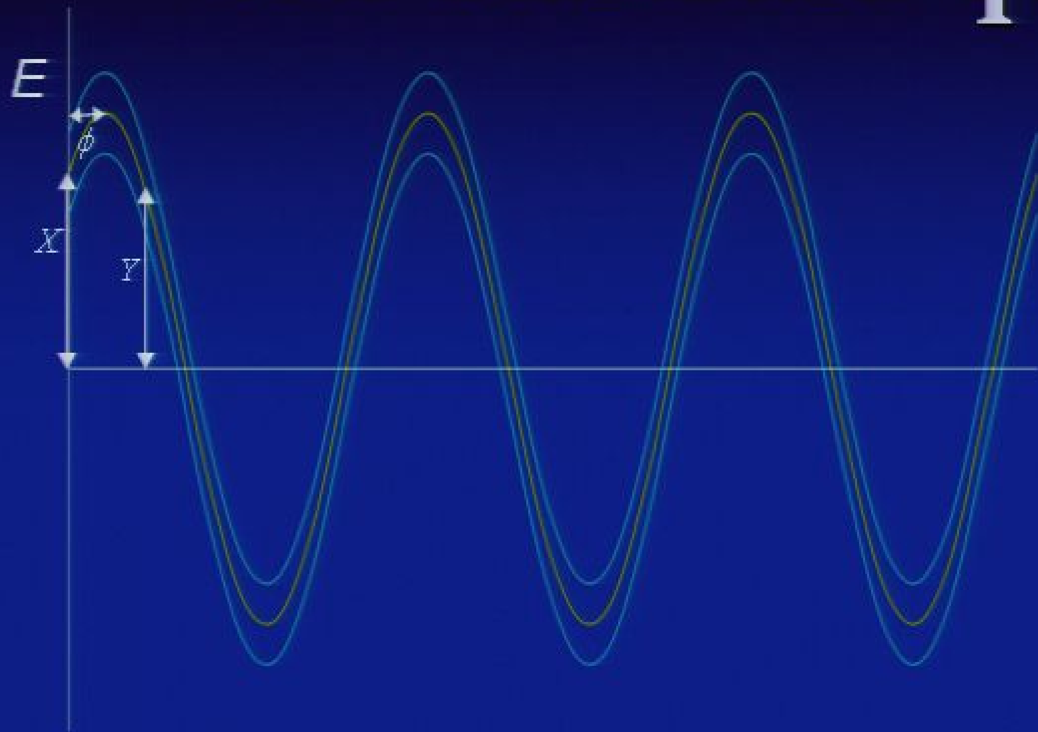
so

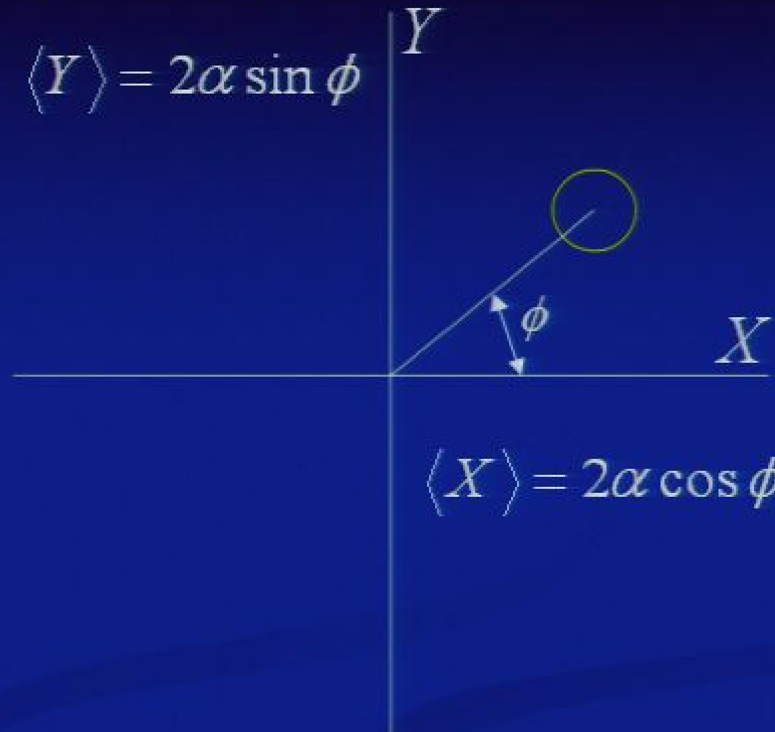
$$\Delta \phi \geq \frac{1}{2N}$$

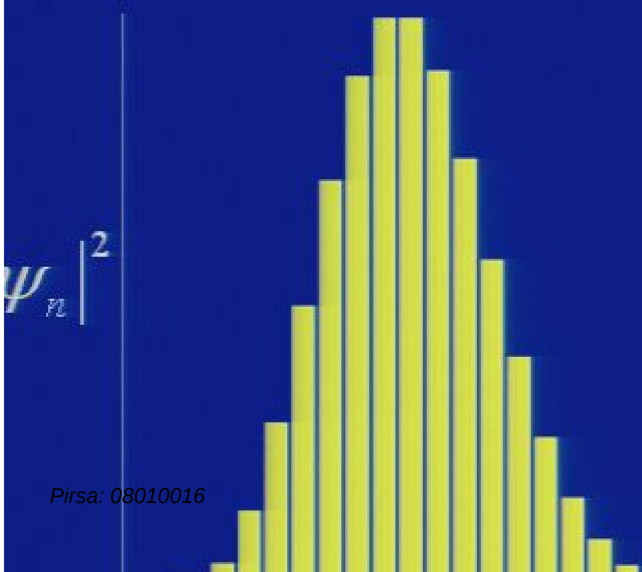
- The exact limit is

$$\Delta \phi \geq \tan\left( \frac{\pi}{N+2} \right) \sim \frac{\pi}{N}$$

# The standard quantum limit




- The uncertainty principle gives

$$\Delta X \, \Delta Y \geq 1$$

- For coherent states

$$\Delta X = \Delta Y = 1$$

- Estimating $\phi$ by measuring $Y$ gives the uncertainty

$$\Delta\phi \approx \frac{\Delta Y}{2\alpha} = \frac{1}{2\sqrt{\langle n \rangle}}$$

# The Heisenberg limit vs the standard quantum limit

- The standard quantum limit (SQL) is

$$\Delta\phi \propto 1/\sqrt{N}$$

- The fundamental limit is the Heisenberg limit of

$$\Delta\phi \propto 1/N$$

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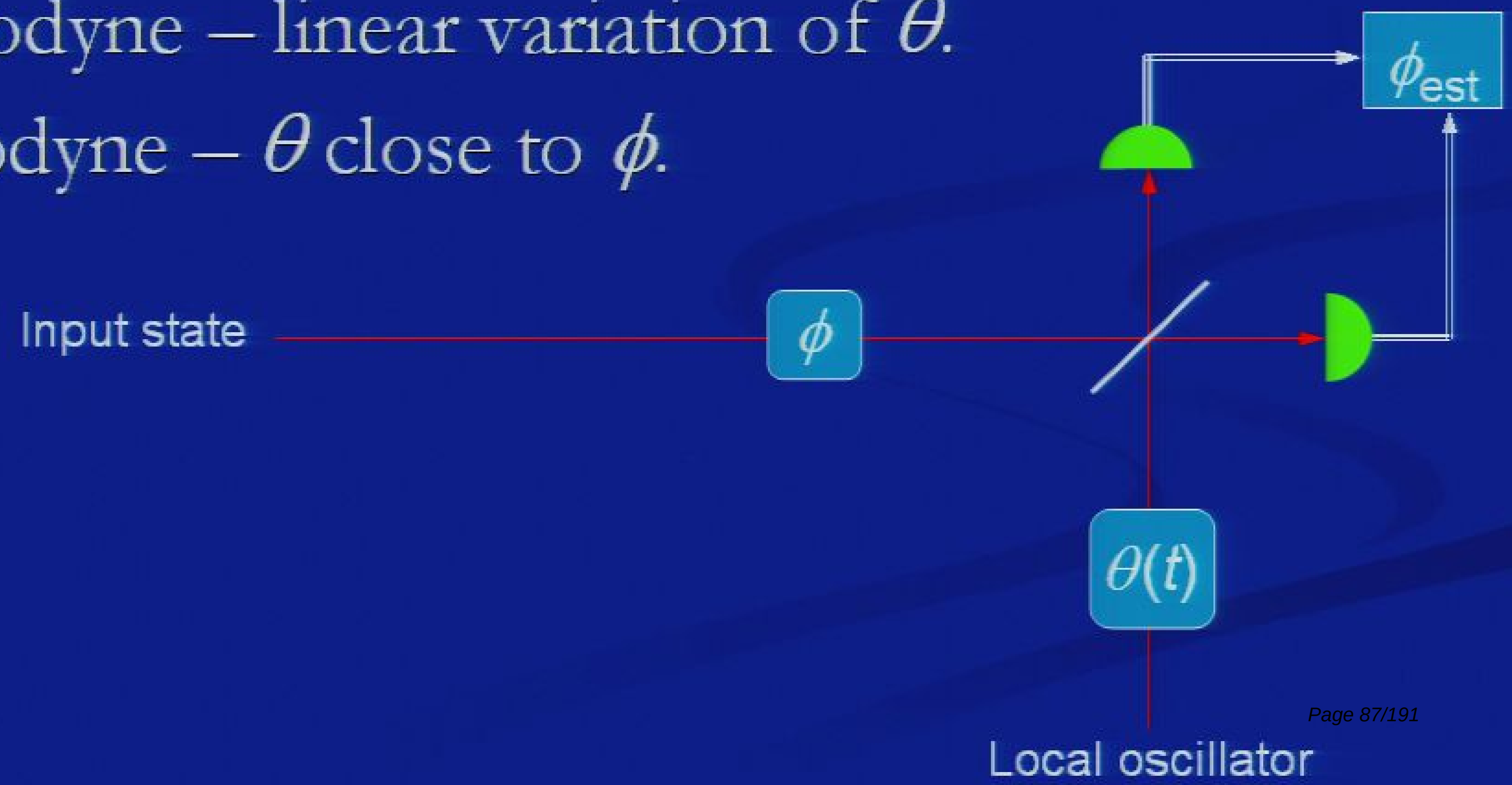
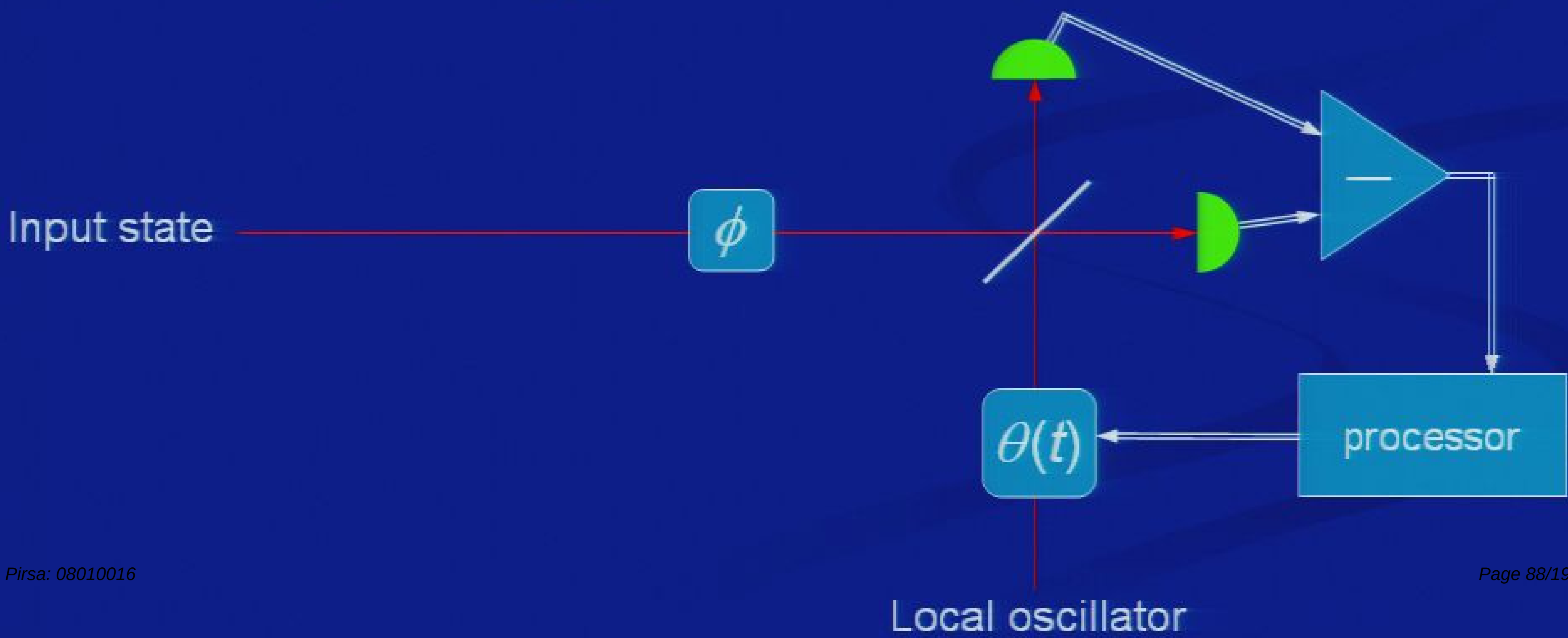
# Dyne measurements

- Phase is measured relative to a local oscillator. This is a "dyne" measurement.
- Heterodyne – linear variation of $\theta$.
- Homodyne – $\theta$ close to $\phi$.



# Adaptive phase measurements

- The results during the measurement are used to obtain an estimate of the phase.
- That estimate of the phase is used to feed back to the local oscillator phase to approximate a homodyne measurement.



# Adaptive phase measurements

- The difference of the photocounts, gives the photocurrent

$$I(t) = \lim_{\delta t \to 0} \lim_{\beta \to 0} \frac{\delta n_+(t) - \delta n_-(t)}{\beta \delta t}$$

- We then define the quantities, in terms of scaled time $v$,

$$A_v = \int_0^v I(u) e^{i\theta(u)} du, \qquad B_v = -\int_0^v e^{2i\theta(u)} du$$

- We also use the quantity

$$C_v = A_v v + B_v A_v^*$$

- The POVM for the measurement is[1]

$$G(A,B) = Q(A,B) \left| \tilde{\psi}(A,B) \right\rangle \left\langle \tilde{\psi}(A,B) \right|$$

Probability distribution for vacuum state.

$$\left| \tilde{\psi}(A,B) \right\rangle = \exp[\tfrac{1}{2} B (a^\dagger)^2 + A a^\dagger] \left| 0 \right\rangle$$

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Probability distribution for vacuum state.

$$\left| \tilde{\psi}(A,B) \right\rangle = \exp[\tfrac{1}{2} B (a^\dagger)^2 + A a^\dagger] \left| 0 \right\rangle$$

# The theoretical limit

- The state is proportional to a squeezed state

$$|\tilde{\psi}(A,B)\rangle \propto |\alpha,\xi\rangle = \exp(\alpha a^\dagger - \alpha^* a)\exp[\tfrac{1}{2}\xi^* a^2 - \tfrac{1}{2}\xi(a^\dagger)^2]|0\rangle$$

$$\alpha = \frac{A+BA^*}{1-|B|^2}, \qquad \xi = -\frac{B\,\mathrm{atanh}|B|}{|B|}$$

- The probability distribution for $A$ and $B$ is therefore given by

$$P(A,B) = Q'(A,B)\left|\langle\psi|\alpha,\xi\rangle\right|^2$$

- The phase variance is approximately the sum of the intrinsic phase variance and the phase variance for $|\alpha,\xi\rangle$.
- The limit to the introduced phase variance is the same as that for an optimal squeezed state,

$$\frac{\ln\bar{n} + 2.43\ldots}{4\bar{n}^2}$$

# Achieving the theoretical limit

- The type of phase estimate used determines $Q'$. Expressed as a function of $\bar{n}$ and $\xi$, this function is peaked along a line, effectively giving $\xi$ as a function of $\bar{n}$.
- To achieve the theoretical limit, $Q'$ should give values of $\xi$ corresponding to optimally squeezed states.
- This means giving

$$|A| \propto \frac{\ln \bar{n}}{\sqrt{\bar{n}}}$$

- It can be shown that the expected increment in $|A_v|^2$ is

$$\langle d|A_v|^2 \rangle = [1 - 4|A_v||\alpha|\sin(\theta - \phi_v)\sin(\arg A_v - \theta)]dv$$

- We want $\theta$ to be between the actual phase and the phase of $A_v$.
- Use

$$\theta(v) = \arg[C_v^{1-\varepsilon} A_v^{\varepsilon}]$$

D. W. Berry and H. M. Wiseman,

# Adaptive phase measurements

- The difference of the photocounts, gives the photocurrent

$$I(t) = \lim_{\delta t \to 0} \lim_{\beta \to 0} \frac{\delta n_+(t) - \delta n_-(t)}{\beta \delta t}$$

- We then define the quantities, in terms of scaled time $v$,

$$A_v = \int_0^v I(u) e^{i\theta(u)} du, \qquad B_v = -\int_0^v e^{2i\theta(u)} du$$

- We also use the quantity

$$C_v = A_v v + B_v A_v^*$$

- The POVM for the measurement is[1]

$$G(A,B) = Q(A,B) \left| \tilde{\psi}(A,B) \right\rangle \left\langle \tilde{\psi}(A,B) \right|$$

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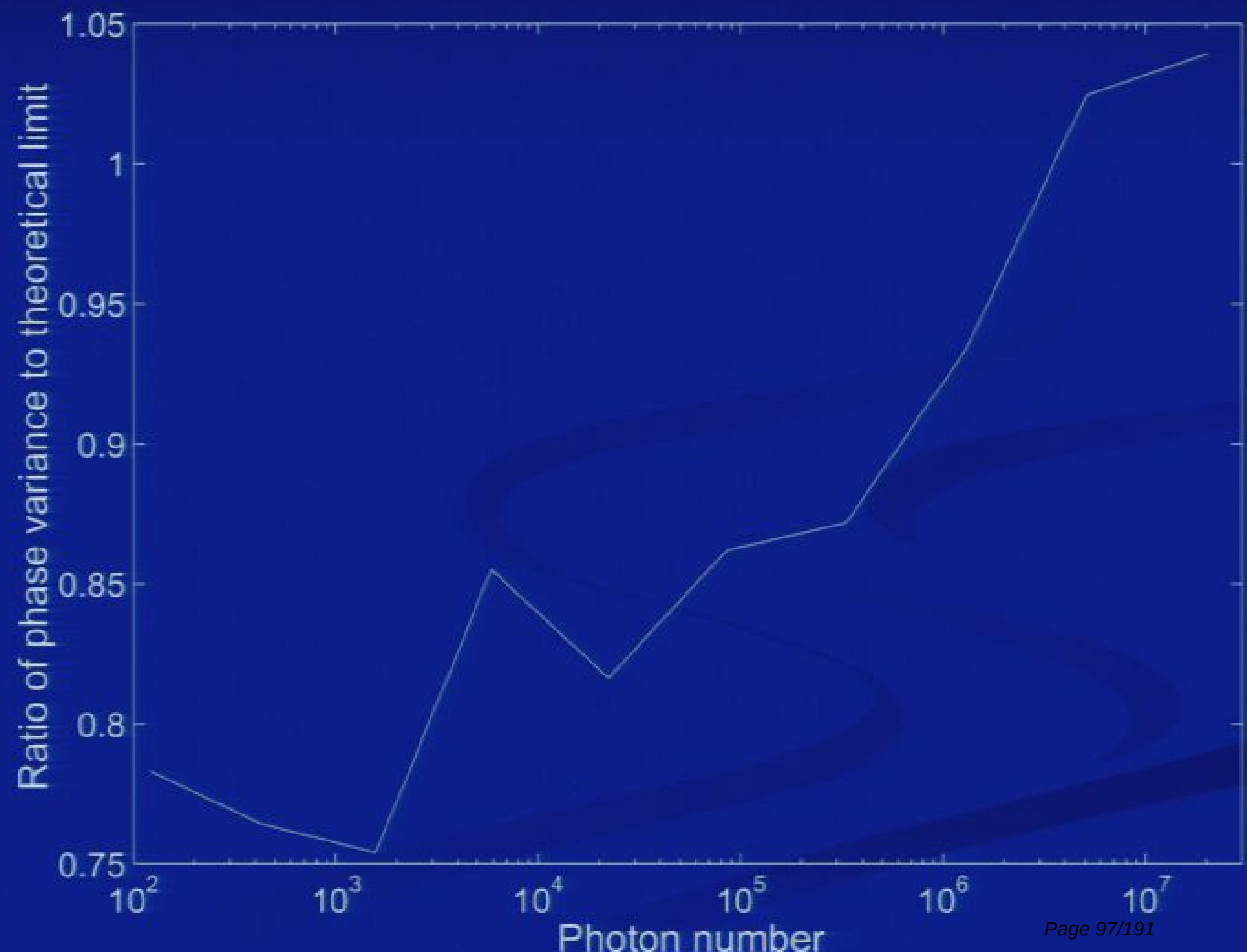
D. W. Berry and H. M. Wiseman,

# Achieving the theoretical limit

- The only way of making $|A|$ small enough is to use a time-varying $\varepsilon$.

$$\varepsilon(v) = \frac{v^2 - |B_v|^2}{|C_v|^2}\sqrt{\frac{v}{1-v}}$$



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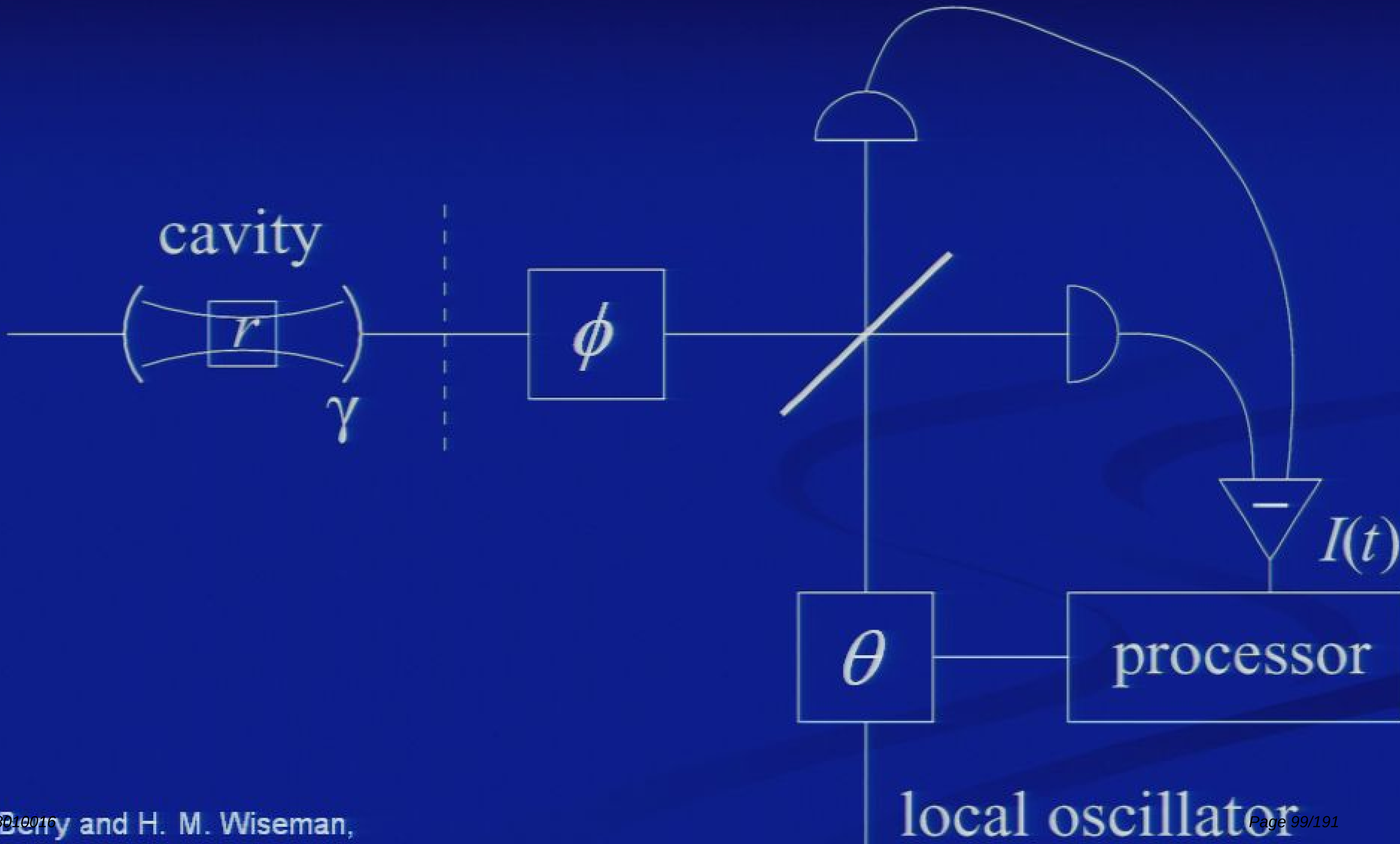
# Continuous phase measurements




D. W. Berry and H. M. Wiseman,
Phys. Rev. A **73**, 063824 (2006).

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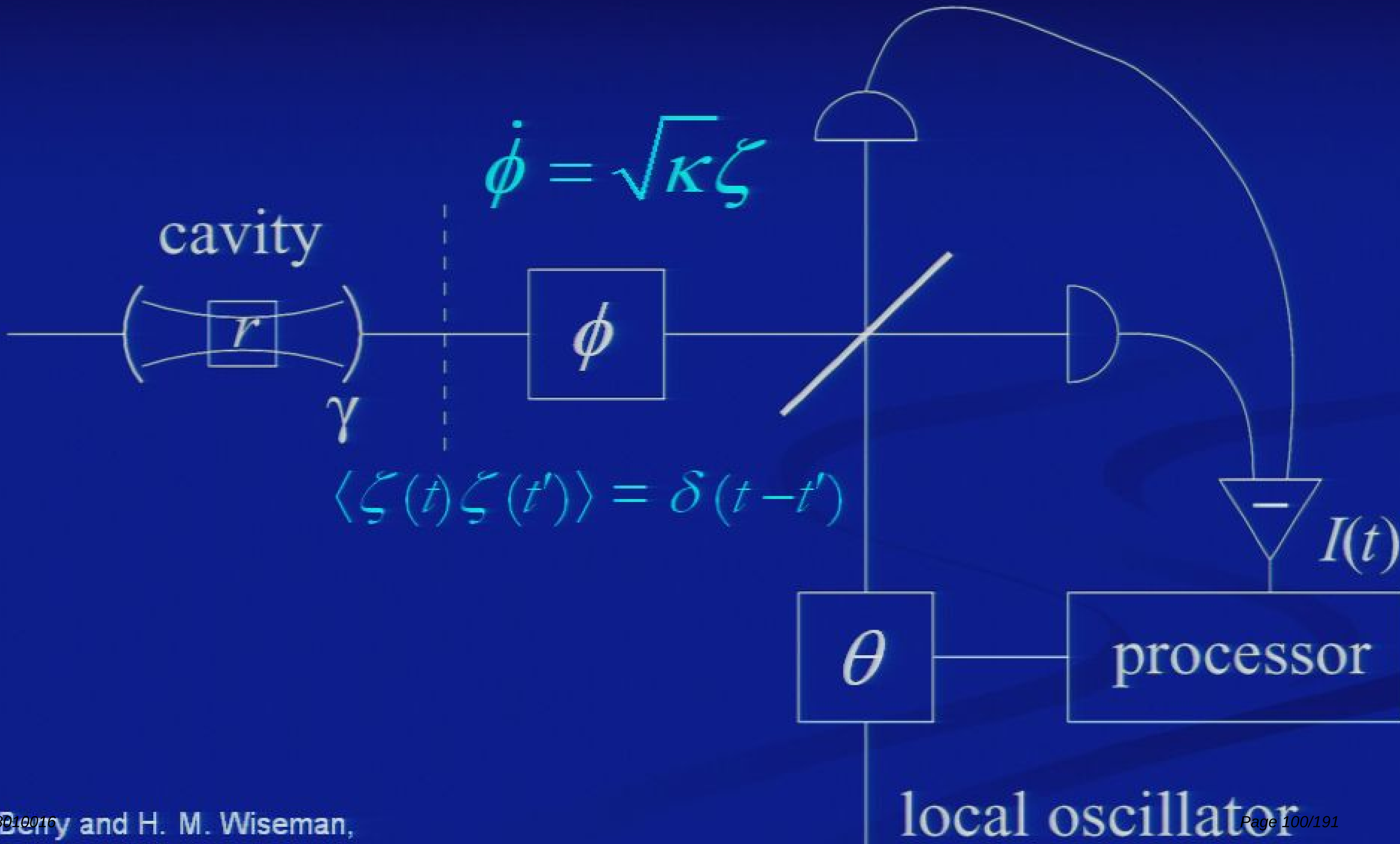


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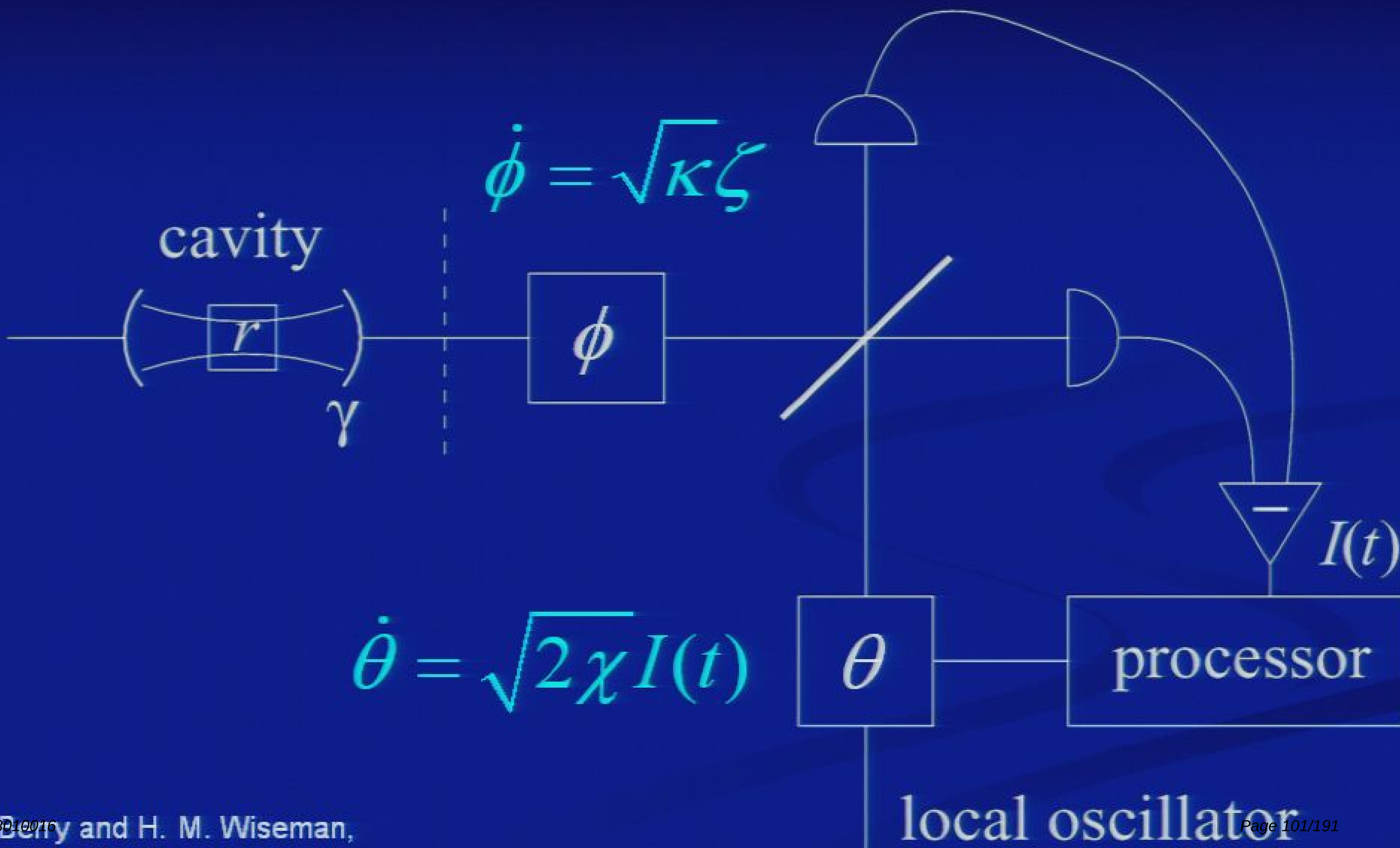


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# Coherent state scaling

- The canonical phase variance is $1/4\bar{n}$.
- For continuous beam with flux $N$, the variance for measurements over time $\Delta t$ is $\sigma^2 = 1/4N\Delta t$.
- Taking a weighted average of measurements up to time $t$ and measurements between time $t$ and $t+\Delta t$ gives

$$\sigma_{t+\Delta t}^2 \approx \frac{1}{1/\sigma_t^2 + 4N\Delta t}$$

- If we now take account of the change in the system phase, we get

$$\sigma_{t+\Delta t}^2 \approx \frac{1}{1/\sigma_t^2 + 4N\Delta t} + \kappa\Delta t$$

- For $\kappa\Delta t << \sigma_t^2 << 1/4N\Delta t$ we have the approximate change in inverse variance

$$\Delta\left(\frac{1}{\sigma^2}\right) \approx -\Delta t \frac{\kappa}{\sigma^4} + 4N\Delta t$$

- This gives the steady state variance

$$\sigma^2 \approx \tfrac{1}{2}\sqrt{\kappa/N}$$

# Squeezed state scaling – Part 1

- We have coherent amplitude $E$, squeezing parameter $r$ and bandwidth $\gamma$.
- The flux is given by

$$N = \frac{E^2}{4} + \frac{\gamma}{2}\sinh^2 r$$

- For moderate squeezing, the variance for measurements over time $\Delta t$ is

$$\sigma^2 \approx \frac{1}{e^{2r} E^2 \Delta t}$$

- Carrying through the previous derivation gives

$$\Delta\left(\frac{1}{\sigma^2}\right) \approx -\Delta t \frac{\kappa}{\sigma^4} + e^{2r} E^2 \Delta t$$

- The phase variance is therefore

$$\sigma^2 \approx \tfrac{1}{2} e^{-r} \sqrt{\kappa / N}$$

# Squeezed state scaling – Part 3

- Now we take account of the fact that the time scale over which measurements are taken needs to be greater than approximately $e^r/\gamma$.
- The time interval over which measurements are used is approximately $\sigma^2/\kappa$ (over longer time intervals the phase has varied by more than $\sigma$).

$\Rightarrow$
$$\frac{\sigma^2}{\kappa} > \frac{e^r}{\gamma}$$

- Due to the contribution to the photon flux from squeezing, we have $N > \gamma e^{2r}$.

$\Rightarrow$
$$\sigma^2 > \frac{\kappa e^{3r}}{N}$$

- The largest squeezing compatible with this equation and $\sigma^2 \approx \frac{1}{2} e^{-r} \sqrt{\kappa / N}$ is $e^r \sim (N/\kappa)^{1/8}$.
- This yields scaling of

$$\sigma^2 \sim \left(\frac{\kappa}{N}\right)^{5/8}$$

# Numerical methods

- We calculate the quantities

$$A_t = \int_{-\infty}^{t} e^{\chi(u-t)} e^{i\Phi} I(u)\,du, \qquad B_t = \int_{-\infty}^{t} e^{\chi(u-t)} e^{2i\Phi}\,du,$$

$$C_t = A_t + \chi B_t A_t^*$$

- The phase of $C_t$ gives a good phase estimate; a poorer phase estimate is given by the phase of $A_t$.
- We use $\arg(C_t)$ for the phase estimate and the feedback is

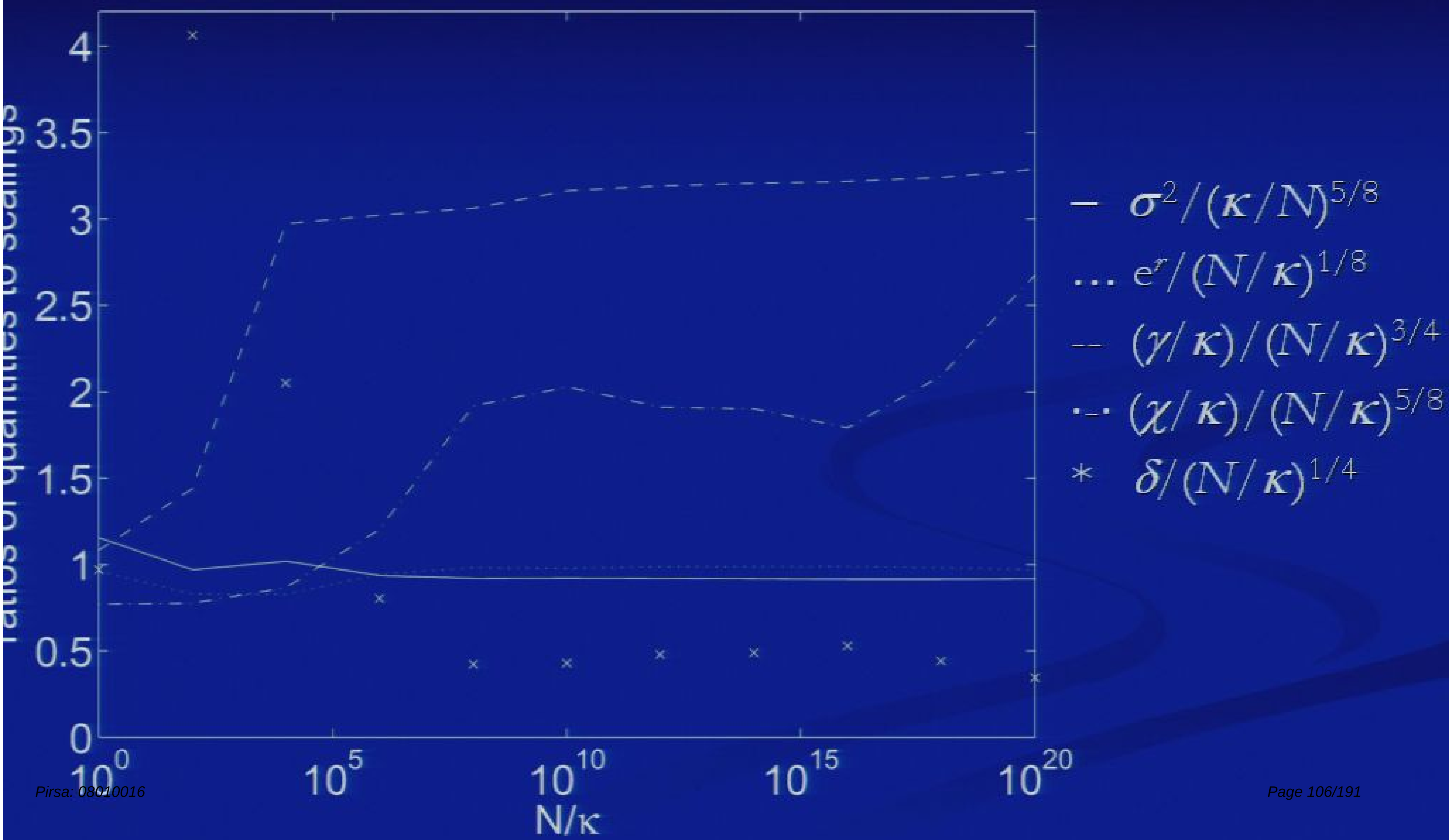
$$\theta(t) = \arg(C_t^{1-\delta} A_t^{\delta})$$

- From the above theory we predict the scalings

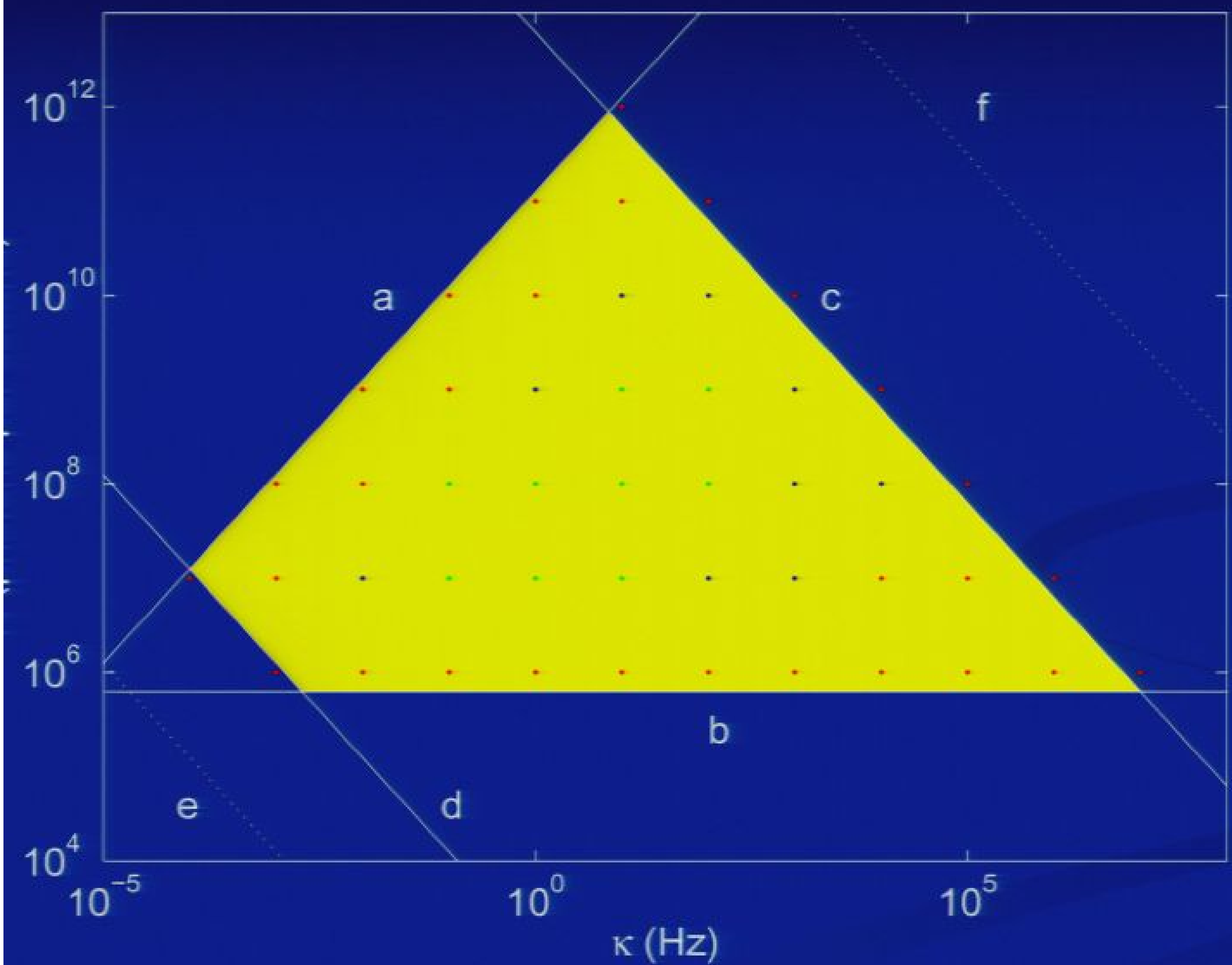
$$\sigma^2 \sim (\kappa / N)^{5/8}, \qquad e^r \sim (N/\kappa)^{1/8},$$

$$\frac{\gamma}{\kappa} \sim (N/\kappa)^{3/4}, \qquad \frac{\chi}{\kappa} \sim (N/\kappa)^{5/8}.$$

# Numerical results



# Practical region




a – interferometer accuracy

b – flux due to squeezing

c – squeezing bandwidth

d – phase locking

e – *RC* time constant

f – analog feedback delay

● – above SQL

● – between 90% and SQL

● – below 90% of SQL

# The Mach-Zehnder interferometer


- $N$ photons are sent independently.

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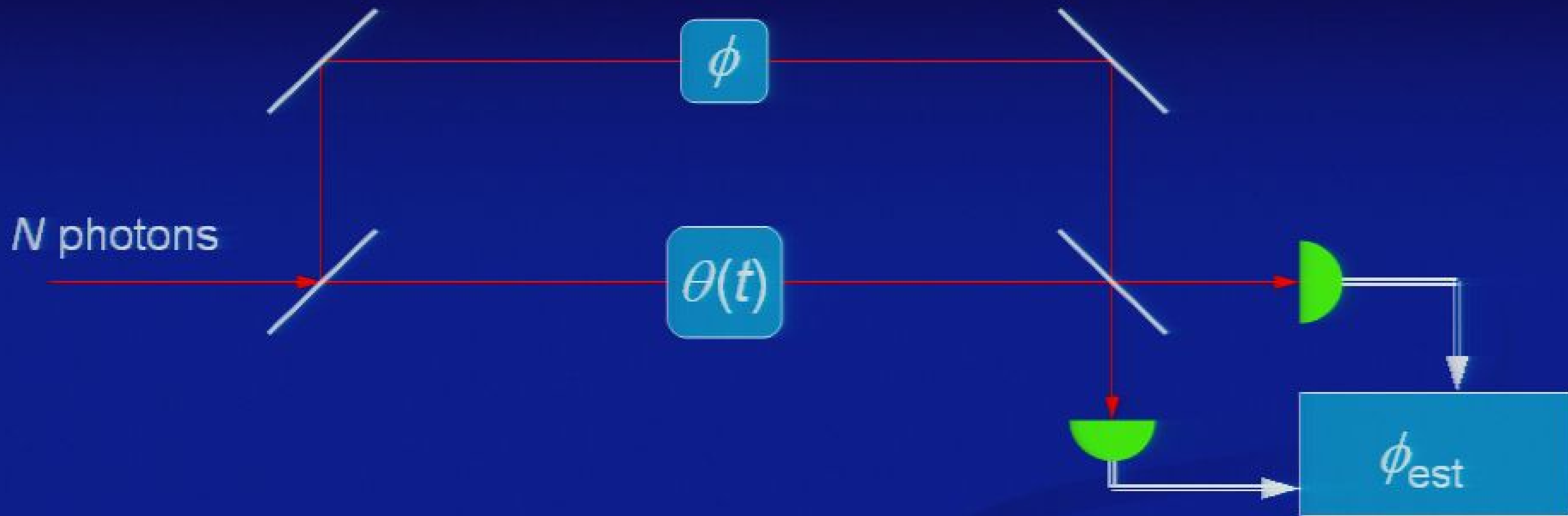


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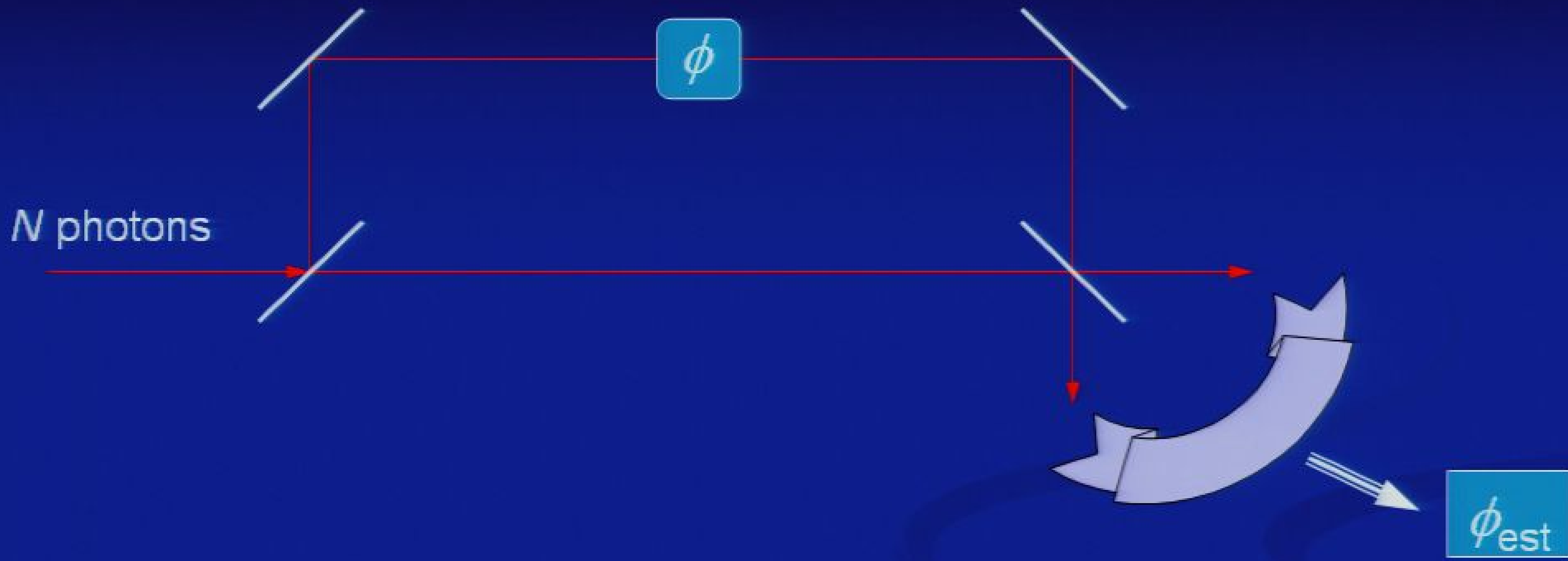
# The Mach-Zehnder interferometer


- $N$ photons are sent independently.
- The estimate of $\phi$ is based on which outputs the photons are detected in.
- Phase $\theta(t)$ is varied in steps of $\pi/N$.
- The uncertainty scales as $1/N^{1/2}$ – SQL scaling.

# How to improve on this?




Optimal two-mode joint measurement[2]



[2] Sanders and Milburn, PRL **75**, 2944 (1995).

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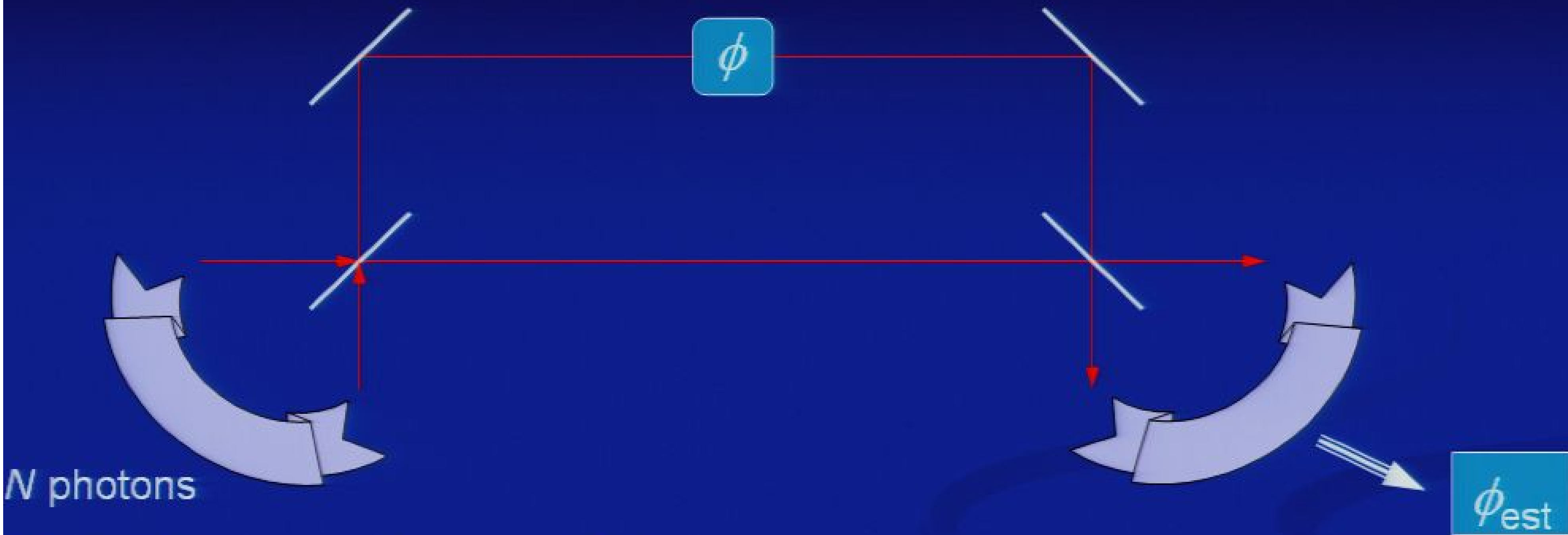


Optimal two-mode entangled input state[1]

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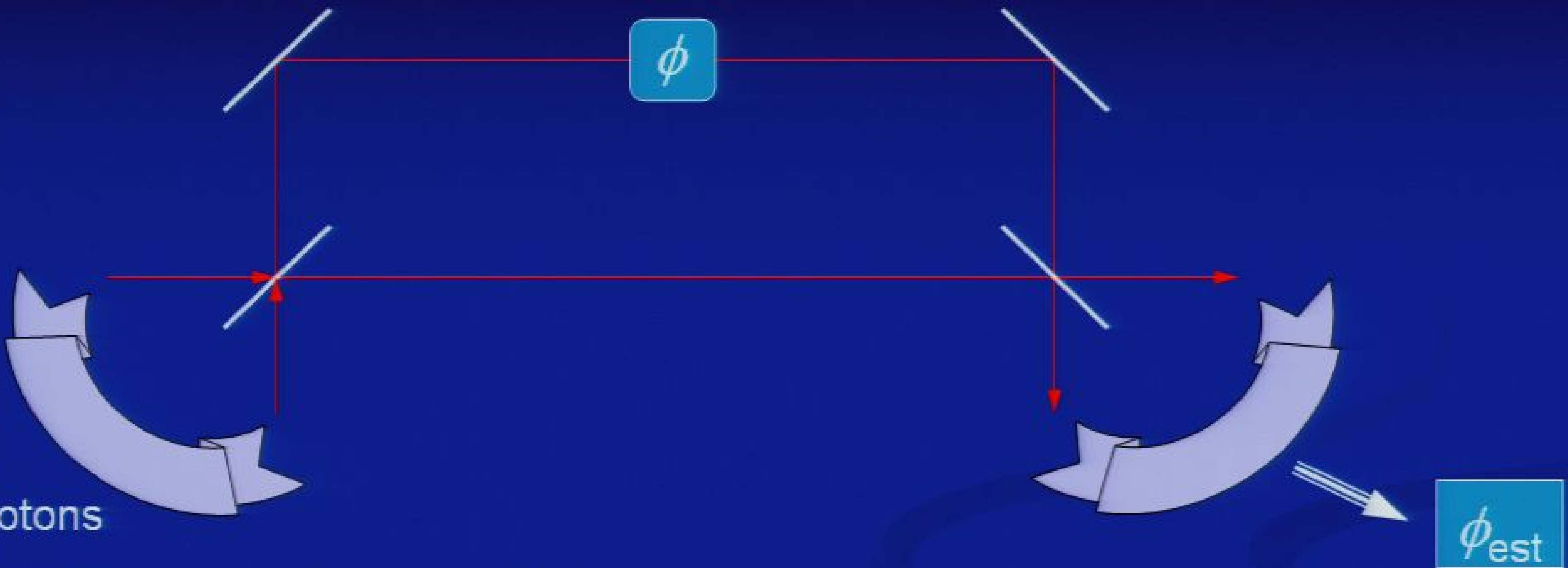


*N* photons

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$$\Delta\phi \approx \pi/N$$

The Heisenberg limit!



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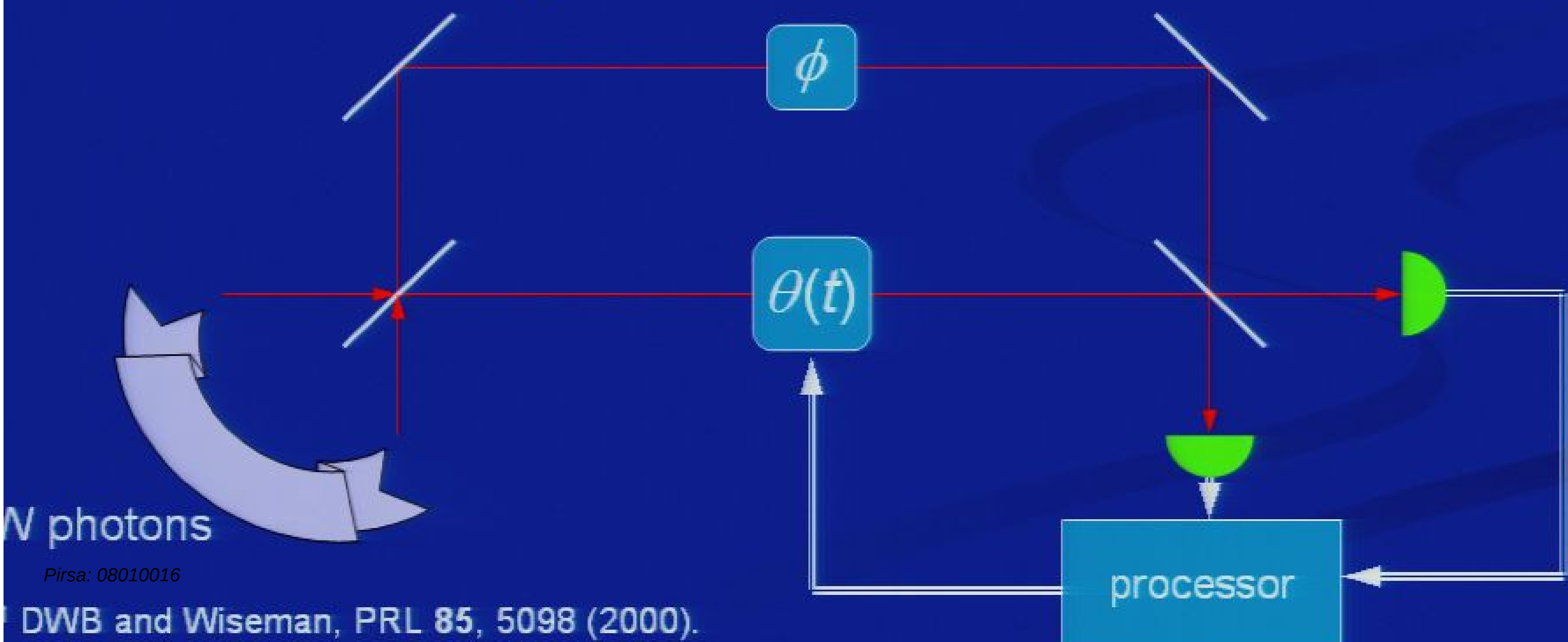
# How to perform the measurement?

- Feed measurement results into processor which adjusts feedback phase $\theta(t)$ in real time.
- After each detection the phase $\theta(t)$ is adjusted to minimise the expected variance after the next detection.
- Phase uncertainty $\propto 1/N$ is obtained[1].

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Two problems:

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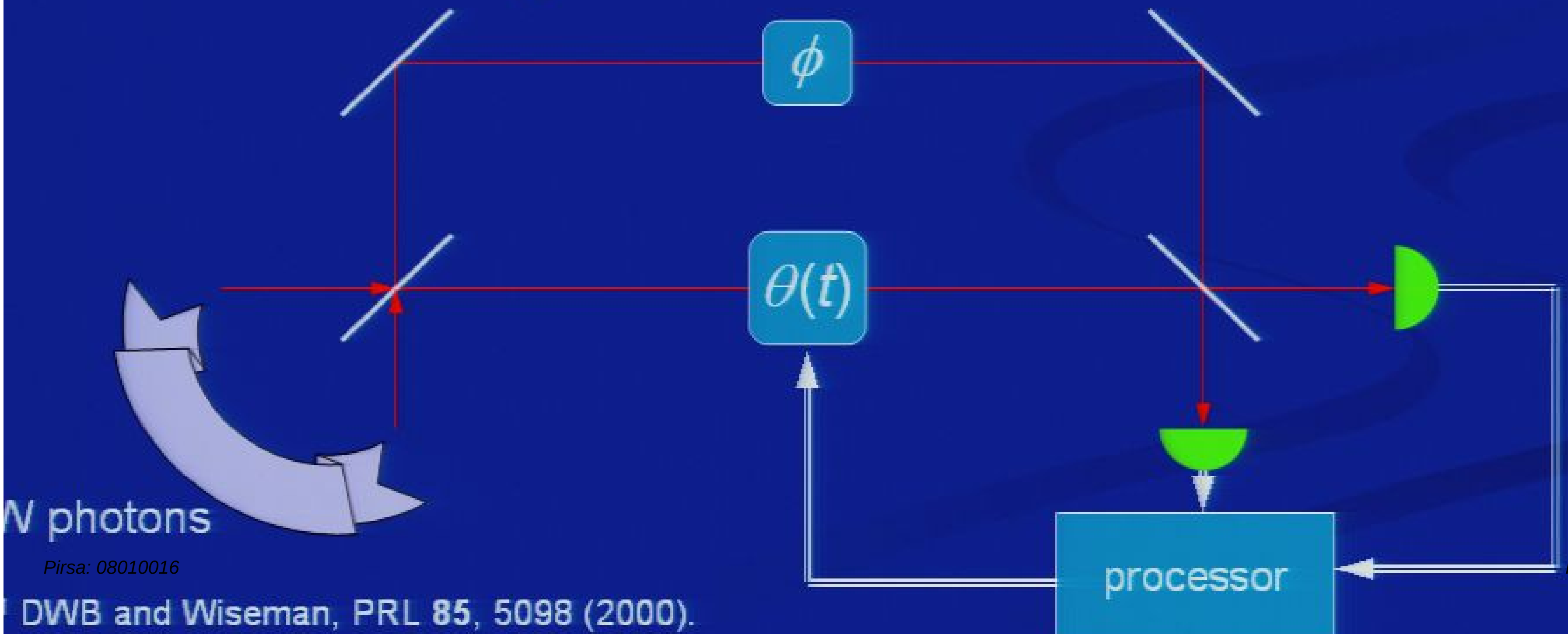






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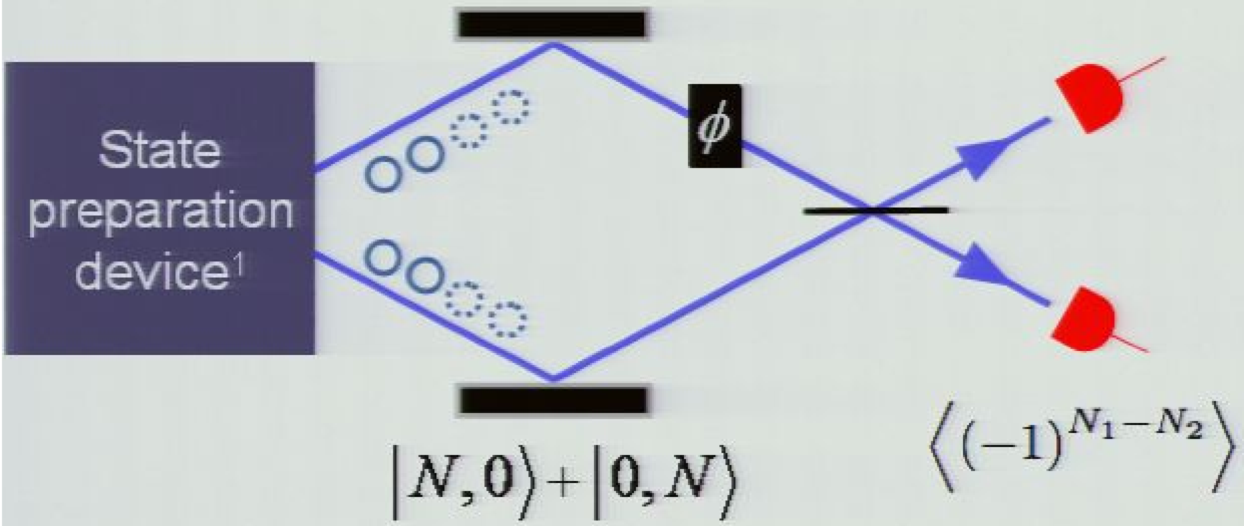
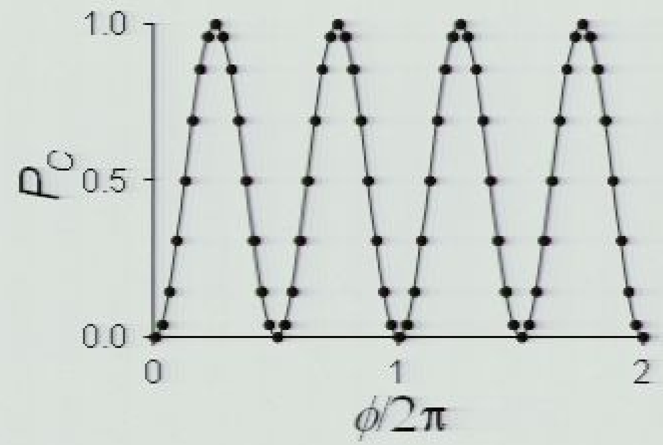
# NOON states




"Period 1/N" fringes

- In experiments[1] no feedback phase is used; it is not necessary to adjust a phase between detections.
- Two problems with NOON states:

i. NOON states are still extremely hard to generate. Experiments are limited to about $N = 6$.

ii. The phase is only distinguished over an interval of $[0, 2\pi/N)$.



[1]Mitchell *et al*, Nature **429**, 161 (2004); Walther *et al*, Nature **429**, 158 (2004); Resch *et al*, PRL **98**, 223601 (2007)

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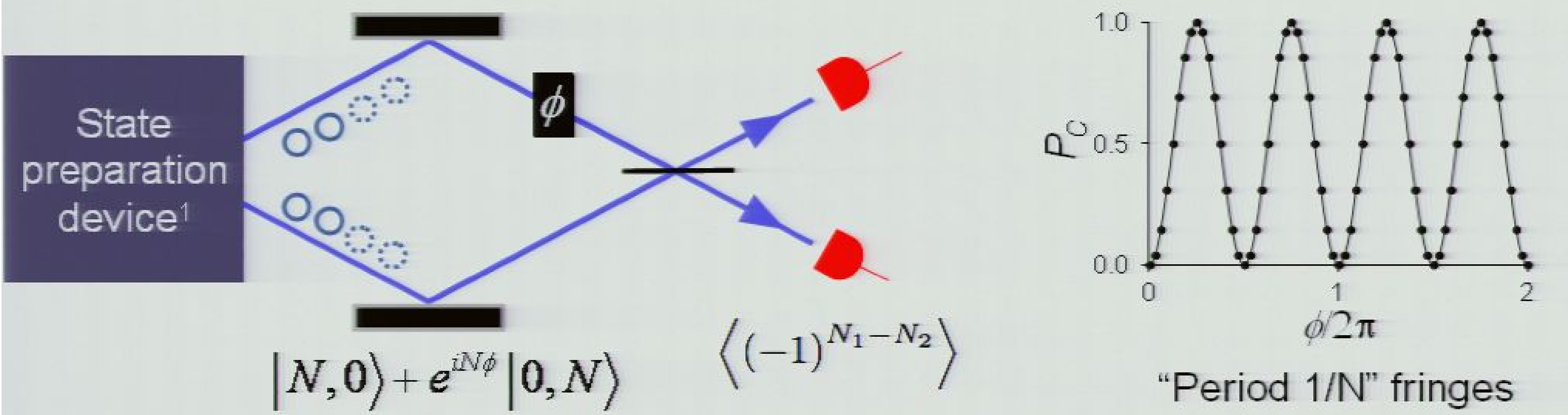


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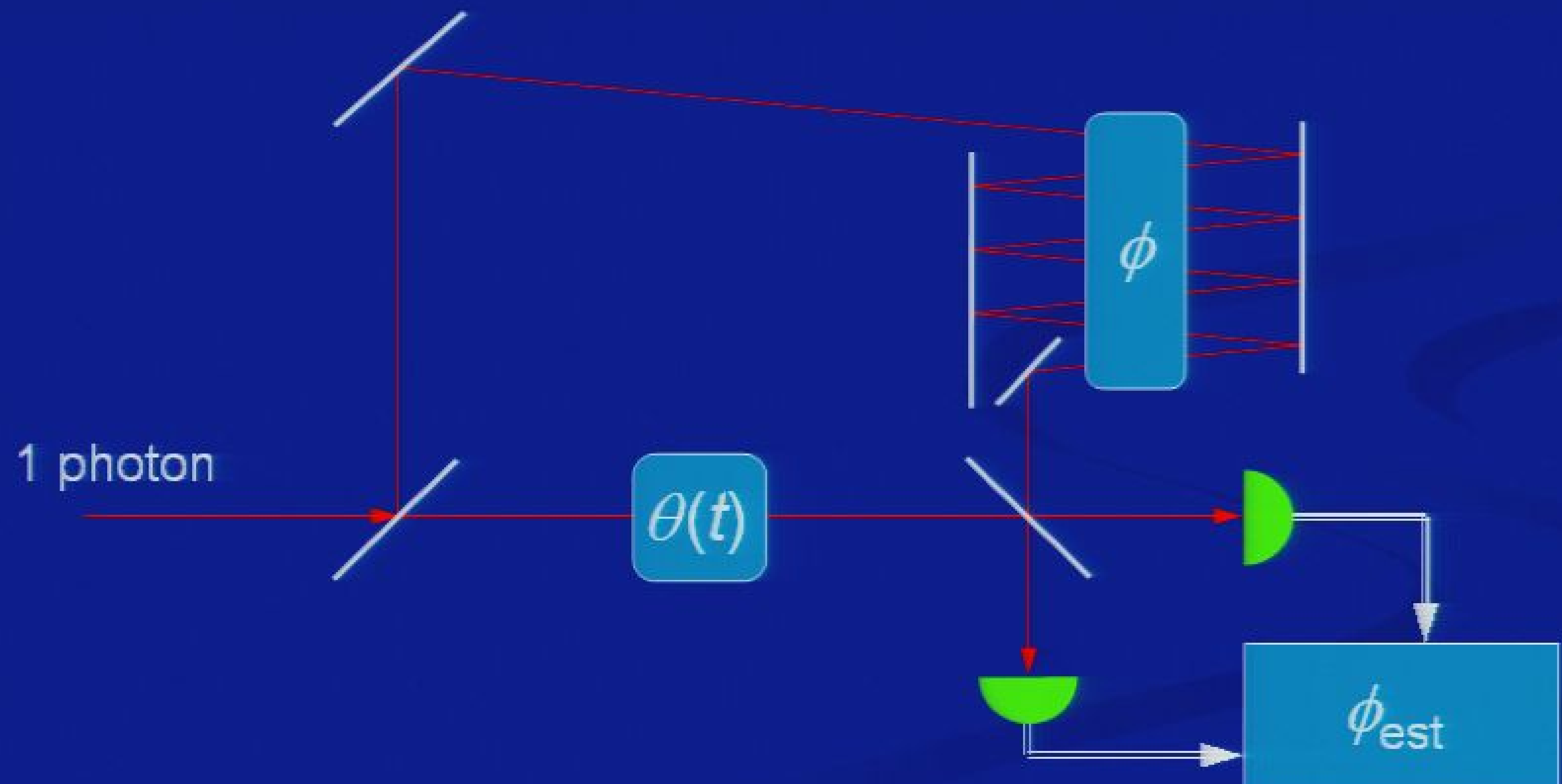
# Multiple passes

The NOON state may be replaced with *N* passes through a phase shift.



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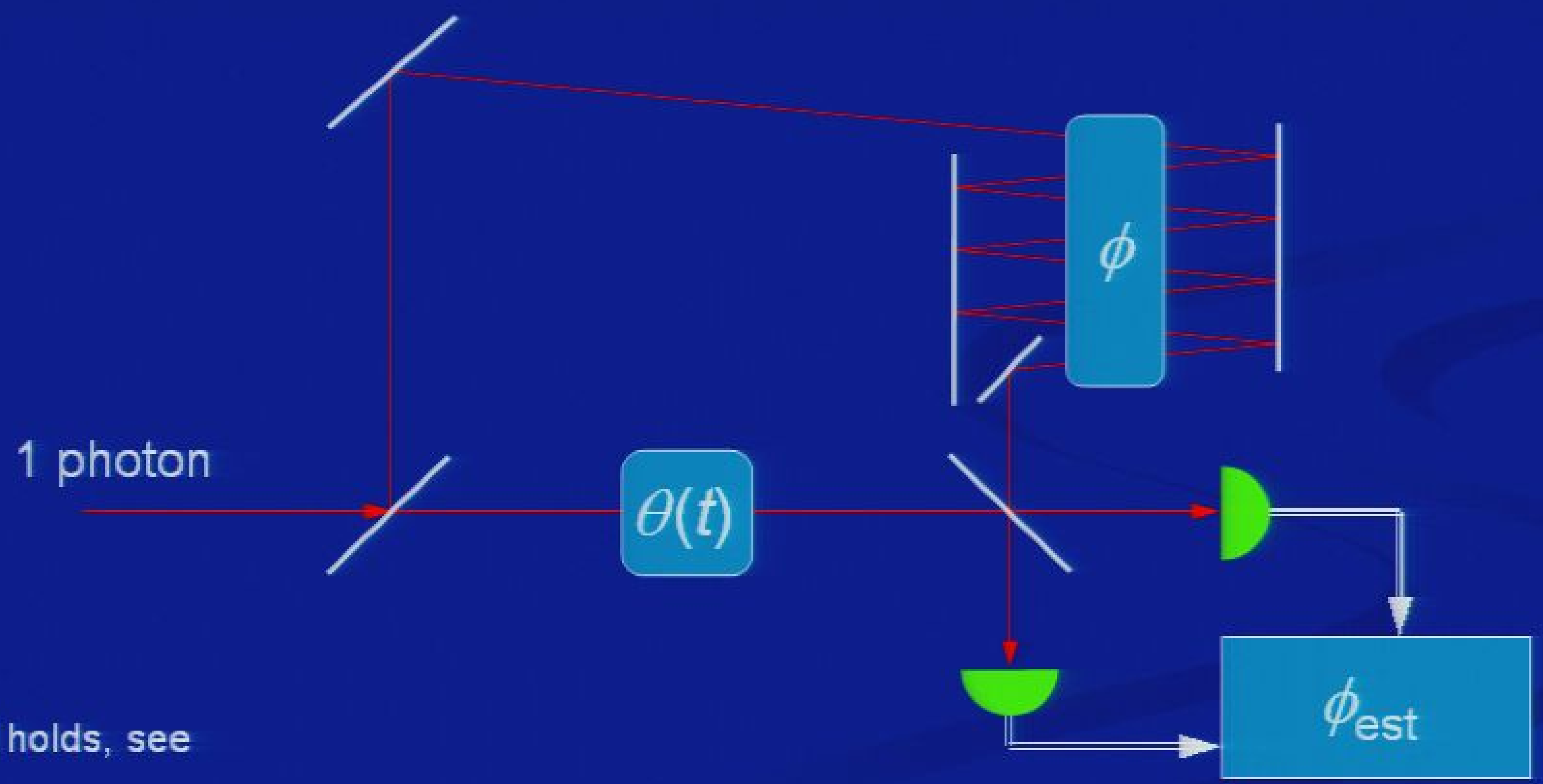
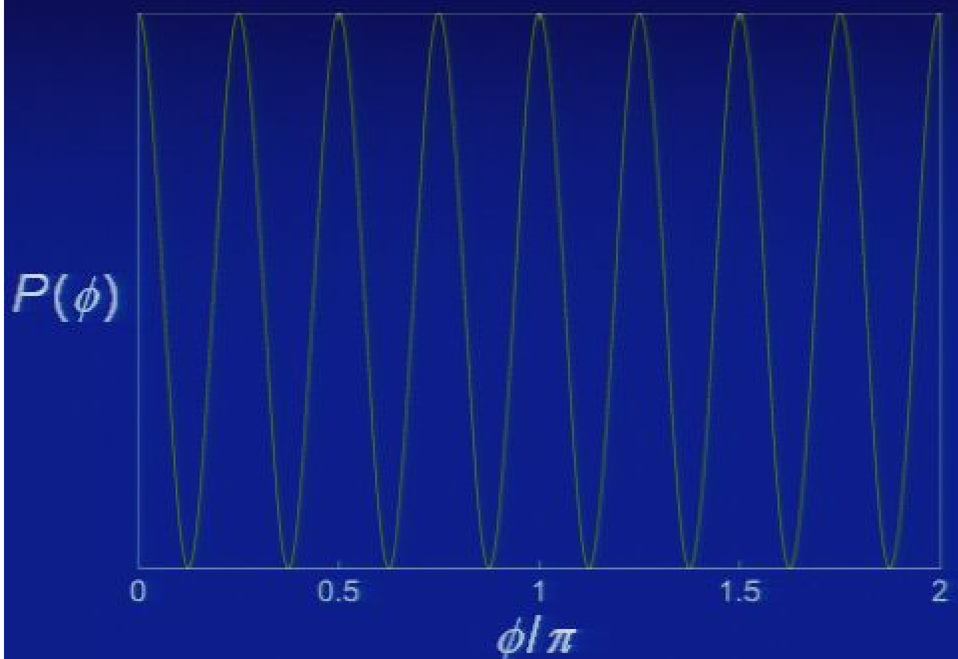



Limit still holds, see

W. van Dam, G. M. D'Ariano, A. Ekert, C. Macchiavello, M. Mosca, Phys. Rev. Lett. **98**, 090501 (2007).

# Probability distribution




For one detection result…

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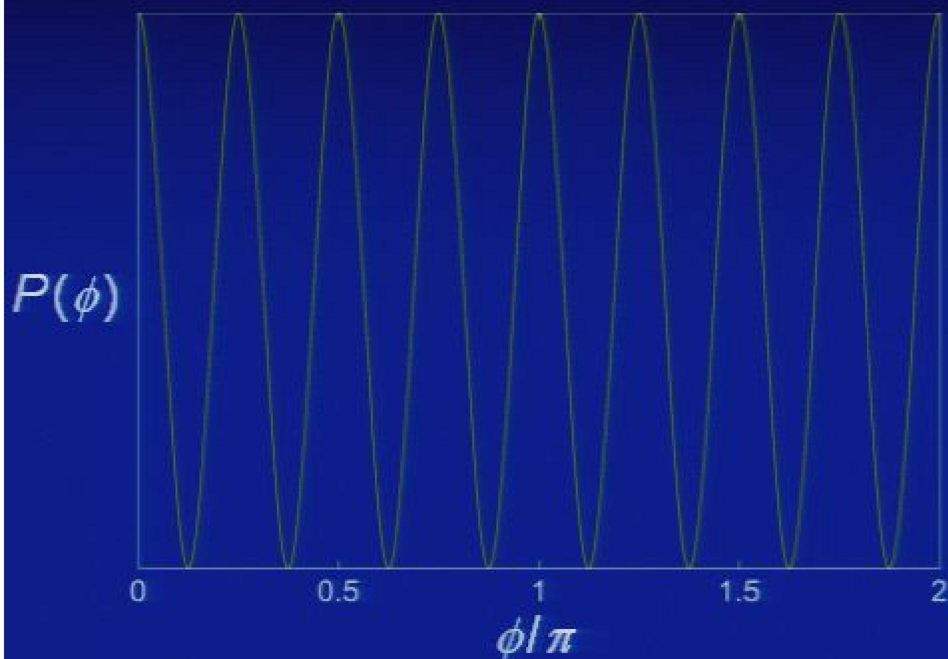


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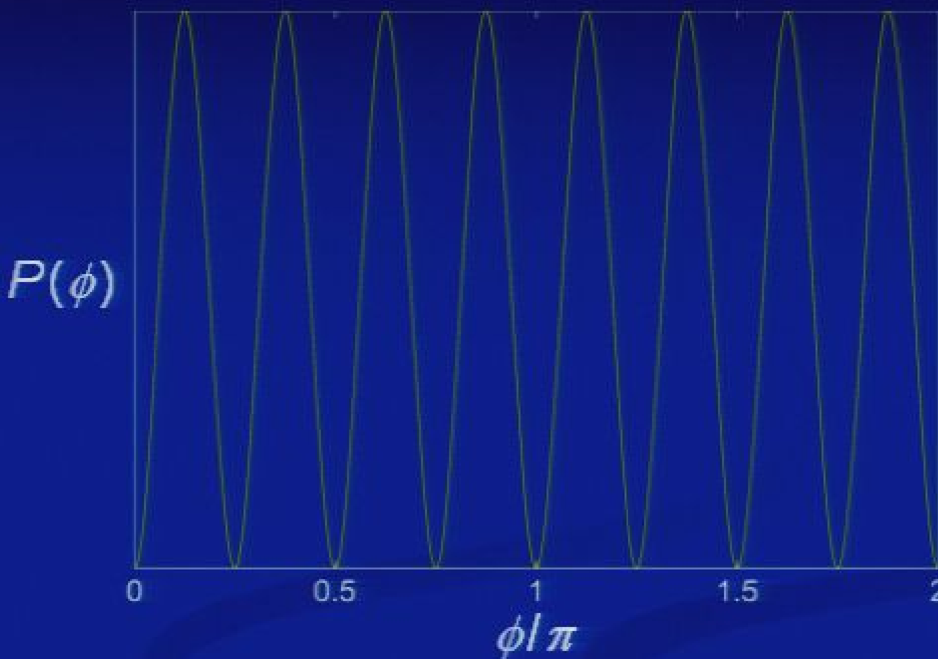


and for the other detection result…

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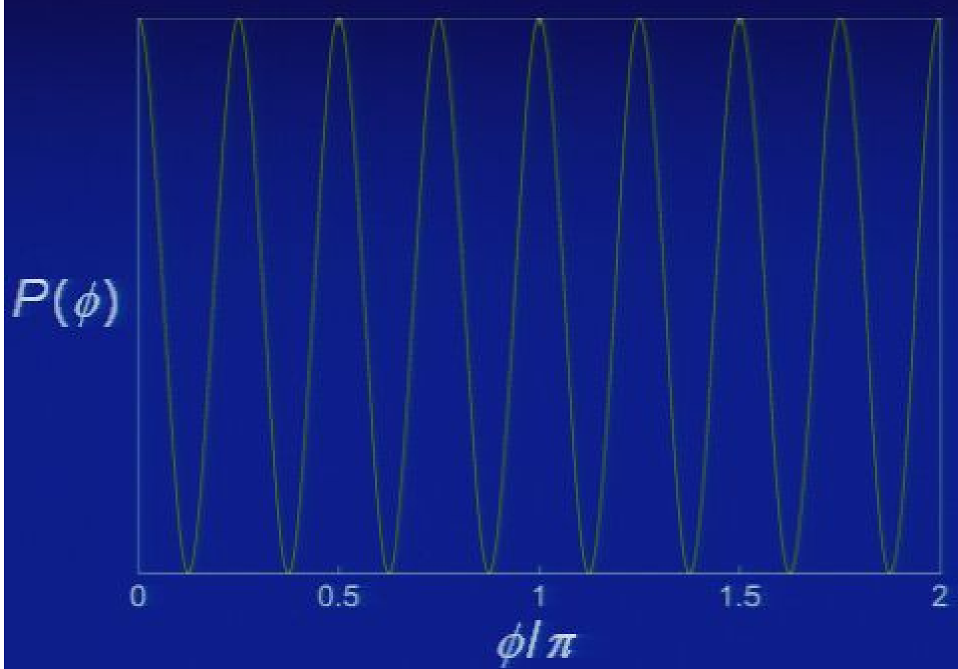


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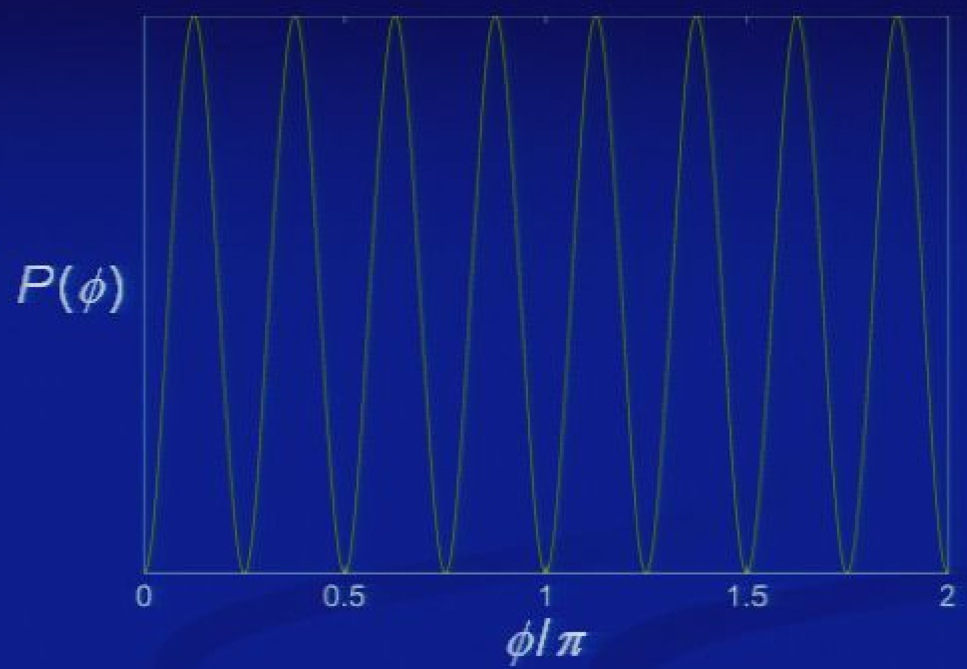


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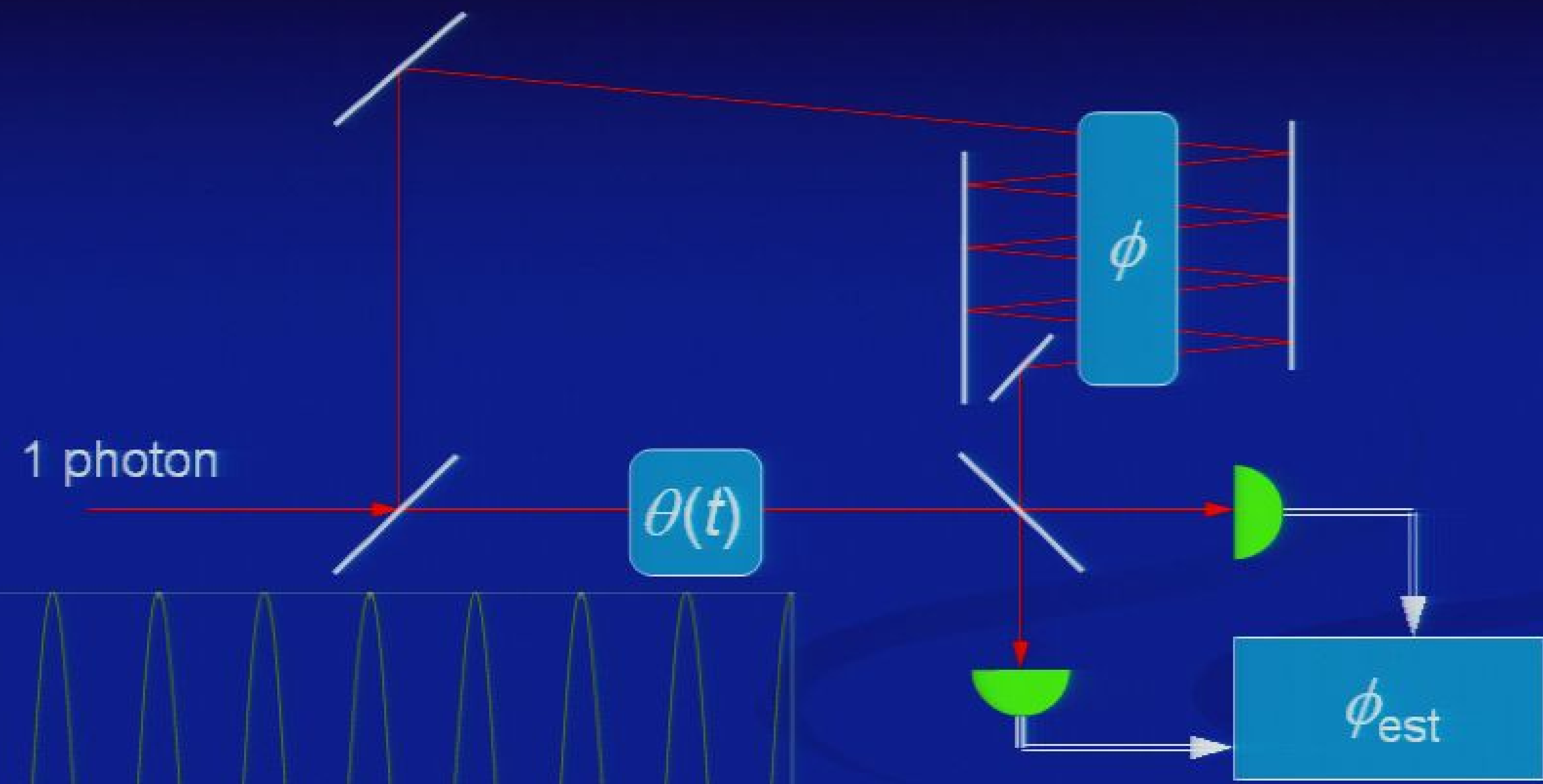
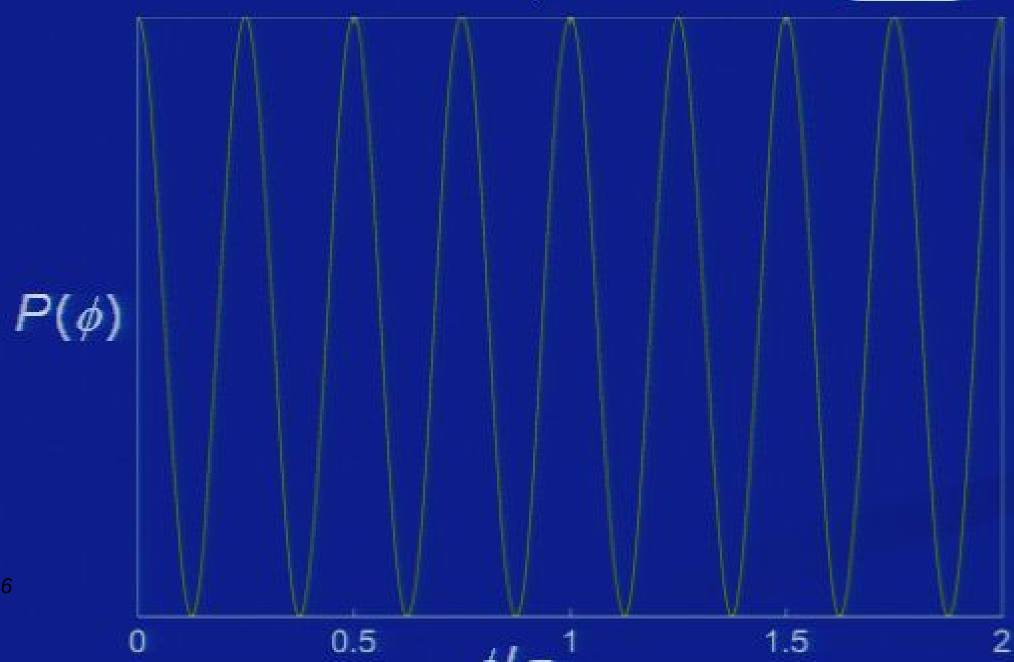
# Eliminating the fringes

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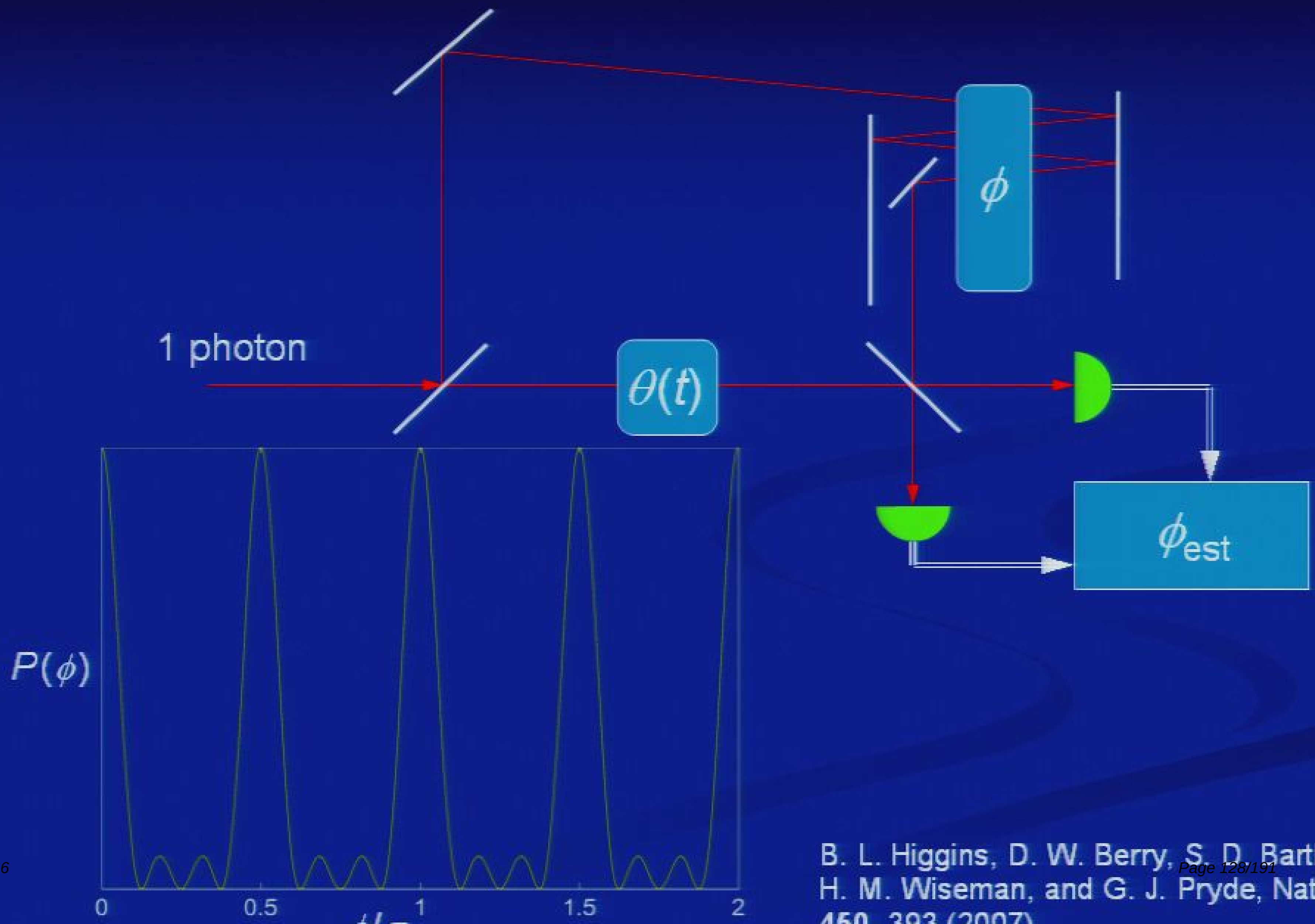


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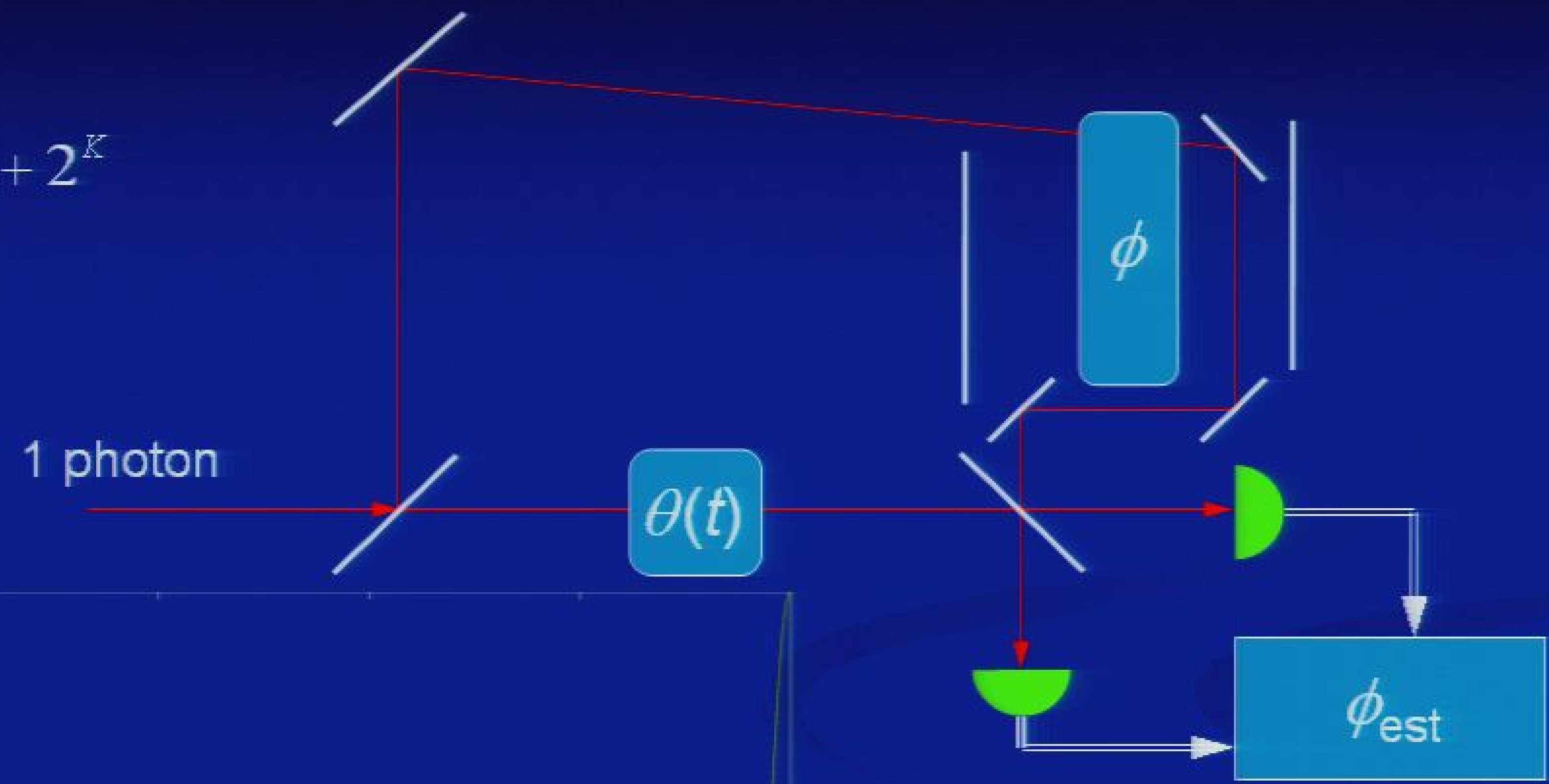
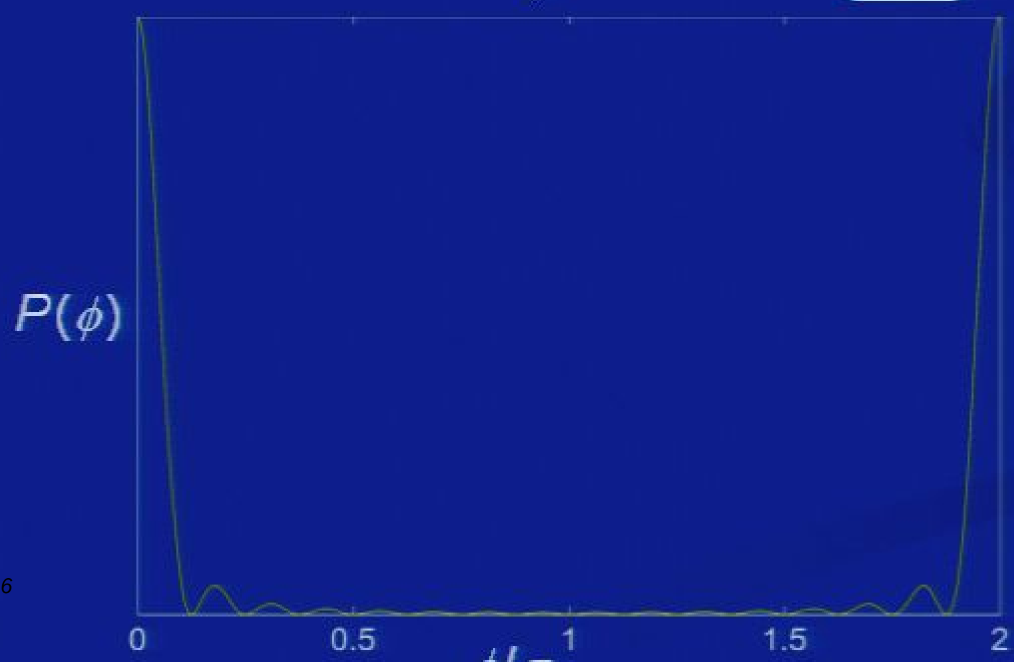
# Eliminating the fringes

$$N = 1 + 2 + \ldots + 2^K$$
$$= 2^{K+1} - 1$$

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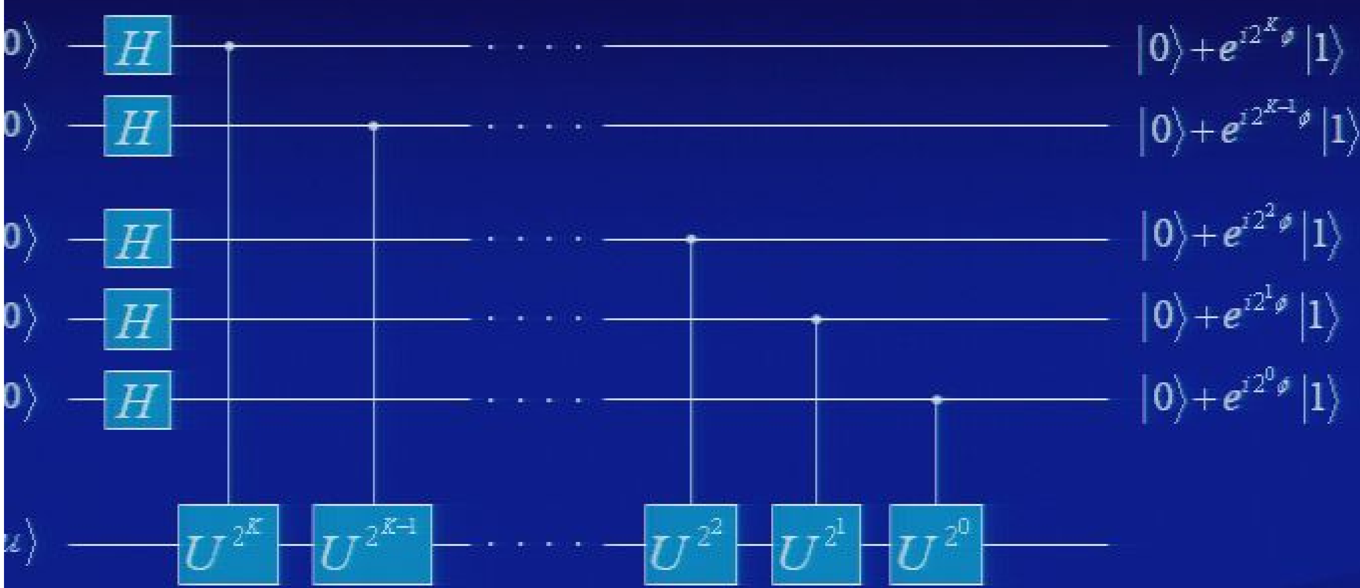
# Kitaev phase estimation




- The phase shifts are obtained from unitary $U$ satisfying

$$U|u\rangle = e^{i\phi}|u\rangle$$

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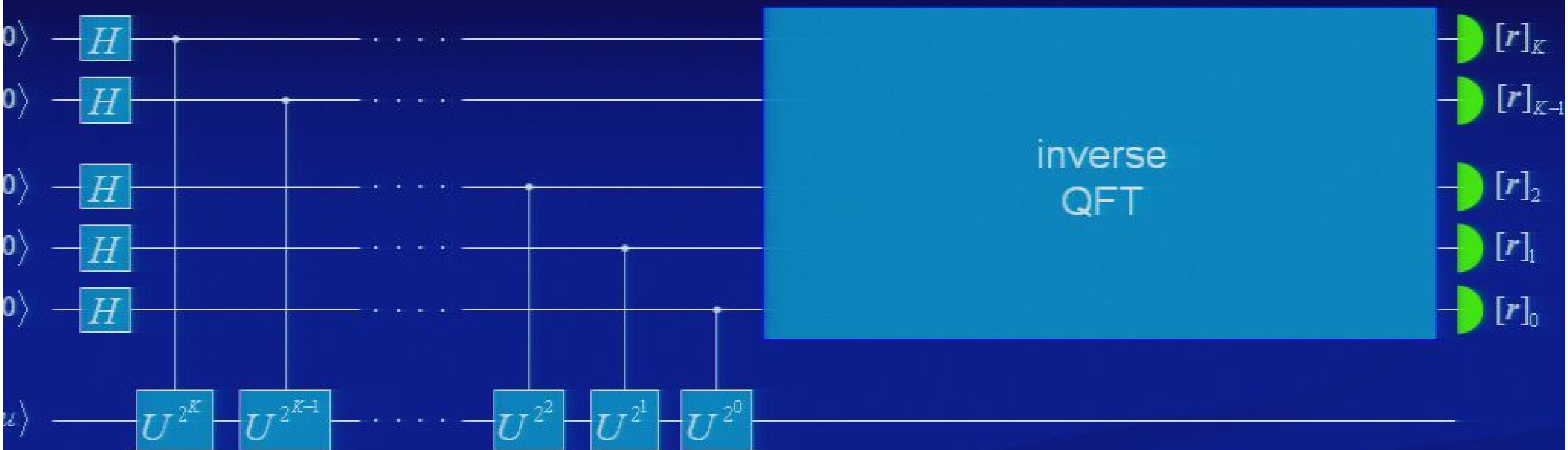


- Provided $\phi$ is of the form $\phi = \pi r / 2^K$, the inverse quantum Fourier transform gives the bits of $r$ at the output.

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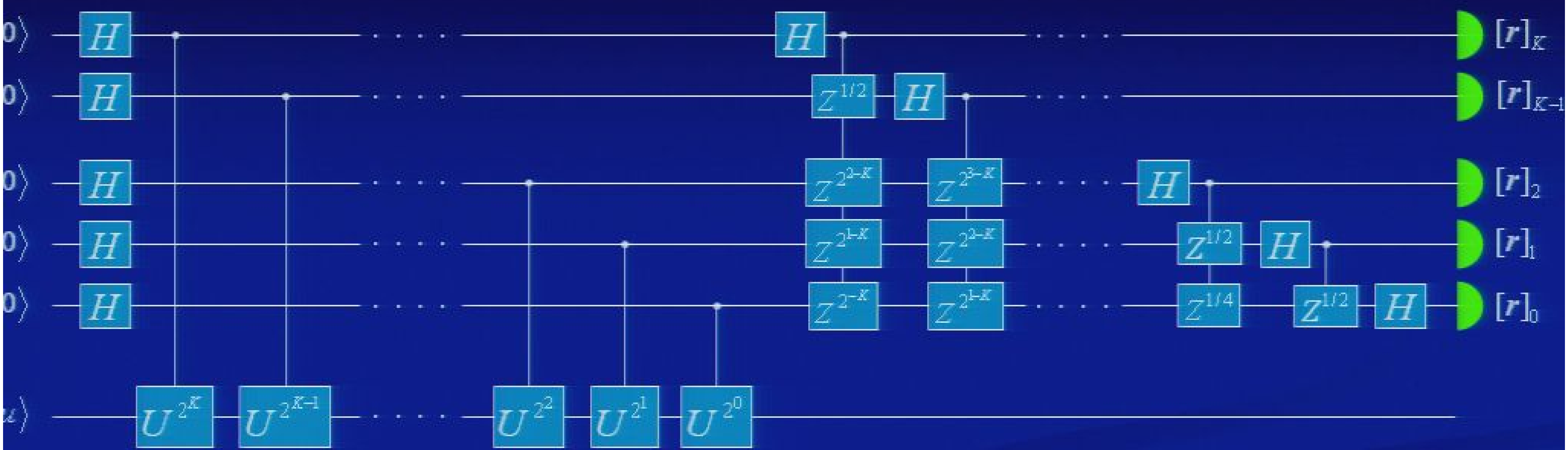


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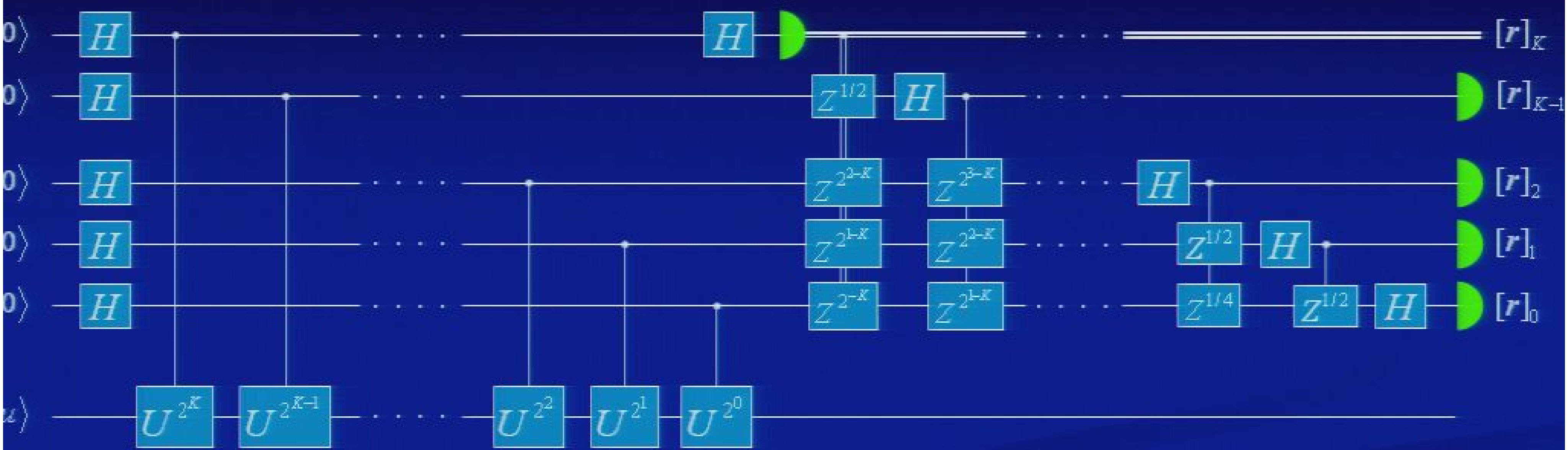


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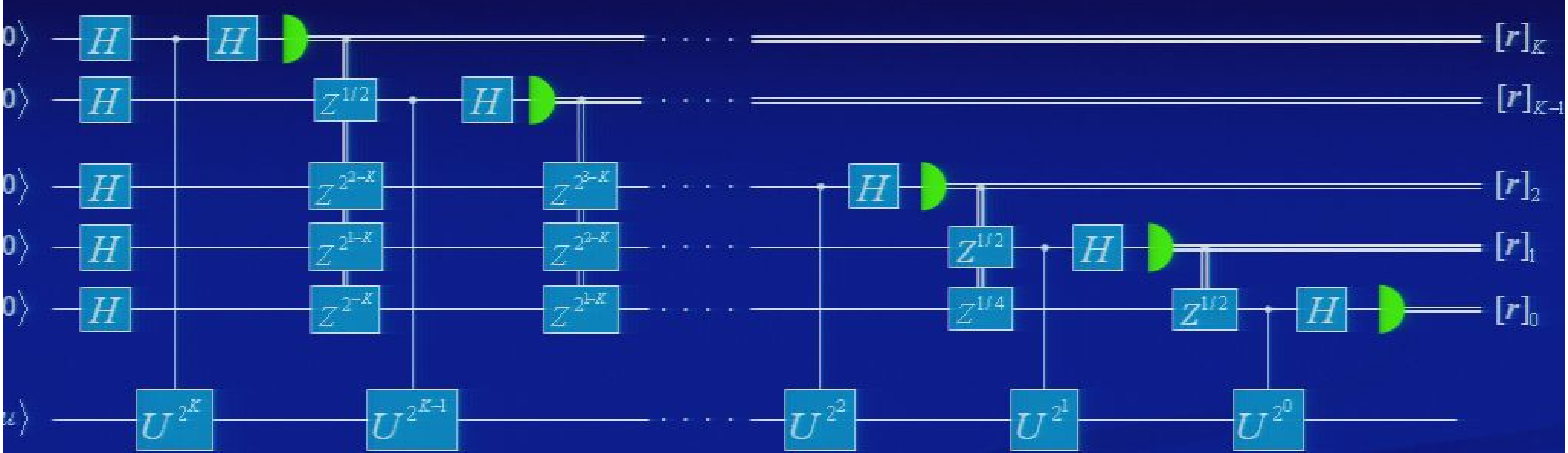


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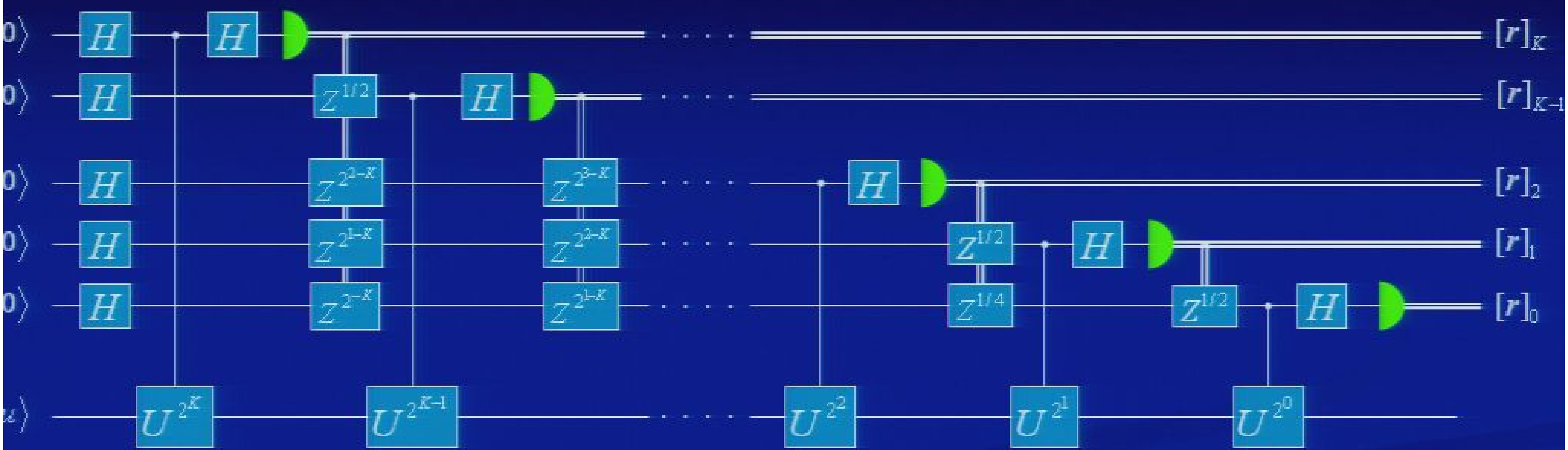

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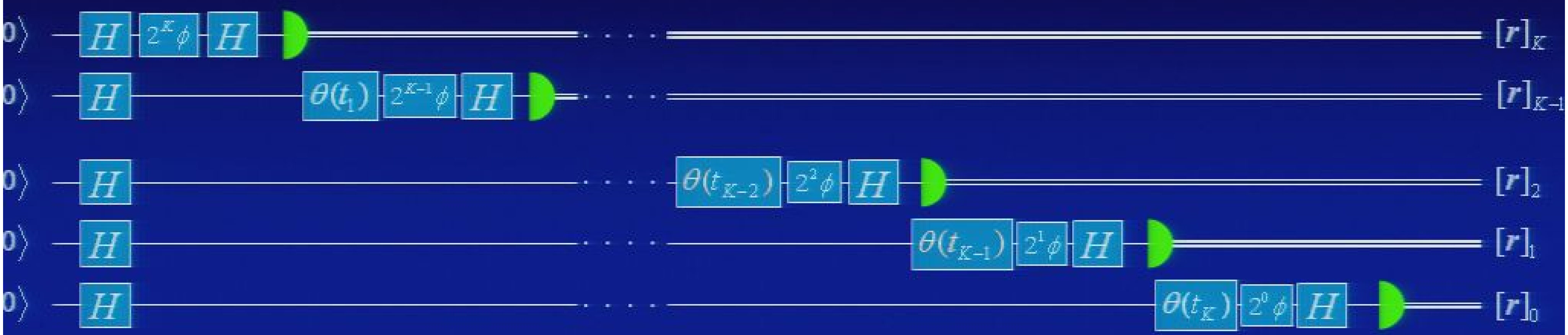
# Kitaev phase estimation

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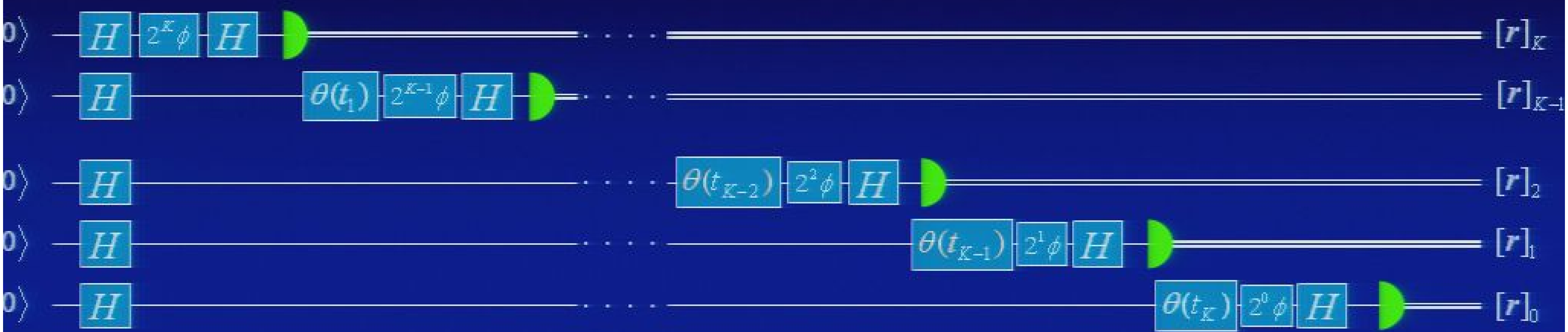


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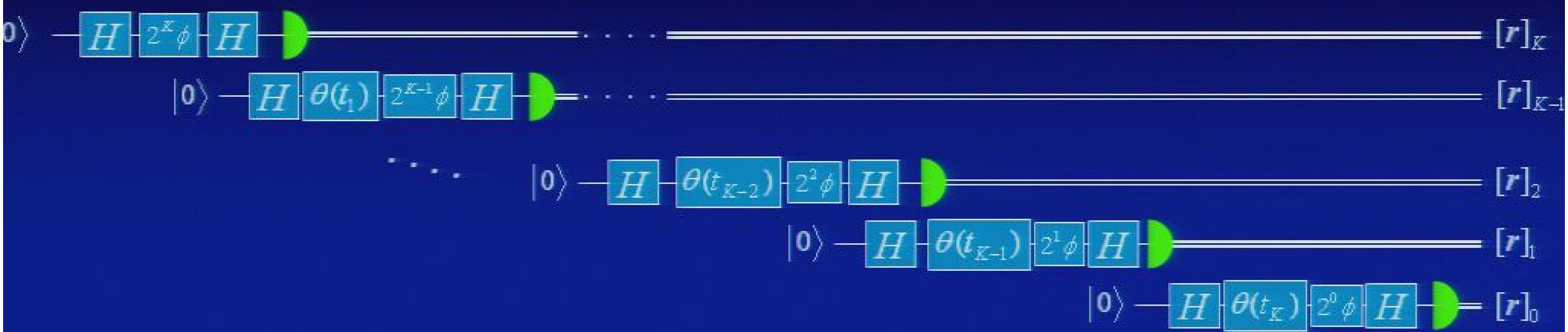


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# The equivalent state

- The sequence of different numbers of passes is equivalent to a tensor product of NOON states:

$$\left(\left|2^K,0\right\rangle+\left|0,2^K\right\rangle\right)\otimes\ldots\otimes\left(\left|2^1,0\right\rangle+\left|0,2^1\right\rangle\right)\otimes\left(\left|1,0\right\rangle+\left|0,1\right\rangle\right)$$

- This is equivalent to

$$\sum_{n=0}^{N}\left|n,N-n\right\rangle$$

for $N = 2^{K+1}-1$.

# How to create the input state?

Two problems:

1. The state needs to be a special coherent superposition of the form

$$\sum_{n=0}^{N} \psi_n |n\rangle |N-n\rangle$$

   There is no known way of producing such a state.

2. The input mode needs to be very long so that $\theta(t)$ can be adjusted between detections.

# How to create the input state?

1. Using multiple passes of single photons we obtain an effective state of the form

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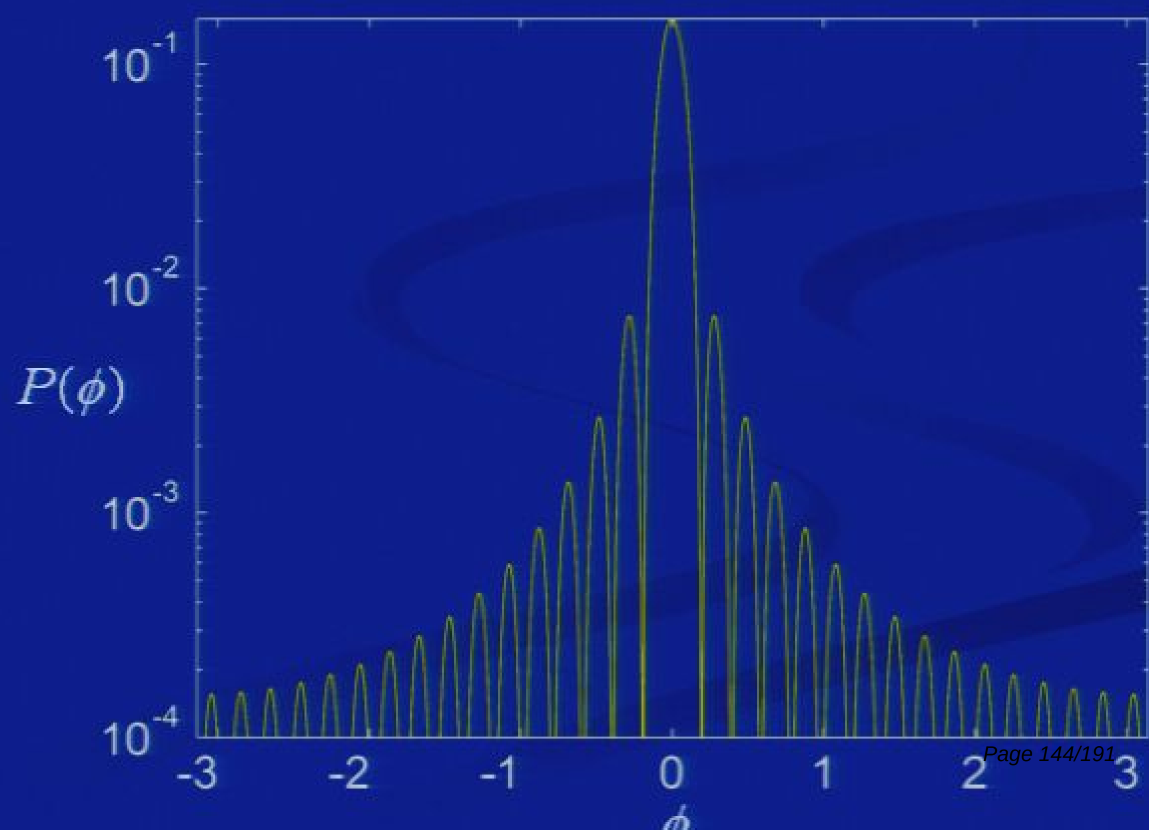
2. The input mode does not need to be long – we can send photons through one at a time.

# The variance

- For a phase of exactly $\pi r/2^K$, the exact phase is obtained.
- For a general phase, we get variance of $V \approx 2/N$.
- This does not beat the SQL!
- The problem is high tails in the distribution.



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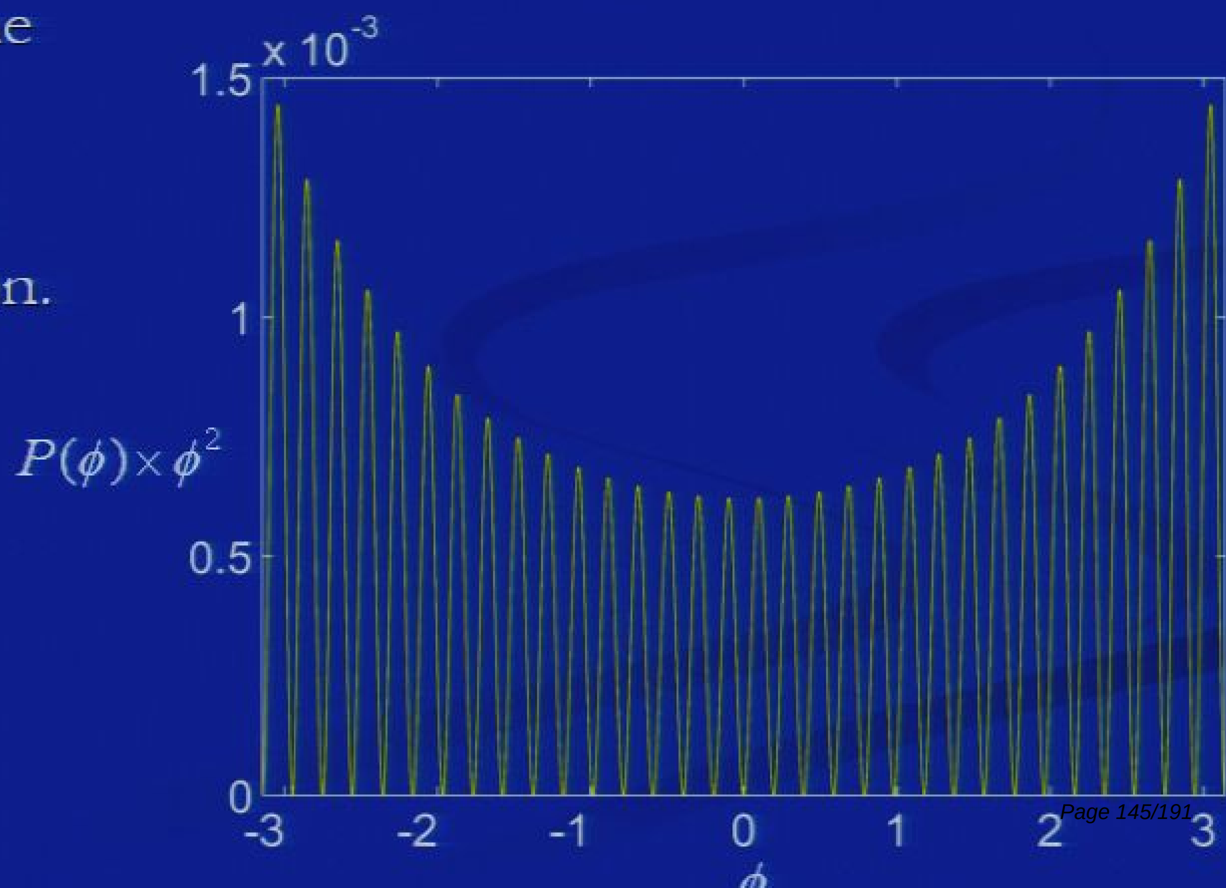

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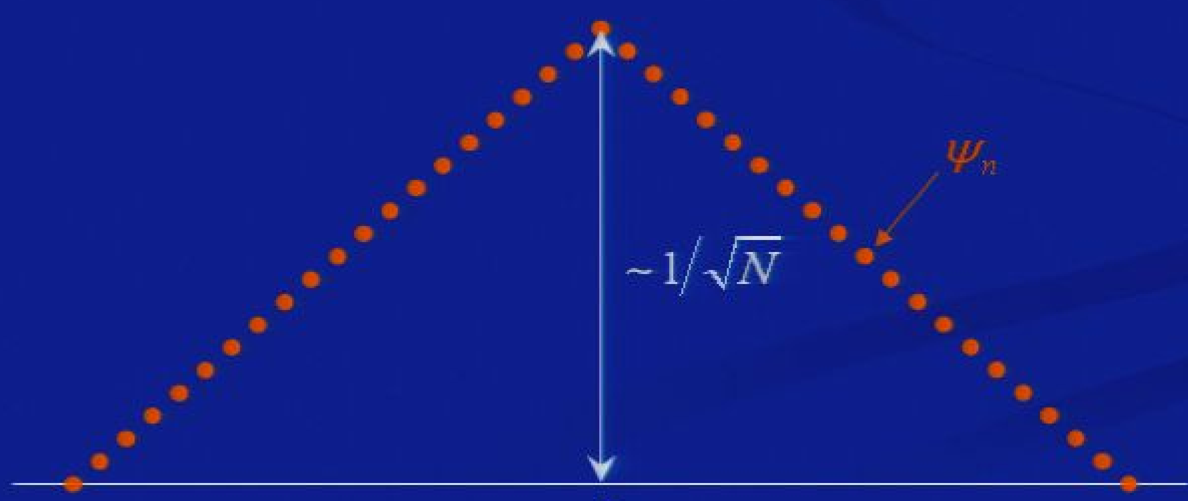

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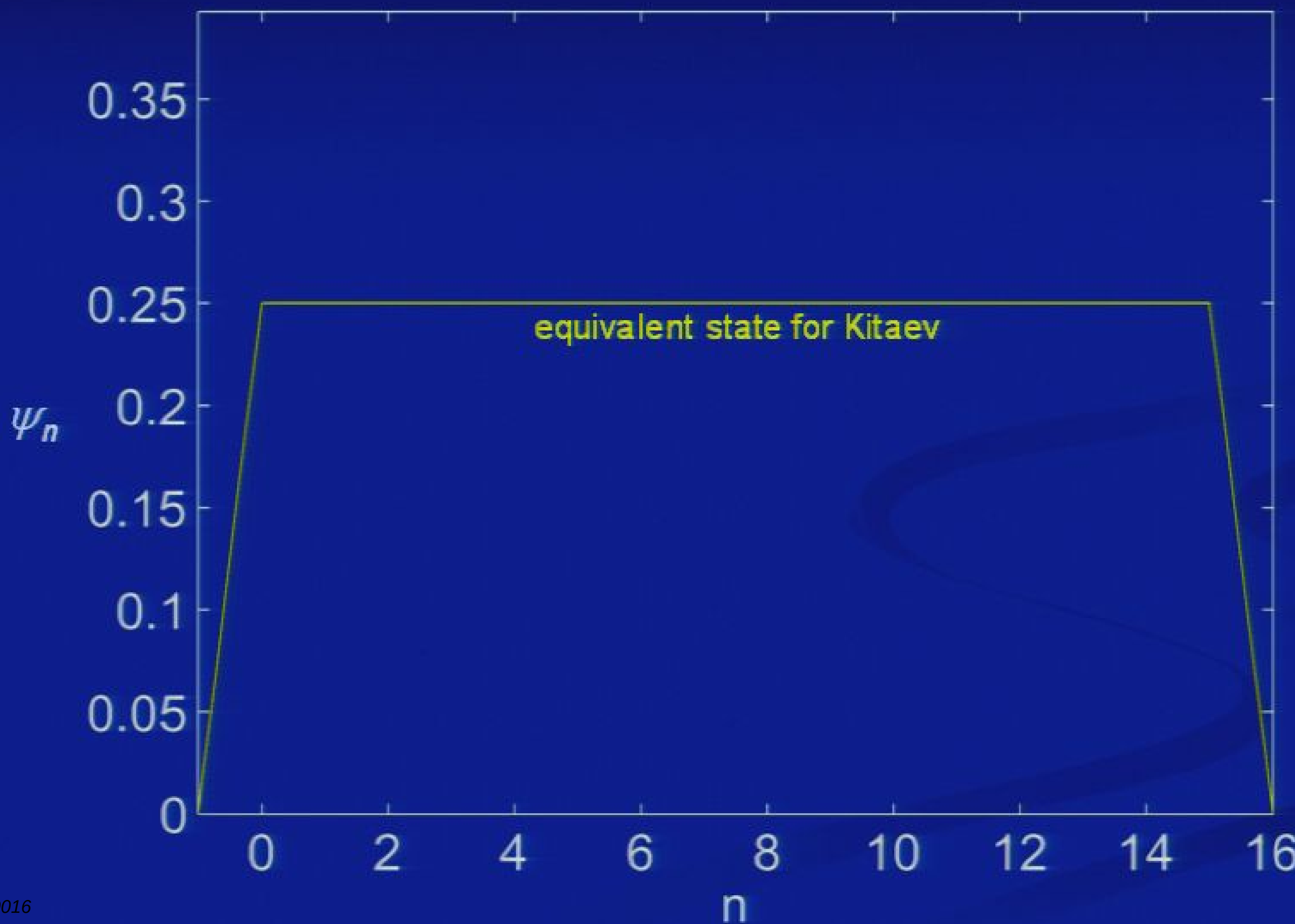

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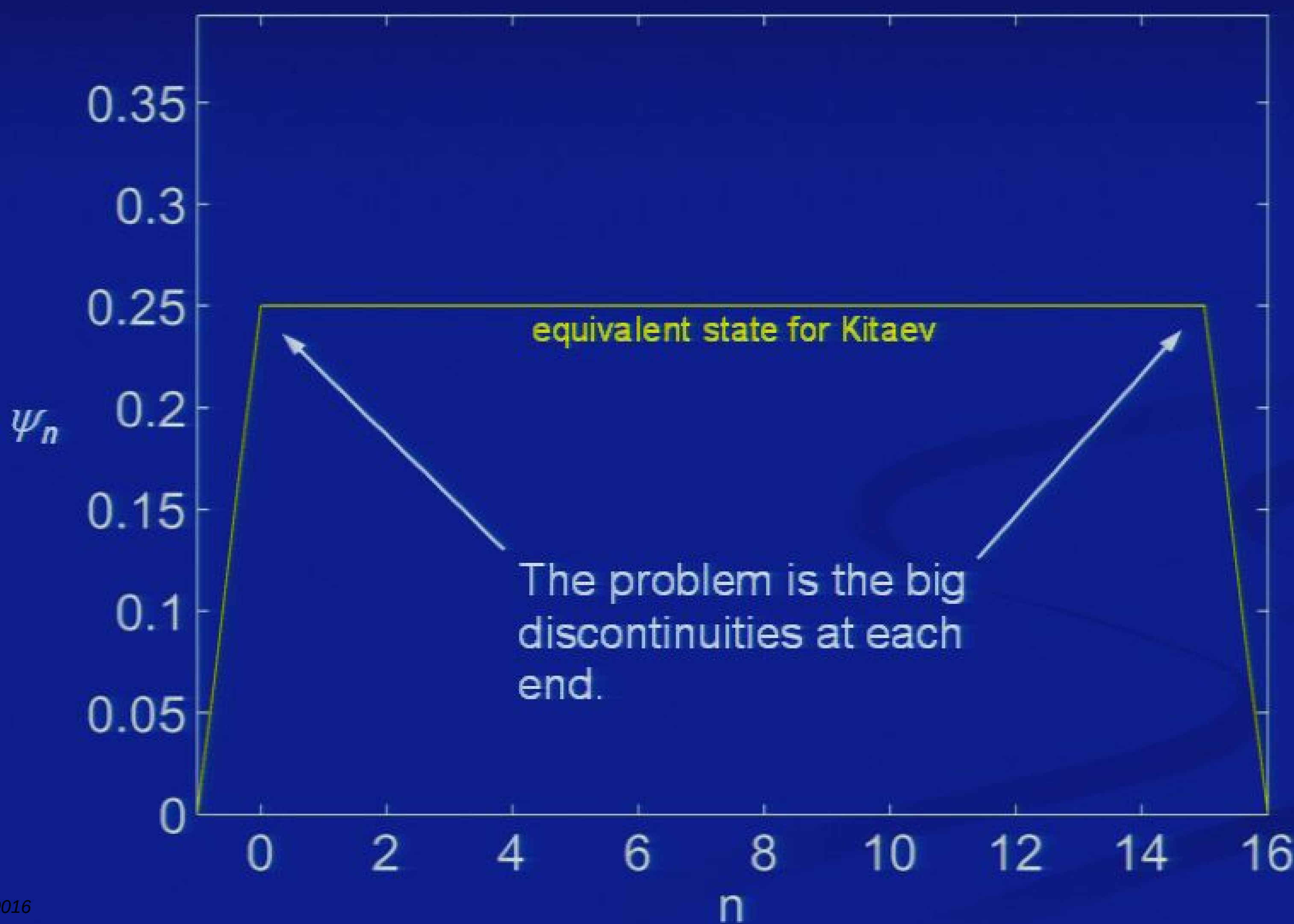

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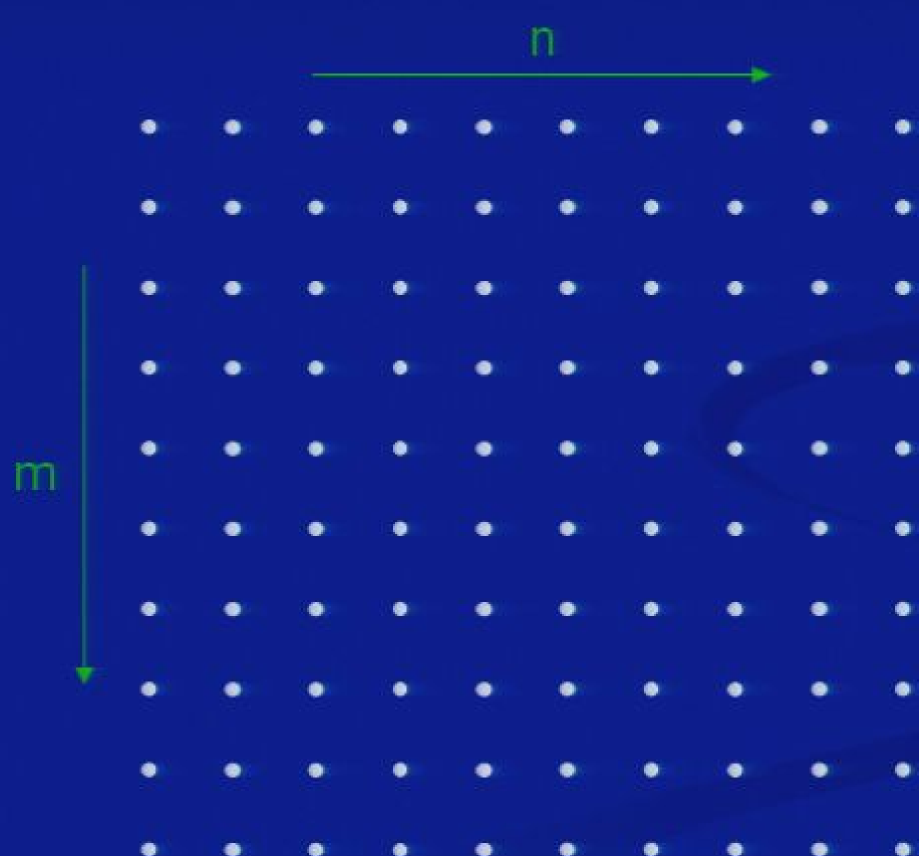

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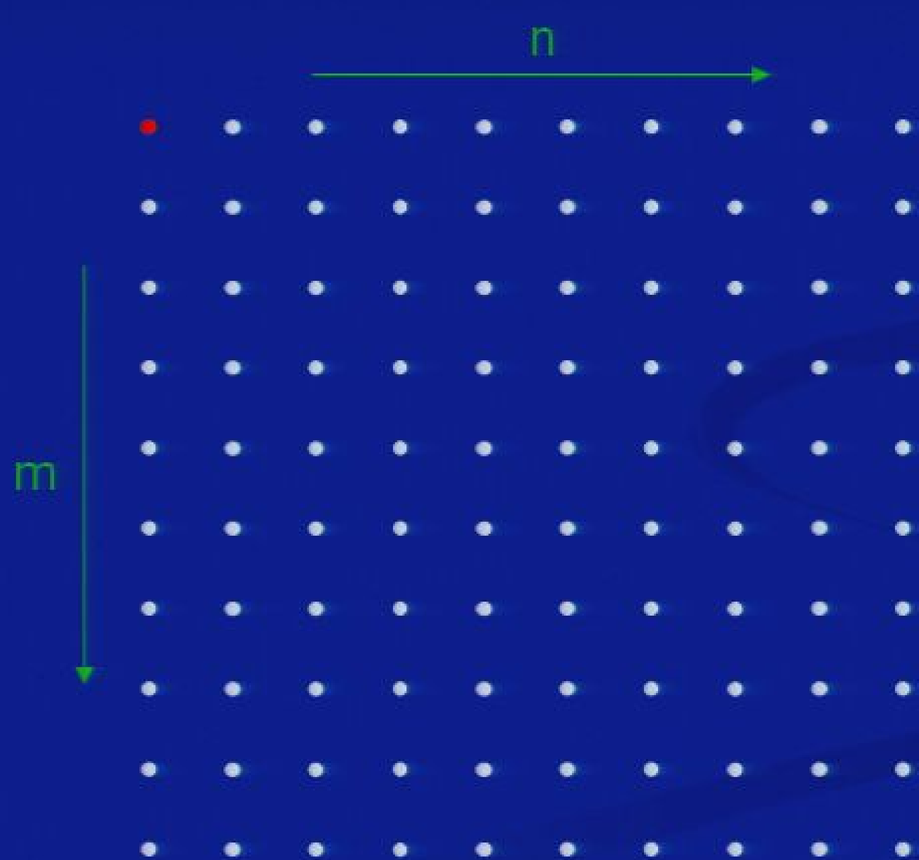

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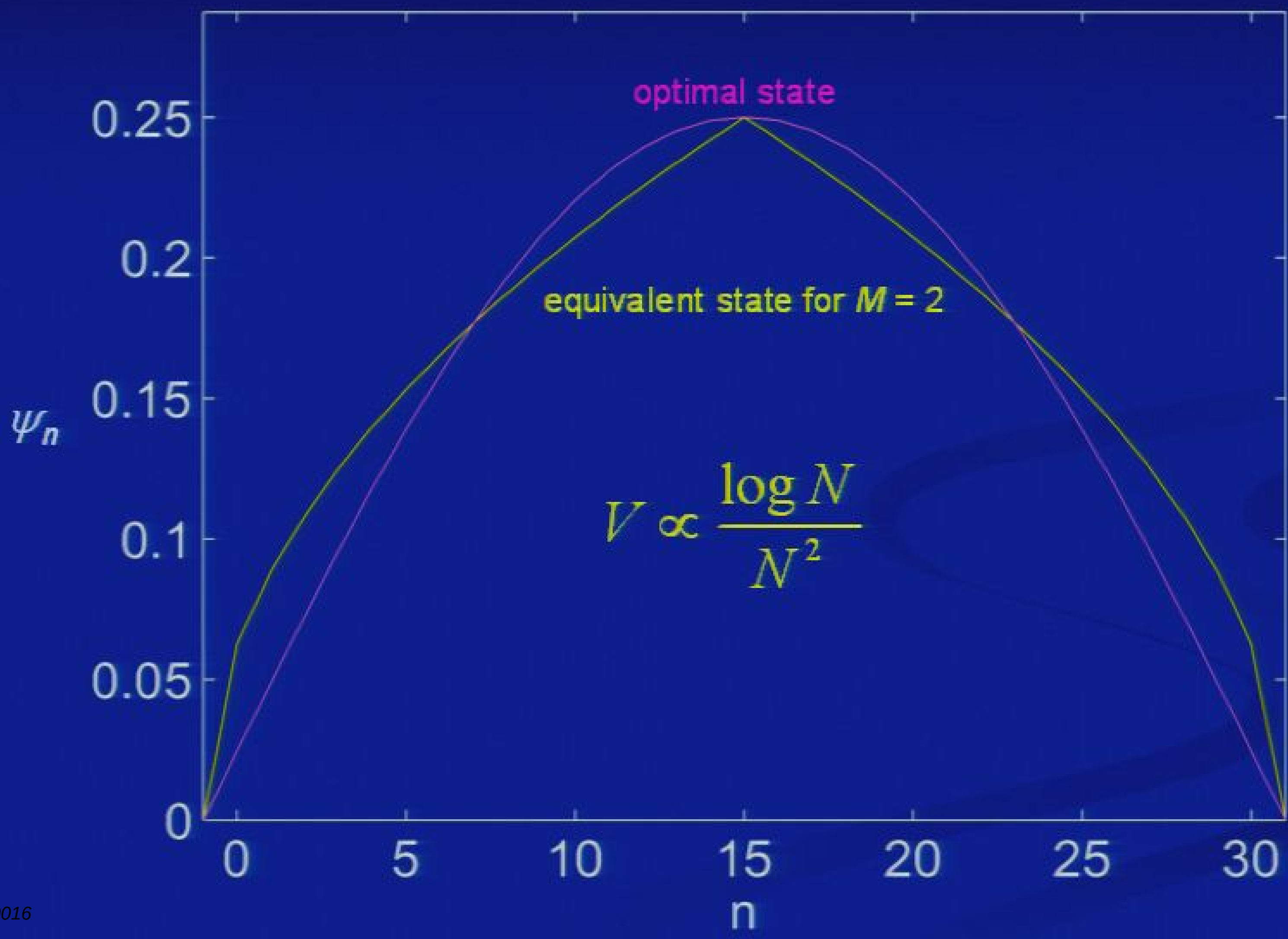

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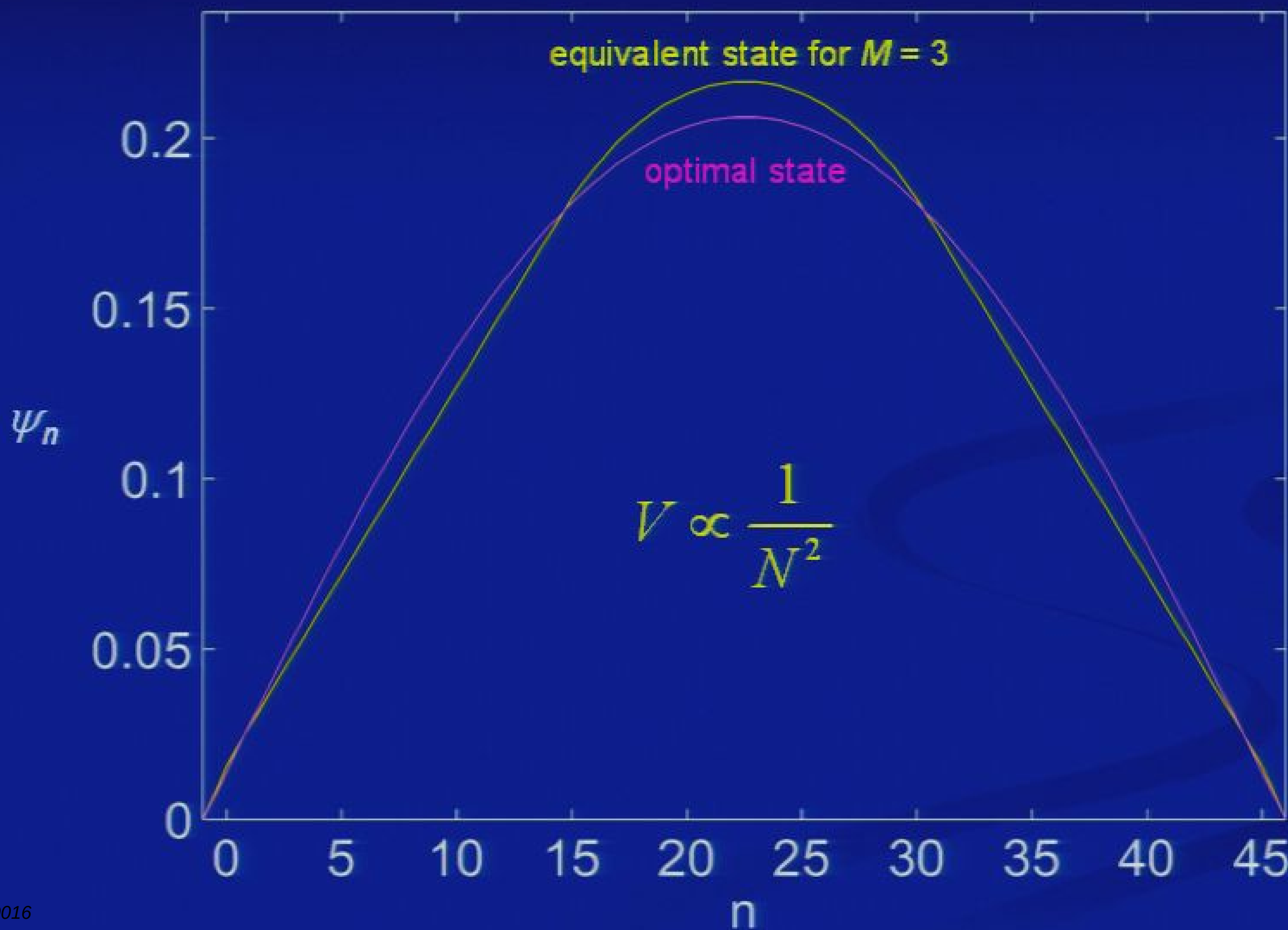

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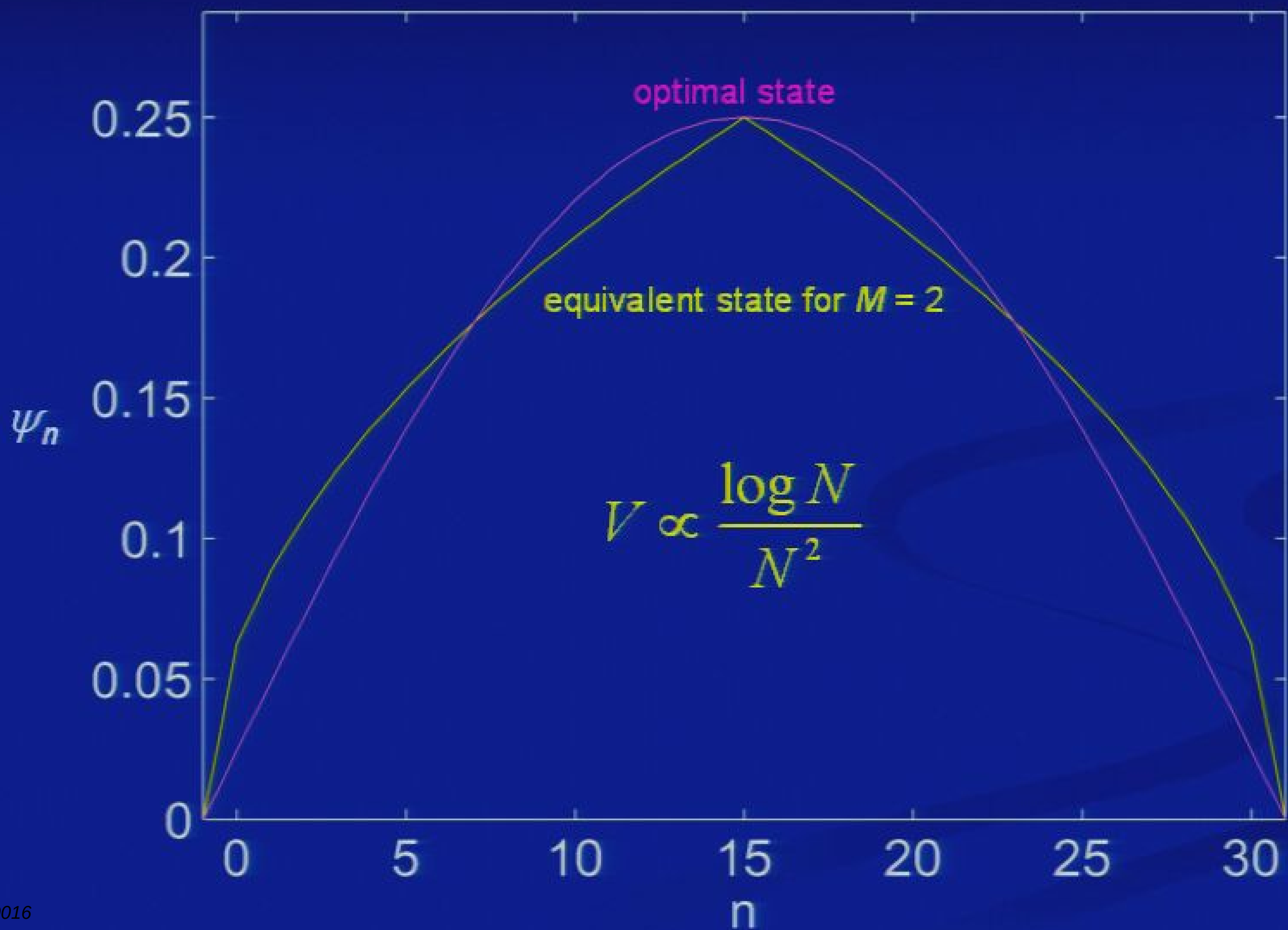

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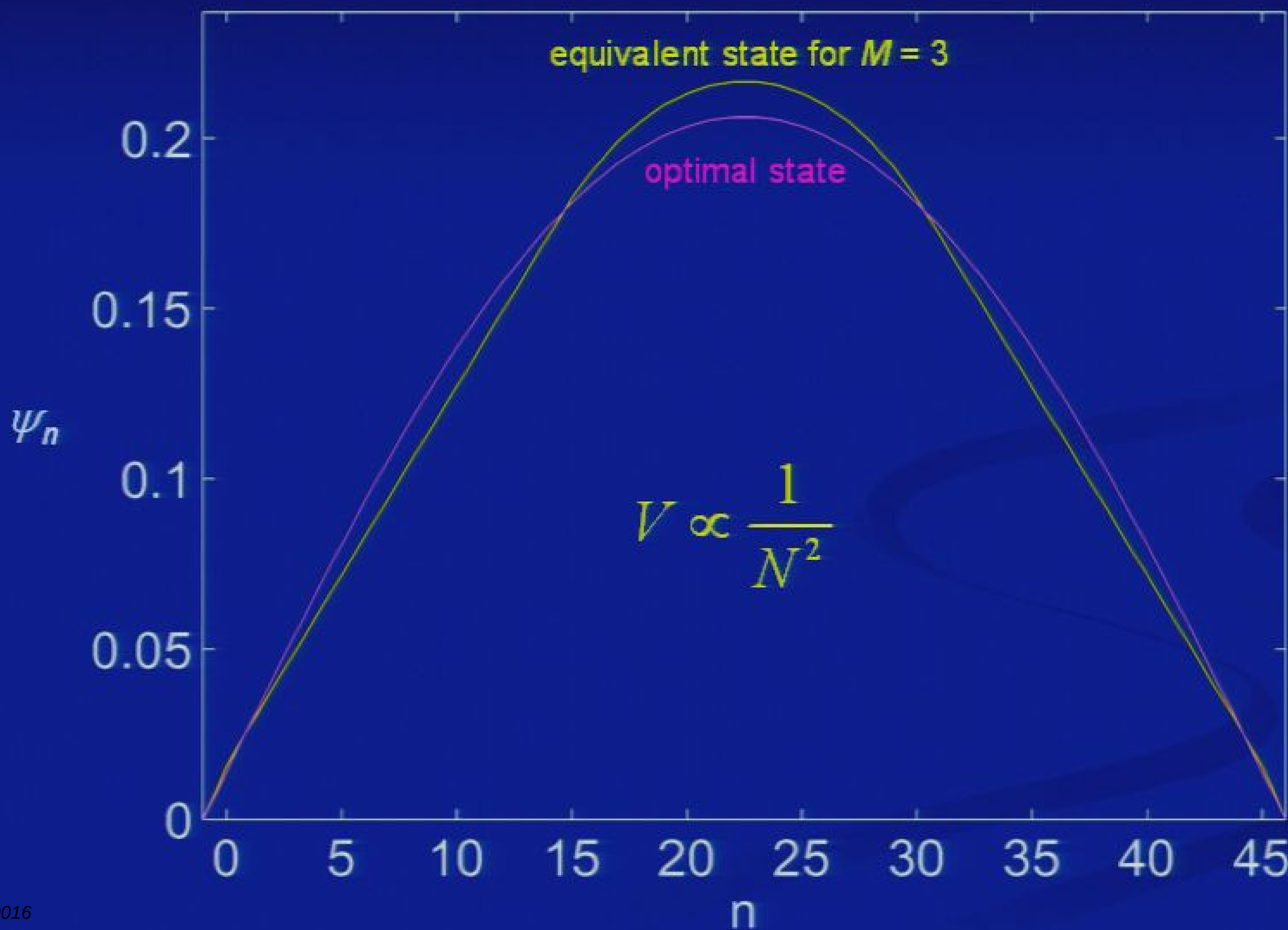

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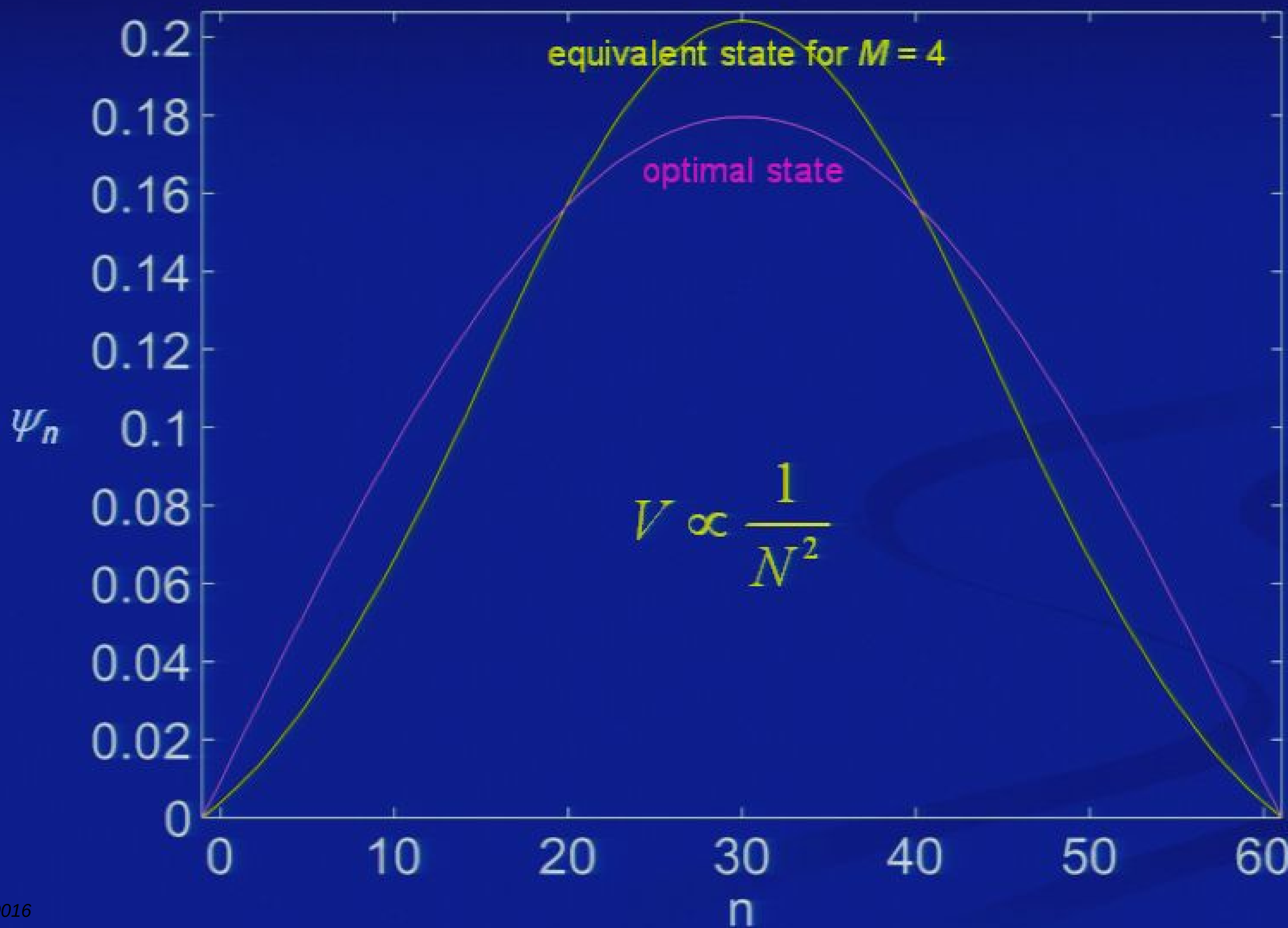

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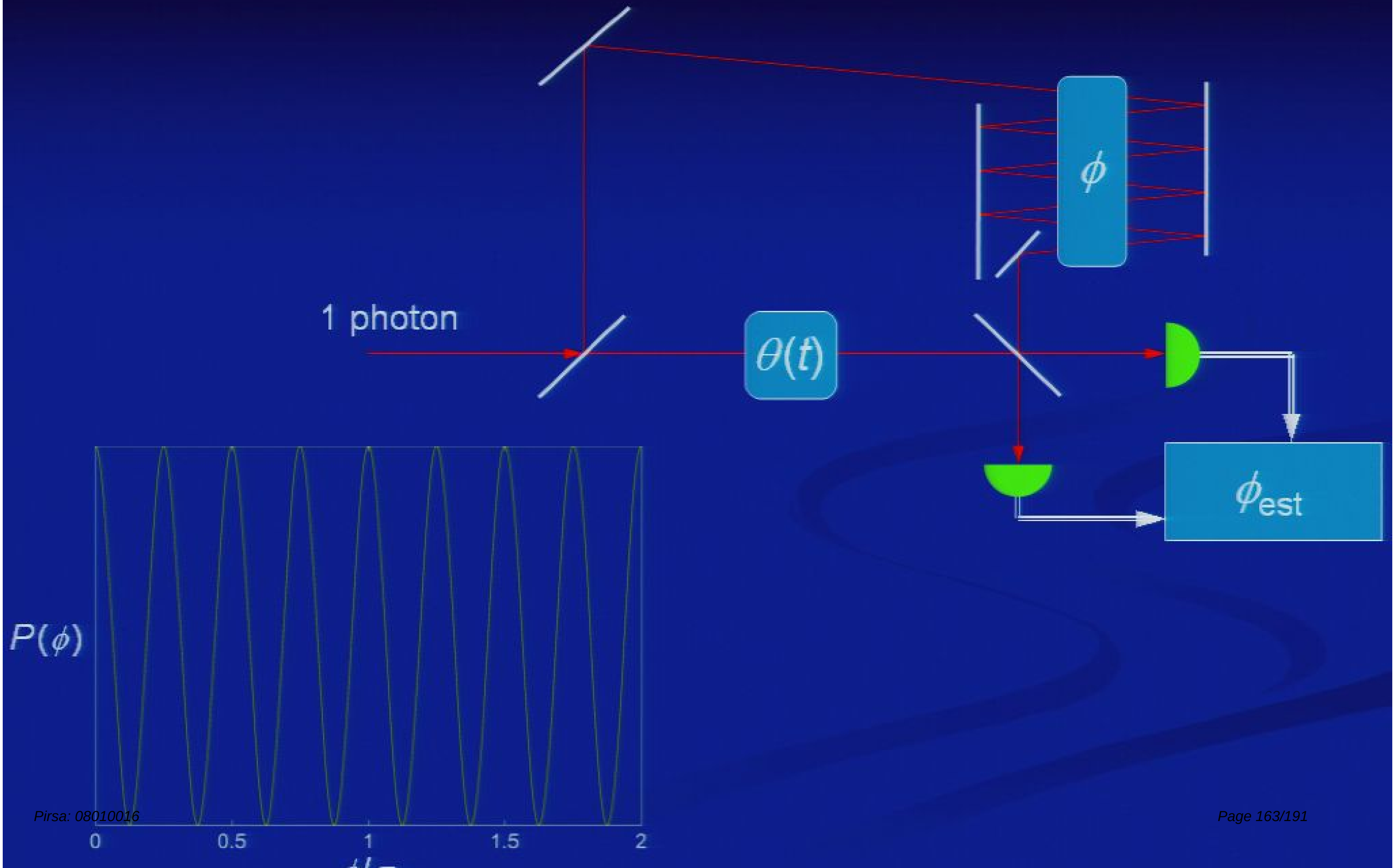

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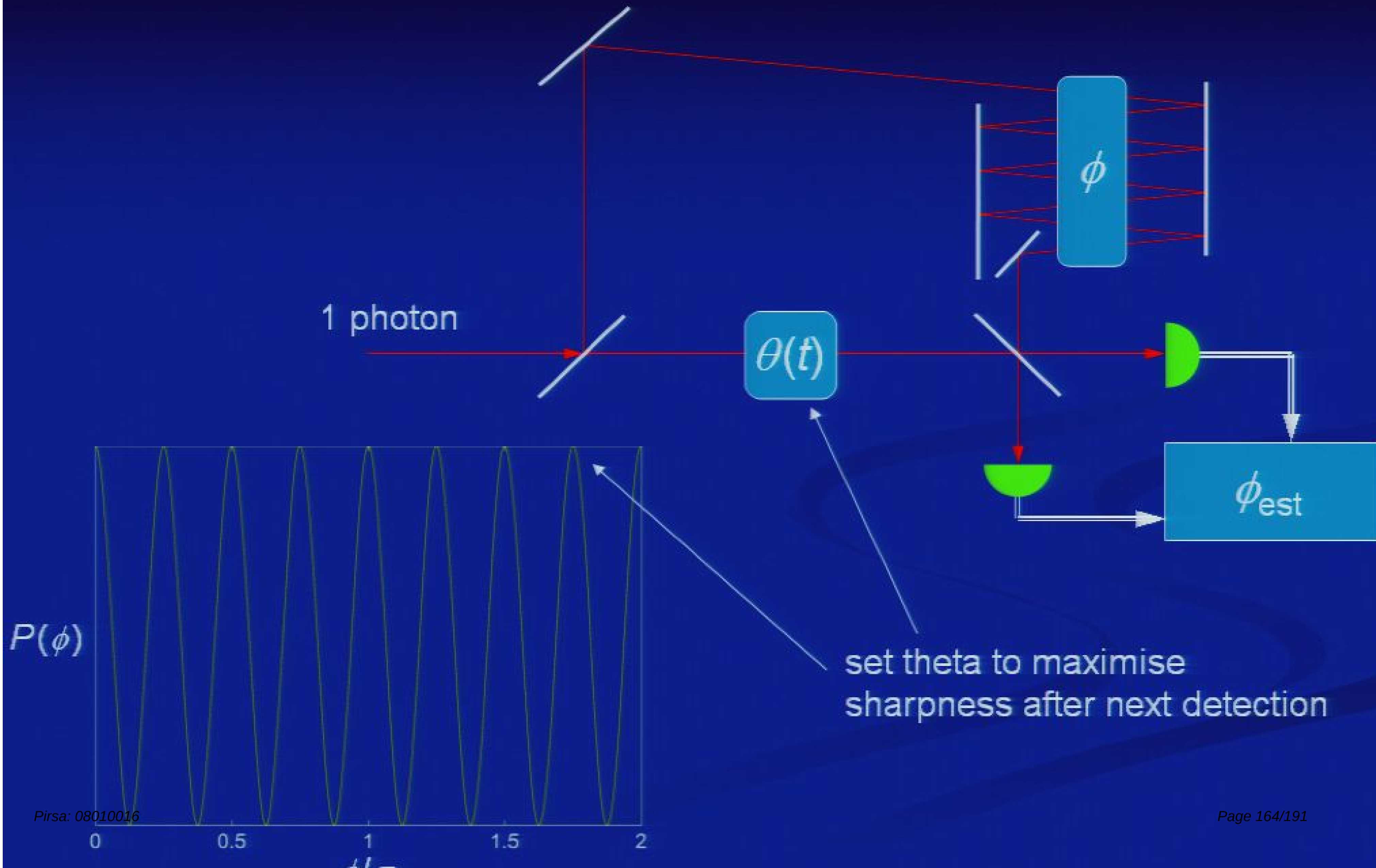

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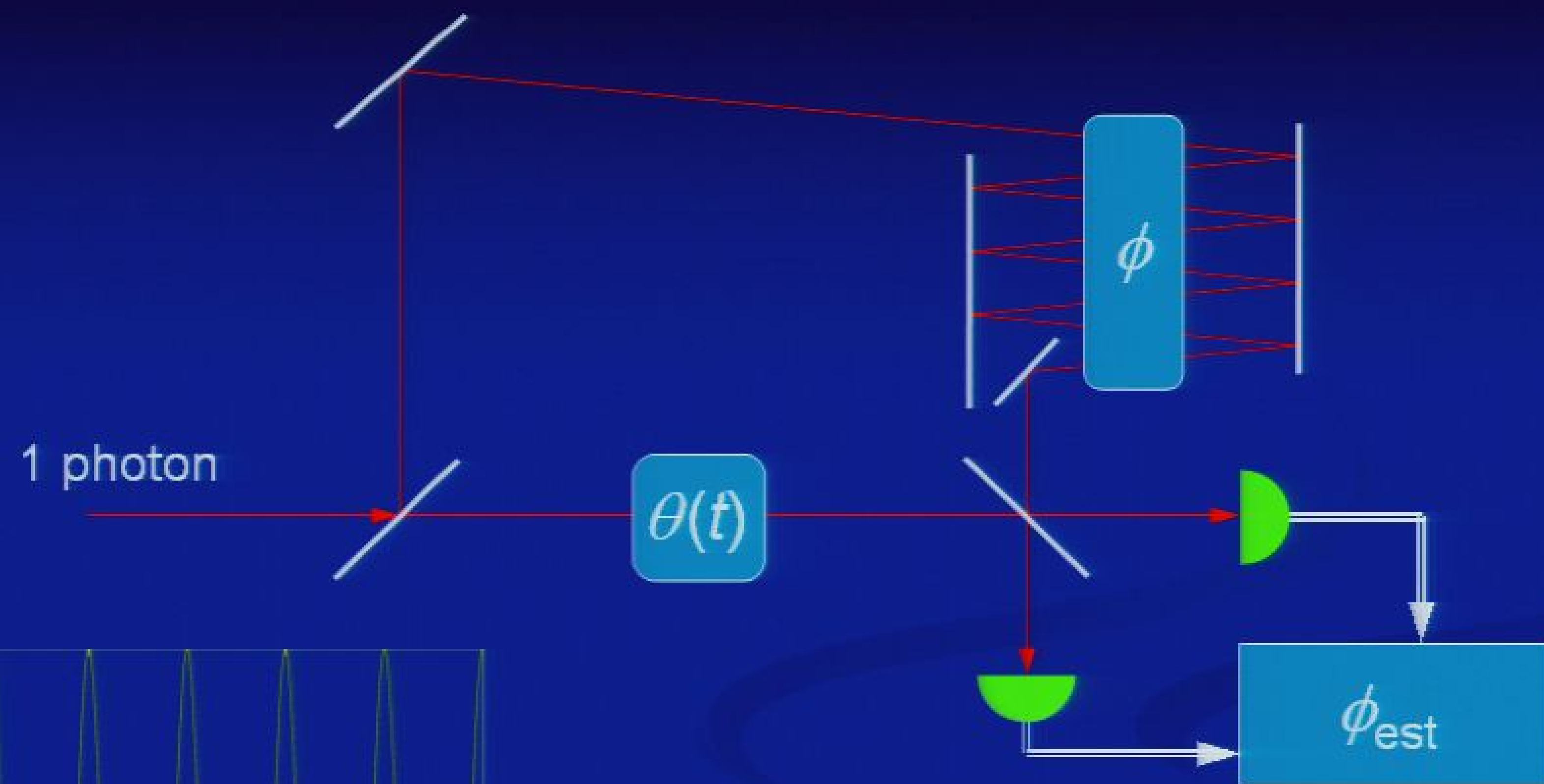
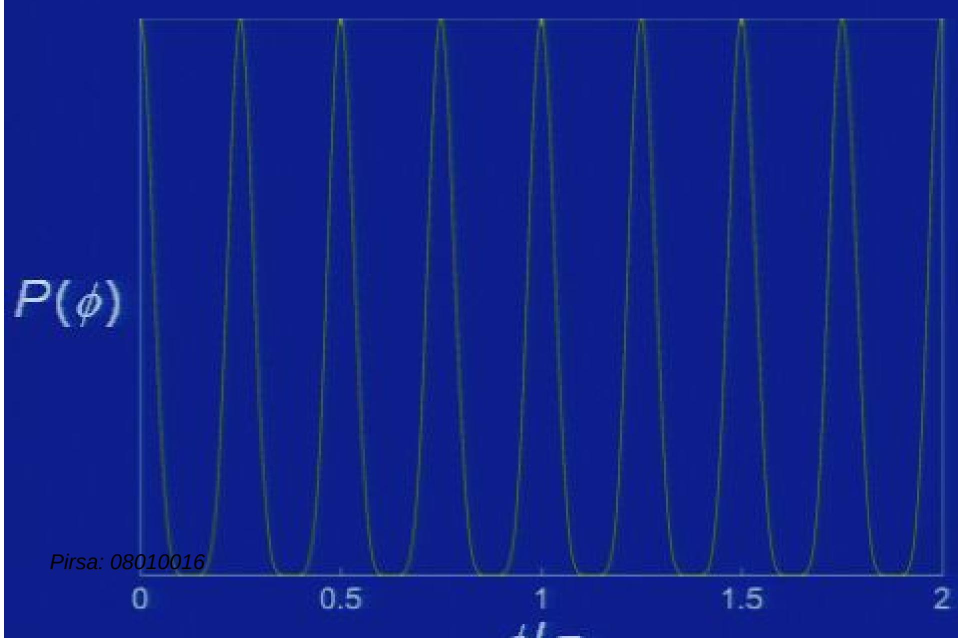
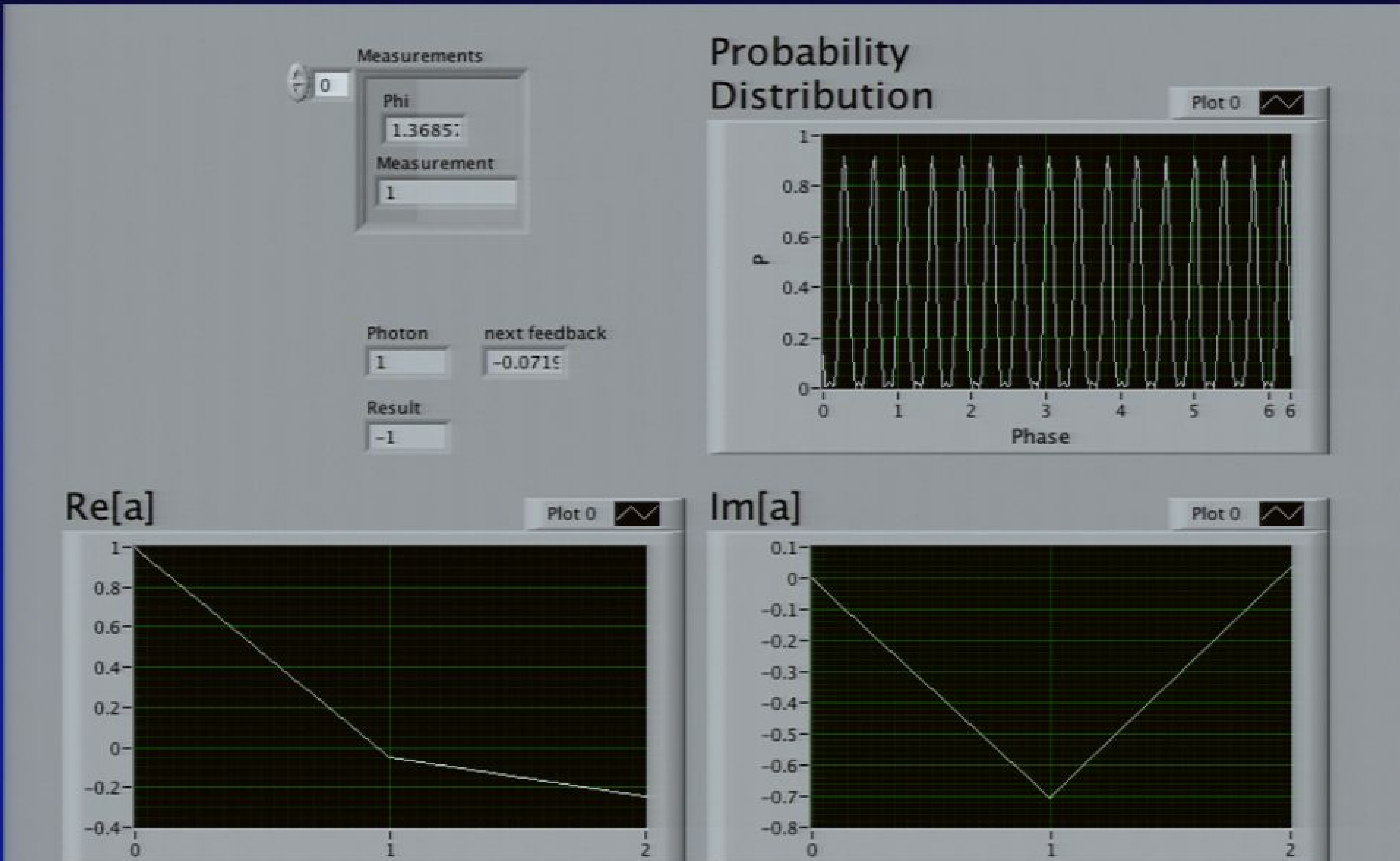



$$N = M(2^{K+1} - 1)$$

# Updating the probability distribution



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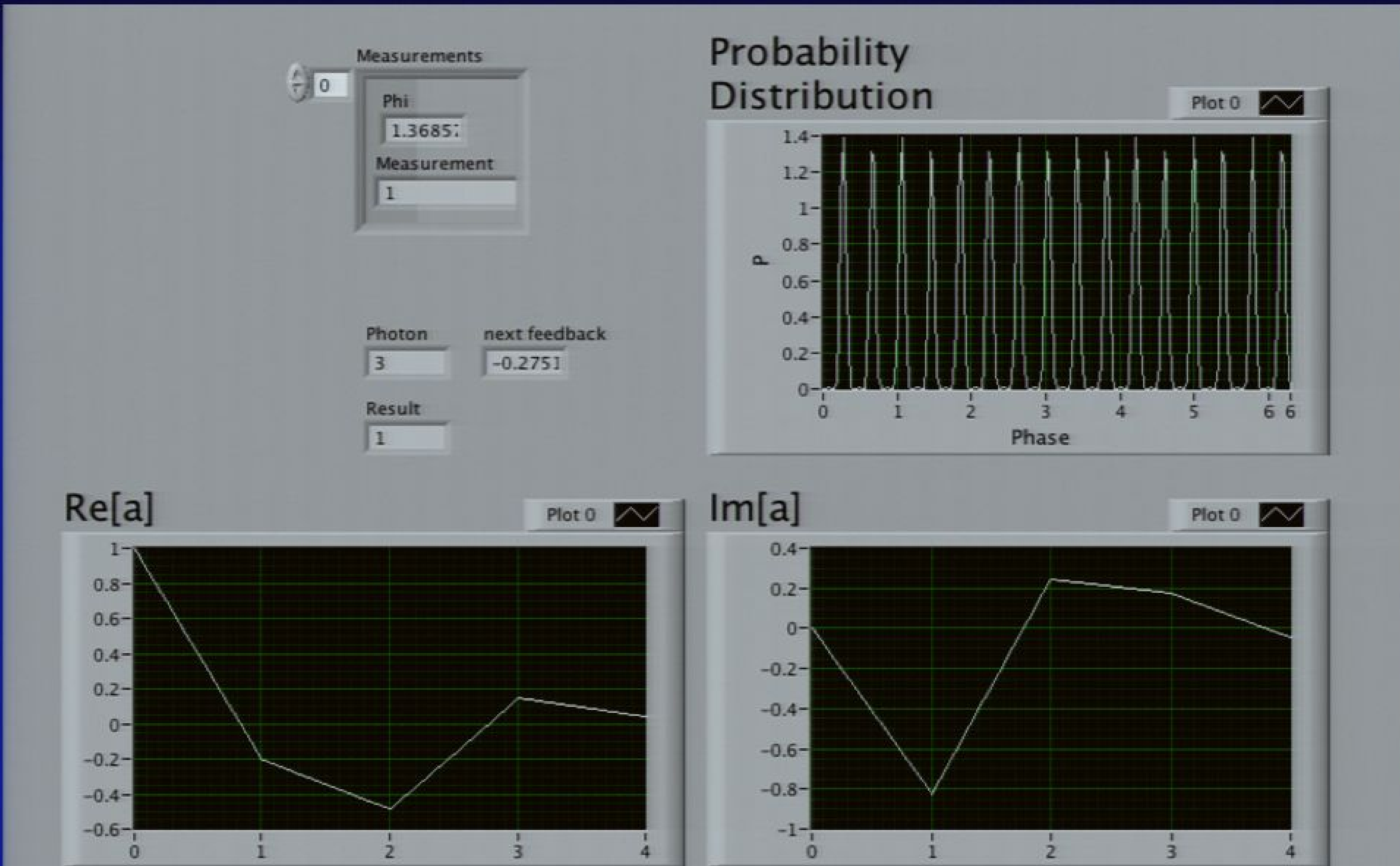

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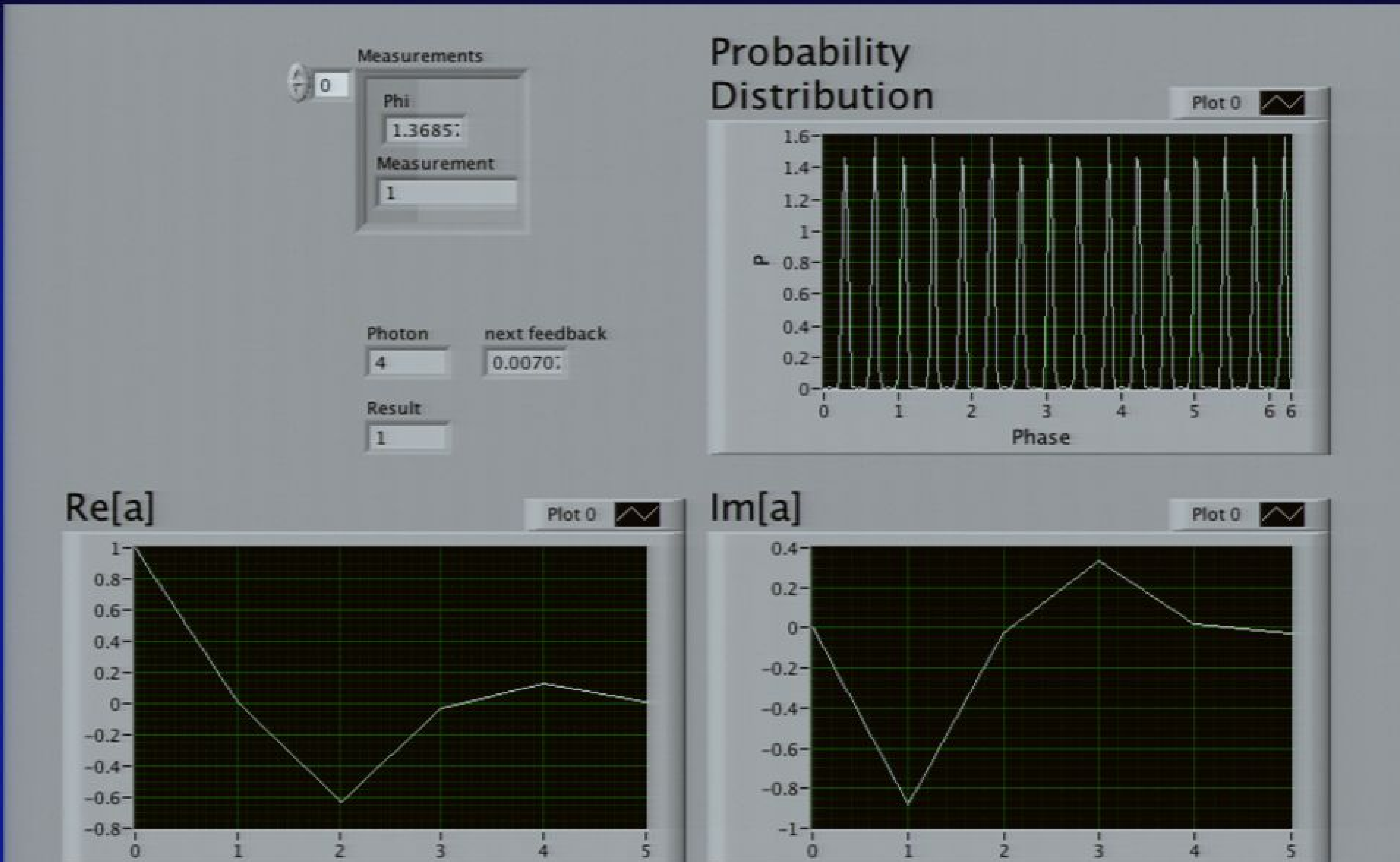

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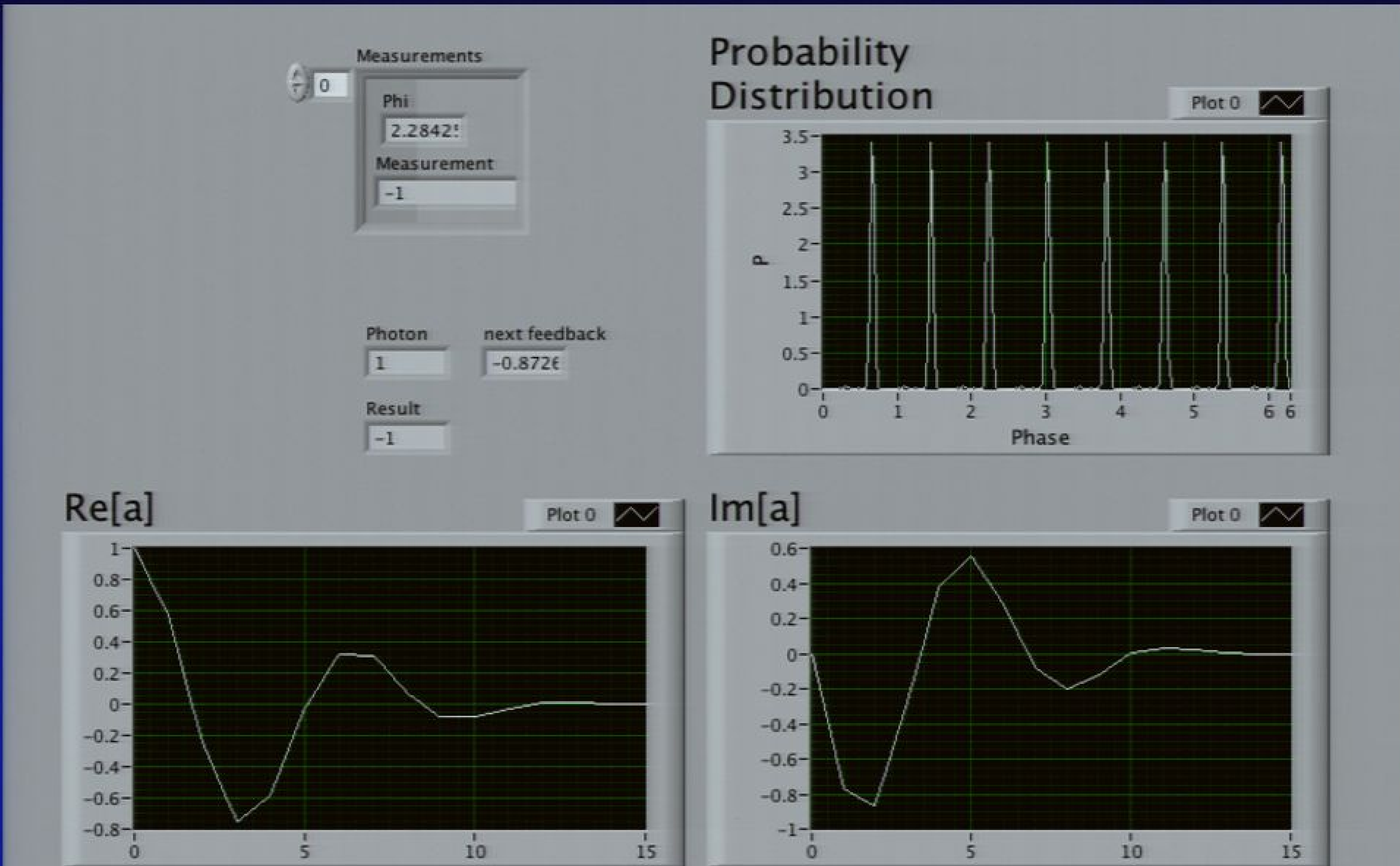

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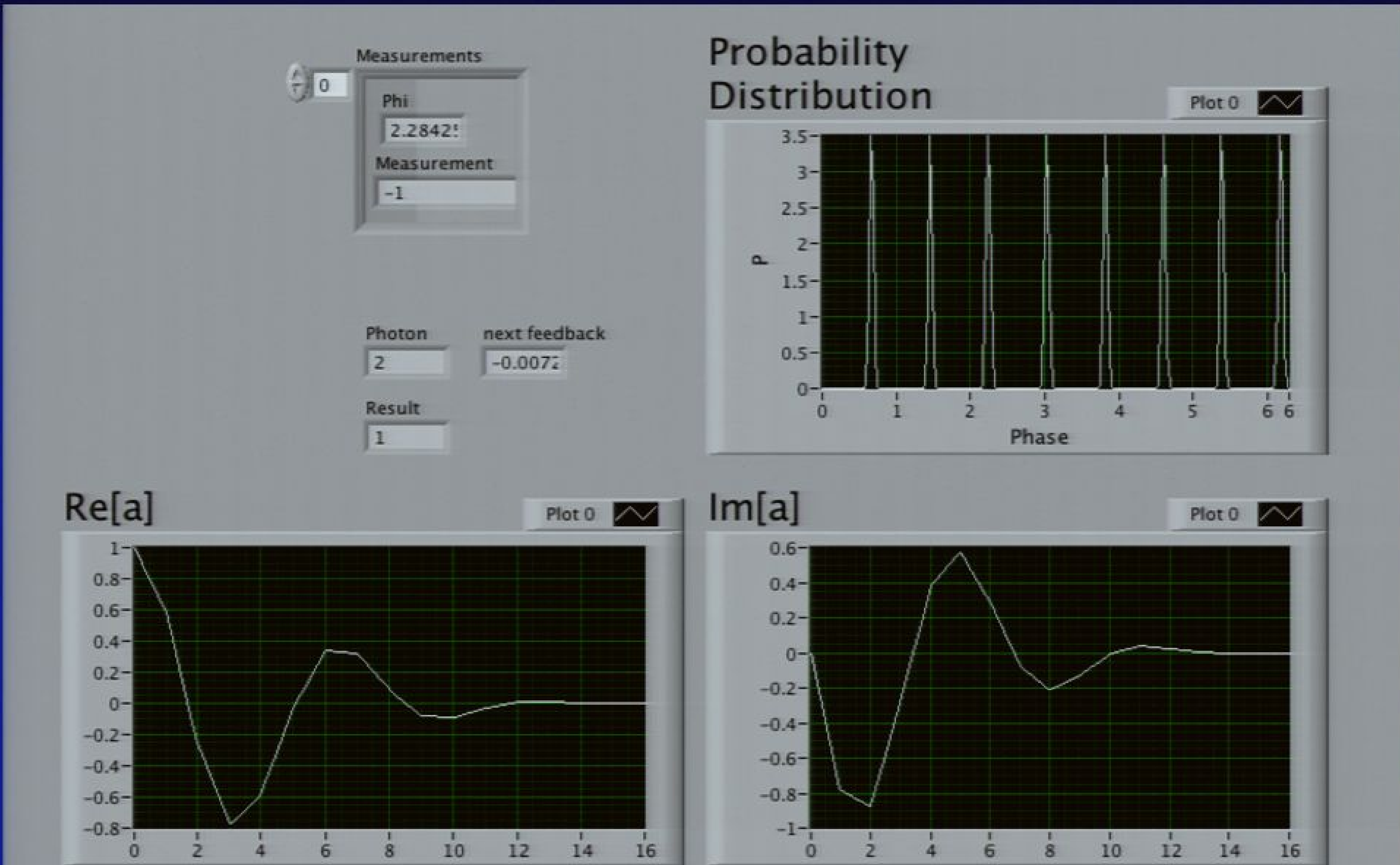

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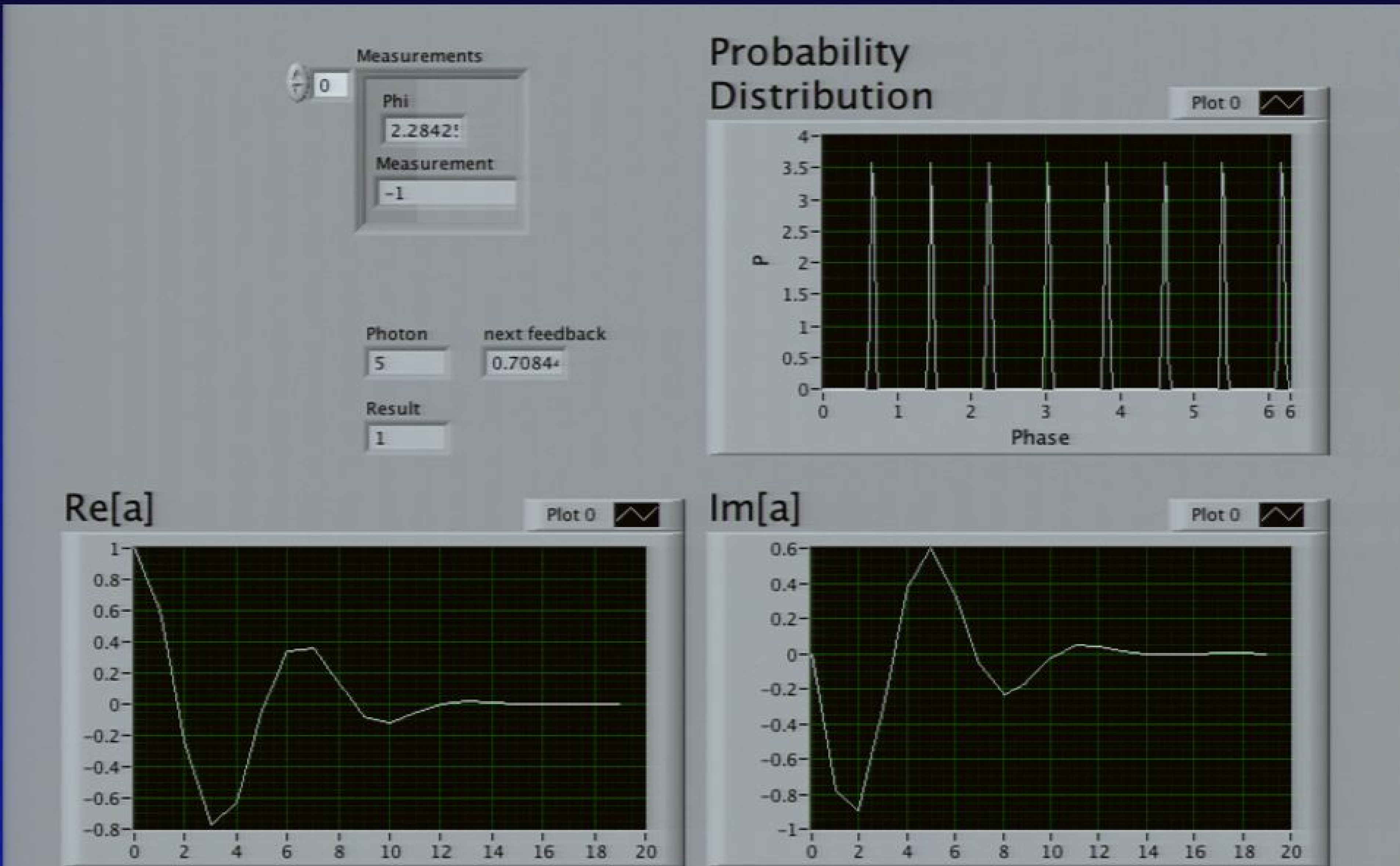

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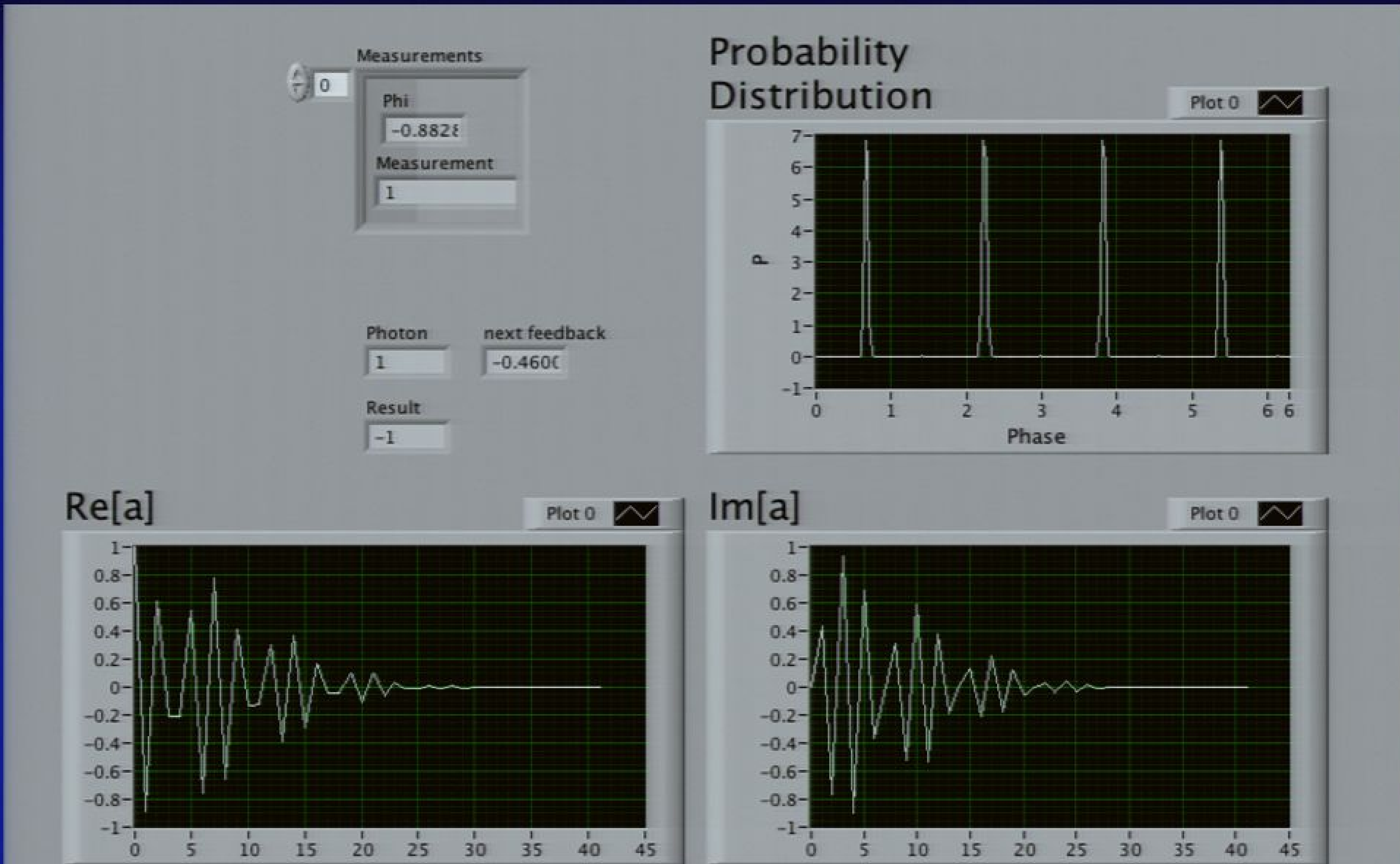

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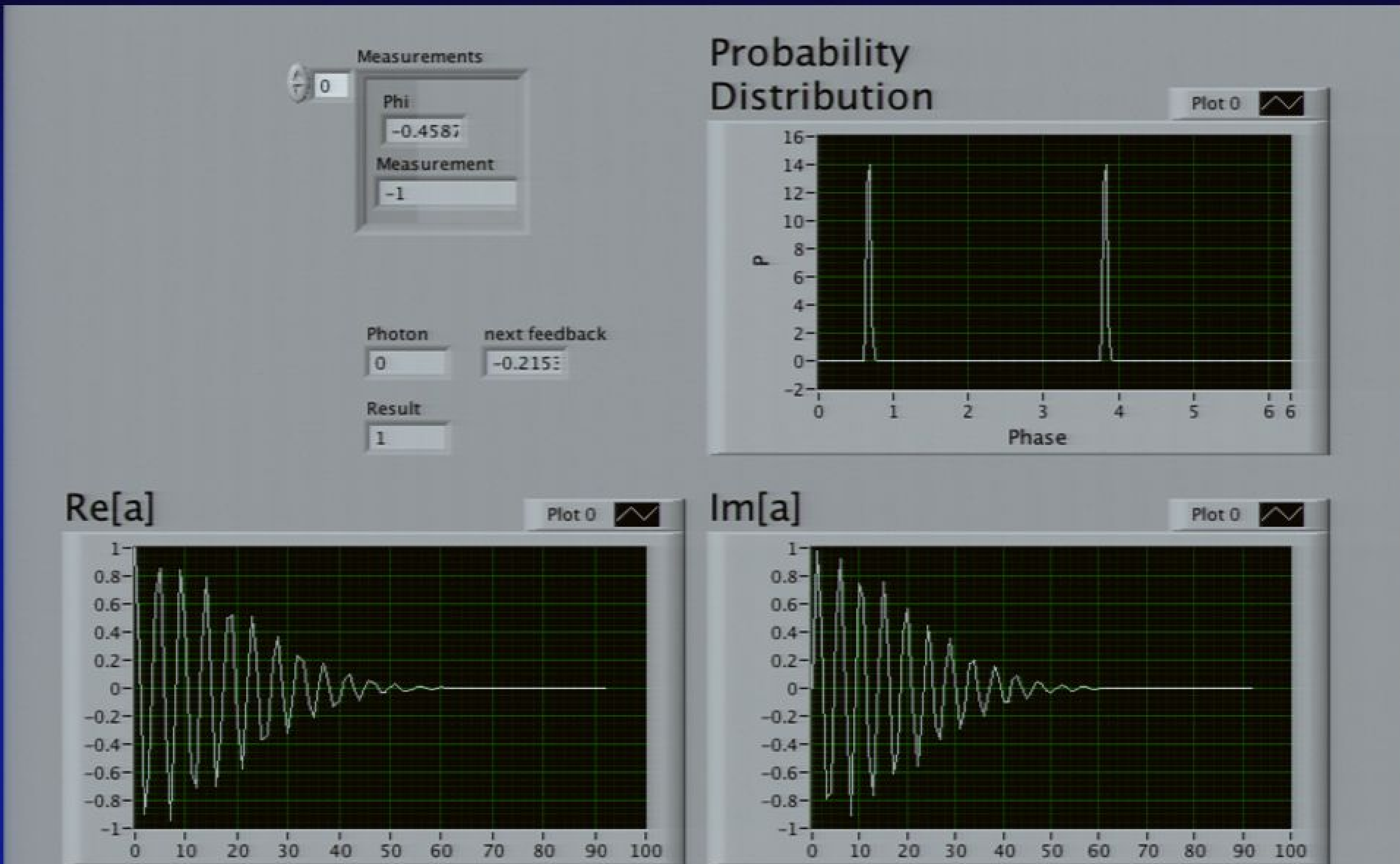

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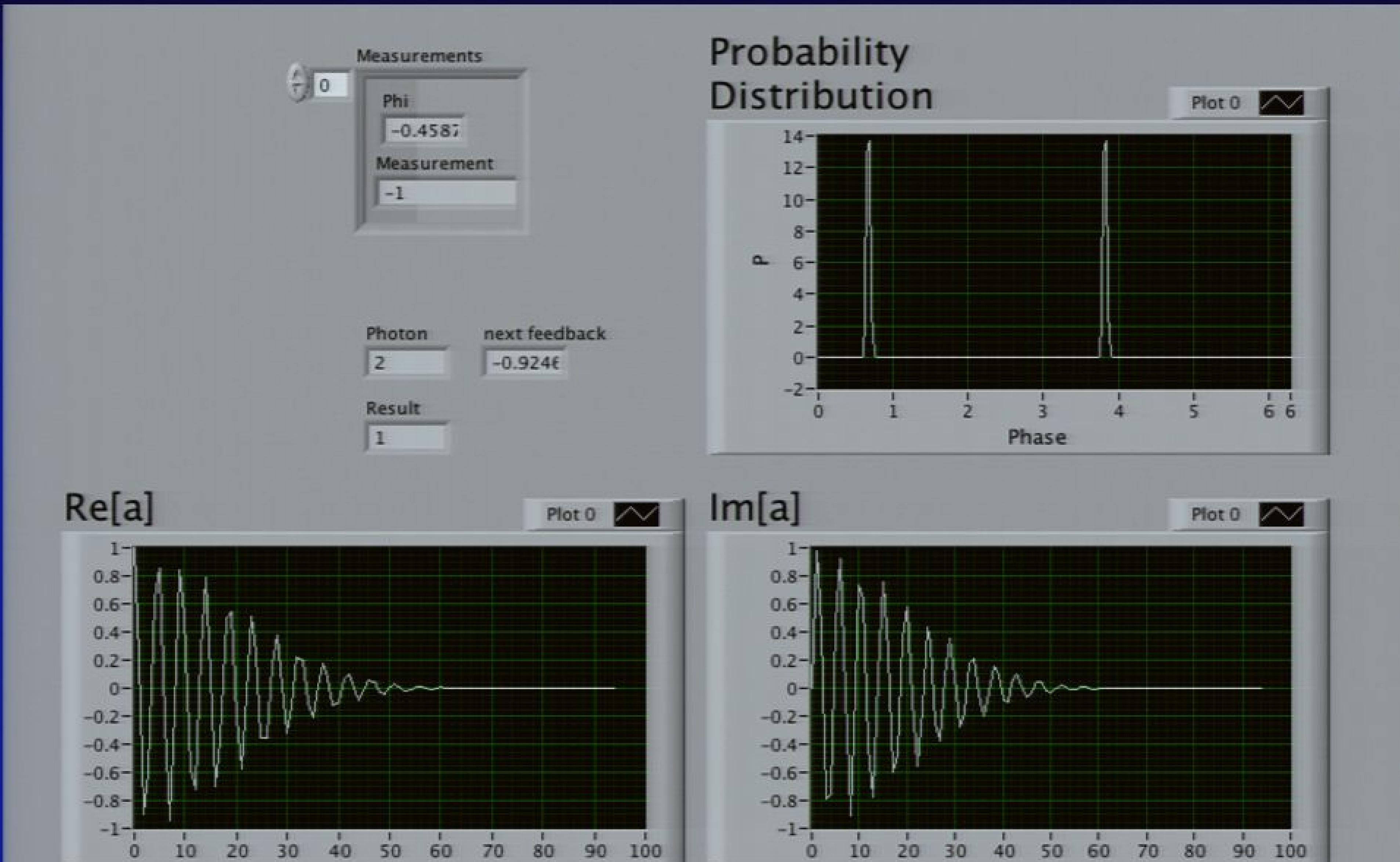

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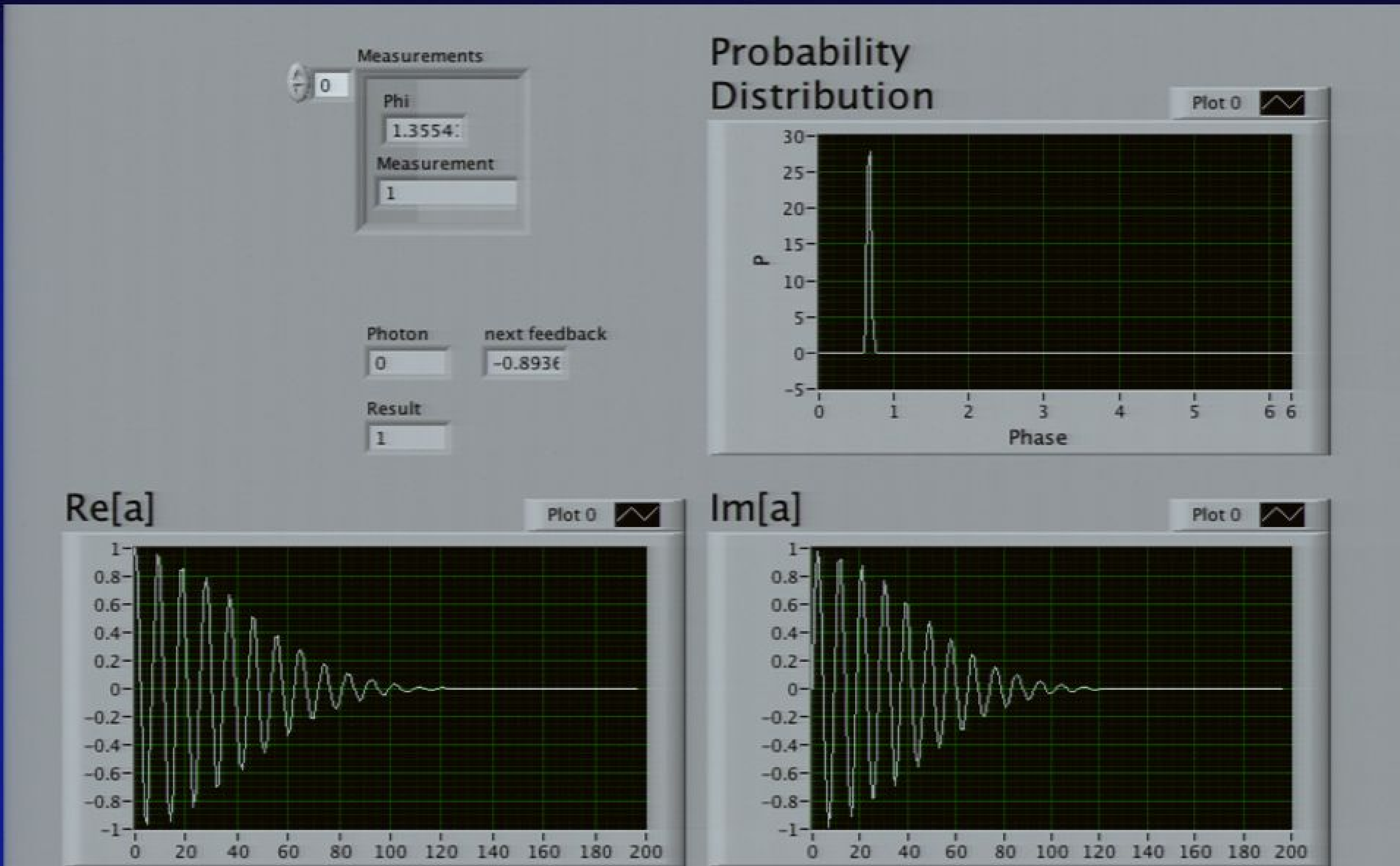

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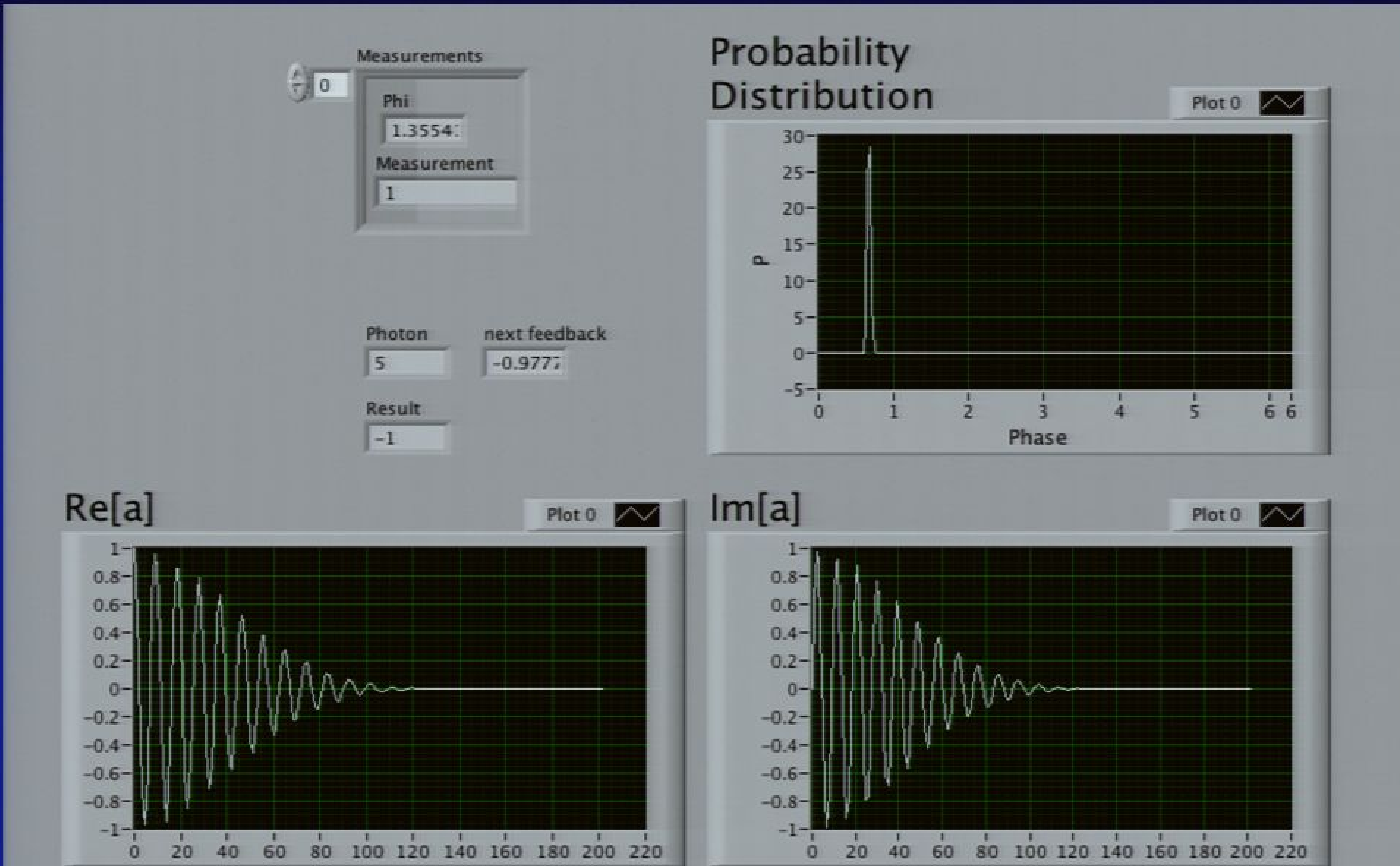
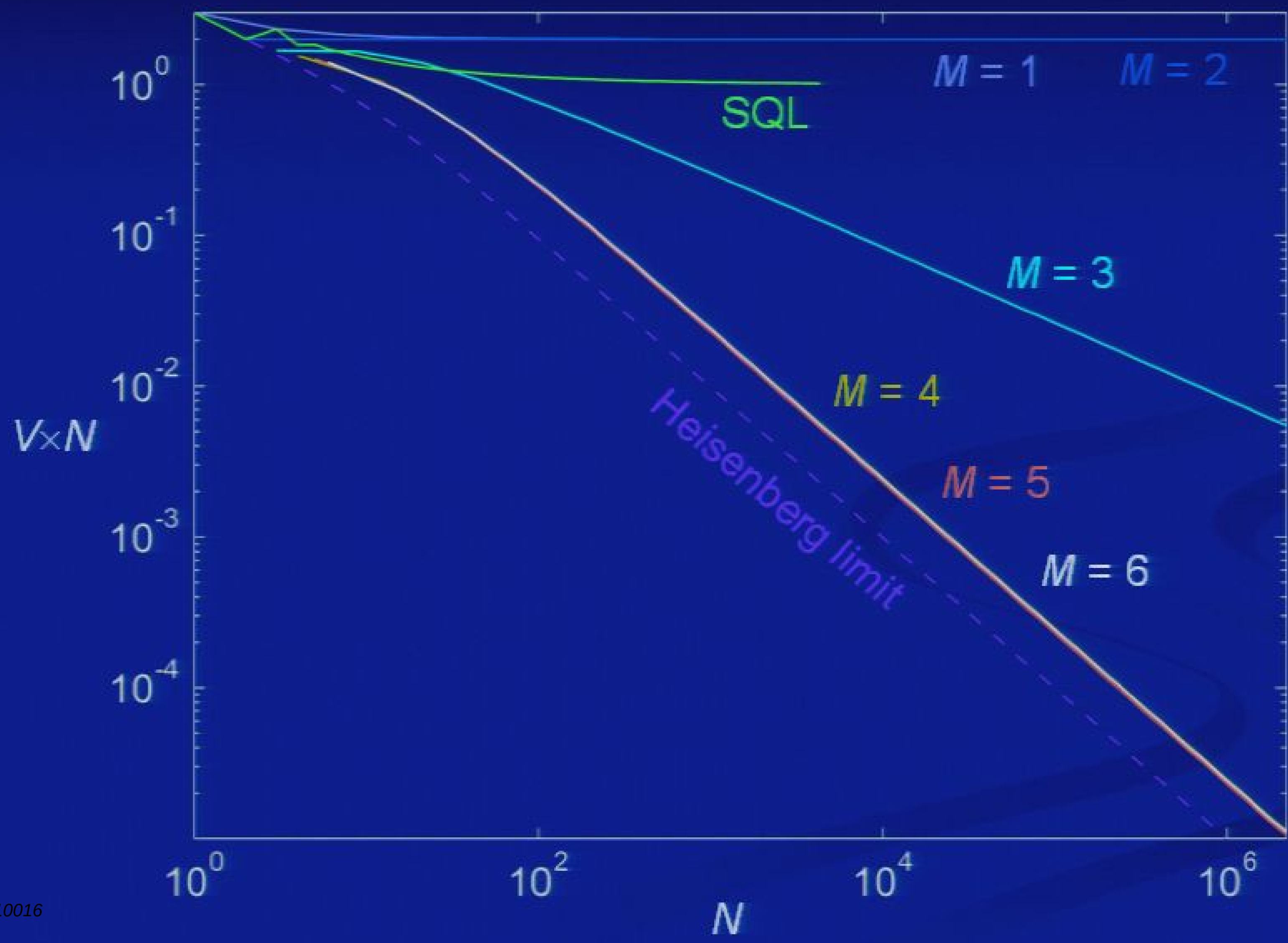
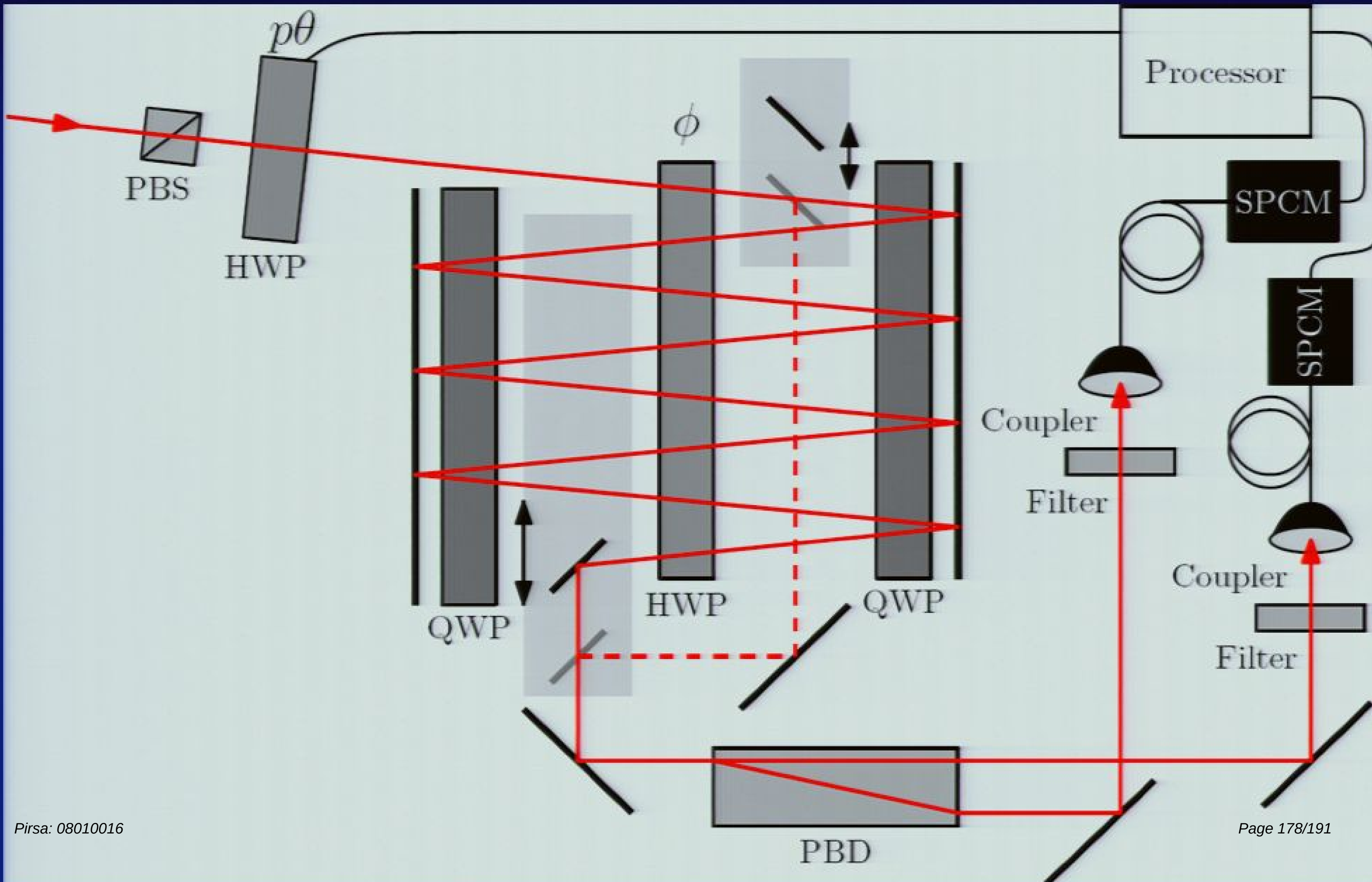


# Predicted variances



# The experiment



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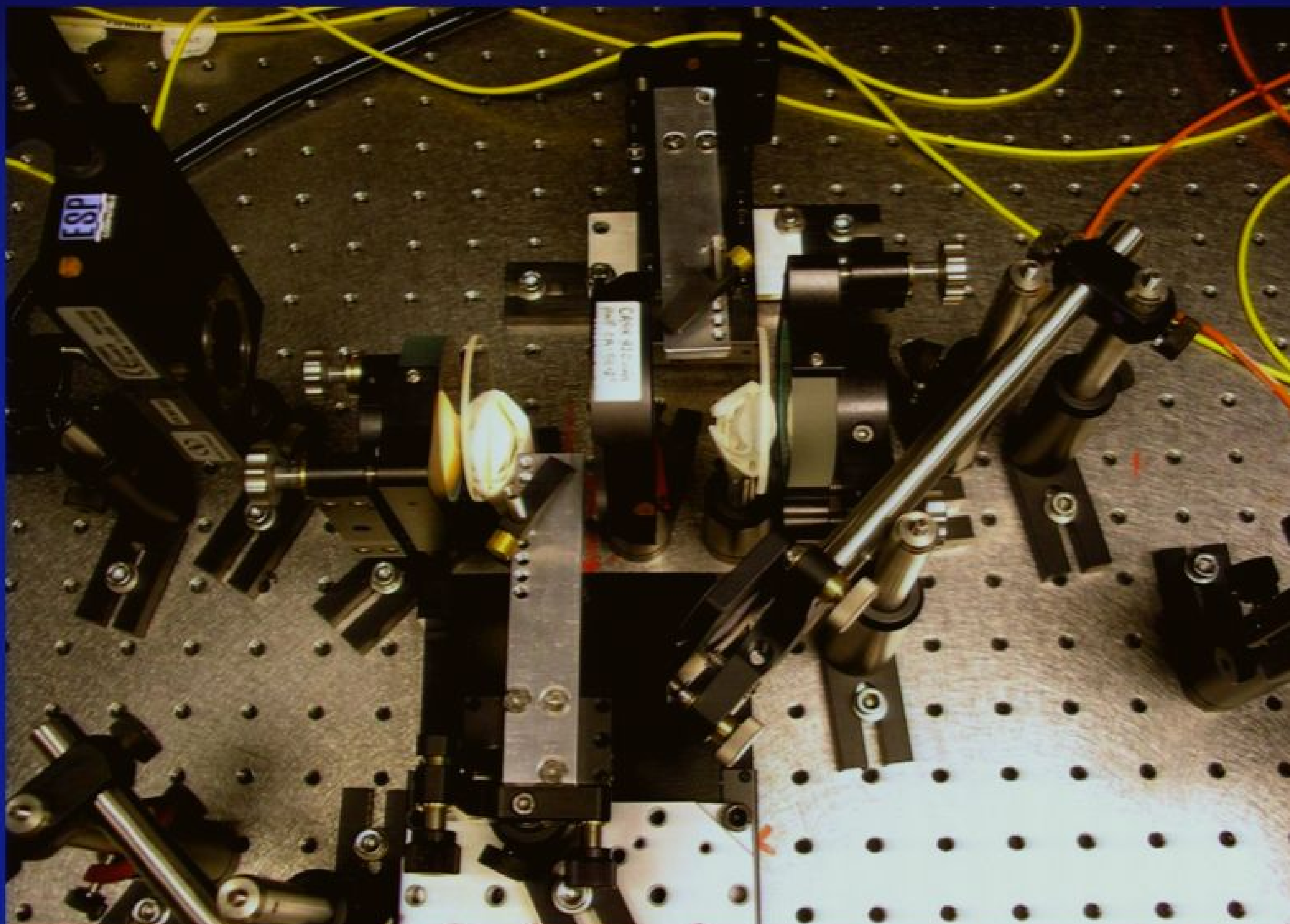
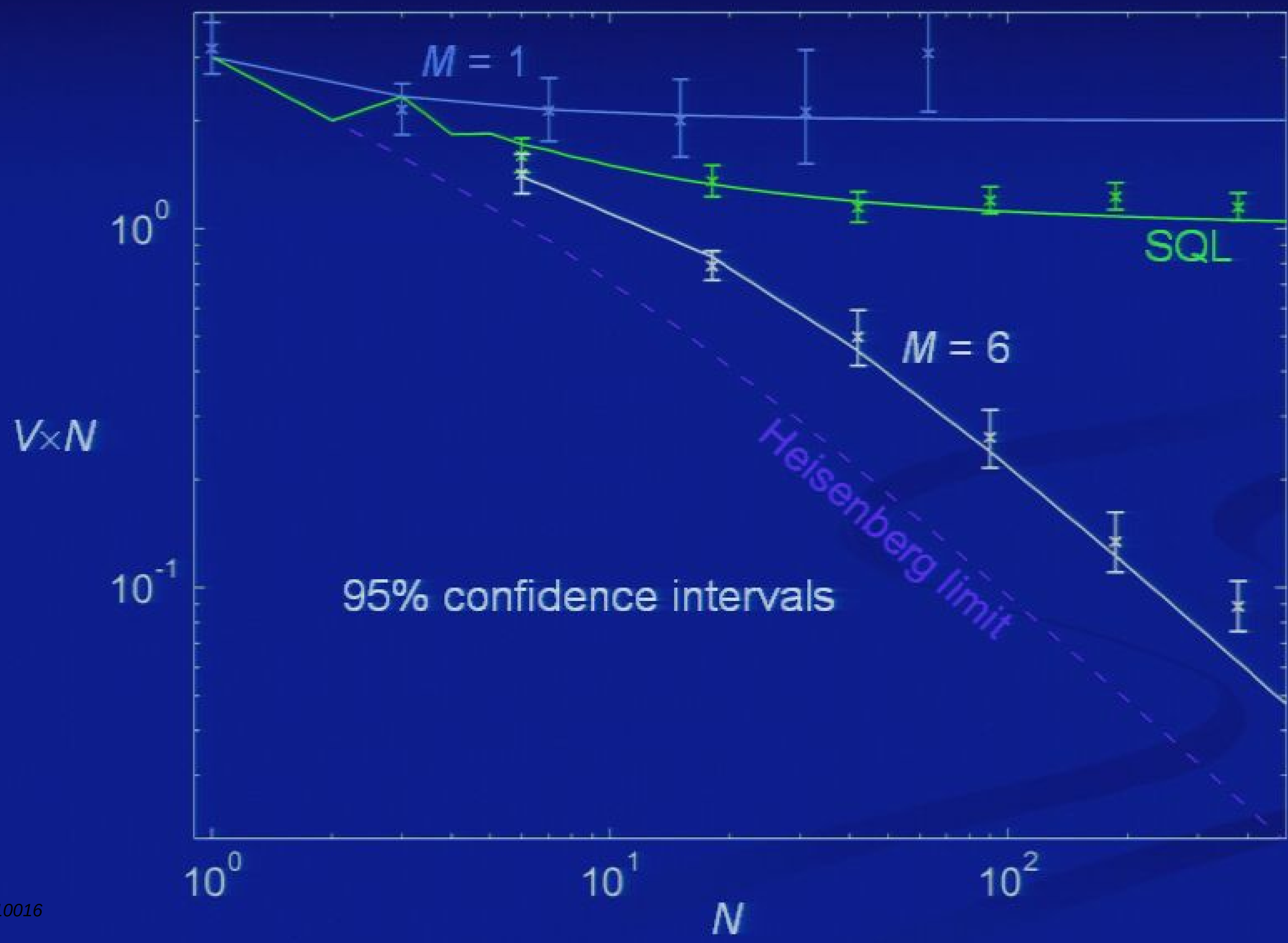


# Experimental results



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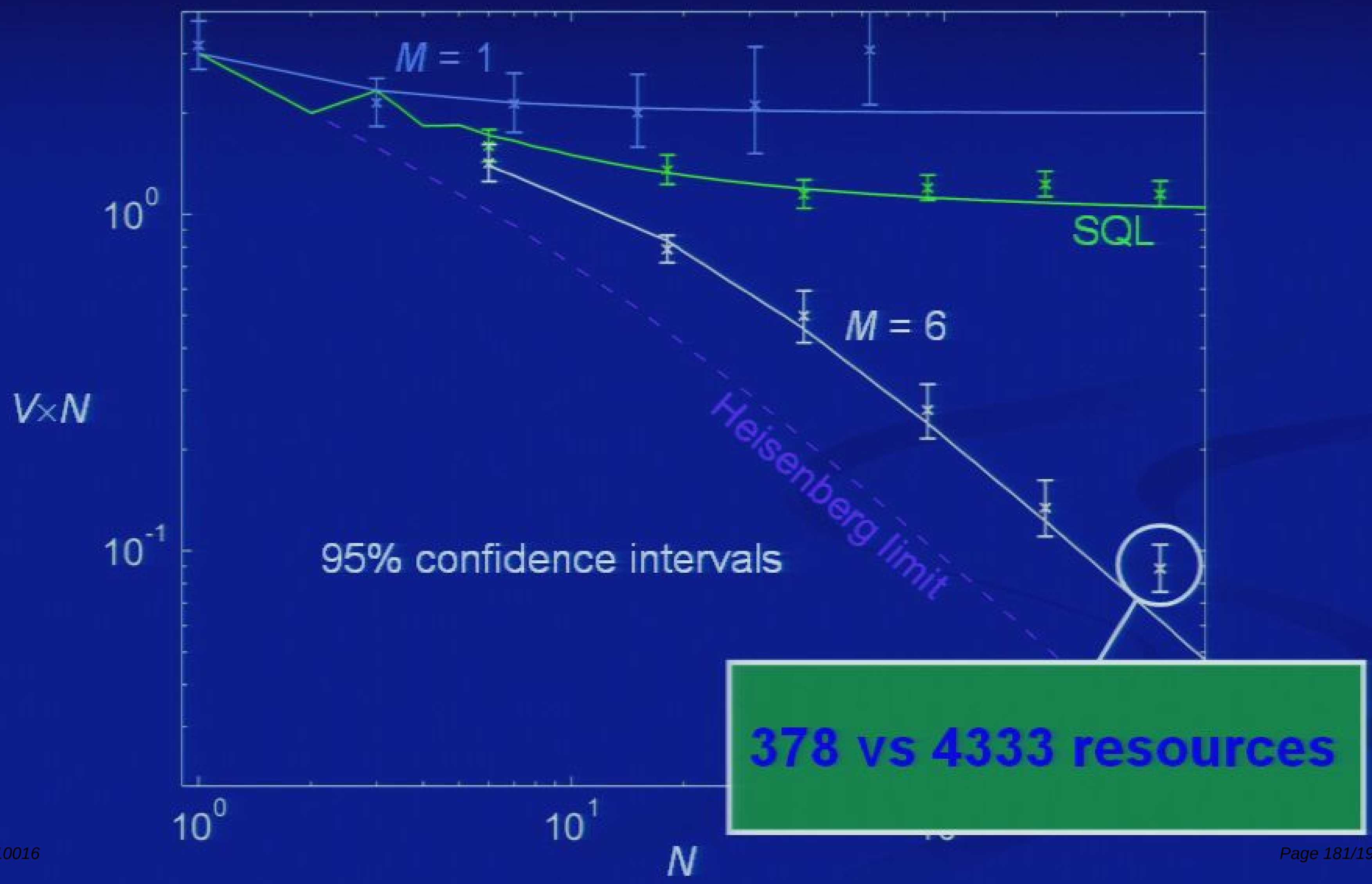


# Summary of phase section

- For single mode adaptive measurements the theoretical limit is the same for optimal squeezed states.
- This limit can be achieved for appropriately chosen feedback.
- For measurements of a variable phase the uncertainty is the square root of that for a static phase.
- For narrowband squeezed states the scaling is as $(\kappa/N)^{5/8}$.
- In interferometry adaptive measurements approximate an ideal entangled measurement.
- By using multiple passes of single photons, we obtain the equivalent of an entangled input state.
- This allows the Heisenberg limit to be achieved.

# Summary of Hamiltonian section

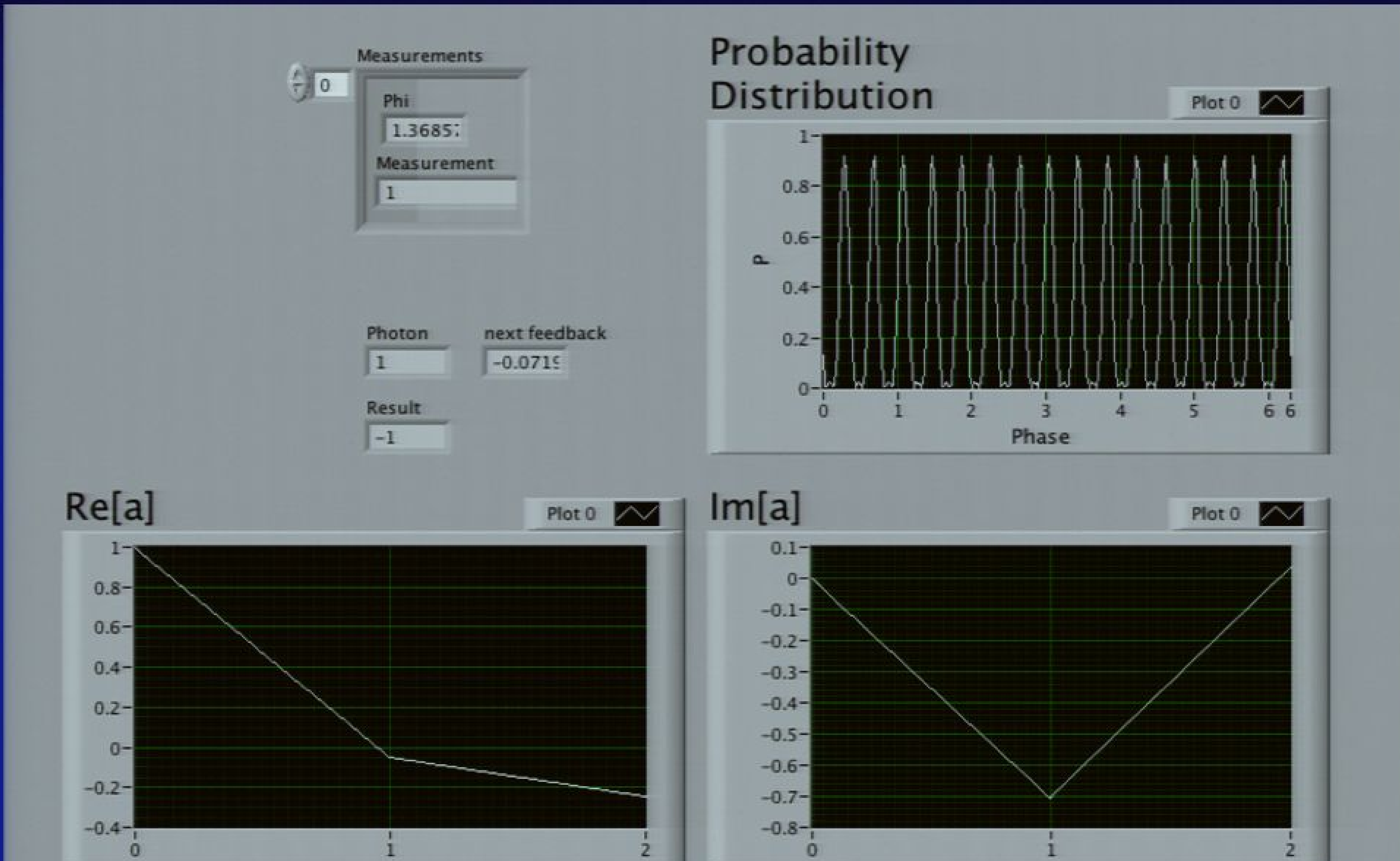
The method for implementing Hamiltonians is improved in the following ways:

1. We have simplified the ancilla.
2. The entanglement required is reduced.
3. We may implement multipartite unitaries.
4. The communication required may be reduced to the entanglement consumed except for the communication from the final party.

The method for simulating Hamiltonians is improved in the following ways:

1. We use higher order integrators to improve the scaling to be very close to linear in $t$.
2. The method of decomposing the Hamiltonian into a sum is improved to be very close to linear in the number of qubits.

# Updating the probability distribution



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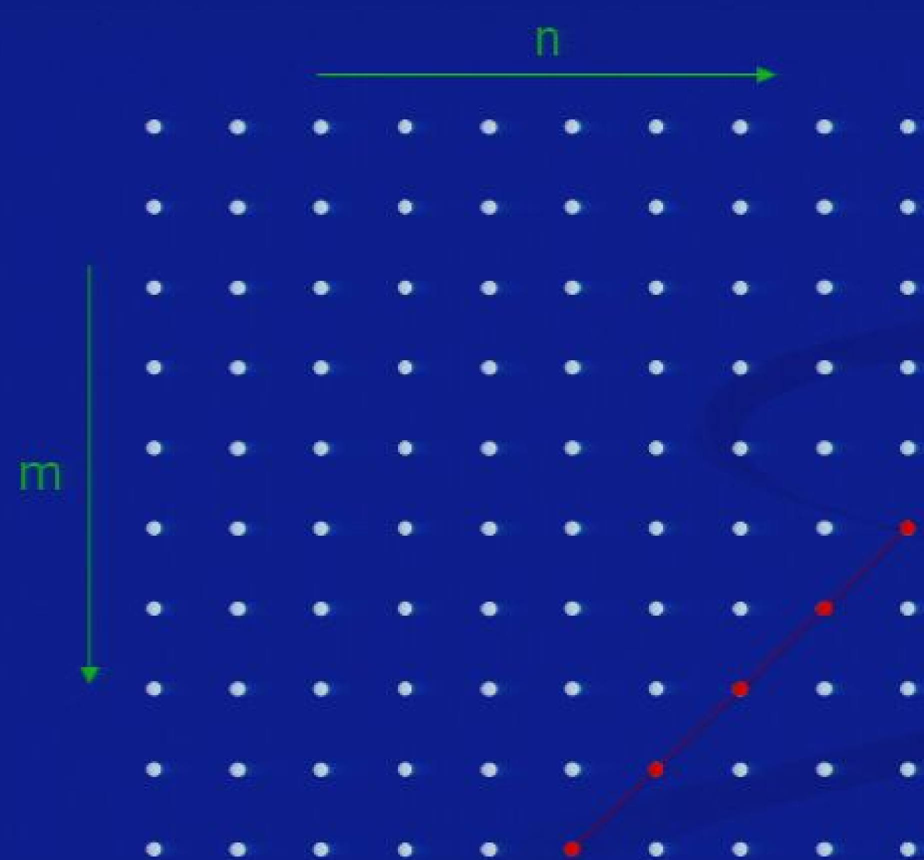

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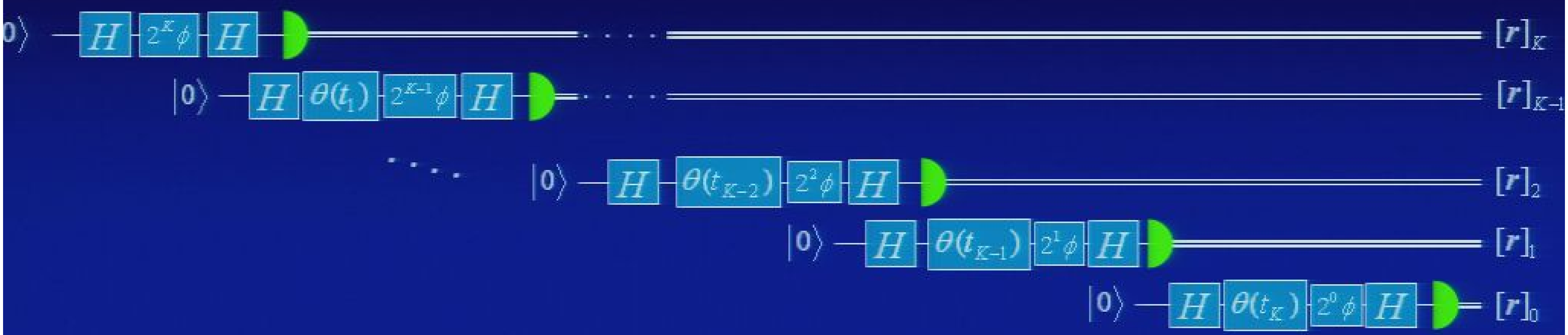


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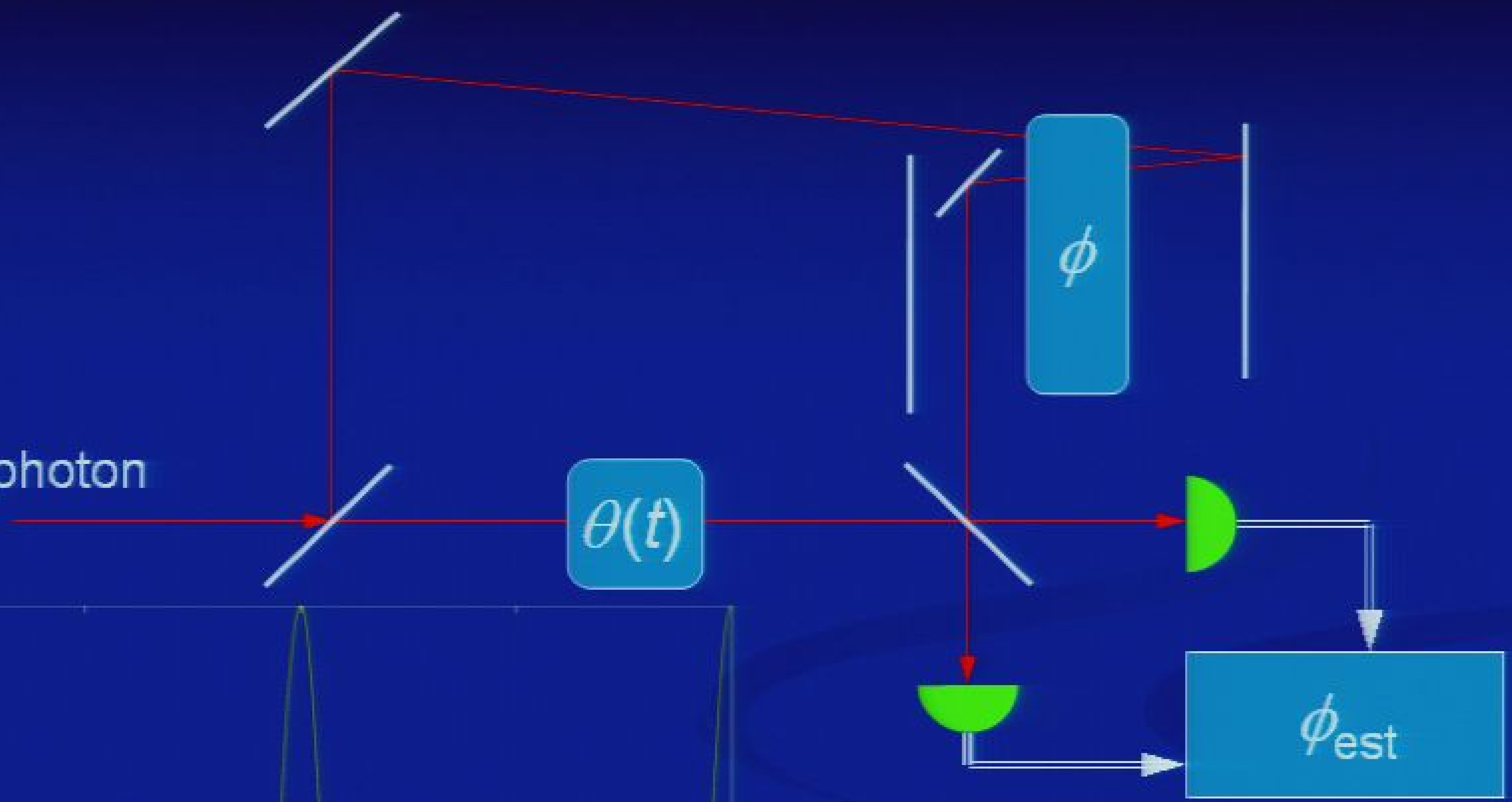
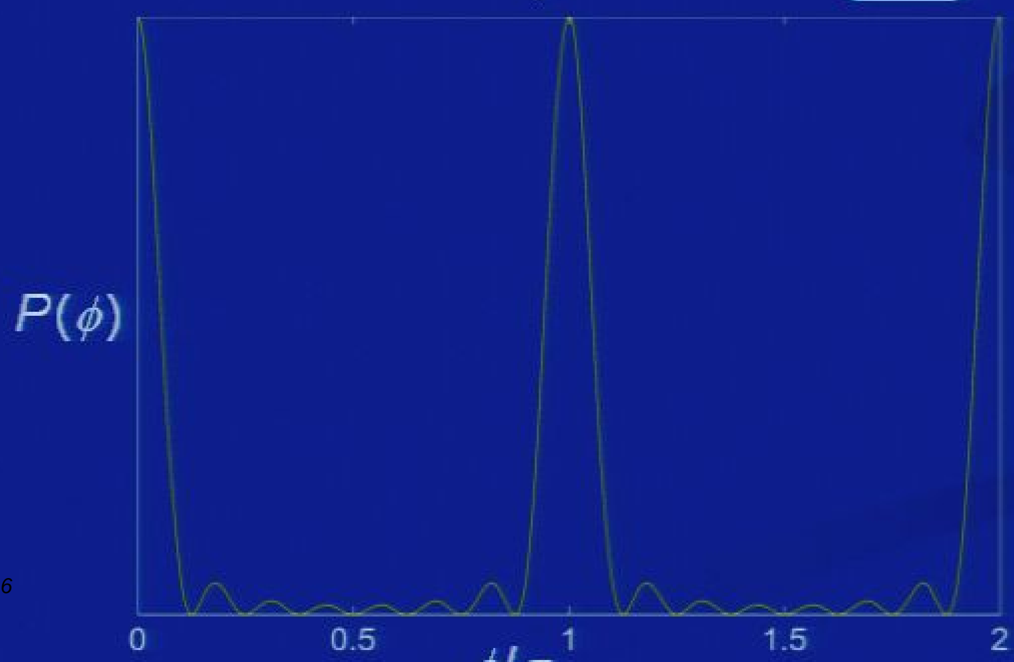
# Eliminating the fringes

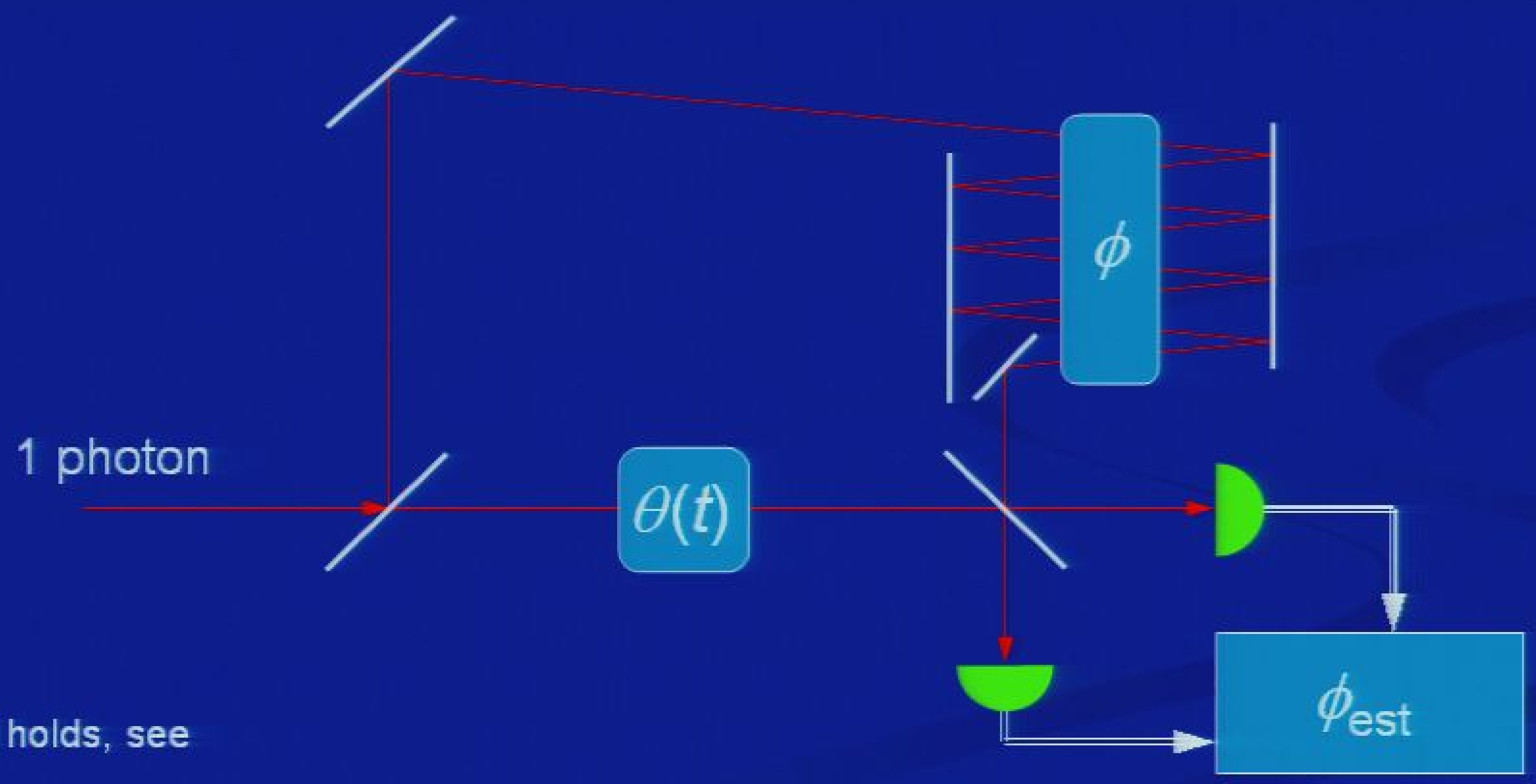



B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature **450**, 393 (2007).

# Multiple passes

The NOON state may be replaced with $N$ passes through a phase shift.




Limit still holds, see

W. van Dam, G. M. D'Ariano, A. Ekert, C. Macchiavello, M. Mosca, Phys. Rev. Lett. **98**, 090501 (2007).

# Further Reading

- Hamiltonian implementation:

  D. W. Berry, Phys. Rev. A **75**, 032349 (2007).

- Hamiltonian simulation:

  D. W. Berry, G. Ahokas, R. Cleve, and B. C. Sanders, Comm. Math. Phys. **270,** 359 (2007).

- Optimal single-mode phase measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. A **63**, 013813 (2001).

- Continuous phase measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. A **73**, 063824 (2006).

- Adaptive interferometric measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. Lett. **85**, 5098 (2000).

- Heisenberg limited interferometry:

  B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature **450**, 393 (2007).