Title: Timeless Questions in the Decoherent Histories Approach to Quantum Theory

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Abstract: In any attempt to construct a Quantum Theory of Gravity, one has to deal with the fact that Time in Quantum Mechanics appears to be very different from Time in General Relativity. This is the famous (or actually
notorious!) \"Problem of Time\", and gives rise to both conceptual and technical problems. The decoherent histories approach to quantum theory, is an alternative formulation of quantum theory specially designed to deal with closed (no-external observer or environment) systems. This approach has been considered particularly promising, in dealing with the problem of time, since it puts space and time in equal footing (unlike standard QM) . This talk develops a particular implementation of the above expectations, i.e.
we construct a general set of \"Class Operators\" corresponding to questions that appear to be \"Timeless\" (independent of the parameter time), but correspond to physically interesting questions. This is similar to finding a general enough set of timeless observables, in the evolving constants approach to the problem of time.

Timeless Questions in the Decoherent Histories Approach to Quantum Theory

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- J.J. Halliwell & P.W.: PRD 73, 024011 (2006).
- P.W.: to appear in IJTP, gr-qc/0607072.
- P.W.: J. Phys.: Conf. Ser. 67, 012043 (2007).

Contents

- Introduce the Problem of Time, conceptual and technical aspects. Motivation for constructing timeless theories.

- Introduce the decoherent histories approach to Quantum Theory.

- Examine the decoherent histories analysis of the Problem of Time.

- Construct reparametrization invariant Class Operators.

- Examine the probabilities and decoherent conditions that arise from this proposal.

- Give a simple example
- Discussion and Summary

Problem of Time

Diffeomorphism invariance in GR Vs
Fixed parameter time in Newtonian Physics.

- Time in Quantum Theory:
  - Not Observable
  - Appears as a parameter
  - Physical clocks run backwards in abstract Newtonian Time
- Time in General Relativity:
  - How does 'change' appears?
  - Time is locally defined
  - How to make it compatible with QT that is based on Newtonian Time?

Technical Problem

- Constrained Systems:

  Less degrees of freedom

  e.g. in E.M.: $A_\mu \to A_\mu + \partial_\mu \phi$

  Physics cannot depend on choice of gauge

  Physical states are *equivalences classes*

- To quatnize:

  (a) Constrain and THEN quantize

  (b) Quantize and THEN impose the constraint

- If the constraint in the Classical Theory is:

  $\phi(p, x) = 0$

  in the Quantum Theory becomes:

  $\hat{\phi}|\psi\rangle = 0$

  and also require that observables $\hat{A}$ obey:

  $[\hat{A}, \hat{\phi}] = 0$

  We start with the *kinematical* Hilbert space (unconstrained) $\mathcal{H}_{kin}$.

  The physical states that obey the above condition form the *physical* Hilbert space $\mathcal{H}_{phys}$.

GR as Constrained System

The "gauge" in general relativity is the invariance of the theory under diffeomorphisms, Diff($\mathcal{M}$), which breaks into:

(a) Spatial "three"-dimensional diffeomorphisms

(b) *Hamiltonian* constraint: $\hat{H}|\psi\rangle = 0$

$$i\hbar\frac{d\hat{A}(t)}{dt} = [\hat{H}, \hat{A}(t)] = 0$$

for any $\hat{A}$ observable, due to the constraint:

Any observable $\hat{A}$, is independent of time!

• General feature of ANY theory that has vanishing Hamiltonian (e.g. relativistic particle)

Timeless Theories

• Need to construct a Quantum Theory that time does not have any fundamental role.

• Time "emerges" as a coarse grained property of the relative field configurations.

> All physical questions can be translated to questions about the possible relative configurations of the universe and its material content.

(a) Evolving Constants

(b) Decoherent Histories

The Decoherent Histories Approach to QT

An alternative formulation of Quantum Theory designed to deal with closed systems.

(a) The mathematical aim is to assign probabilities to histories of closed system.

(b) Supply a QM framework of reasoning about closed physical systems.

(c) Stress Classical Logic.

(d) Understand approcimate classical universe from underlying quantum one.

(e) Deal with time-extended questions.

(f) Put space and time in equal footing.

- No macroscopic-microscopic distinction.
- No a-priori system-environment split.
- No "classical world" is assumed.
- Time is no longer in a preferred position, since we are dealing with whole histories of the system (rather than single time propositions).

Decoherent Histories: Non-relativistic QM

Copenhagen probabilities for sequential measurements:

$P(\alpha_{t_1}$ at t_1 and α_{t_2} at $t_2 \cdots \alpha_{t_n}$ at $t_n; \rho(t_0))=$

$\mathrm{Tr}(\alpha_{t_n}(t_n) \cdots \alpha_{t_1}(t_1)\rho(t_0)\alpha_{t_1}(t_1) \cdots \alpha_{t_n}(t_n))$

This is NOT probability for closed system, fails to satisfy the "additivity of disjoint regions of the sample space", due to interference.

Under certain conditions this probability CAN be assigned to histories of closed systems.

Class operator: $C_{\underline{\alpha}} = \alpha_{t_n}(t_n) \cdots \alpha_{t_1}(t_1)$

$$p(\alpha_1 \sqcup \alpha_2) \neq p(\alpha_1) + p(\alpha_2)$$

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Decohernece Functional (measures interference):

$\mathcal{D}(\underline{\alpha}, \underline{\alpha}') = \text{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}'})$

A set of histories $\{\underline{\alpha}_i\}$, that is *disjoint* and *exhaustive* is called *complete*.

Probabilities are assigned to a history α_i, provided it belongs to a complete set such that:

$\mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_j) = 0$, for all $i \neq j$.

The probability is then $p(\underline{\alpha}_i) = \mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_i)$.

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Generalized Quantum Theory

(1) Fine grained Histories {f}: Most refined description (e.g. paths in extended configuration space). Encodes the KINEMATICS.

(2) Coarse grainings{h}: All possible partitions of fine grained sets into exhaustive and exclusive sets (complete sets).

(3) Decoherence functional $\mathcal{D}$: A complex-valued function on pair of histories obeying:

(i) Hermiticity: $\mathcal{D}(h', h) = \mathcal{D}^*(h, h')$

(ii) Positivity: $\mathcal{D}(h, h) \geq 0$

(iii) Normalization: $\sum_{h,h'} \mathcal{D}(h', h) = 1$

It encodes DYNAMICS and INITIAL CONDITION. Can be expressed via Sum Over Histories, or in Operator language:

(a) $\mathcal{D}(h', h) = \int_h \delta q \int_{h'} \delta q' \exp(i\,(S[q] - S[q'])\,/\hbar) \times \rho(q_0, q'_0)$

(b) $\mathcal{D}(h', h) = \mathrm{Tr}(C_h \rho C_{h'}^\dagger)$, where ρ is the initial state, the class operator C_h is suitably defined, and care is taken in the inner product used.

• When a coarse grained set (complete) obeys $\mathcal{D}(h, h') \approx \delta_{h,h'} p(h)$ for every pair (decoherence condition), we can assign probabilities on this set.

• Typically, there exist more than one coarse grained set that obey the decoherence condition. There is some interpretational ambiguity.

Decoherent Histories and the Problem of Time

Histories and Classical Timeless Questions:

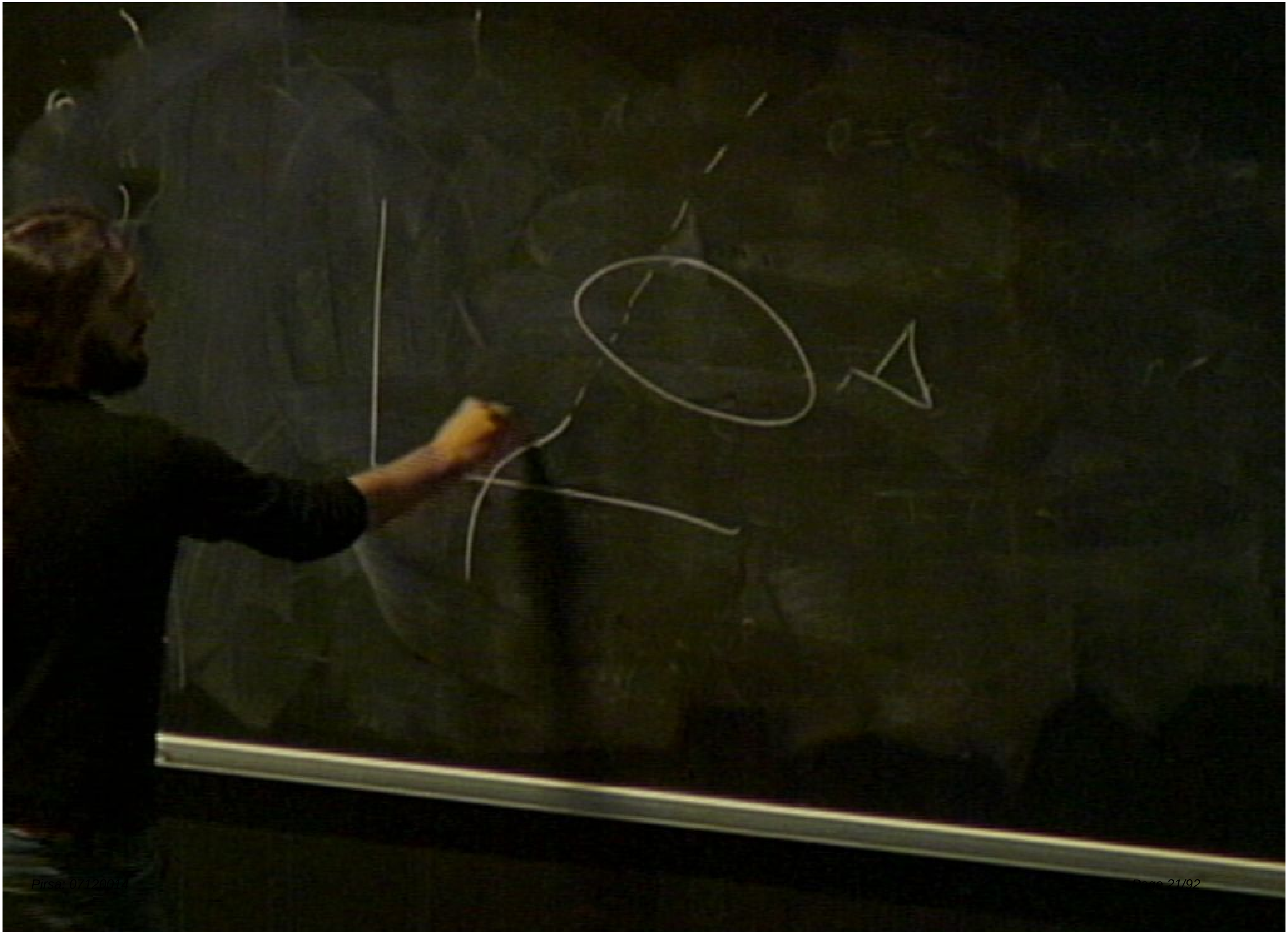
> Does a (classical) trajectory cross a given region Δ of the configuration space?

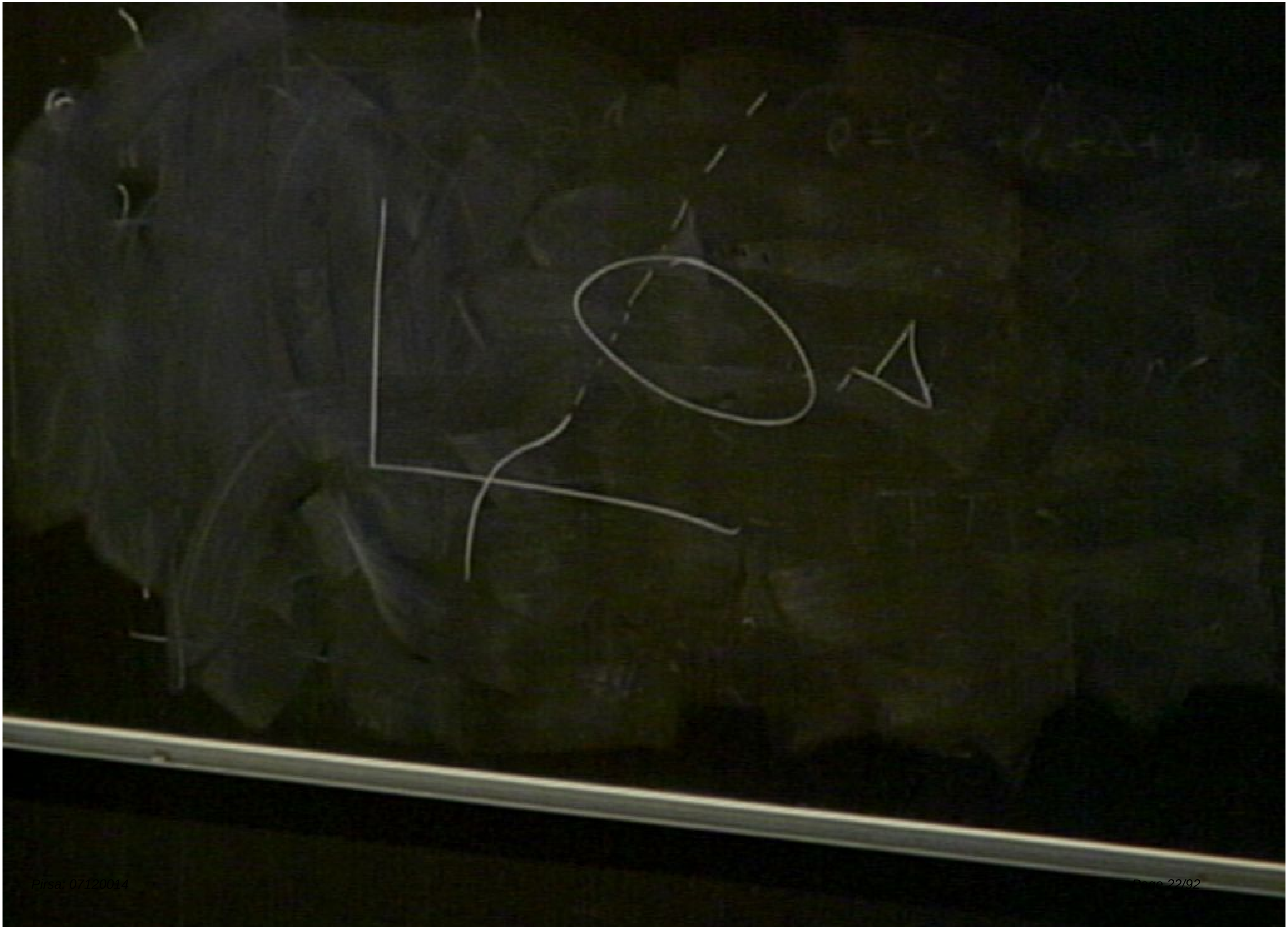
If the trajectory is infinitely extended to the past and future (i.e. the full trajectory), then it is indeed reparametrization invariant.

In the Quantum Case, we require also:

(i) Initial state has to obey: $\hat{H}|\psi\rangle = 0$

(ii) Class operator: $[\hat{C}_\alpha, H] = 0$





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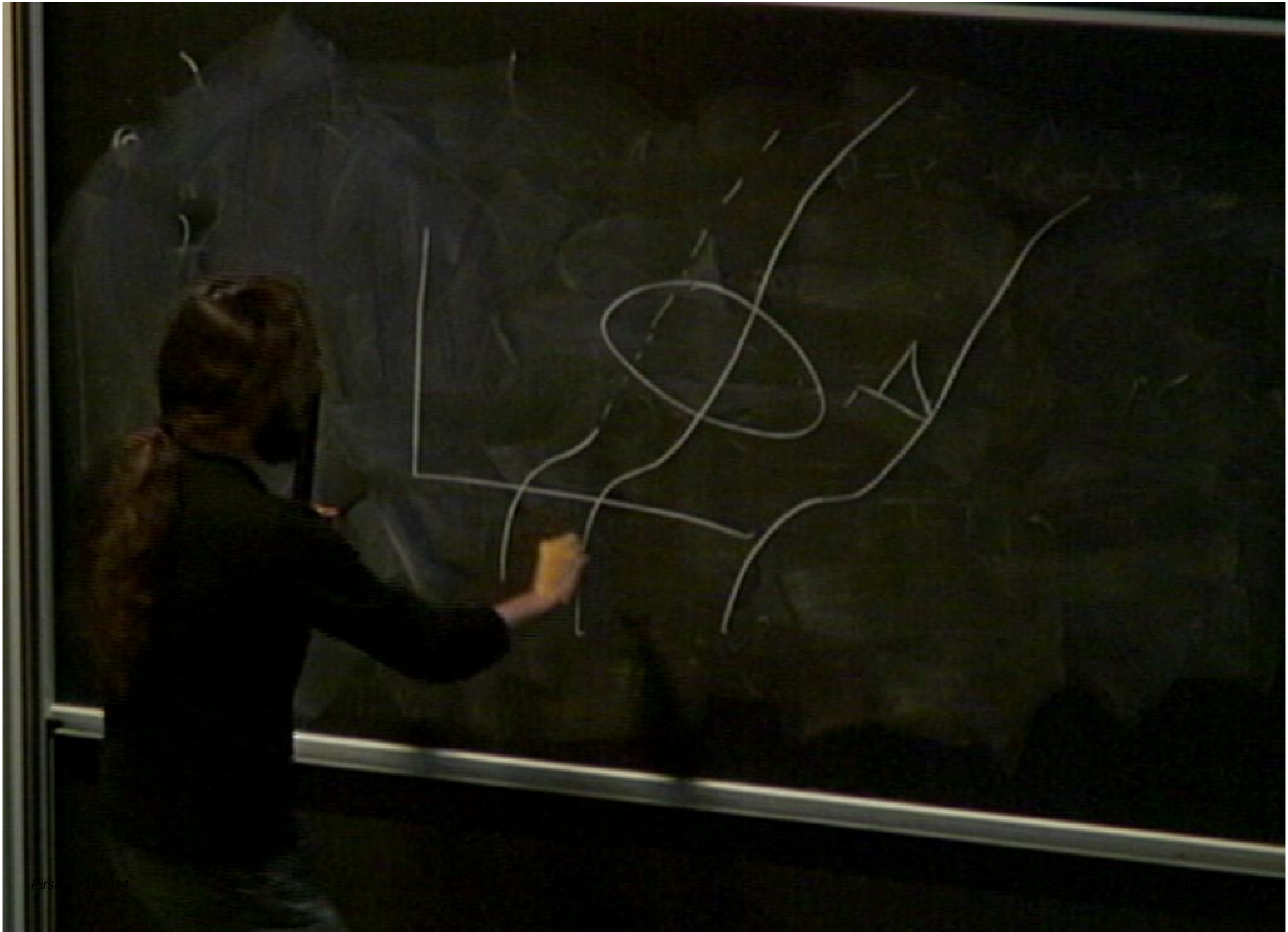

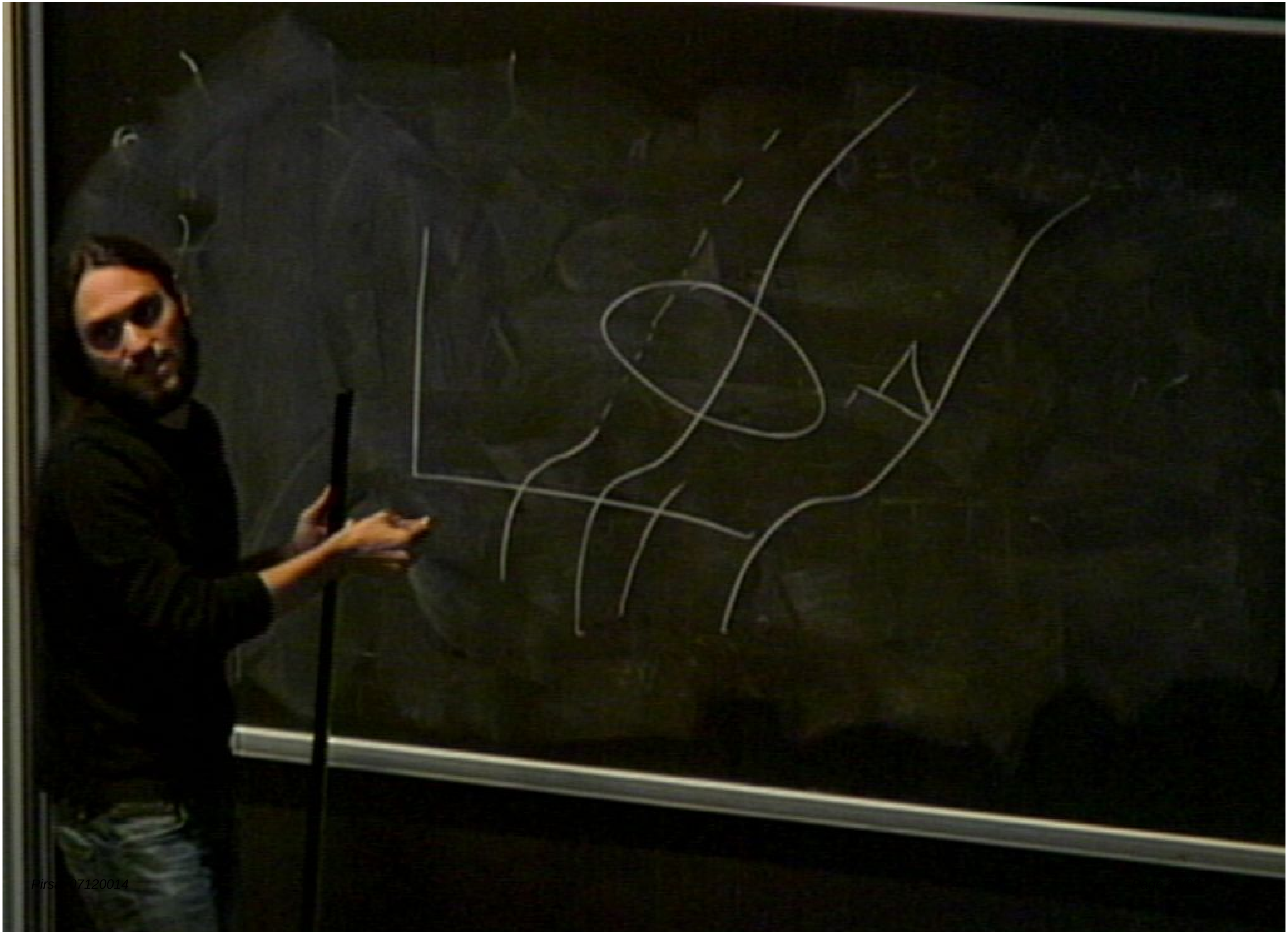

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(i) Initial state has to obey: $\hat{H}|\psi\rangle = 0$

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(iii) We have to use the *induced* (or Rieffel) inner product.

- In reparametrization invariant systems, states are not normalized on the Schrödinger inner product.
- Use inner product defined on solutions of the constraint.
- Loosely speaking:

$$H|\psi_{E_k}\rangle = E|\psi_{E_k}\rangle$$

$$\langle\psi_{E'_{k'}}|\psi_{E_k}\rangle_S = \delta(E-E')\delta(k-k')$$

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Decoherent Histories Vs Evolving Constants

- Evolving Constants:

  (i) Need to find observables (relational) that correspond to physically interesting questions.

  (ii) These observables $\hat{A}$, have to be self-adjoint operators in the Hilbert Space.

  (iii) The operators need to commute with the Hamiltonian: $[\hat{A}, \hat{H}] = 0$

- Decoherent Histories:

  (i) Need to find physically interesting coarse-grained histories (Class operators).

(ii) Those class operators $\hat{C}_\alpha$, do NOT need to be self-adjoint operators.

(iii) They have to commute with the Hamiltonian: $[\hat{C}_\alpha, \hat{H}] = 0$

(iv) The Class operators, (along with the appropriate 'initial' state, $|\Psi\rangle$) should DECOHERE. This is required to interpret $p_\alpha = \langle\Psi|C_\alpha^\dagger C_\alpha|\Psi\rangle$, as probability.

Note that the possible existence of RECORDS corresponding to histories that decohere, brings a closer relation with the evolving constant approach.

Constructing the Class Operators

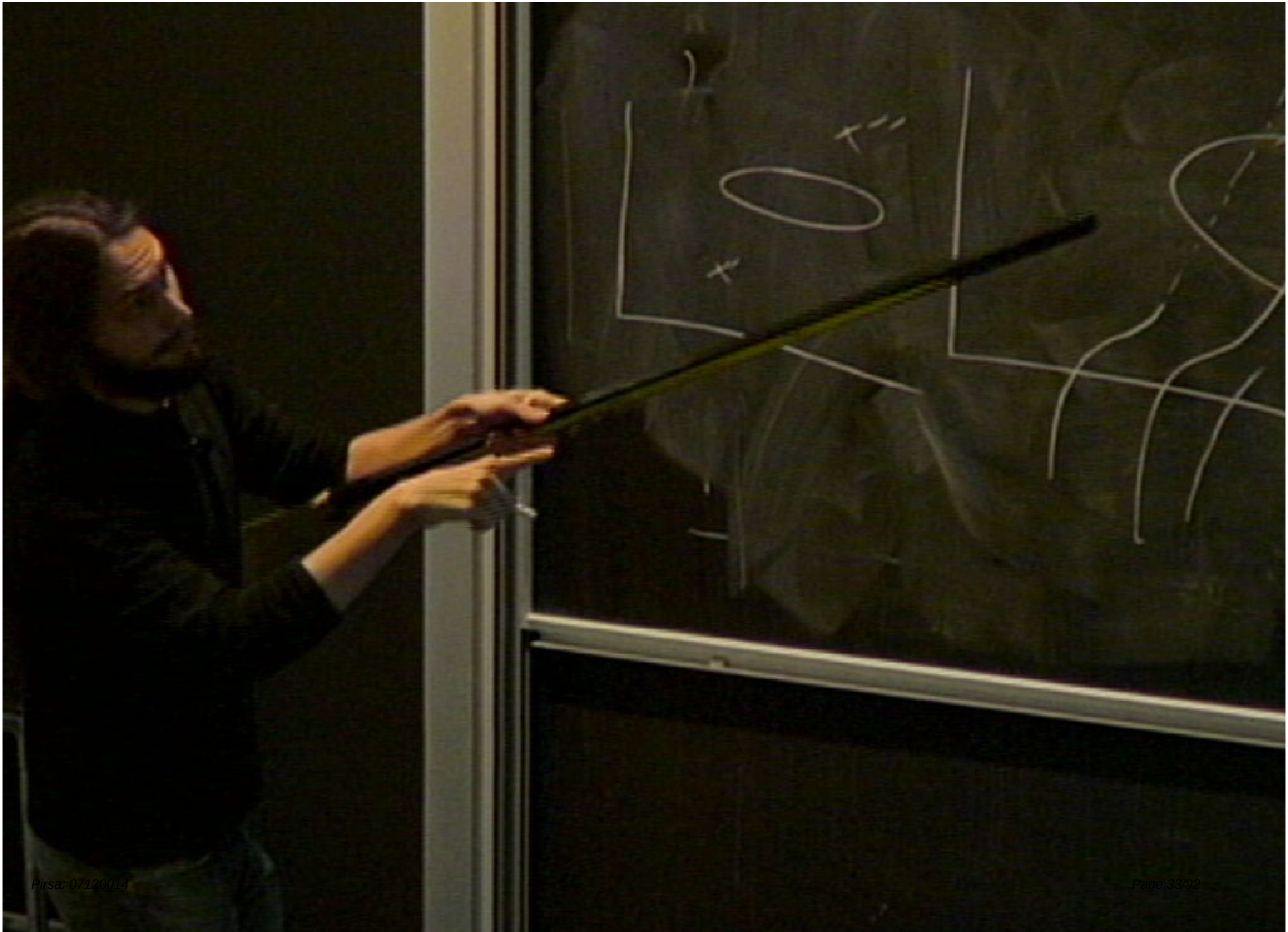
QUESTION: What is the probability that the system crosses the region Δ of the configuration space, with no reference in time.

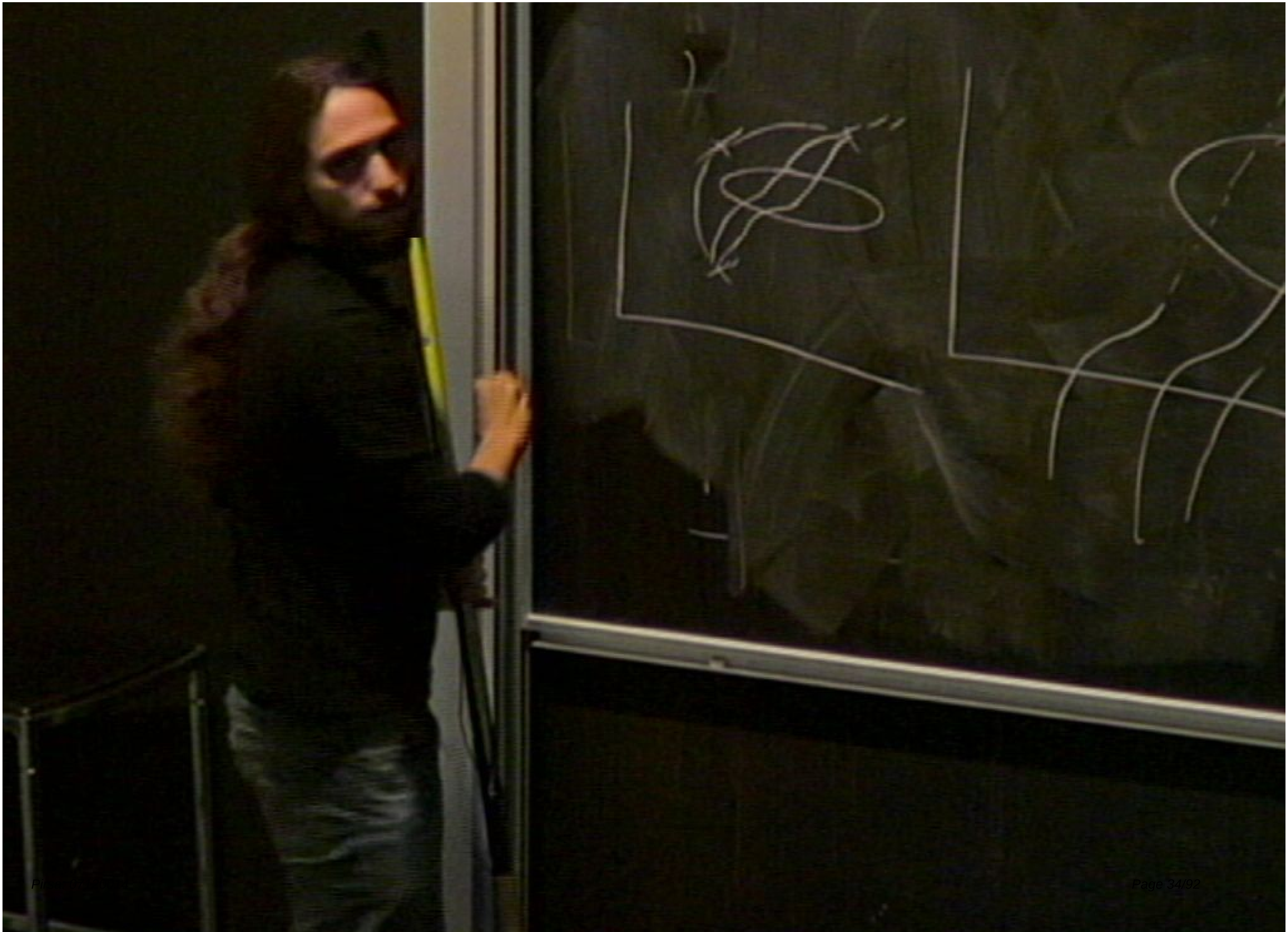
Earlier Attempts:

$$\langle x''|C_\Delta|x'\rangle = \int_0^\infty dT\, g_\Delta(x'', T|x', 0)$$

$$g_\Delta(x'', T|x', 0) = \int_\Delta \mathcal{D}x \exp(iS[x(t)]),$$

and the sum is over all the paths from x' to x'' within (parameter) time T, that crosses the region Δ.





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Time of Arrival Problem

What is the probability of a particle entering a given region Δ of space at any time between t_1 and t_2?

The class operator (in the most intuitive approach) is given by:

$$
\begin{aligned}
\langle x_f | C_\Delta | x_i \rangle &= g(x_f, T | x_i, 0) - g_r(x_f, T | x_i, 0) \\
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J.J. Halliwell & E. Zafiris, PRD 57, 3351 (1998).
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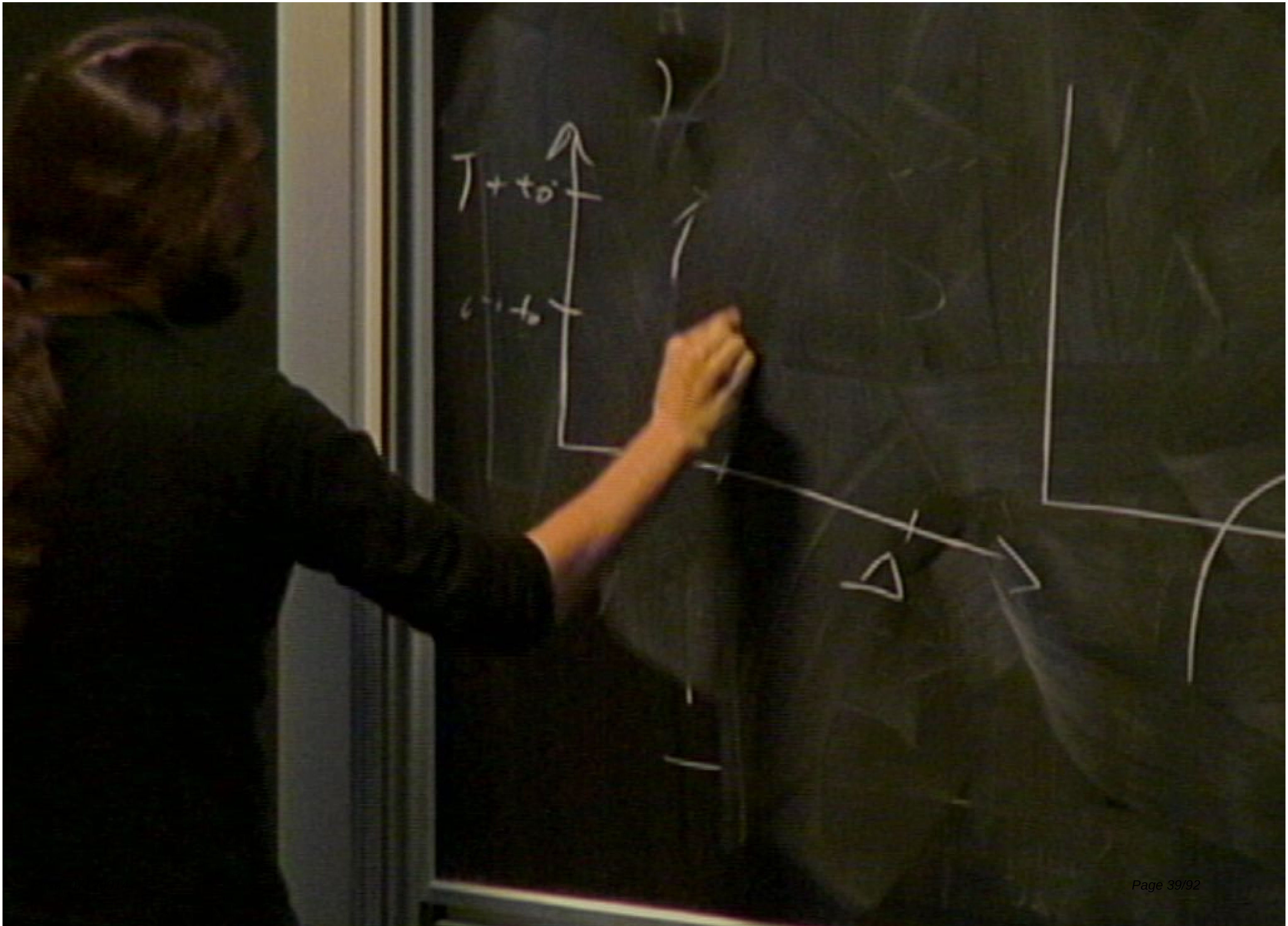
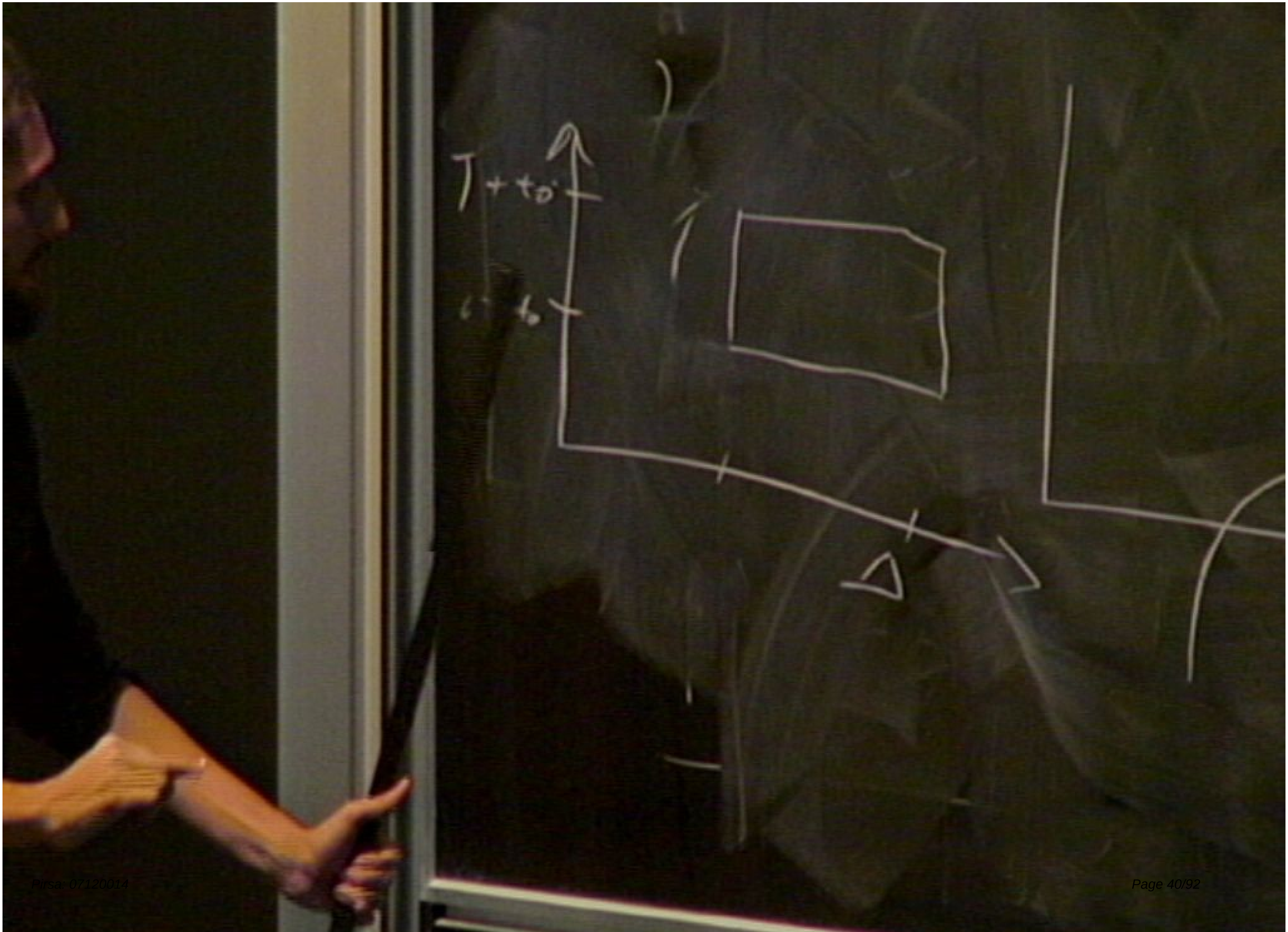

T + t0
t + t0
Δ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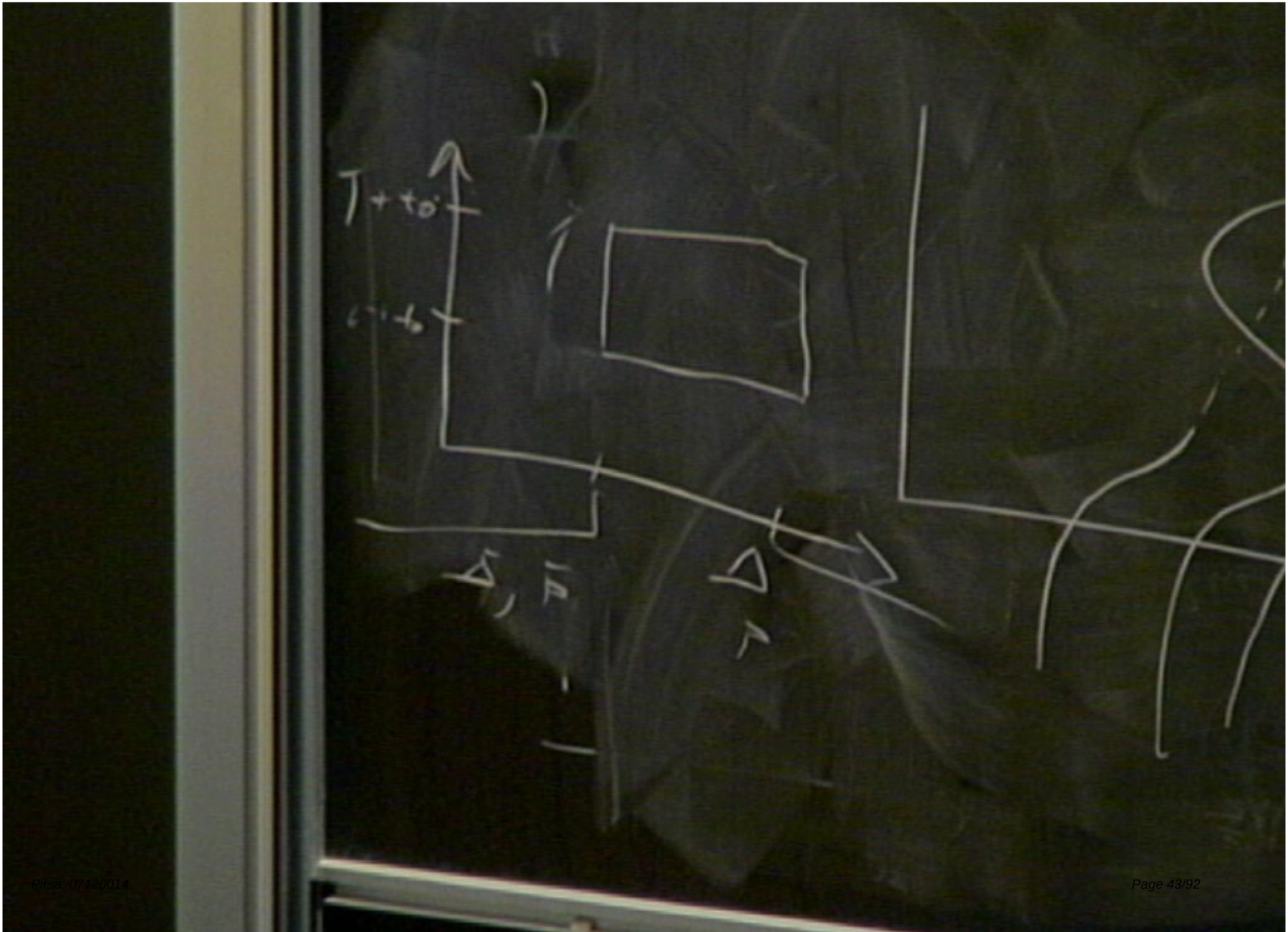

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Proposed Class Operator

Need to find a Class Operator (C.O.) that commutes with the Hamiltonian and gives (semi-classically) sensible results.

Since the relevant classical reparametrization invariant object is the full trajectory we will consider propagators that run from $-\infty$ to $+\infty$ in the unphysical parameter time.

C.O. Crossing Δ = 1− C.O. Never in Δ
= 1- C.O Always in $\bar{\Delta}$

- $$C_{\bar{\Delta}} = \prod_{t=-\infty}^{t=+\infty} \bar{P}(t)$$

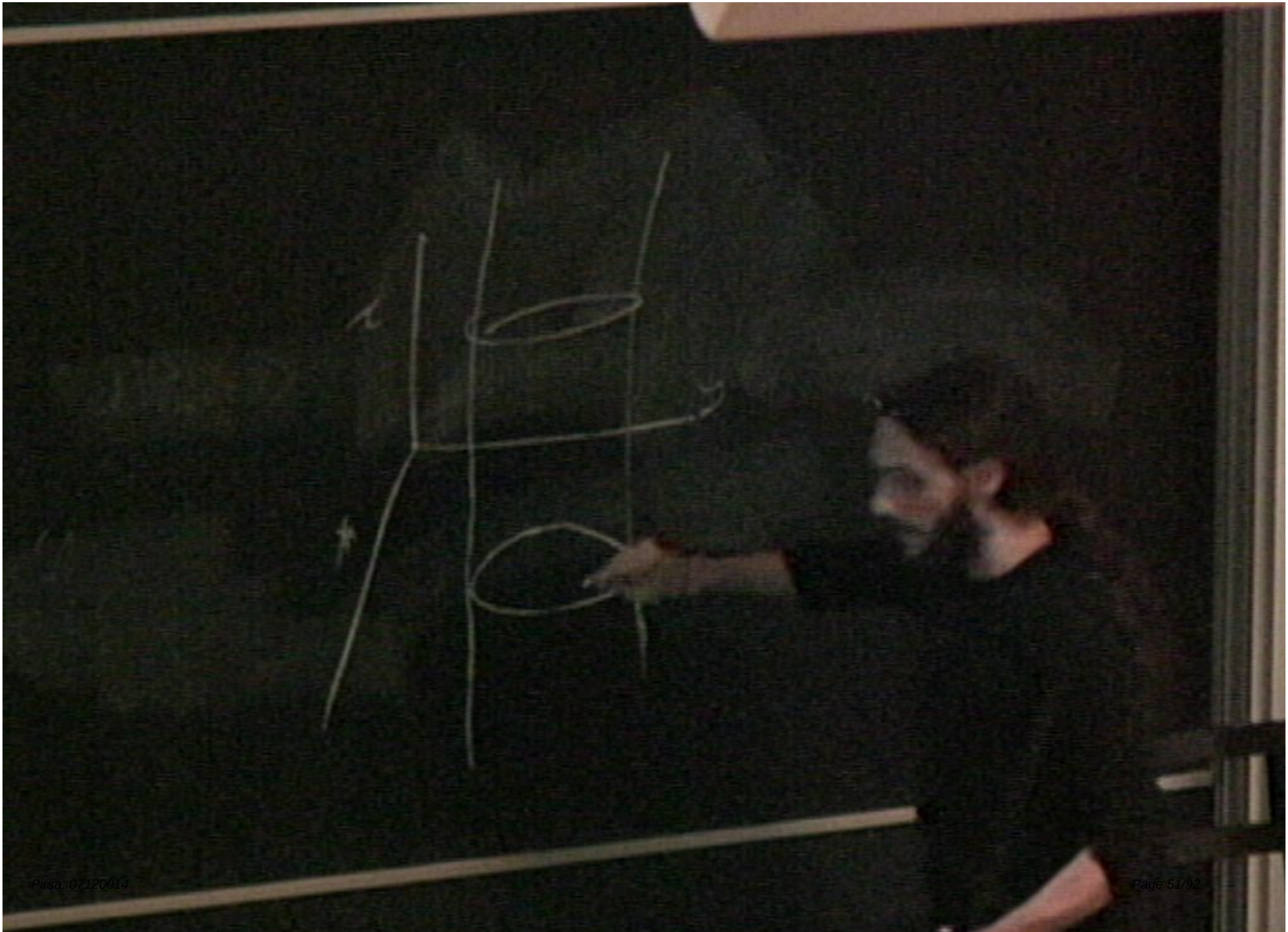
- $$[C_{\bar{\Delta}}, H] = 0$$

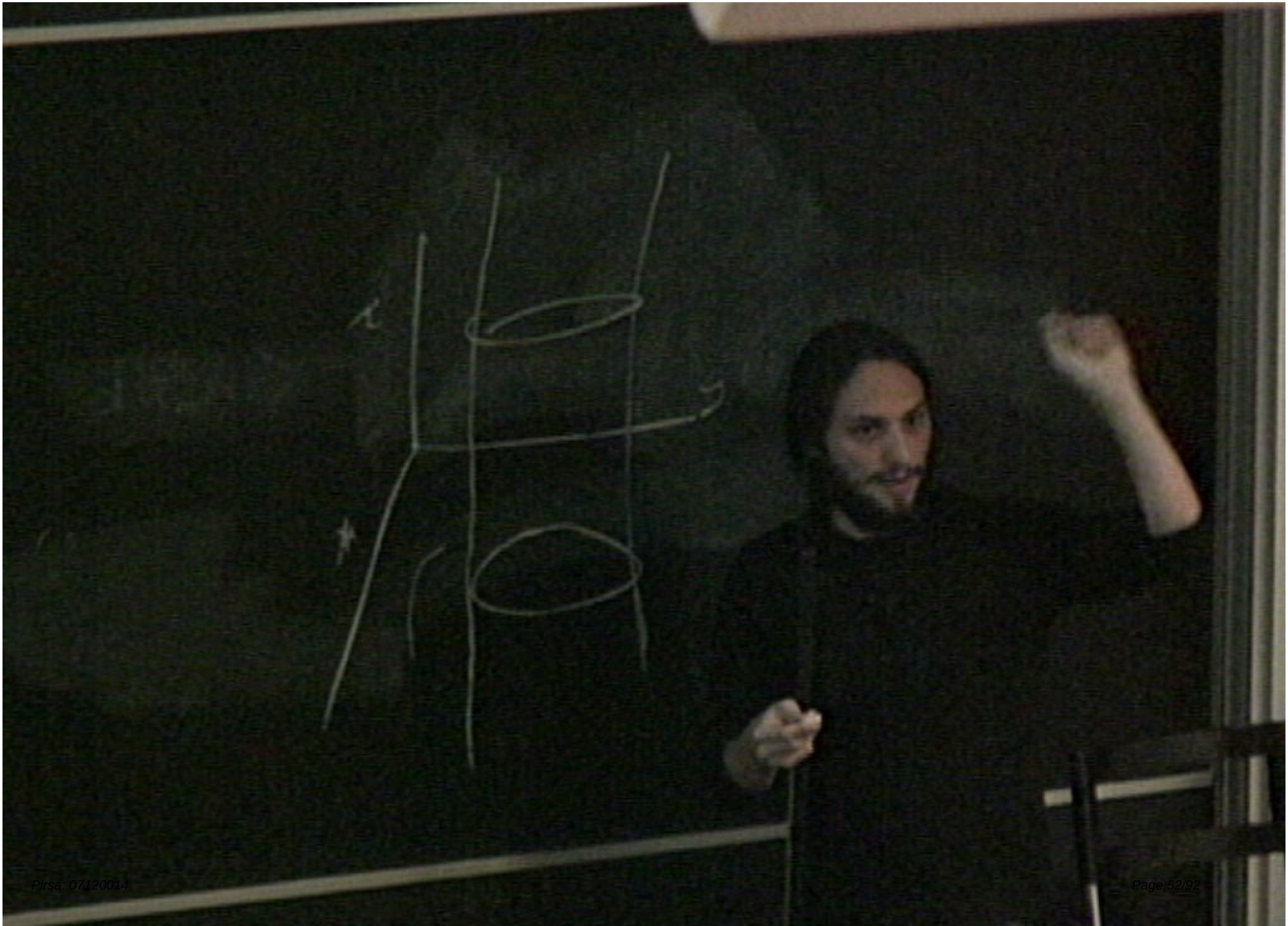
- $$C_{\bar{\Delta}} = \lim_{t'' \to \infty, t' \to -\infty} \exp(-iHt'') g_r(t'', t') \exp(iHt')$$

- $$C_{\Delta} = 1 - C_{\bar{\Delta}}$$

This expression resembles the arrival time problem in standard non-relativistic quantum mechanics.

Note that for periodic Hamiltonians (bounded systems), the above analysis changes slightly (see details in J.J.Haliwell & PW).





Restricted Propagator

- Path Integral definition:

$$g_r(x, t \mid x_0, t_0) = \int_{\bar{\Delta}} \mathcal{D}x \exp(iS[x(t)]) = \langle x | g_r(t, t_0) | x_0 \rangle \quad (1)$$

(the integral is over paths restricted to the region $\bar{\Delta}$)

- Operator definition (more general):

$$g_r(t, t_0) = \lim_{\delta t \to 0} \bar{P} e^{-iH(t_n - t_{n-1})} \bar{P} \ldots \bar{P} e^{-iH(t_1 - t_0)} \bar{P} \quad (2)$$

$\delta t \to 0$, $n \to \infty$ keeping $\delta t \times n = (t - t_0)$.

- Differential Equation:

$$(i\frac{\partial}{\partial t} - H)g_r(t,t_0) = [\bar{P},H]g_r(t,t_0) \qquad (3)$$

- 'Zeno' Expression:

$$g_r(t,t_0) = \bar{P}\exp\left(-i(t-t_0)\bar{P}H\bar{P}\right)\bar{P} \qquad (4)$$

- It is the "full" propagator with Hamiltonian $H_r = \bar{P}H\bar{P}$. To see this we pre-multiply Eq. (3) with $\bar{P}$ and use the fact that $\bar{P}[H,\bar{P}]\bar{P} = 0$ to end up with the Schröndinger equation with $H = H_r$.

- Important property that is transparent from the 'Zeno' expression:

$$g_r^\dagger(t,t_0)g_r(t,t_0) = \bar{P} \qquad (5)$$

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• It is the "full" propagator with Hamiltonian $H_r = \bar{P}H\bar{P}$. To see this we pre-multiply Eq. (3) with $\bar{P}$ and use the fact that $\bar{P}[H, \bar{P}]\bar{P} = 0$ to end up with the Schröndinger equation with $H = H_r$.

• Important property that is transparent from the 'Zeno' expression:

$$g_r^\dagger(t, t_0)g_r(t, t_0) = \bar{P} \quad (5)$$

General No-Crossing Probabilities

Using the last property we have (candidate) probability for not crossing:

$$p_{\bar{\Delta}} = \langle\psi|C^{\dagger}_{\bar{\Delta}} C_{\bar{\Delta}}|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle$$

IF we had *decoherence*, then the crossing probability would be:

$$p_{\Delta} = 1 - p_{\bar{\Delta}} = \langle\psi|P|\psi\rangle$$

This has quite striking consequence for the decoherent histories analysis of the arrival time problem. It doesn't allow for particle starting at $\bar{\Delta}$ crossing to the region Δ!!

$$(i\frac{\partial}{\partial t} - H)g_r(t, t_0) = [\bar{P}, H]g_r(t, t_0) \quad (3)$$

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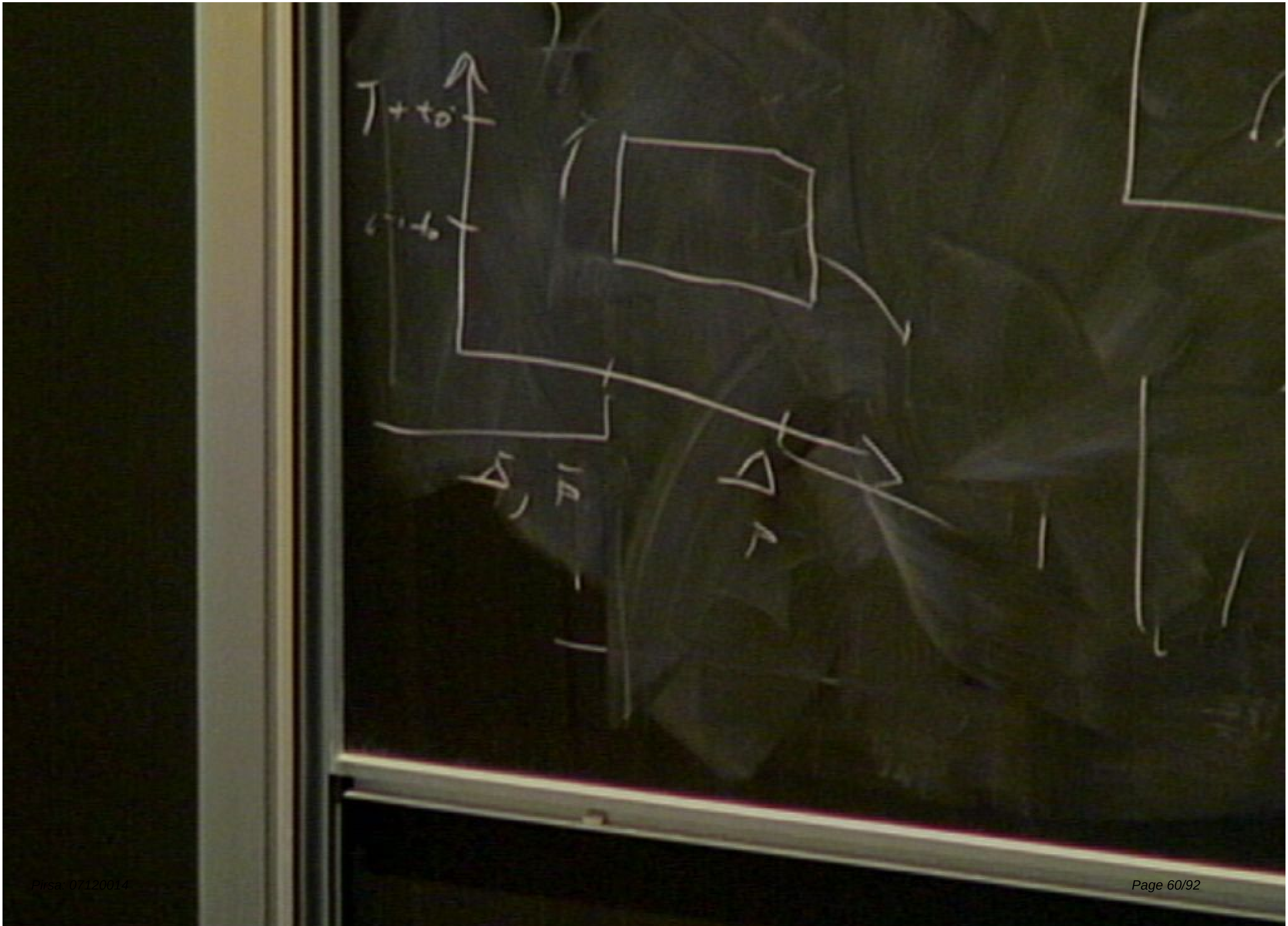

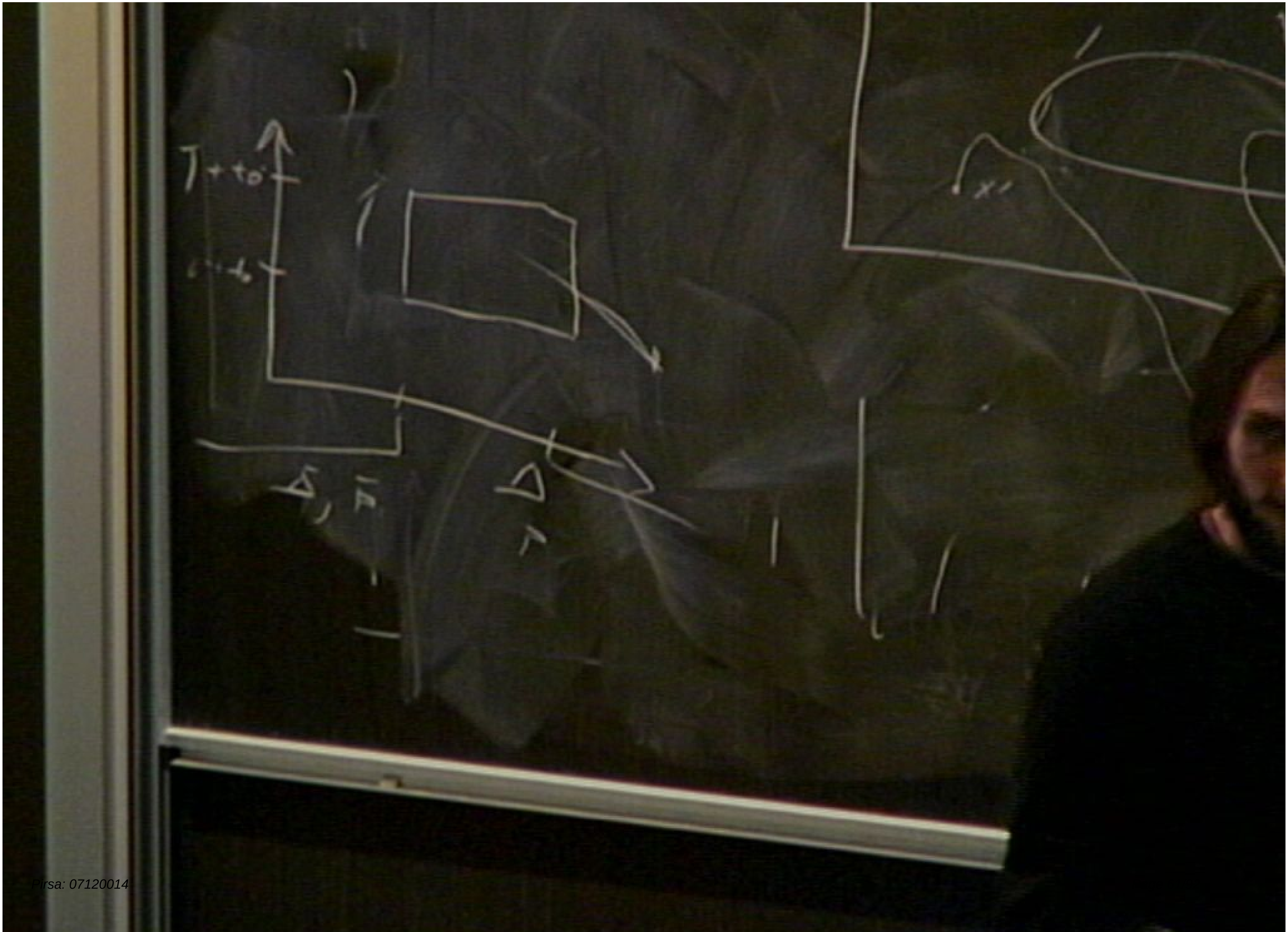

Such counter-intuitive results are avoided because at cases where we would expect the system to cross, the above coarse-graining would NOT give a decoherent set (fail to satisfy the decoherent condition).

To those cases, in order to get a physical answer, we need to employ different partition of the history space (coarse-grainings).

E.g.: Ask whether the system has crossed the region Δ at a discrete (but frequent) number of times (i.e. not taking the limit $\delta t \to 0$ in the restricted propagator). Or consider unsharp measurements (POVM's) instead of Projections.

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Constructing the Class Operators

QUESTION: What is the probability that the system crosses the region Δ of the configuration space, with no reference in time.

Earlier Attempts:

$$\langle x''|C_\Delta|x'\rangle = \int_0^\infty dT\, g_\Delta(x'', T|x', 0)$$

$$g_\Delta(x'', T|x', 0) = \int_\Delta \mathcal{D}x \exp(iS[x(t)]),$$

and the sum is over all the paths from x' to x'' within (parameter) time T, that crosses the region Δ.

Decoherent Histories Vs Evolving Constants

- Evolving Constants:

  (i) Need to find observables (relational) that correspond to physically interesting questions.

  (ii) These observables $\hat{A}$, have to be self-adjoint operators in the Hilbert Space.

  (iii) The operators need to commute with the Hamiltonian: $[\hat{A}, \hat{H}] = 0$

- Decoherent Histories:

  (i) Need to find physically interesting coarse-grained histories (Class operators).

(ii) Those class operators $\hat{C}_\alpha$, do NOT need to be self-adjoint operators.

(iii) They have to commute with the Hamiltonian: $[\hat{C}_\alpha, \hat{H}] = 0$

(iv) The Class operators, (along with the appropriate 'initial' state, $|\Psi\rangle$) should DECOHERE. This is required to interpret $p_\alpha = \langle\Psi|C_\alpha^\dagger C_\alpha|\Psi\rangle$, as probability.

Note that the possible existence of RECORDS corresponding to histories that decohere, brings a closer relation with the evolving constant approach.

An Example

Free Particle in an Energy Eigenstate

• The constraint we impose is $(\hat{H} - E)|\psi\rangle = 0$

with $\hat{H} = \frac{\hat{p}^2}{2m}$. Note that the constraint is quadratic in all variables, unlike the parametrized non-relativistic particle, and thus resembles more the Wheeler-DeWitt equation.

• Let us ask the question whether the system crosses to the $x \geq 0$ region.

$P = \int_0^\infty |x\rangle\langle x| dx$

Solutions of the constraint are of the form:

$\psi(x) = \psi_1 e^{ikx} + \psi_2 e^{-ikx}$ where $k = \sqrt{2mE}$.

Decoherence Conditions

(a) Special Initial States

(b) Addition of environment

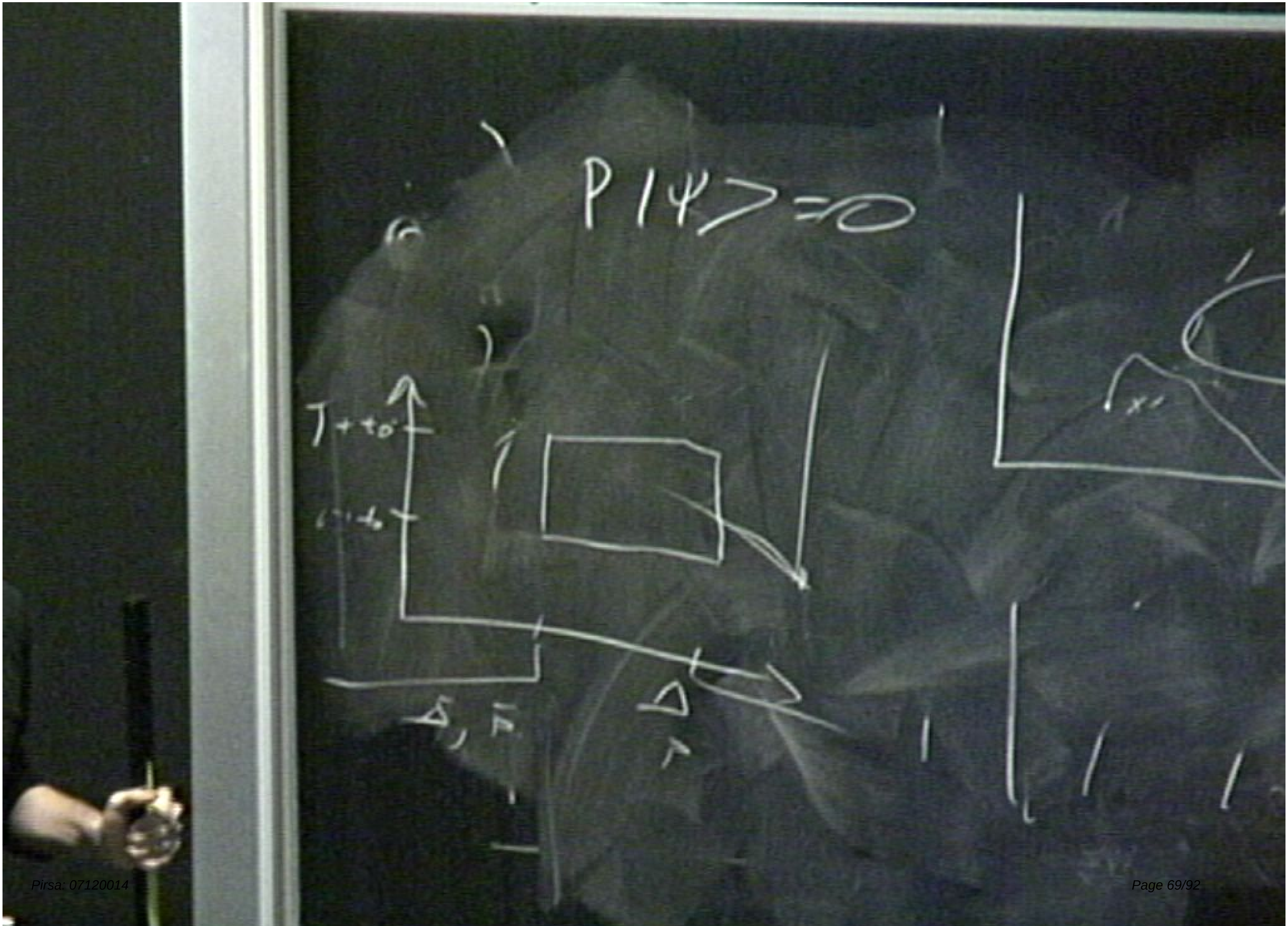
$\langle\psi|C^{\dagger}_{\Delta}C_{\Delta}|\psi\rangle = 0$

However using the property of restricted propagator we get:

$$\lim_{t\to\infty,t_0\to-\infty} e^{iE(t-t_0)}\langle\psi|g_r(t,t_0)|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle$$

this condition reduces to the requirement that the state $|\psi\rangle$ (apart from being an energy eigenstate), is required to vanish on the boundary of the region, i.e.:

$\langle x|\psi\rangle = 0, \quad \forall x \in \partial\bar{\Delta}$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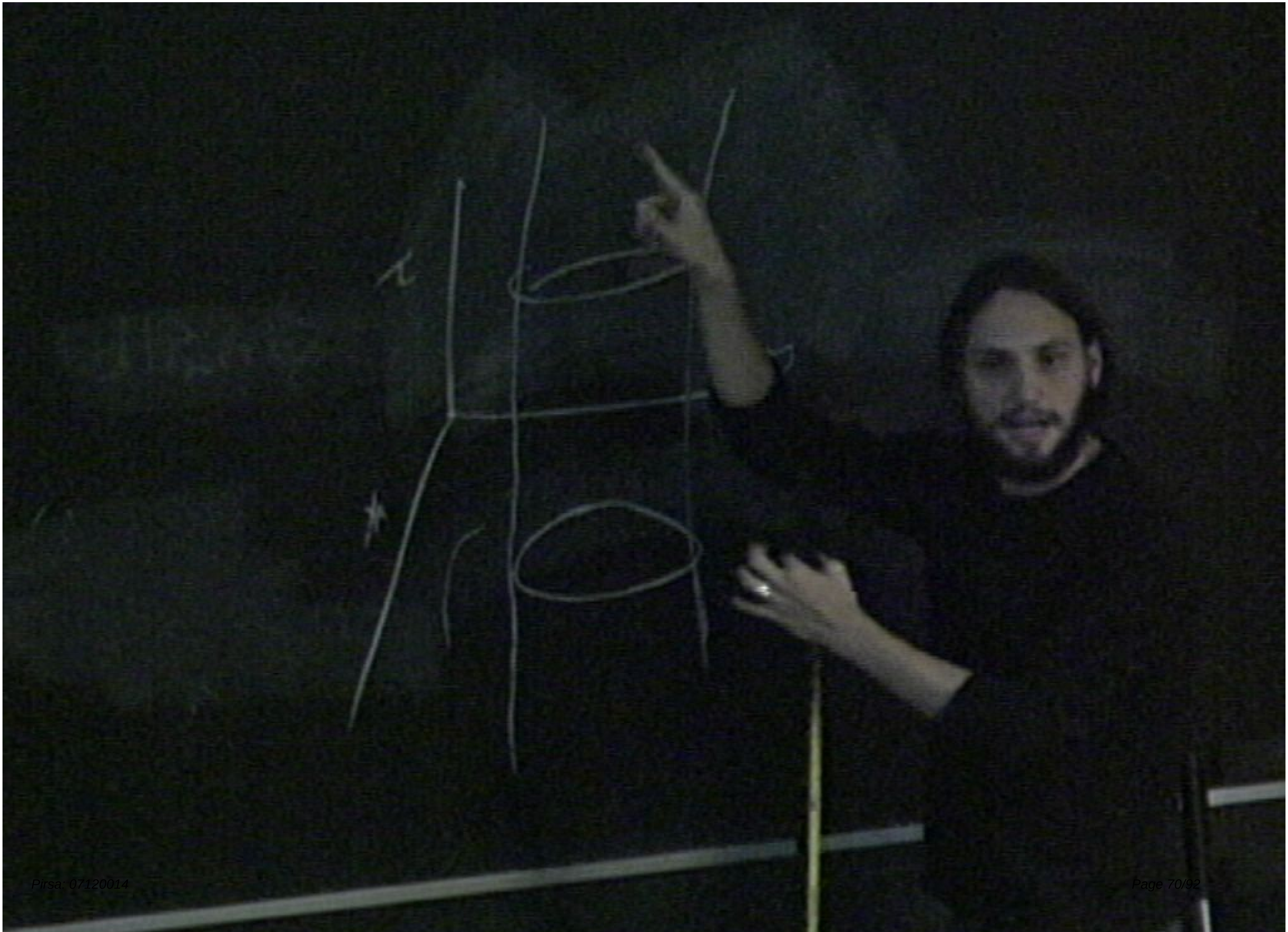

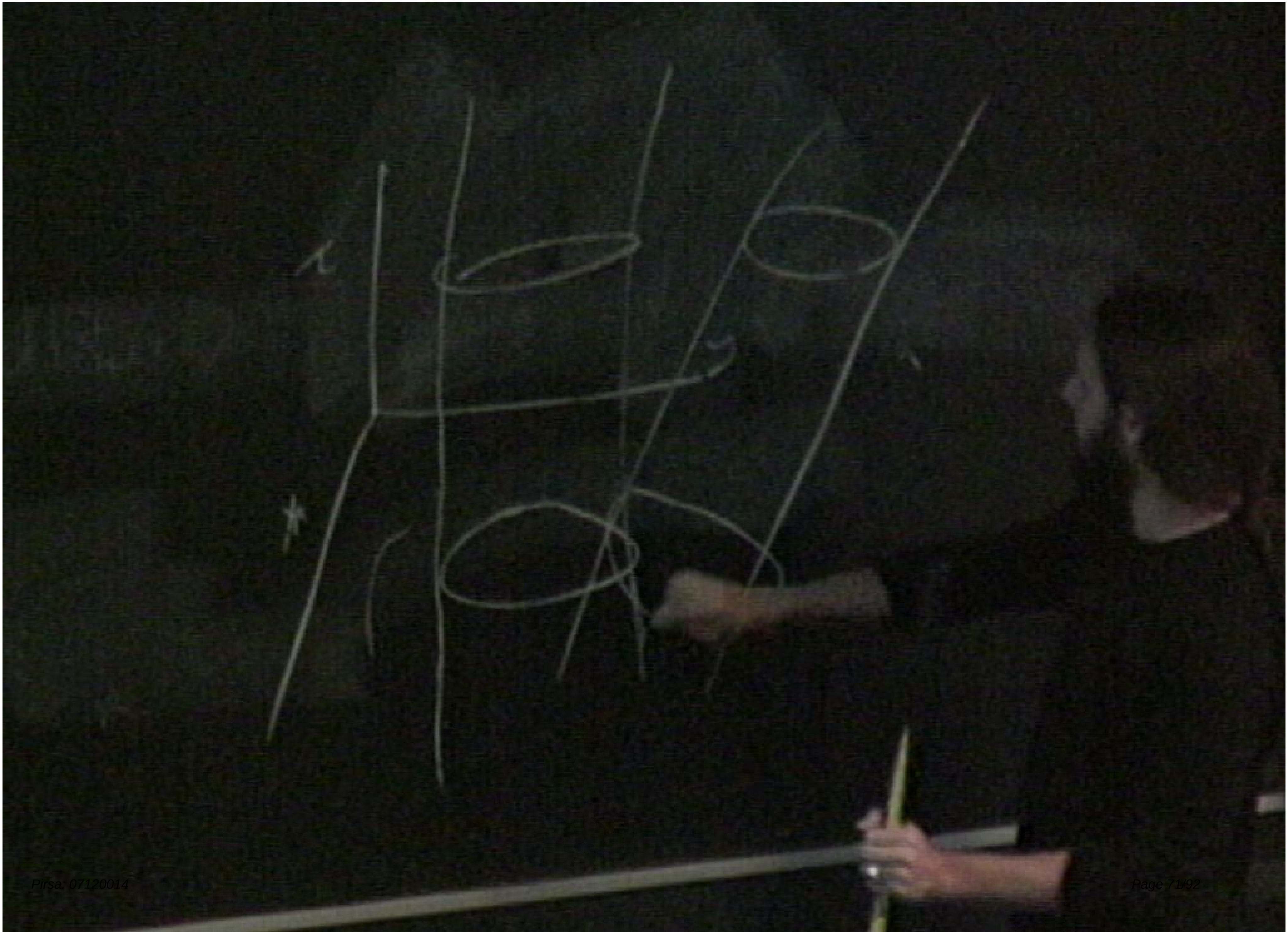

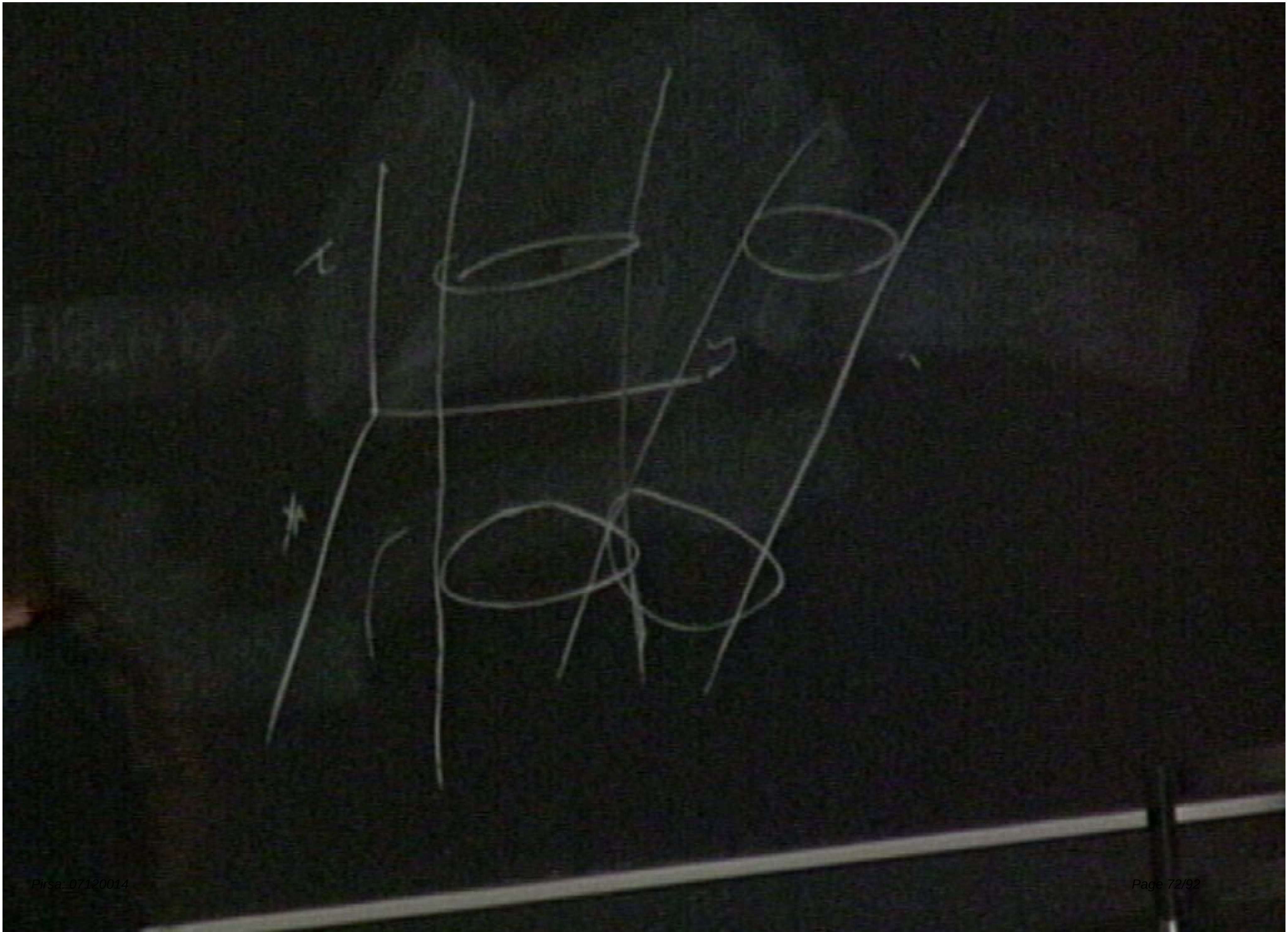

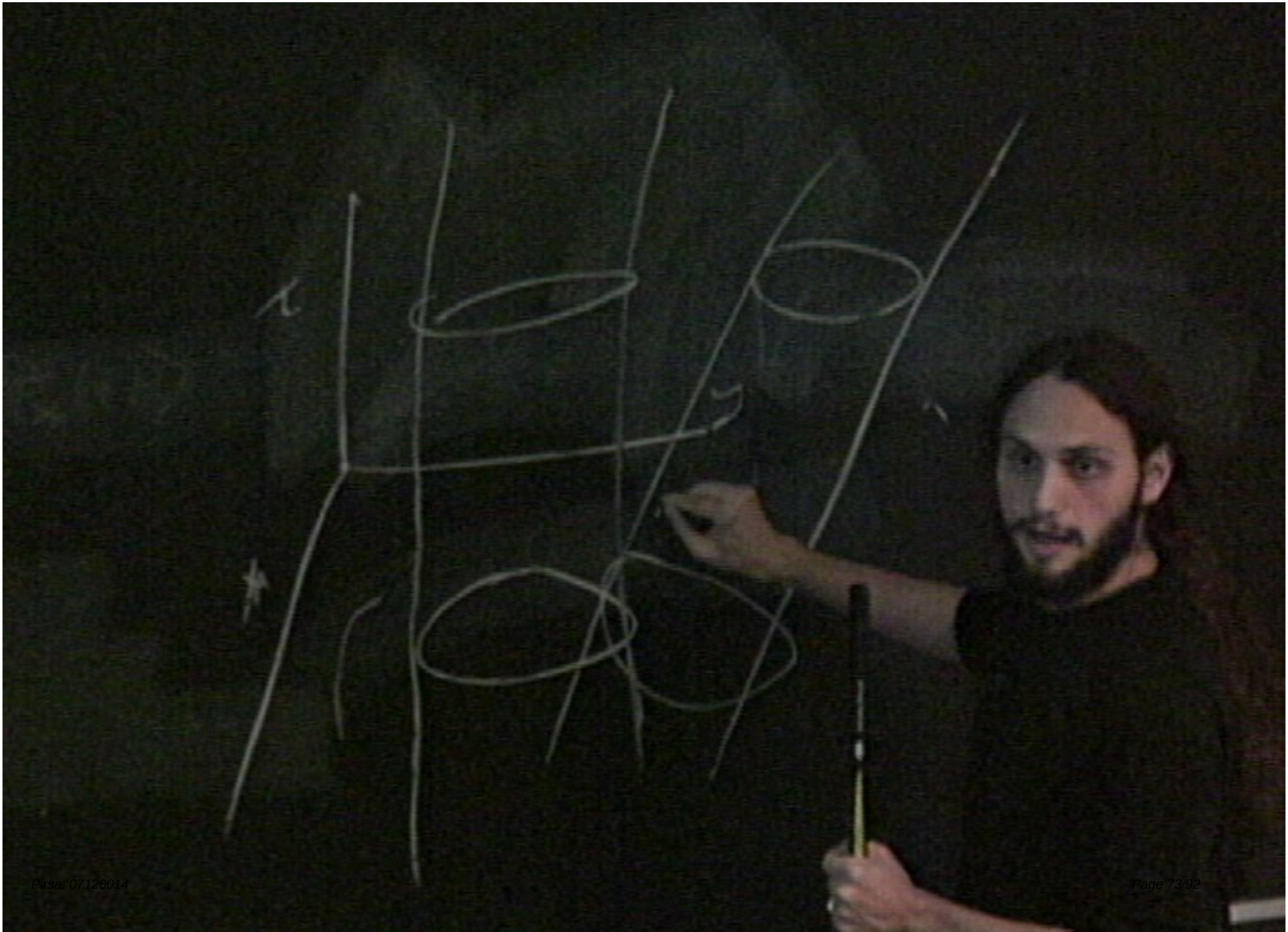

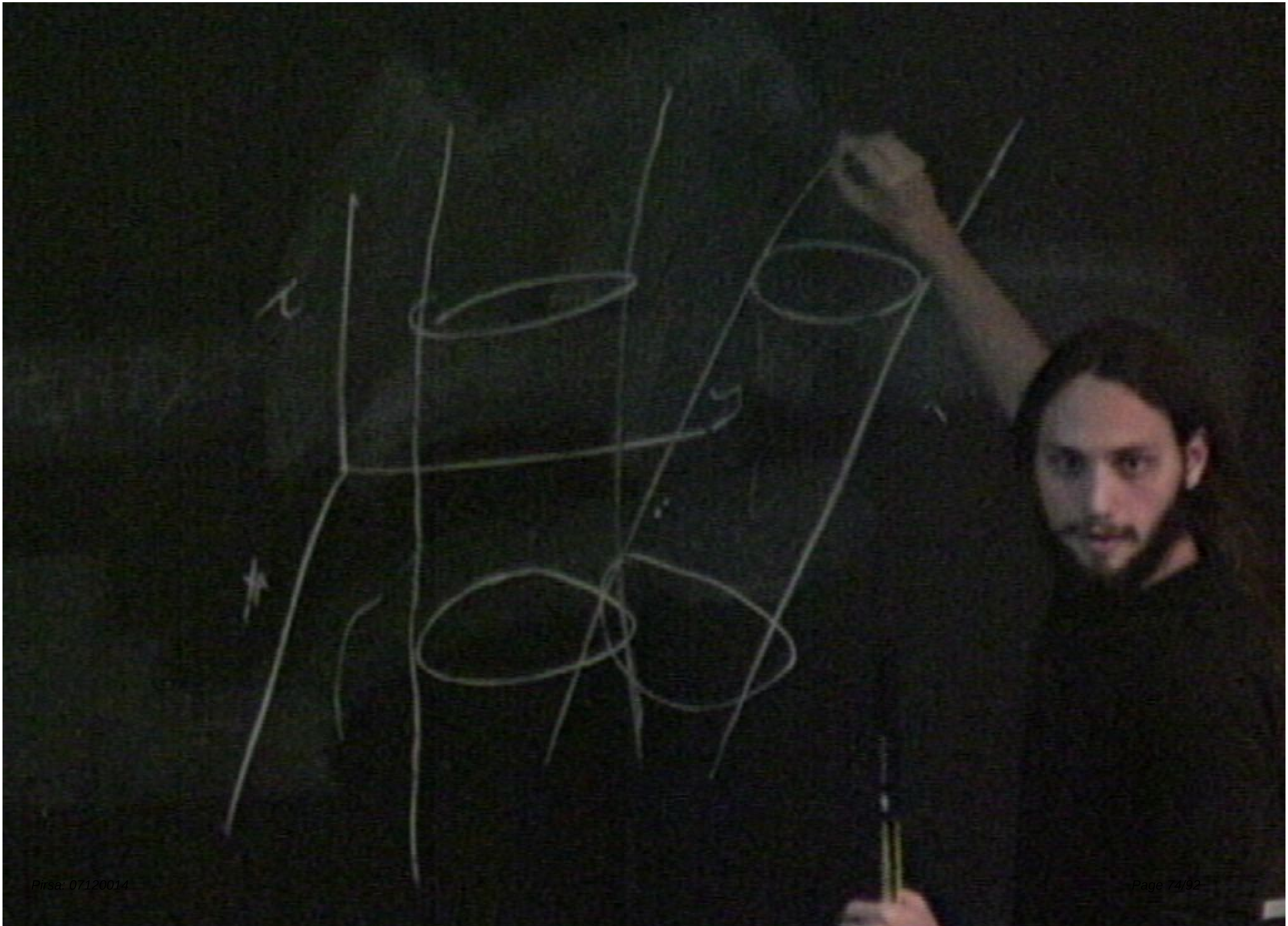

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Solutions of the constraint are of the form:

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The requirement that the wavefunction vanish at the boundary forces $\psi_1 = -\psi_2$ and therefore

$\psi(x) = \sin(kx)$.

Note, also that any fiducial state $\psi(x)$ that is anti-symmetric, gives the correct state when projected to the solution (of the constraint) space, i.e.:

$\delta(\hat{H} - E)\psi(x) = \sin(kx)$

For those states, the probability to be found at the $x \geq 0$ region turns out to be $1/2$, if we use the induced inner product.

$$P_\Delta = \frac{\int_0^\infty \sin(kx)}{\int_{-\infty}^{+\infty} \sin(kx)} = 1/2$$

2-D Free particle in Energy Eigenstate

Similar with the Relativistic Particle.

Consider region Δ the wedge between $0 \leq \phi \leq \pi/b = \beta$.

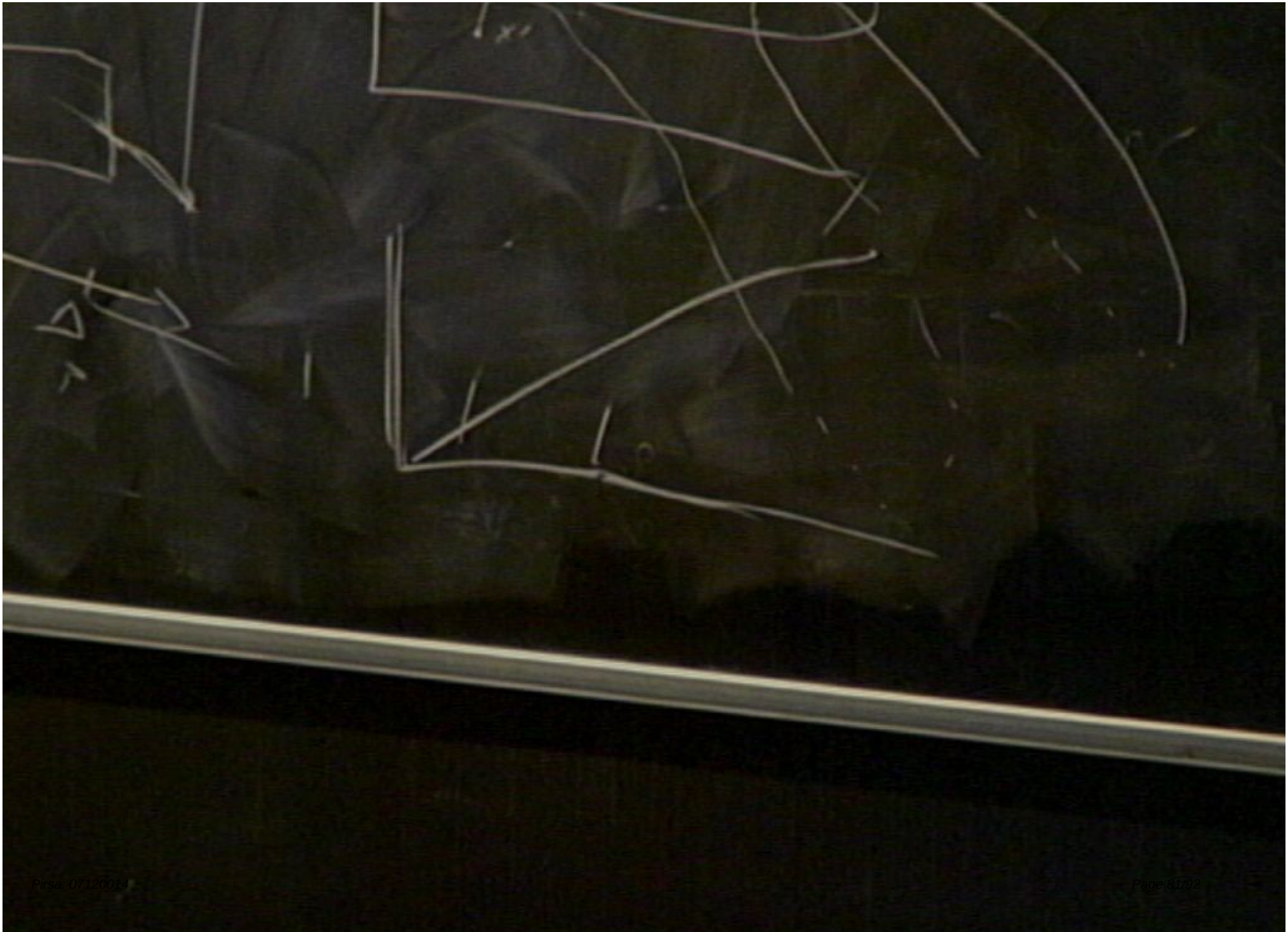
Energy eigenstate vanishing on this boundary:

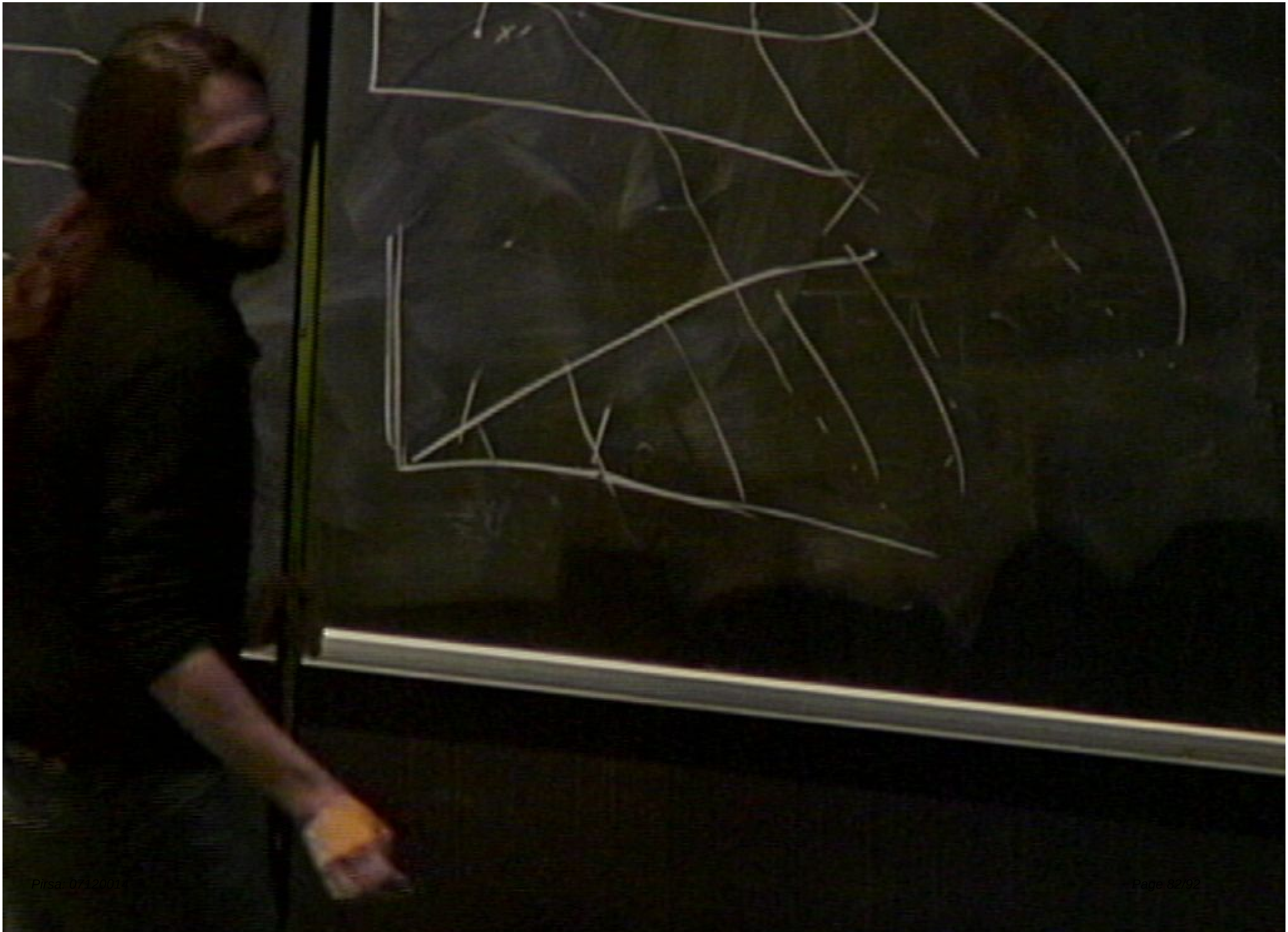
$$\psi(r, \phi) = \sin(b\phi) J_b(r\sqrt{2mE})$$

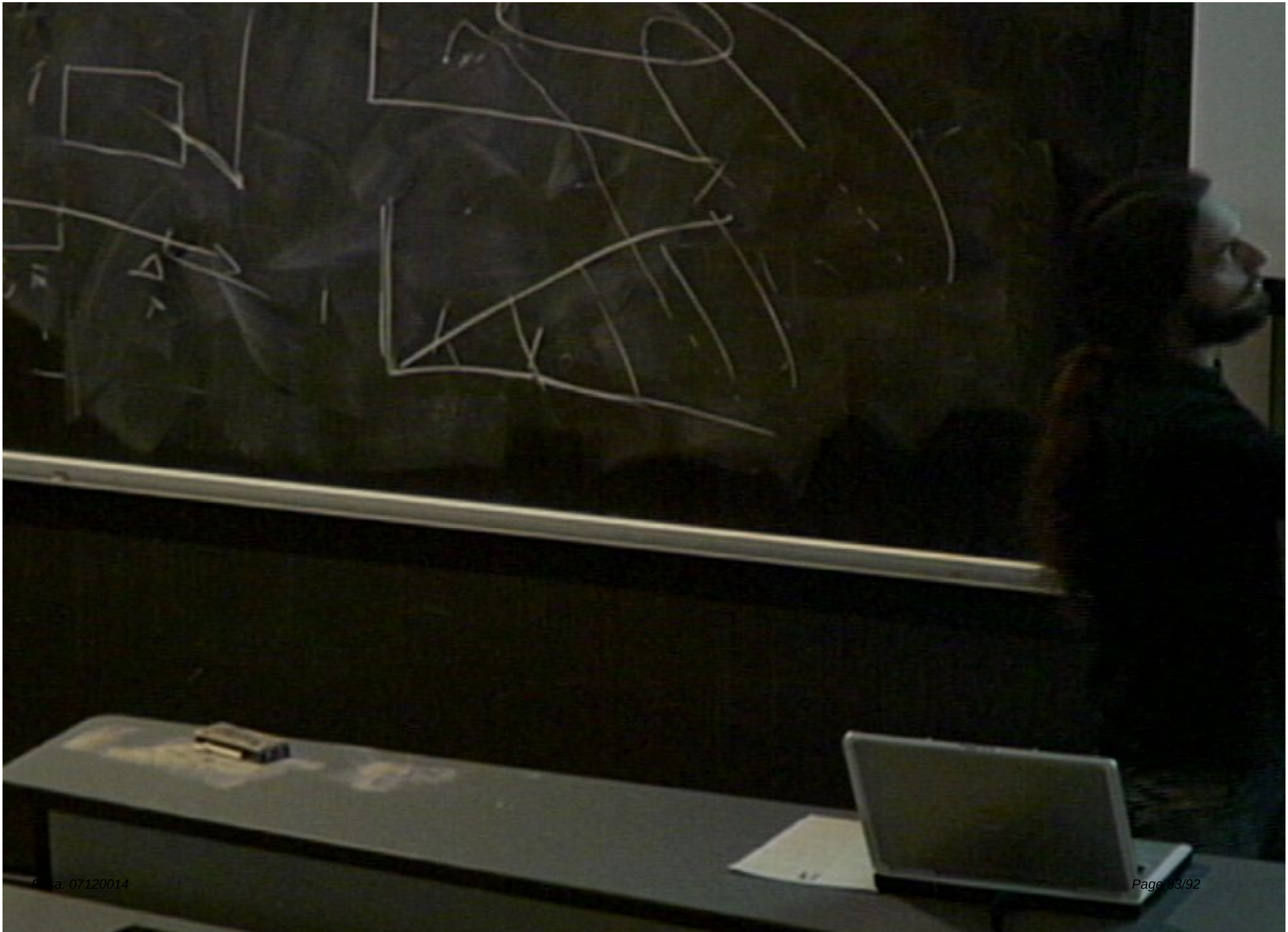
Gives rise to probability of crossing the wedge:

$$p_\beta = \beta/2\pi$$

We can use the method of images to compute the restricted propagator and this initial state indeed decoheres, i.e. obeys $\langle\psi|C_\Delta^\dagger C_\Delta|\psi\rangle = 0$.







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Summary & Conclusions

- The Problem of Time in Quantum Gravity

- The Decoherent Histories Approach to QT

- We examined the Decoherent Histories analysis of timeless QT by constructing appropriate Class Operators that respect the Hamiltonian constraint.

- They consisted of a general enough set of physical questions of the type:

"Which is the probability that a system crosses a region Δ in the configuration space with no reference in time"

Mathematically, it was a (continuous) product of projections operators.

• We have got an easy but VERY RESTRICTIVE decoherence condition (only to those few cases proper probabilities can be assigned).

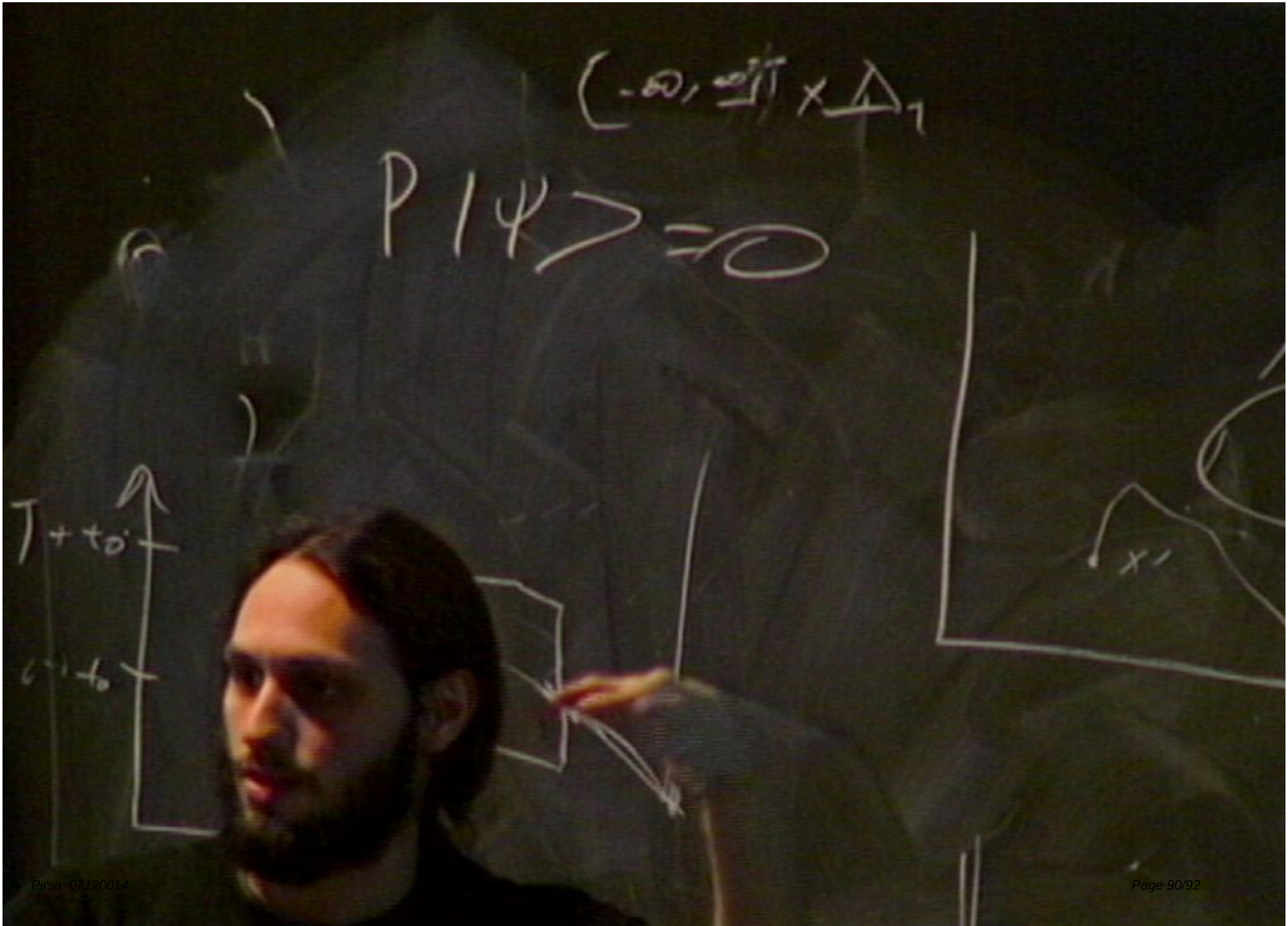
It requires that the "initial" state has to vanish on the boundary of the region considered.

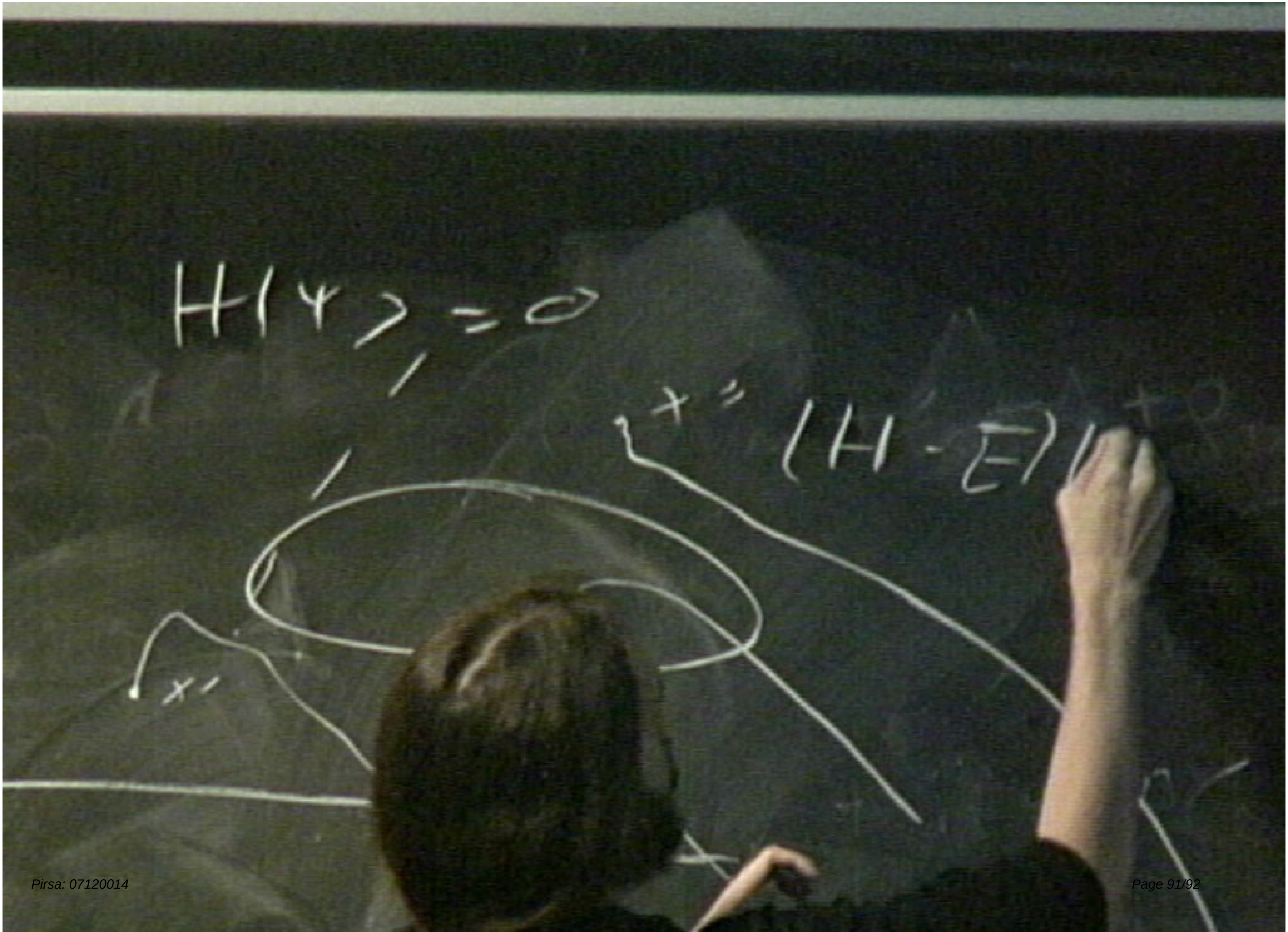
• The probabilities for those histories are easily calculated.

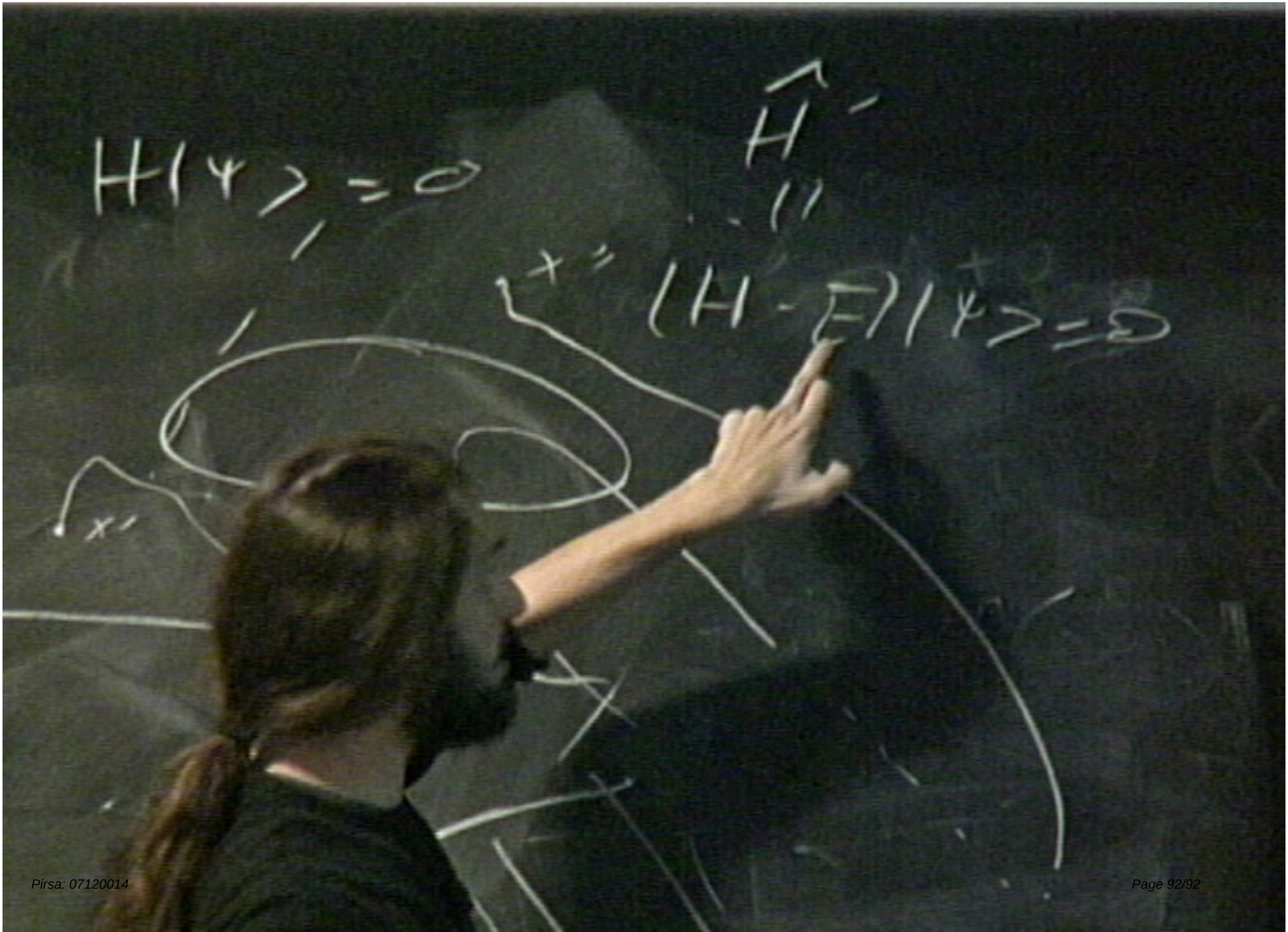
• In the case of Quantum Cosmology (in principle), given a state-solution of the Wheeler-DeWitt equation we can find the locus of zero's and then deduce what is the probability that the universe has crossed these regions.

- Due to the restricted decoherence condition, the set of questions that can be answered decreases drastically.

- We can attempt constructing other class operators i.e. look for different partitions of the history space, that could give answer to more physical questions AND satisfy the constraint (this last part is difficult, since most alternative attempts fail).




H|ψ> = 0
x'' (H - E)|
x'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