Title: Distinguishability of Quantum Operations

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Abstract: In this talk, we will investigate the distinguishability of quantum operations from both discrete and continuous point of view. In the discrete case, the main topic is how we can identify quantum measurement apparatuses by considering the patterns of measurement outcomes. In the continuous case, we will focus on the efficiency of parameter estimation of quantum operations. We will discuss several methods that can achieve Heisenberg Limit and prove in some other cases the impossibility of breaking the Standard Quantum Limit. The general treatment of estimation of quantum operations also allows an investigation of the effect of noise on estimation efficiency.

Distinguishability of Quantum Operations

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January 16th, 2008 @ PI

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Quantum Distinguishability

Quantum Postulates

State	Composition
Evolution	Measurement

Quantum Distinguishability

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- State Distinguishability

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  ρ_0 vs. ρ_1

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  $\mathcal{E}_0$ vs. $\mathcal{E}_1$

- With Many Copies

Quantum Distinguishability

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- With Many Copies
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  ρ_θ or $\mathcal{E}_\theta$

- Differences

State Distinguishability

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Acin and D' Ariano et al. 2001

Distinguishing Unitary Operations

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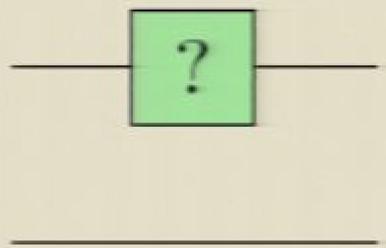

U_0/U_1 Discrimination

Lemma 1. *U_0, U_1 are perfectly distinguishable using the circuit on the left iff there exists a ρ such that*

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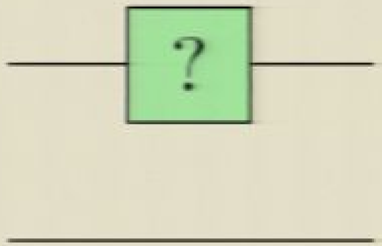

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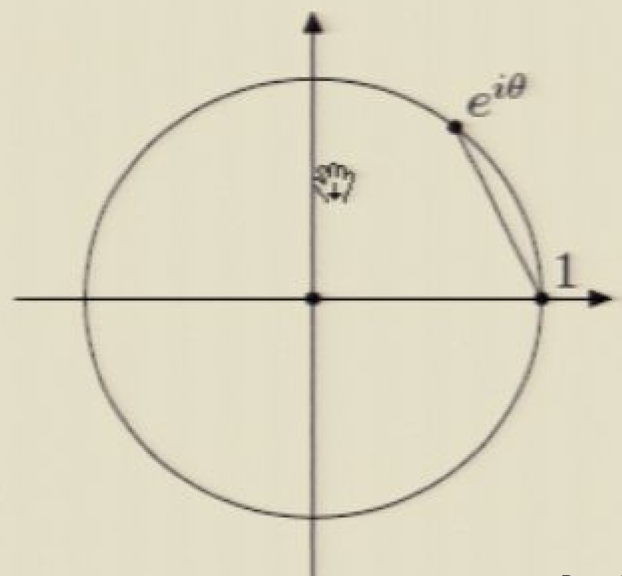

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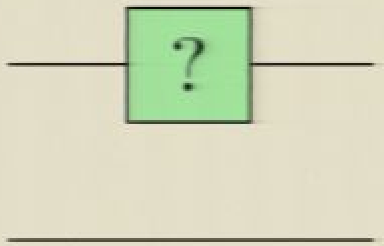

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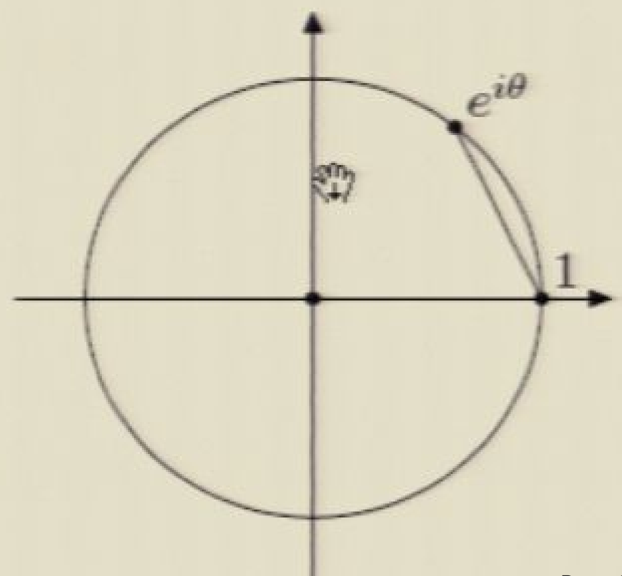
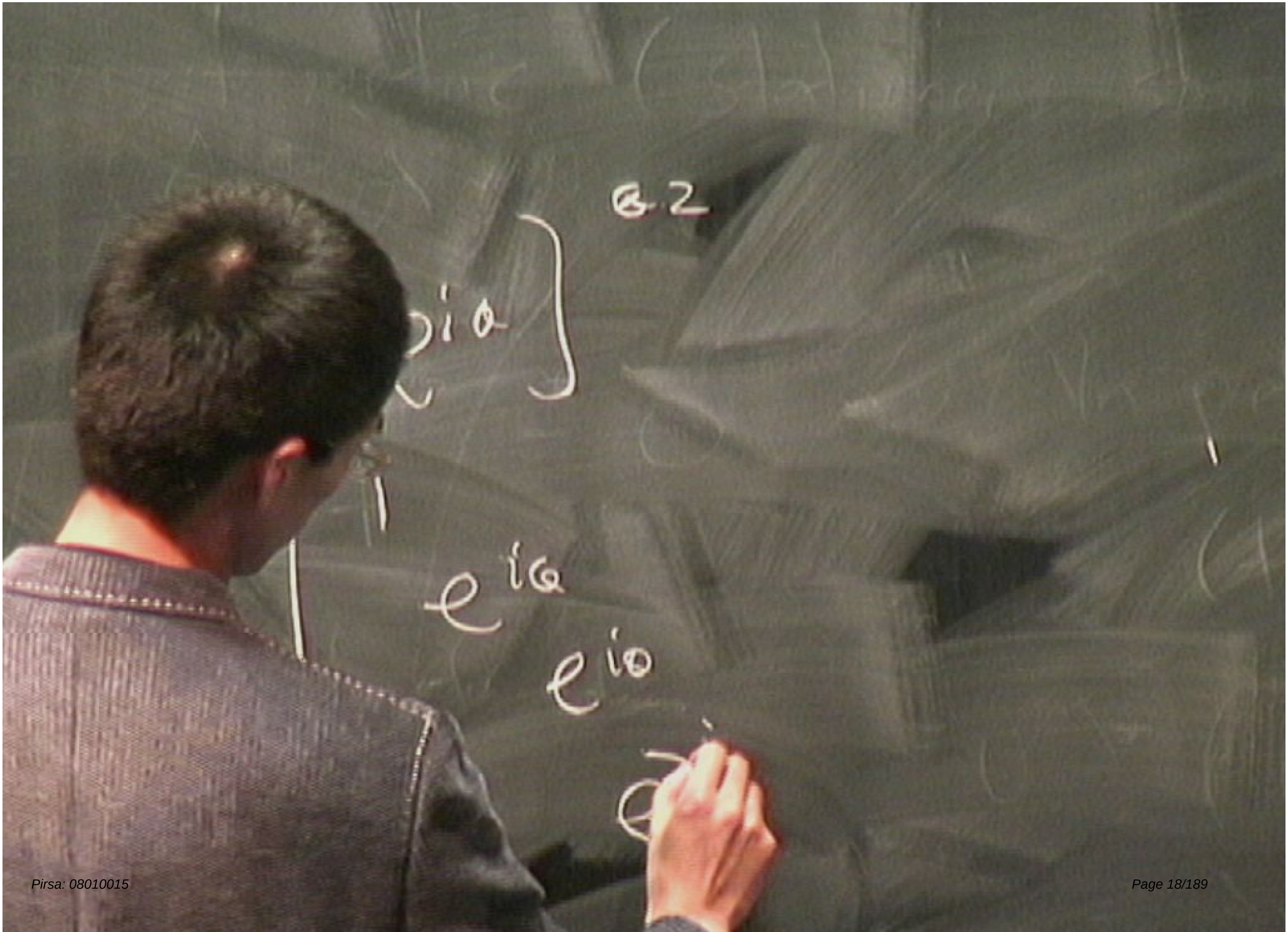
Goal: 0 is in the convex hull of $e^{i\theta_j}$.





$$\begin{bmatrix} 1 & \\ & e^{i\theta} \end{bmatrix}^{\otimes 2} = \begin{bmatrix} 1 & & & \\ & e^{i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{2i\theta} \end{bmatrix}$$

Distinguishing Measurement Apparatuses

Measurement M is either $M_0 : \{P_m\}$ or $M_1 : \{Q_m\}$.
Recover the lost label "0" or "1".

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Recipe:

1. Prepare n-qubit W state;
2. Measure the unknown apparatus on every qubits;
3. The label is "0" if there is exactly one "-1" in the outcome and "1" otherwise.

Completely Projective Measurements

$M_0 : \{|\phi_m\rangle\langle\phi_m|\}$ vs. $M_1 : \{|\psi_m\rangle\langle\psi_m|\}$.

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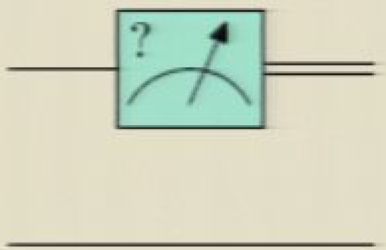

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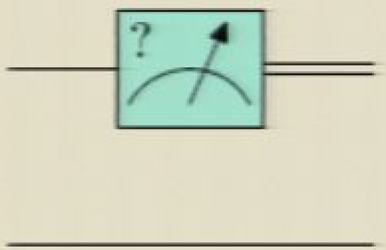

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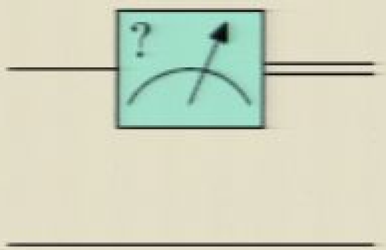

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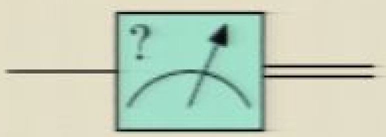

Theorem 2. *M_0 and M_1 are perfectly distinguishable without ancilla iff there exists a pure state $|\xi\rangle$ such that $|\xi\rangle\langle\xi|V$ has zero diagonal.*

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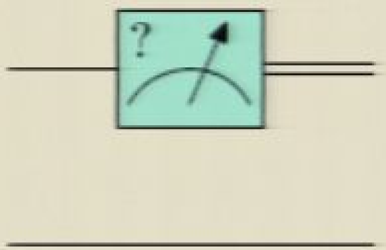

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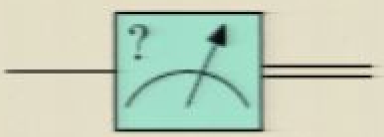

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Theorem 3. *M_0 and M_1 are perfectly distinguishable without ancilla iff matrix V has a zero principal minor.*

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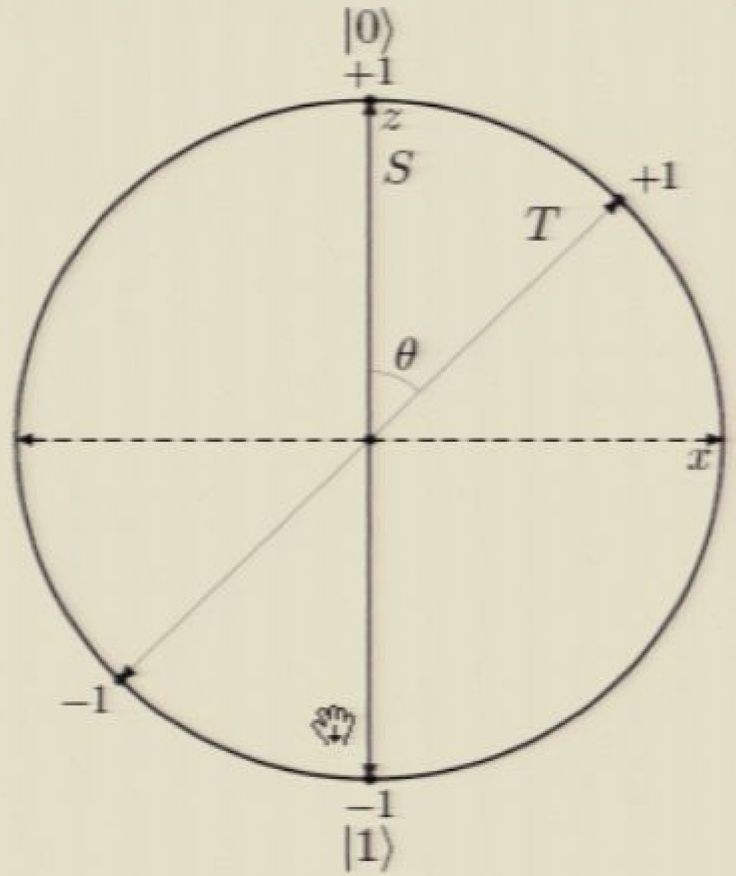

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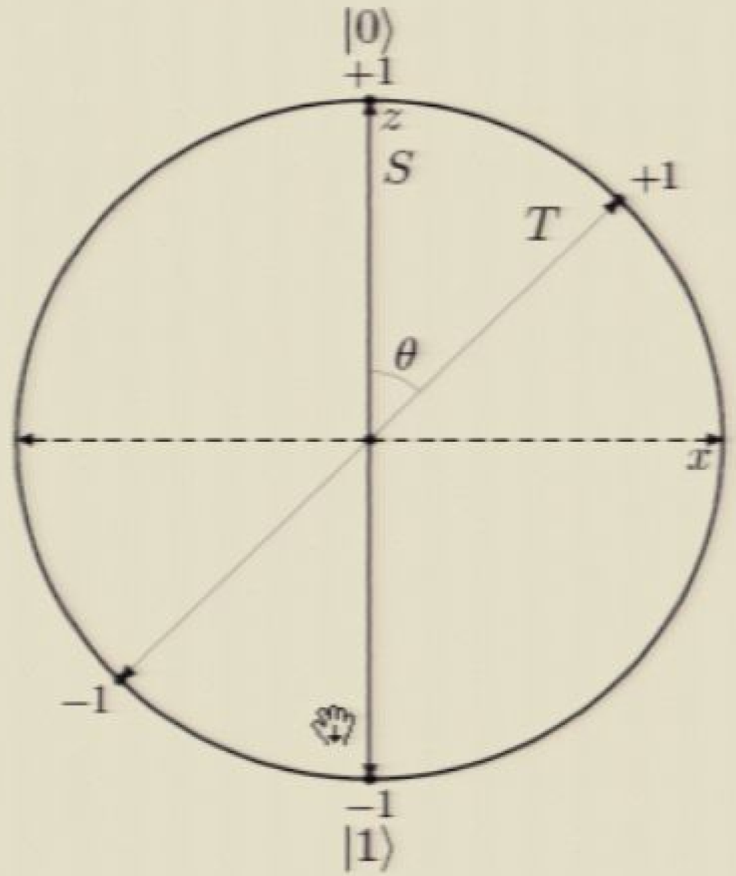

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Reduced to the case of unitary discrimination.

Lower Bound: $n \geq \lceil \pi/\theta \rceil$.

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Construction of $n = \lceil \pi/\theta \rceil$

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Construction of $n = \lceil \pi/\theta \rceil$

1. Define $E_n = \{j \mid w(j) \text{ is even}, 0 \leq j < 2^n\}$, and state $|E_n\rangle = \sum_{j \in E_n} (-1)^{w(j)/2} |j\rangle / \sqrt{2^{n-1}}$.

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2. The j-th diagonal element of $|E_n\rangle\langle E_n| V^{\otimes n}$ is 0 if $j \notin E_n$ and $\frac{1}{2^{n-1}} \cos \frac{n\theta}{2}$ otherwise.

For $j \in E_n$, the j-th element of $|E_n\rangle\langle E_n|V^{\otimes n}$ is

$$\frac{1}{2^{n-1}} \sum_{k \in E_n} (-1)^{w(j \cdot k)} a^{n-d(j,k)} b^{d(j,k)} (-1)^{(w(j)+w(k))/2}$$

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= \; & \frac{1}{2^{n-1}} \sum_{l \text{ is even}} (-1)^{l/2} \binom{n}{l} \cos^{n-l}\frac{\theta}{2} \sin^{l}\frac{\theta}{2} = \frac{1}{2^{n-1}} \cos\frac{n\theta}{2}
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Construction of $n = \lceil \pi/\theta \rceil$

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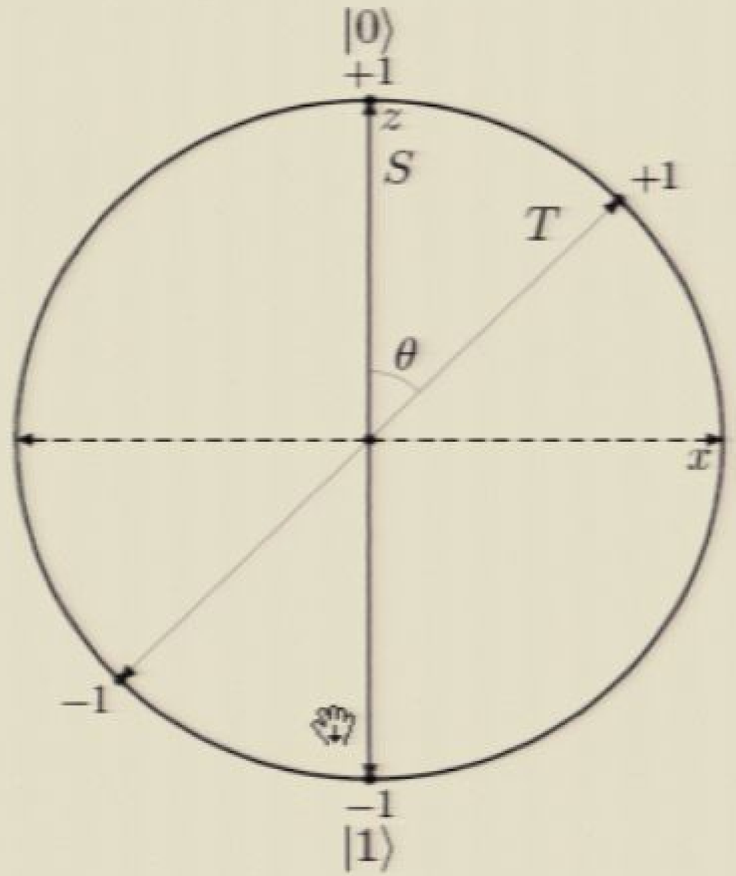
5. Choose ρ to be a convex combination of $|E_n\rangle\langle E_n|$ and all $|E_n^k\rangle\langle E_n^k|$'s.

Single Qubit Case: Optimal Solution

$M_0 : \sigma_z$ vs. $M_1 : \cos\theta\sigma_z + \sin\theta\sigma_x$.

$$V = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad a = \cos\frac{\theta}{2}, b = \sin\frac{\theta}{2}.$$

$$n = \lceil \pi/\theta \rceil$$



Distinguishing Projective Measurements

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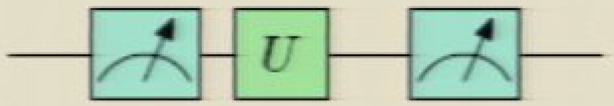

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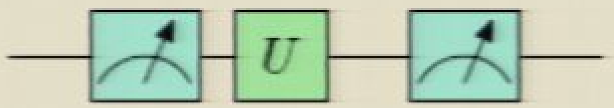

1. Prepare $|\psi\rangle \in P_k$ and measure;
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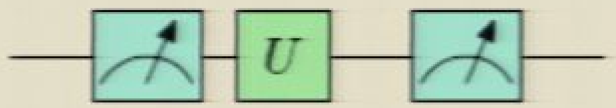

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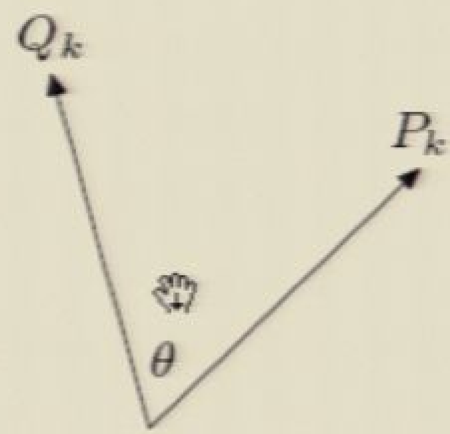

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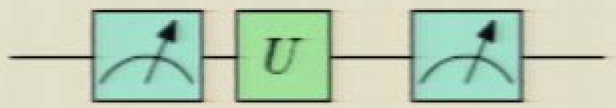

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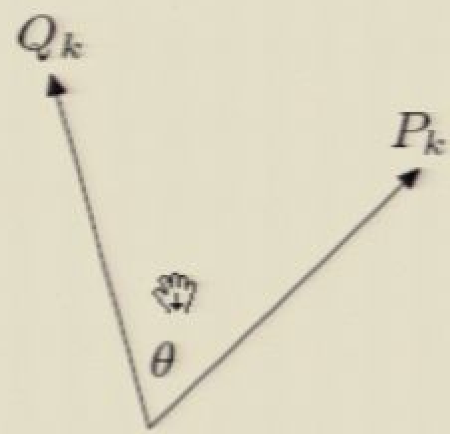

Lemma 2. *Let r be the same rank of P_k and Q_k. Such a U exists if $\|P_kQ_k\| \leq 1/\sqrt{2}$ and $d \geq 3r$.*

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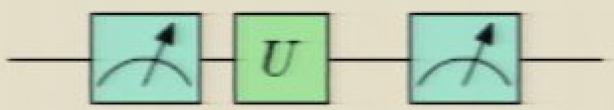
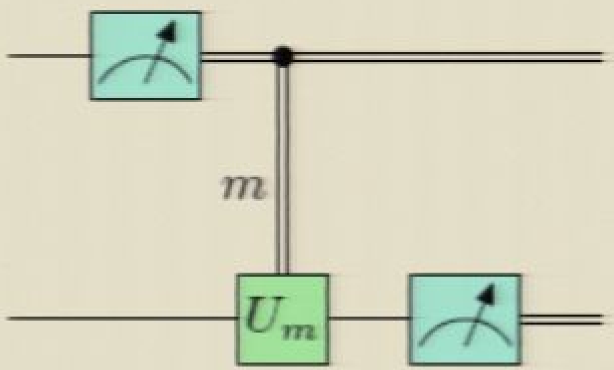


The use of post-measurement state can be avoided.



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Operation Distinguishability

- All unitary evolutions are perfectly distinguishable.
- All projective measurements are perfectly distinguishable.
- Some (not all) non-unitary evolutions are perfectly distinguishable.

$$\mathcal{M} : \rho \mapsto \mathrm{tr}(\rho P)|0\rangle\langle 0| + (1 - \mathrm{tr}(\rho P))\,|1\rangle\langle 1|$$

Continuous Case

Parameter Estimation

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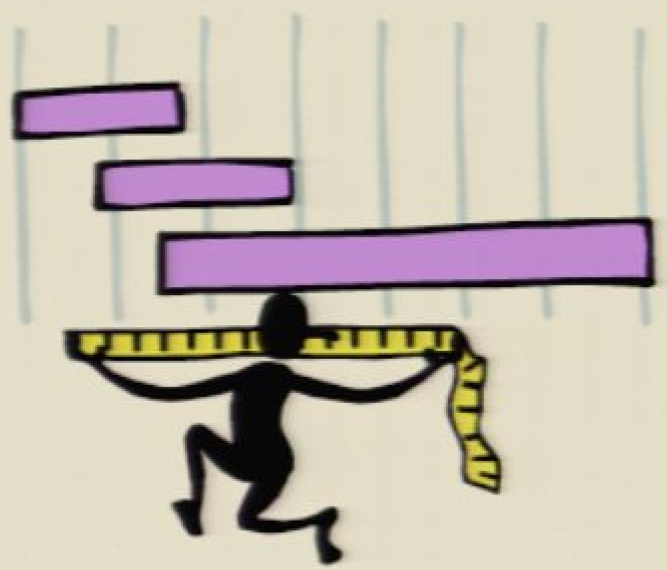
- Parameter: length, angle, time, . . . , or something abstract.



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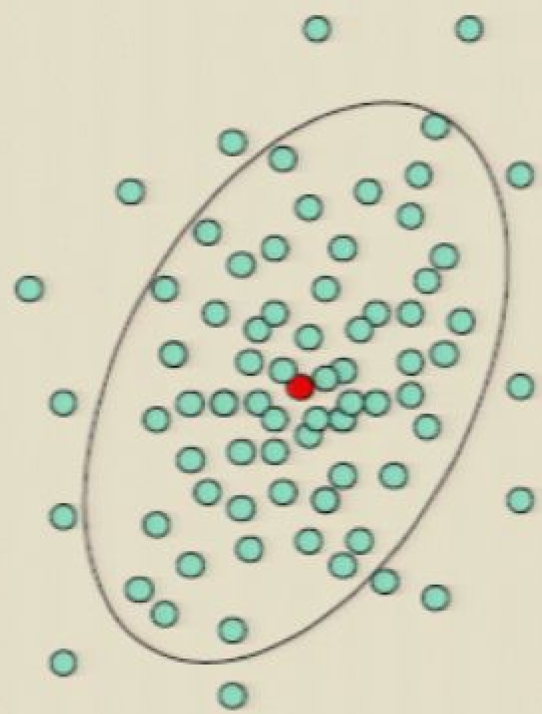

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  - Quadratic speedup

$$\frac{1}{\sqrt{N}} \longrightarrow \frac{1}{N}$$

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It is unbiased, consistent, and with MSE $p(1-p)/N$.

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Theorem 5 (The Cramér-Rao Bound)**.** *For unbiased estimator* $\hat{\theta}(x_1, x_2, \ldots, x_N)$*,*

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Quantum Detection and Estimation Theory

Helstrom, 1976

Probabilistic and Statistical Aspects of Quantum Theory

Holevo, 1982

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Let $\rho = \sum_i p_i |i\rangle\langle i|$; $\mathcal{L}_\rho$ is the superoperator

$$\mathcal{L}_\rho(O) = \sum_{\{j,k|p_j+p_k\neq 0\}} \frac{2}{p_j + p_k} O_{jk} |j\rangle\langle k|. \quad (3)$$

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Heisenberg Limit (HL)

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Problem: Estimate $\theta \in [0, 1)$ of $U_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{bmatrix}$.

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Two Different Ways

1. Covariant Method

   Holevo

2. Amplification Method

   Rudolph and Grover

Covariant Method

1. Prepare Input $\sum_{k=0}^{N} a_k|\hat{k}\rangle$

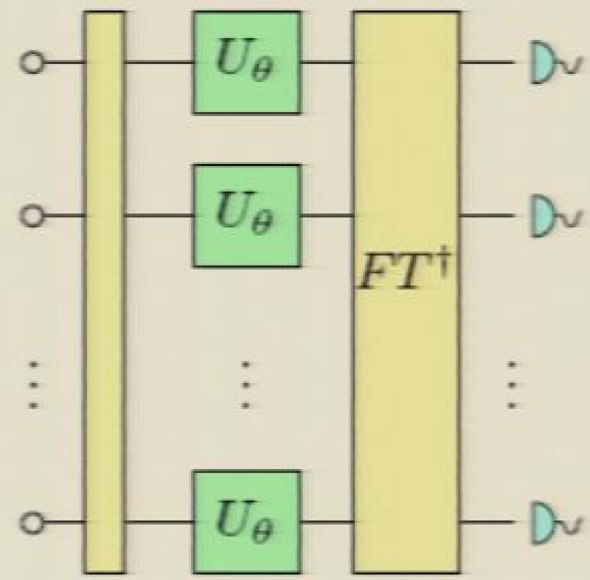

Illustration of
Holevo's Covariant Method

Covariant Method

1. Prepare Input $\sum_{k=0}^{N} a_k|\hat{k}\rangle$

$$|\hat{k}\rangle = \frac{1}{\sqrt{\binom{N}{l}}} \sum_{l:w(l)=k} |l\rangle$$

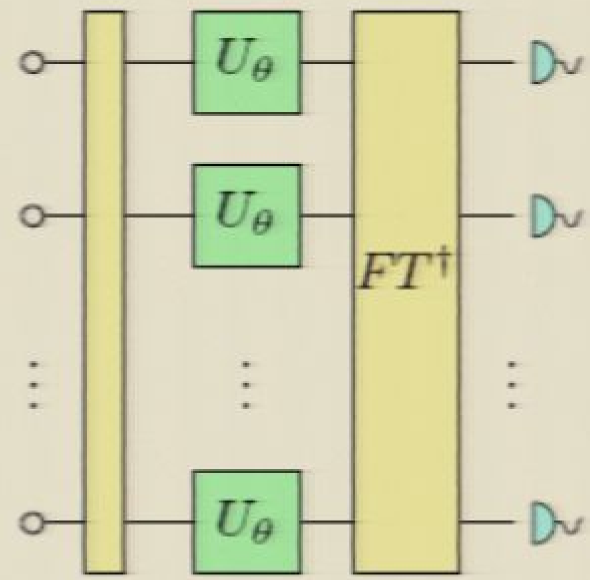

Illustration of
Holevo's Covariant Method

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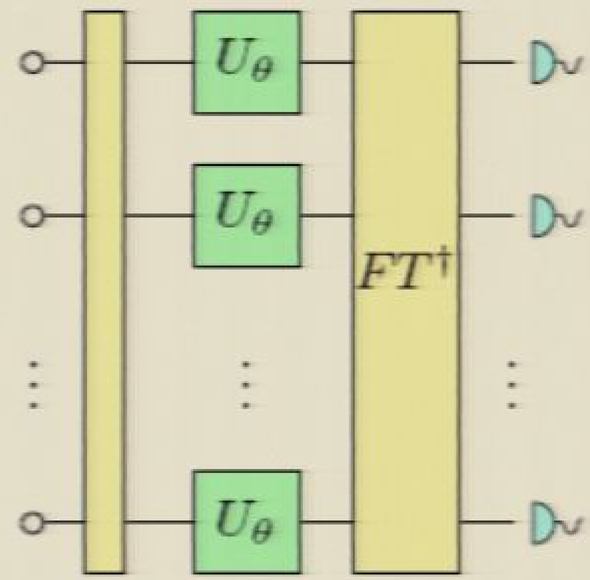

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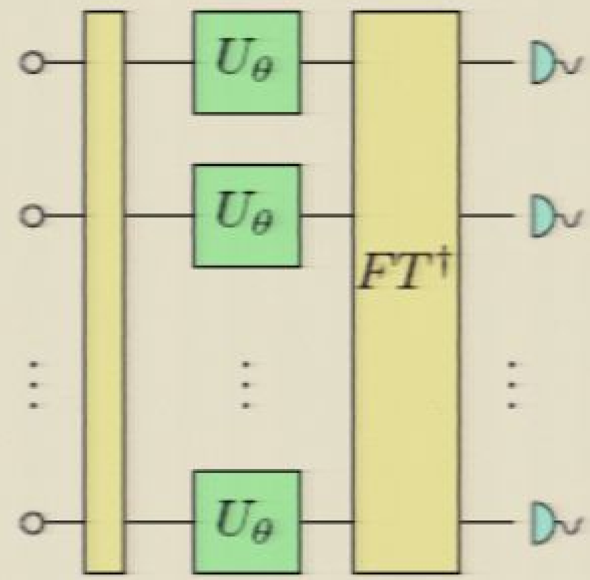

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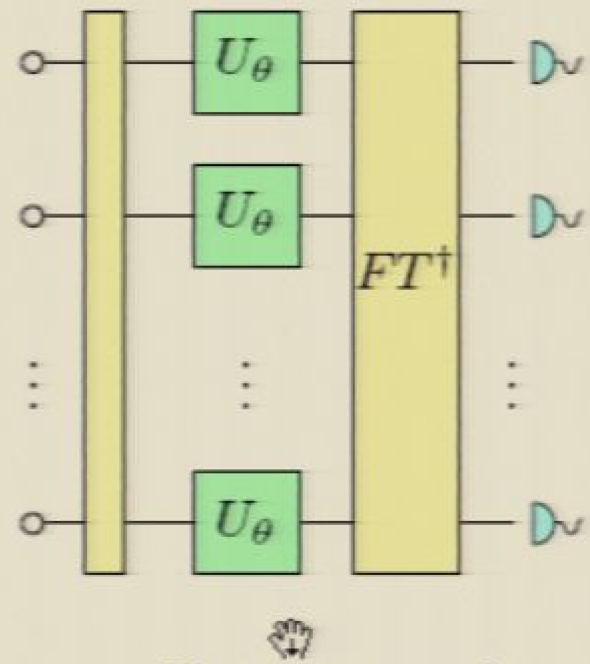

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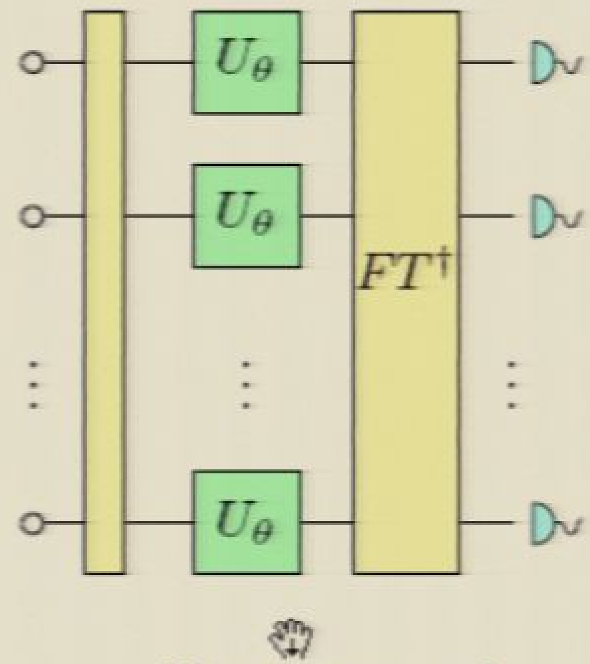

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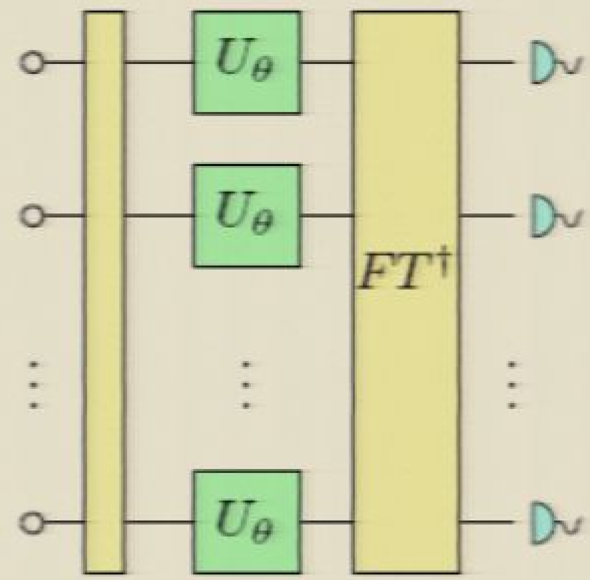

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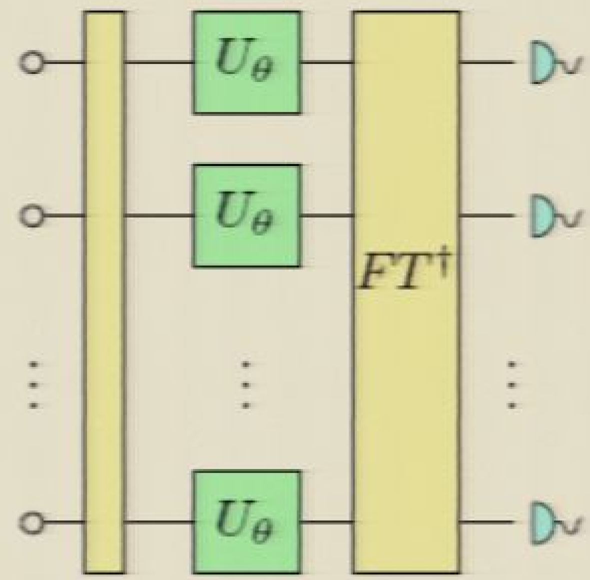

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Analysis of the Covariant Method

- Key Point: Choose

$$W = E\big(1 - \cos(2\pi(\hat{\theta} - \theta))\big)$$

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- Simplifies further calculations …

Analysis of the Covariant Method

Probability of observing l 1's

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Bužek, Derka and Massar

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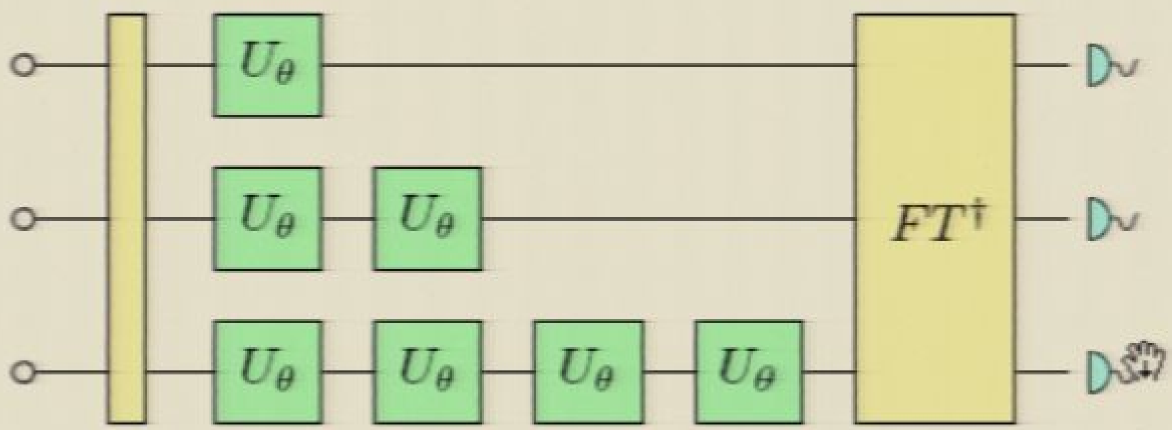

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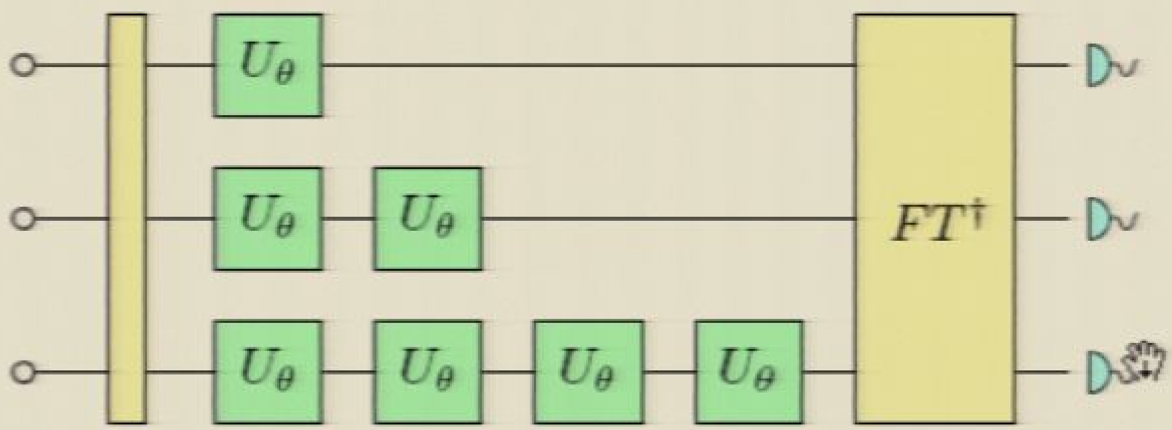

van Dam, D' Ariano, Ekert, Macchiavello and Mosca

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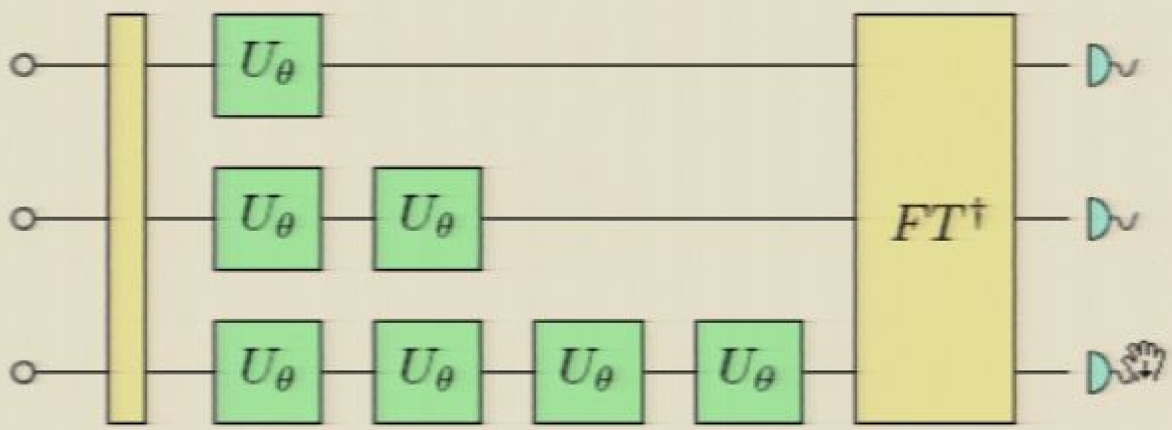

van Dam, D' Ariano, Ekert, Macchiavello and Mosca

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Key Idea: Amplify θ to $k\theta$ with cost linear in k.

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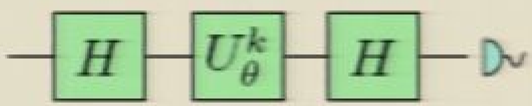

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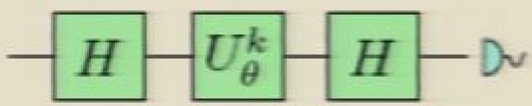

$\cos^2(k\pi\theta)$-coin

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$$n\theta \pm 0.1$$

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A "bit-wise" estimation protocol of RMSE $O(\log N/N)$.

Rudolph and Grover

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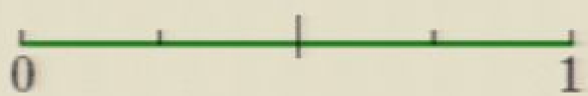

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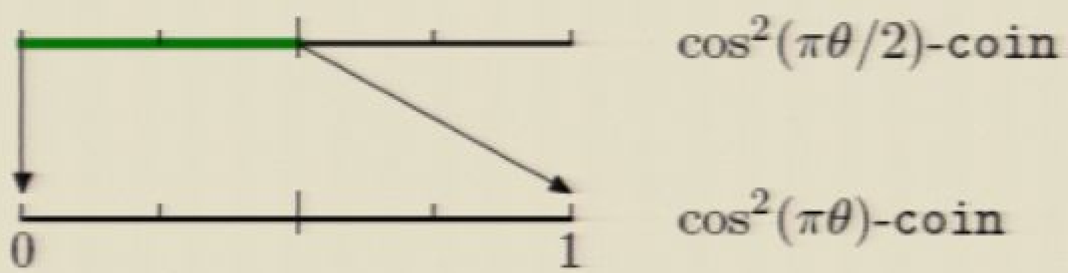

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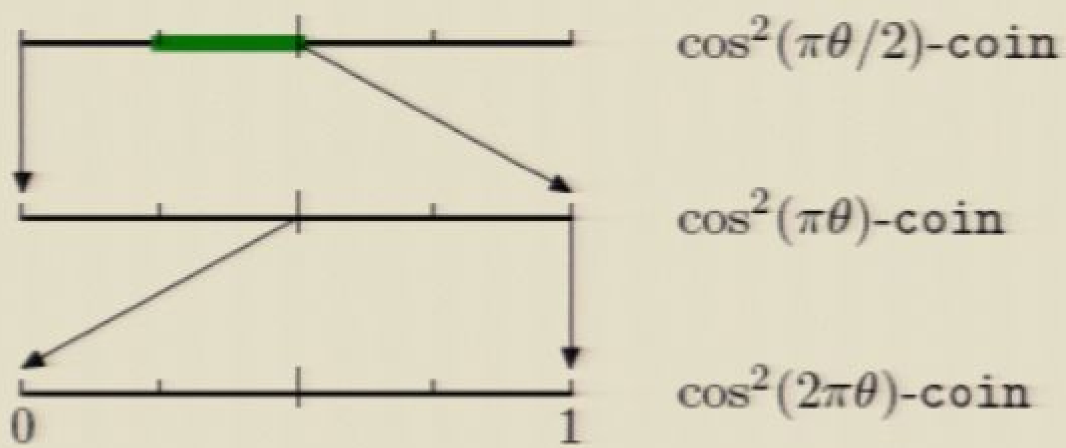

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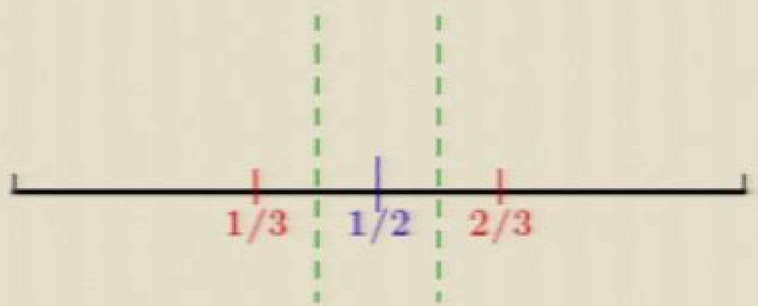

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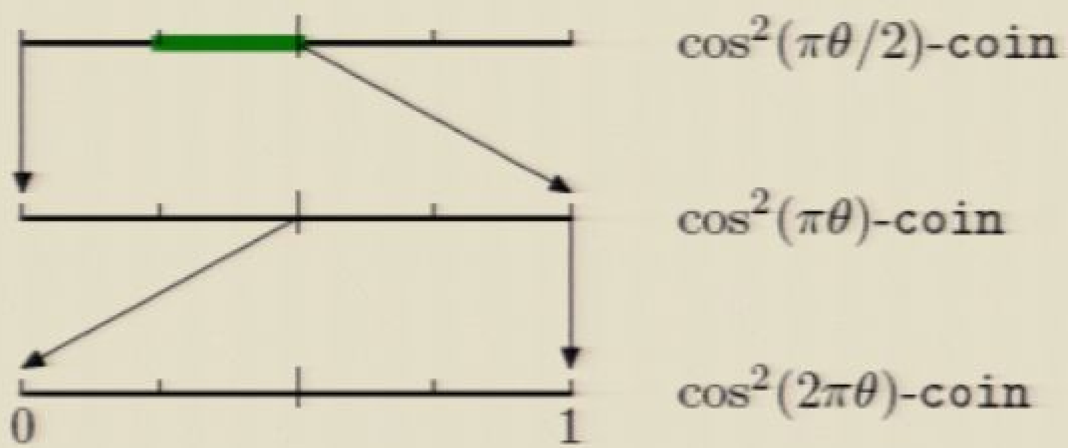

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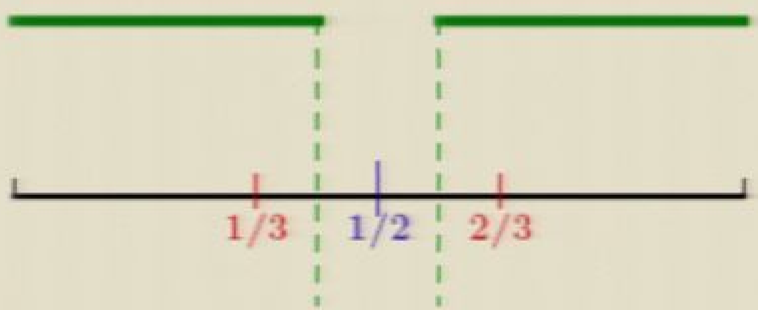

$|\hat{\theta} - \theta| < 1/12$ with high probability

Case 1: $\hat{\theta} < 5/12$, $\theta \in [1, 1/2]$ w.h.p.
Case 2: $\hat{\theta} > 7/12$, $\theta \in [1/2, 1]$ w.h.p.

Estimation of General Quantum Operations

How noisy is the depolarizing channel $\mathcal{D}_\theta(\rho) = \theta\frac{I}{2} + (1-\theta)\rho$?

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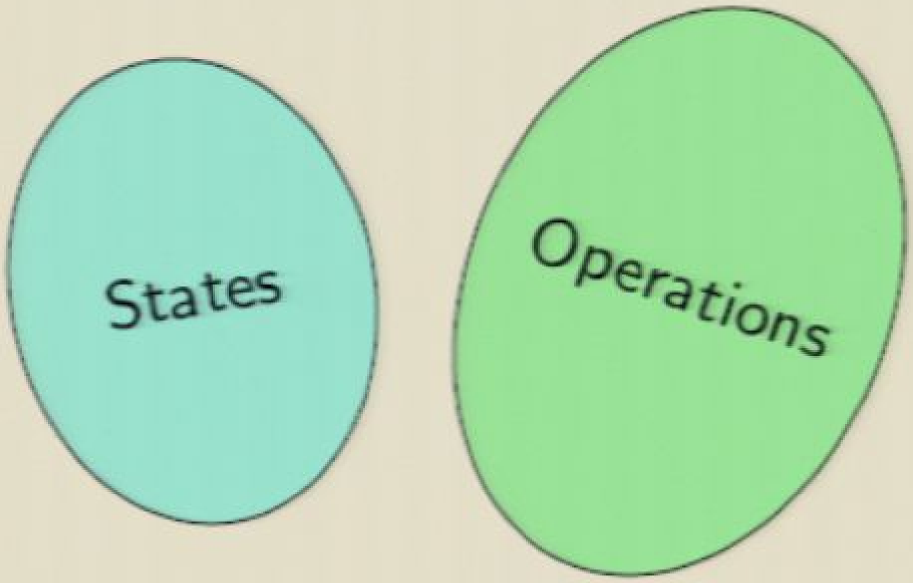

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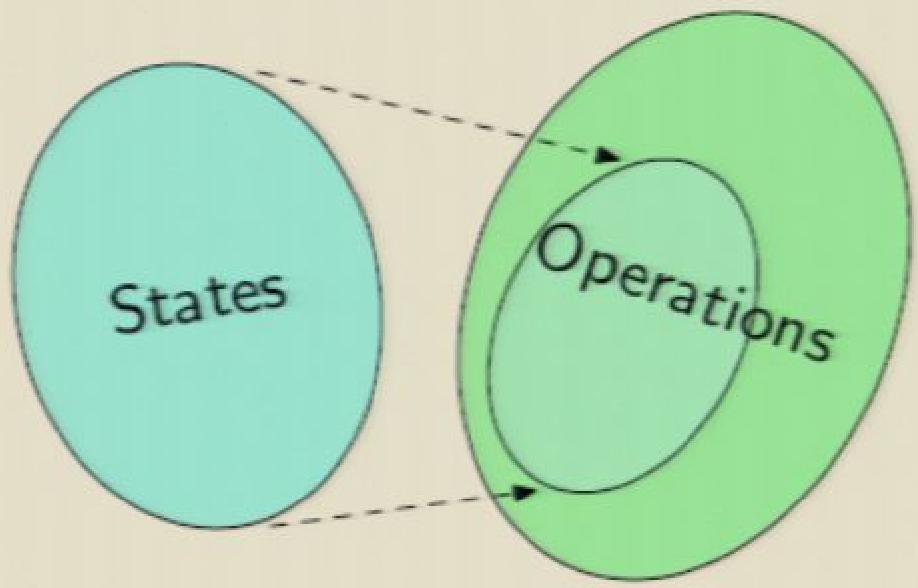
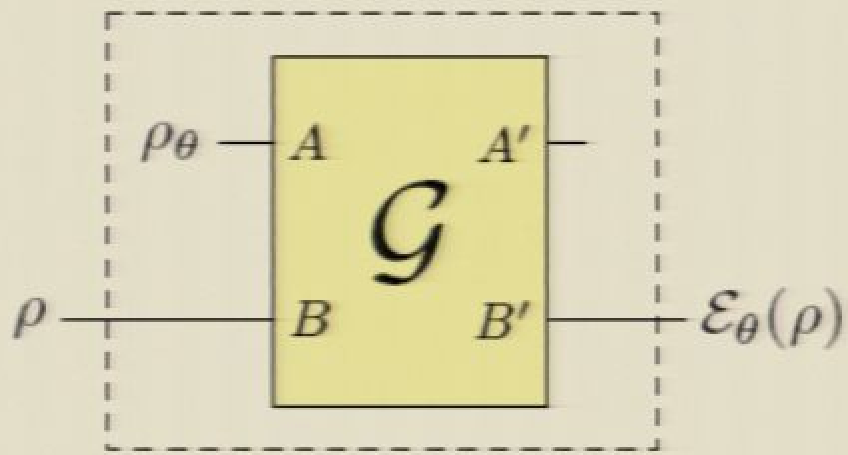


Programmability and Efficiency

Programmable Quantum Operations



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$$\mathcal{E}_\theta(\rho) = \mathrm{tr}_{A'}(\mathcal{G}(\rho_\theta^A \otimes \rho^B)).$$

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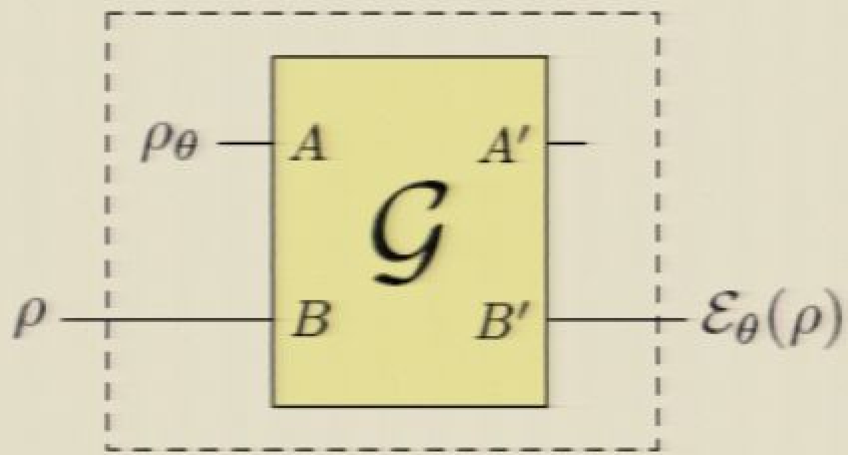

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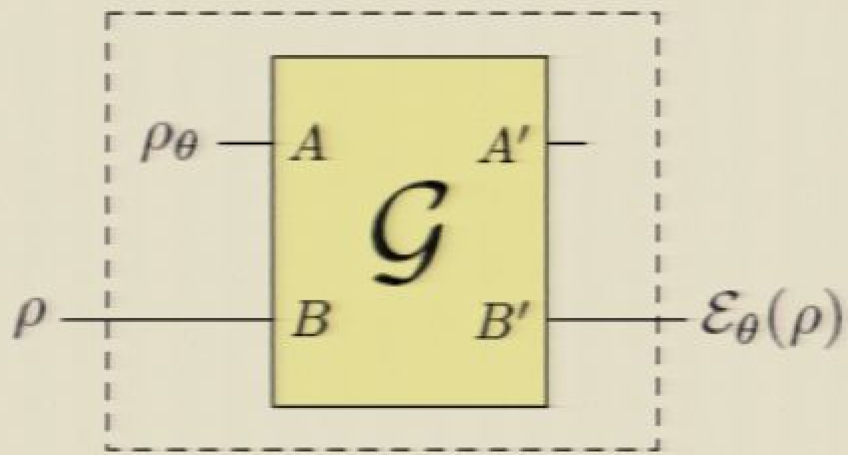

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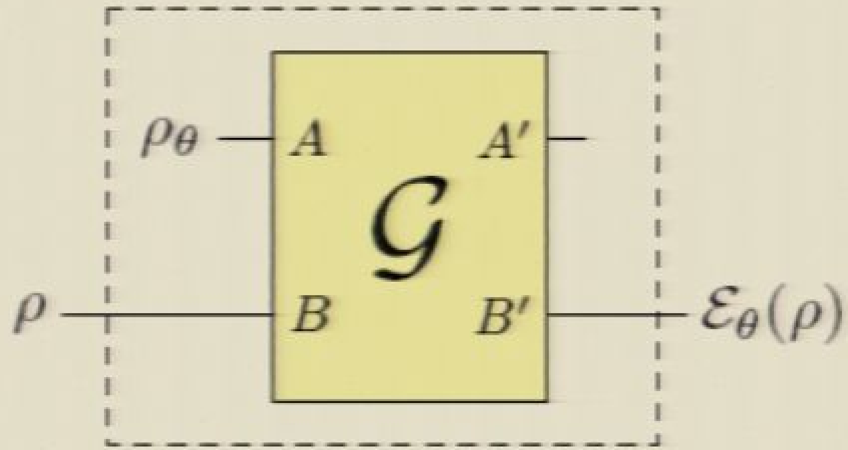

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Criterion: Programmable parameters obey **SQL**

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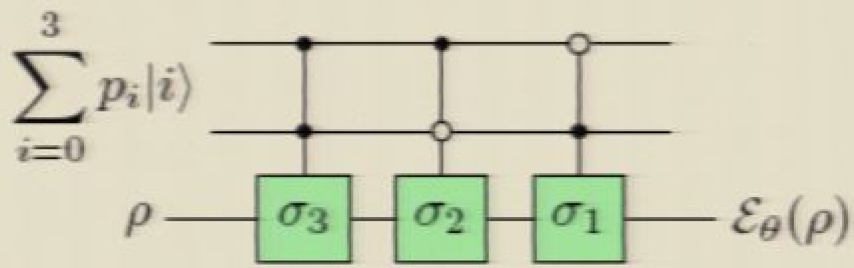

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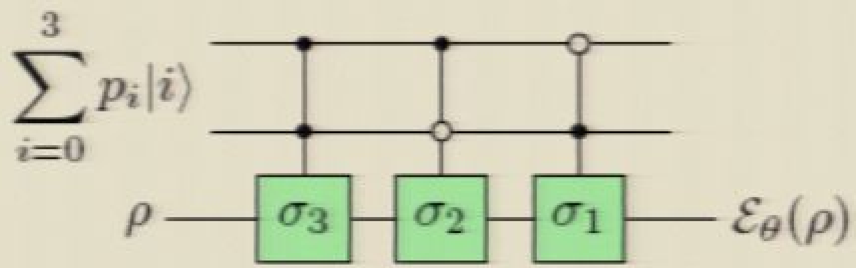

- Corollary: Estimation of θ is of order $\Omega(1/\sqrt{N})$.

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- As a quantum channel:

$$\rho \mapsto \langle 0|\rho|0\rangle \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix} + \langle 1|\rho|1\rangle \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}.$$

- Corollary: Estimation of DMC is of order $\Omega(1/\sqrt{N})$.
- Classical channels are quantum programmable.
  Quantum channels are not always quantum programmable.

Application 3: Effect of Depolarizing Noise

- Depolarizing noise:

$$\mathcal{E}_\theta(\rho) = \epsilon\rho_0 + (1-\epsilon)\mathcal{U}_\theta(\rho)$$

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- No threshold theorem for estimation!

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$\mathcal{M}_\theta = \mathcal{M} \circ \mathcal{U}_\theta$ where $\mathcal{U}_\theta$ corresponds to $|0\rangle\langle\theta| + |1\rangle\langle\theta^\perp|$.
Superefficient estimation of $\mathcal{U}_\theta$ can tolerant phase flip noise.

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- For quantum operations, parameter estimation can beat the SQL,
- But estimation of programmable parameters cannot.
- Depolarizing noise ruins superefficient estimation universally.
- Todo:
  - Estimation of amplitude damping channels.
  - Estimation of general qubit measurements. (Effect of flip noise on estimation)
  - Characterize more accurately the set of superefficient parameters.

Thank You!

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The Estimation Game

In the so-called estimation game, you are required to give the best estimate of a real parameter θ in $[0, 1)$. Each access to the $\cos^2(m\pi\theta/2)$-`coin` once will cost you m dollars.

What is the best you can do with N dollars?