

Title: Phase transition of computational power of measurement-based quantum computer

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Abstract: One of the most significant questions in quantum information is about the origin of the computational power of the quantum computer; namely, from which feature of quantum mechanics and how does the quantum computer obtain its superior computational potential compared with the classical computer?

In my talk, I address this open question more concisely through the study of measurement-based quantum computer, in which all the quantum resource is attributed to entanglement since computation is carried through its consumption by local measurements. I also show a simple model of the phase transition of quantum computer occurring at some threshold, below which the quantum computer comes to allow an efficient classical simulation in accordance with an exponential drop in the amount of entanglement.

YRC@PI , December 4th (2007)

Phase transition of computational power of measurement-based quantum computer

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Motivation

A fundamental question: origin of computational power of QC
from which feature of quantum mechanics and how does the quantum computer obtain its superior computational power compared with the classical computer?

Measurement-based quantum computer

the role of entanglement is highlighted!

- the amount of entanglement reflects computational power
- phase transition (exponential change of entanglement)

Outline

Motivation

origin of computational power of quantum computer

Measurement-based quantum computer

role of entanglement for universal quantum computation

Entanglement criterion for universality

the amount of entanglement must grow faster than polylogarithmic in the system size for universality

Phase transition of computational power of a faulty cluster state.

exponential change of entanglement

Summary

Measurement-based quantum computer

[Raussendorf & Briegel, PRL 86, 5188 (2001)]

Resources

- preparation of a multipartite entangled state called **2D cluster state** $|\phi\rangle_C$, which exhibits the following quantum correlation:

$$K^{(a)} |\phi\rangle_C = |\phi\rangle_C, \quad \text{eigenvalue equations}$$

$$K^{(a)} \equiv \sigma_x^{(a)} \prod_{\langle a', a \rangle} \sigma_z^{(a')}$$

set of correlation operators

- single-qubit projective measurements
- classical communication for feedforward of measurement outcomes



Graph states

For a graph G ,
 vertices = qubits, edges = Ising-type interaction pattern.
 degree is the number of edges from a vertex.

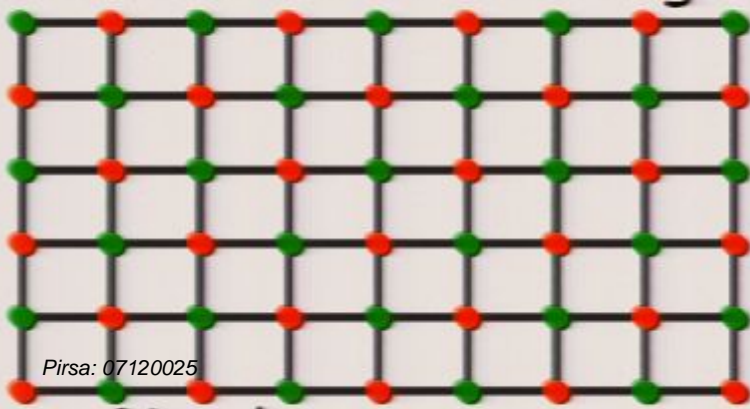
[review: Hein et al., quant-ph/0602096]

$$|G\rangle = \prod_{(a,b) \in \text{edges}} CZ^{(a,b)} |+\rangle^N, \quad CZ = \text{diag}(1,1,1,-1), \quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

joint eigenstate of N commuting correlation operators
 for N qubits.

$$K_a |G\rangle = |G\rangle, \quad K_a = \sigma_x^{(a)} \otimes_{b \in N_a} \sigma_z^{(b)}$$

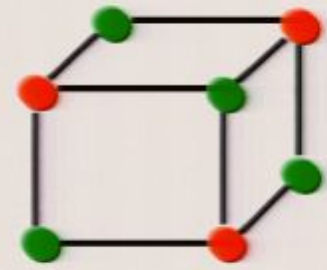
stabilizer states = graph state, up to local unitaries.



2D cluster state



GHZ state



7-qubit error correction codeword

Measurement-based quantum computer

[Raussendorf & Briegel, PRL 86, 5188 (2001)]

I. Computational process

According to quantum algorithms, the directions of single-qubit projective measurements are determined.

$$P_{0,\vec{n}} = |0\rangle_{\vec{n}} \langle 0|$$

$$P_{1,\vec{n}} = |1\rangle_{\vec{n}} \langle 1|$$

$$P_{j,\vec{n}}^{(a)} = \frac{1 + (-1)^j \vec{n}^{(a)} \cdot \vec{\sigma}^{(a)}}{2}$$



$\{\vec{n}^{(a)} \mid a \in C\}$ $\begin{cases} \text{fixed} \\ \text{adaptive} \end{cases}$

The set of rules for adapting measurement directions and processing their outcomes by classical communication carries computation

Measurement-based quantum computer

[Raussendorf & Briegel, PRL 86, 5188 (2001)]

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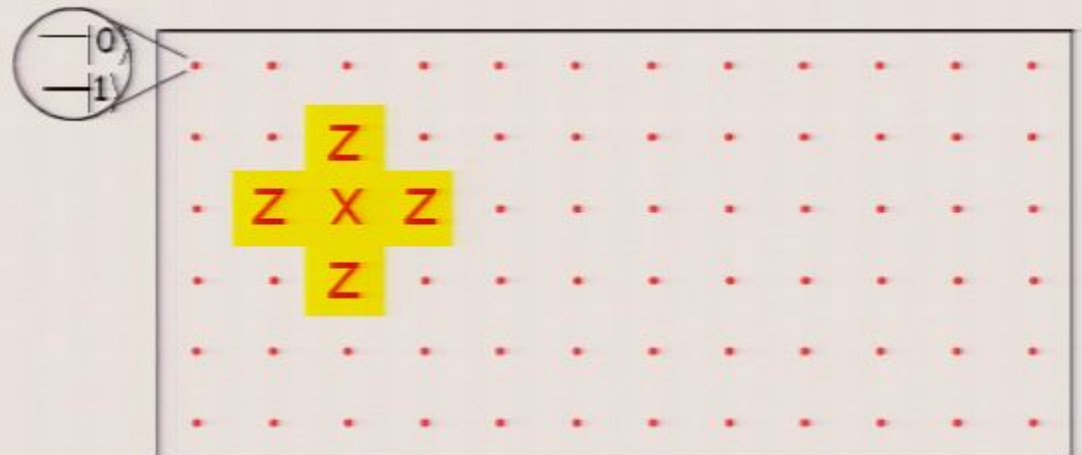
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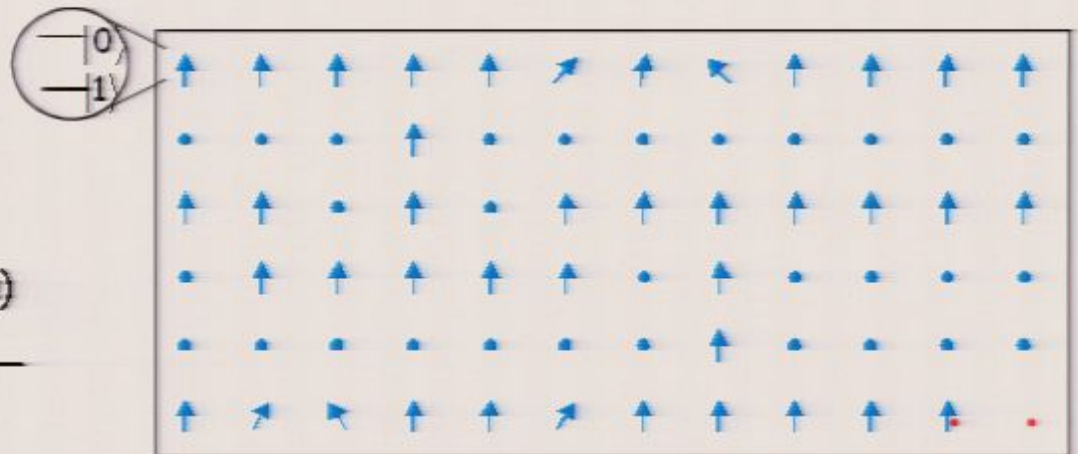
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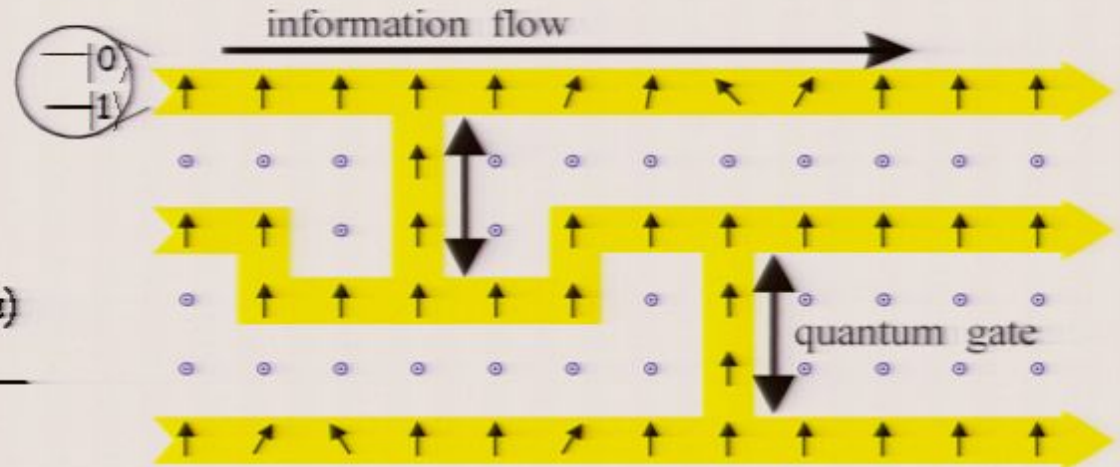
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measurements: \circ in Z direction
 \uparrow in X direction
 \nearrow in X-Y plane

The set of rules for adapting measurement directions and processing their outcomes by classical communication carries computation

Role of entanglement

The role of entanglement in the initial resource state is highlighted in measurement-based quantum computer, since computation is carried through its consumption by local measurements and classical communication.

role of entanglement for universality ("most powerful potential") of measurement-based quantum computation

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role of entanglement for universality ("most powerful potential") of measurement-based quantum computation

merits to consider universality as computational power:

- taking advantage of entanglement theory
- no direct consideration on speed-up (complexity) in comparison with classical computation
- but, some non-universal quantum computation may allow efficient classical simulation, ...indeed true in our case!

Universality of quantum computation

Quantum computation is universal in the standard circuit model, for any n -qubit unitary gate operation U can be realized (in arbitrary accuracy) for the n logical quantum wires.

conventional (bottom-up) approach

To show a capability of a universal set of elementary gates and their composability

exact universality

$$\{\text{CNOT}, \forall \text{SU}(2)\}$$

approximate universality

$$\left\{ \underbrace{\text{CNOT, Hadmard, Phase, } \frac{\pi}{8}}_{\text{Clifford}} \right\}$$

entanglement (top-down) approach

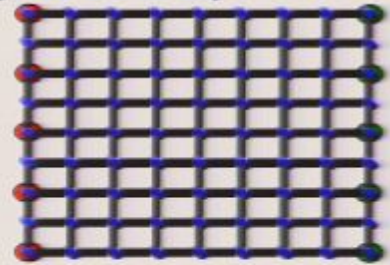
To consider entanglement convertibility by local measurements from an initial entangled resource state in measurement-based

QC

Universality in one-way QC

Using single-qubit projective measurements and classical feedforward of measurement outcomes,

$$\underbrace{|\psi\rangle}_N \xrightarrow{\text{LOCC}} \underbrace{|\phi\rangle}_n \underbrace{|\text{measured}\rangle}_{N-n}$$



- it's capable to simulate any unitary gate operation on (known) input states deterministically
- it's capable to produce any corresponding output state $|\phi\rangle$

2D cluster states $\{|\mathcal{C}_k\rangle, k = 1, 2, \dots\}$ have been shown to be universal in the sense of the above (CQ-) definition.

definition of Universal resource:

A set of states $\Psi = \{|\psi_1\rangle, |\psi_2\rangle, \dots\}$ is universal for measurement-based QC, if

$$\exists \underbrace{|\psi_k\rangle}_N \xrightarrow{\text{LOCC}} \forall \underbrace{|\phi\rangle}_n \underbrace{|\text{measured}\rangle}_{N-n}$$

symbolically,

$$|\Psi\rangle \geq_{\text{LOCC}} |\phi\rangle$$

General framework for universal resources

universal resource (exact deterministic):

$$\exists \underbrace{|\psi_k\rangle}_{\substack{N \\ \text{with fidelity } F \text{ one}}} \xrightarrow{\text{LOCC}} \forall \underbrace{|\phi\rangle}_{\substack{n \\ \text{with success probability } p \text{ one}}} \underbrace{|\text{measured}\rangle}_{N-n}$$

with fidelity F one, with success probability p one.

fidelity

- exact
- approximate ($1-\epsilon$)

probability

- deterministic
- quasi-deterministic ($1-\delta$)
- stochastic (nonzero, finite)

encoding

- logical subspace

efficiency

- universal resource is called efficient if it is capable of preparing efficiently as well any state preparable efficiently (**poly time and size**) in a standard universal (circuit/2D cluster) model

same entanglement scaling, up to a polynomial overhead, must hold true among the families of efficiently universal resources

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exponential change of entanglement

Summary

Entanglement criterion for universality

idea: if any significant entanglement feature exhibited by set of universal resource states (ex. the 2D cluster states) is not available from another set Ψ , then it cannot be universal

Condition for proper measures

$E(|\phi\rangle) \geq E(|\phi'\rangle)$ whenever $|\phi\rangle \geq_{\text{LOCC}} |\phi'\rangle$

e.x. by entanglement monotones such that $E(|\phi\rangle \otimes |0\rangle) = E(|\phi\rangle)$

- entanglement width: distinguish 1D and 2D cluster states
- geometric measure
- Schmidt measure
- ...

Criterion for universality

A set of states Ψ cannot be universal for one-way QC if

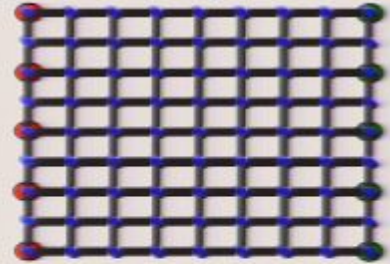
$$\sup_{|\phi\rangle} E(|\phi\rangle) > \sup_{|\psi\rangle \in \Psi} E(|\psi\rangle)$$

$$\sup_{|\phi\rangle} E(|\phi\rangle) = \sup_k E(|C_k\rangle)$$

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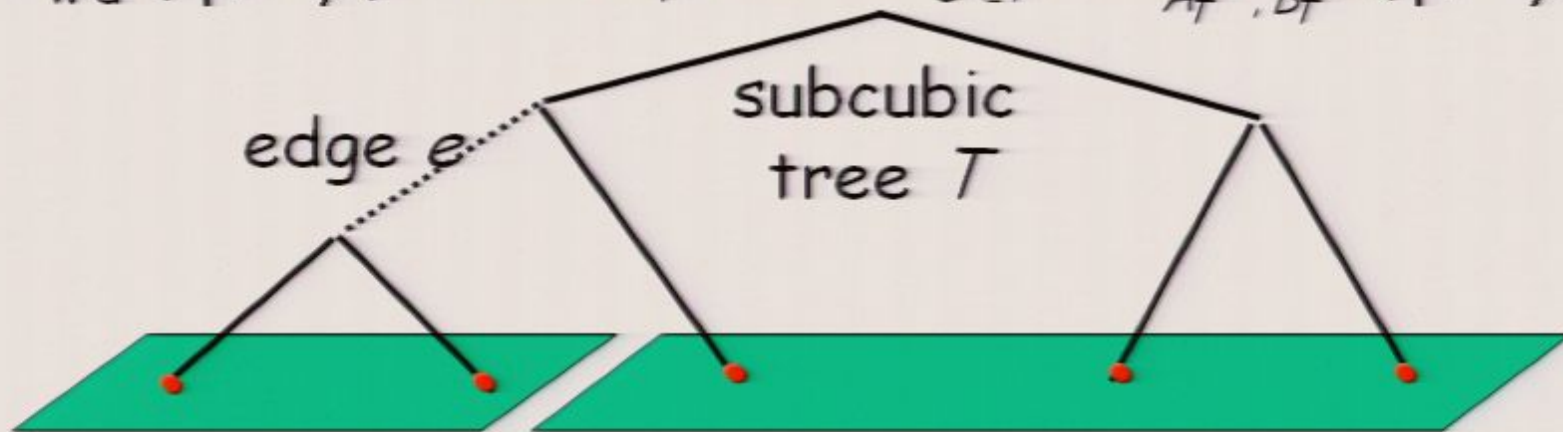
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Entanglement width

Entanglement-width

maximal entropy of entanglement associated with certain bipartitions (cf. area law of entanglement)

$$E_{wd}(|\psi\rangle) = \min_T \max_{e \in T} E_{A_T^e, B_T^e}(|\psi\rangle)$$



- non-increasing under deterministic LOCC
- equivalent to the rank-width in graph theory

1D cluster states show the divergence such as $E_{wd}(|C_N\rangle) \geq O(\sqrt{N})$

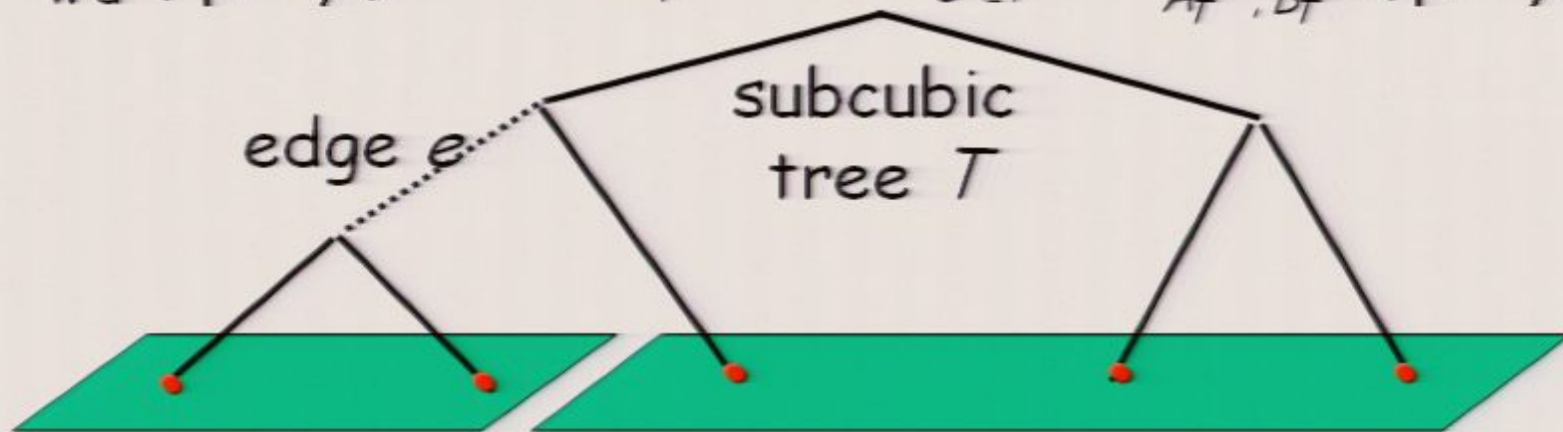
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any (efficient) universal resource must have a diverging entanglement width which scales faster than logarithmic in N

Non-universal graph states

Theorem 1:

Any set of the graph states whose entanglement width is at most polylogarithmic in the number of qubits N is not an efficient universal resource.

"interaction geometry determines the value as a resource"

• 1D cluster states ($E_{wd} = 1$) 

• GHZ (tree) states, fully connected graphs

• graphs with bounded tree-width or clique-width



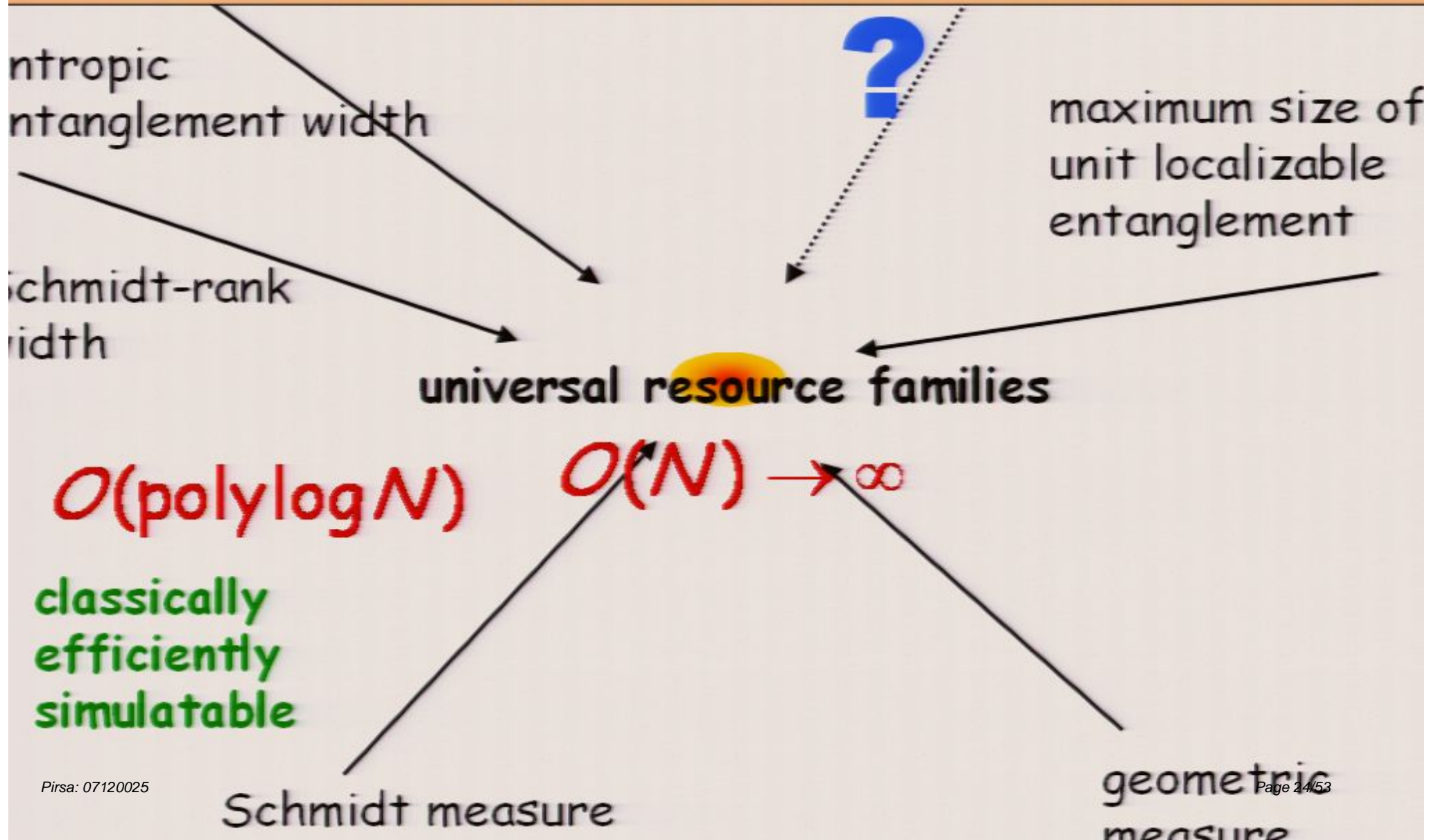
Efficient classical simulation of QC:

For graph states, if $E_{wd} = \ll \text{polylog}(N)$, then measurement-based QC is efficiently simulatable by classical computers

One-way computation on 1D cluster states and GHZ states:

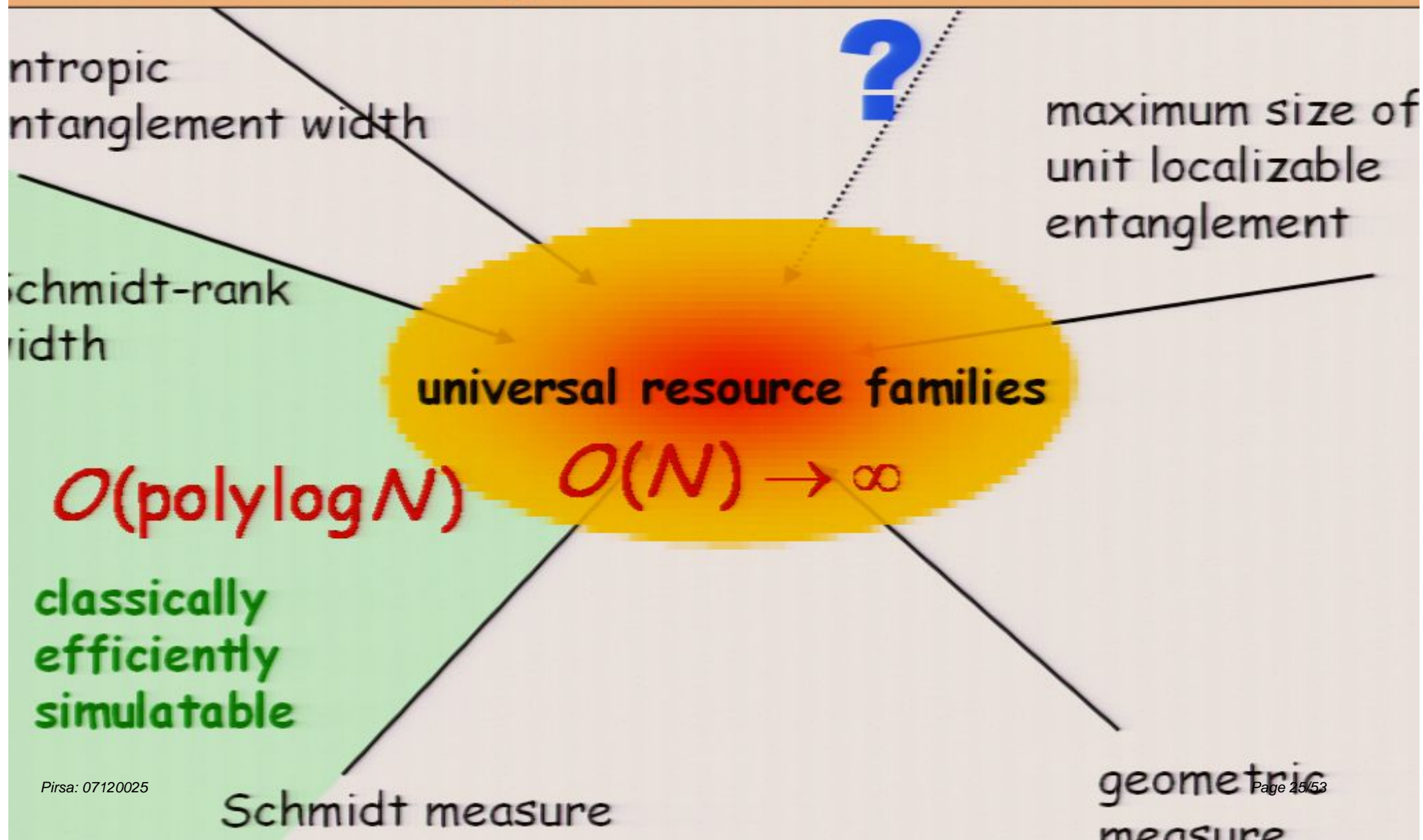
"Universe" of entanglement resources

The amount of entanglement must scale fast unboundedly!



"Universe" of entanglement resources

The amount of entanglement must scale fast unboundedly!



Blowing-up universal resource states

Observation:

A set of states Ψ is a universal resource for one-way QC, if and only if $\Psi \geq_{\text{LOCC}} |\mathcal{C}_n\rangle$ for every n .

2D cluster states

$\dots, |\mathcal{C}_n\rangle, \dots, |\mathcal{C}_N\rangle, \dots$

single-qubit
measurements

other universal
resource states

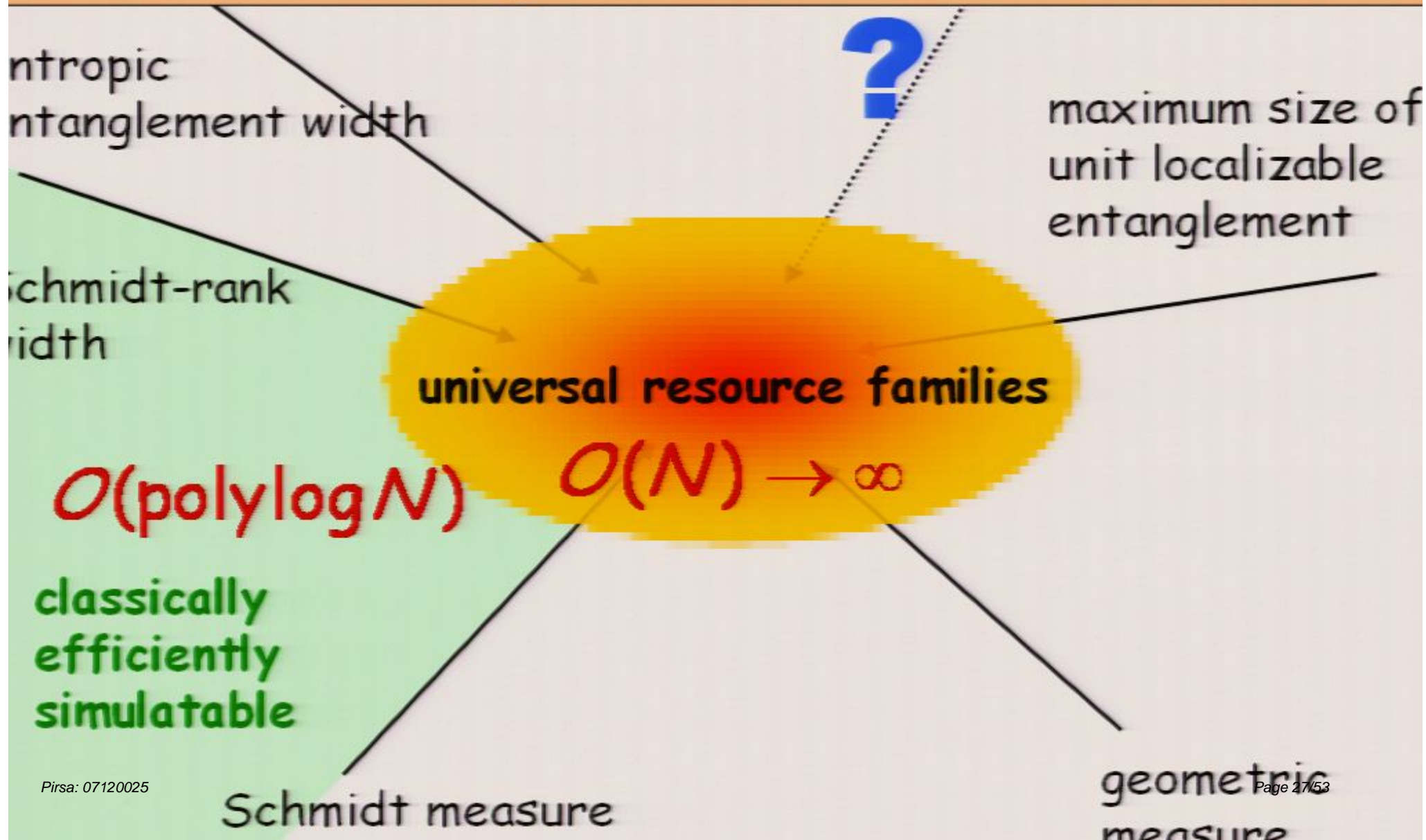
$\Psi = \{\dots, |\psi_n\rangle, \dots, |\psi_N\rangle, \dots\}$

efficient $N = \text{poly}(n)$

show equivalence between two families of entanglement resource states under LOCC (constructive proof).

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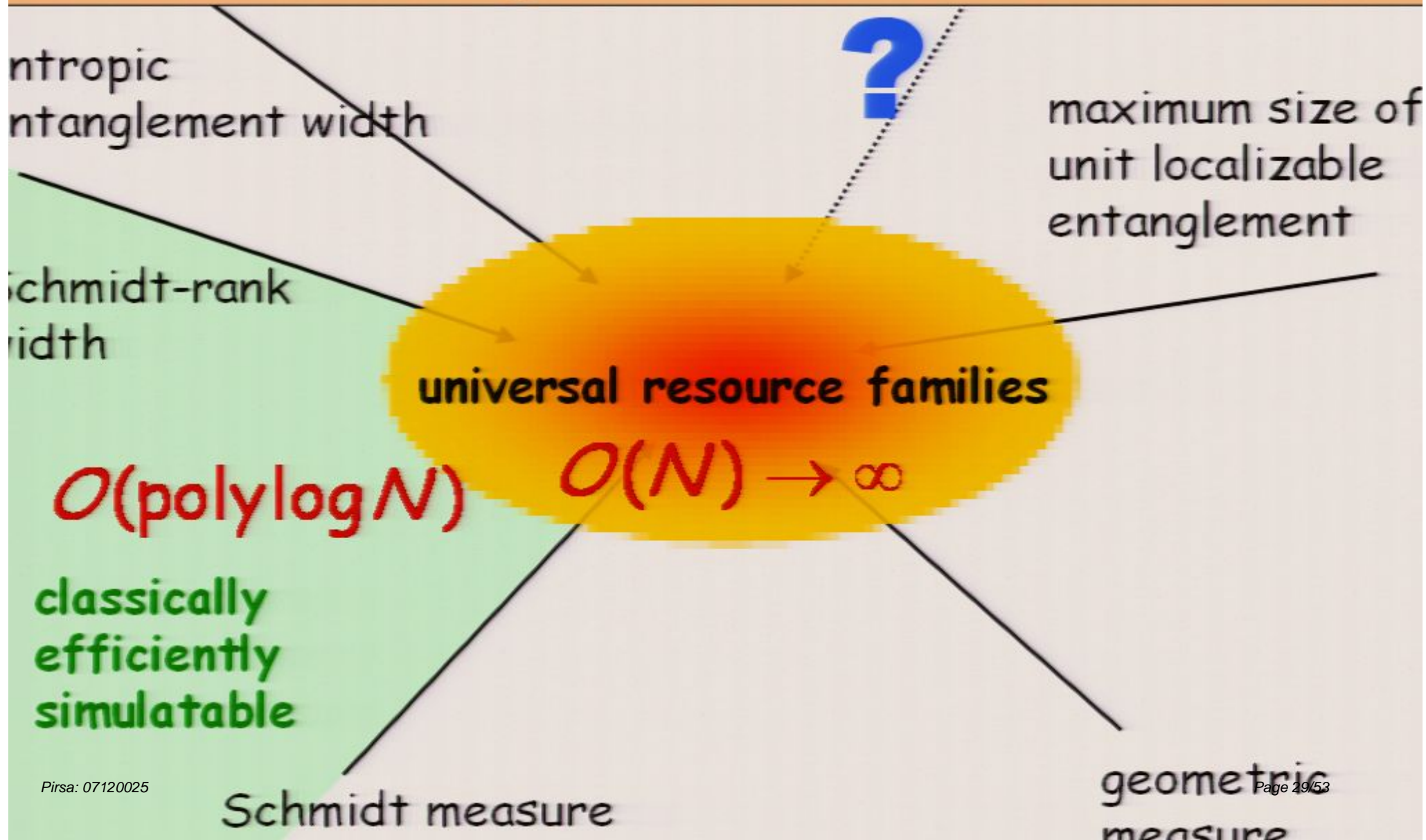
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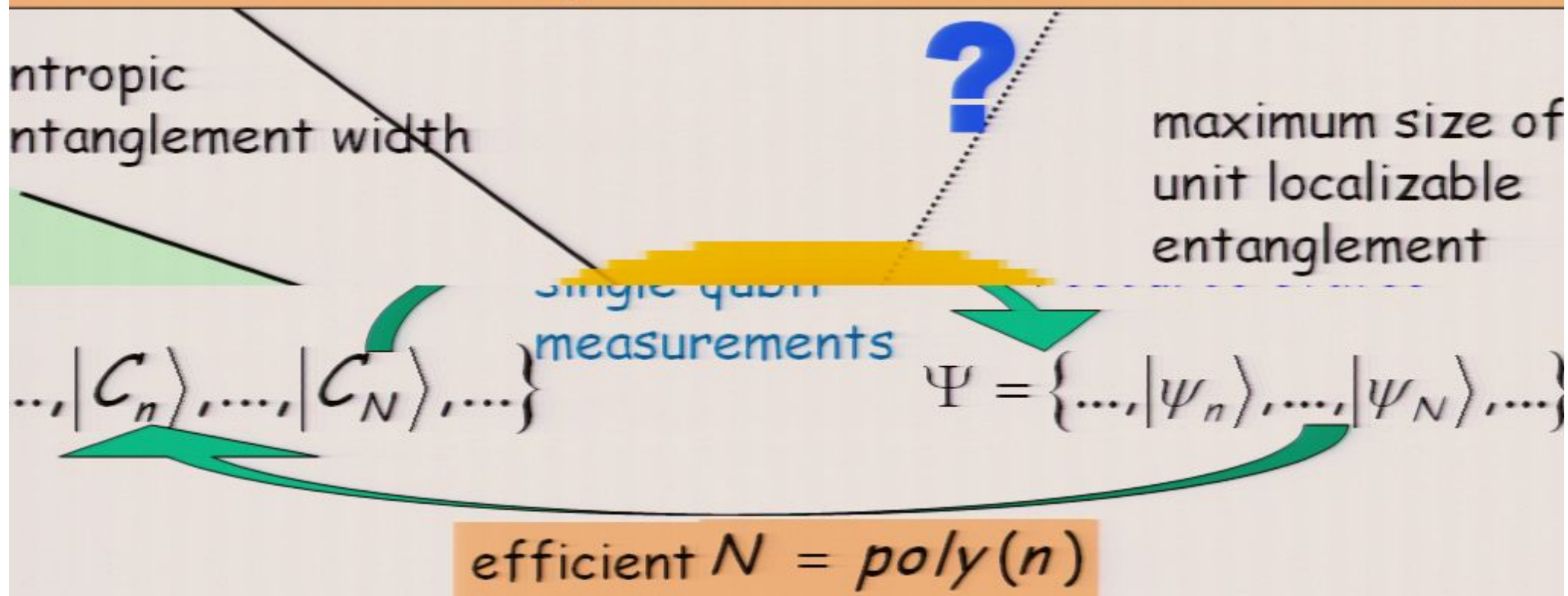
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note: it is often easier than proving universality with other (e.g. circuit) universal models, as seen soon

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Examples of universal resources

Theorem 2:

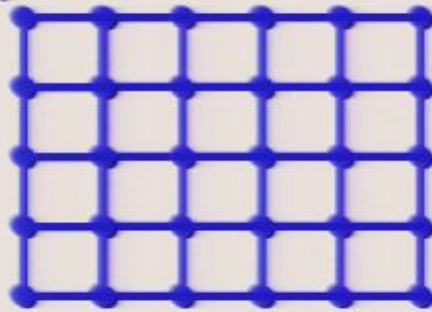
All graph states by 2D regular lattices are universal resources.

hexagonal lattice



degree = 3

square lattice
(2D cluster state)



degree = 4

triangular lattice



degree = 6

minimum possible degree for
uniform lattices to be universal

merits of the lower degree

- increased robustness for local decoherence
- convenience for bottom-up methods (ex. by polarizing beam splitters in Linear Optics QC matter-photon hybrid approach)

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Motivating observations

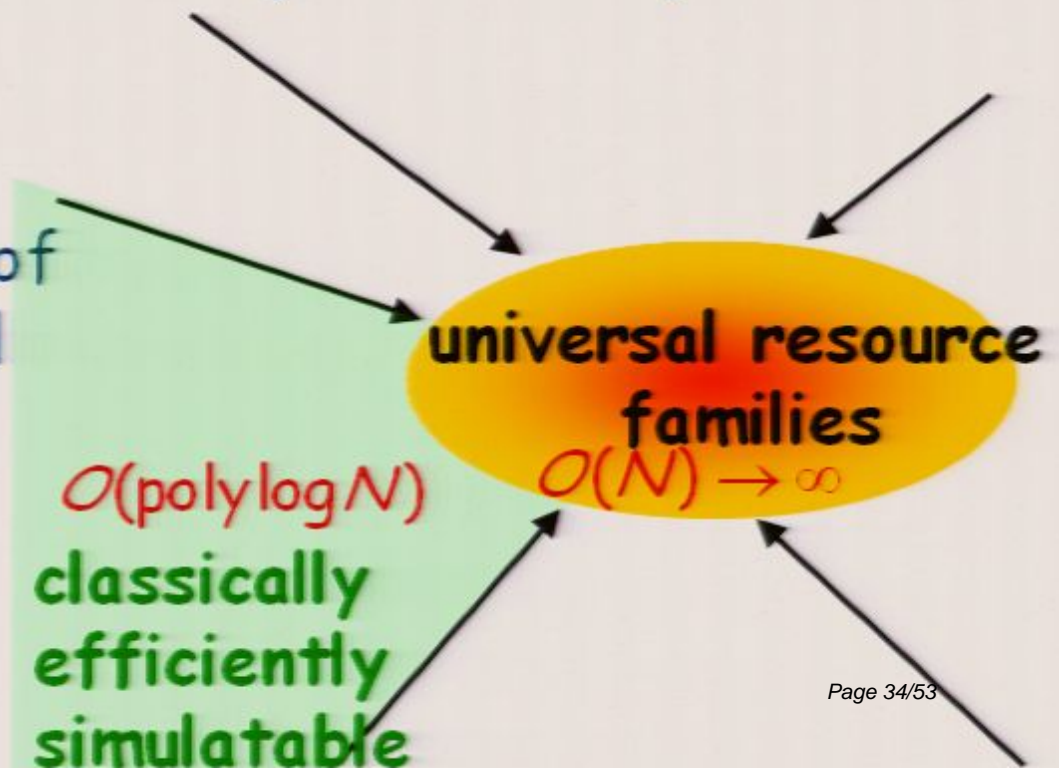
dimensionality was a key for computational power

- all 2D regular lattices: efficiently universal $E_{wd} \geq O(\sqrt{N})$

- 1D chain: non-universal, & classically simulatable $E_{wd} = 1$

What will happen at an "intermediate" (fractal-like) dimension?

Is the change ("boundary") of computational power gradual physically?



$$O((\sqrt{N})^{\frac{1}{2}})$$

Motivating observations

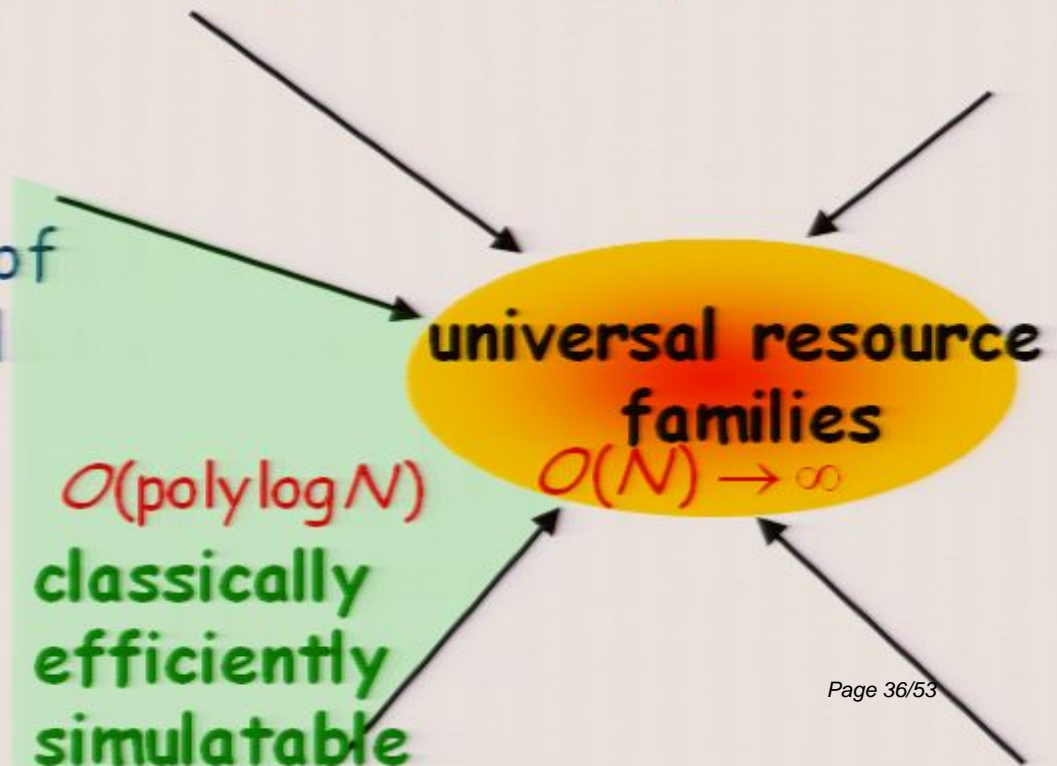
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Simple model with phase transition

2D square lattice with holes

(cf. physical implementation by the cold atoms stored in the optical lattice in the Mott-insulator regime)

every site is occupied independently with probability p_{site} , followed by the controlled-phase operations.

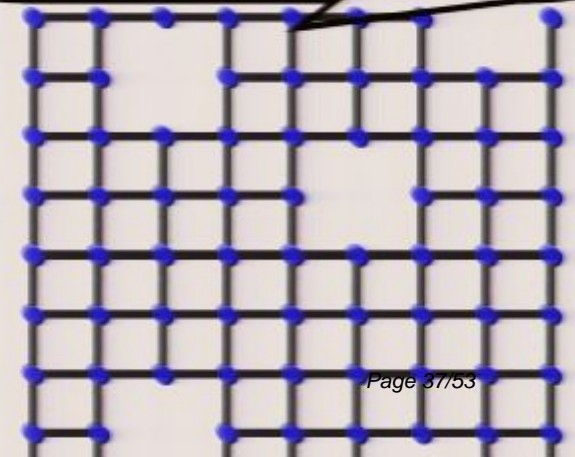
(empty site becomes a "hole" without adjacent edges)
the locations of holes are heralded.

effective dimension is decreasing from 2.

p_{site} corresponds to physical imperfection in filling the 2D square lattice.

"Is the value as the resource getting worse gradually?" **NO!**

background lattice
size $N = L \times L$



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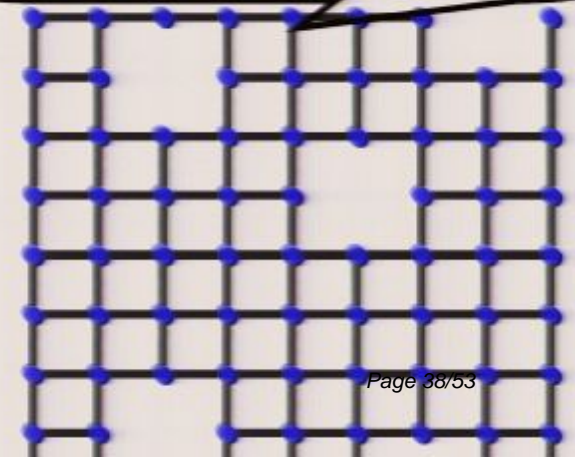
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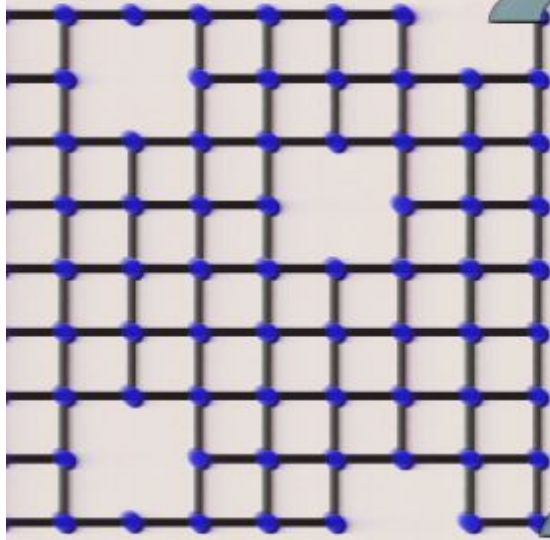


Percolation

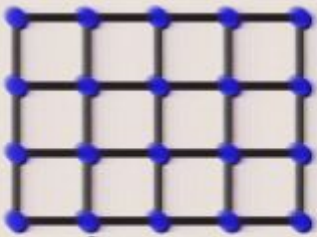
percolation (existence of $O(N)$ giant cluster) above $p_{th} = 0.592$.

Note: the $O(N)$ connected giant graph state is not necessarily universal (cf. 1D cluster and tree graphs with size N).

poly-time classical and quantum algorithm



$$p_{site} > p_{th}$$



$$E_{wd} \geq O(\sqrt{N})$$

$O(N)$ -qubit universal resource state (with a constant overhead depending on p_{site})

any LOCC

$$p_{site} < p_{th}$$



$$E_{wd} \leq O(\log N)$$

not efficiently universal (one-way QC is classically simulatable efficiently)

$$p_{site} > p_{th}$$

Supercritical phase

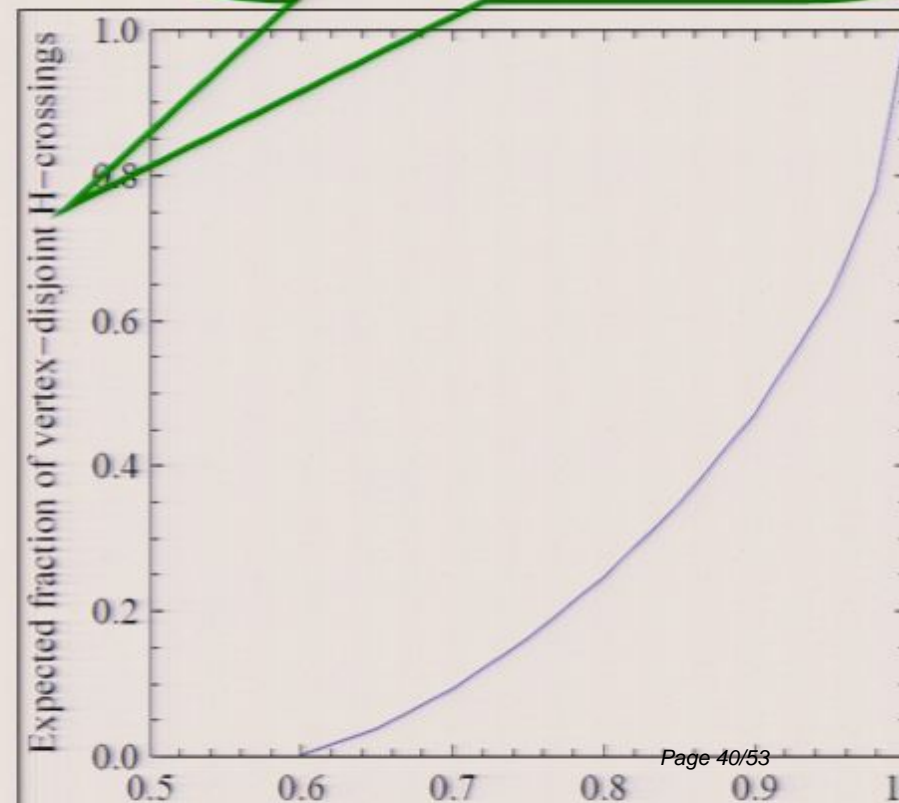
polynomial-time (quasi-deterministic) algorithm to concentrate perfect 2D cluster state with a constant overhead c

1) Lattice identification by classical poly-time algorithm

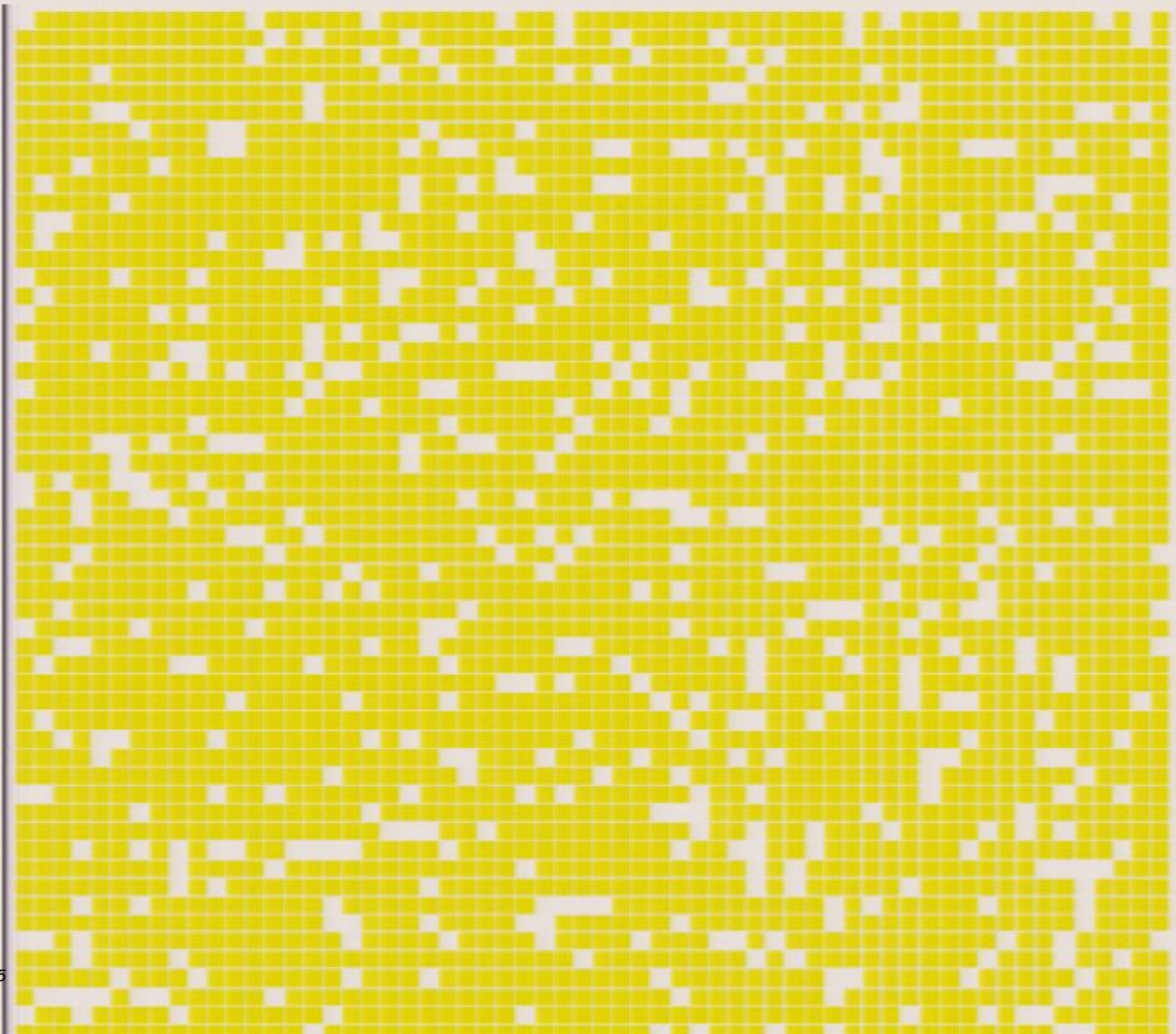
- identification of disjoint paths
- localization and correction of errors

$O(L)$ disjoint paths
(constant overhead)

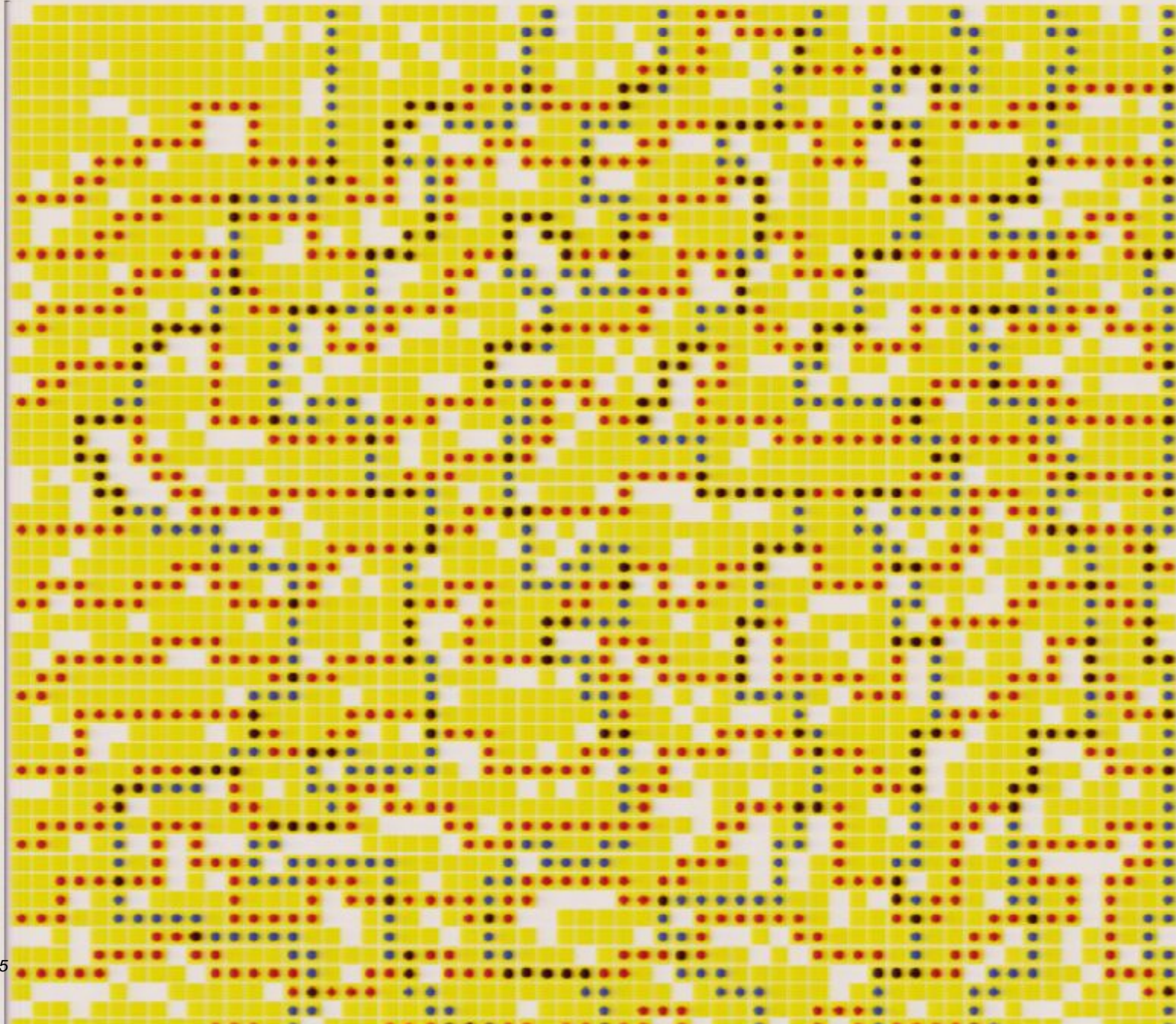
2) Lattice contraction by quantum measurements



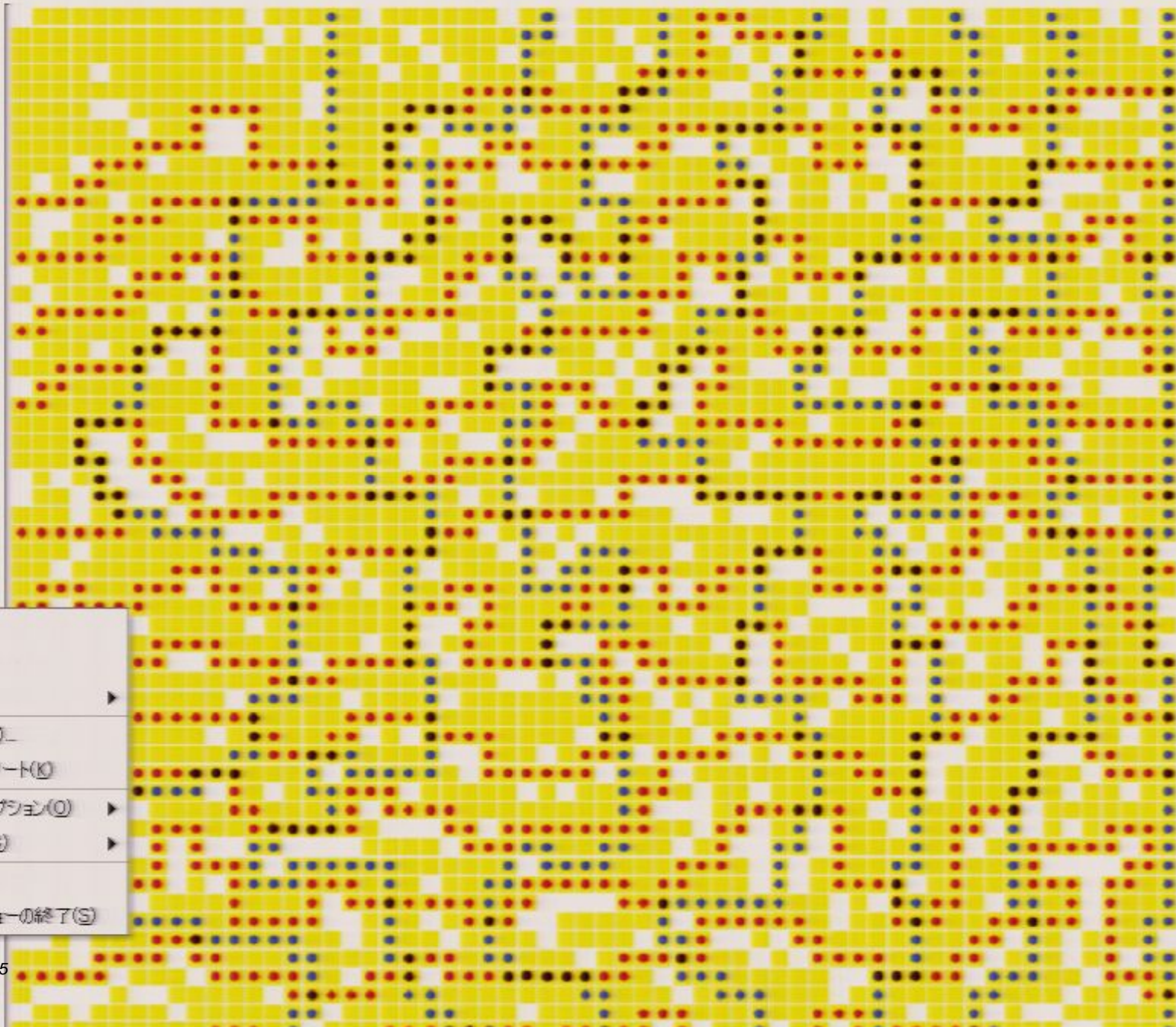
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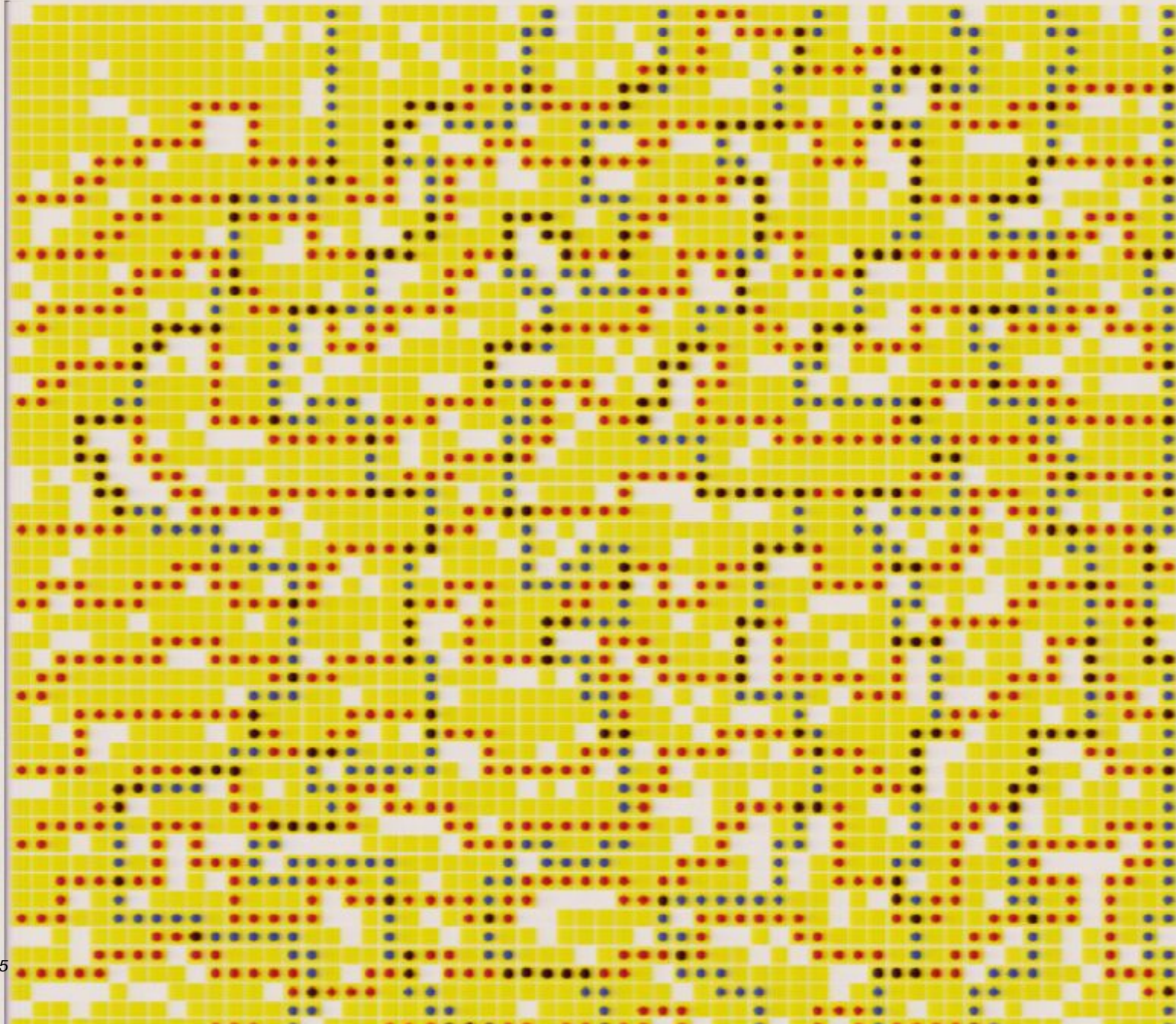


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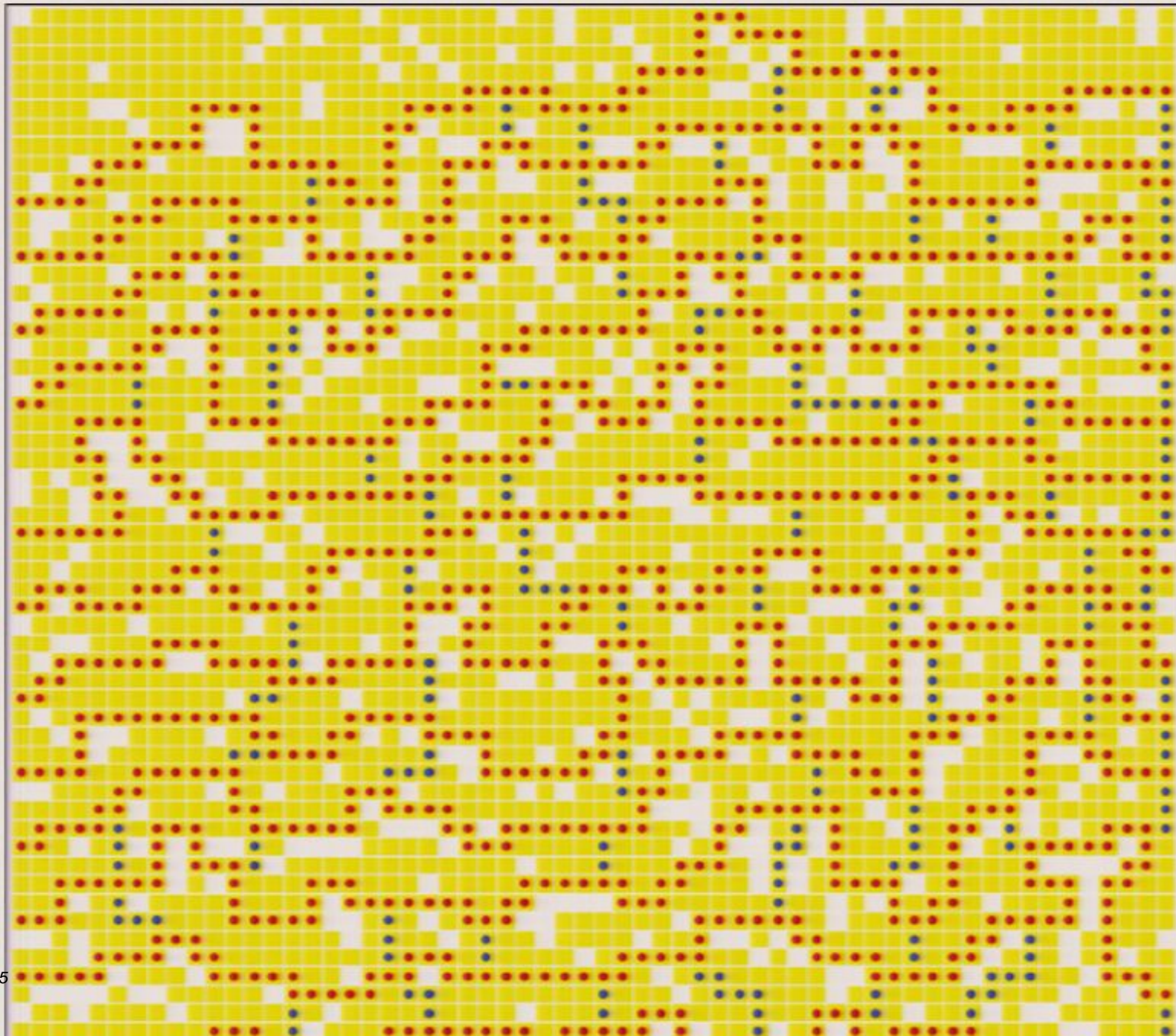


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- 前へ(P)
- 移動(G) ▶
- 会議メモ(T)
- 発表者のノート(K)
- ポインタ オプション(O) ▶
- スクリーン(C) ▶
- ヘルプ(H)
- スライドショーの終了(S)

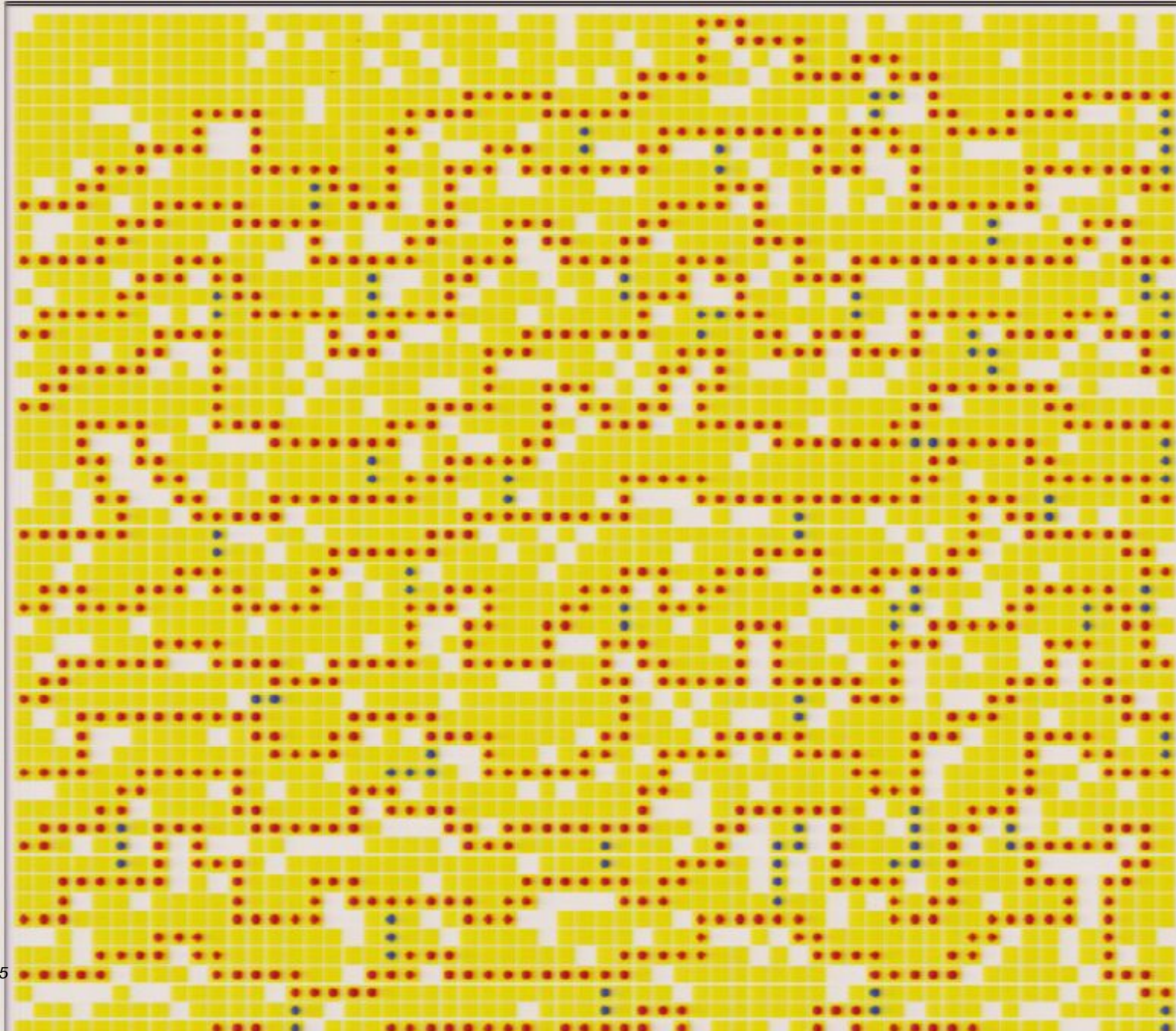
Concentration algorithm at $p=0.85$



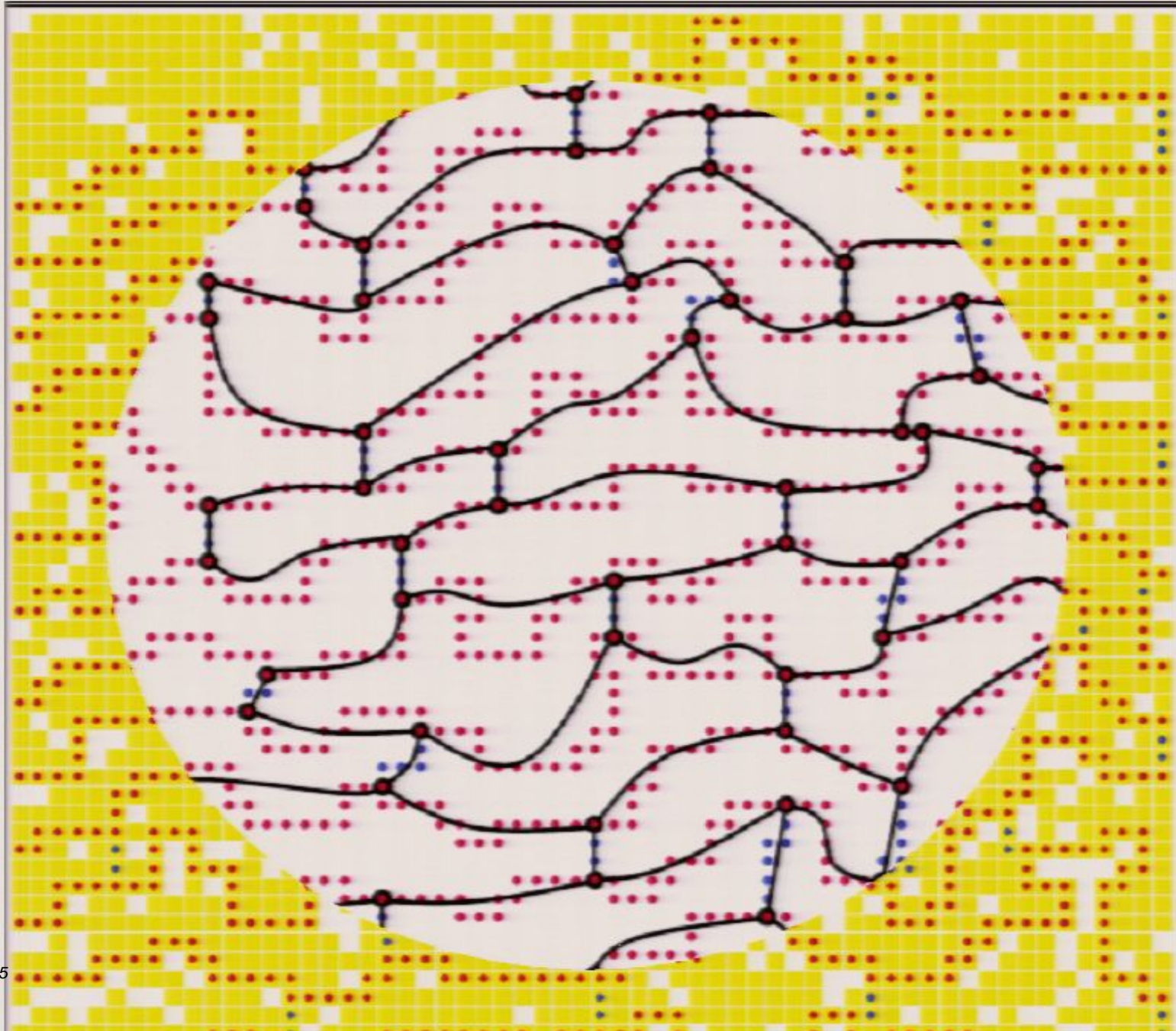
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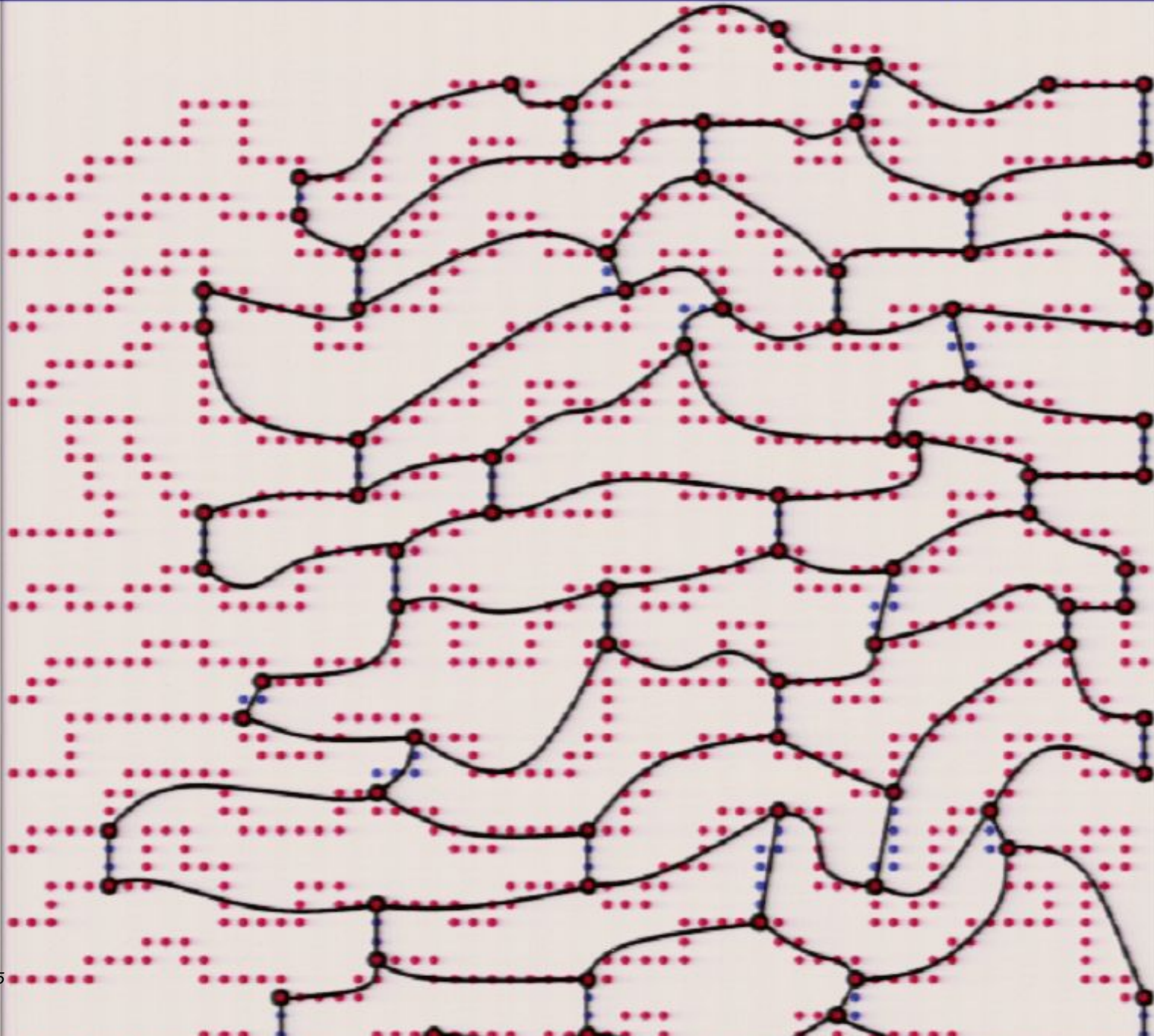
Concentration algorithm at $p=0.85$



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Concentration algorithm at $p=0.85$



$$p_{site} > p_{th}$$

Supercritical phase

polynomial-time (quasi-deterministic) algorithm to concentrate perfect 2D cluster state with a constant overhead c

1) Lattice identification by classical poly-time algorithm

- identification of disjoint crossings
- localization and correction of errors

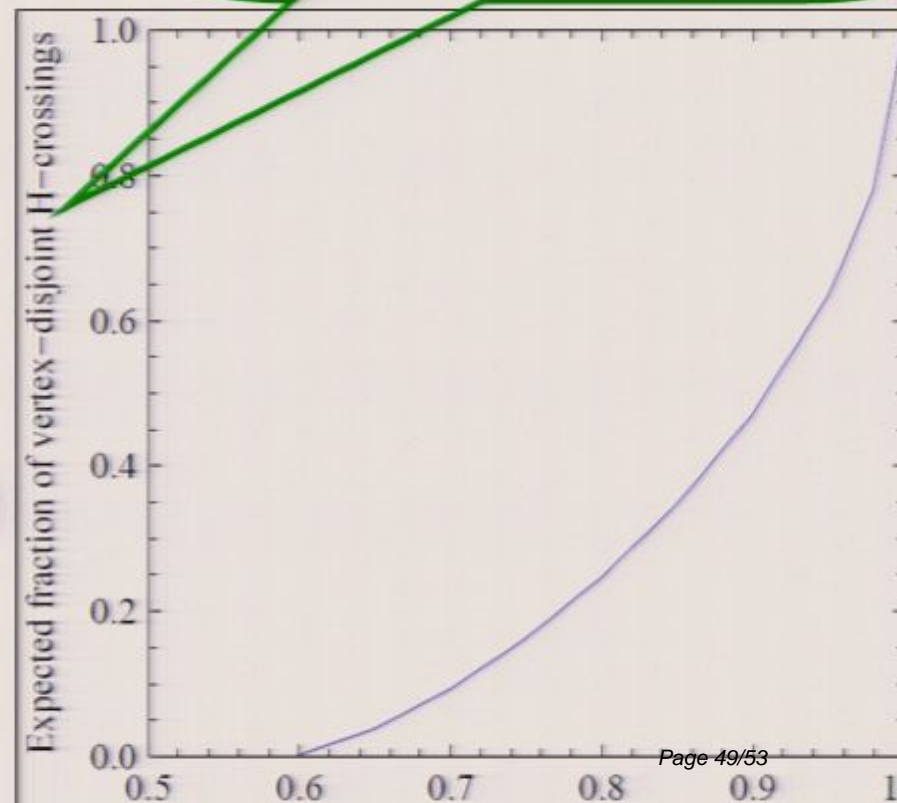
$O(L)$ disjoint paths
(constant overhead)

2) Lattice contraction by quantum measurements

E_{wd} (faulty lattice with size N)

E_{wd} (perfect 2D cluster with size cN)

$$O(\sqrt{N})$$



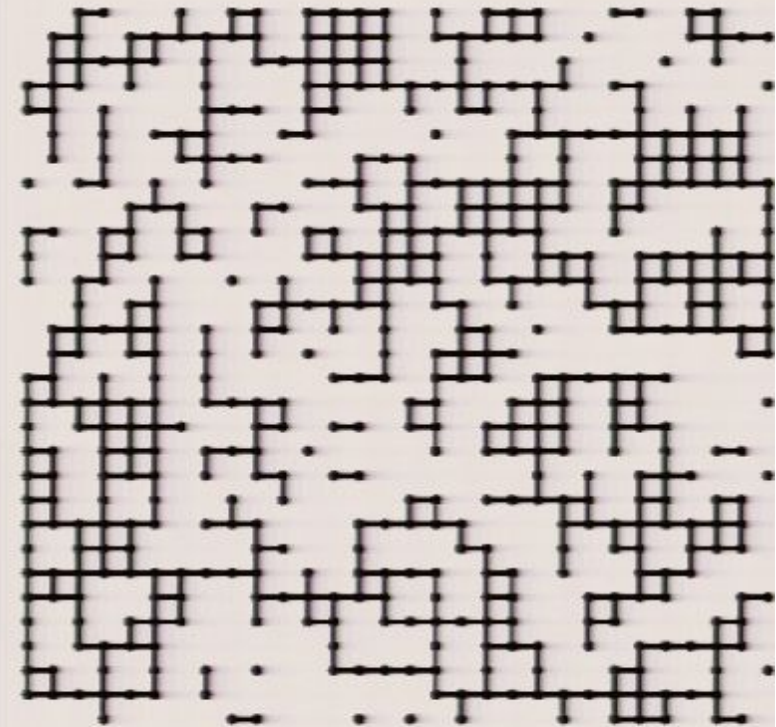
Subcritical phase

$$P_{site} < P_{th}$$

$$E_{wd}(\otimes_j |G_j\rangle) = \max_j E_{wd}(|G_j\rangle) \leq O(\log N)$$

The total amount of entanglement is determined by that of the most entangled connected component.

largest component is almost surely $O(\log N)$



quantum computational power is not simply additive!"

The computational power is not efficiently universal, and actually any measurement-based computation on it is efficiently simulatable by classical computer

Phase transition of computational power

2D square lattice with holes

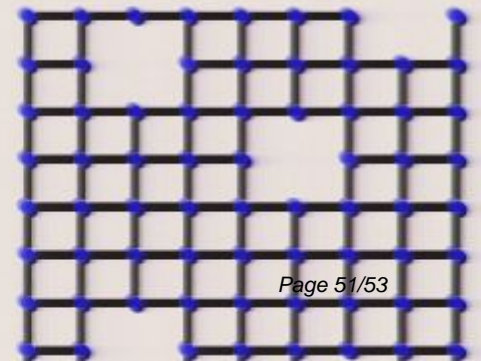
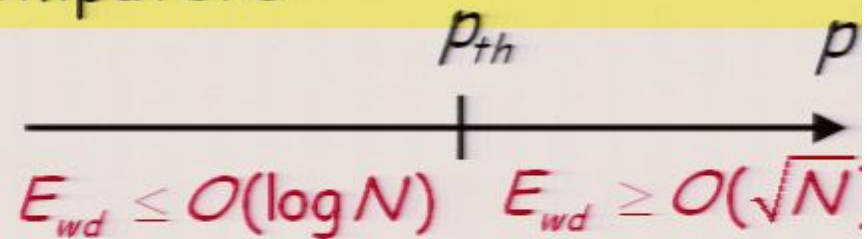
The 2D cluster state with holes undergoes the phase transition in its computational power at the percolation threshold $p_{th} \sim 0.5927\dots$

$p_{site} > p_{th}$: it is as efficiently universal as the perfect 2D cluster with size N

$p_{site} < p_{th}$: measurement-based QC on it is efficiently simulatable by classical computers

Entanglement as an order parameter

amount of entanglement measured by entanglement width changes **exponentially** at the threshold



Summary

From which entanglement feature and how does the measurement-based quantum computer obtain its superior computational power compared with the classical computer?

- scaling of entanglement reflects computational power
- **phase transition** (exponential change of entanglement)

- Van den Nest, Miyake, Dür, & Briegel,
Phys. Rev. Lett. **97**, 150504 (2006).

- Van den Nest, Dür, Miyake, & Briegel,
New J. Phys. **9**, 204 (2007), in the special issue on the
measurement-based quantum information processing.

- Browne, Elliott, Flammia, Merkel, Miyake, & Short,
arXiv:0709.1729

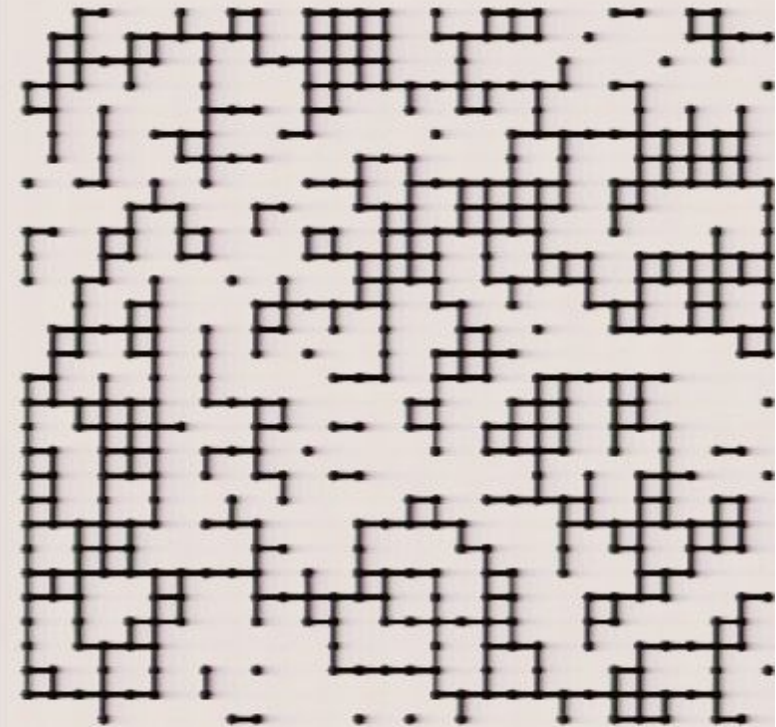
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