Title: Binary black hole merger: symmetry and the spin expansion

Date: Nov 29, 2007 05:30 PM

URL: http://pirsa.org/07110079

Abstract:

Binary black hole (BBH) mergers

- The *Holy Grail* of numerical relativity
- Two BHs lose *E* and *L* to GWs, form a common horizon, ring down
- Frans Pretorius chose his gauge and ICs *wisely*
- Welcome to the Golden Age of Relativity

Binary black hole (BBH) mergers

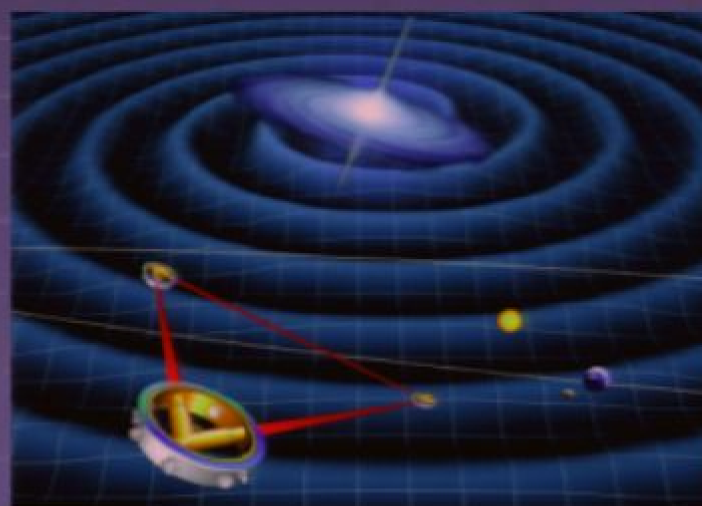
- The *Holy Grail* of numerical relativity
- Two BHs lose *E* and *L* to GWs, form a common horizon, ring down
- Frans Pretorius chose his gauge and ICs *wisely*
- Welcome to the Golden Age of Relativity

Near-Term Observations

LIGO

LISA

AGN



images courtesy of www.nasa.gov

From Kerr to Kerr: you *can* go home again

- Relativists can now simulate the entire merger:

  inspiral → merger → ringdown
- 2 widely separated Kerr BHs → 1 final Kerr BH
- "No hair" theorem → Kerr spacetime fully described by M, **S** (and **k** in general inertial frame)
- How much can we learn about BBH mergers purely from the symmetry of these simple initial and final states?

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

$$\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$$

$$\mathbf{e}^{(3)} \propto \mathbf{L}$$

$$\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$$

- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \bullet \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \bullet \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \to 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \to -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

$$\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$$

$$\mathbf{e}^{(3)} \propto \mathbf{L}$$

$$\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$$

- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \bullet \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \bullet \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \rightarrow 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \rightarrow -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

  $\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$

  $\mathbf{e}^{(3)} \propto \mathbf{L}$

  $\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$
- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \cdot \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \cdot \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \rightarrow 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \rightarrow -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order:

$$\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products **X** • $\mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \to 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \to -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order:

$$\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$$

  - At 2nd order:

$$\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3) + \mathbf{k}_\perp^{200|000}(a_1^2 - b_1^2) + \mathbf{k}_\perp^{020|000}(a_2^2 - b_2^2) + \mathbf{k}_\perp^{002|000}(a_3^2 - b_3^2) + \mathbf{k}_\perp^{110|000}(a_1 a_2 - b_1 b_2) + \mathbf{k}_\perp^{100|010}(a_1 b_2 - a_2 b_1)$$

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

  $\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$

  $\mathbf{e}^{(3)} \propto \mathbf{L}$

  $\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$
- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \bullet \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \bullet \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \to 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \to -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

$$\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$$

$$\mathbf{e}^{(3)} \propto \mathbf{L}$$

$$\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$$

- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \bullet \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \bullet \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \rightarrow 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_{\perp} \rightarrow -\mathbf{e}_{\perp}$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_{\perp} = \mathbf{k}_{\perp}^{001|000}(a_3 - b_3)$

Spin Expansion

- Non-spinning mergers *comparatively* well understood. Can we understand the spinning case by a judicious Taylor expansion of final-state quantities in the initial spin components?
- We define an orthonormal triad

$$\mathbf{e}^{(1)} \propto (\mathbf{x}_B - \mathbf{x}_A)$$

$$\mathbf{e}^{(3)} \propto \mathbf{L}$$

$$\mathbf{e}^{(2)} \equiv \mathbf{e}^{(3)} \times \mathbf{e}^{(1)}$$

- Final mass *m*, kick **k**, and spin **s** are expanded in terms of $a_i \equiv \mathbf{a} \bullet \mathbf{e}^{(i)}$, $b_i \equiv \mathbf{b} \bullet \mathbf{e}^{(i)}$:

$$\mathbf{k}_\perp = \sum \mathbf{k}_\perp^{m_1 m_2 m_3 | n_1 n_2 n_3}(q) a_1^{m_1} a_2^{m_2} a_3^{m_3} b_1^{n_1} b_2^{n_2} b_3^{n_3}$$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \rightarrow 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_\perp \rightarrow -\mathbf{e}_\perp$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_\perp = \mathbf{k}_\perp^{001|000}(a_3 - b_3)$

How does symmetry help us?

- Identify symmetries of each IC/final observable
  - Rotation *R*: inner products $\mathbf{X} \cdot \mathbf{e}^{(i)}$ preserved
  - Parity *P*: (pseudo)scalars keep (change) sign
  - Exchange *X*: relabeling BBHs *A* and *B* sends $q \rightarrow 1/q$, $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{e}_{\perp} \rightarrow -\mathbf{e}_{\perp}$, others unchanged
- Use symmetries to eliminate terms
  - At 1st order: $\mathbf{k}_{\perp} = \mathbf{k}_{\perp}^{001|000}(a_3 - b_3)$
  - At 2nd order:

$$\mathbf{k}_{\perp} = \mathbf{k}_{\perp}^{001|000}(a_3 - b_3) + \mathbf{k}_{\perp}^{200|000}(a_1^2 - b_1^2) + \mathbf{k}_{\perp}^{020|000}(a_2^2 - b_2^2)$$
$$+ \mathbf{k}_{\perp}^{002|000}(a_3^2 - b_3^2) + \mathbf{k}_{\perp}^{110|000}(a_1 a_2 - b_1 b_2) + \mathbf{k}_{\perp}^{100|010}(a_1 b_2 - a_2 b_1)$$

What do we gain from the spin expansion?

- It is *simple*. It provides quantitative understanding about *m*, **k** and **s** --- even for NON-experts.
- It *complements* post-Newtonian and numerical approaches. Symmetries carry through the entire inspiral, merger, ringdown.
- It *separates* the dynamics of Einstein's equations (HARD) from more elementary geometry (MUCH EASIER).
- It is *efficient*. Calibrating with a handful of simulations, 7 parameter space (*q*, **a**, **b**) reduces to just 1 (*q*).

Why is this an improvement?

Kidder formulae

- developed from PN expansions for *instantaneous* loss of *E*, *L*, *P*
- can only be applied at *merger*
- *linear* in the initial spins

$$\vec{V}_{\rm recoil}(q,\vec{\alpha}_i) = v_m\,\hat{e}_1 + v_\perp(\cos(\xi)\,\hat{e}_1 + \sin(\xi)\,\hat{e}_2) + v_\parallel\,\hat{e}_z,$$

$$v_m = A\frac{q^2(1-q)}{(1+q)^5}\left(1 + B\,\frac{q}{(1+q)^2}\right),$$

$$v_\perp = H\frac{q^2}{(1+q)^5}\left(\alpha_2^\parallel - q\alpha_1^\parallel\right),$$

$$v_\parallel = K\cos(\Theta-\Theta_0)\frac{q^2}{(1+q)^5}\left(\alpha_2^\perp - q\alpha_1^\perp\right), \quad (1)$$

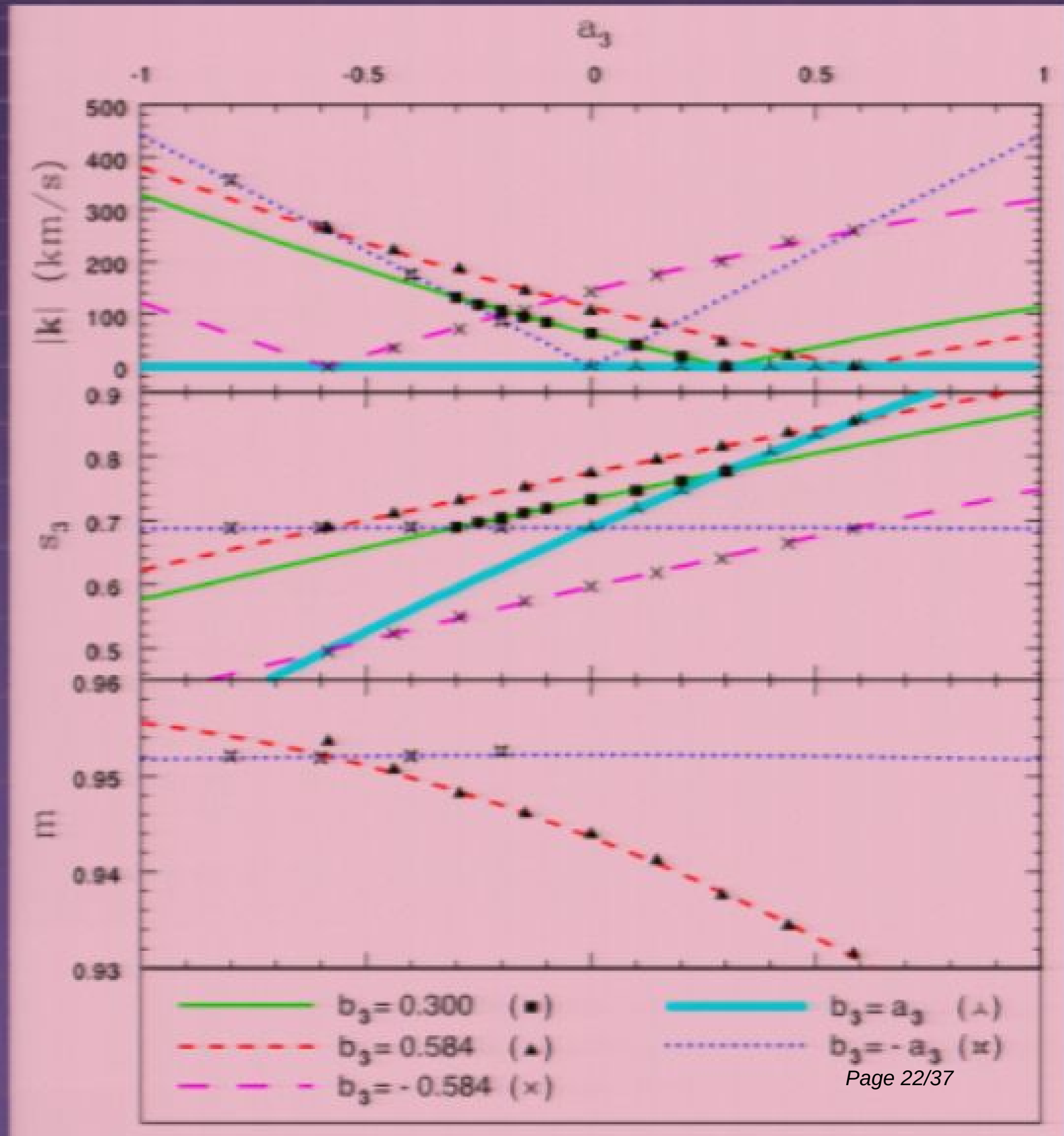
Campanelli *et al.*, 2007

Brügmann *et al.*, 2007

(Anti-)aligned spins

Many simulations available:

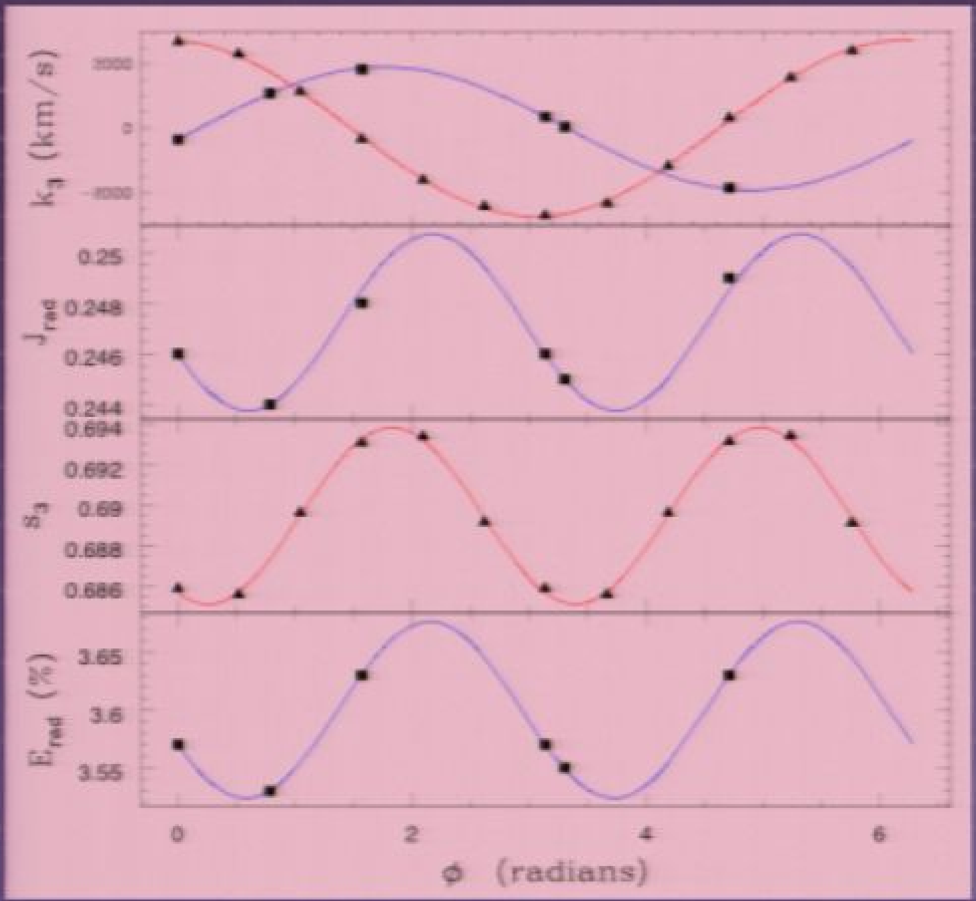
- Pretorius, 2005
- Herrmann *et al.*, 2007
- Baker *et al.*, 2007
- Pollney *et al.*, 2007
- Rezzolla *et al.*, 2007
- Marronetti *et al.*, 2007



$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}.$$

Superkicks

- 18 simulations
  - Brügmann *et al.*, 2007
  - Campanelli *et al.*, 2007
- Spin dependence of E_{rad} and J_{rad} *unnoticed*
- Spin dependence of s_3 *discounted*

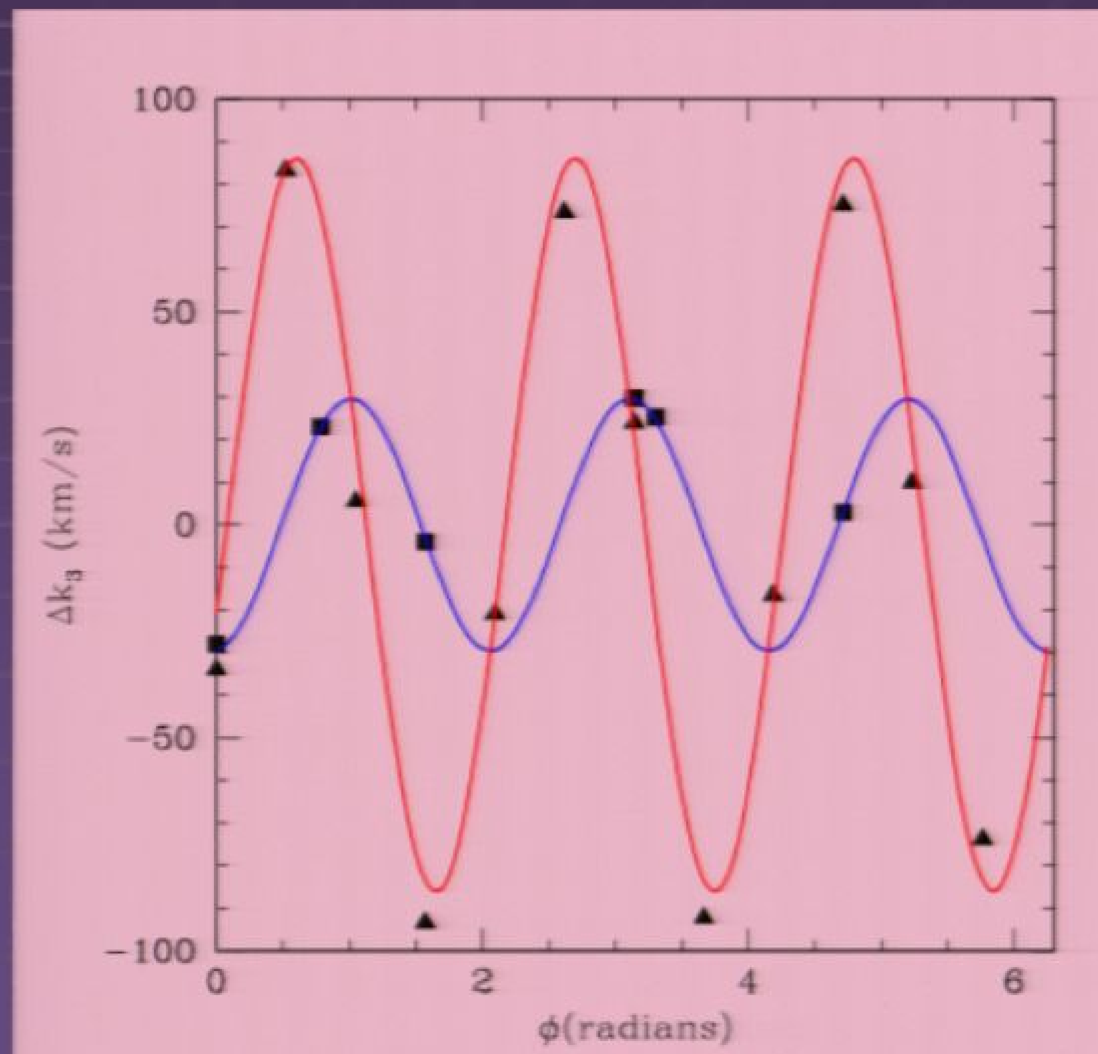


$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} +a\cos\phi \\ +a\sin\phi \\ 0 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -a\cos\phi \\ -a\sin\phi \\ 0 \end{pmatrix}$$

Higher-order Corrections

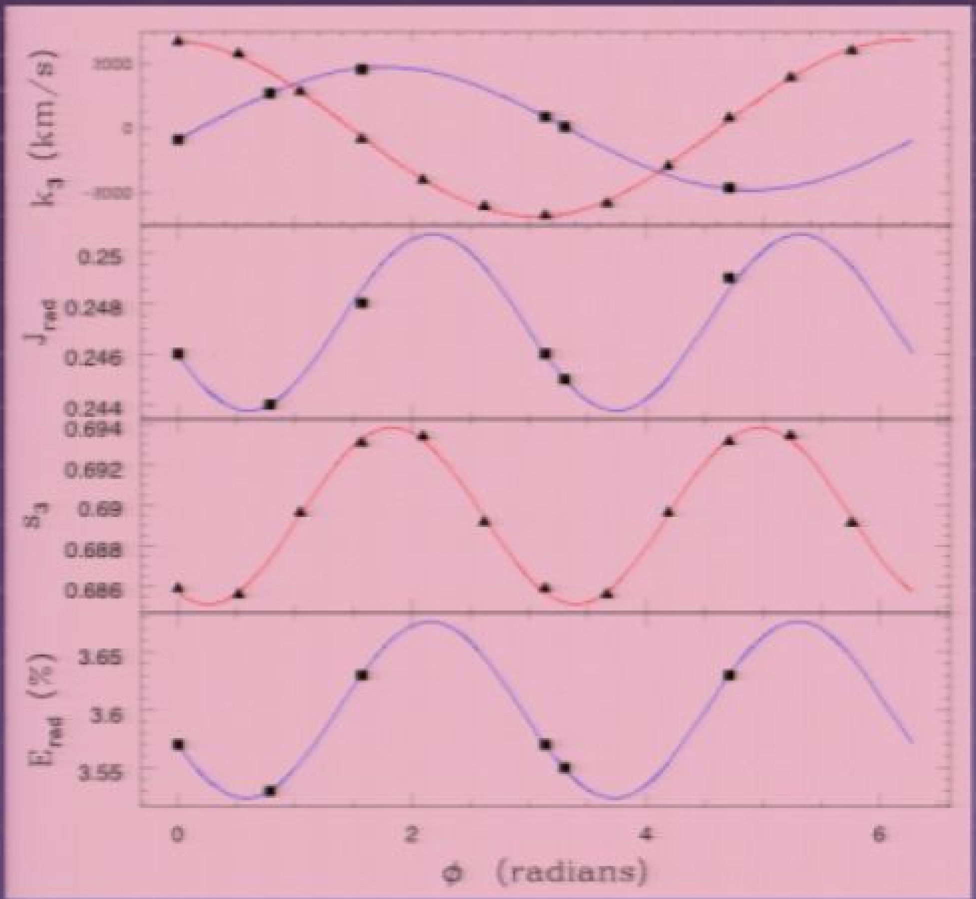
Spin expansion predicts higher-order corrections to the superkicks

- $\propto a^3$
- 3× superkick frequency



Superkicks

- 18 simulations
  - Brügmann *et al.*, 2007
  - Campanelli *et al.*, 2007
- Spin dependence of E_{rad} and J_{rad} *unnoticed*
- Spin dependence of s_3 *discounted*

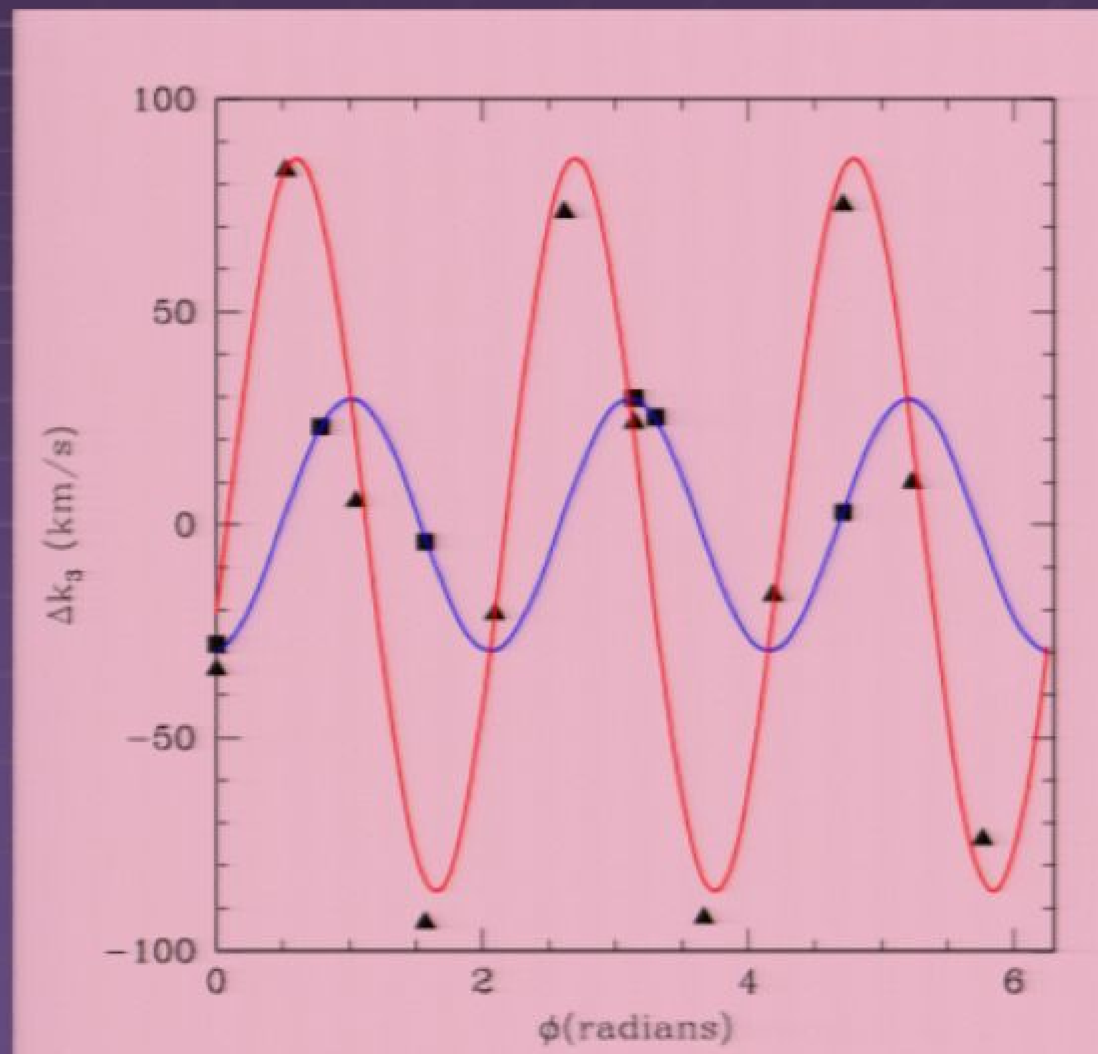


$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} +a\cos\phi \\ +a\sin\phi \\ 0 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -a\cos\phi \\ -a\sin\phi \\ 0 \end{pmatrix}$$

Higher-order Corrections

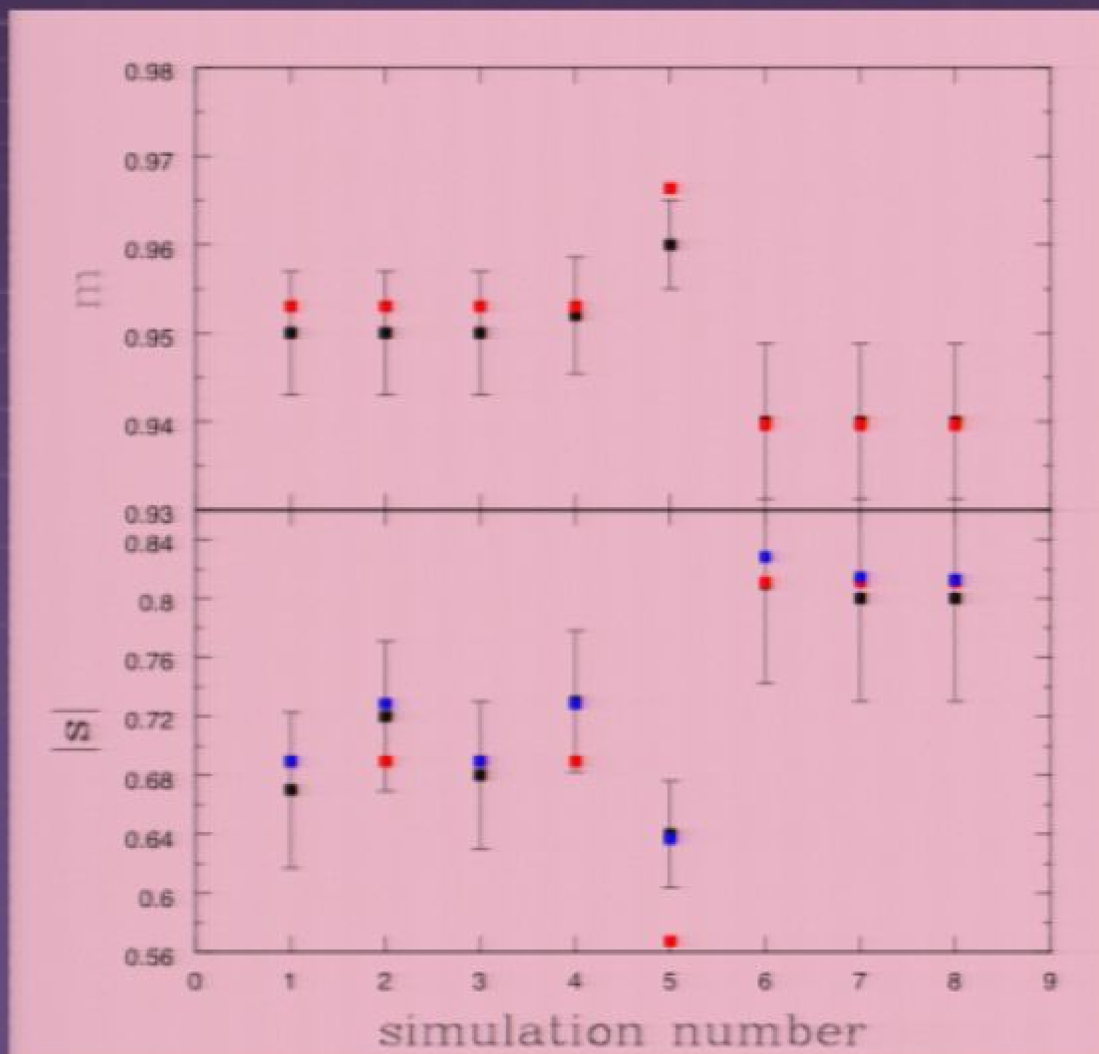
Spin expansion predicts higher-order corrections to the superkicks

- $\propto a^3$
- 3× superkick frequency



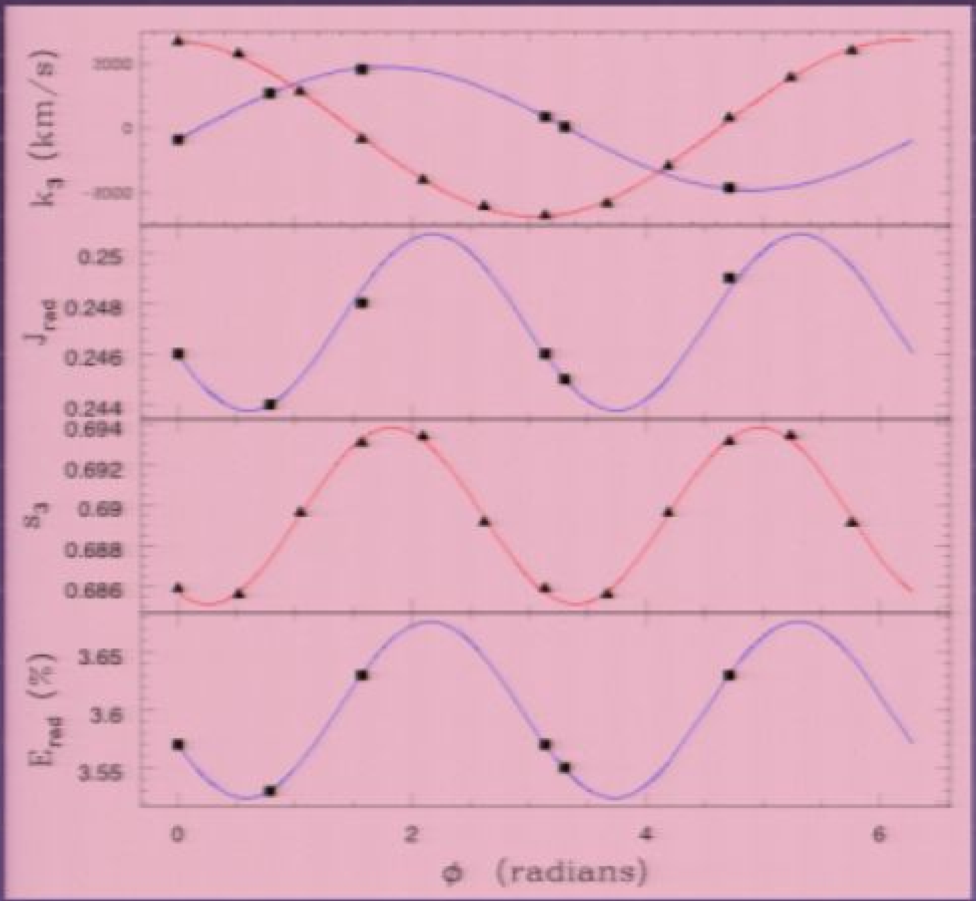
Generic Orientations

- 8 simulations published in Tichy and Marronetti, 2007



Superkicks

- 18 simulations
  - Brügmann *et al.*, 2007
  - Campanelli *et al.*, 2007
- Spin dependence of E_{rad} and J_{rad} *unnoticed*
- Spin dependence of s_3 *discounted*

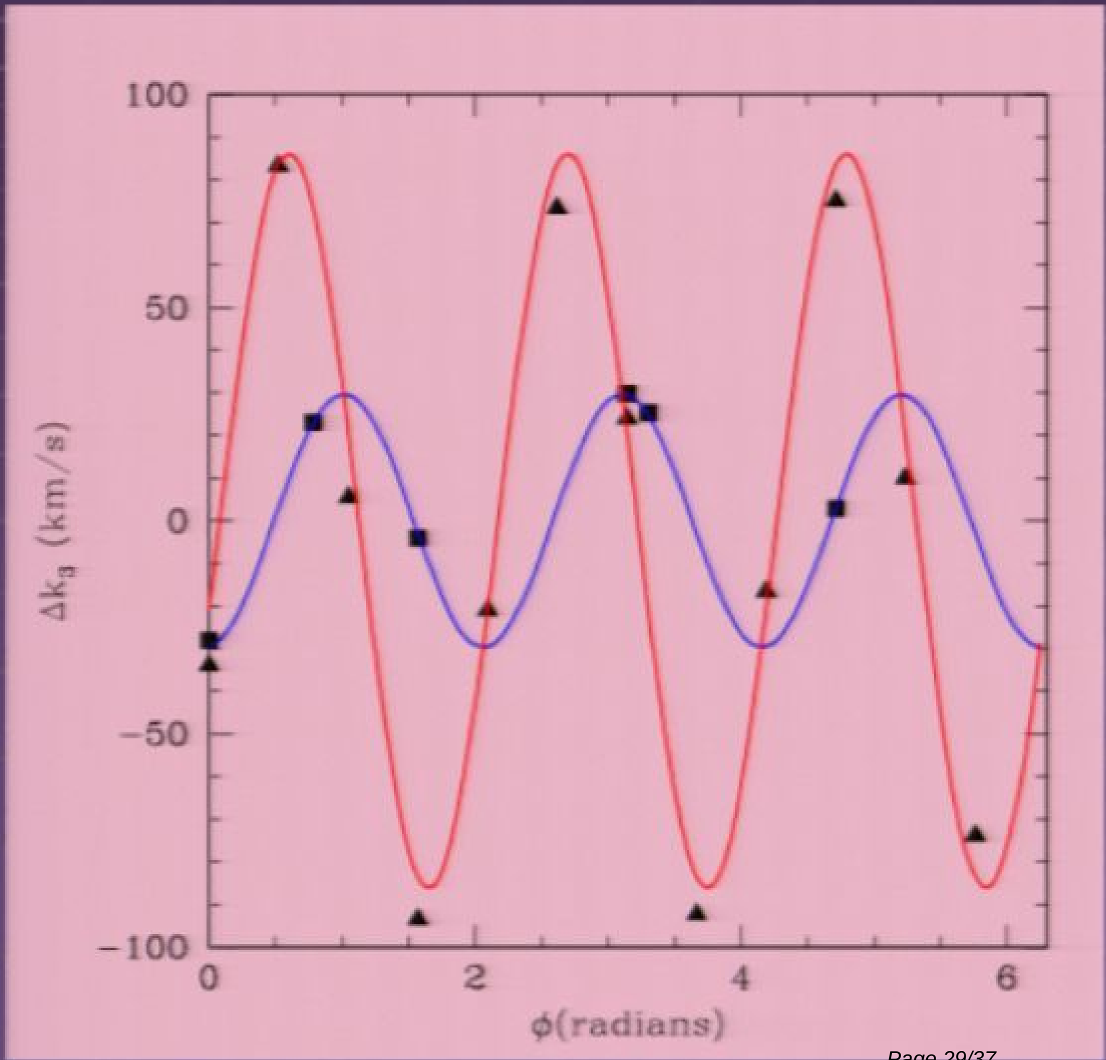


$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} +a\cos\phi \\ +a\sin\phi \\ 0 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -a\cos\phi \\ -a\sin\phi \\ 0 \end{pmatrix}$$

Higher-order Corrections

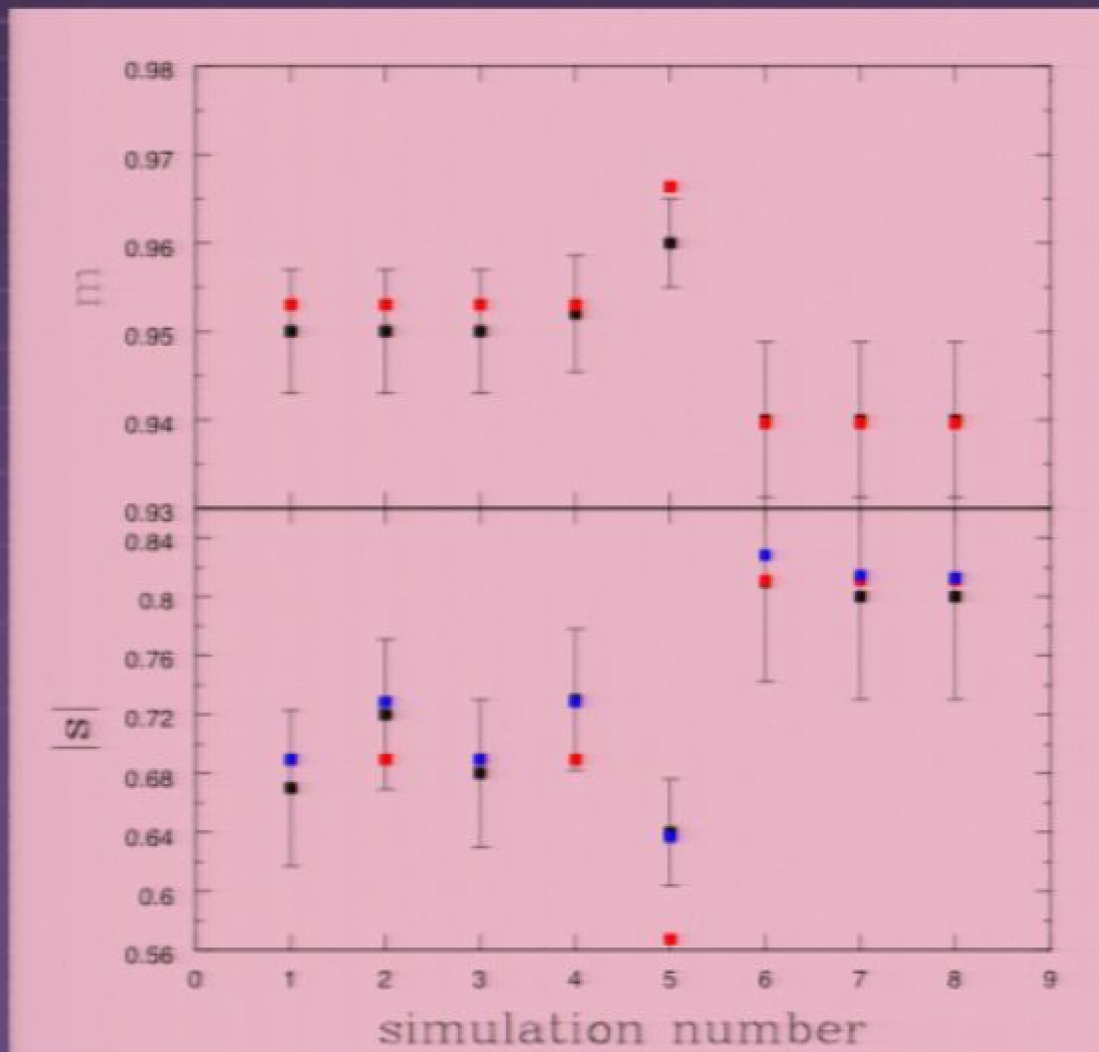
Spin expansion predicts higher-order corrections to the superkicks

- $\propto a^3$
- 3× superkick frequency



Generic Orientations

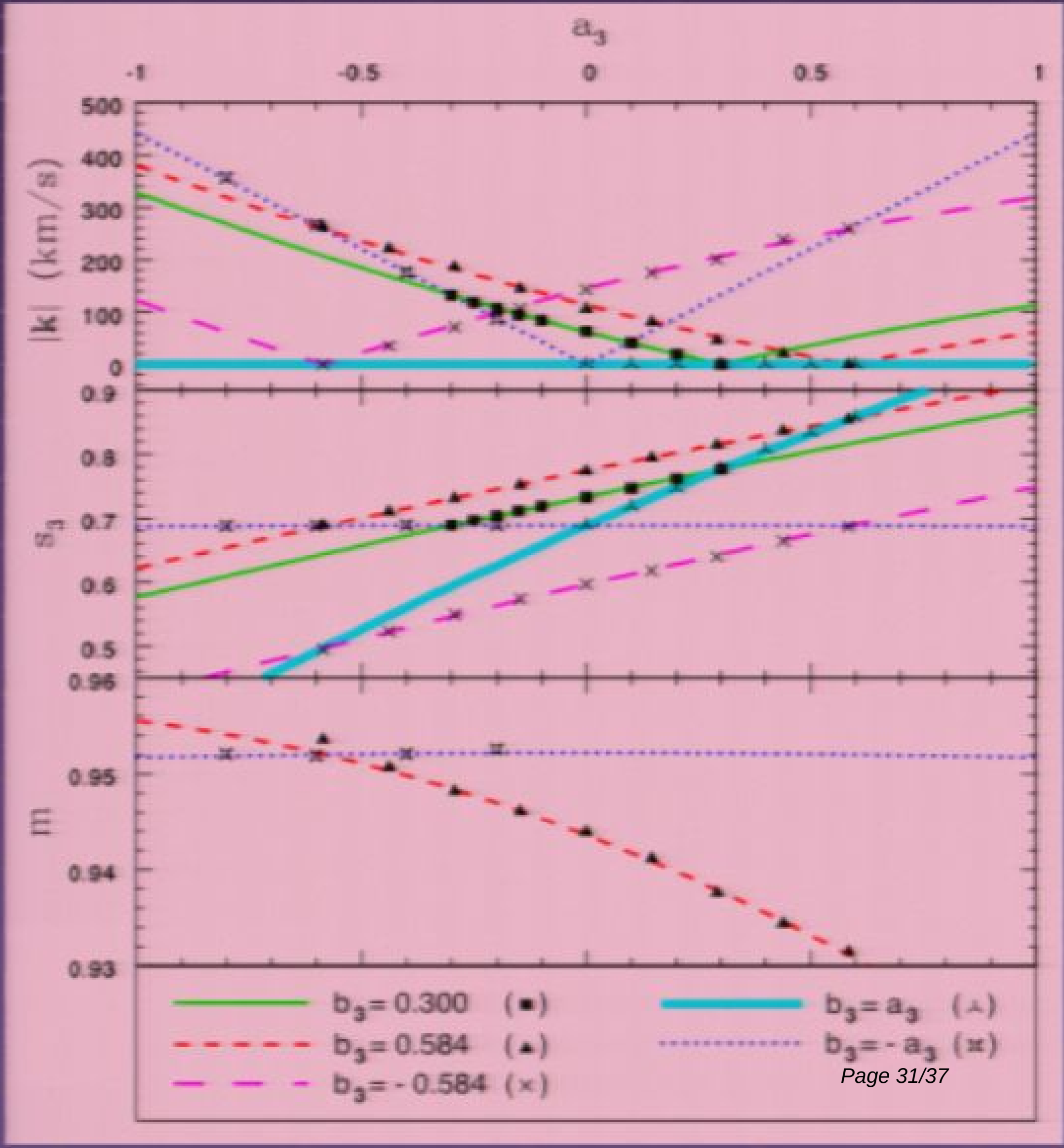
- 8 simulations published in Tichy and Marronetti, 2007



(Anti-)aligned spins

Many simulations available:

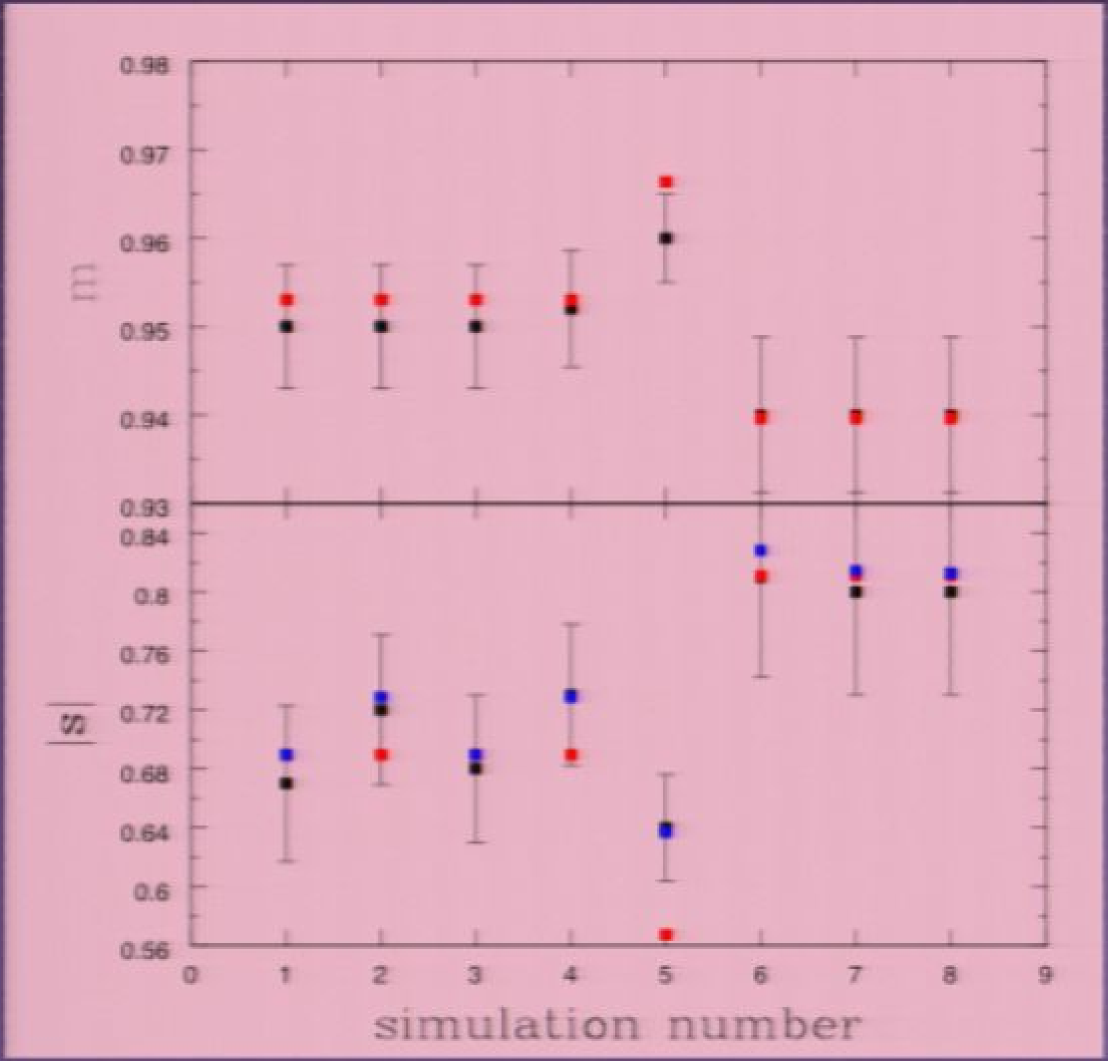
- Pretorius, 2005
- Herrmann *et al.*, 2007
- Baker *et al.*, 2007
- Pollney *et al.*, 2007
- Rezzolla *et al.*, 2007
- Marronetti *et al.*, 2007



$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}.$$

Generic Orientations

- 8 simulations published in Tichy and Marronetti, 2007



Where do we go from here?

- Existing simulations are not ideal for testing/calibrating the spin expansion
  - simulated configurations are *too* symmetric
  - different orientations have *same* spin magnitudes
- We propose *new* simulations that uniquely determine *all* coefficients to 2nd order

eq0: none	
eq1: a_1	eq2: a_2
eq3: a_1, a_3	eq4: b_1, a_3
eq5: a_1, a_2, a_3	eq6: a_1, b_2, a_3
eq7: a_1, b_1	
eq8: a_2, b_2	
eq9: a_3, b_3	
N.B. $a_3^{(3)} \neq a_3^{(4)}$	

uneq0: none	
uneq1: a_1	uneq2: a_2
uneq3: b_1	uneq4: b_2
uneq5: a_1, a_3	uneq6: a_2, a_3
uneq7: b_1, b_3	uneq8: b_2, b_3
uneq9: a_1, b_1, a_3	uneq10: a_2, b_2, a_3
uneq11: a_1, b_2, b_3	uneq12: b_1, a_2, b_3
uneq13: a_1, a_2	
uneq14: b_1, b_2	
uneq15: a_3, b_3	
N.B. $a_3^{(5)} \neq a_3^{(6)}$, $b_3^{(7)} \neq b_3^{(8)}$	

Conclusions

- Numerical breakthroughs inaugurated a "Golden Age of Relativity"
- We introduce a "spin expansion" that is:
  - Simple
  - Complementary
  - Informative
  - Efficient
- We make a host of *quantitative* predictions waiting to be tested, e.g. for kick components in the case equal and opposite spins:

$$\mathbf{k}_\perp = \hat{\mathbf{k}}_\perp^{(1)}\cos(\theta) + \hat{\mathbf{k}}_\perp^{(3)}\cos(3\theta) + \hat{\mathbf{k}}_\perp^{(5)}\cos(5\theta) + \ldots$$
$$k_3 = \hat{k}_3^{(1)}\sin(\theta) + \hat{k}_3^{(3)}\sin(3\theta) + \hat{k}_3^{(5)}\sin(5\theta) + \ldots$$

Where do we go from here?

- Existing simulations are not ideal for testing/calibrating the spin expansion
  - simulated configurations are *too* symmetric
  - different orientations have *same* spin magnitudes
- We propose *new* simulations that uniquely determine *all* coefficients to 2nd order

eq0: none	
eq1: a_1	eq2: a_2
eq3: a_1, a_3	eq4: b_1, a_3
eq5: a_1, a_2, a_3	eq6: a_1, b_2, a_3
eq7: a_1, b_1	
eq8: a_2, b_2	
eq9: a_3, b_3	
N.B. $a_3^{(3)} \neq a_3^{(4)}$	

uneq0: none	
uneq1: a_1	uneq2: a_2
uneq3: b_1	uneq4: b_2
uneq5: a_1, a_3	uneq6: a_2, a_3
uneq7: b_1, b_3	uneq8: b_2, b_3
uneq9: a_1, b_1, a_3	uneq10: a_2, b_2, a_3
uneq11: a_1, b_2, b_3	uneq12: b_1, a_2, b_3
uneq13: a_1, a_2	
uneq14: b_1, b_2	
uneq15: a_3, b_3	
N.B. $a_3^{(5)} \neq a_3^{(6)}$, $b_3^{(7)} \neq b_3^{(8)}$	

Where do we go from here?

- Existing simulations are not ideal for testing/calibrating the spin expansion
  - simulated configurations are *too* symmetric
  - different orientations have *same* spin magnitudes
- We propose *new* simulations that uniquely determine *all* coefficients to 2nd order

eq0: none	
eq1: a_1	eq2: a_2
eq3: a_1, a_3	eq4: b_1, a_3
eq5: a_1, a_2, a_3	eq6: a_1, b_2, a_3
eq7: a_1, b_1	
eq8: a_2, b_2	
eq9: a_3, b_3	
N.B. $a_3^{(3)} \neq a_3^{(4)}$	

uneq0: none	
uneq1: a_1	uneq2: a_2
uneq3: b_1	uneq4: b_2
uneq5: a_1, a_3	uneq6: a_2, a_3
uneq7: b_1, b_3	uneq8: b_2, b_3
uneq9: a_1, b_1, a_3	uneq10: a_2, b_2, a_3
uneq11: a_1, b_2, b_3	uneq12: b_1, a_2, b_3
uneq13: a_1, a_2	
uneq14: b_1, b_2	
uneq15: a_3, b_3	
N.B. $a_3^{(5)} \neq a_3^{(6)}$, $b_3^{(7)} \neq b_3^{(8)}$	

Where do we go from here?

- Existing simulations are not ideal for testing/calibrating the spin expansion
  - simulated configurations are *too* symmetric
  - different orientations have *same* spin magnitudes
- We propose *new* simulations that uniquely determine *all* coefficients to 2nd order

eq0: none	
eq1: a_1	eq2: a_2
eq3: a_1, a_3	eq4: b_1, a_3
eq5: a_1, a_2, a_3	eq6: a_1, b_2, a_3
eq7: a_1, b_1	
eq8: a_2, b_2	
eq9: a_3, b_3	
N.B. $a_3^{(3)} \neq a_3^{(4)}$	

uneq0: none	
uneq1: a_1	uneq2: a_2
uneq3: b_1	uneq4: b_2
uneq5: a_1, a_3	uneq6: a_2, a_3
uneq7: b_1, b_3	uneq8: b_2, b_3
uneq9: a_1, b_1, a_3	uneq10: a_2, b_2, a_3
uneq11: a_1, b_2, b_3	uneq12: b_1, a_2, b_3
uneq13: a_1, a_2	
uneq14: b_1, b_2	
uneq15: a_3, b_3	
N.B. $a_3^{(5)} \neq a_3^{(6)}, b_3^{(7)} \neq b_3^{(8)}$	