Title: Moment Problem and Homodyne Detection

Date: Dec 05, 2007 02:45 PM

URL: http://pirsa.org/07120029

Abstract: We describe the measurement statistics of the balanced homodyne detection scheme in terms of the moment operators of the associated positive operator measures. In particular, we give a mathematically rigorous proof for the fact that the high amplitude limit in the local oscillator leads to a measurement of a rotated quadrature operator of the signal _eld. Using these results, we also show that each covariant phase space observable can be measured with the eight-port homodyne detector.

Moment problem and homodyne detection

J. Kiukas (with P. Lahti) arXiv:0706.4436

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Department of Physics,
University of Turku, Finland

YRC
Perimeter Institute
5. December 2007

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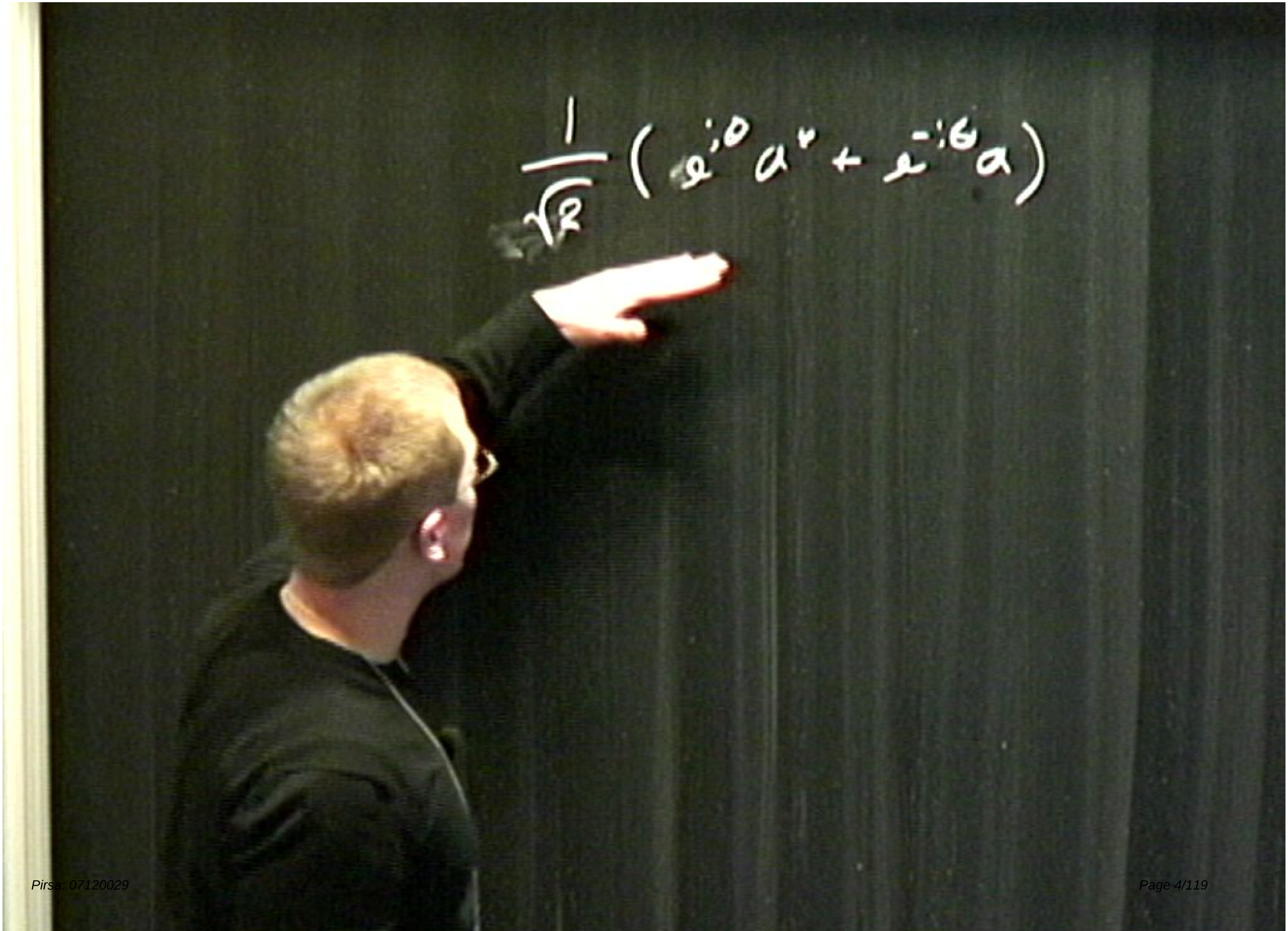
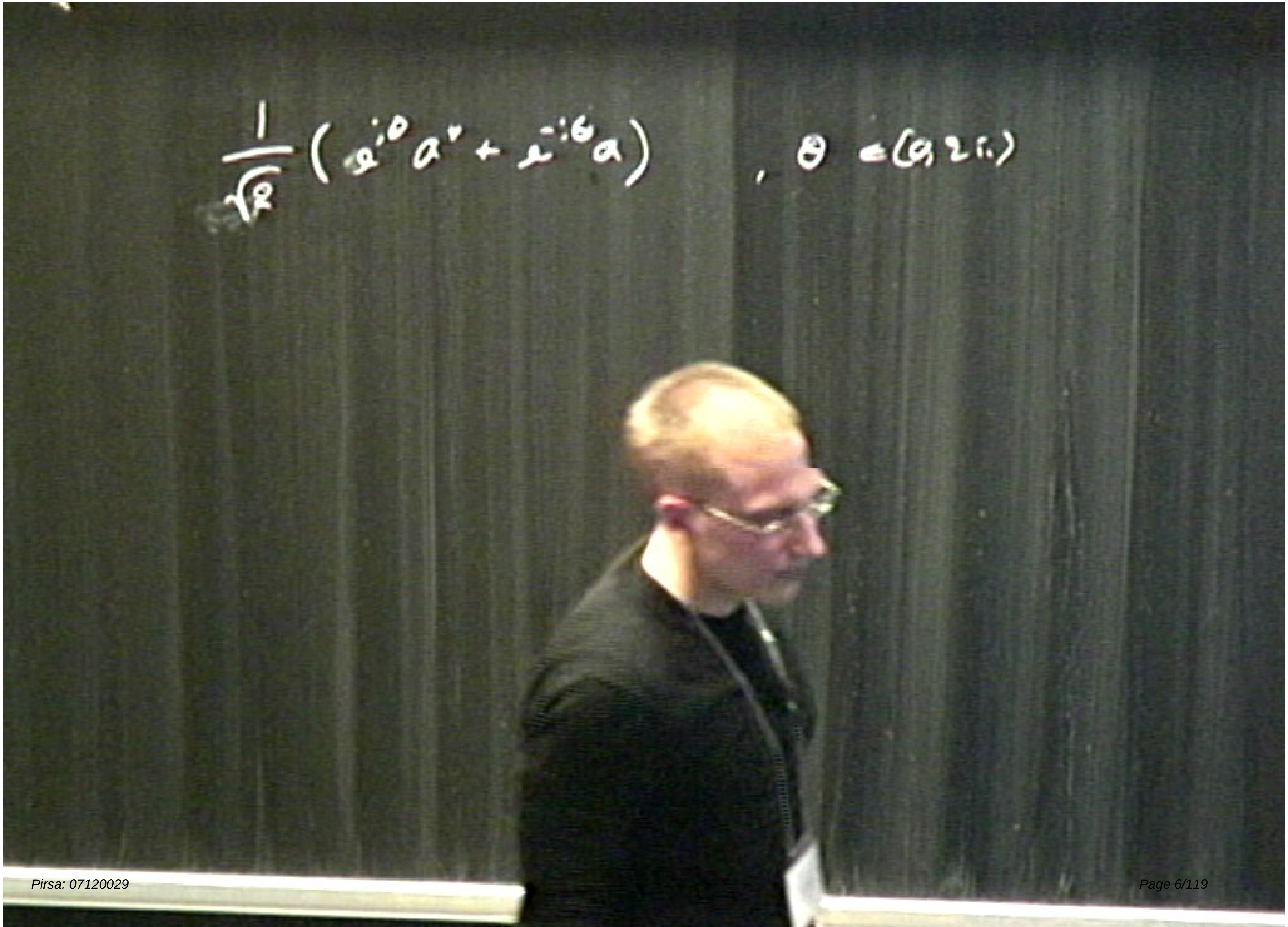
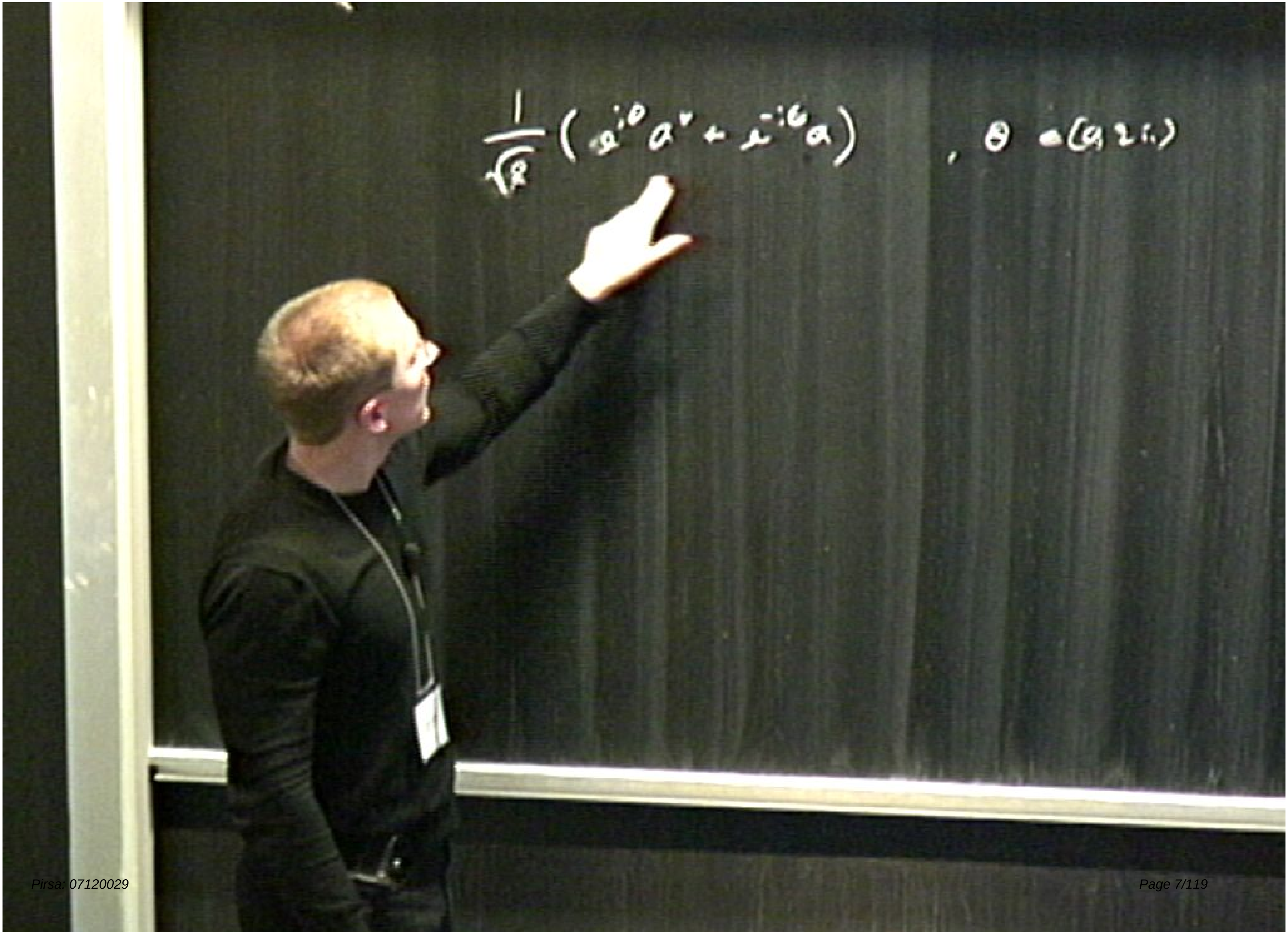

$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger}+e^{-i\theta}a\right)$

$$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger} + e^{-i\theta}a\right) \quad , \; \theta \in (0, 2\pi)$$


$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger}+e^{-i\theta}a\right)$, $\theta \in [0, 2\pi)$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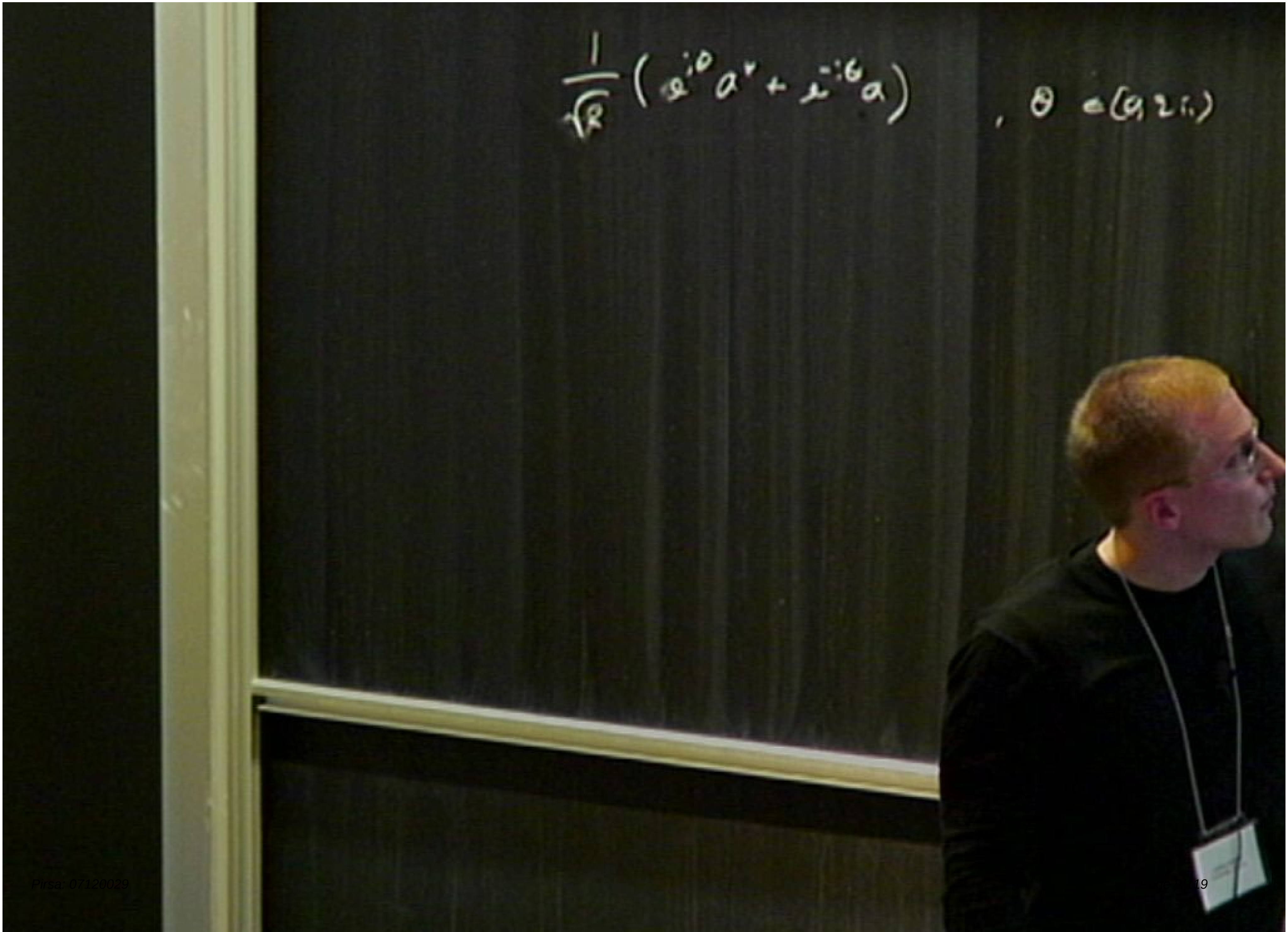
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The balanced homodyne detection

- is an established measurement technique in quantum optics (first introduced by Yuen & Chan 1983).
- used e.g. in quantum state reconstruction.
- The technique:
  - signal beam mixed with an auxiliary coherent beam via a beam splitter
  - photons counted at the output ports $\rightsquigarrow n_1, n_2$
  - The scaled difference $(n_1 - n_2)/|z|$ recorded, $|z|$ being the aux field amplitude.
  - result depends on the phase difference θ between the input beams.
- Claim: For strong aux field (large $|z|$), this amounts to measuring the rotated quadrature operators $Q_\theta = \frac{1}{\sqrt{2}}(e^{-i\theta}a + e^{i\theta}a^*)$ of the signal field.
- The purpose of this talk is to give a rigorous mathematical meaning to this claim.

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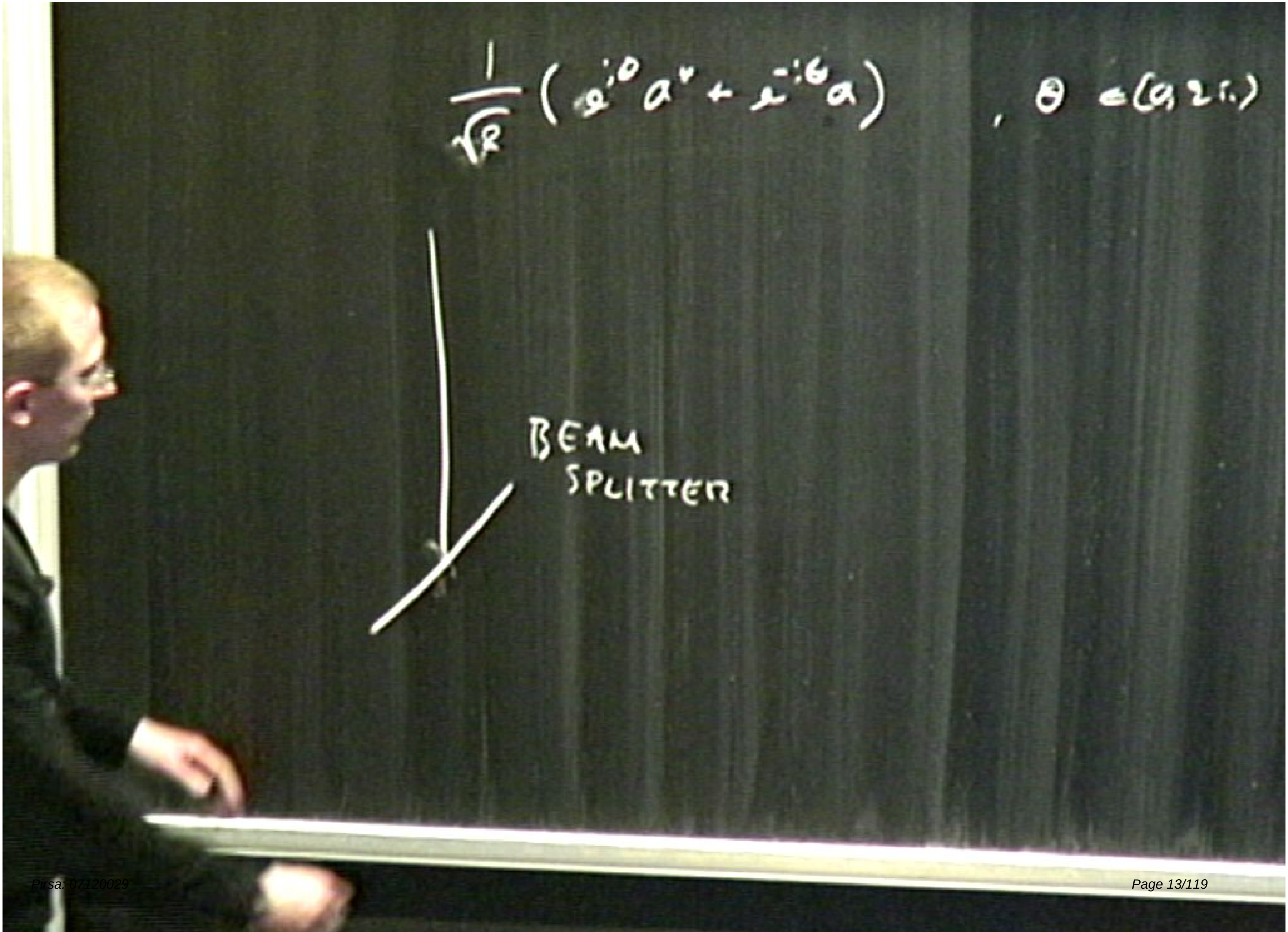
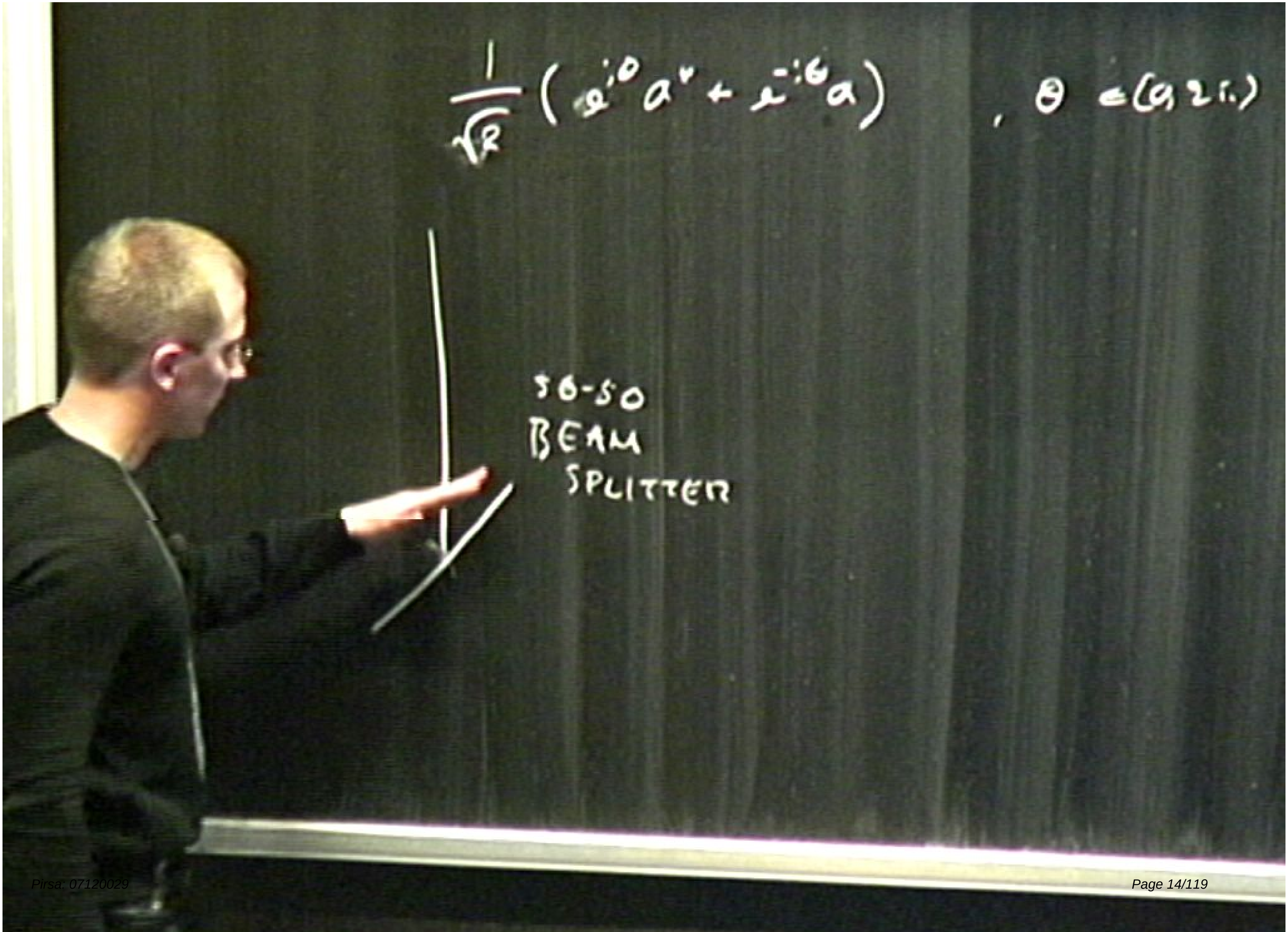
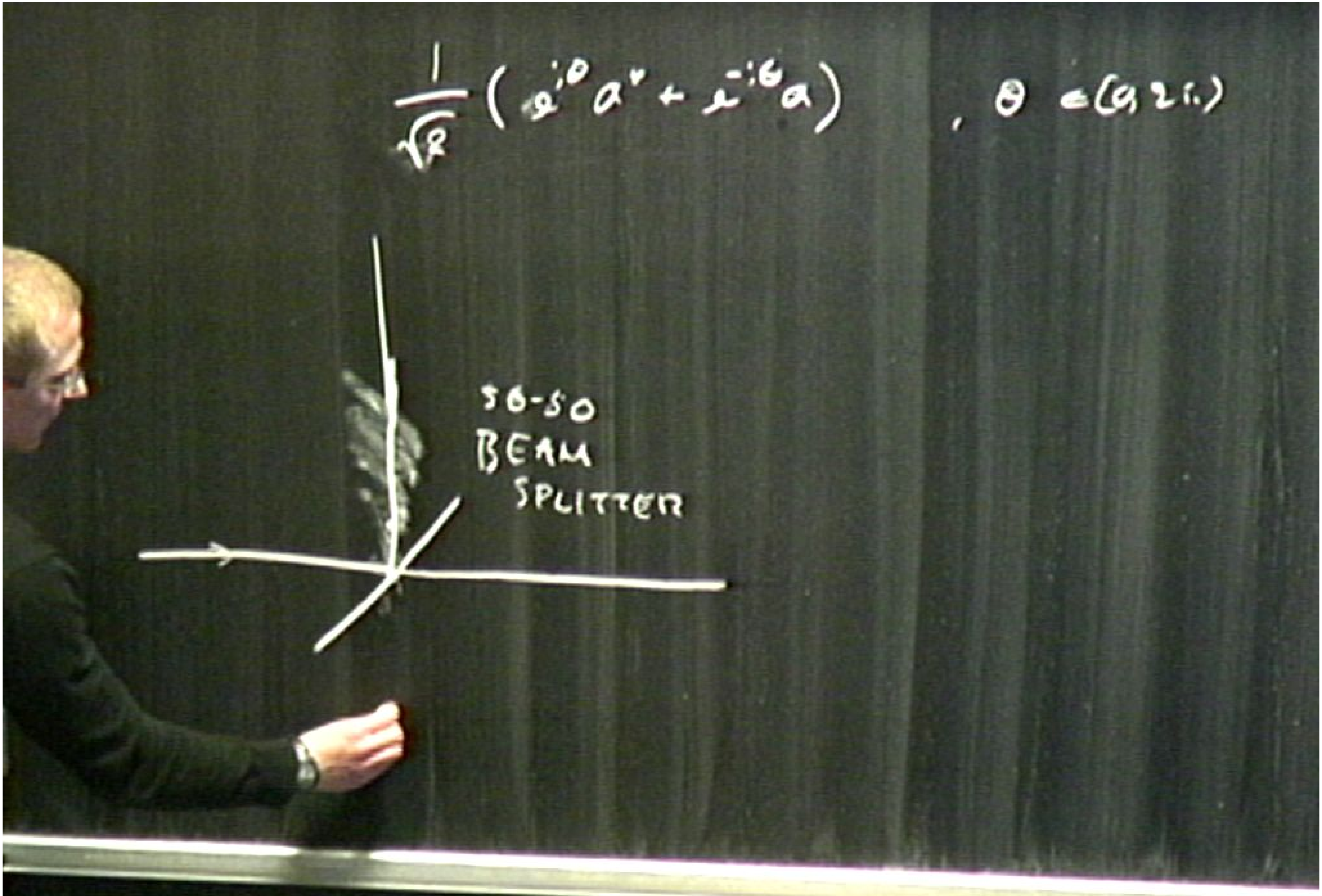
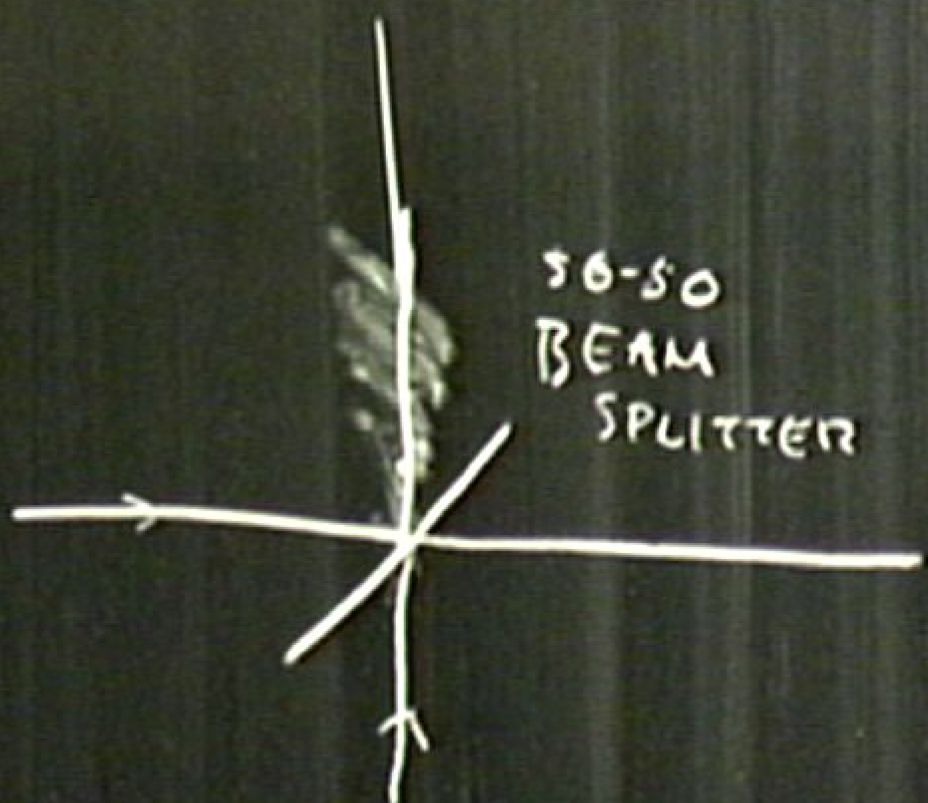

$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger} + e^{-i\theta}a\right)$, $\theta \in (0, 2\pi)$
BEAM
SPLITTER



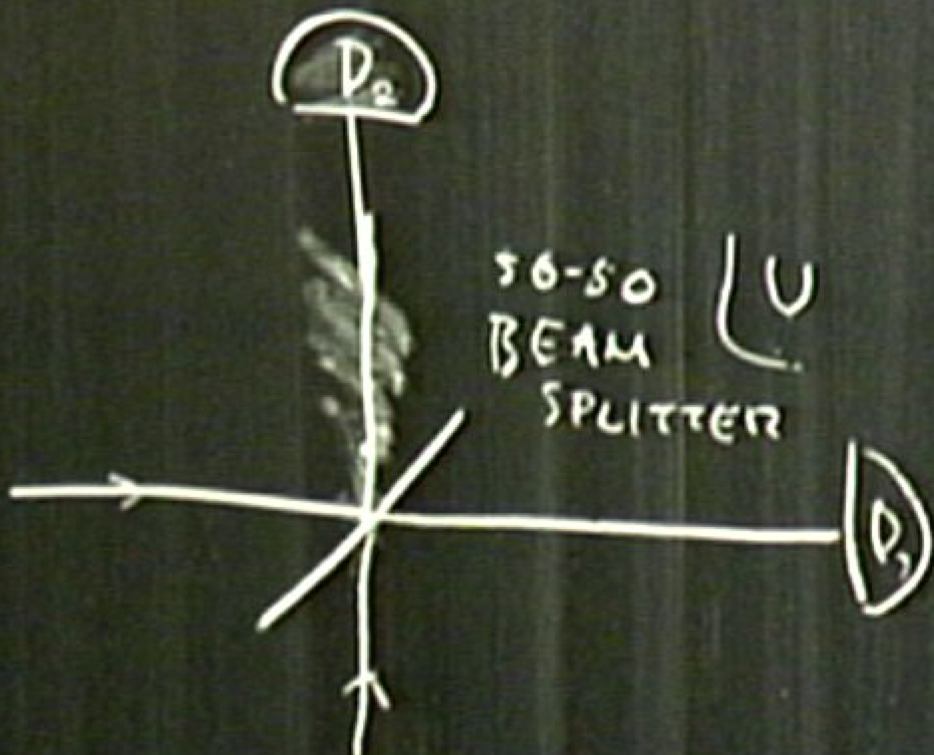
$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger}+e^{-i\theta}a\right)$, $\theta \in (0, 2\pi)$
50-50
BEAM
SPLITTER

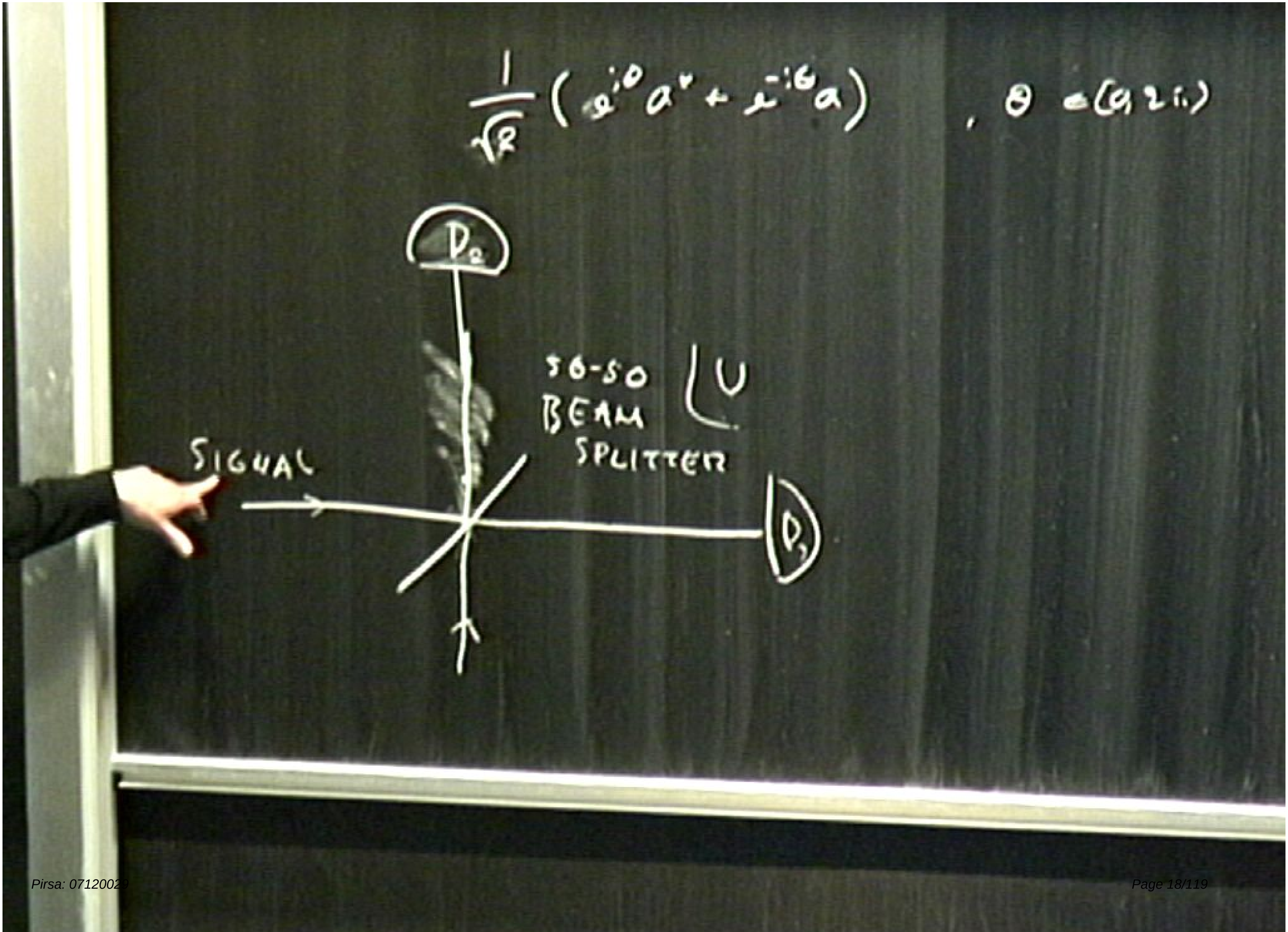

$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^\dagger + e^{-i\theta}a\right), \theta \in [0, 2\pi)$
50-50
BEAM
SPLITTER

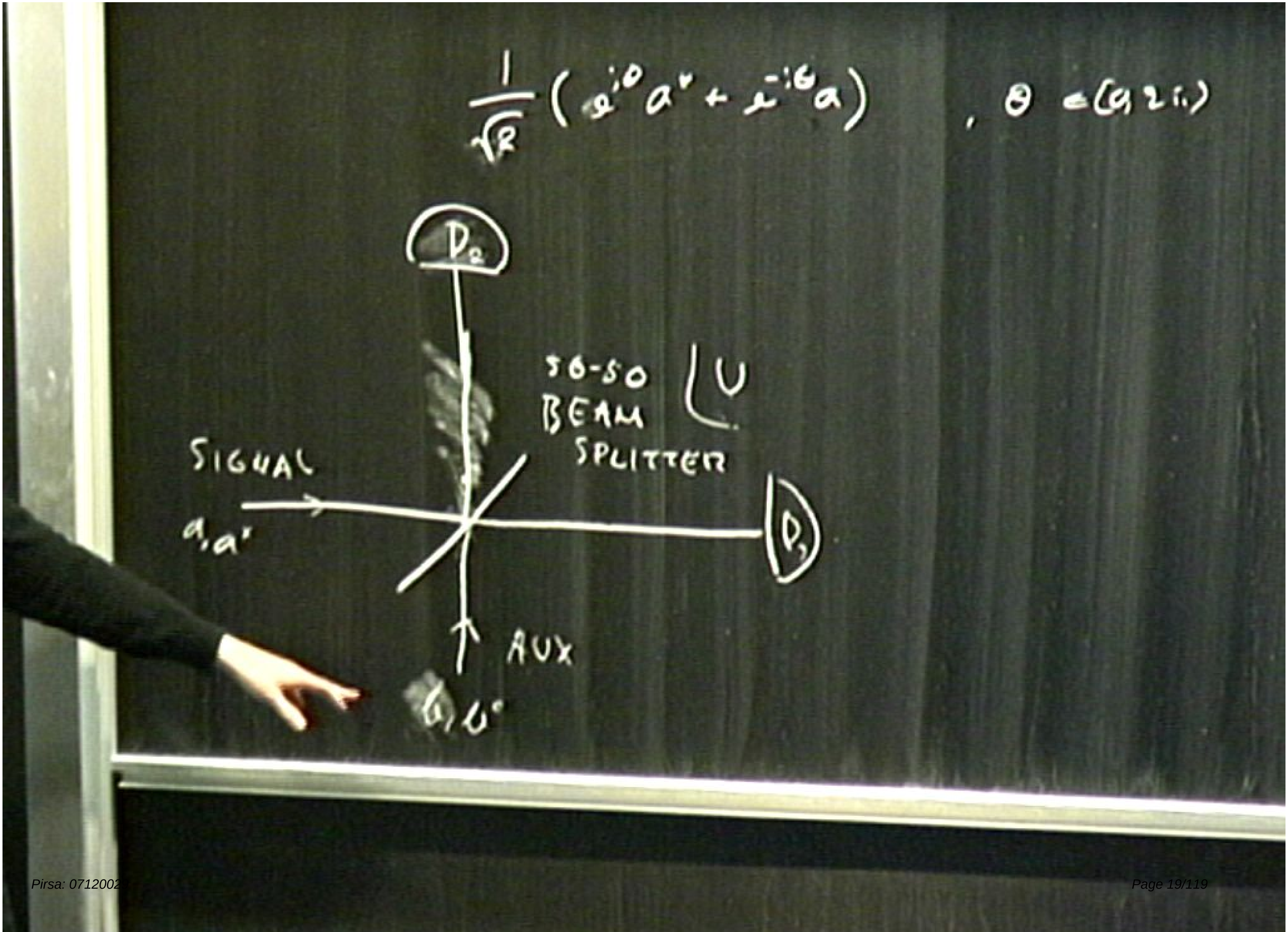
$$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger} + e^{-i\theta}a\right), \quad \theta \in [0, 2\pi)$$

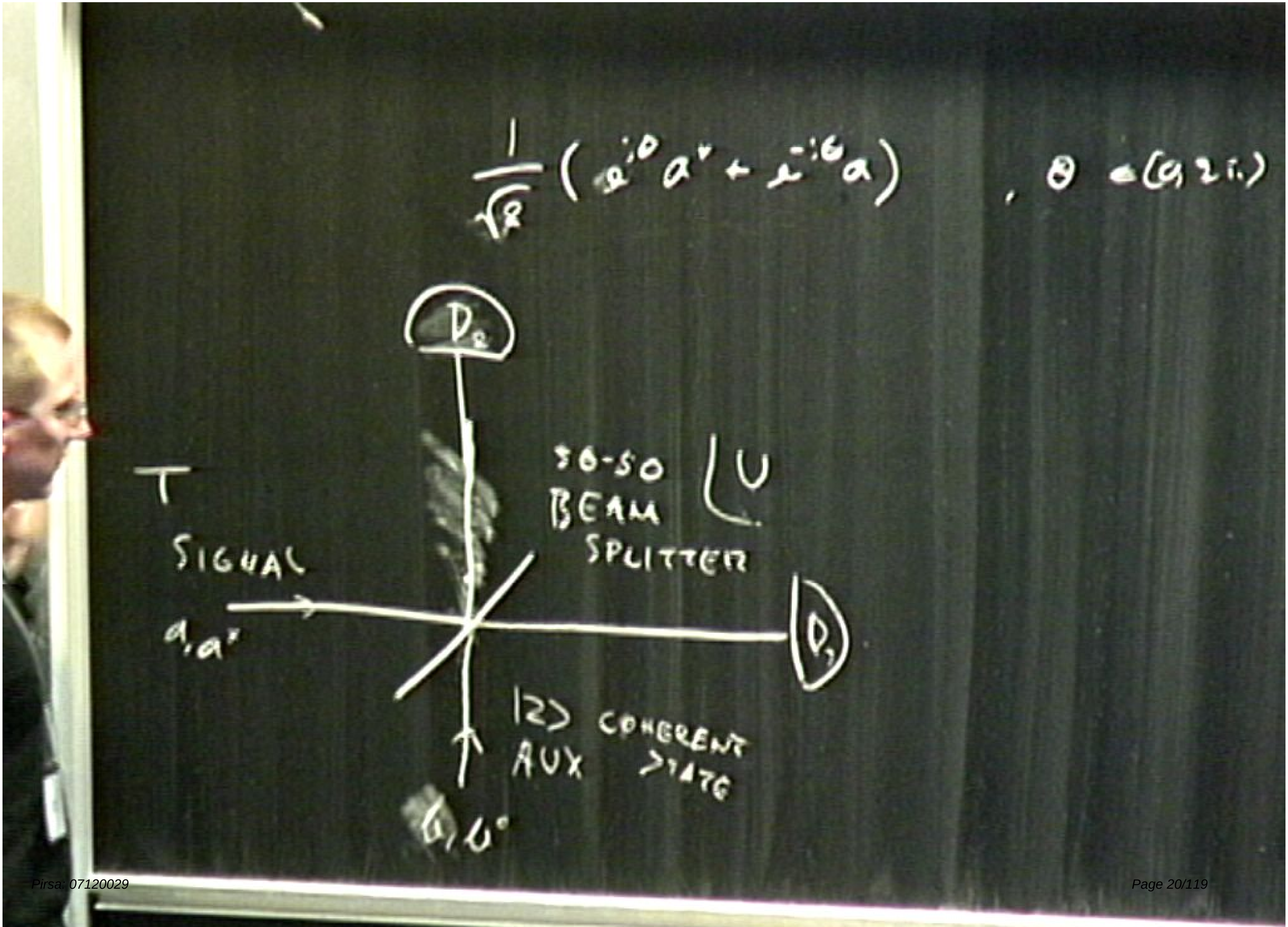


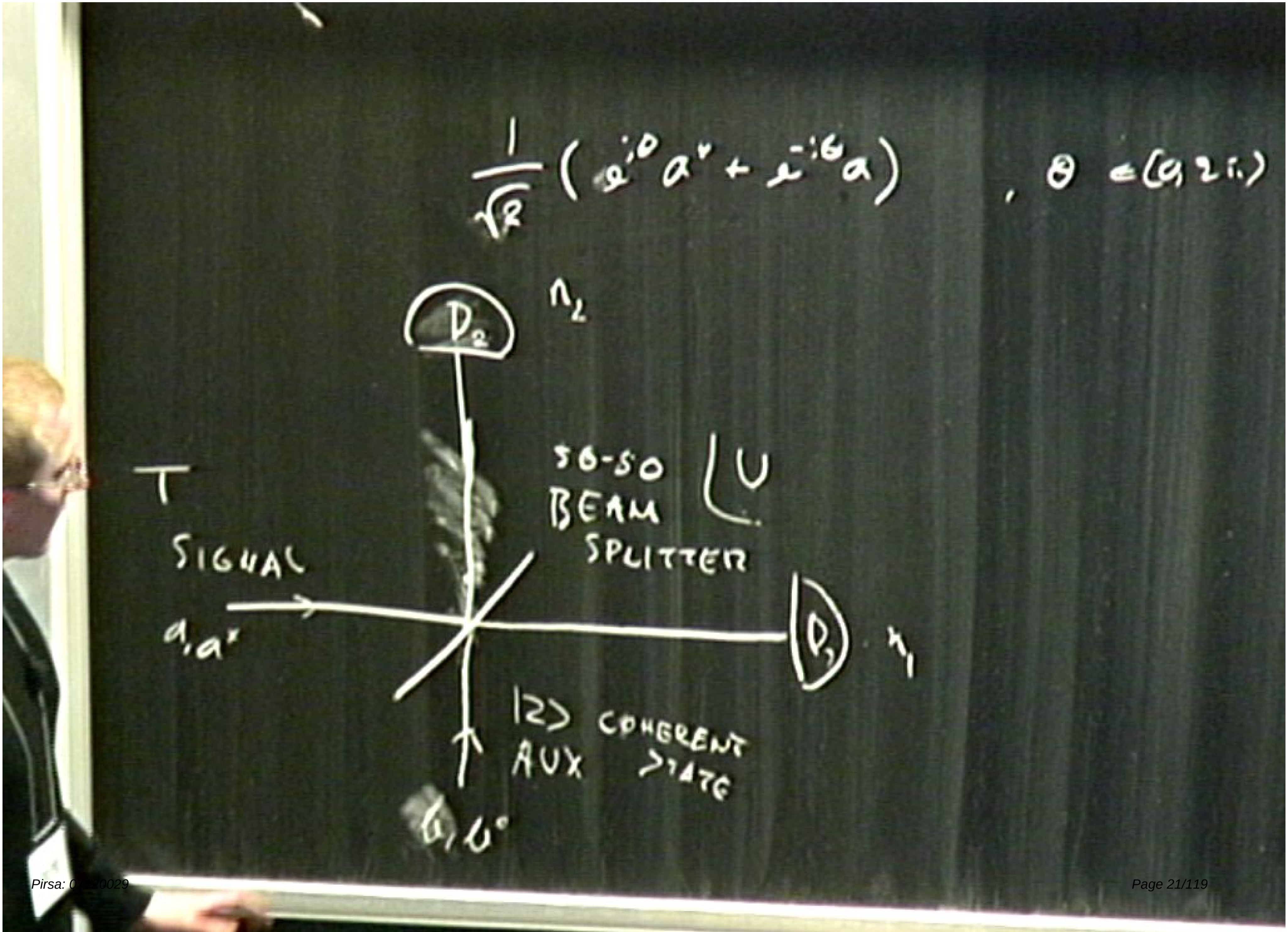
$$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger} + e^{-i\theta}a\right), \quad \theta \in (0, 2\pi)$$

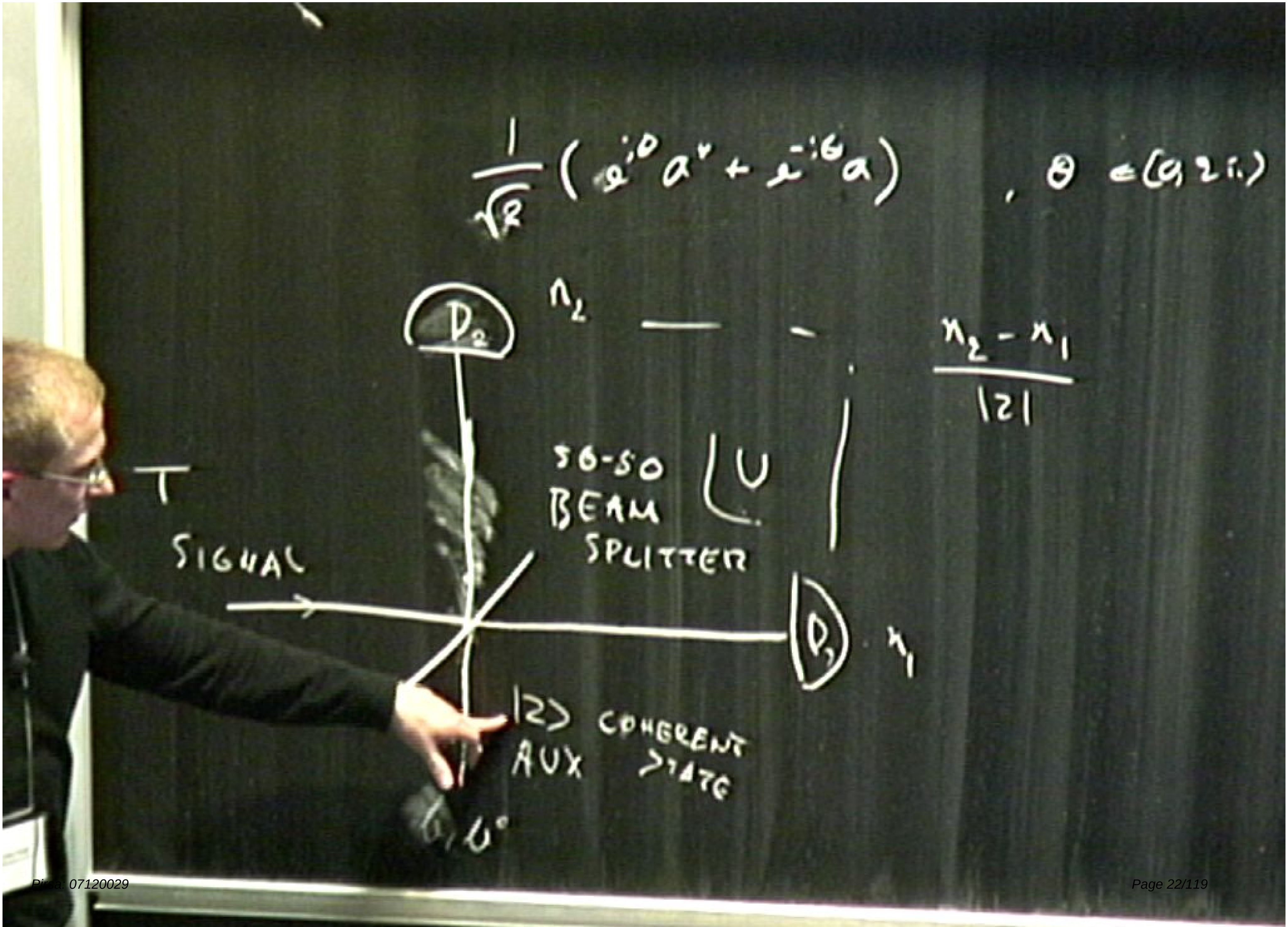



$\frac{1}{\sqrt{2}}(e^{i\theta}a^\dagger + e^{-i\theta}a)$, $\theta \in (0, 2\pi)$
D2
50-50
BEAM
SPLITTER
U
SIGNAL
D1

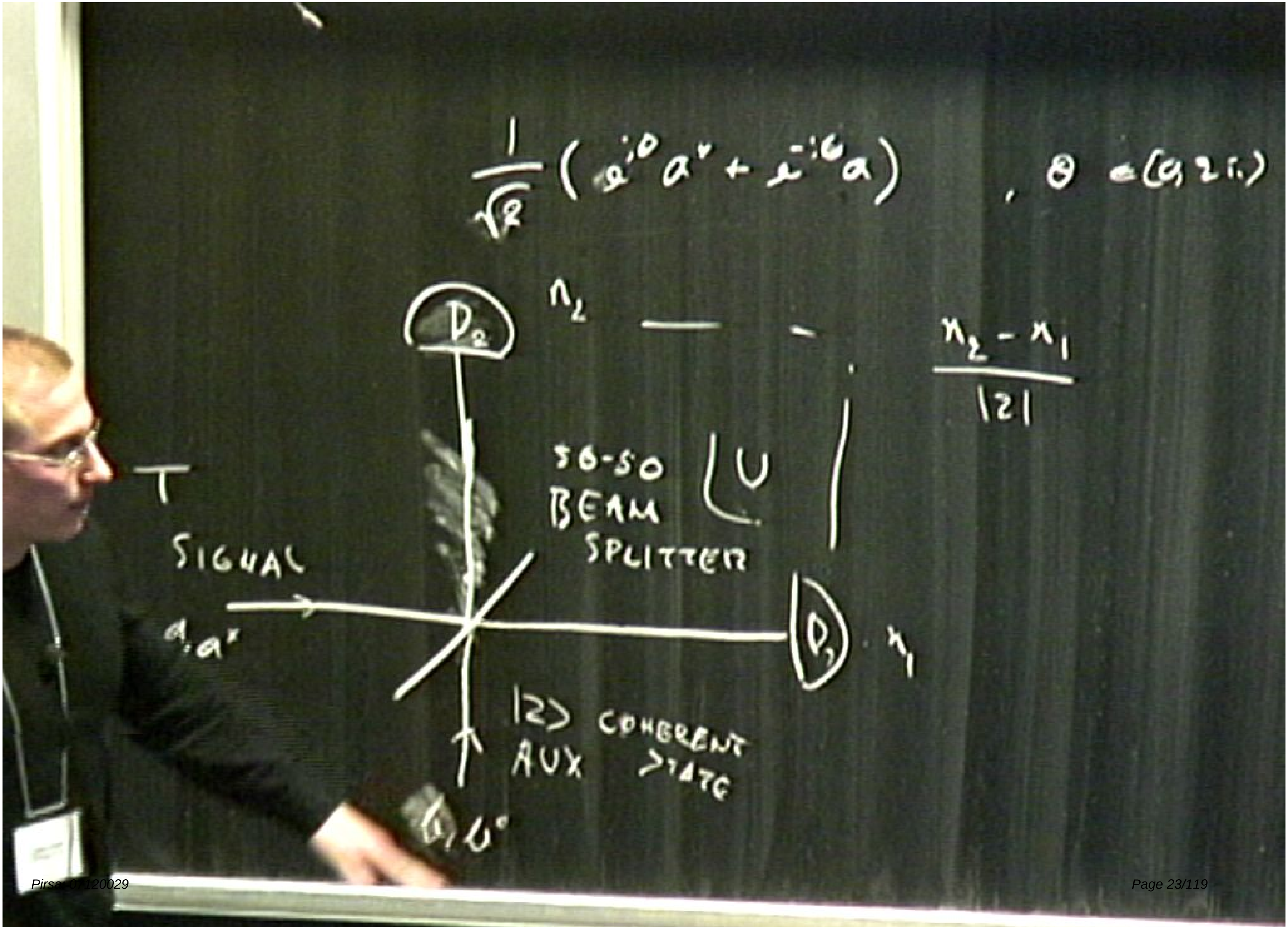

$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^{\dagger} + e^{-i\theta}a\right)$, $\theta \in [0, 2\pi)$
D2
50-50
BEAM
SPLITTER
U
SIGNAL
a, a†
D1
AUX
b, b†

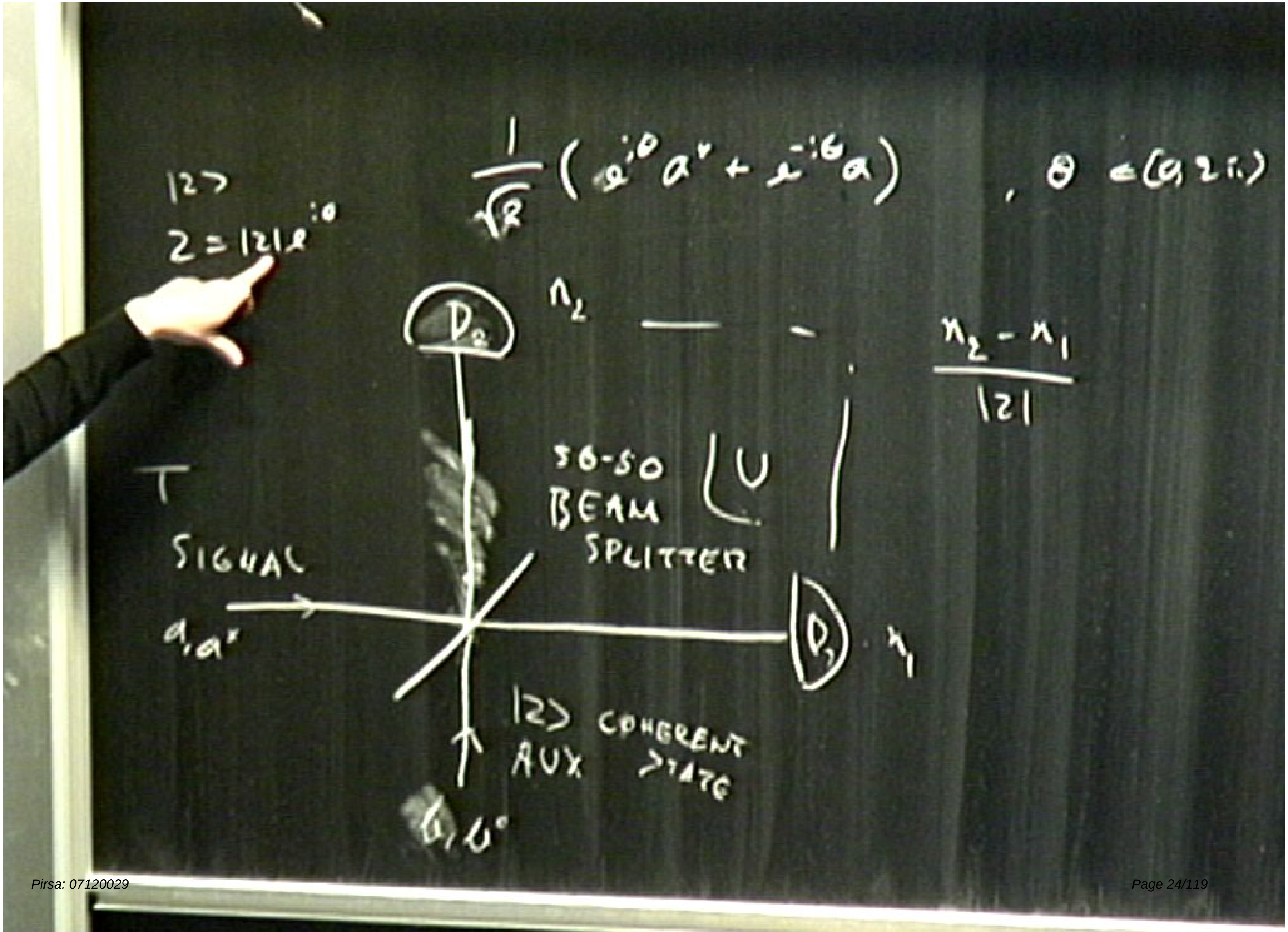

$\frac{1}{\sqrt{2}}(e^{i\theta}a^{\dagger} + e^{-i\theta}a)$, $\theta \in (0, 2\pi)$
D_2
50-50
BEAM
SPLITTER
U
T
SIGNAL
a, a†
D_1
|z> COHERENT
AUX STATE
b, b†

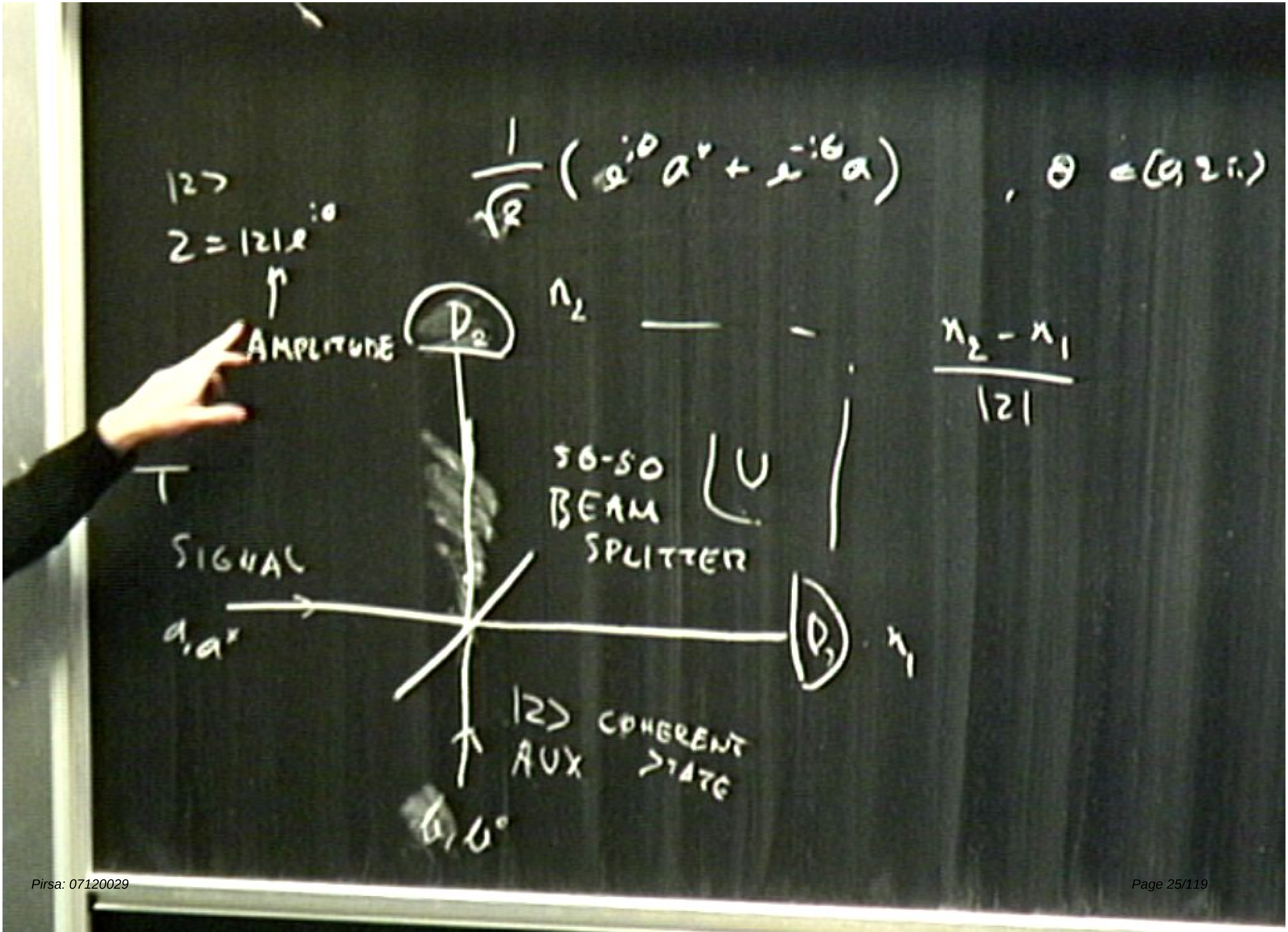

$\frac{1}{\sqrt{2}}\left(e^{i\theta}a^\dagger + e^{-i\theta}a\right)$, $\theta \in [0, 2\pi)$
n_2
D_2
50-50
BEAM
SPLITTER
U
T
SIGNAL
$a, a^\dagger$
D_1
n_1
$|z\rangle$ COHERENT
AUX STATE
$b, b^\dagger$



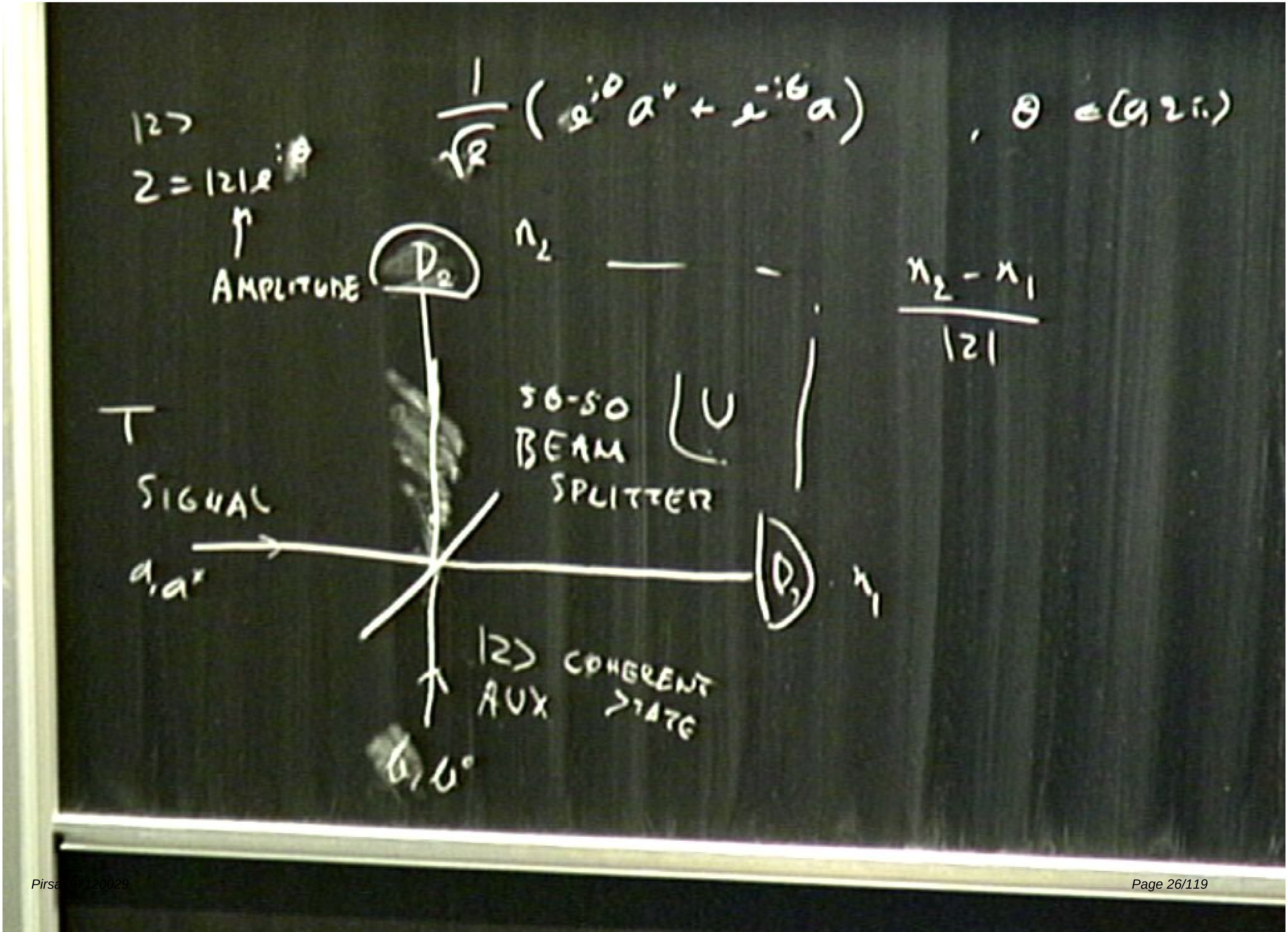
1/√2 (e^{iθ} a† + e^{-iθ} a) , θ ∈ (0, 2π)
D2
n2
(n2 - n1)/|z|
50-50
BEAM
SPLITTER
U
T
SIGNAL
D1
n1
|z> COHERENT
AUX STATE

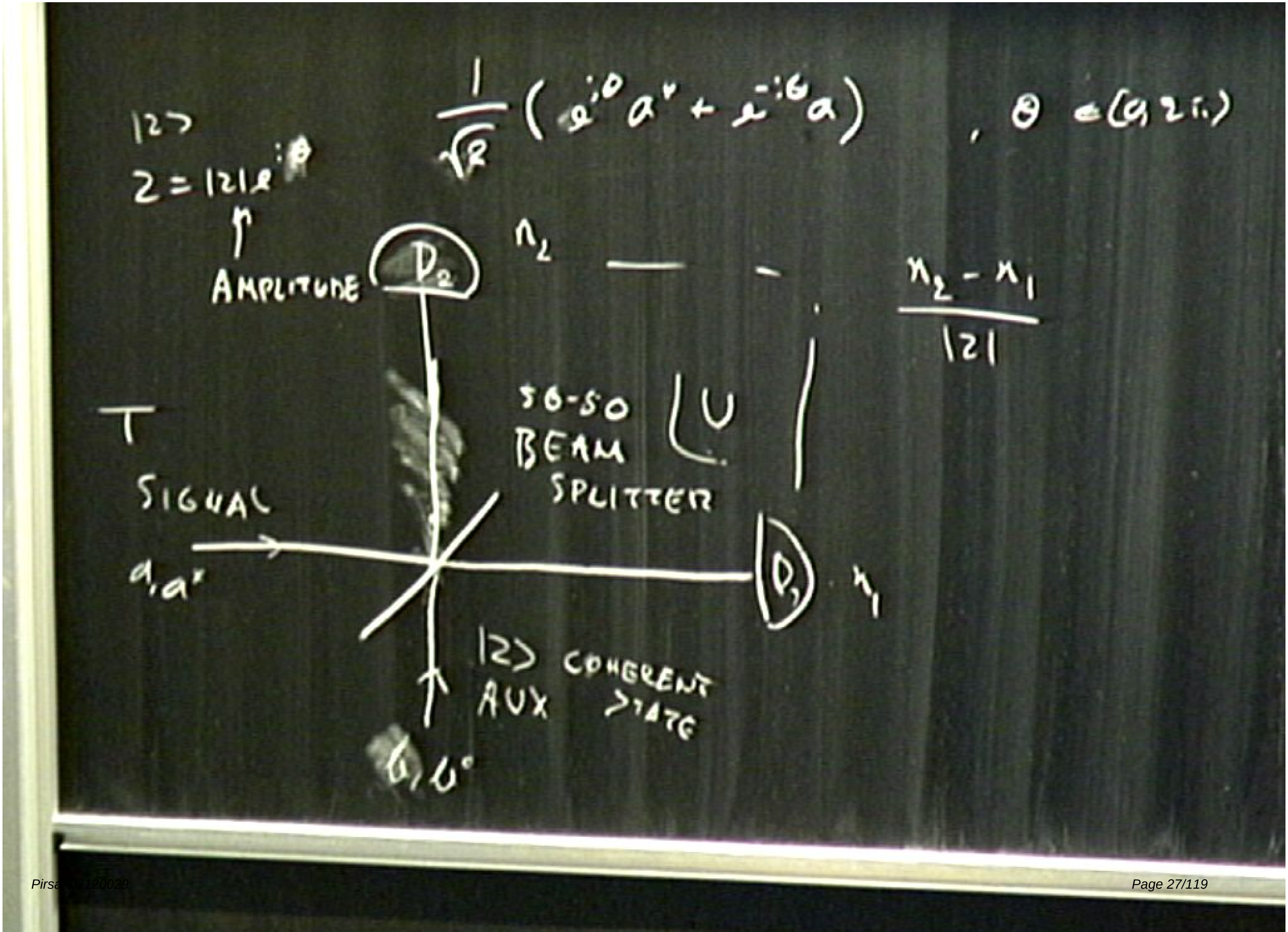

T
SIGNAL
50-50
BEAM
SPLITTER
U
D2
D1
n2
n1
|z> COHERENT
AUX STATE


|z>
z = |z|e^{iθ}
1/√2 (e^{iθ} a† + e^{-iθ} a) , θ ∈ (0, 2π)
D2
n2
n2 - n1 / |z|
50-50 BEAM SPLITTER
U
SIGNAL
a, a†
D1
n1
|z> COHERENT AUX STATE
b, b†



|z>
z = |z| e^{iθ}
AMPLITUDE
SIGNAL
a, a†
D2
n2
50-50
BEAM
SPLITTER
U
D1
n1
|z> COHERENT
AUX STATE
b, b†
θ ∈ (0, 2π)


|z⟩
z = |z|e^{iθ}
AMPLITUDE
1/√2 (e^{iθ} a† + e^{-iθ} a) , θ ∈ (0, 2π)
n_2
(n_2 - n_1)/|z|
D_2
50-50
BEAM
SPLITTER
SIGNAL
a, a†
D_1
n_1
|z⟩ COHERENT
AUX STATE
b, b†



|z>
z = |z| e^{iθ}
AMPLITUDE
1/√2 (e^{iθ} a† + e^{-iθ} a) , θ ∈ (0, 2π)
D2
n2
(n2 − n1)/|z|
50-50
BEAM
SPLITTER
U
SIGNAL
a, a†
D1
n1
|z> COHERENT
AUX STATE
b, b†

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The balanced homodyne detection

- The signal field (with annihilation operator a) is coupled via 50-50-beam splitter with the auxiliary field (b), which is in a coherent state $|z\rangle$.
- Detection observable is the amplitude-scaled photon difference $D_z := \frac{1}{|z|\sqrt{2}}(\overline{I \otimes b^* b - a^* a \otimes I})$.
- The beam splitter, a unitary operator U, transforms D_z into

$$U^* D_z U = \frac{1}{\sqrt{2}|z|}(\overline{a \otimes b^* + a^* \otimes b}) =: |z|^{-1} A$$

- Heuristic argument: When the amplitude $|z|$ is large, the auxiliary field is classical; $b \sim |z| e^{i\theta}$. Then $|z|^{-1} A \sim \frac{1}{\sqrt{2}}(\overline{e^{-i\theta} a + e^{i\theta} a^*})$, so quadrature is measured.

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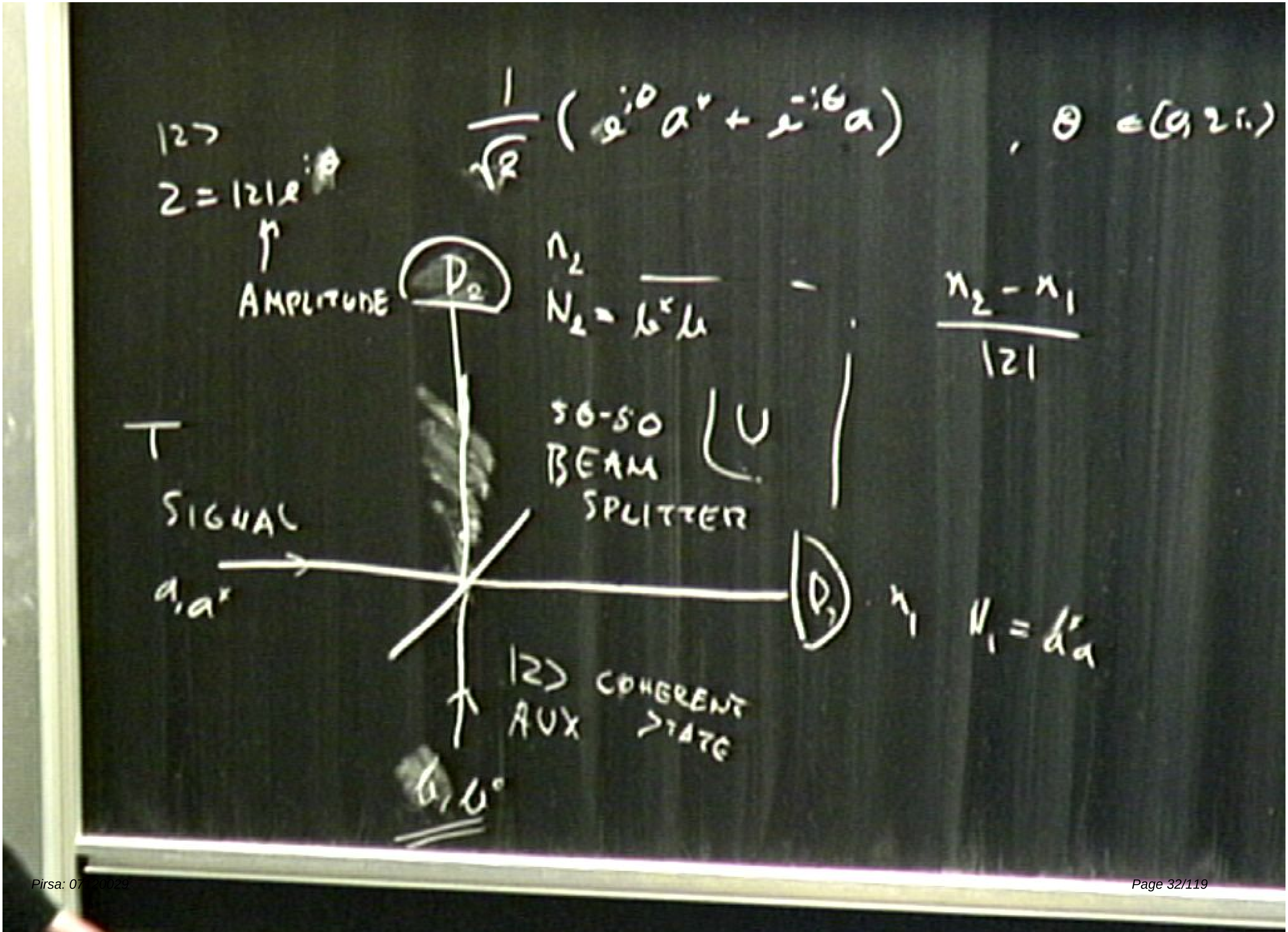
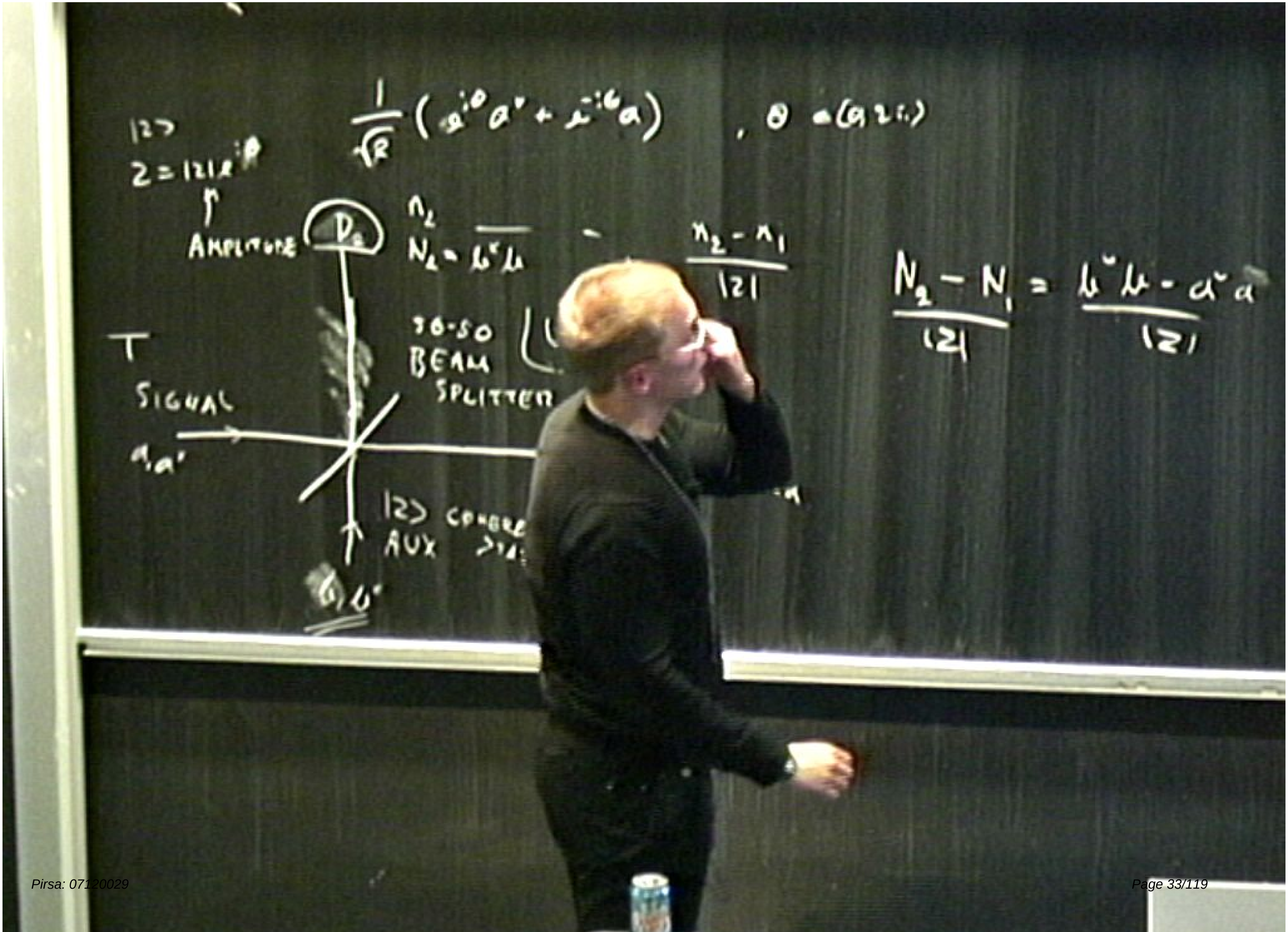
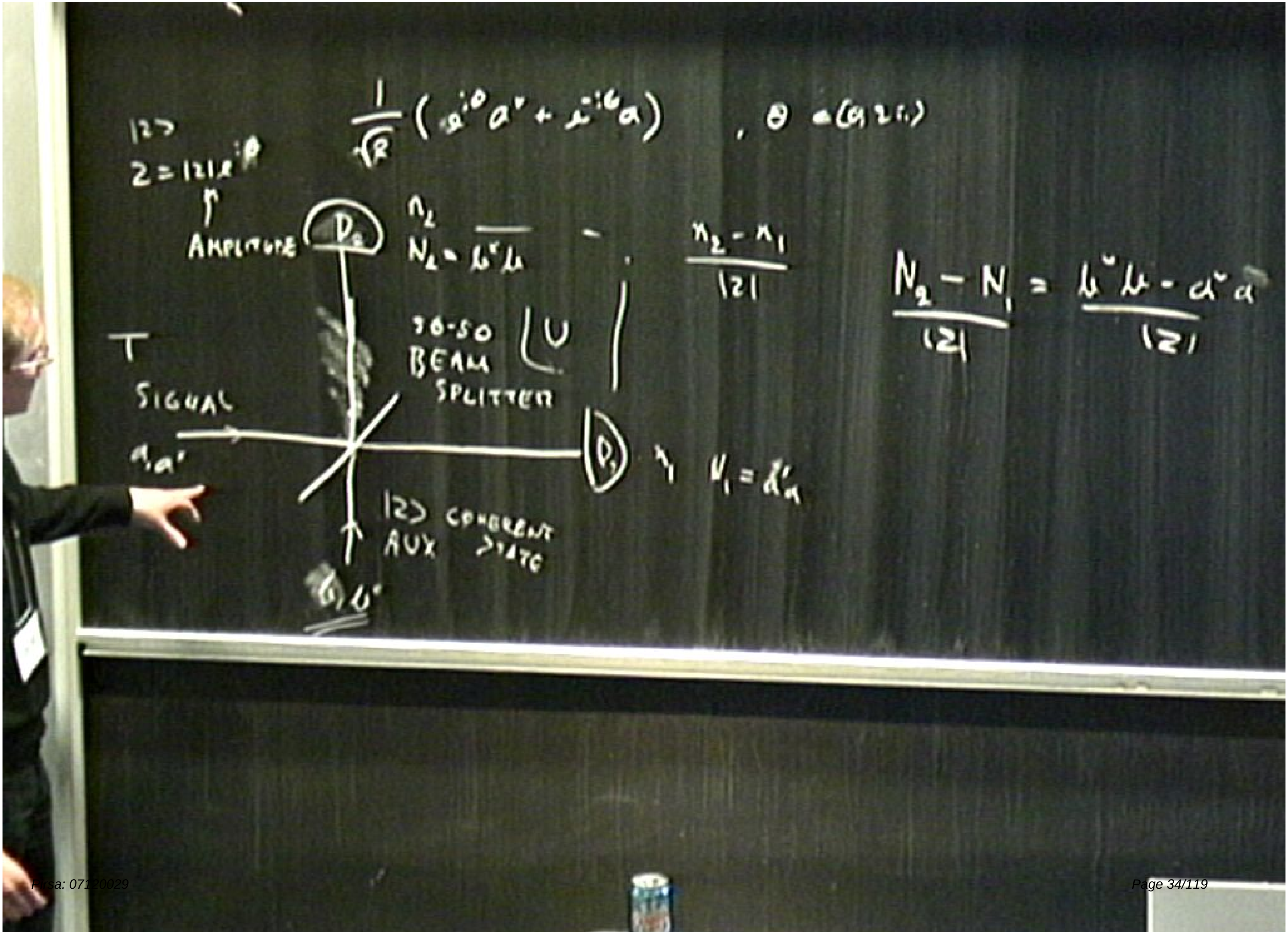

|z⟩
z = |z| e^{iθ}
AMPLITUDE
D2
n2
N2 = b†b
50-50
BEAM
SPLITTER
SIGNAL
a, a†
D1
n1
N1 = b†a
(n2 − n1) / |z|
|z⟩ COHERENT
AUX STATE
b, b†
T


|z>
AMPLITUDE
50-50
BEAM
SPLITTER
SIGNAL
|z> COHERE
AUX
N2 − N1 = b†b − a†a
|z| |z|



|z>
z = |z|e^{iφ}
AMPLITUDE
n2
N2 = b†b
n2 - n1 / |z|
N2 - N1 / |z| = b†b - a†a / |z|
50-50
BEAM
SPLITTER
SIGNAL
a, a†
|z> COHERENT
AUX STATE
b, b†
N1 = a†a

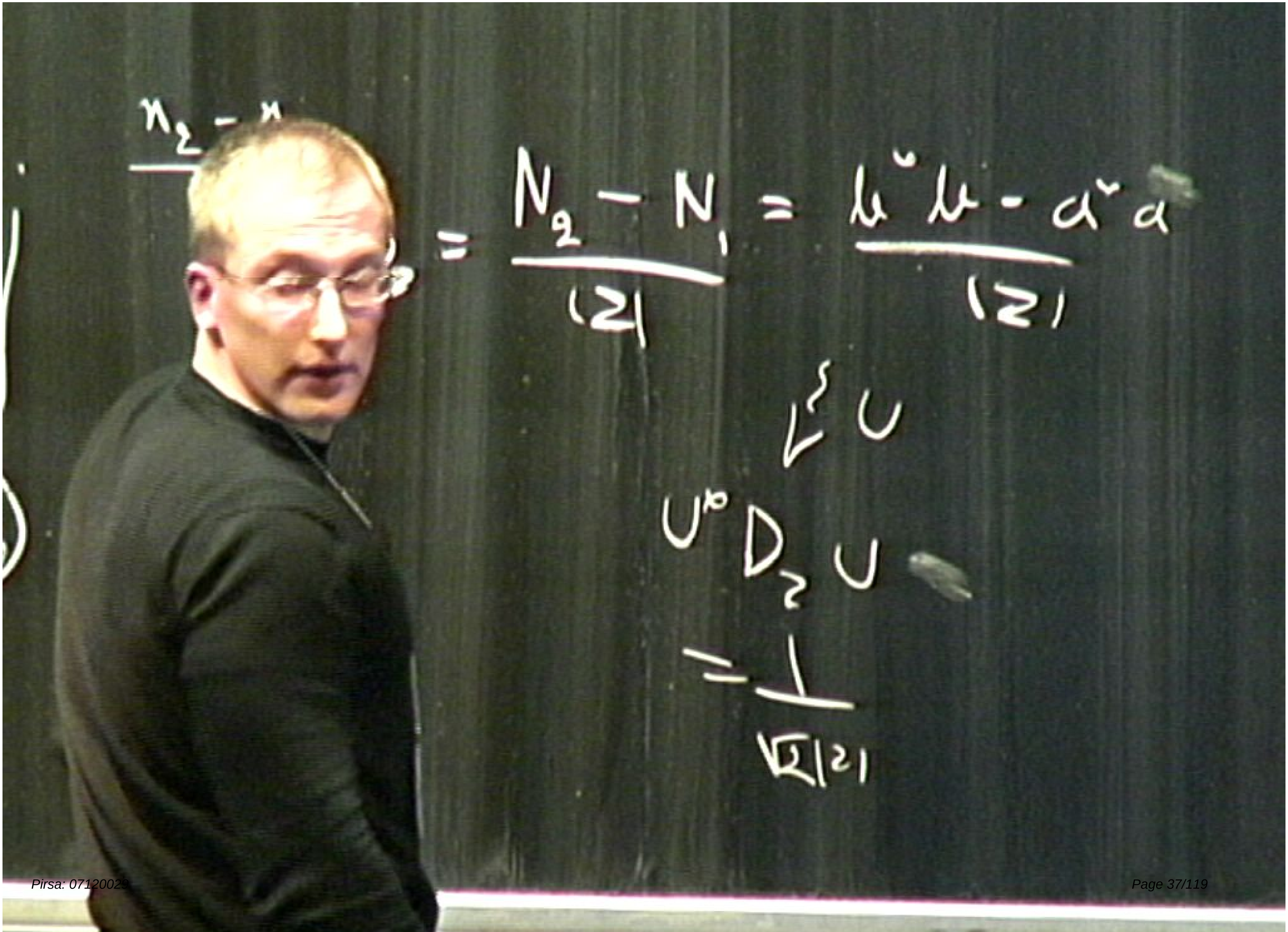
$$\frac{N_2 - N_1}{2} = \frac{b^* b - a^* a}{2}$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

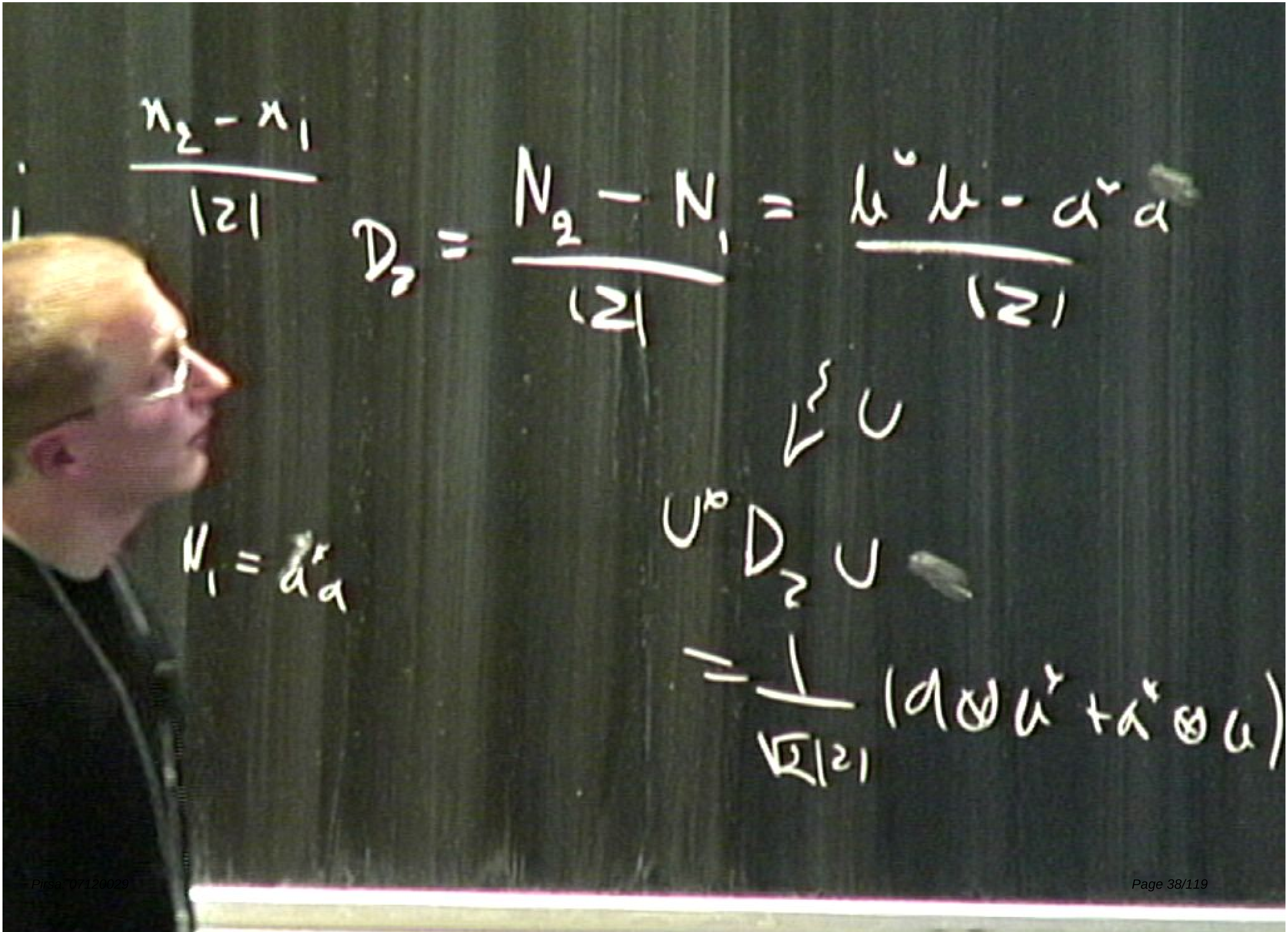
$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

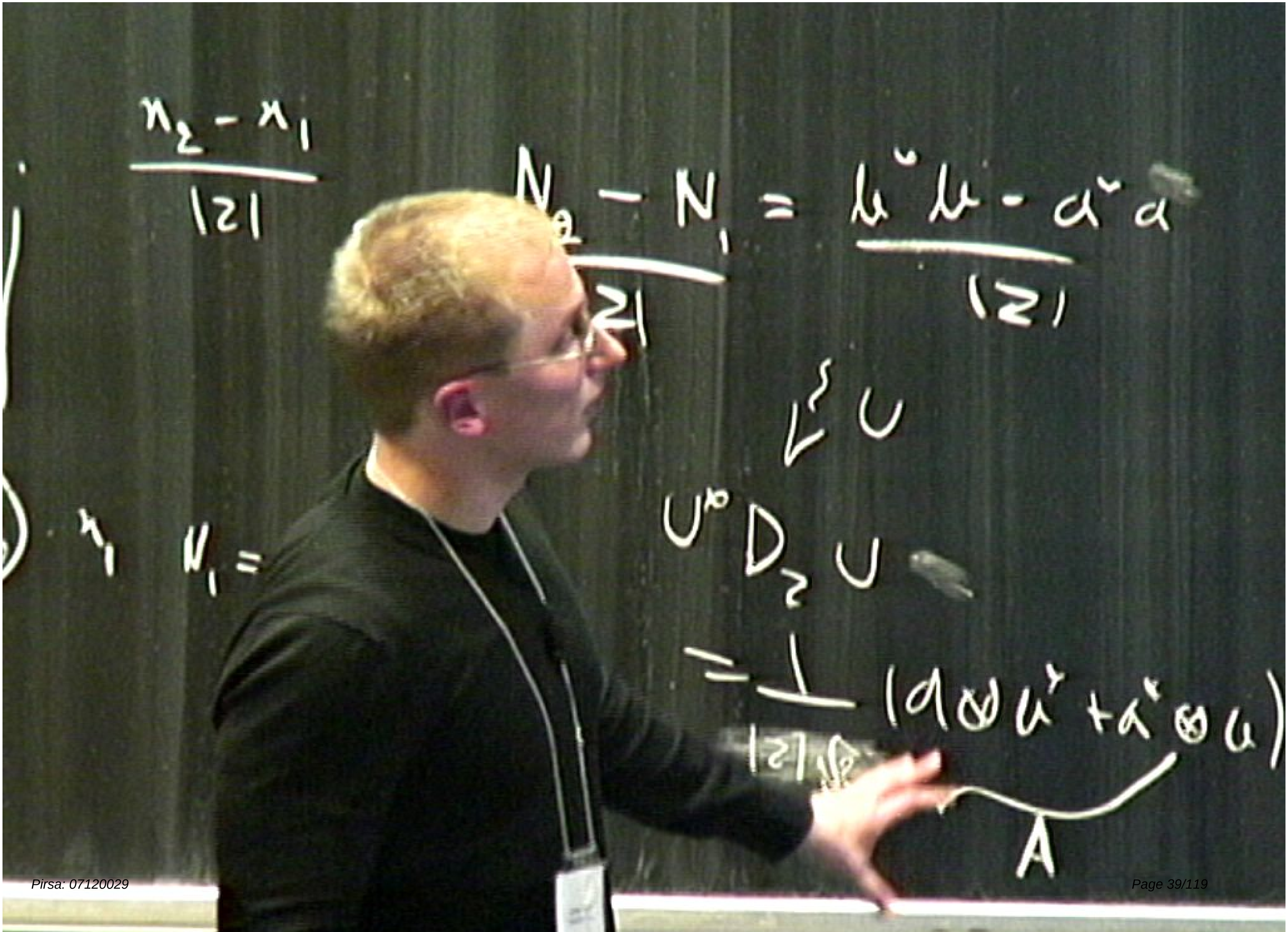
$$\theta \in [0, 2\pi)$$

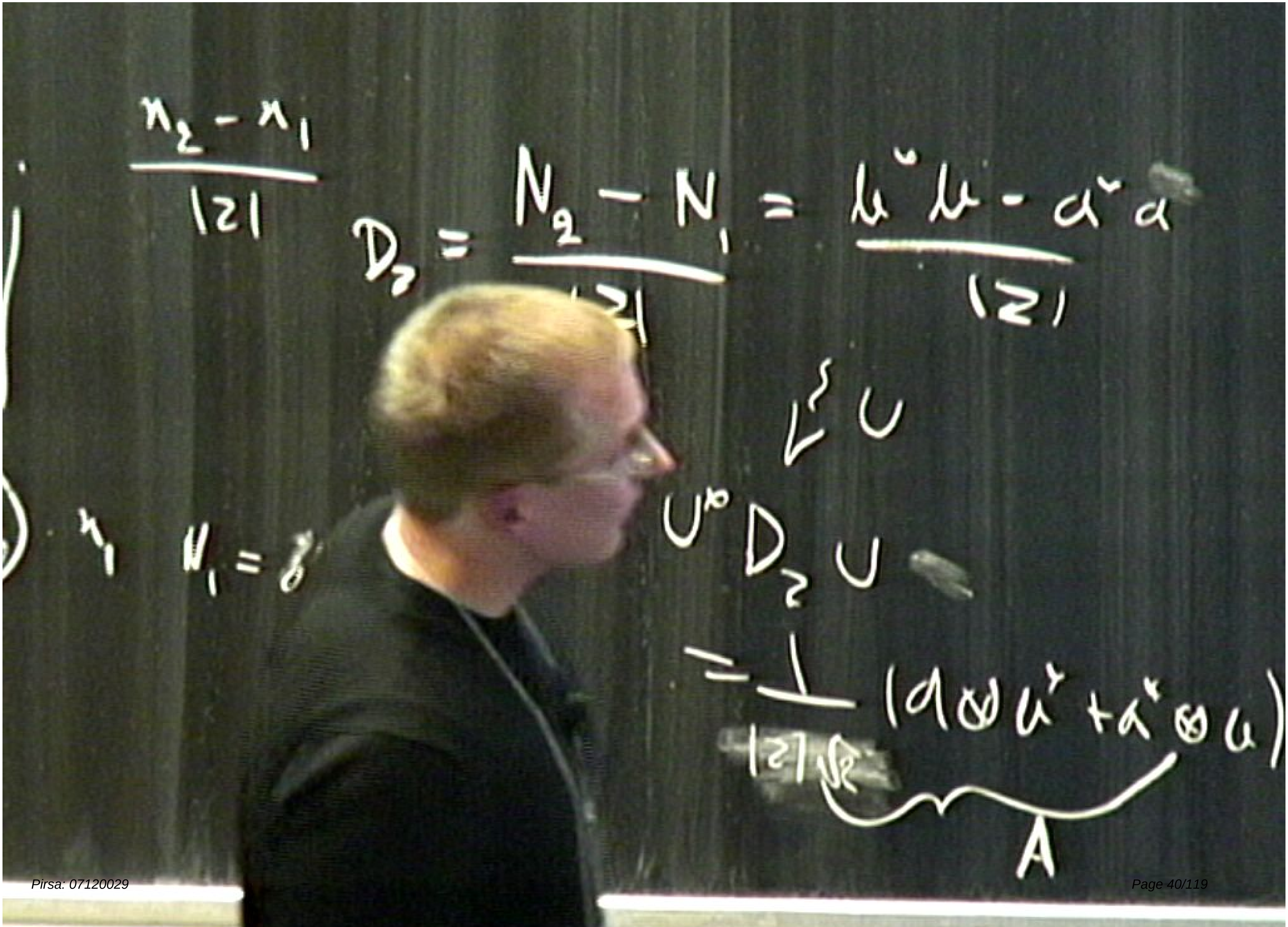
$$\frac{x_2 - x_1}{\;}$$



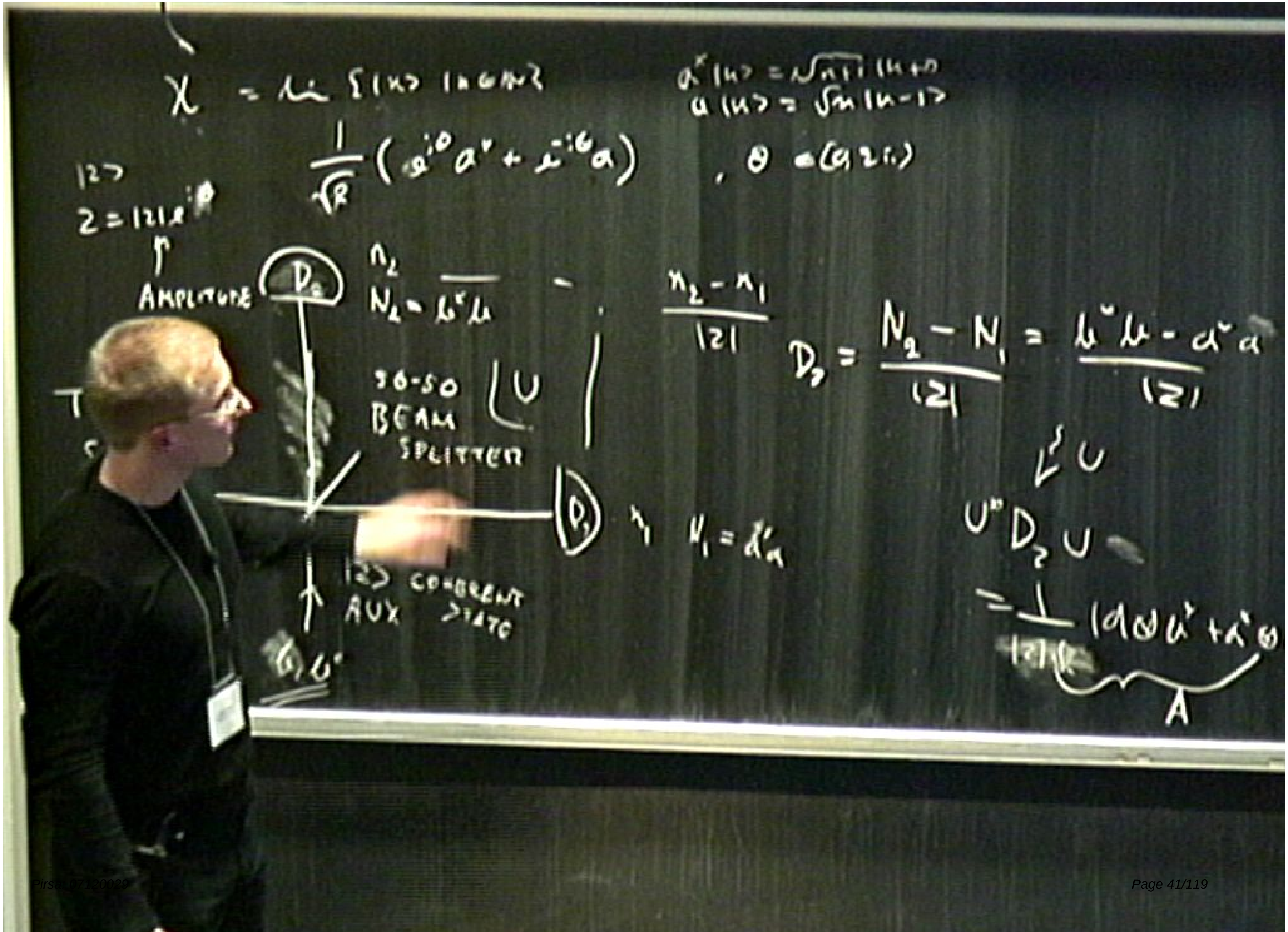
$= \frac{N_2 - N_1}{|z|} = \frac{b^* b - a^* a}{|z|}$
$U^* D_2 U$
$= \frac{1}{\sqrt{2}|z|}$

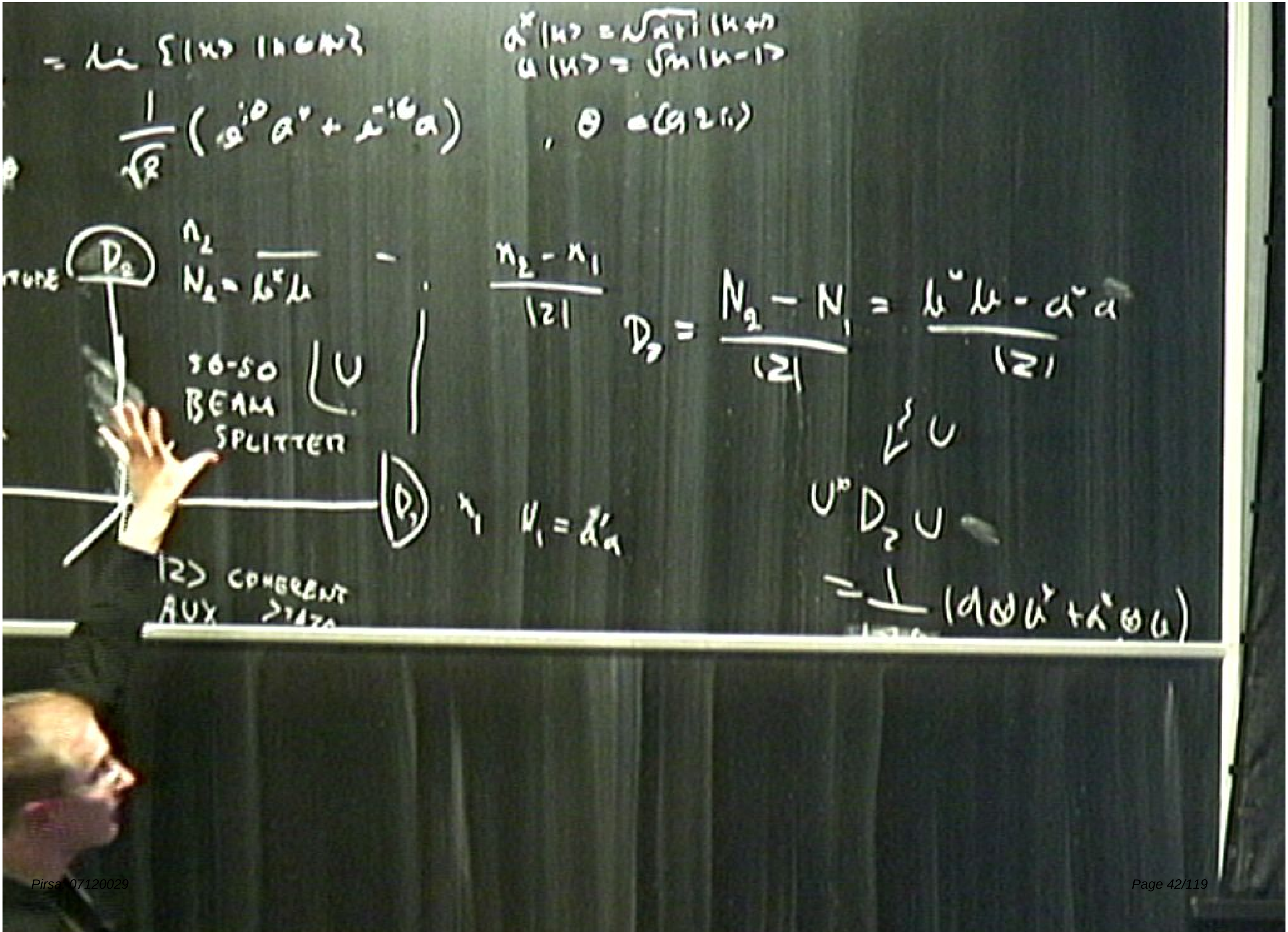

x_2 - x_1 / |z|
D_z = (N_2 - N_1) / |z| = (b*b - a*a) / |z|
U
N_1 = a*a
U* D_z U
= 1/(√2 |z|) (a ⊗ a* + a* ⊗ a)

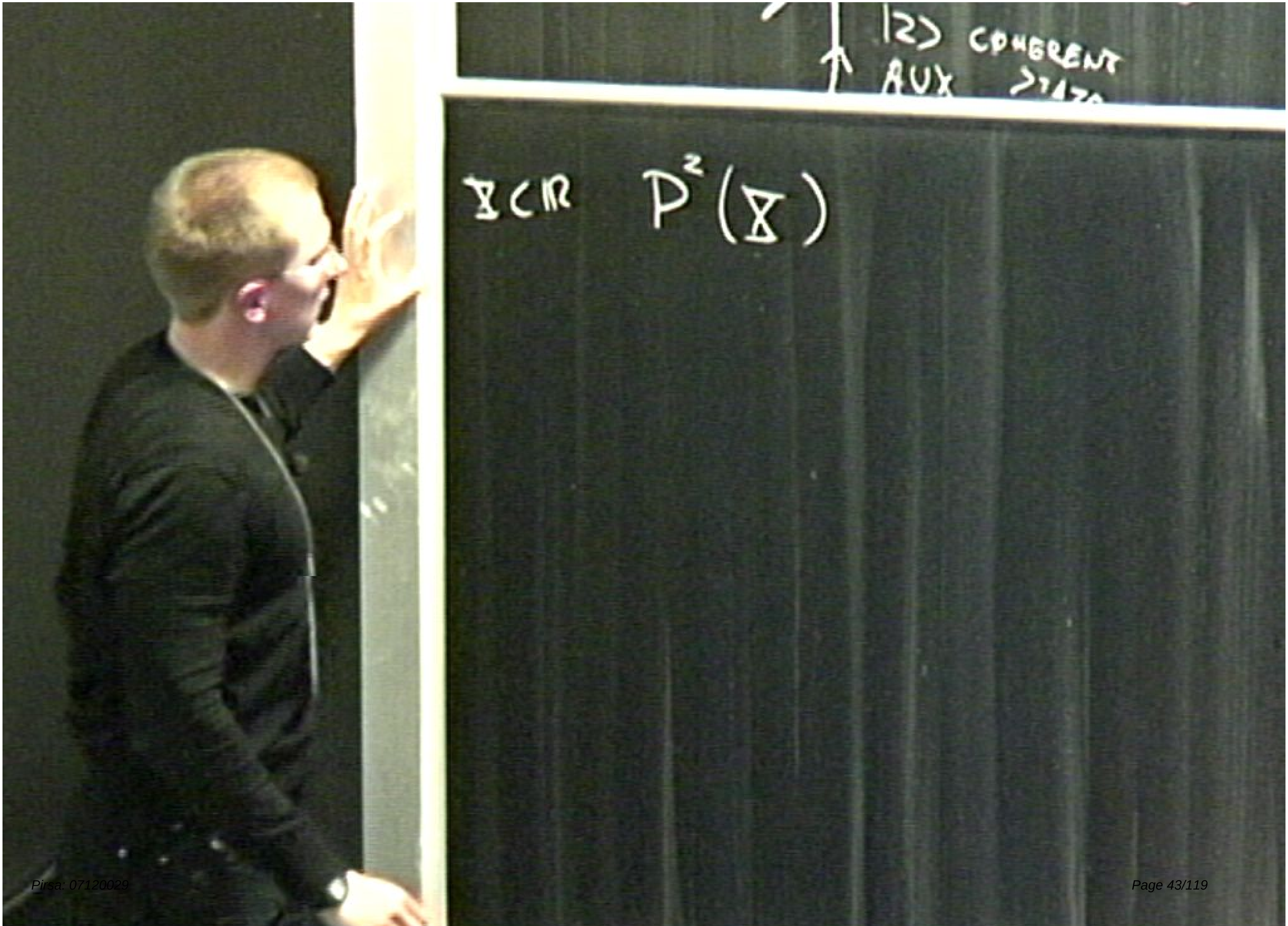

$\frac{x_2 - x_1}{|z|}$
$N_2 - N_1 = \frac{b^* b - a^* a}{|z|}$
$U^* D_2 U$
A

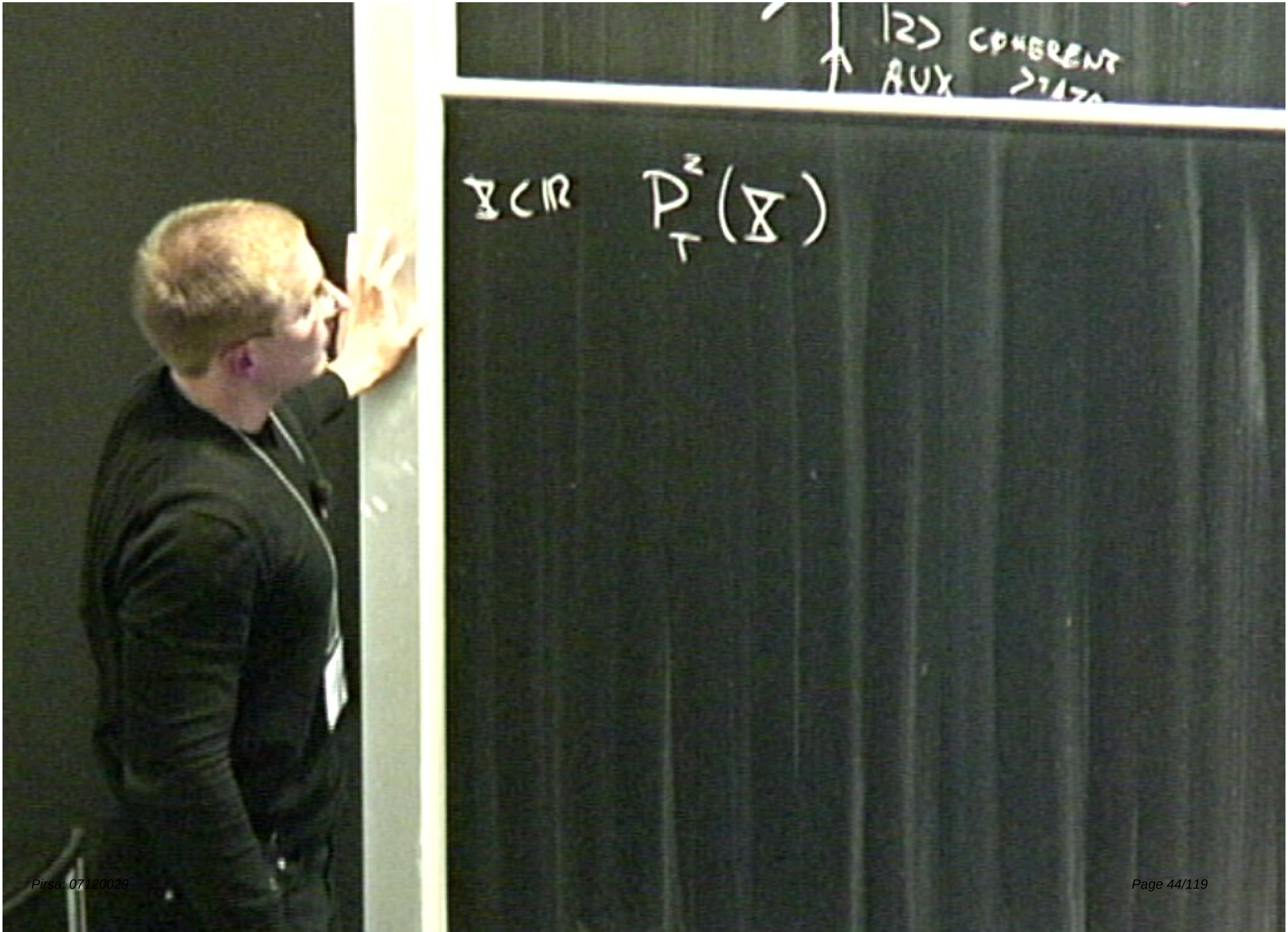


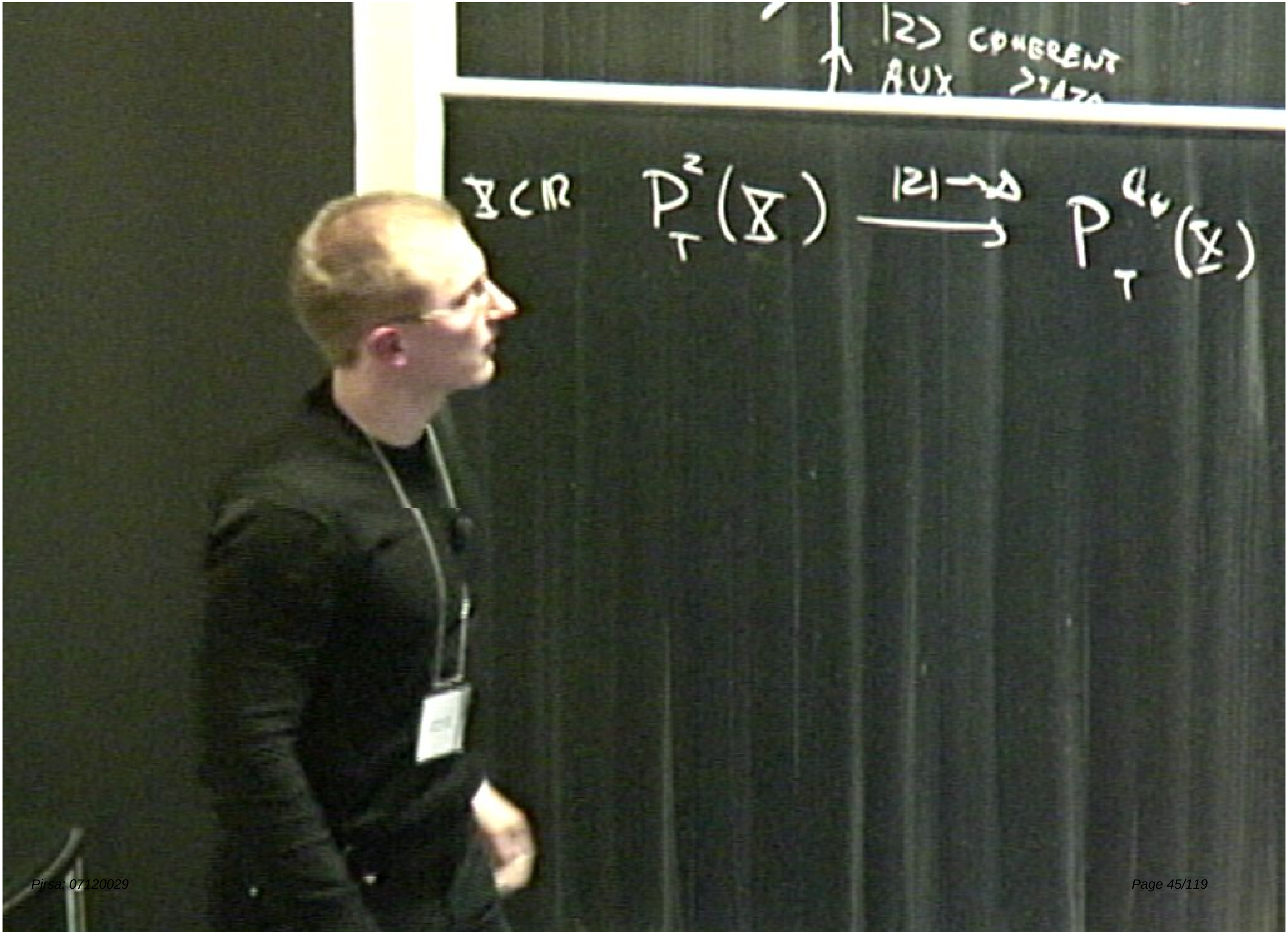
x_2 - x_1 / |z|
D_2 = N_2 - N_1 / |z| = b*b - a*a / |z|
N_1 = a*a
U* D_2 U
= 1/(|z|√2) (a⊗a* + a*⊗a)
A


AMPLITUDE
50-50 BEAM SPLITTER
COHERENT AUX STATE
A


a* |n> = √(n+1) |n+1>
a |n> = √n |n-1>
50-50
BEAM
SPLITTER
N₂ = b*b
N₁ = a*a
D₂ = (N₂ - N₁)/|z| = (b*b - a*a)/|z|
U* D₂ U
|z> COHERENT
AUX


|z> COHERENT
AUX
P²(X)


|z> COHERENT
AUX
X ⊂ R
P_T^2(X)

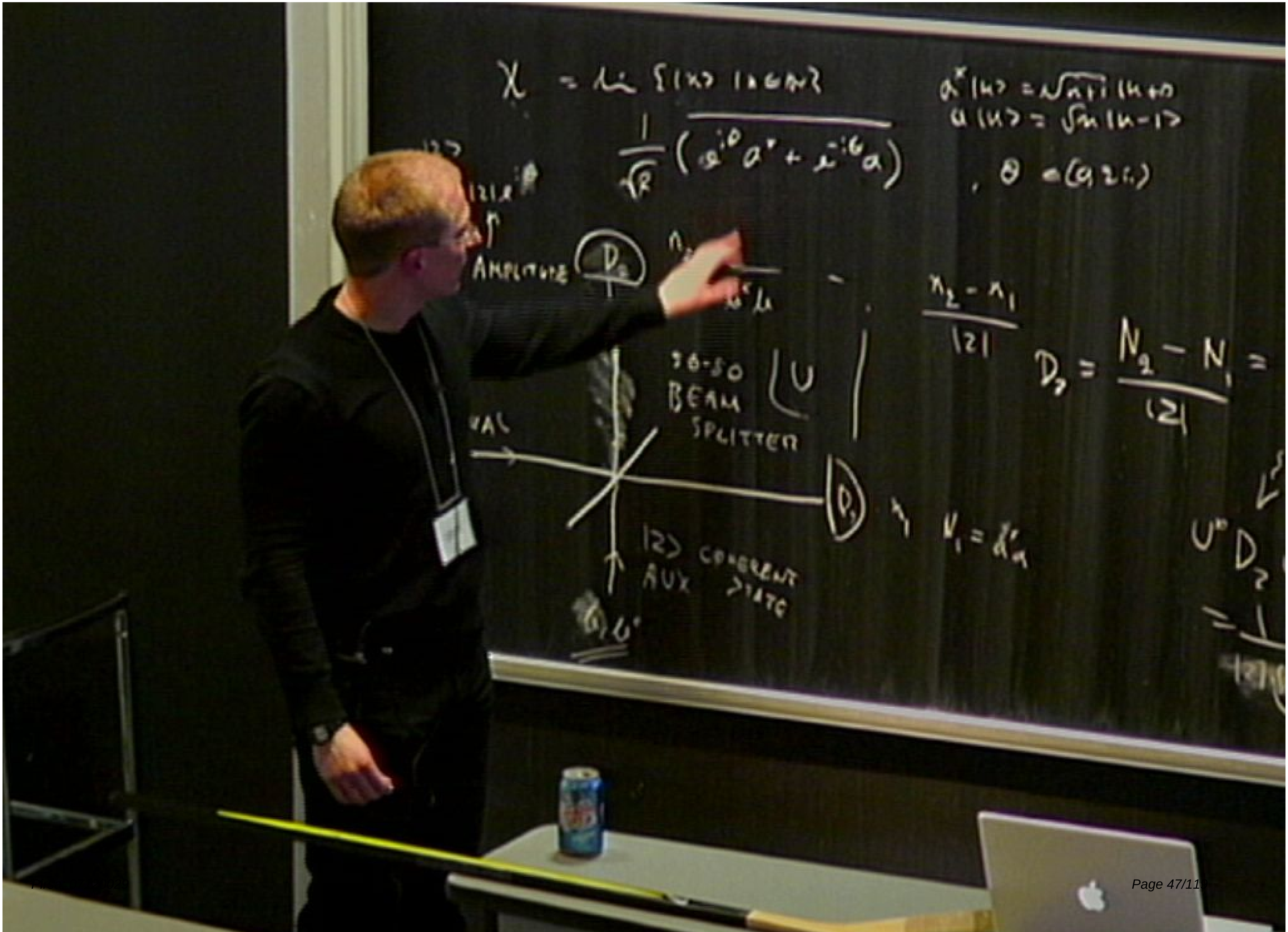

|z> COHERENT
AUX
X ⊂ R   P_T^2(X) —|z|→∞→ P_T^{q_u}(X)

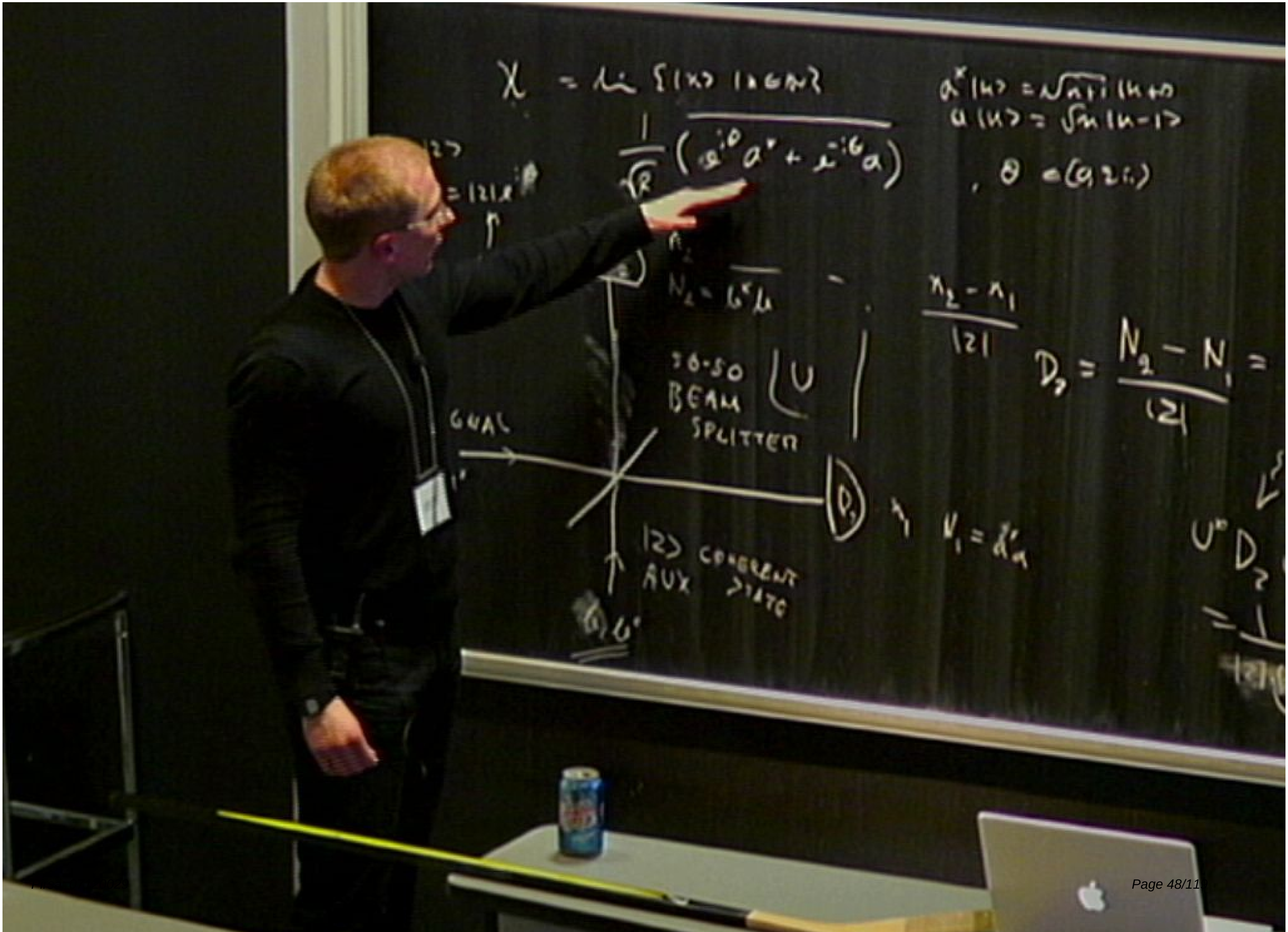
Some notations (and standard conventions)

Moment problem and homodyne detection

J. Kiukas
(with P. Lahti)
arXiv:0706.4436

- $\mathcal{H}$ - a complex separable Hilbert space.
- $L(\mathcal{H})$ - the set of bounded operators on $\mathcal{H}$.
- $\overline{A}$ - the closure of a symmetric operator A on $\mathcal{H}$.
- $\mathcal{B}(\mathbb{R})$ - the Borel σ-algebra of $\mathbb{R}$.
- Observable (or measurement) - a normalized POM (positive operator (valued) measure) $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$.
- State of a quantum system - positive operator $T \in L(\mathcal{H})$ of trace one (density operator).
- Measurement outcome statistics of E in state T - the probability measure $X \mapsto \mathrm{Tr}[TE(X)]$.
- $P^A : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ - the spectral measure of a selfadjoint operator A (projection valued measurement).


AMPLITUDE
50-50
BEAM
SPLITTER
|z> COHERENT
AUX STATE


50-50
BEAM
SPLITTER
COHERENT
AUX
STATE

Some notations (and standard conventions)

Moment problem and homodyne detection

J. Kiukas (with P. Lahti) arXiv:0706.4438

Introduction

Balanced homodyne detector observables and their moment operators

Limit of strong auxiliary field

An application to eight-port homodyne detector

- $\mathcal{H}$ - a complex separable Hilbert space.
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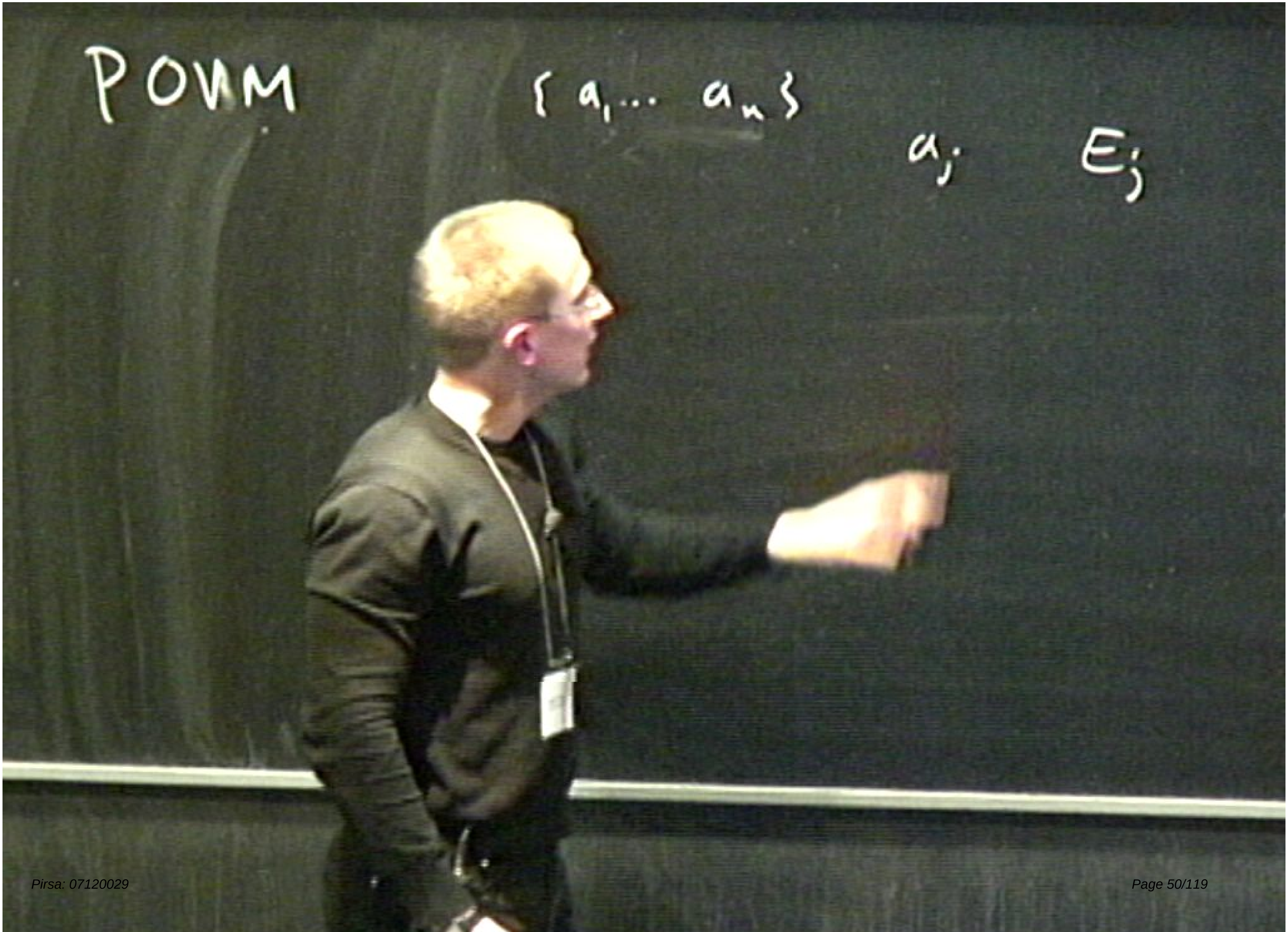
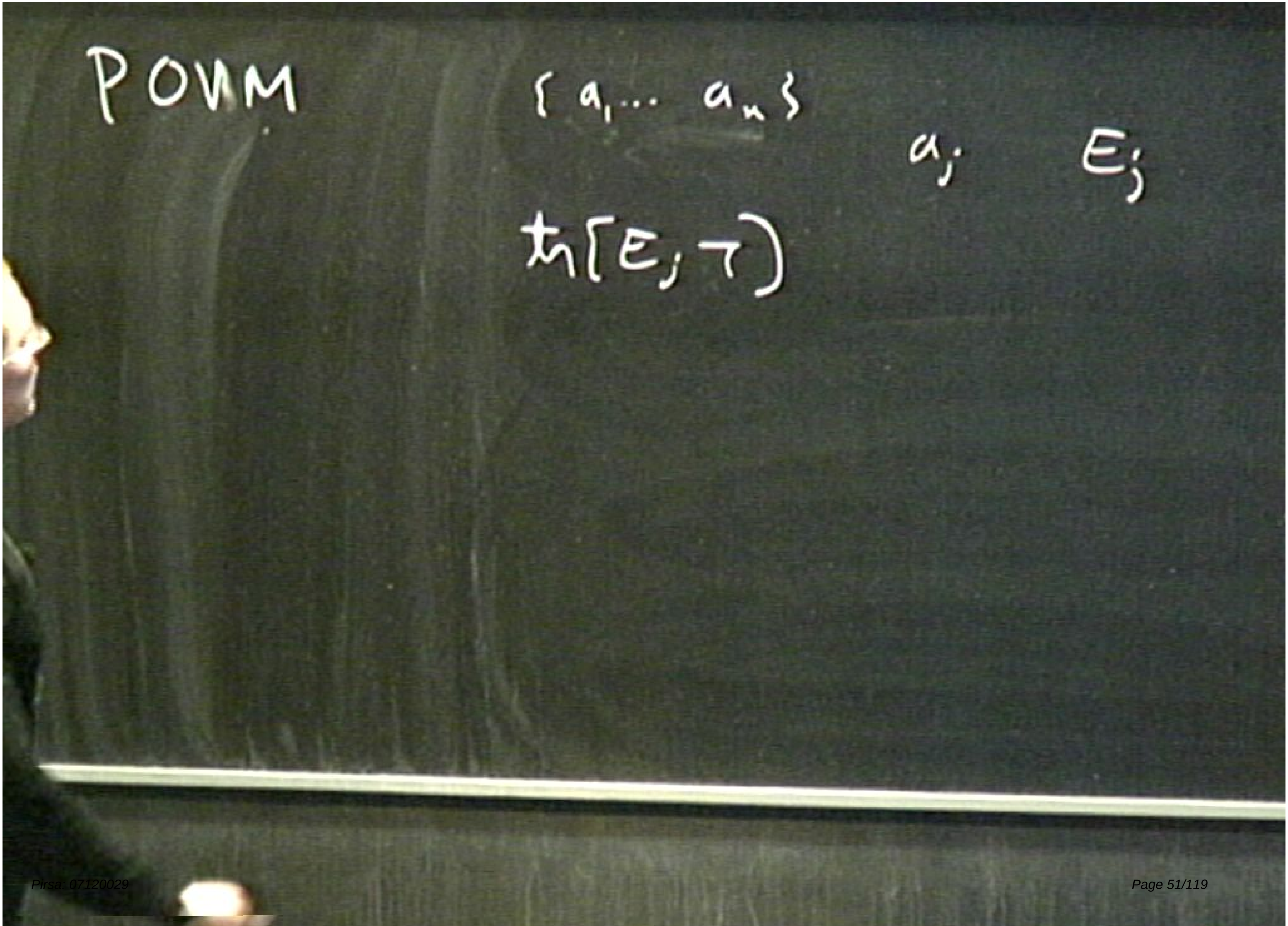
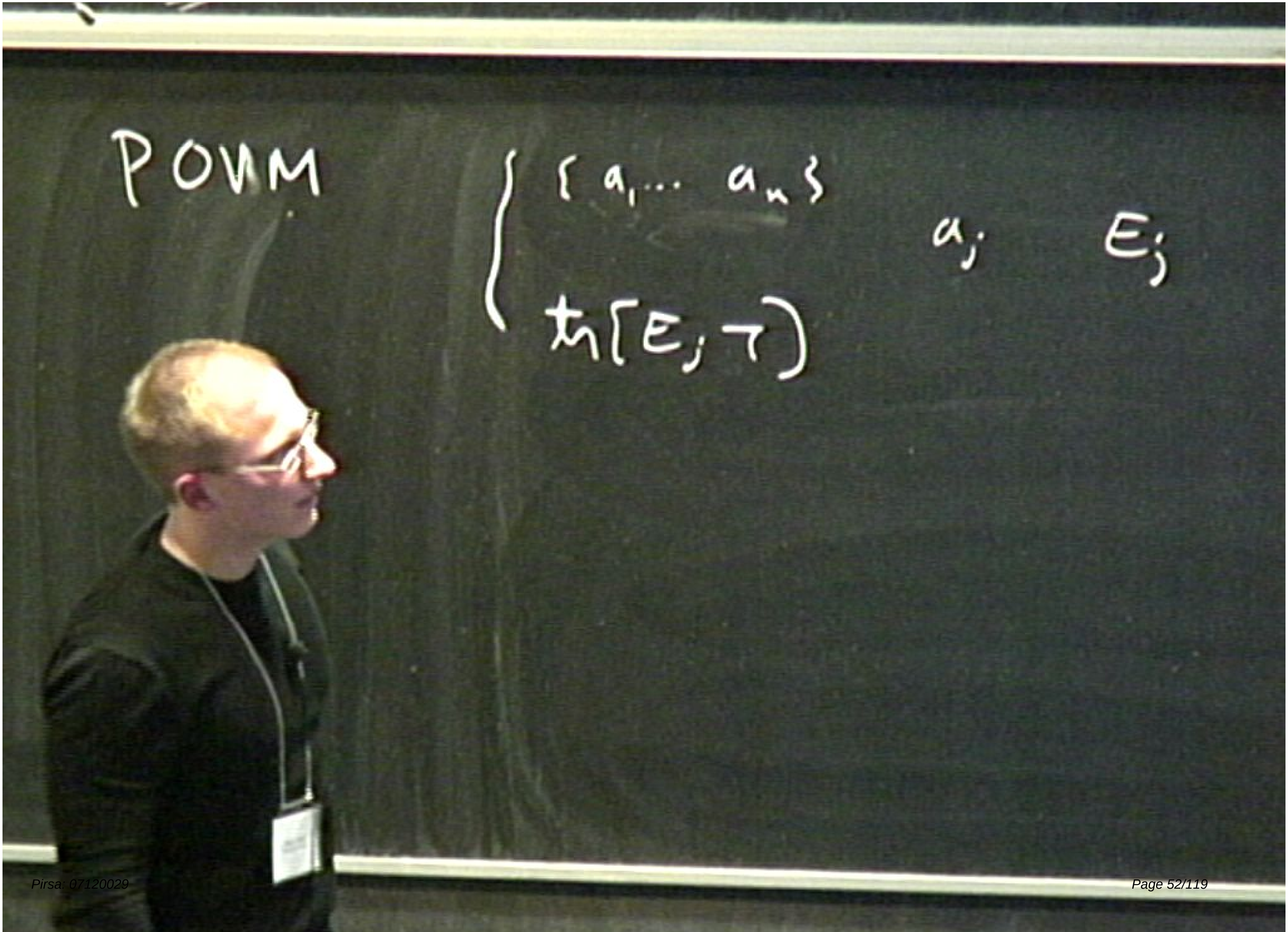
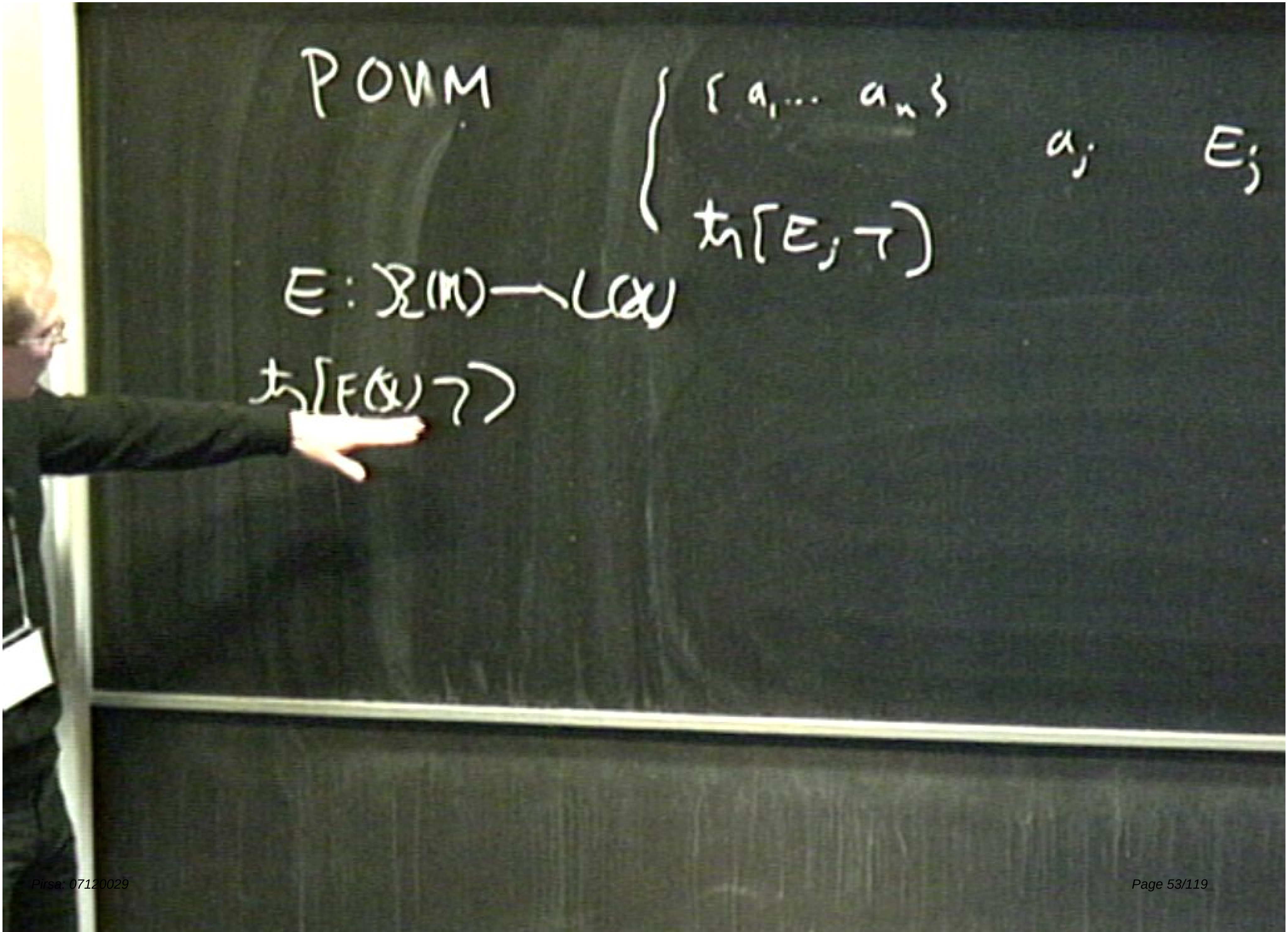
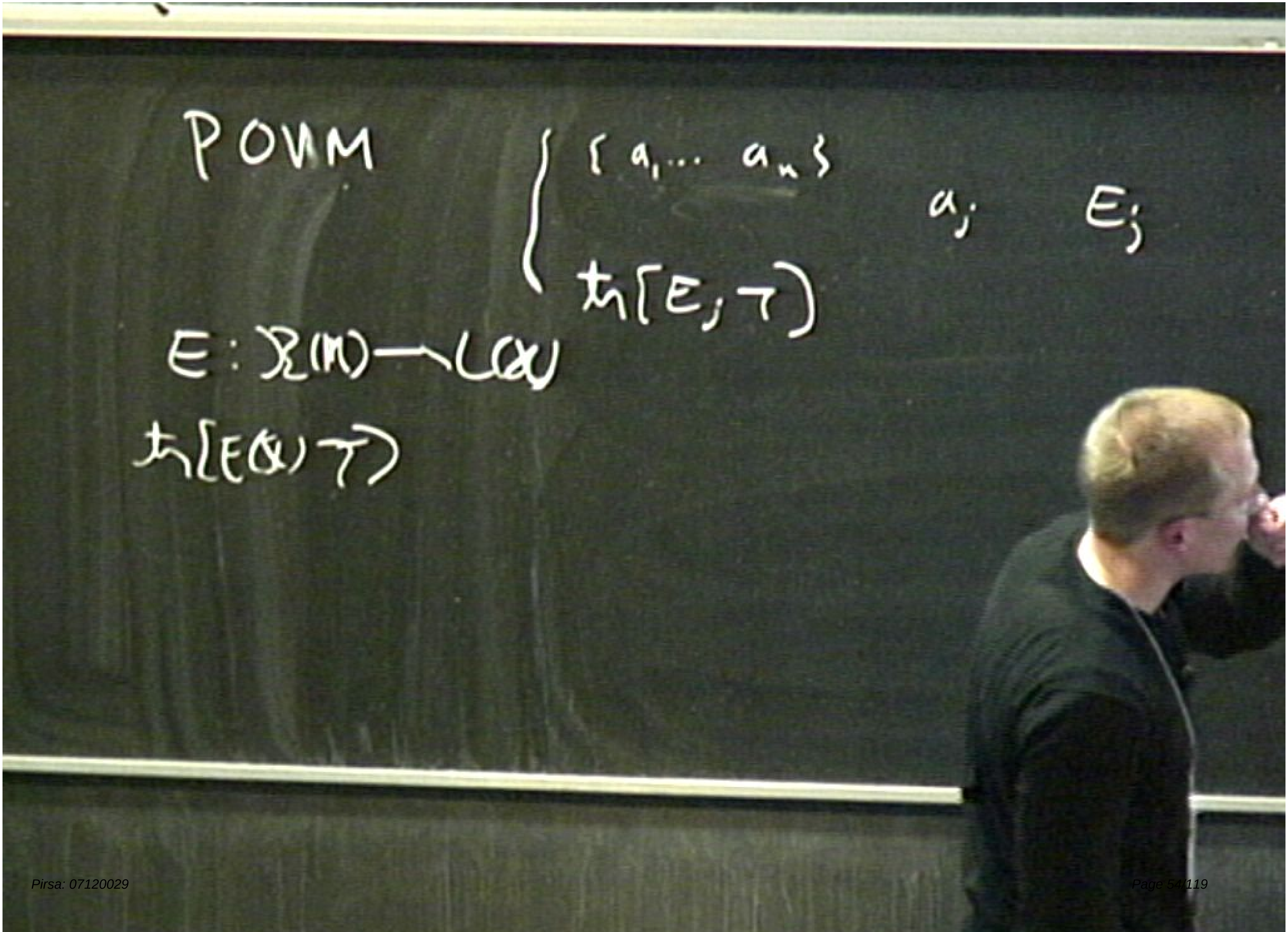
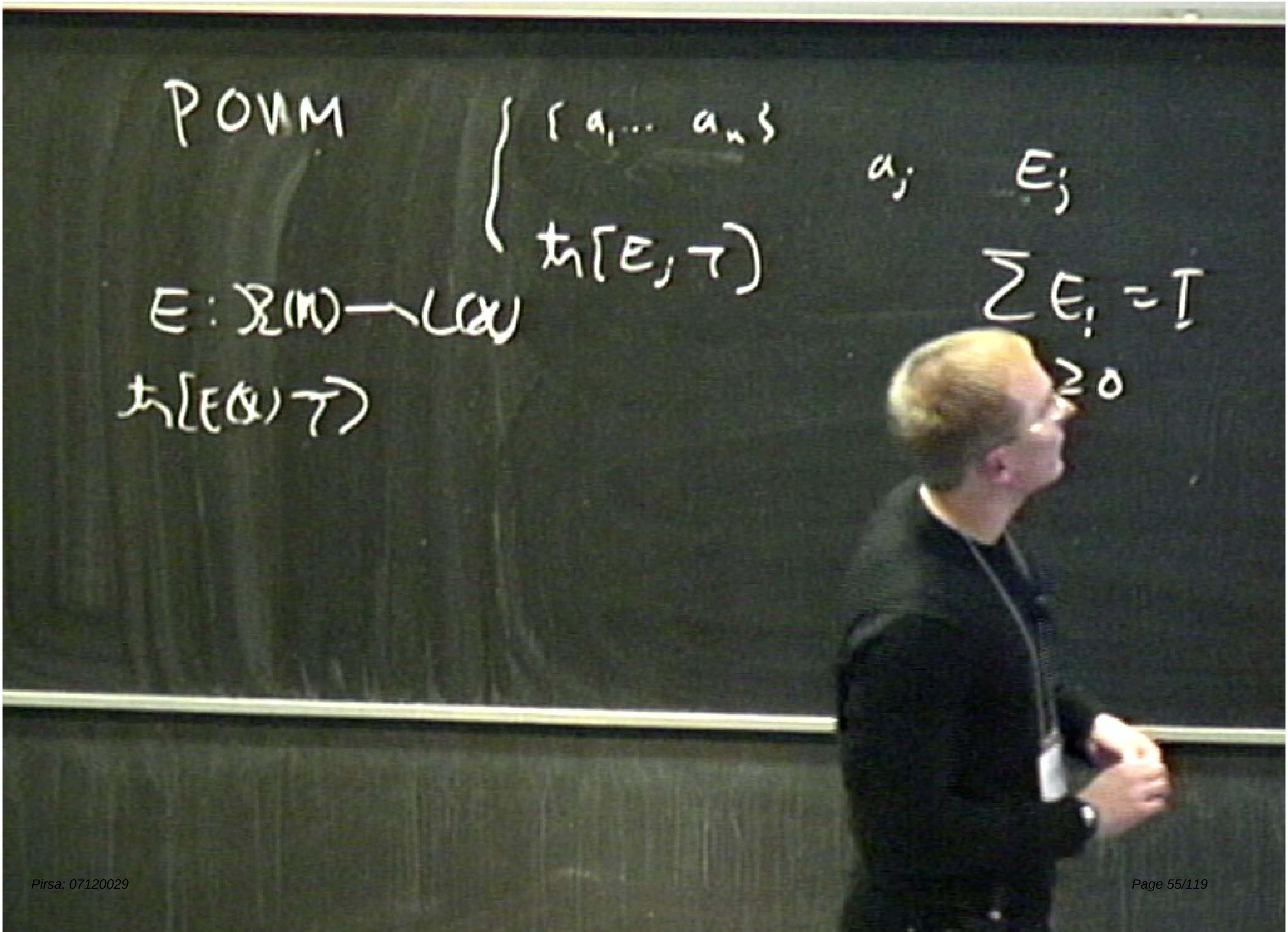

POVM
$\{a_1 \ldots a_n\}$
a_j
E_j

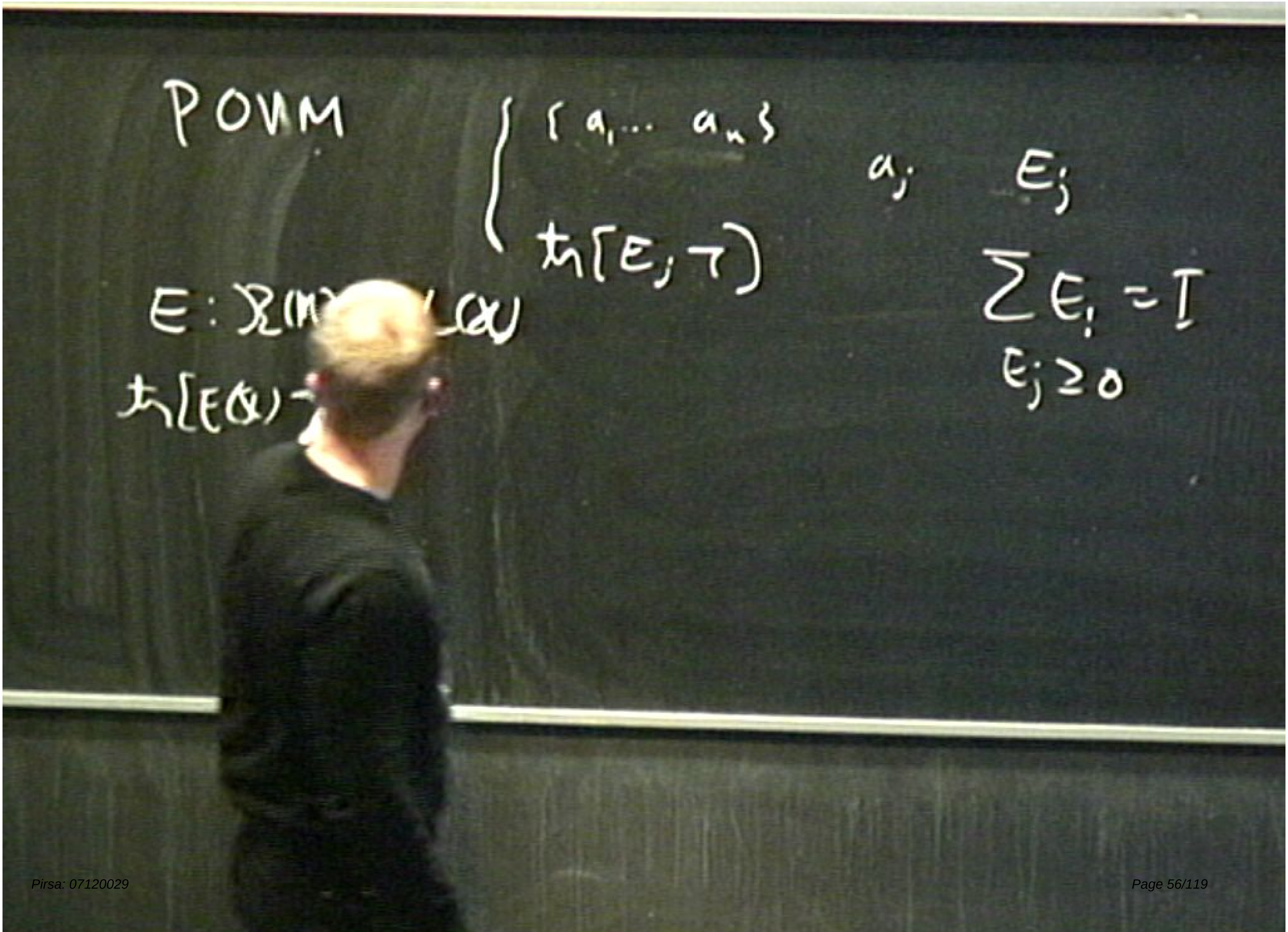

POVM
{a1 ... an}
aj   Ej
tr(Ej τ)

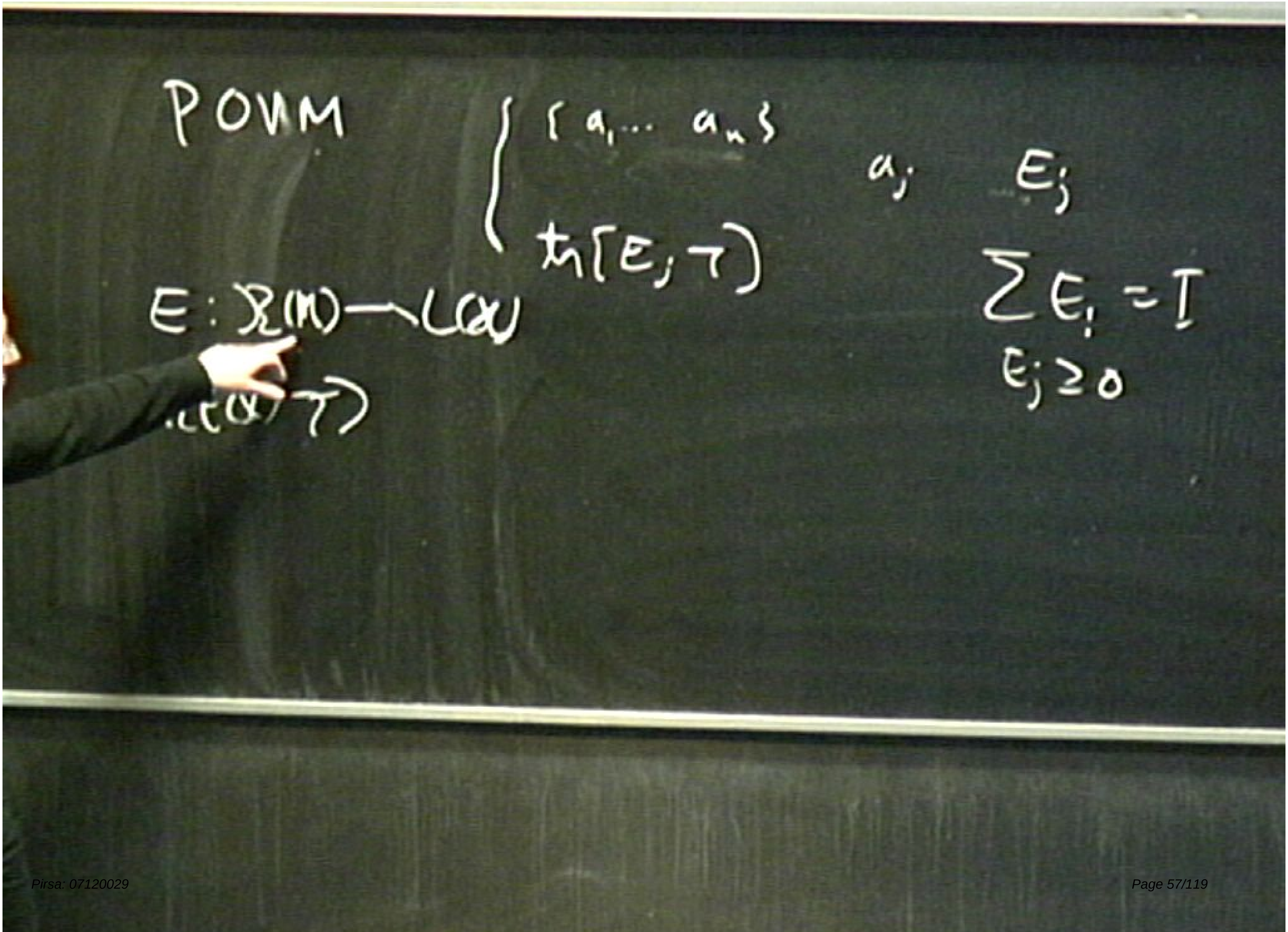

POVM
{a_1 … a_n}
tr[E_j τ)
a_j
E_j


POVM
{a_1 ... a_n}
a_j
E_j
tr[E_j ρ]
tr[E(Δ) ρ]


POVM
{a_1 ... a_n}
a_j
E_j
tr[E_j τ)
E : Σ(M) → L(H)
tr[E(X) τ)


POVM
{a_1 ... a_n}
a_j E_j
tr[E_j τ]
Σ E_i = I
≥ 0
E : Σ(Ω) → L(H)
tr[E(X) τ]


POVM
{a₁ … aₙ}
tr[Eⱼ τ]
aⱼ
Eⱼ
Σ Eᵢ = I
Eⱼ ≥ 0



POVM
{a_1 ... a_n}
a_j
E_j
tr[E_j τ]
∑ E_i = I
E_j ≥ 0

Normalized POM

Definition

A set function $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ is a *(normalized) POM*, if

- $E(\emptyset) = 0$, $E(\mathbb{R}) = I$;
- $0 \leq E(X) \leq I$ for all $X \in \mathcal{B}(\mathbb{R})$;
- $E(\bigcup_{n\in\mathbb{N}} X_n) = \sum_{n\in\mathbb{N}} E(X_n)$ in the weak operator topology, for each disjoint sequence $(X_n) \subset \mathcal{B}(\mathbb{R})$.

For $\varphi, \psi \in \mathcal{H}$, denote $E_{\psi,\varphi}(X) := \langle \psi | E(X)\varphi \rangle$, $X \in \mathcal{B}(\mathbb{R})$. $E_{\psi,\varphi}$ is a complex measure.

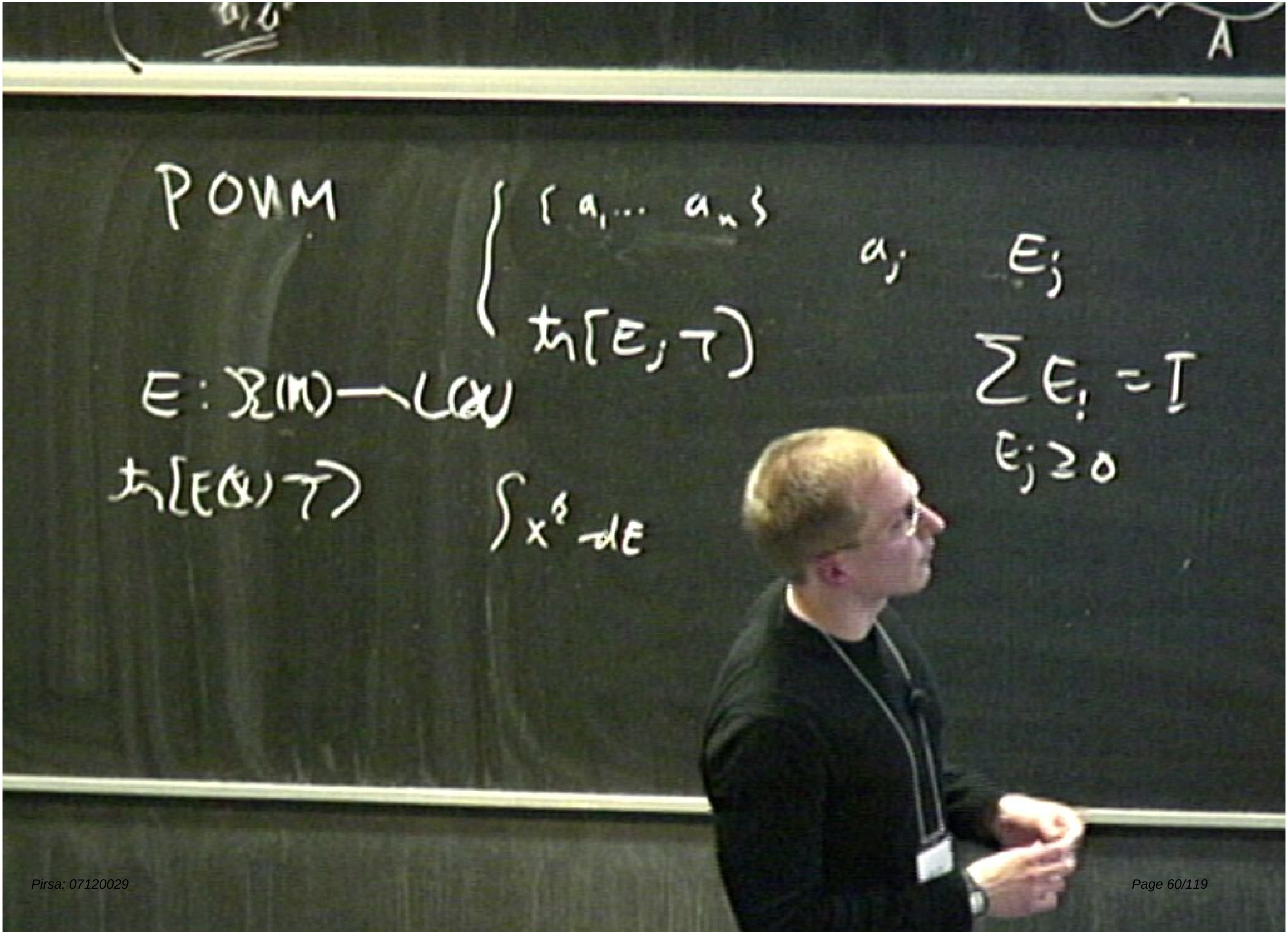
The moment operators of a POM

- The kth moment operator $\int x^k \, dE$ of a POM $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ is defined by

$$\langle \psi | (\int x^k \, dE) \varphi \rangle = \int x^k \, dE_{\psi,\varphi}(x) \quad \psi \in \mathcal{H}, \varphi \in D(x^k, E),$$

on the domain

$$D(\int x^k \, dE) := \{\varphi \in \mathcal{H} | \; x^k \text{ is } E_{\psi,\varphi}\text{-integrable for all } \psi \in \mathcal{H}\}.$$


POVM
{a1 ... an}
aj
Ej
tr[Ej τ)
ΣEi = I
Ej ≥ 0
tr[E(X) τ)
∫x² dE

Observables measured by the detector

Moment problem and homodyne detection

J. Kiukas (with P. Lahti) arXiv:0706.4436

Introduction

Balanced homodyne detector observables and their moment operators

Limit of strong auxiliary field

An application to eight-port homodyne detector

- For any input state T, the detection statistics are

$$\mathrm{Tr}[(UT \otimes |z\rangle\langle z|U^*)P^{D_z}(X)], \quad X \in \mathcal{B}(\mathbb{R}).$$

- This is of the form $\mathrm{Tr}[TE^z(X)]$, with POM E^z defined as

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where $V_z : \mathcal{H} \ni \varphi \mapsto \varphi \otimes |z\rangle \in \mathcal{H} \otimes \mathcal{H}_{aux}$.

- In the limit $r \to \infty$, $z = re^{i\theta}$, POM E^z should become the spectral measure P^{Q_θ} of the quadrature $Q_\theta = \frac{1}{\sqrt{2}}(e^{-i\theta}a + e^{i\theta}a^*)$. But in what sense?

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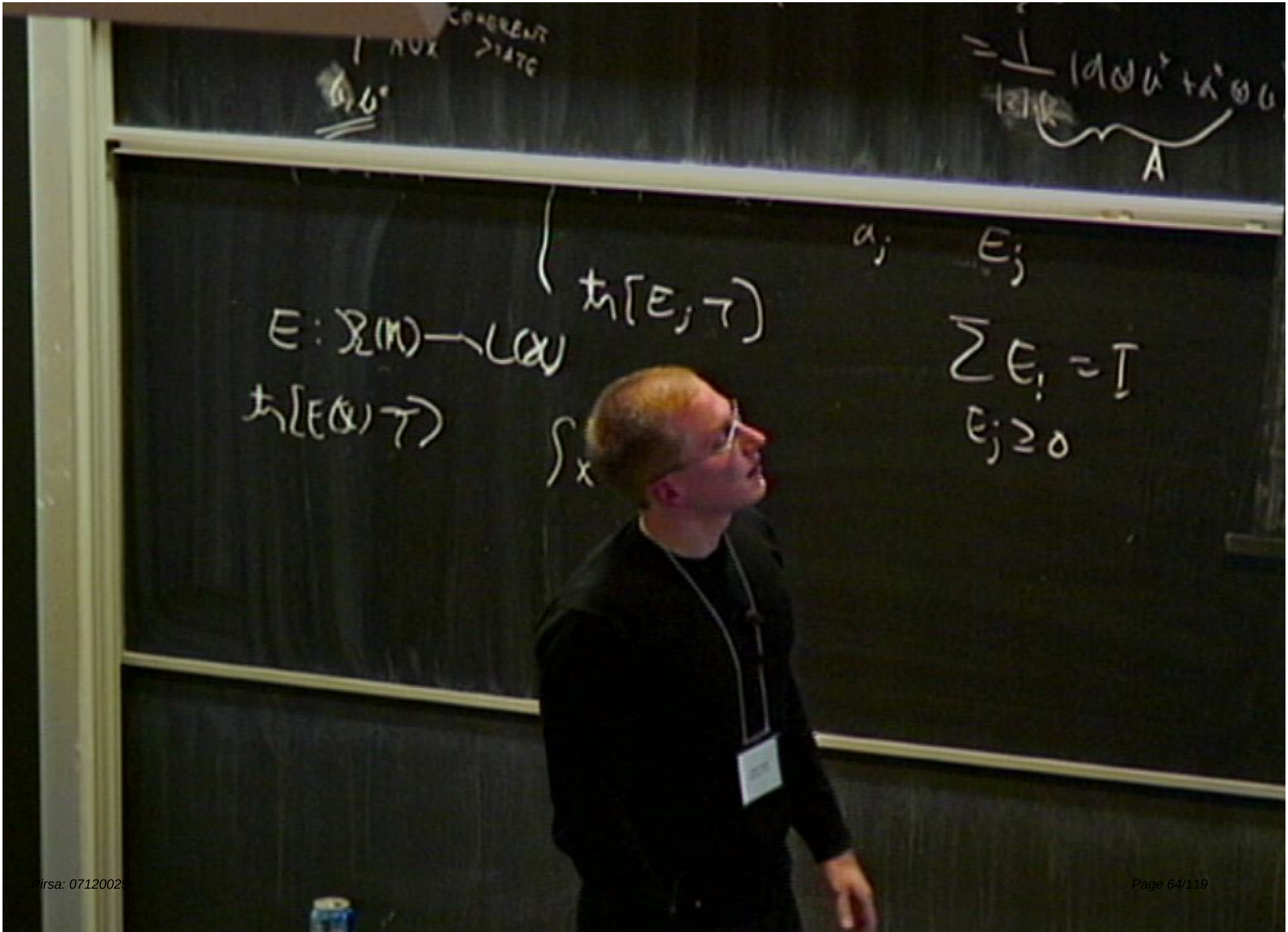
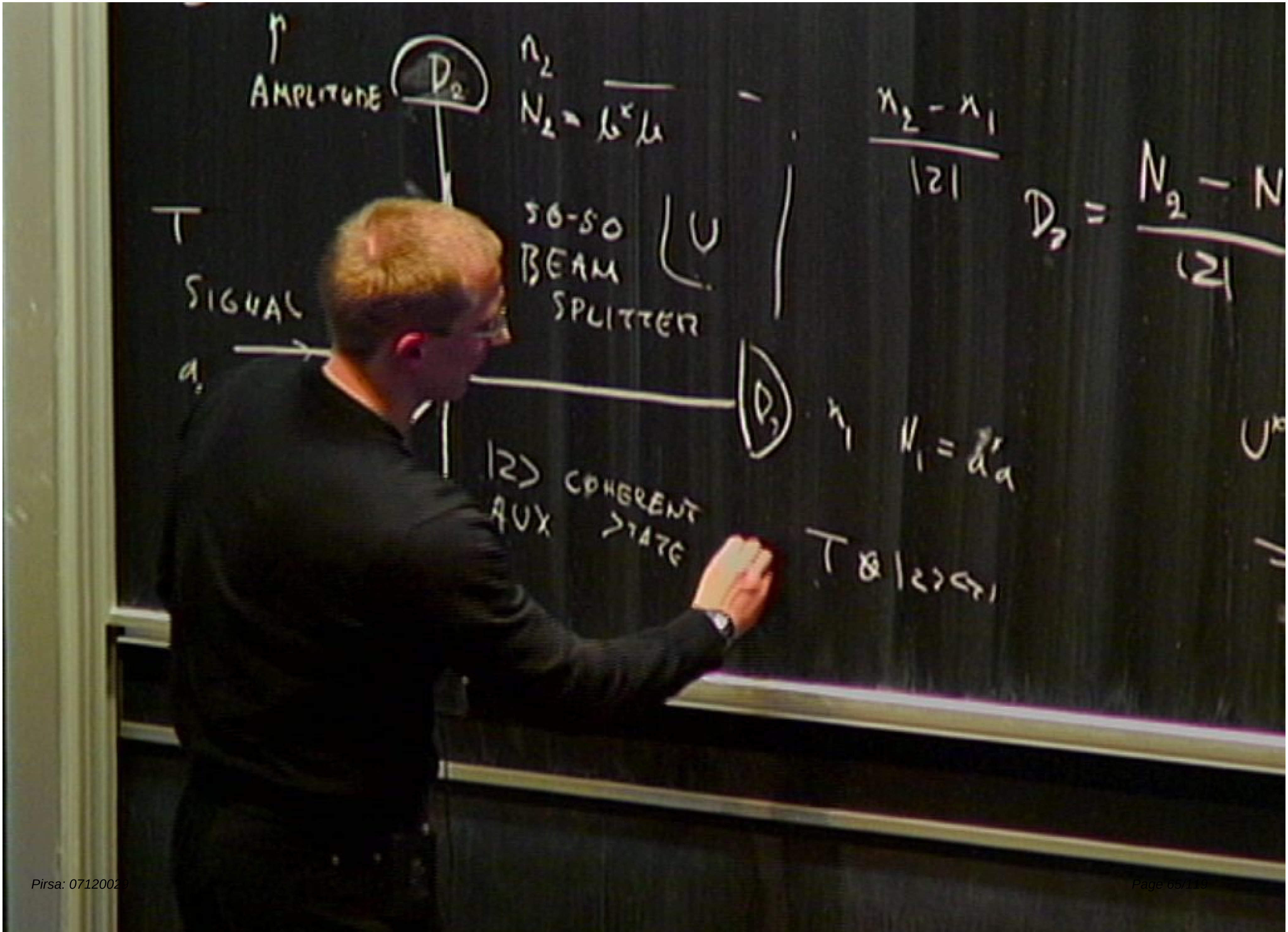
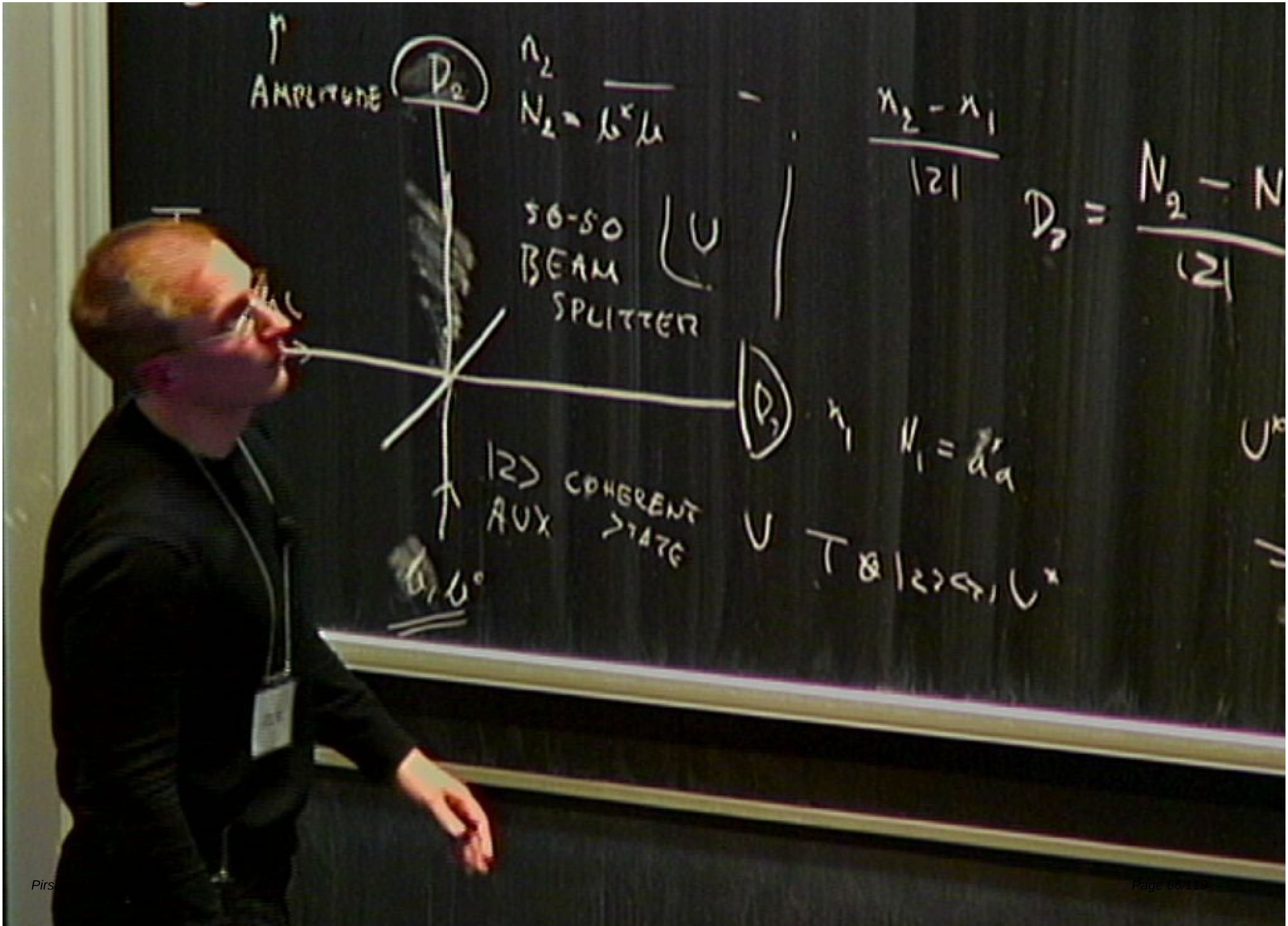

COHERENT
AUX STATE
A
E
E_j
Σ E_i = I
E_j ≥ 0


AMPLITUDE
50-50
BEAM
SPLITTER
SIGNAL
COHERENT
AUX
STATE


AMPLITUDE
50-50
BEAM
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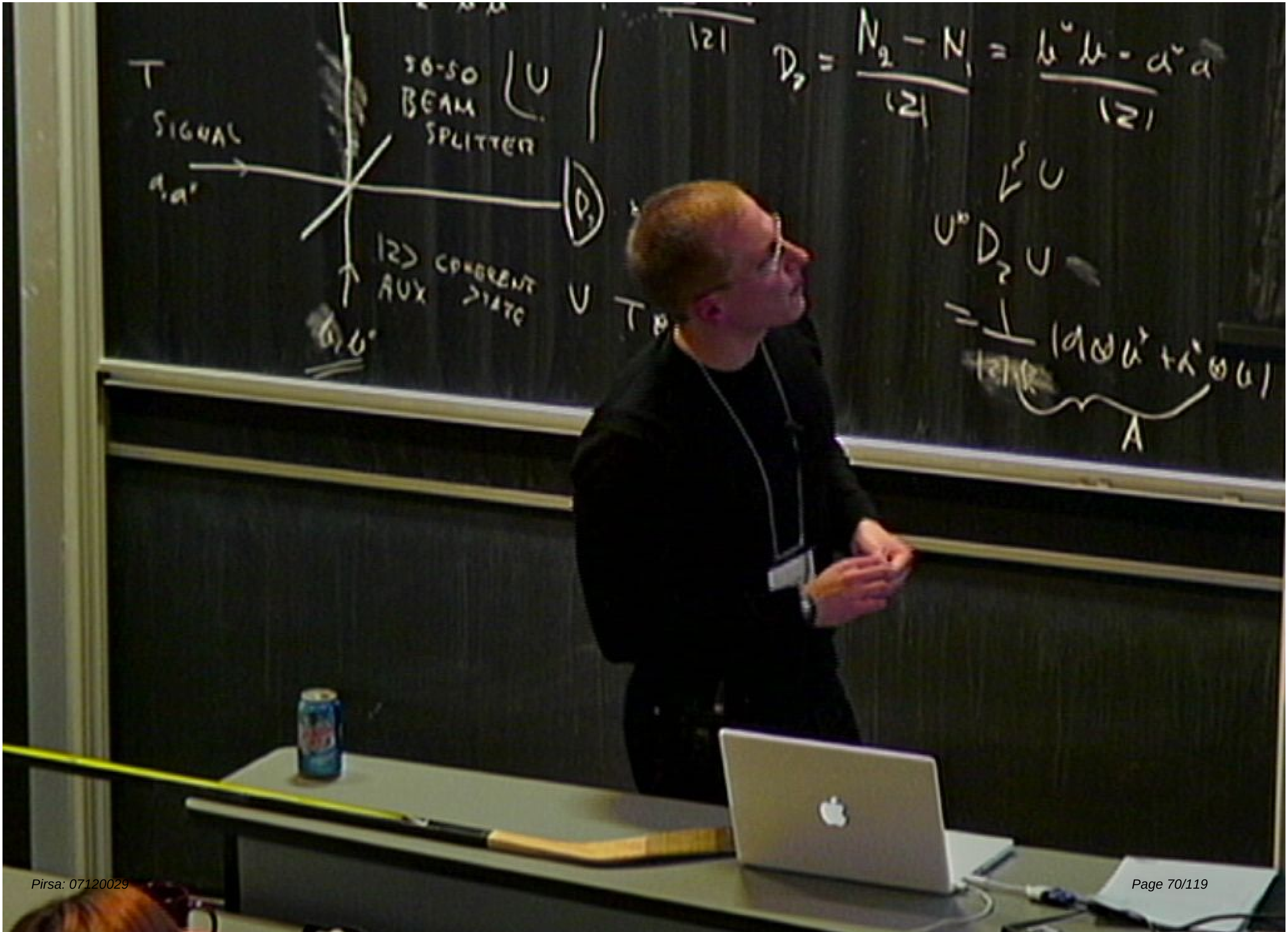
SIGNAL
50-50
BEAM
SPLITTER
|z> COHERENT
AUX STATE
$D_3 = \frac{N_2 - N_1}{|z|} = \frac{b^\dagger b - a^\dagger a}{|z|}$


SIGNAL
50-50
BEAM
SPLITTER
COHERENT
AUX
STATE
A

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if $X \in \mathcal{B}(\mathbb{R})$ is such that $\mathrm{Tr}[TP^{Q_\theta}(\partial X)] = 0$
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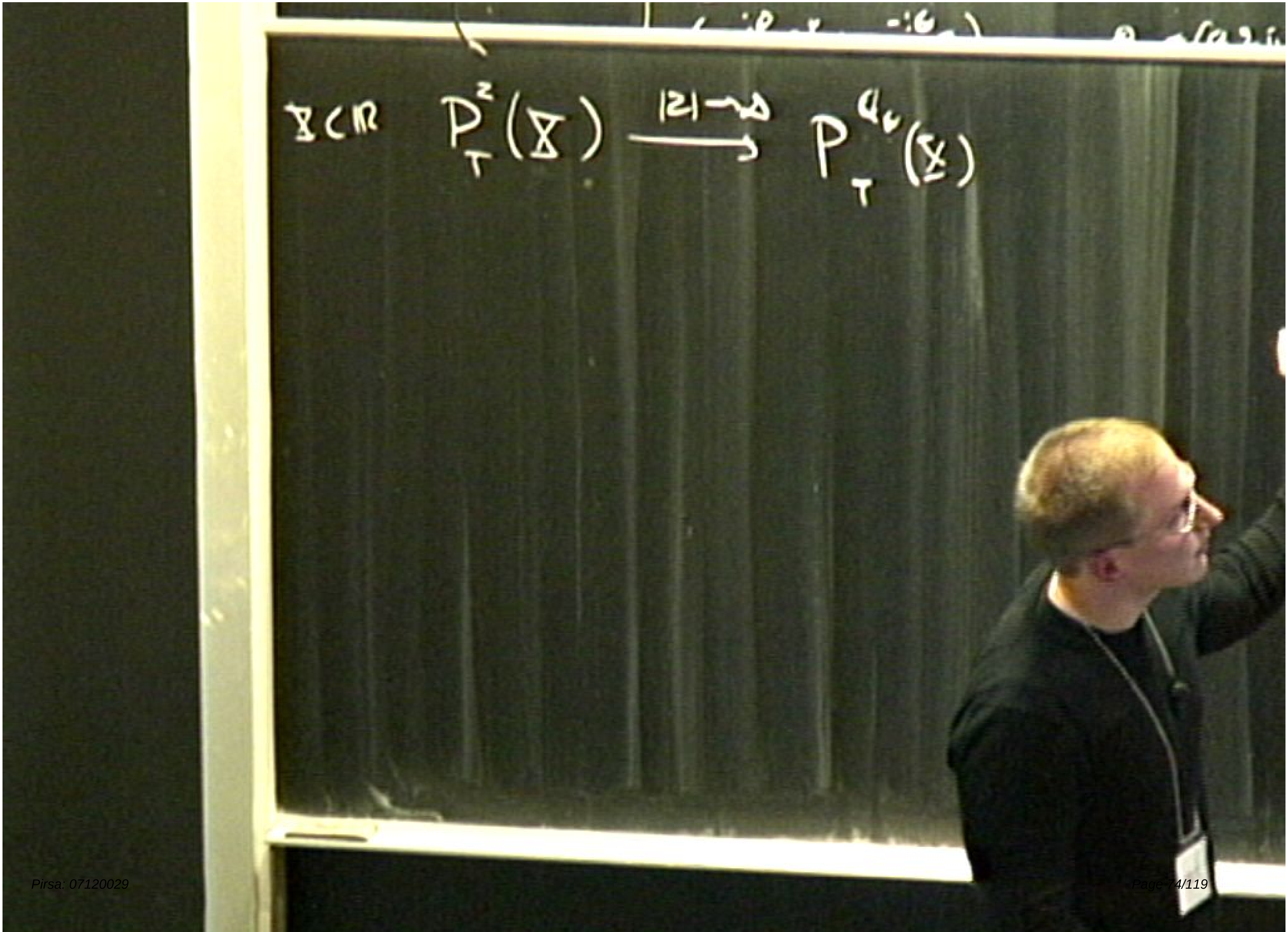
X ⊂ ℝ
$P_T^2(X)$
|z|→∞
$P_T^{4_v}(X)$

Moment operators

- $\int x^k \, dE^z \supset \frac{1}{|z|^k} V_z^* A^k V_z, \quad D(a^k) \subset D(\int x^k \, dE)$.
- $\int x \, dE^z \supset Q_\theta|_{D(a)}, \quad z = |z| e^{i\theta}$.
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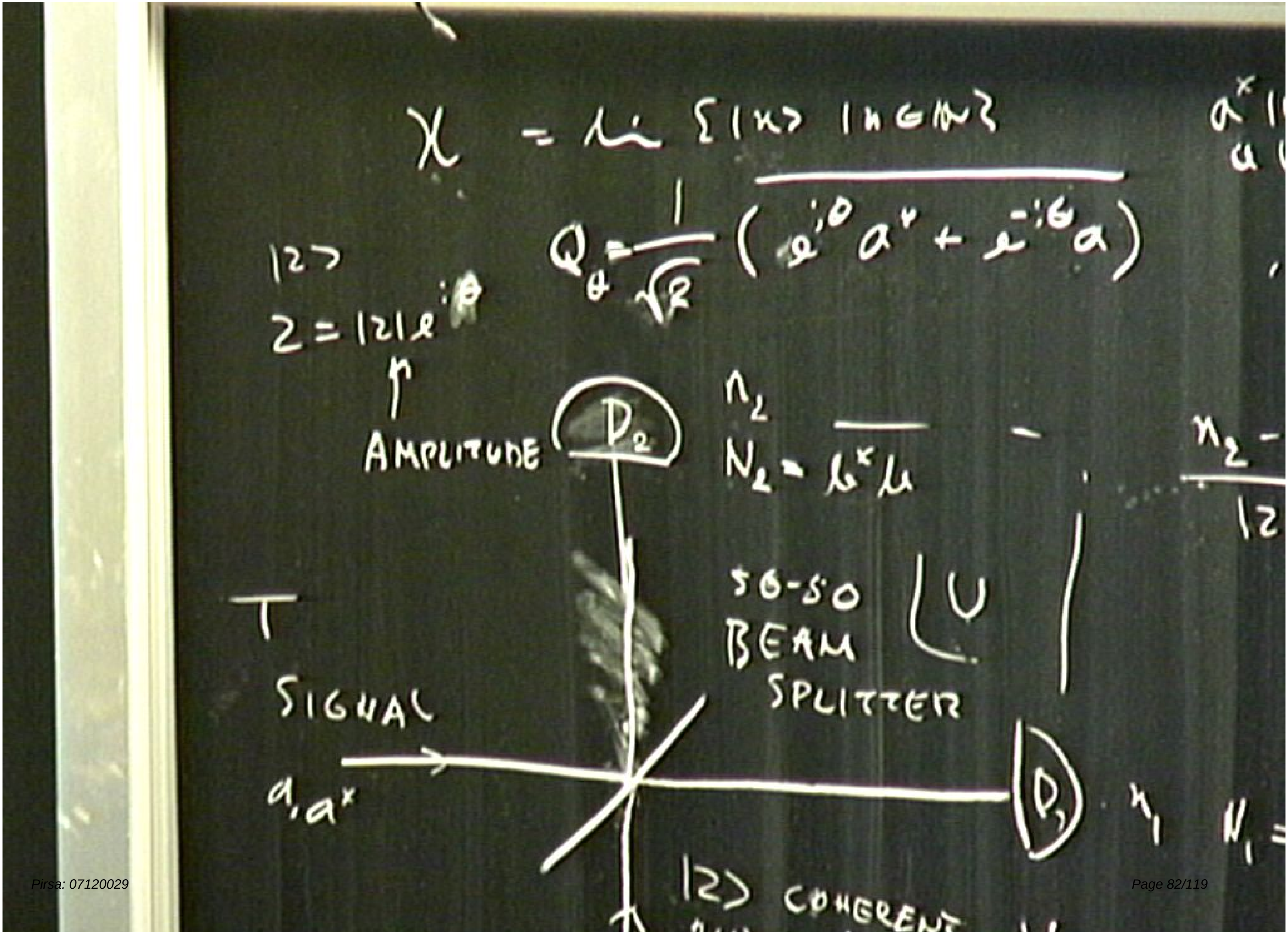
$Q_\theta = \frac{1}{\sqrt{2}}\left(e^{i\theta}a^\dagger + e^{-i\theta}a\right)$
$|z\rangle$
$z = |z|e^{i\theta}$
AMPLITUDE
D_2
n_2
$N_2 = b^\dagger b$
50-50
BEAM
SPLITTER
SIGNAL
$a, a^\dagger$
D_1
n_1
$|z\rangle$ COHERENT

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A general theorem

Theorem

Let $E^n : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a POM for each $n \in \mathbb{N}$, and let $\mathcal{D} \subset \mathcal{H}$ be a dense subspace. Assume that

$$\lim_{n\to\infty} \langle \varphi | (\int x^k \, dE^n) \varphi \rangle \text{ exists for all } k \in \mathbb{N}, \varphi \in \mathcal{D}.$$

Then there exists a POM E such that

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If measures $X \mapsto \langle \varphi | E(X) \varphi \rangle$, $\varphi \in \mathcal{D}$ are determinate, then

(1) *E is the only POM with the property (1).*

(2) *$\lim_{n\to\infty} E^n(X) = E(X)$ (weak op. top.) whenever $X \in \mathcal{B}(\mathbb{R})$, and $E(\partial X) = 0$.*

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(2) $\lim_{n\to\infty} E^n(X) = E(X)$ *(weak op. top.) whenever $X \in \mathcal{B}(\mathbb{R})$, and $E(\partial X) = 0$.*

A general theorem

Moment problem and homodyne detection

J. Kiukas (with P. Lahti) arXiv:0706.4436

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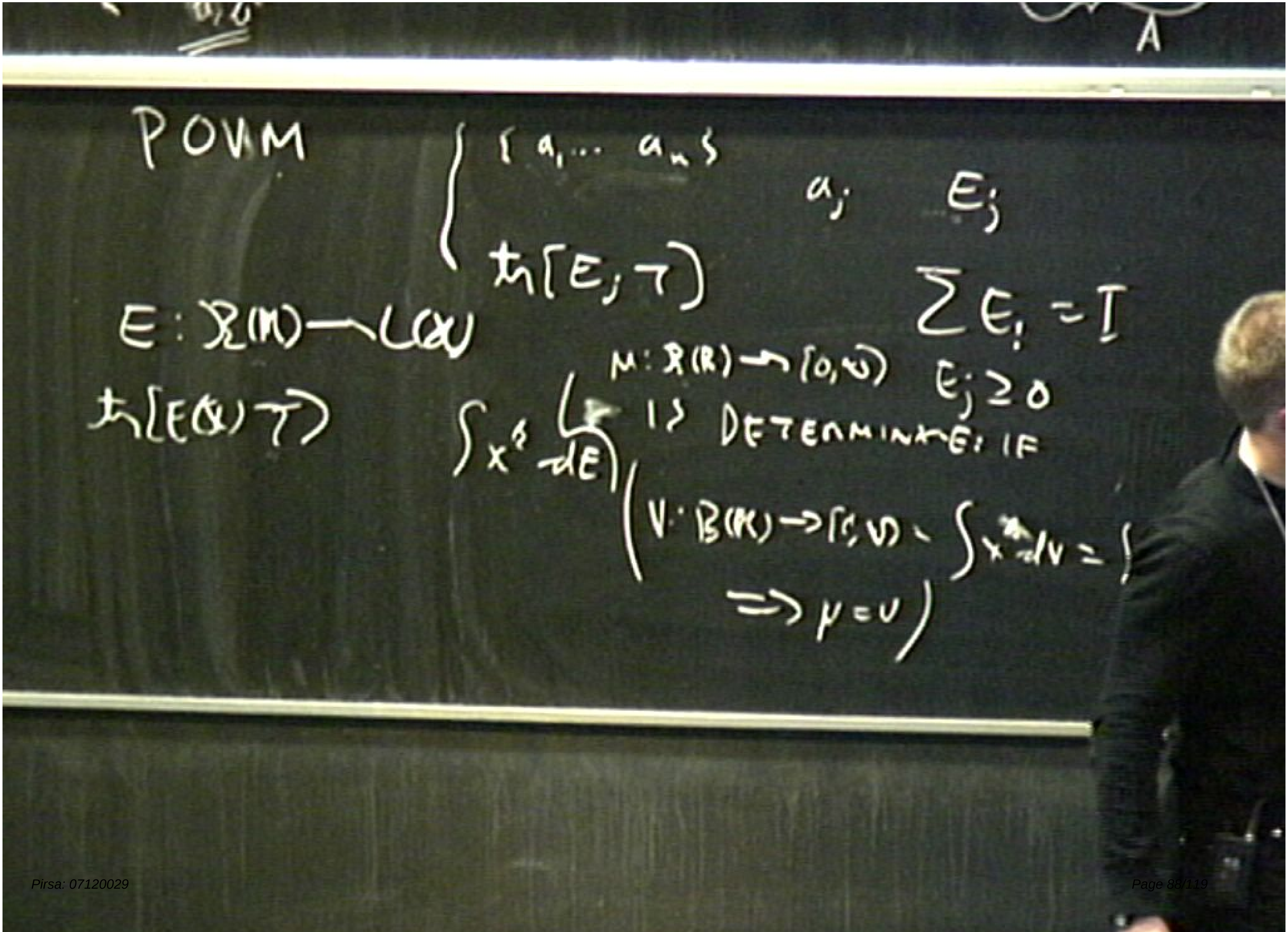
POVM
{a1 … an}
aj Ej
tr[Ej ρ]
∑ Ei = I
Ej ≥ 0
E : B(R) → L(H)
tr[E(Δ) ρ]
μ : B(R) → [0,∞)
IS DETERMINATE IF
(ν : B(R) → [0,∞) ∫x^n dν = ∫ …
⇒ μ = ν)

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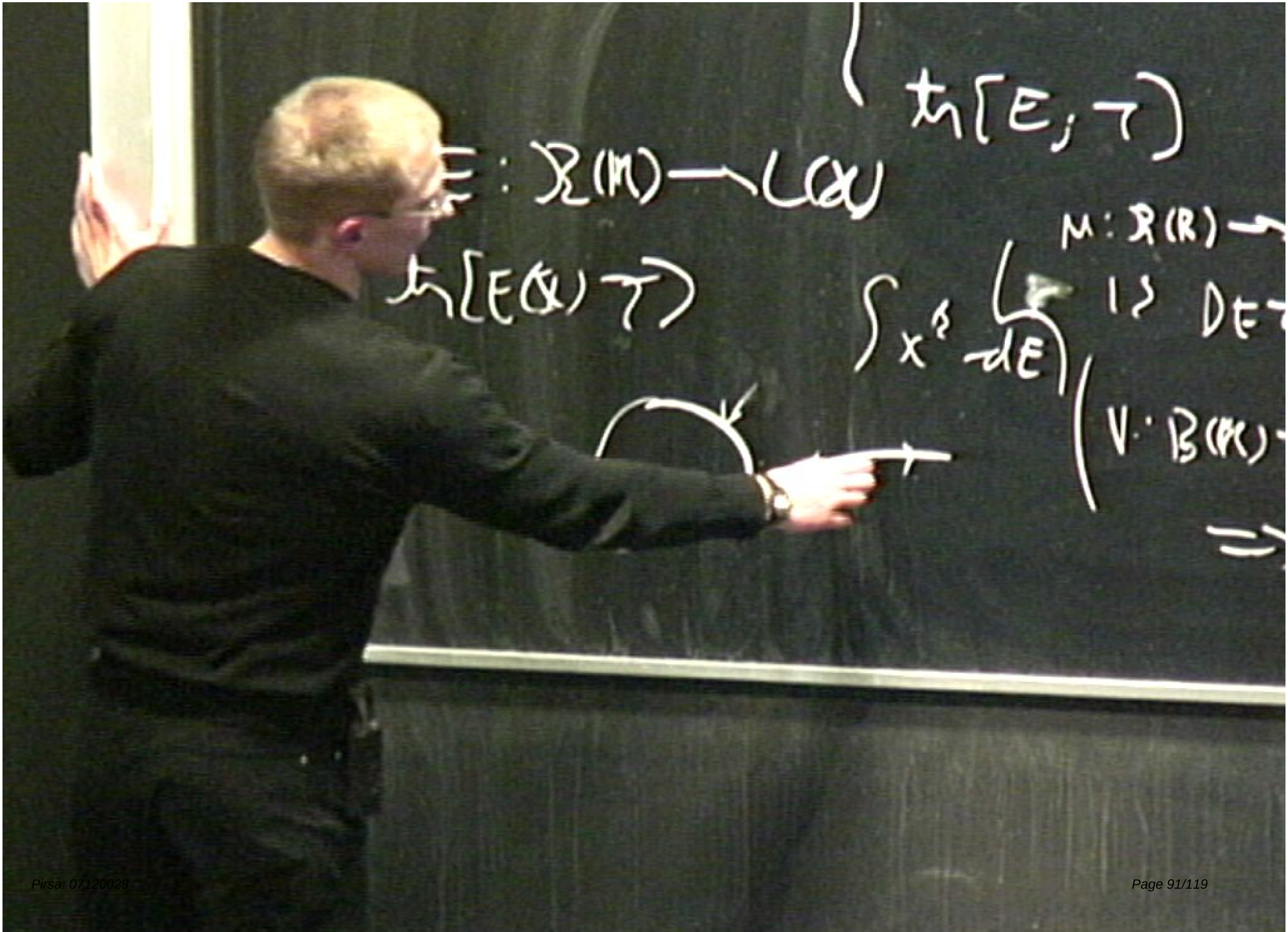
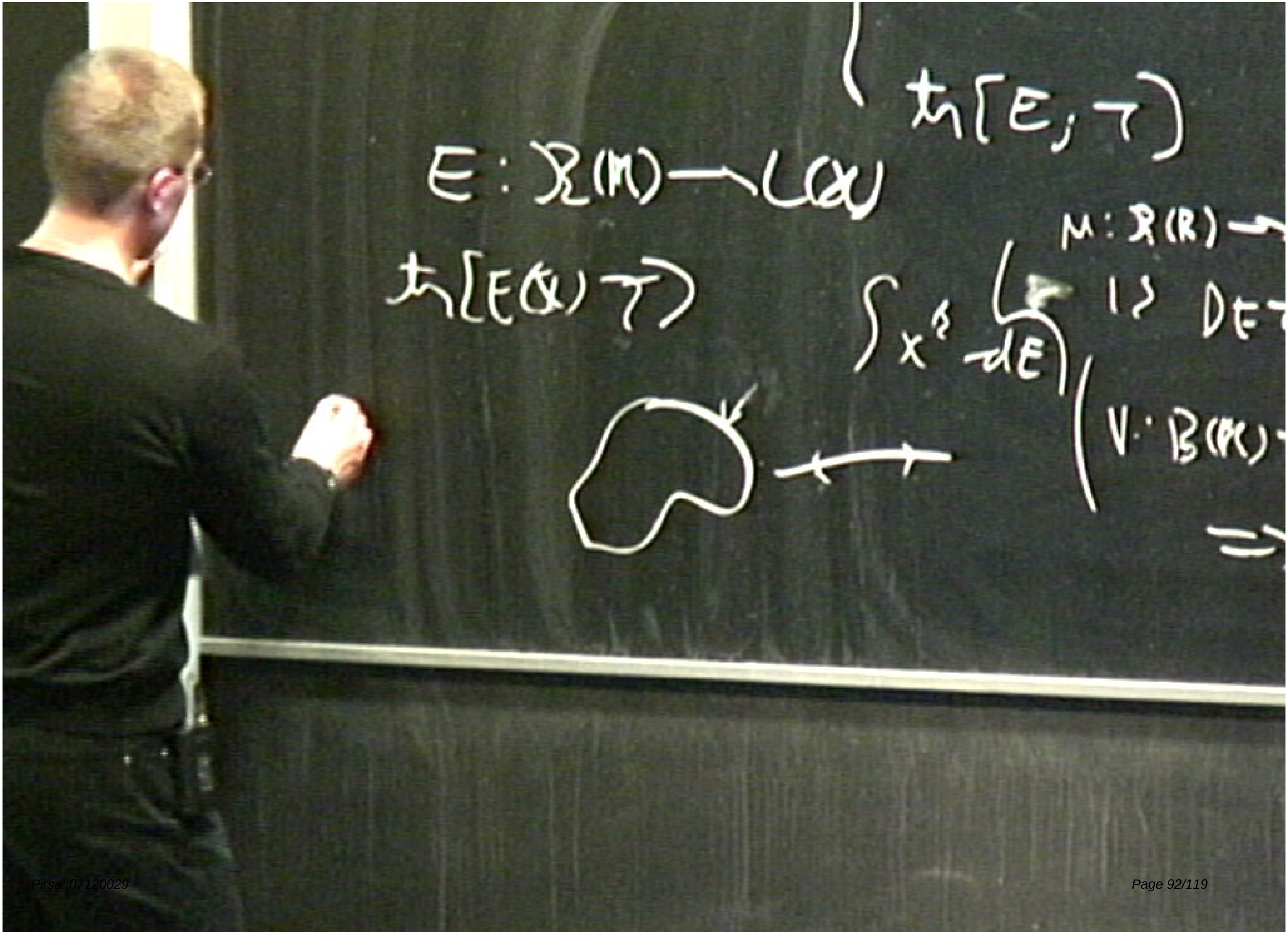
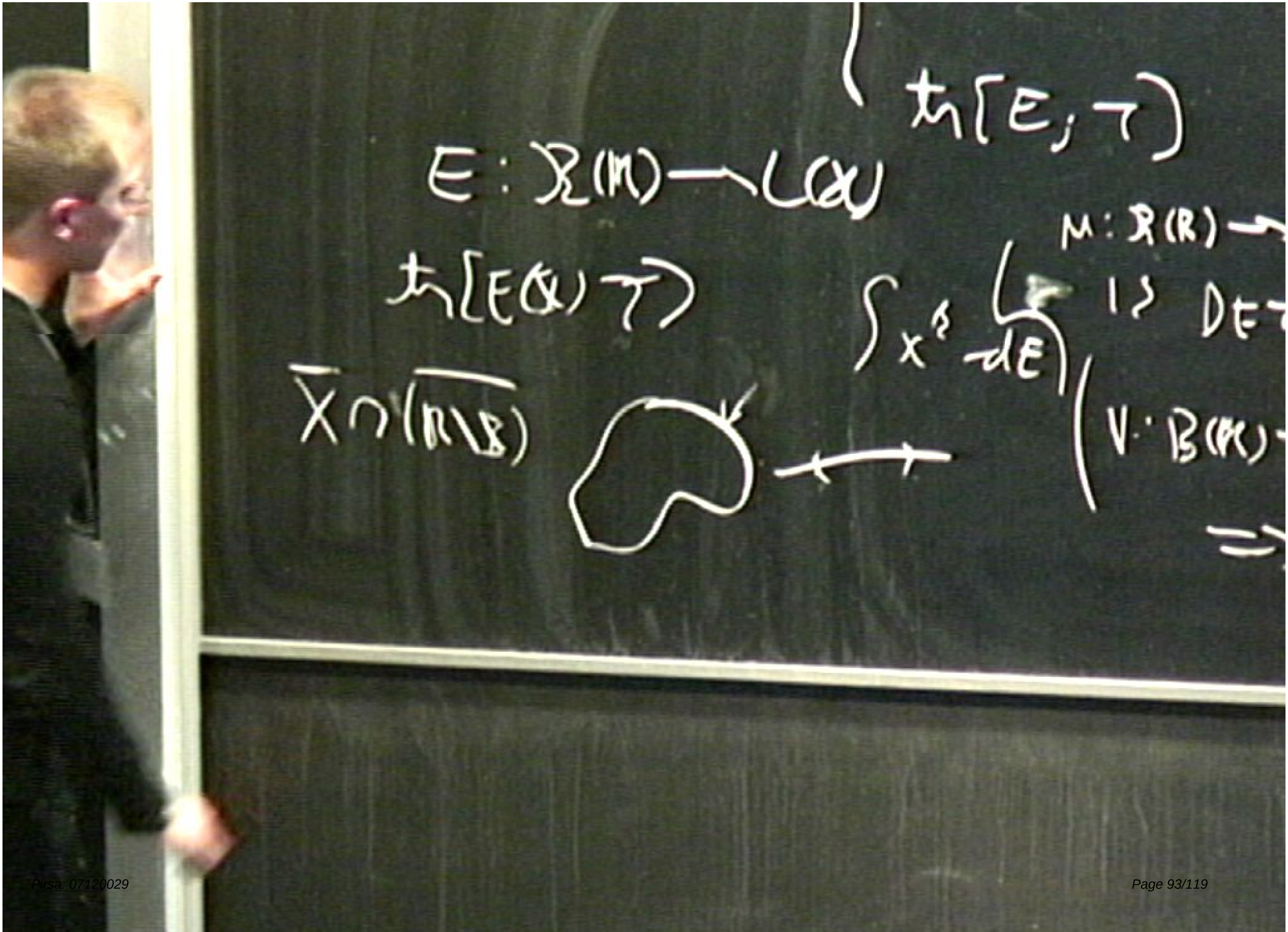



tr[E; τ]
E : B(M) → L(H)
tr[E(X) τ]
M : B(R) →
∫ x² dE
V · B(R)



$\mathrm{tr}[E; \tau]$

$E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$

$\mathrm{tr}[E(X)\tau]$

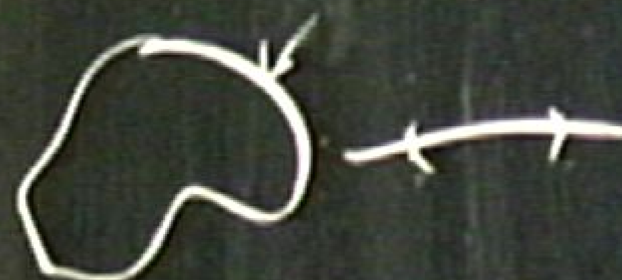
$\int x^n \, dE$

$M : \mathcal{B}(\mathbb{R}) \to$

$1 \geq DE$

$V : \mathcal{B}(\mathbb{R})$

$\overline{X} \cap \overline{(\mathbb{R}\setminus X)} =: \partial X$



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Fix $\theta \in [0, 2\pi)$, and take $z_n(\theta) = r_n e^{i\theta}$, with $\lim_{n\to\infty} r_n = \infty$.
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Then

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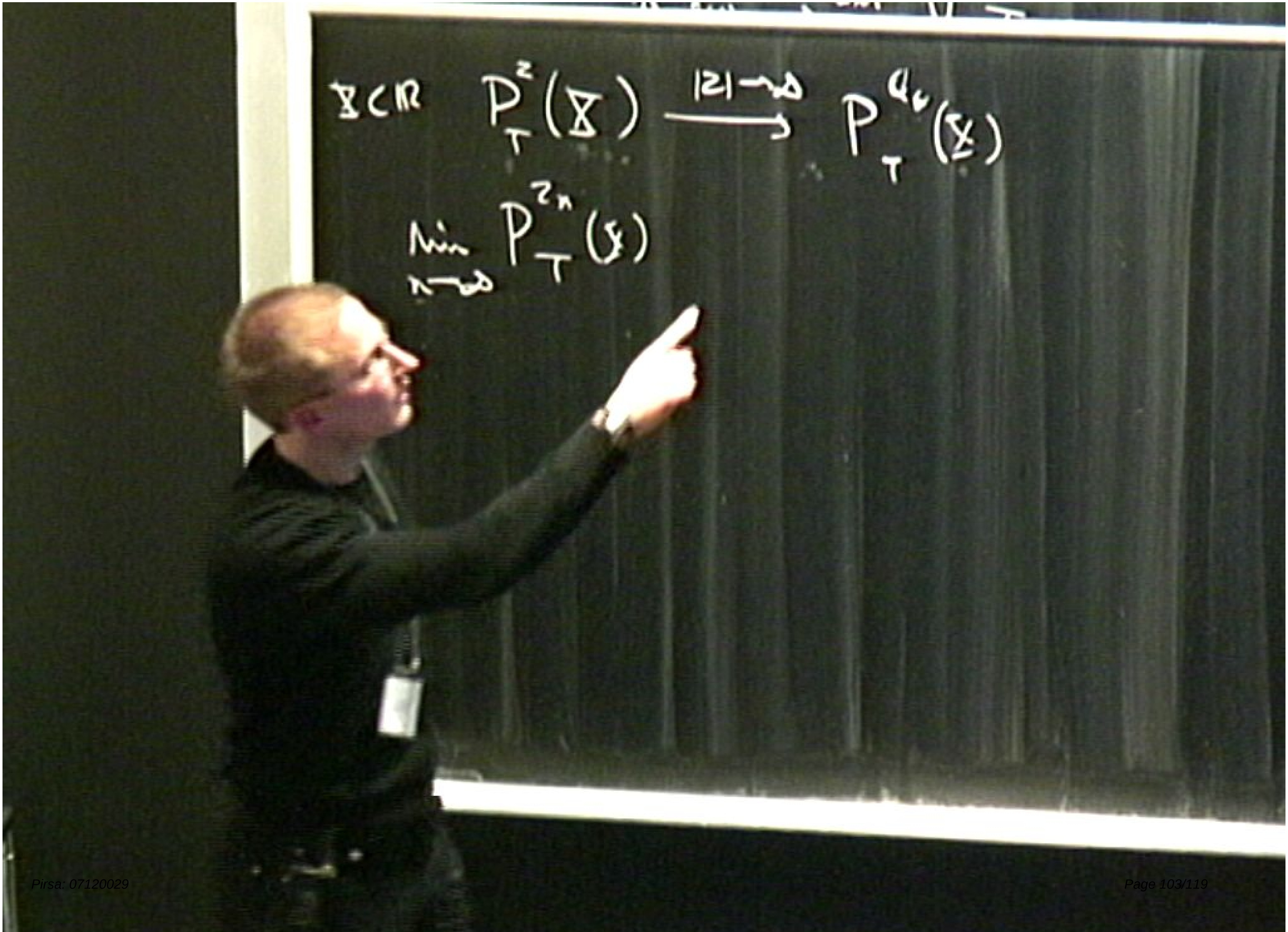
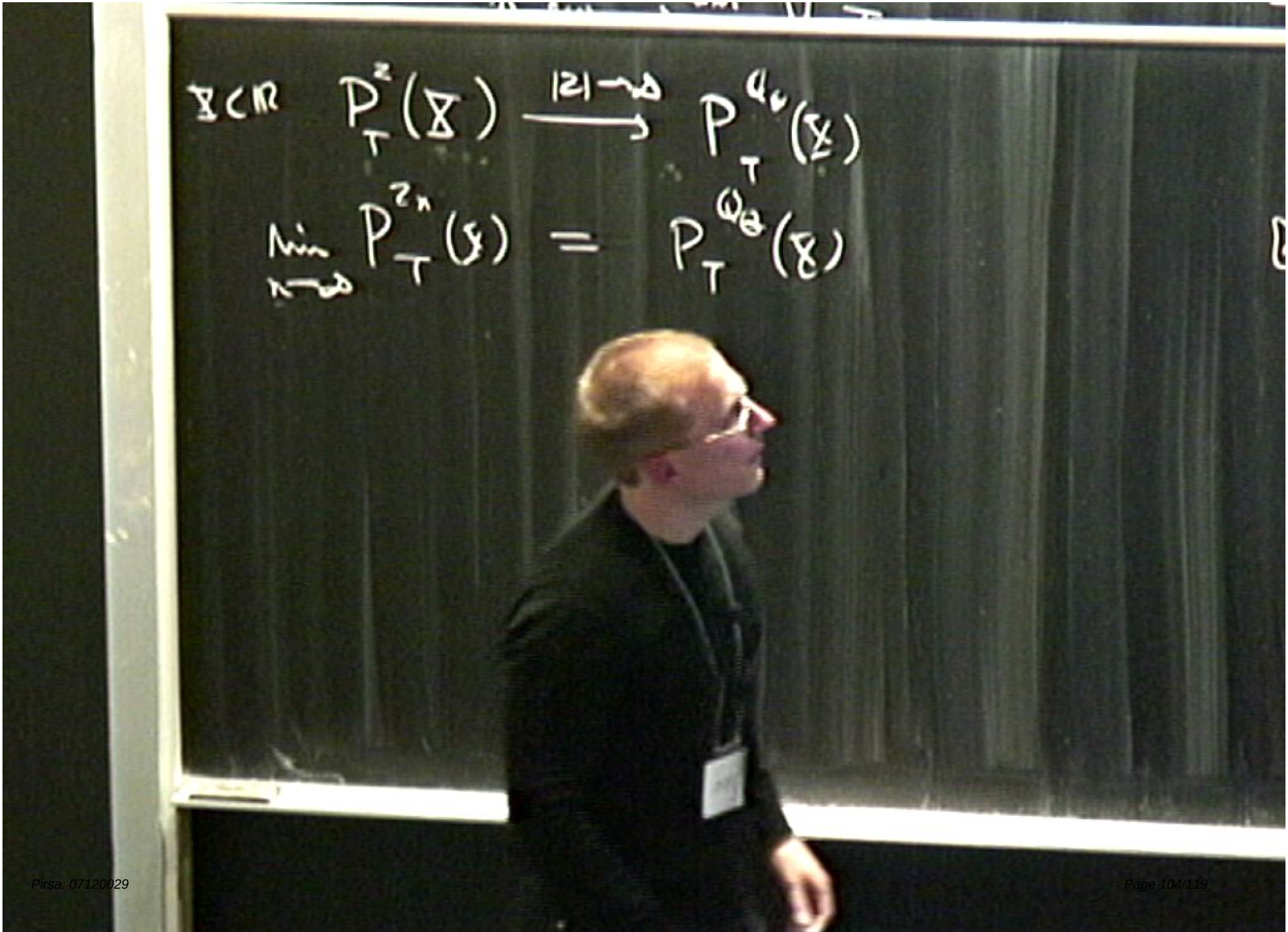
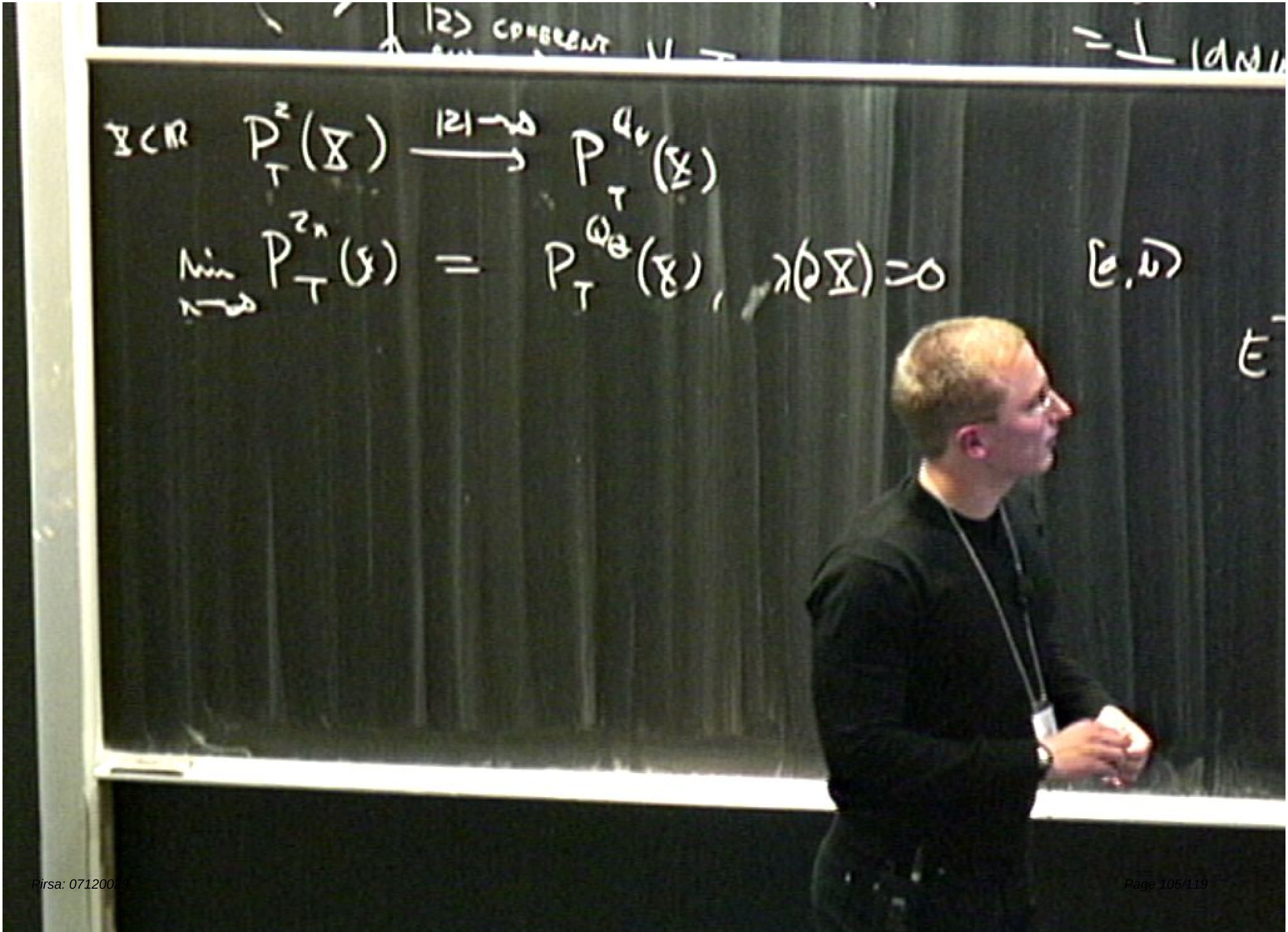
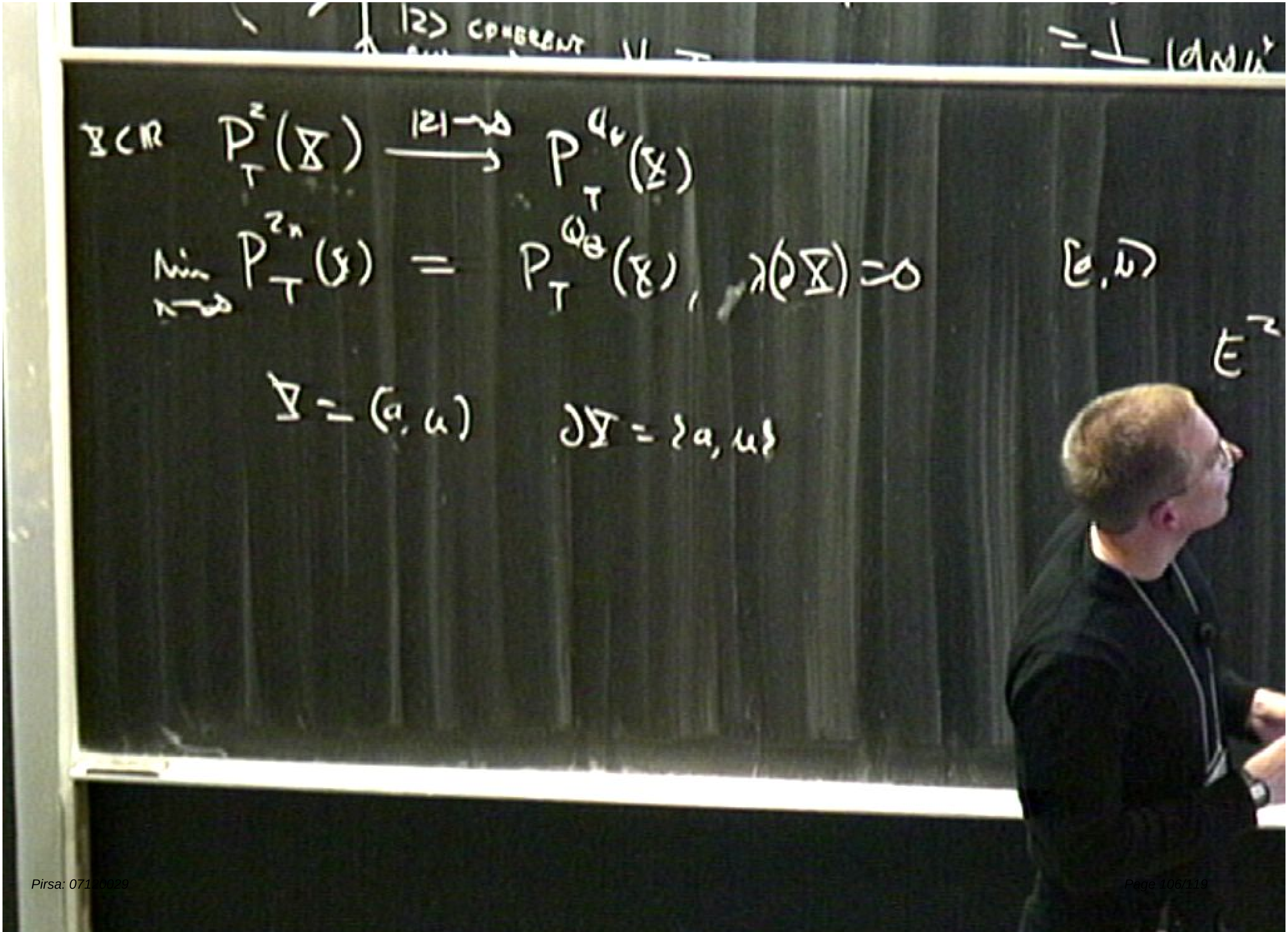

X ⊂ ℝ $P_T^z(X) \xrightarrow{|z|\to\infty} P_T^{q_v}(X)$
$\lim_{n\to\infty} P_T^{z_n}(X)$


$X \subset \mathbb{R} \quad P_T^{z}(X) \xrightarrow{|z| \to \infty} P_T^{q_v}(X)$
$\lim_{n \to \infty} P_T^{z_n}(X) = P_T^{\omega_\theta}(X)$


|z> COHERENT
X ⊂ ℝ   P_T^z(X) —|z|→∞→ P_T^{ω_v}(X)
lim_{n→∞} P_T^{z_n}(X) = P_T^{ω_θ}(X), λ(∂X)=0
[0,∞)


|z> COHERENT
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λ(∂Ɣ)=0
Ɣ = (a, u)
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- Eight-port homodyne detector is argued to provide a way to measure covariant phase space observables

$$E^S(Z) = \int_Z W(q,p) S W(q,p)^* \, dqdp, \quad Z \in \mathcal{B}(\mathbb{R}^2).$$

  We make this statement precise.
- Input 4 is in a coherent state $|\sqrt{2}z\rangle$, and we have two detection observables: $|z|^{-1}(a_3^* a_3 - a_1^* a_1)$ and $|z|^{-1}(a_4^* a_4 - a_2^* a_2)$.
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Summary

- Moments of the observables measured in the balanced homodyne detector converge to the moments of a quadrature operator, in the high amplitude limit of the auxiliary oscillator.
- According to a general theorem, this ensures that (most of) the POM elements of the observables converge weakly to the corresponding elements of the quadrature spectral measure. $\rightsquigarrow$ quadrature measurement is indeed achieved.
- Each covariant phase space observable is obtained similarly as the high amplitude limit in the eight-port homodyne detector.

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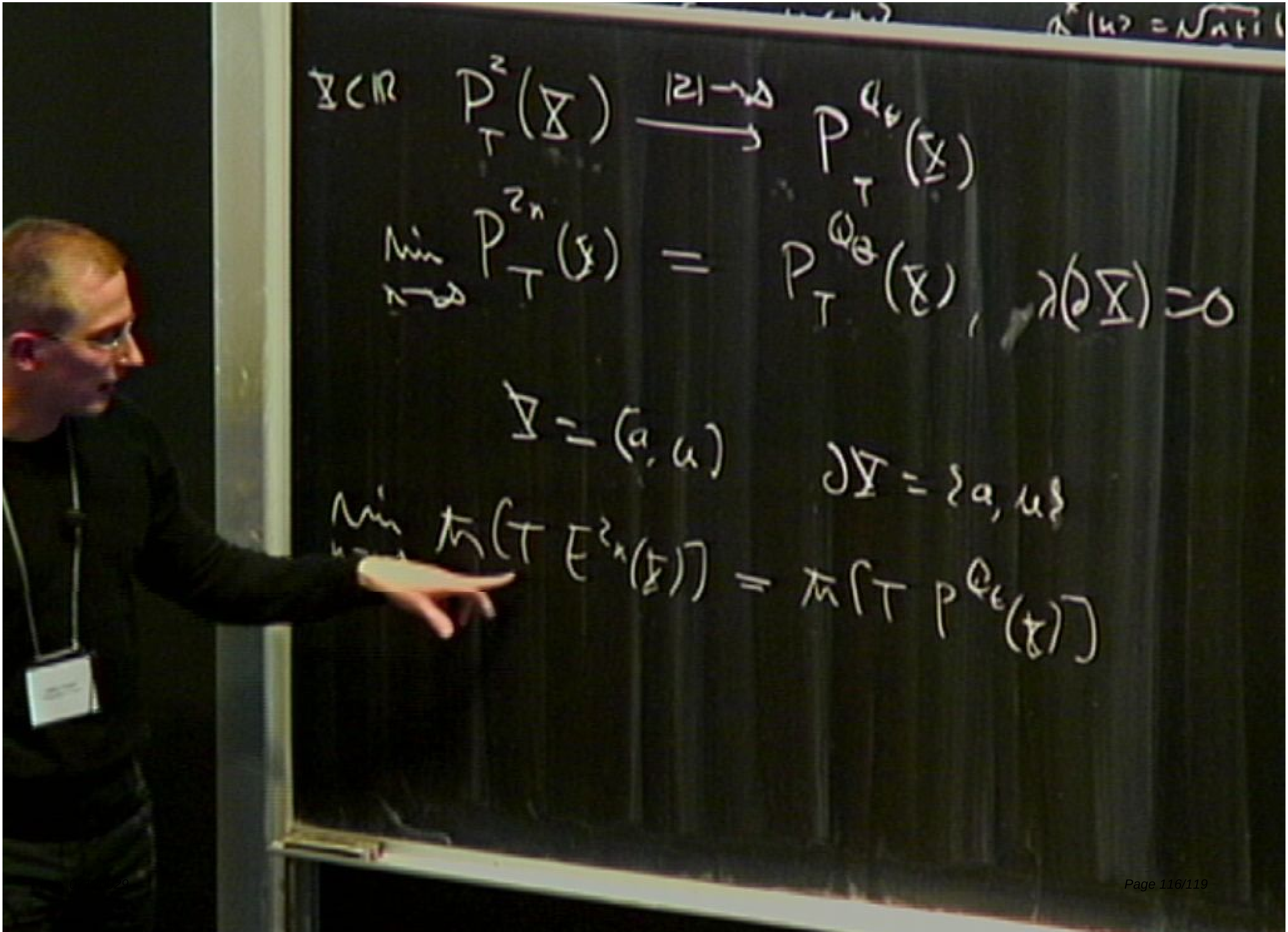
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In other words,

- The limits on the moments of the measurement statistics for the (superpositions of) coherent states determine the limiting observable as Q_θ.
- For any state T, the quadrature probability $\mathrm{Tr}[TP^{Q_\theta}(X)]$ is obtained as the limit of a sequence of actual measurement outcome probabilities, provided that ∂X has zero Lebesgue measure.