Title: Relational Particle Mechanics

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Abstract: Relational particle mechanics are theories of relative angles and relative (ratios of) separations only. These bear a number of resemblances to the geometrodynamical formulation of general relativity and as such are useful analogues for at least some approaches to the notorious problem of time in quantum gravity. I have recently provided a fairly complete study of the configuration spaces of these theories in spatial dimension 1 and 2, am subsequently studying the redused forms of these theories at the quantum level, and this shall provide a number of useful examples for the conceptual discussion of various problem of time strategies

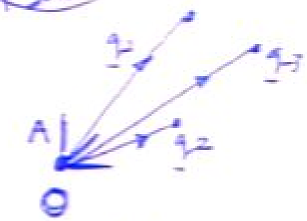
(page 1)

<u>Temporal Relationalism</u>: no overall notion of time for the whole system (e.g. universe).

- Implement by considering reparametrization-invariant actions that do not contain extraneous variables that are related to time, either (e.g. lapse).
- reparametrization invariance $\Rightarrow$ there's **at** least 1 primary constraint.

<u>Configurational Relationalism</u>: A certain group of transformations $G$, that act on theory's configuration space $Q$, are physically meaningless.

- One widely-useable implementation is to use arbitrary $G$-frame corrected quantities rather than 'bare' $Q$ quantities.
- Though at first this augments $Q$ to $Q \times G$, variation wrt each indep. adjoined $G$-variable produces a secondary constraint that uses up <u>both</u> 1 $G$ degree of freedom and 1 redundant $Q$ degree of freedom, so that one ends up dealing with the desired (partly) reduced configuration space: the quotient space $\frac{Q}{G}$.




Absolute positions

with masses $m_i$


A basis of relative separations


basis of relative interparticle (cluster) separations.

w. interparticle (cluster) masses $\mu_i$


Euclidean-relational variables

$\Phi$, $R = \sqrt{\frac{\mu_1}{\mu_2}}\left|\frac{R_1}{R_2}\right|$

Similarity-relational variables.

Relational particle models: $Q = \{ \underline{q}_i \quad i = 1 \text{ to } N \}$

$G$ = Euclidean group, or Similarity group.

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## RELATIONAL PARTICLE MODELS

$$I_{\text{Euler-Lagrange}} = \int dt \left( T\left(\substack{\text{in terms}\\ \text{of } dx/dt}\right) - V \right)$$

$\downarrow$ parametrization

$$I_{\text{param}} = \int d\lambda\, \dot{t} \left( \frac{T}{\dot{t}^2} - V \right)$$

Routhian reduction / elimination of $\dot{t}$

$$I_{\text{Jacobi type}} = 2\int d\lambda \sqrt{T\,(U + E)}$$

$U$: − potential; $E$: energy

$$T = M^{ij\alpha\beta}\, \&R_{i\alpha}\, \&R_{j\beta}/2$$

$M$: kinetic metric $\mu_i \delta^{ij}\delta^{\alpha\beta}$

$R_{i\alpha}$: Jacobi relative coords

$$\dot{R}_{j\beta} - (\dot{\underline{b}} \times \underline{R}_j)_\beta + \dot{c} R_{j\beta}$$

$\frac{d}{d\lambda}$, $\lambda$ a label time

$b$, $c$: rot, dil auxiliaries

[ $G = \text{Rot}$ [ $\times$ Dil ] ]

- Reparametrization invariance
$\Rightarrow$ primary constraint

$$\mathcal{H} = \frac{1}{2} N_{ij\alpha\beta}\, P^{i\alpha} P^{j\beta} + V - E = 0$$

$N$: inverse of $M$; $P$: momenta conjugate to $R_{i\alpha}$

(Energy constraint)

- $\underline{b}$ - variation
$\Rightarrow$ secondary constraint

$$L_\alpha \equiv (\underline{R}_{i\alpha} \times \underline{P}^{i\alpha})_\alpha = 0$$

(Zero total AM constraint)

[ • $c$ - variation
$\Rightarrow D \equiv \underline{R}_i \cdot \underline{P}^i = 0$

## GENERAL RELATIVITY

$$I_{\text{Einstein-Hilbert}} = \int d^4x \sqrt{|g|}\, R$$

$R$: Ricci scalar

3+1 splits

$${}^{(4)}g = \begin{pmatrix} \alpha^2 - \beta_\gamma\beta^\gamma & \beta_\gamma \\ \beta_\delta & h_{\gamma\delta} \end{pmatrix}$$

$$I = \int d\lambda \int d^3x \sqrt{h}\, \dot{I} \left( \frac{T}{\dot{I}^2} + R - 2\Lambda \right)$$

similar to $I_{ADM}$ (has lapse $\alpha$ in place of vel of instant)

Routhian reduction / elimination of $\dot{I}$

multiplier elim of $\alpha$

$$I_{\text{BFO-A}} = 2\int d\lambda \int d^3x \sqrt{T(R - 2\Lambda)}$$

similar to $I_{BSW}$

$R$: Ricci 3-scalar; $\Lambda$: cc

$$T = \mathcal{M}^{\mu\nu\rho\sigma}\, \&h_{\mu\nu}\, \&h_{\rho\sigma}/4$$

$\mathcal{M}$: undensitized inverse DeWitt supermetric

$h_{\mu\nu}$: the dyn 3 metric variables

$$\dot{h}_{\alpha\beta} - \pounds_{\underline{F}}\, h_{\alpha\beta}$$

[ BSW has $\dot{h}_{\alpha\beta} - \pounds_\beta h_{\alpha\beta}$ ]

$\pounds$: Lie derivative; $F$: frame (grid) auxiliary; $\beta$: shift auxiliary.

$G = \text{Diff}$

- Reparametrization invariance
$\Rightarrow$ primary constraint

$$\mathcal{H} \equiv \frac{1}{\sqrt{h}} \mathcal{N}_{\mu\nu\rho\sigma}\, \pi^{\mu\nu}\pi^{\rho\sigma} - \sqrt{h}(R - 2\Lambda) = 0$$

$\mathcal{N}$: DeWitt supermetric; $\pi$: momenta conjugate to $h_{\mu\nu}$.

(Hamiltonian constraint)

- $\underline{F}$ - variation
$\Rightarrow$ secondary constraint

$$\mathcal{H}_\mu \equiv -2 D_\nu \pi_\mu{}^\nu = 0$$

(momentum constraint)

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$G = \mathrm{Rot} \times \mathrm{Dil}$ particle model

& $\dot{R}_{i\alpha} = \dot{R}_{i\alpha} - \dot{b} \times R_{i\alpha} + \dot{c} R_{i\alpha}$

$c$-variation yields

$$R_{i\alpha} P^{i\alpha} = 0$$

(Zero dilational momentum constraint)

$G = \mathrm{Rot}$
but in shape-scale variables
$\sqrt{\text{Moment of inertia}}$

has $R_{i\alpha} P^{i\alpha} \neq 0$ in general
as its conjugate quantity.

Try GR with
$G = \mathrm{Diff} \times$ (some form of Conf).

& $\dot{h}_{\alpha\beta}$ is now $\phi^4 \left( \dot{h}_{\alpha\beta} - \pounds_{\dot{F}} h_{\alpha\beta} + \frac{4\dot{\phi}}{\phi} h_{\alpha\beta} \right)$

$\phi$ shows up elsewhere in action
($h_{\alpha\beta} \longrightarrow \phi^4 h_{\alpha\beta}$).

Straightforward Conf:

$\phi$-variation yields 2 pieces:

$h_{\alpha\beta} \Pi^{\alpha\beta} = 0$ (maximal slice)

and some 'lapse fixing' like eq$^n$.
Frozenness or departure from GR.

With VP Conf

$$\phi \longrightarrow \hat{\phi} = \phi \Big/ \left( \int d^3x \sqrt{h}\, \phi^6 \right)^{1/6}$$

then $\phi$-variation yields
$h_{\alpha\beta} \Pi^{\alpha\beta} / \sqrt{h} = C$ spatially const.
(constant mean curvature slice)
and some 'lapse fixing eq'.
$\sim$ recover York's formulation
from an action principle.

CONFIGURATION SPACES $\rightarrow$ fixed spatial topology

$$\begin{tikzcd}
& \text{Relative Space } R(N,d) \arrow[dl, "\text{take out Rot}(d)"'] \arrow[dr, "\text{take out Dil}"] & \\
\text{Relational Space } \mathcal{R}(N,d) \arrow[dr, "\text{take out Dil}"'] & & \text{Preshape Space } P(N,d) \arrow[dl, "\text{take out Rot}"] \\
& \text{Shape Space } S(N,d) &
\end{tikzcd}$$

$\mathrm{Riem}(\Sigma,3)$ of fixed 3-metrics.

$$\begin{tikzcd}
& \mathrm{Riem}(\Sigma,3) \arrow[dl, "\text{take out Diff}(\Sigma,3)"'] \arrow[d] \arrow[dr, dashed] & \\
\mathrm{Superspace}(\Sigma,3) \arrow[d, "\text{take out Conf}(\Sigma,3)"'] & \mathrm{CRiem}(\Sigma,3) \arrow[dl, "\text{take out Diff}(\Sigma,3)"] \arrow[dr, dashed, "\text{take out VPConf}(\Sigma,3)"] & \{\mathrm{CRiem}+V\}(\Sigma,3) \arrow[d, "\text{take out Diff}(\Sigma,3)"] \\
\text{Conformal Superspace}(\Sigma,3) & & \{CS+V\}(\Sigma,3)
\end{tikzcd}$$

## The Problem of Time (in Quantum GR)

'Time' takes a different meaning in each of GR and ordinary quantum theory. This has many facets, a notable one being the frozen formalism: canonically quantizing, the Hamiltonian constraint $\mathcal{H} = 0$ becomes

$$0 = \hat{\mathcal{H}}\Psi = -\hbar^{2}\frac{1}{\sqrt{h}}\frac{\delta}{\delta h^{\mu\nu}}\left\{\sqrt{h}\,\mathcal{N}^{\mu\nu\rho\sigma}\frac{\delta\Psi}{\delta h^{\rho\sigma}}\right\} - \sqrt{h}R\Psi + 2\sqrt{h}\Lambda\Psi$$

(Wheeler-DeWitt eq (WDE))

which is a timeless / stationary Schrödinger eq rather than the time-dependent Schrödinger eq's that one would 'ordinarily' get.

The Problem of time has a long history of putting forward many conceptual strategies that attempt to solve it, but which do <u>not</u> work when examined in detail.

[ specific examples of strategies mentioned overleaf are ones that my present program is capable of producing insights for; there are others, for which e.g. minisuperspace, 2+1 GR, parametrized relativistic particles / fields ..., rather, are illuminating]

(page 4b)

## TEMPUS ANTE QUANTUM

Find a fundamental time for the full theory at cl. level.
It is 'hidden' rather than obvious,
find it by canonically transforming $(h_{\mu\nu}, \pi^{\mu\nu})$
to $(\underbrace{X_\Delta, P^\Delta}_{\text{embedding variables}}, \underbrace{\phi^I_{\text{true}}, \pi^{\text{true}}_{\phi_I}}_{\text{true dynamical degrees of freedom}})$ Then get $\to$ $P^\Delta = h^{\text{true}}(\phi, \pi \ldots)$ $\to$ $i\hbar \frac{\partial \Psi}{\partial t} = h^{\text{true}} \Psi$

embedding variables true dynamical degrees of freedom.

Eg. $X_0 = T =$ York Time $\propto h_{\mu\nu}\pi^{\mu\nu}/\sqrt{h}$.

## TEMPUS POST QUANTUM

There is no fundamental time, but ~~the~~ notion of time can emerge from quantum theory in certain circumstances.

E.g. in semiclassical universes:

$$\Psi = \underbrace{\psi(H)}_{\text{heavy, slow}} \underbrace{|\chi(H, L)\rangle}_{\text{fast, light}} \quad \text{and} \quad \psi(H) = e^{iW(H)/\hbar}$$

requires nonseparability

Born–Oppenheimer approx WKB approx.

Then $\hbar^2 N^{\mu\nu\rho\sigma} \frac{\delta^2 \Psi}{\delta h^{\mu\nu} \delta h^{\rho\sigma}}$ contains $\hbar^2 N^{\mu\nu\rho\sigma} \frac{i}{\hbar} \frac{\delta W}{\delta h^{\mu\nu}} \frac{\delta|\chi\rangle}{\delta h^{\rho\sigma}}$ 'cross term'

which is equal to $i\hbar N^{\mu\nu\rho\sigma} \pi_{\mu\nu} \frac{\delta|\chi\rangle}{\delta h^{\rho\sigma}}$ by HJT relation for momentum

$= i\hbar N^{\mu\nu\rho\sigma} \frac{\delta h^{\rho\sigma}}{\delta t^{\text{WKB}}} \frac{\delta|\chi\rangle}{\delta h^{\rho\sigma}}$ by mom–vel relation and defining $\frac{\partial}{\partial t^{\text{WKB}}} = \frac{\partial}{\partial h}$

$= i\hbar \frac{\delta|\chi\rangle}{\delta t^{\text{WKB}}}$ by chain-rule in reverse.

so NDE becomes a time-dependent Schrödinger eq $i\hbar \frac{\delta|\chi\rangle}{\delta t^{\text{WKB}}} = \left(\begin{array}{c}\text{other contribs forming an}\\ \text{effective } \hat{H}\end{array}\right)\Psi$ for L-subsystem

This framework is additionally useful for quantum cosmology
$H$ : expansion of universe

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heavy, slow — requires nonseparability — fast, light

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$= i\hbar \frac{\delta |\chi\rangle}{\delta t^{\text{WKB}}}$ by chain-rule in reverse.

So WDE becomes a time-dependent Schrödinger eq $i\hbar \frac{\delta |\chi\rangle}{\delta t^{\text{WKB}}} = \left(\begin{matrix}\text{other contributions}\\ \text{forming an}\\ \text{effective } \hat{H}\end{matrix}\right)\Psi$ for L-subsystem.

This framework is additionally useful for quantum cosmology
H : expansion of universe
L : small inhomogeneities (localized bumps). $\rightsquigarrow$ Halliwell–Hawking like calcs for CMB inhomo / galaxy formation.

page 4c

## 3) TEMPUS NIHIL EST

There is no fundamental time.

One family of such strategies are Records Theories

- Page-Wootters conditional probabilities interpretation
- Barbour's approach and similar work by (Bell; Castagnino; Halliwell)
- Records within Histories theory (Gell-Mann & Hartle; Halliwell)
- my own approach tries to have the 'best of each'.

The primary objects are records: information-containing subconfigurations of a single instant.

Records theory: one then seeks to construct a semblance of dynamics / history from the correlations between such records.

## 4 NON TEMPUS SED HISTORIA

Perhaps it is histories that are primary

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[Records Theory, the semiclassical approach and Histories theory may be inter-related.

Examples:

a) semblance of dynamics through semiclassicality?

b) semiclassicality due to histories decohering?

c) I built ground-level structure of Records theory in parallel to that for Histories theory ] (page 4d) SUMMARY OF WHY RPM'S ARE DESIRABLE MODELS

*RPM's
- nontrivial linear constraints
- notion of small localized clusters paralleling small inhomogeneous bumps in quantum cosmology.

*RPM's ~~can~~ have an internal time // to York's, semiclassical approach works out for them, Records Theory is + straightforward for them (+ve def kinetic term gives natural distance between configurations (shapes), have some chance of knowing microstates so that we can compute measures of information (≈ - entropy, computible from partition function).

## IDEA OF THIS TALK

To show that some nontrivial examples of RPM's permit considerable control / explicit computation, so as to (eventually) further our conceptual understanding of some of the Problem of Time strategies.

## CONFIGURATION SPACE GEOMETRY FOR RPM's

Take $\underline{L}$ (and where applicable, D),
and write in Lagrangian form.
Use these to eliminate $\dot{\underline{b}}$ (and $\dot{c}$).
Find that this can be completely carried out for
(N, 1) and (N, 2).
particle number — dimension
(note: Need $N \geq 3$ for RPM to be meaningful).

+ Cast in terms of fully non-redundant variables.

e.g. Find that (N, 1) reduced configuration spaces for similarity (scale-free) RPM are $\mathbb{S}^{N-2}$ both topologically and metrically w. std spherical metric

and (N, 2) reduced configuration spaces are $\mathbb{CP}^{N-2}$ both topologically and metrically w. std Fubini-Study metric

so $$T_{\text{reduced}} = \frac{1}{2}\frac{\{1 + \sum_a |Z_a|^2\}\sum_b |\dot{Z}_b|^2 - |\sum_a \bar{Z}_a \dot{Z}_a|^2}{\{1 + \sum_c |Z_c|^2\}^2}.$$

for $Z_a$ inhomogeneous coords ($z_i / z_1$ for $z$ the cplx form of polar Jacobi coords).

Thus, building on Kendall et al, I can compute essentially anything of relevance for these config spaces.

To show [illegible]
permit considerable control /explicit computation, so as to (eventually) further our conceptual understanding of some of the Problem of Time strategies.

## CONFIGURATION SPACE GEOMETRY FOR RPM's

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and write in Lagrangian form.
Use these to eliminate $\dot{\underline{b}}$ ( and $\dot{c}$ ).
Find that this can be completely carried out for
$(N, 1)$ and $(N, 2)$.

$N$ ← particle number; $1$, $2$ ← dimension (note: Need $N \geq 3$ for RPM to be meaningful).

+ Cast in terms of fully non-redundant variables.

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Thus, building on Kendall et al, I can compute essentially anything of relevance for these config spaces.

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# LET'S CONCENTRATE ON A PARTICULAR EXAMPLE FOR THIS TALK...

the (3, 2) SRPM (so still have 1 nontriv cpt of ZAM constraint)

For this, benefit from the 'accident' $\mathbb{CP}^1 = \mathbb{S}^2$.

Then can write

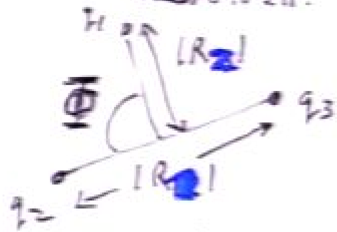
"stereographic coords" "spherical coords"

$$2T_{\text{red}} = \frac{|\dot{\underline{\mathcal{Z}}}|^2}{(1+|\underline{\mathcal{Z}}|^2)^2} = \frac{\dot{\mathcal{R}}^2 + \mathcal{R}^2\dot{\Phi}^2}{(1+\mathcal{R}^2)^2} = \frac{1}{4}\left(\dot{\Theta}^2 + \sin^2\Theta\,\dot{\Phi}^2\right)$$

$\mathcal{R} = \sqrt{\frac{\mu_1}{\mu_2}}\frac{R_1}{R_2}$ Takes 0 to $\infty$

$\Phi$ is the relative angle Takes 0 to $2\pi$.

$\Theta$ is the std azimuthal angle on a sphere. ($\mathcal{R} = \tan\frac{\Theta}{2}$). Takes 0 to $\pi$.




can think of as a theory of triangleland (with no scale – a dynamics of pure shape).

can conformally transform this one to have

$$2T_{\text{flat}} = \dot{\mathcal{R}}^2 + \mathcal{R}^2\dot{\Phi}^2$$

and $\widetilde{V} - \widetilde{E} = \dfrac{V-E}{(1+\mathcal{R}^2)^2}$.

Next, need to choose a particular type of potential...

# Triple HO potential


$$2V = h_{12}|q_1 - q_2|^2 + h_{13}|q_1 - q_3|^2 + h_{23}|q_2 - q_3|^2$$

becomes, in Jacobi coordinates

$$2V = H_1 R_1^2 + H_2 R_2^2 + G\, R_1 \cdot R_2$$

for $H_1 = h_{23} + \dfrac{h_{13} m_2^2 + h_{12} m_3^2}{(m_2 + m_3)^2}$, $H_2 = h_{12} + h_{13}$

$$G = \frac{2(h_{13} m_2 - h_{12} m_3)}{m_2 + m_3}$$

$G = 0$ is a considerable simplification "Special case"
- amounts to $h_{12} m_3 = h_{13} m_2$.

Then in flat version of $R$-coords,

$$2\tilde{V} = \frac{\frac{H_1}{\mu_1} R^2 + \frac{H_2}{\mu_2} + \frac{G}{\sqrt{\mu_1 \mu_2}} R \cos\Phi}{(1 + R^2)^3}$$

And in $\Theta$ coordinates,

$$V = A + B\cos\Theta + C \sin\Theta \cos\Phi$$

for $A = \frac{1}{4}\left(\frac{H_1}{\mu_1} + \frac{H_2}{\mu_2}\right)$, $B = \frac{1}{4}\left(\frac{H_2}{\mu_2} - \frac{H_1}{\mu_1}\right)$

and $C = \dfrac{G}{4\sqrt{\mu_1 \mu_2}}$.

$B = C = 0$ is an even bigger simplification
- the constant potential. $h_{12} m_3 = h_{23} m_1 = h_{31} m_2$.
"very special case" (even more 'symmetric')

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## Classical analysis

In general, $\Phi$-indep ~~potential~~ potential means there is a 1st $\int$ $\quad P_\Phi = R^2\Phi' = J$, (1) const.

Then the even more ubiquitous 1st $\int$ of 'energy type' takes the form

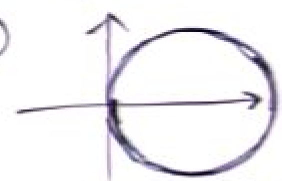
$$\frac{1}{2}\left\{R'^2 + \underbrace{\frac{J^2}{R^2}\right\} + \tilde{V}(R)}_{V_{eff}} - \tilde{E}(R) = 0 \quad (2)$$

$\tilde{E}(R) \mapsto E/(1+R^2)^2$.

which can be rearranged to an integral.

Also in general, large-$R$ asymptotics goes like $\tilde{V} = \dfrac{Q}{R^4}$.

For this, the orbit integral (2)/(1)$^2$ rearranged gives $R = \sqrt{\dfrac{2Q}{J}}\cos(\Phi - \Phi_0)$



~things return rather than being outflung forever.

that's in terms of $V_{orb} = R^4 V_{eff}$

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# The Quantum Problem

Laplacian ordering $\quad -\frac{\hbar^2}{2}\frac{1}{\sqrt{M}}\frac{\partial}{\partial Q^A}\left\{\sqrt{M}\,N^{AB}\frac{\partial \Psi}{\partial Q^B}\right\} + V\Psi = E\Psi$

so for the 3-particle 2-d SRPM in spherical coords

$$-\frac{\hbar^2}{2}\cdot 4\left\{\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left\{\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right\} + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\Phi^2}\right. + V(\Theta,\Phi)\Psi = E\Psi$$

Then for a triple HO potential

( $\div 2\hbar^2$ and A here is (old A $- E)/2\hbar^2$

B ——— $B/2\hbar^2$

C ——— $C/2\hbar^2$ )

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left\{\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right\} + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\Phi^2} = \left\{\begin{matrix}A + B\cos\Theta + \\ C\sin\Theta\cos\Phi\end{matrix}\right\}\Psi$$

Useful methods

$\mathcal{R} = \tan\frac{\Theta}{2}$ ‘simple ratio variable’

$X = \cos\Theta$ ‘Legendre variable’

and $C = 0$ : separable

or B, C small.

or $\mathcal{R}$ large, small (compared to 1)

(Reminds to $\pi - \Theta$ & $\Theta$ being small angles).

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I'll give some indications that this is a problem over which a great deal of control can be attained.

* e.g $\mathcal{R}$ large asymptotics gives

$$\Psi_m \propto J_m(\kappa/\mathcal{R})\, e^{im\Phi} = J_m\!\left(\kappa \cot\tfrac{\Theta}{2}\right) e^{im\Phi}$$

Bessel function

$$= J_m\!\left(\kappa \sqrt{\tfrac{\mu_2}{\mu_1}}\tfrac{R_2}{R_1}\right) \exp\!\left(im \arccos\!\left(\tfrac{R_1\cdot R_2}{|R_1||R_2|}\right)\right)$$

in terms of 'original' Jacobi variables

are 'reciprocals'

* $\mathcal{R}$ small asymptotics gives

$$\Psi_m \propto J_m(\kappa \mathcal{R})\, e^{im\Phi} = J_m\!\left(\kappa \cot\tfrac{\Theta}{2}\right) e^{im\Phi}$$

$$= J_m\!\left(\kappa \sqrt{\tfrac{\mu_1}{\mu_2}}\tfrac{R_1}{R_2}\right) \exp\!\left(im \arccos\!\left(\tfrac{R_1\cdot R_2}{|R_1||R_2|}\right)\right).$$

or, to better approximation (keep $O(\mathcal{R}^3)$)

$$\Psi_{nm} \propto \mathcal{R}^{|m|} \exp(-\ell \mathcal{R}^2/2)\, L_n^{|m|}(\ell \mathcal{R}^2)\, e^{im\Phi}$$

Assoc. Laguerre polynomial

$$= \tan^{|m|}\tfrac{\Theta}{2} \exp\!\left(-\ell \tan^2\tfrac{\Theta}{2}/2\right) L_n^{|m|}\!\left(\ell \tan^2\tfrac{\Theta}{2}\right) e^{im\Phi}$$

$$= \left(\sqrt{\tfrac{\mu_1}{\mu_2}}\tfrac{R_1}{R_2}\right)^{|m|} \exp\!\left(-\ell \tfrac{\mu_1}{\mu_2}\tfrac{R_1^2}{R_2^2}\right) L_n^{|m|}\!\left(\ell \tfrac{\mu_1 R_1^2}{\mu_2 R_2^2}\right) \exp\!\left(im \arccos\!\left(\tfrac{R_1\cdot R_2}{|R_1||R_2|}\right)\right)$$

* For $B = C = 0$, have an obvious solution in $\Theta$-variable:

$$\Psi_{nm} \propto P_n^m(\cos\Theta)\, e^{im\Phi} = P_n^m\!\left(\frac{1-\mathcal{R}^2}{1+\mathcal{R}^2}\right) e^{im\Phi}$$

$$= P_n^m\!\left(\frac{\mu_2 R_2^2 - \mu_1 R_1^2}{\mu_2 R_2^2 + \mu_1 R_1^2}\right) \exp\!\left(im \arccos\!\left(\frac{R_1\cdot R_2}{|R_1||R_2|}\right)\right).$$

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$V = A + BX + C\sqrt{1-x^2}\cos\Phi$

Treat as: An exactly solvable background | A $\Phi$-indep perturbation | A $\Phi$-dep. perturbation

Time-independent QM perturbation theory

$H_0 |n\rangle = E_0^n |n\rangle$ 0th order

$E_1^n = \langle n | H' | n\rangle$, $|\psi_1^n\rangle = \sum_{r \neq n} \frac{|r\rangle\langle r|H'|n\rangle}{E_0^n - E_0^r}$ 1st order...

so $\langle n|H'|r\rangle$ are key objects to be able to compute.

For us,

$$|n\rangle = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^m(\cos\Theta)\, e^{im\Phi}$$

so the key object is $\int_{-1}^{1} P_{n'}^{m'}(X)\, X\, P_n^m(x)\, dX$

but there is a recurrence relation

$$x P_n^m(x) = \frac{(n-m+1)P_{n+1}^m(x) + (n+m)P_{n-1}^m(X)}{2n+1}$$

which is a linear combination of $P_{n\pm 1}^{m}(x)$,

so that the key objects are exactly computable using orthonormality.

Thus have control of this relational problem to first order in perturbation theory.

[eigenvalues & eigenfunctions evaluated]

$\Phi$-dependent potential problem is also exactly tractable to first order in perturbation theory.
For, the key object is now

$$\int_{-1}^{1} P_{n'}^{m'}(x)\sqrt{1-x^2}\,P_n^m(x)\,dx$$

but there is also a recurrence relation,

$$\sqrt{1-x^2}\,P_n^m(x) = \frac{P_{n-1}^{m+1}(x) - P_{n+1}^{m+1}(x)}{2n+1},$$

which is a linear combination of $P_{n''}^{m''}(x)$, so that orthonormality can be applied.

## Applications

$\Phi$-dependent potentials:
nonseparable, so semiclassical approach can be applied.

Also, what was $J$ = const for $\Phi$-indep potentials
now changes, so have something 'dynamical' to track.

For Euclidean RPM, this quantity is
the relative angular momentum of 1 subsystem
with respect to the other

(the 2 subsystems)

(but I don't know what meaning $J$ has for Similarity RPM).

**Further work:** Euclidean RPM with
$\sqrt{\text{Moment of inertia}}$ as H-variable
and shape variables as L-variables.

(4, 1) theory: also $\mathbb{S}^2$ but the natural potentials are different.

(N, 1) theory: uplift $\mathbb{S}^2$ maths to $\mathbb{S}^n$ maths.

(N, 2) theory: need $N \geq 4$ to have unambiguously kinetic geometry effects

[For testing Barbour's conjectures about time capsules
and how the arrow of time might emerge
in a records theory context].

# Semiclassical approach sketch

$$-2k\left\{(1-x^2)\partial_x^2\Psi - 2x\partial_x\Psi + \partial_\Phi^2\Psi\right\} + \left(A + Bx + C\sqrt{1-x^2}\cos\Phi\right)\Psi = E\Psi$$

use $\Psi = \psi(x)\,|\chi(x,\Phi)\rangle$ , $\psi(x) = e^{iW(x)/\hbar}$.

heavy slow (for $x$); light, fast (for $\Phi$)

then $\langle\chi|\cdot$ above is H eq.

above $-\left(\langle\chi|\cdot\text{above}\right)|\chi\rangle$ is L eq.

H eq $\approx$ HJE $\quad (\partial_x W)^2 = \dfrac{E-A-Bx}{2(1-x^2)}$

for $E-A=B$ that's simple enough that $t^{WKB} = f(x)$ is invertible,
so can turn all $x$'s in L-eq into $t^{WKB}$'s,
so that L eq is not only a (WKB) time-dependent Schrödinger eq,
(by argument near start of lecture)
but also it's a (WKB) time-dependent perturbation of a
fairly simple background problem (for $C$ small).

It's looking exactly soluble to 1st order.

The difficulty is in relating this to the
whole universe solution obtained without making
semiclassical assumptions.

## Applications

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