Title: Constrained Quantum Dynamics

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Abstract:

Constrained Quantum Dynamics

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Imperial College London

Constrained systems

Scenario: We wish to implement unitary motion, but the system is subject to constraints

Question: How does one formulate the theory of quantum constraints?

Classical theory: One approach to constrained systems is Dirac's approach, formulated on a classical phase space (symplectic manifold)

Question: Is there an analogue of the classical phase space description for quantum mechanics?

Outline

- Phase-space description of quantum mechanics
  - Developed by [Kibble, Marsden, Weinberg, and many others, in the 1970s and 1980s]
- Two distinct approaches to quantum constraints
  - Extension of classical theory (Kibble et. al. + Dirac $\rightarrow$ framework for quantum constraints)
  - A new approach - using metric geometry of quantum phase space
- Mixed states

Phase-space formulation of QM (1)

The expectation value of any observable in quantum mechanics is independent of the norm of the state vector

$$|x\rangle \sim \lambda|x\rangle \qquad \lambda \in \mathbb{C} - \{0\}$$

A pure state in QM $=$ ray through the origin in $\mathcal{H}^{n+1}$

$$\Downarrow$$

space of rays $=$ projective Hilbert space

In real terms this is an even dimensional manifold Γ equipped with Fubini-Study metric g_{ab} and compatible symplectic structure Ω_{ab}

Phase-space formulation of QM (2)

Physical observable $H(x)$ is given by the expectation of $\widehat{H}$ at each point x on Γ:

$$H(x) = \frac{\langle x|\widehat{H}|x\rangle}{\langle x|x\rangle}$$

The eigenstates of $\widehat{H}$ correspond to fixed points of the flow $\Omega^{ab}\nabla_b H$ on Γ. These are the points for which

$$\nabla_a H(x) = 0$$

The eigenvalues are the values of $H(x)$ at the corresponding fixed points

Phase-space formulation of QM (3)

In the phase space formulation of QM the Schrödinger equation takes the form:

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \Omega^{ab}\nabla_b H(x) \qquad \text{where} \qquad H(x) = \frac{\langle x|\widehat{H}|x\rangle}{\langle x|x\rangle}$$

Remarkably we see that this is Hamilton's equation of classical mechanics, for the given phase space.

Thus the Schrödinger equation corresponds to a classical Hamiltonian evolution on the symplectic manifold defined by the quantum phase space.

Phase-space formulation of QM (4)

We can parameterize the state $x^a = \{q_i, p_i\}$ by writing

$$|x\rangle = \sum_{i=1}^{n} \sqrt{p_i} \mathrm{e}^{-\mathrm{i}q_i} |E_i\rangle + \sqrt{1 - \sum_{i=1}^{n} p_i} |E_{n+1}\rangle$$

The Hamiltonian then takes the form:

$$H(q_i, p_i) = E_{n+1} + \sum_{i=1}^{n} \omega_i p_i \quad \text{where} \quad \omega_i = E_i - E_{n+1}$$

Equations of motion become

$$\dot{q}_i = \frac{\partial H(q_i, p_i)}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H(q_i, p_i)}{\partial q_i}$$

with solutions $q_i(t) = q_i(0) + \omega_i t$ and $p_i(t) = p_i(0)$

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Constraining quantum systems

- Recent work on extending classical theory
  - Burić, [arXiv:0704.1359]
  - Corichi, [arXiv:0801.1119]
  - Brody, Gustavsson, Hughston [J. Phys. A 41 (2008)]

  However, the extended version of Dirac's approach is only applicable to a certain class of systems

- Alternative approach
  - [Brody, Gustavsson, Hughston 2008]

Terminology

- **Dirac approach**
  $\rightarrow$ based on the symplectic geometry of the classical phase space

  $\Rightarrow$ "symplectic approach"

- **New approach**
  $\rightarrow$ relies on the metric geometry of the quantum state space

  $\Rightarrow$ "metric approach"

Outline these two approaches

Illustrate some results through examples

Constraints

Family of constraints

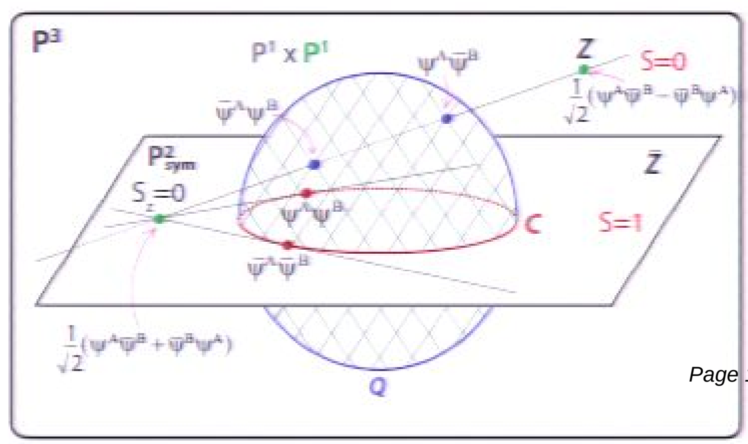
$$\Phi^i(x) = 0 \qquad i = 1, 2, \ldots, N$$

Examples:

- conserved observables

$$\Phi^i(x) = \frac{\langle x|\widehat{\Phi}^i|x\rangle}{\langle x|x\rangle}$$

- algebraic submanifold of Γ
  e.g. subspace of disentangled states



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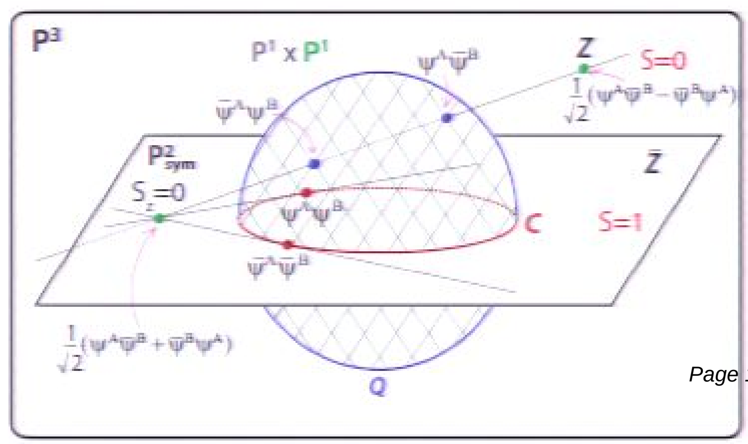

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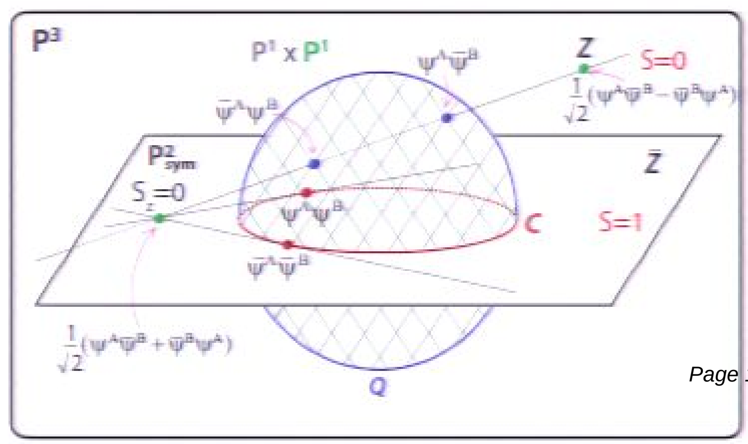

Symplectic approach

Modified equation of motion

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \Omega^{ab}\nabla_b H + \lambda_i \Omega^{ab}\nabla_b \Phi^i$$

In some cases the Lagrange multipliers λ_i can be obtained explicitly by considering $\dot{\Phi}^i = 0$

$$\begin{aligned}\dot{\Phi}^i &= \dot{x}^a \nabla_a \Phi^i \\ &= \Omega^{ab}\nabla_a \Phi^i \nabla_b H + \lambda_j \Omega^{ab}\nabla_a \Phi^i \nabla_b \Phi^j\end{aligned}$$

The Lagrange multiplier is given by

$$\lambda_i = \omega_{ji}\Omega^{ab}\nabla_a \Phi^j \nabla_b H$$

where

$$\omega^{ij} = \Omega^{ab}\nabla_a \Phi^i \nabla_b \Phi^j$$

Symplectic approach (cont'd)

The modified equation of motion becomes

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \tilde{\Omega}^{ab}\nabla_b H$$

where $\tilde{\Omega}^{ab}$ is induced symplectic structure on the constraint subspace:

$$\tilde{\Omega}^{ab} = \Omega^{ab} + \Omega^{ac}\Omega^{bd}\omega_{ij}\nabla_c\Phi^i\nabla_d\Phi^j$$

and

$$\omega^{ij} = \Omega^{ab}\nabla_a\Phi^i\nabla_b\Phi^j$$

Constraints

Family of constraints

$$\Phi^i(x) = 0 \qquad i = 1, 2, \ldots, N$$

Examples:

- conserved observables

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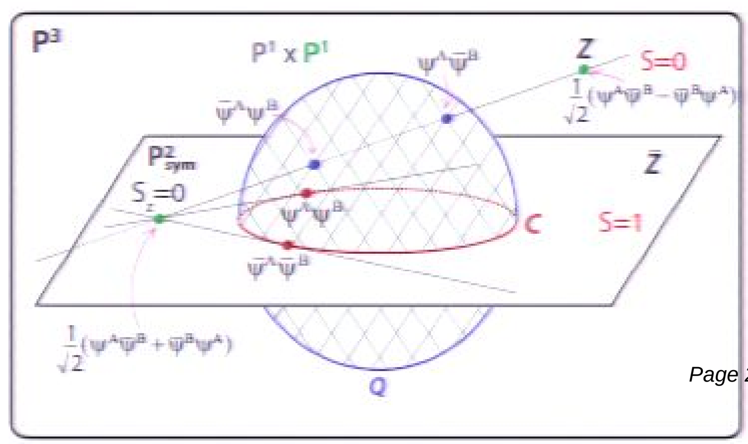
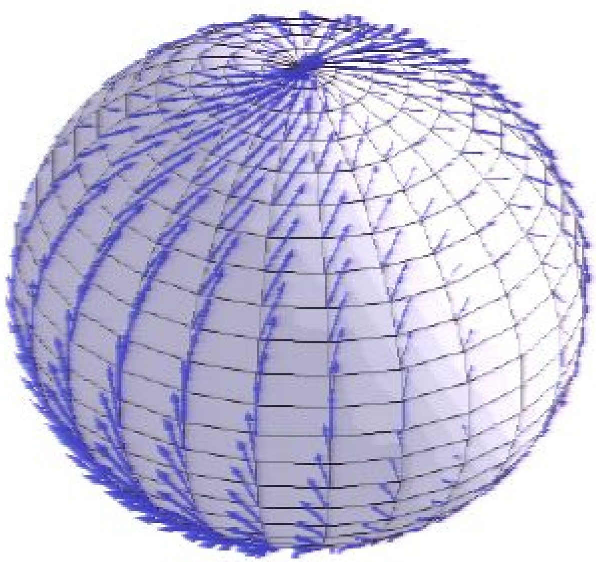
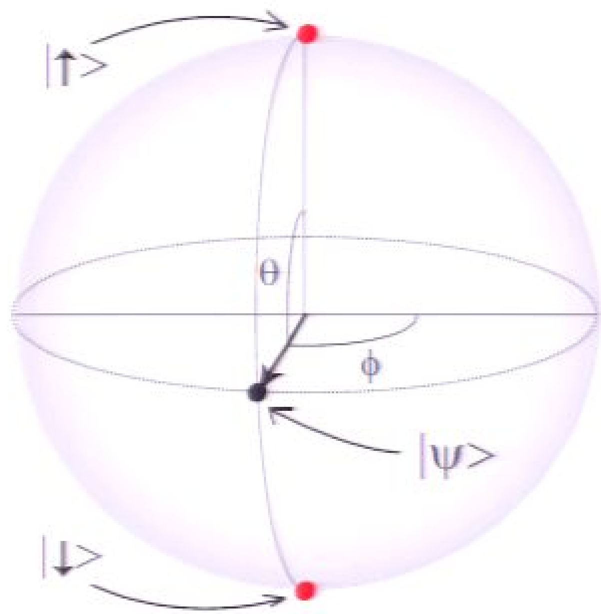


Symplectic approach (cont'd)

Example: 2 spin-$\frac{1}{2}$ particles constrained to subspace of disentangled states with Hamiltonian:

$$\hat{H} = -J\hat{\boldsymbol{\sigma}}_1 \otimes \hat{\boldsymbol{\sigma}}_2 - B(\hat{\sigma}_1^z \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{\sigma}_2^z)$$





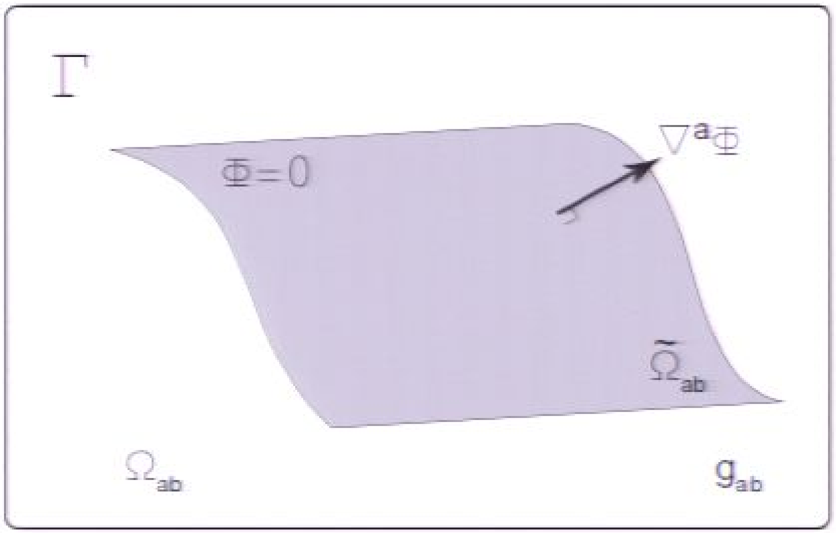
Example of a 'snapshot' of the vector field on S_1^2 when motion on S_2^2 is fixed to the point $\theta_2 = \phi_2 = \frac{1}{2}\pi$

Metric approach

Modified equation of motion

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \Omega^{ab}\nabla_b H - \lambda_i g^{ab}\nabla_b\Phi^i$$

- remove from the unitary vector field the components orthogonal to the constraint surface
- with weaker condition λ_i can be solved explicitly to obtain dynamical equations
- with stronger condition this reduces to the symplectic approach



Metric approach (cont'd)

Modified equation of motion

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \Omega^{ab}\nabla_b H - \lambda_i g^{ab}\nabla_b \Phi^i$$

Find Lagrange multipliers λ_i by considering $\dot{\Phi}^j = 0$

$$\begin{aligned} \dot{\Phi}^j &= \dot{x}^a \nabla_a \Phi^j \\ &= \Omega^{ab}\nabla_a \Phi^j \nabla_b H - \lambda_i g^{ab}\nabla_a \Phi^j \nabla_b \Phi^i \end{aligned}$$

Define

$$M^{ij} = g^{ab}\nabla_a \Phi^i \nabla_b \Phi^j$$

Metric approach (cont'd)

Note that for family of quantum observables the matrix M^{ij} corresponds to

$$M^{ij} = \langle \Phi^i \Phi^j \rangle - \langle \Phi^i \rangle \langle \Phi^j \rangle$$

Lagrange multiplier obtainable when M_{ij} exists:

$$\lambda_i = M_{ij} \Omega^{ab} \nabla_a \Phi^j \nabla_b H$$

Constrained equation of motion becomes

$$\dot{x}^a = \Omega^{ab} \nabla_b H - M_{ij} \Omega^{cd} \nabla_c \Phi^j \nabla_d H g^{ab} \nabla_b \Phi^i$$

Compatibility condition

Sufficient condition under which metric equation of motion

$$\dot{x}^a = \Omega^{ab}\nabla_b H - M_{ij}\Omega^{cd}\nabla_c\Phi^j\nabla_d H g^{ab}\nabla_b\Phi^i$$

can be written in the same form as the symplectic approach

$$\dot{x}^a = \tilde{\Omega}^{ab}\nabla_b H$$

is given by

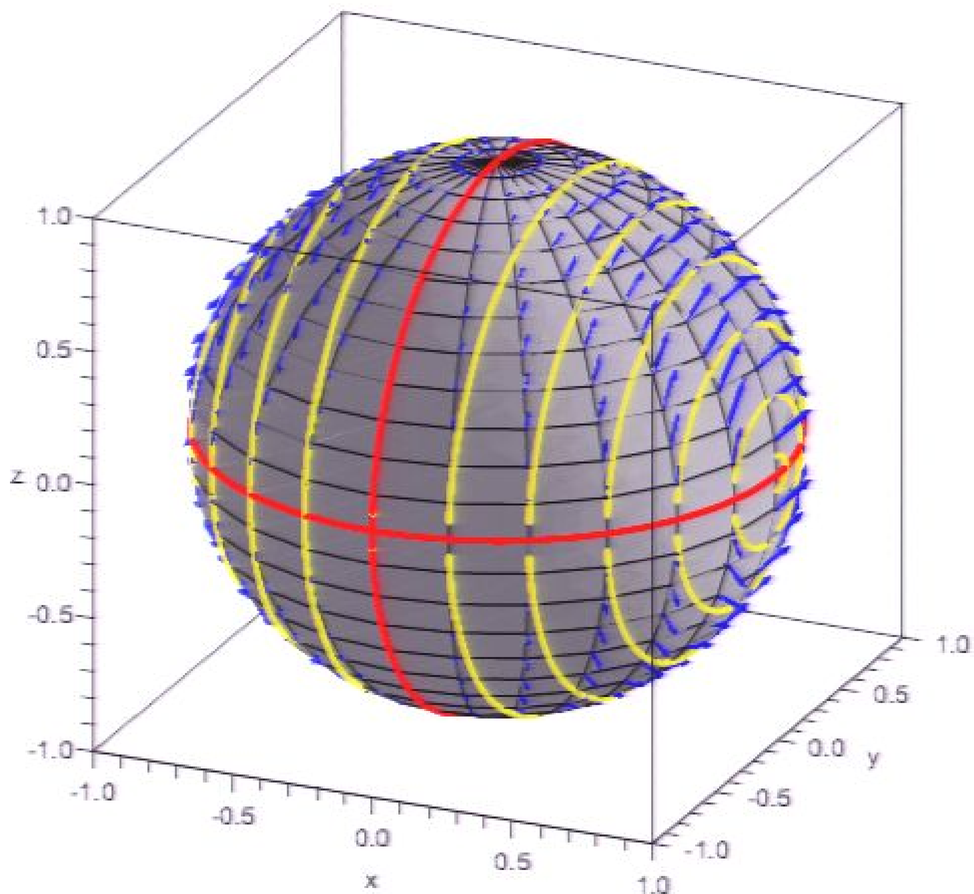
$$J_a^c J_b^d \mu_{cd} = \mu_{ab}$$

where $J_b^a = g^{ac}\Omega_{cb}$ is the complex structure on Γ and where we have defined

$$\mu_{bc} := M_{ij}\nabla_b\Phi^i\nabla_c\Phi^j$$

Metric approach: Example

Example: Single spin-$\frac{1}{2}$ particle with Hamiltonian $\widehat{H} = \widehat{\sigma}_z$ constrained such that $A(x) = \frac{\langle x|\widehat{\sigma}_x|x\rangle}{\langle x|x\rangle}$ is conserved



Mixed states

Equation of motion

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \hat{\rho}]$$

where $\hat{\rho}$ is density matrix representing the state of the system and $\hat{H}$ is the ordinary Hamiltonian operator

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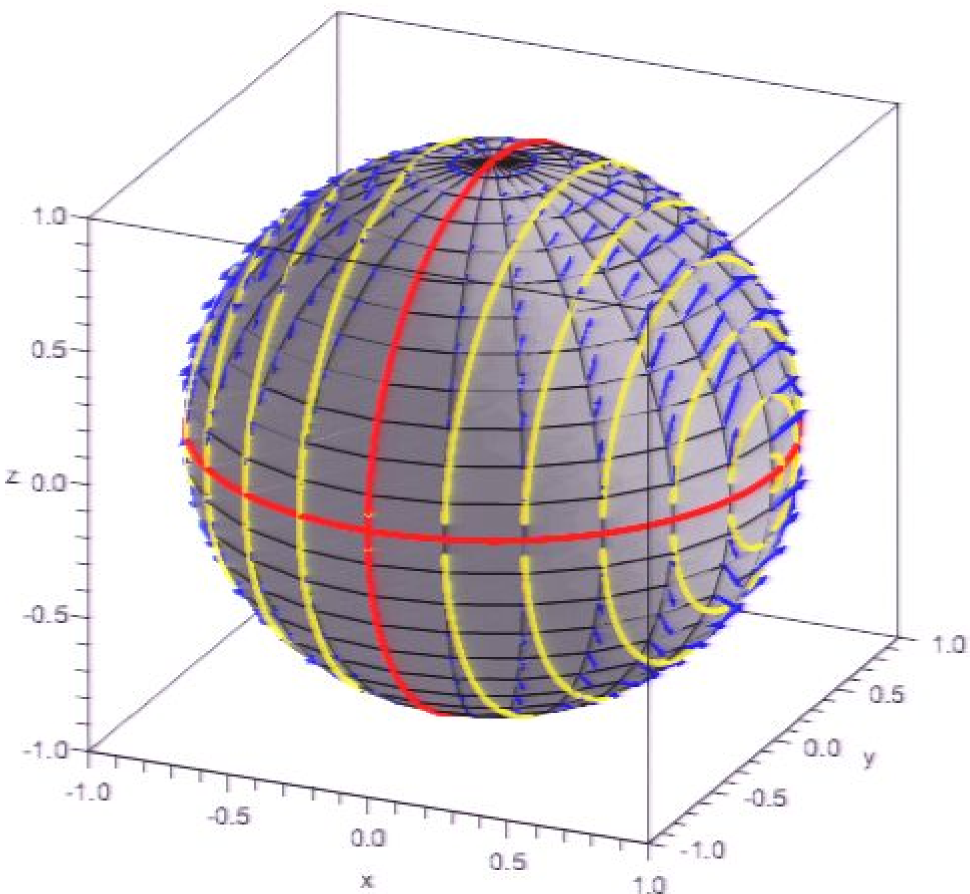

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Mixed states: symplectic approach

Modified equation of motion

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \hat{\rho}] - \mathrm{i}\lambda_j[\hat{\Phi}^j, \hat{\rho}]$$

In some cases it is possible to find λ_j explicitly:

$$\lambda_j = W_{ij}\mathrm{tr}\left(\hat{\rho}[\hat{\mathrm{H}}, \hat{\Phi}^{\mathrm{i}}]\right)$$

where W_{ij} is the inverse of

$$W^{ij} := \mathrm{tr}\left(\hat{\rho}[\hat{\Phi}^i, \hat{\Phi}^j]\right) \qquad \text{such that} \qquad W_{ij}W^{jk} = \delta_i^k$$

Mixed states: metric approach

Modified equation of motion

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \hat{\rho}] + \lambda_j \{\hat{\Phi}^j, \hat{\rho}\}$$

In some cases we can find λ_j explicitly and obtain dynamical equations with

$$\lambda_j = \mathrm{i} N_{ij} \mathrm{tr}\left(\hat{\rho}[\hat{\Phi}^{\mathrm{i}}, \hat{\mathrm{H}}]\right)$$

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Summary

The phase space description of QM gives us a powerful tool for addressing questions in quantum mechanics.

It is possible to treat quantum constraints in two different ways:

(a) The symplectic approach: This is an extension of Dirac's work on constraints to quantum systems

(b) The metric approach: This is a distinct approach that has no classical counterpart

Summary (cont'd)

We have derived a sufficient condition to ensure that the two approaches are equivalent.

It is possible to treat both pure and mixed states, working either in the phase space description of QM or in Hilbert space.

Would like to find an experimental setup that realises specific constraints that can be described in the present framework.

Find out more

- Brody, D. C., Gustavsson, A. C. T. & Hughston, L. P. 2008 Symplectic approach to quantum constraints. J. Phys. A: Math. Theor. 41 (2008) 475301 (arXiv:0808.1068v1)
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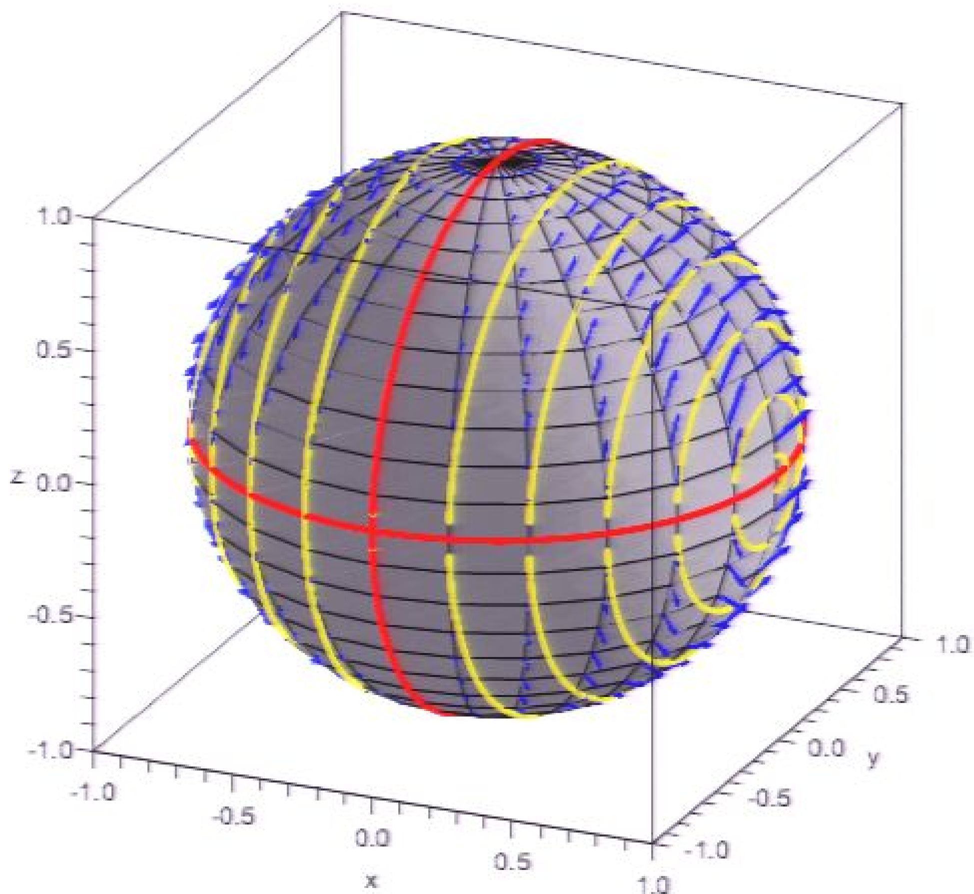

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