Title: Partition Functions of Three Dimensional Gravity

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Abstract: We consider pure three dimensional quantum gravity with a negative cosmological constant. The torus partition function can be computed exactly as a sum over geometries, including all known quantum corrections. The answer provides important clues about the structure of quantum gravity; in particular, in order for the theory to be a proper quantum mechanical system some extra ingredients are needed beyond the usual real geometries considered in general relativity. One possiblity is that complex geometries need to be included; this leads to holomorphically factorized partition functions. These partition functions provide a wealth of information about black hole microphysics. For example, the Hawking-page phase transition can be studied exactly; it is a phase transition of the type described by Lee and Yang, which is associated with a condensation of zeros in the complex temperature plane.

Partition Functions of 3D Gravity

Alex Maloney, McGill University

Perimeter Institute, 11-12-07

A. Maloney & E. Witten, to appear

Overview

The Problem:

Our goal is to quantize three dimensional gravity.

- Locally, the metric g is completely fixed by Einstein's equations. We are left with the global degrees of freedom of constant curvature metrics.
- Although the theory seems trivial, it contains rich physics: black holes, phase transitions, AdS/CFT, ...

We will focus on the case with negative cosmological constant. The theory is parameterized by

- The cosmological constant $\Lambda \sim -1/\ell^2$
- Newton's constant G

which can be combined into a dimensionless ratio $k = \ell/16G$.

We will use two tools: AdS/CFT & Modular Invariance.

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AdS/CFT

The prototypical negative curvature space-time is Anti-de Sitter space AdS_3, which has an asymptotic boundary $R^1 \times S^1$. The isometry group of AdS_3 acts on this boundary by conformal transformations.

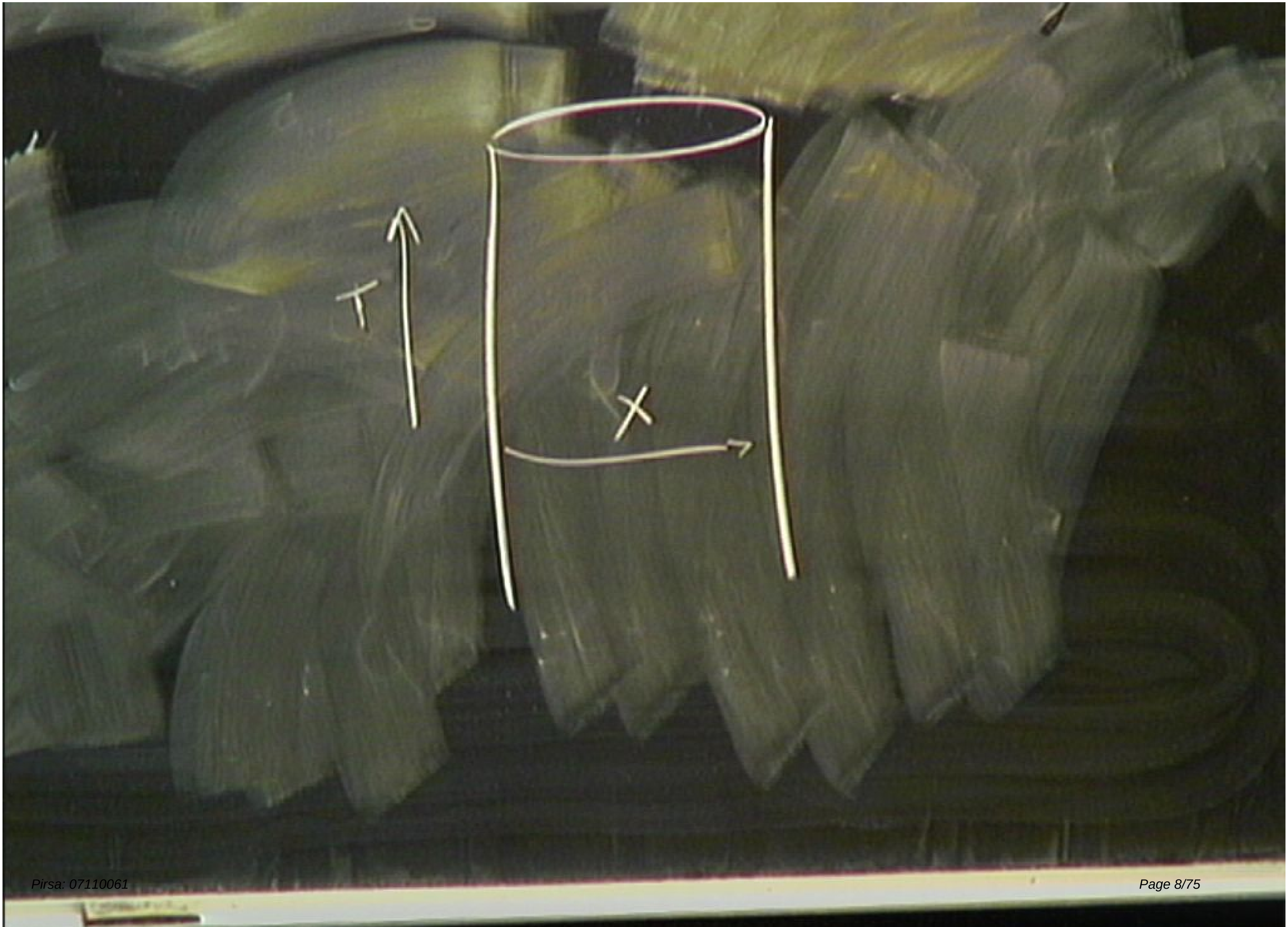
To define a theory of quantum gravity, we must specify a class of metrics. We do this by fixing the conformal structure of the metric at infinity.

Our goal is to integrate over all manifolds with the same conformal boundary at infinity to compute a partition function

$$Z = \int Dg\ e^{-S(g)} = \int Dg\ e^{-kVol(g)}$$

It turns out that Z is the partition function of a two dimensional conformal field theory.

Maldacena; Brown & Henneaux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Modular Invariance

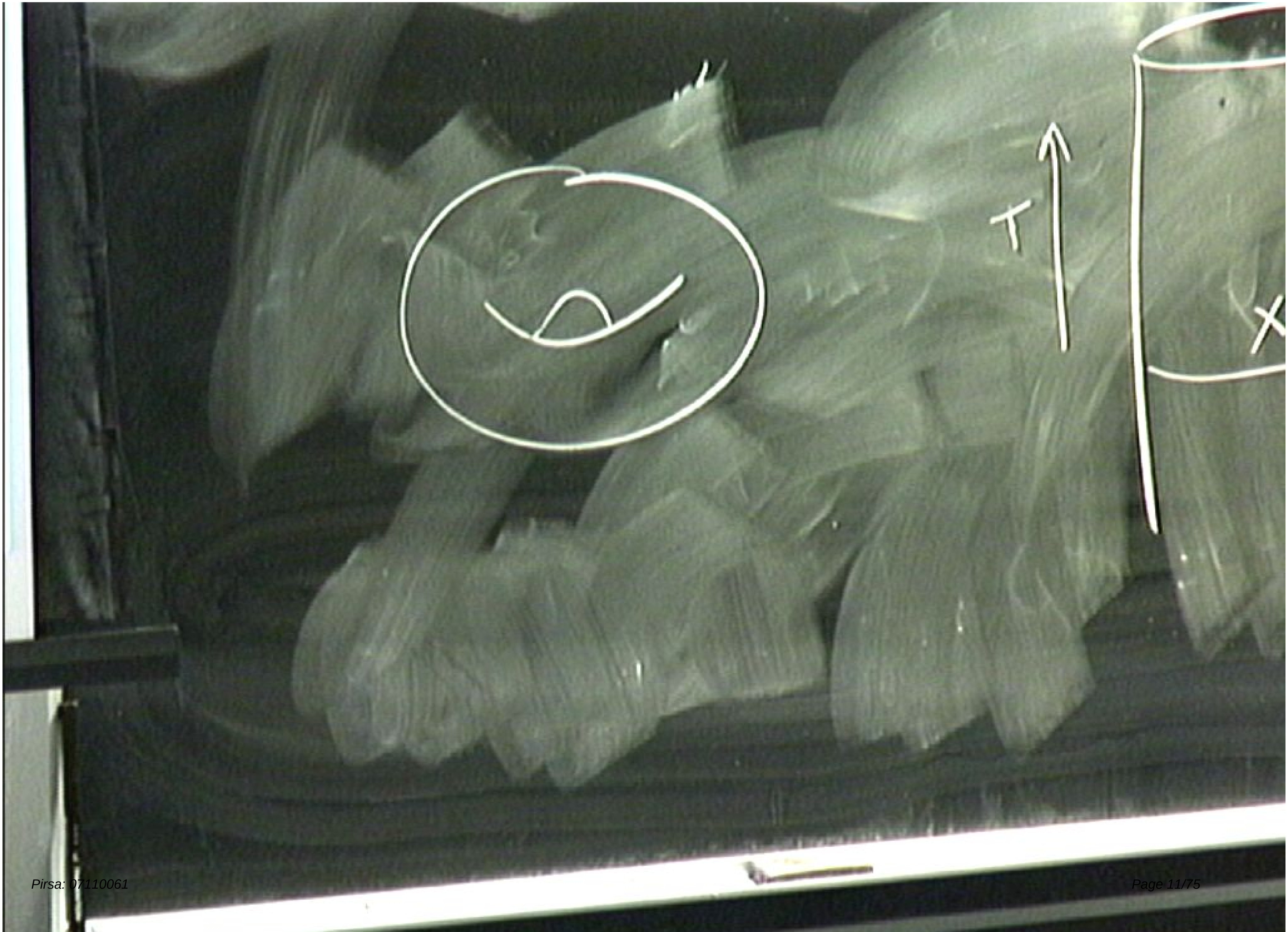
We will study the finite temperature partition function, which is periodic in Euclidean time: $it = t_E \sim t_E + \beta$. This means we are looking for Euclidean manifolds with T^2 boundary.

The conformal structure τ of T^2 is invariant under modular transformations $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$. So $Z = Z(\tau)$ will be a modular function of $\tau = i\beta$.

This is a powerful constraint. For example, if we assume that Z varies holomorphically with τ, modular invariance implies that

- k is an integer $\Rightarrow$ the Cosmological constant ℓ is quantized in Planck units
- all operators have integer dimension $\Rightarrow$ the masses of states (i.e. black holes) are quantized in AdS units.

The \$64,000 question: is Z holomorphically factorized? If so, why?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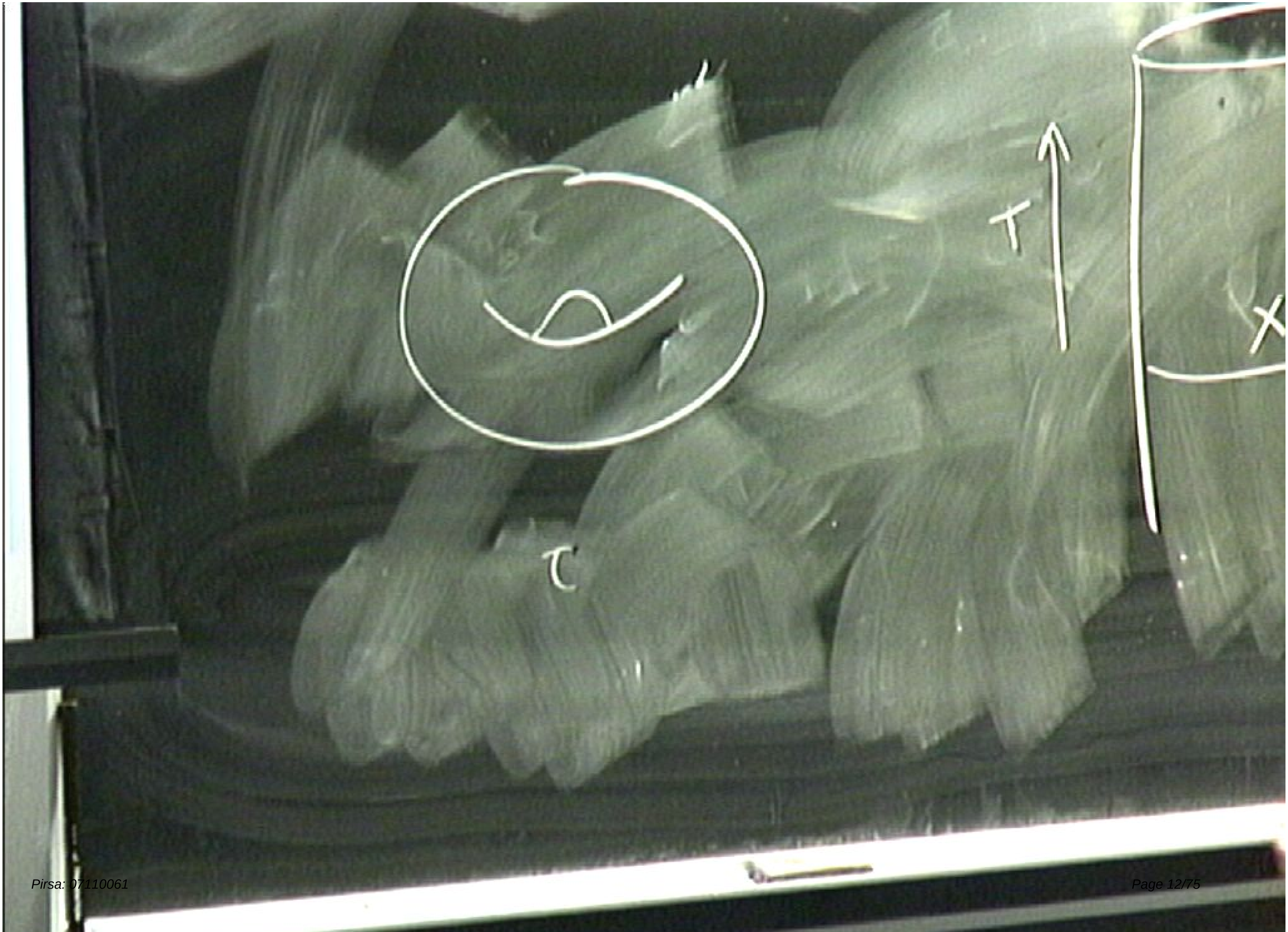
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The Plan for Today:

- Overview: Quantum Gravity in Three Dimensions
- Computing $Z(\tau)$: a Sum over Geometries, with Loop Corrections
- Physical Implications: the Need for Complex Geometries
- Black Hole Microphysics: Entropy
- Black Hole Microphysics: Phase Transitions

Computing $Z(\tau)$

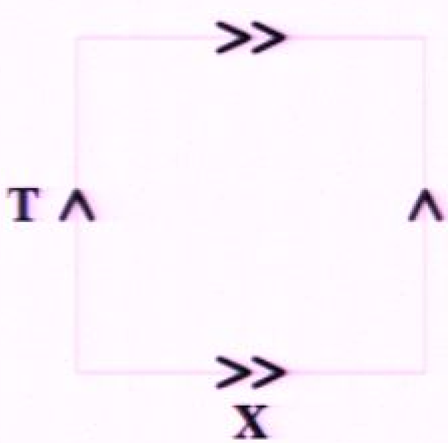
Tree Level

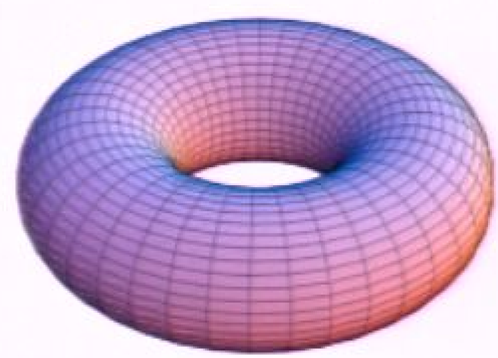
In the saddle point approximation

$$Z(\tau) = \int Dg\; e^{-kVol(g)} \sim e^{-kVol(M_3)}$$

where M_3 is a solution of the equations of motion with T^2 boundary. We won't allow M_3 to have singularities, since we don't know how to define an action in that case.

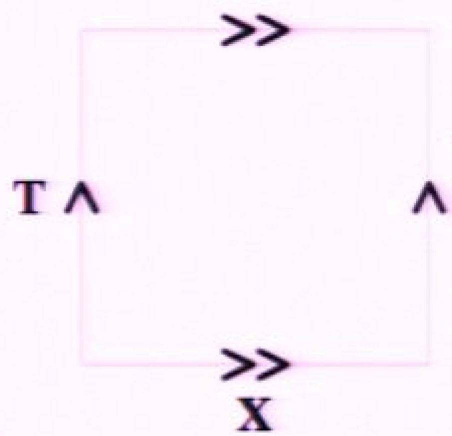
All such three manifolds have been classified. They are solid tori which "fill in" the boundary T^2:





Tree Level II

There are many such $M_3 = M_{c,d}$. One for each choice of cycle $cT + dX$ which is contractible in the interior:



If the complex structure of the boundary T^2 is $\tau = \theta + i\beta$, then

- When X is contractible, the geometry $M_{0,1}$ is thermal AdS with temperature β and angular potiental θ.
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Maldacena & Strominger

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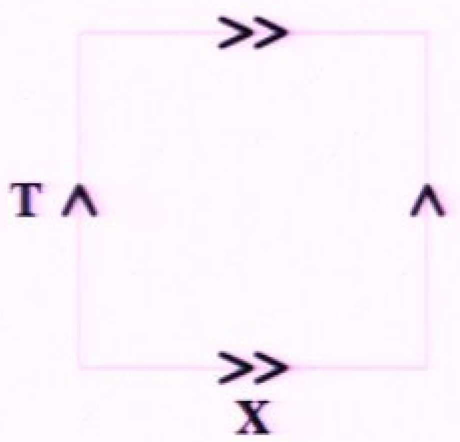

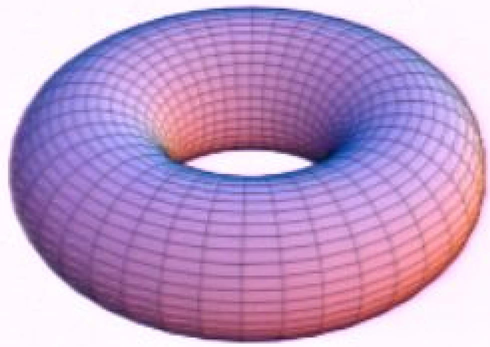

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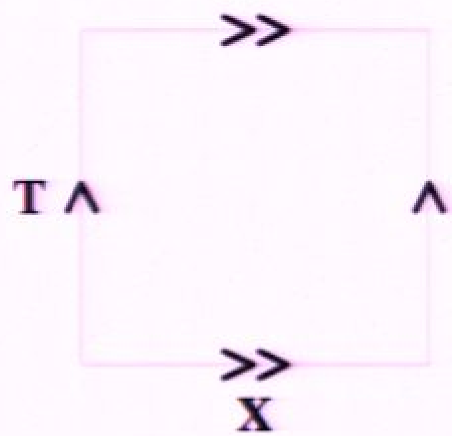

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Sum over Geometries

Each geometry $M_{c,d}$ is associated with an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})/\mathbb{Z}$$

The contribution to the partition function is

$$Vol(M_{c,d}) = \mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \mathrm{Im}\gamma\tau$$

The full partition function is the "Poincaré Series"

$$Z(\tau) \sim \sum_{c,d} \exp\left\{-2\pi k \ \mathrm{Im}\gamma\tau\right\}$$

This sum is badly divergent.

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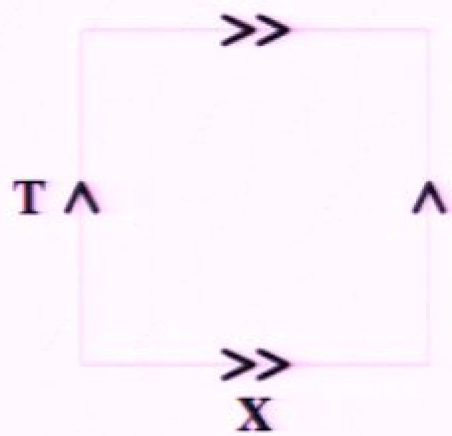

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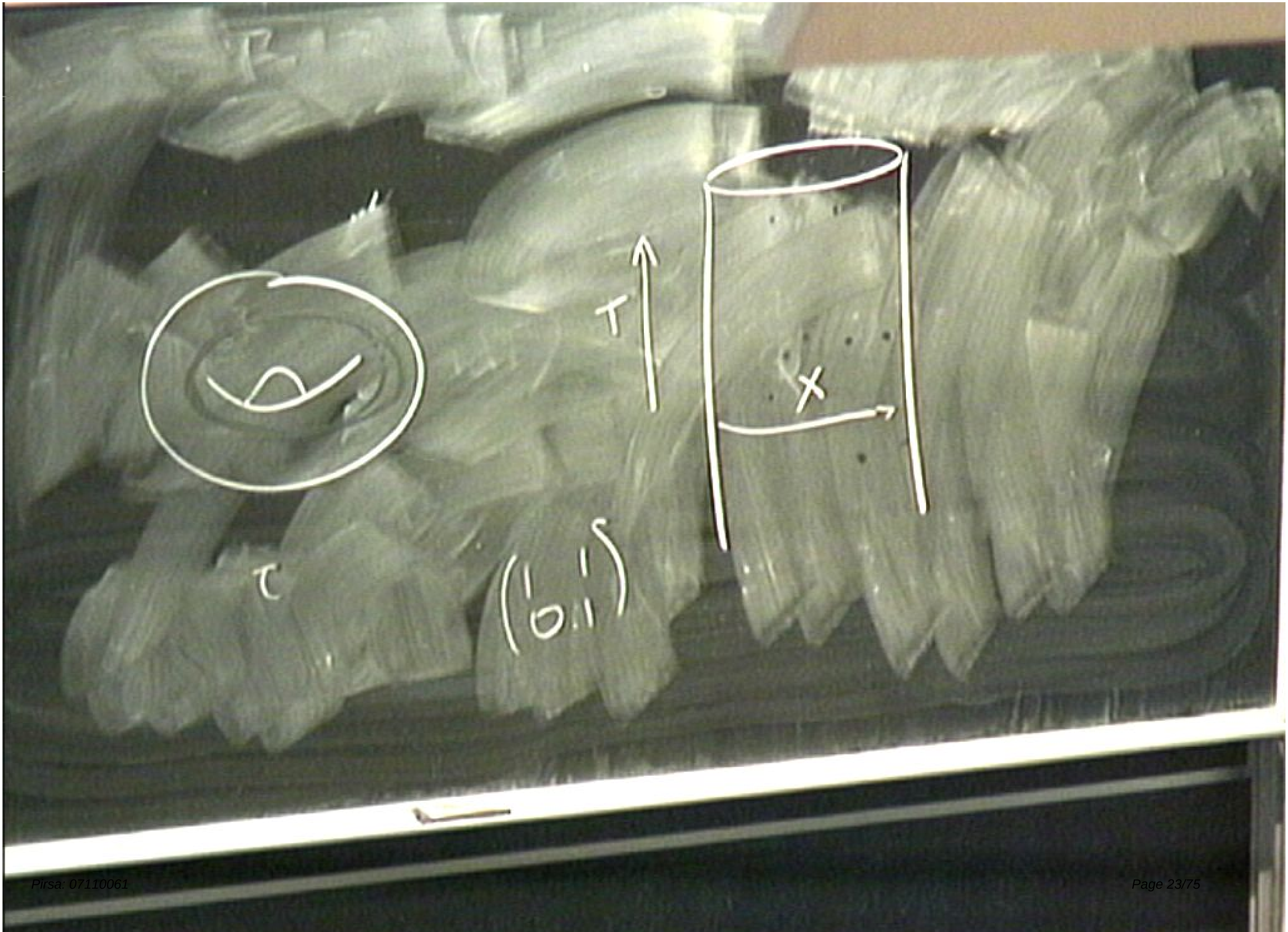
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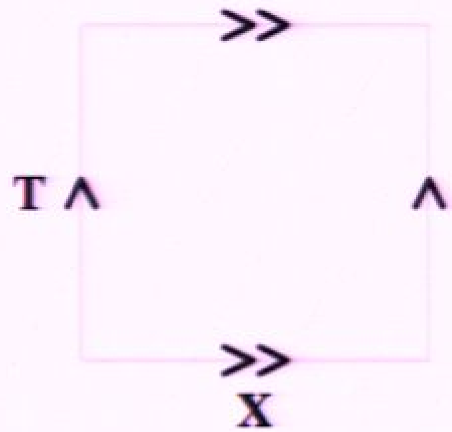

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One Loop

To compute corrections to the saddle point action, we'd like to evaluate the one-loop determinant around $M_{c,d}$. This is difficult to do directly, but we can use a trick...

$M_{0,1}$ is thermal AdS_3, so the partition function $Z_{0,1}$ has a Hamiltonian interpretation

$$Z_{0,1} = \mathrm{Tr}\ e^{-\beta H - i\theta J} = \mathrm{Tr}\ q^{L_0}\bar{q}^{\bar{L}_0}$$

The trace is over the Hilbert space of small fluctuations around AdS_3.

Brown & Henneaux told us exactly what this Hilbert space is. We find it by quantizing the phase space of classical solutions around AdS_3, modulo diffeomorphisms which vanish sufficiently quickly at infinity.

One Loop II

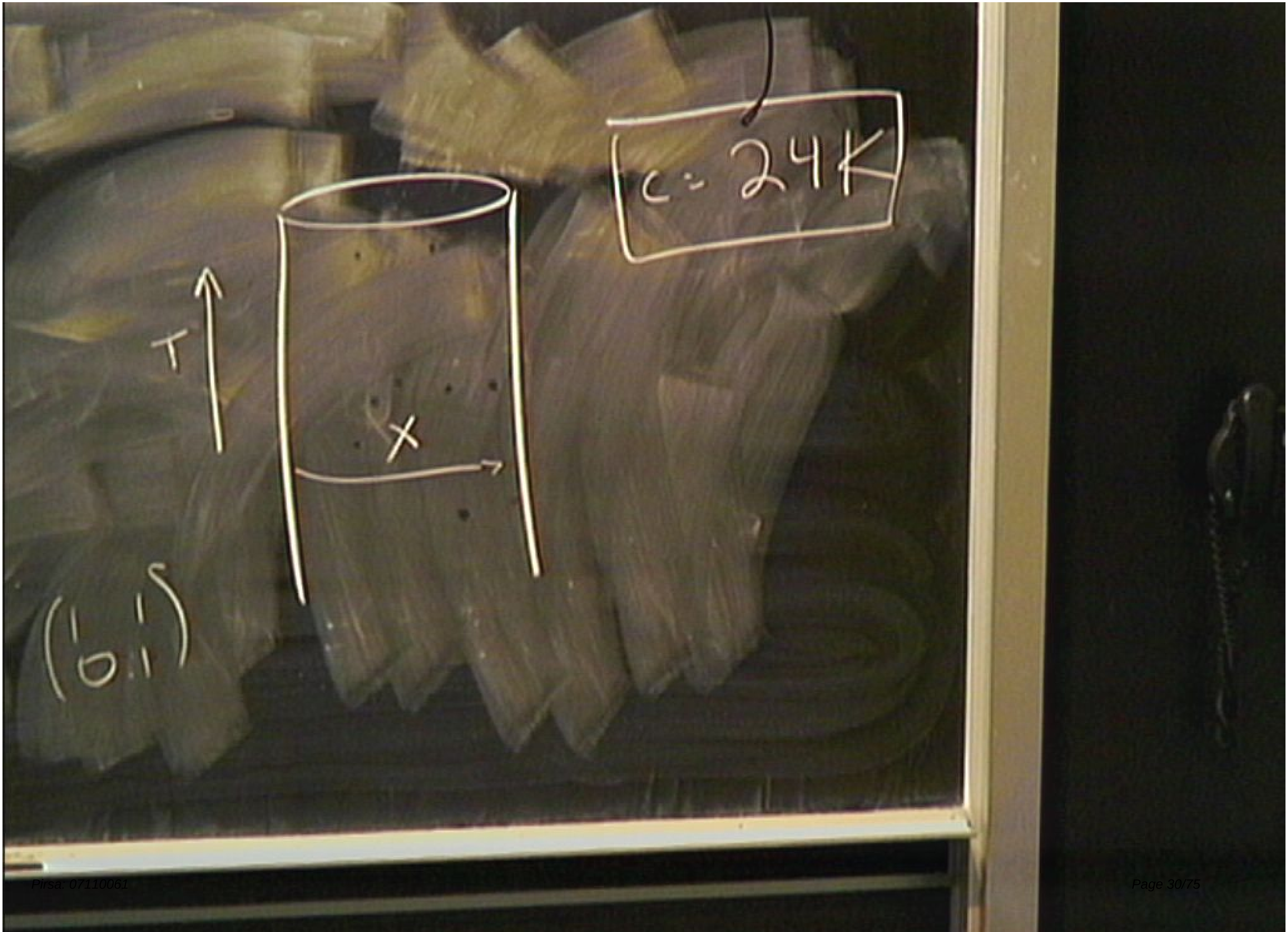
The Hilbert space contains a vacuum state $|0\rangle$, which is empty AdS_3. It also contains Virasoro descendants $L_{-n_1} \ldots L_{-n_i}|0\rangle$, which are excitations of empty AdS_3. Taking the trace gives the one-loop partition function:

$$Z_{0,1}(\tau) = |q|^{-2k} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}$$

In fact, the answer is one-loop exact...

The Hilbert space is a representation of the symmetry group of the theory, which is generated by diffeomorphisms that preserve the boundary. This is a central extention of the Virasoro algebra.

The family of states described above transform in an irreducible representation of this symmetry group, which is fixed in terms of the central charge. The energy levels can not be corrected, so there are no further perturbative corrections to $Z_{0,1}$.


c = 24K
T
X

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Sum over Geometries II

So the sum over all known contributions to the partition function (including all perturbative corrections) is the Poincaré series

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau)$$

This sum is still divergent, but can be regularized:

- Athough $Z(\tau)$ diverges, $(\Delta - 1/4)Z(\tau)$ converges.
- The operator $\Delta - 1/4$ is invertible on $H^2/SL(2,\mathbb{Z})$.

So there is a unique, regularized partition function $Z(\tau)$. It can be computed using zeta function regularization:

$$Z(\tau) = \lim_{s\to 0} \sum_{c,d} \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} Z_{0,1}(\gamma\tau)$$

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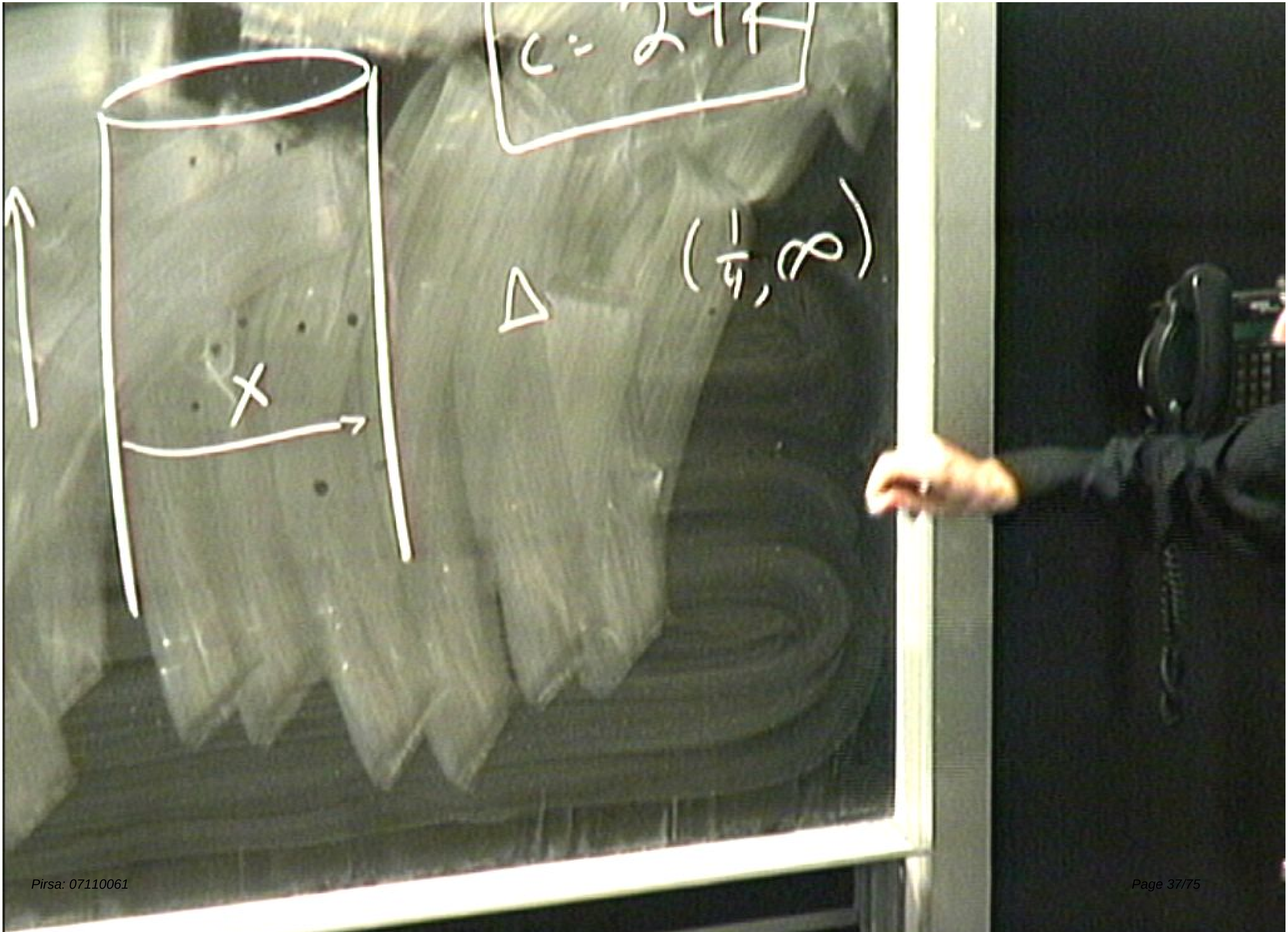
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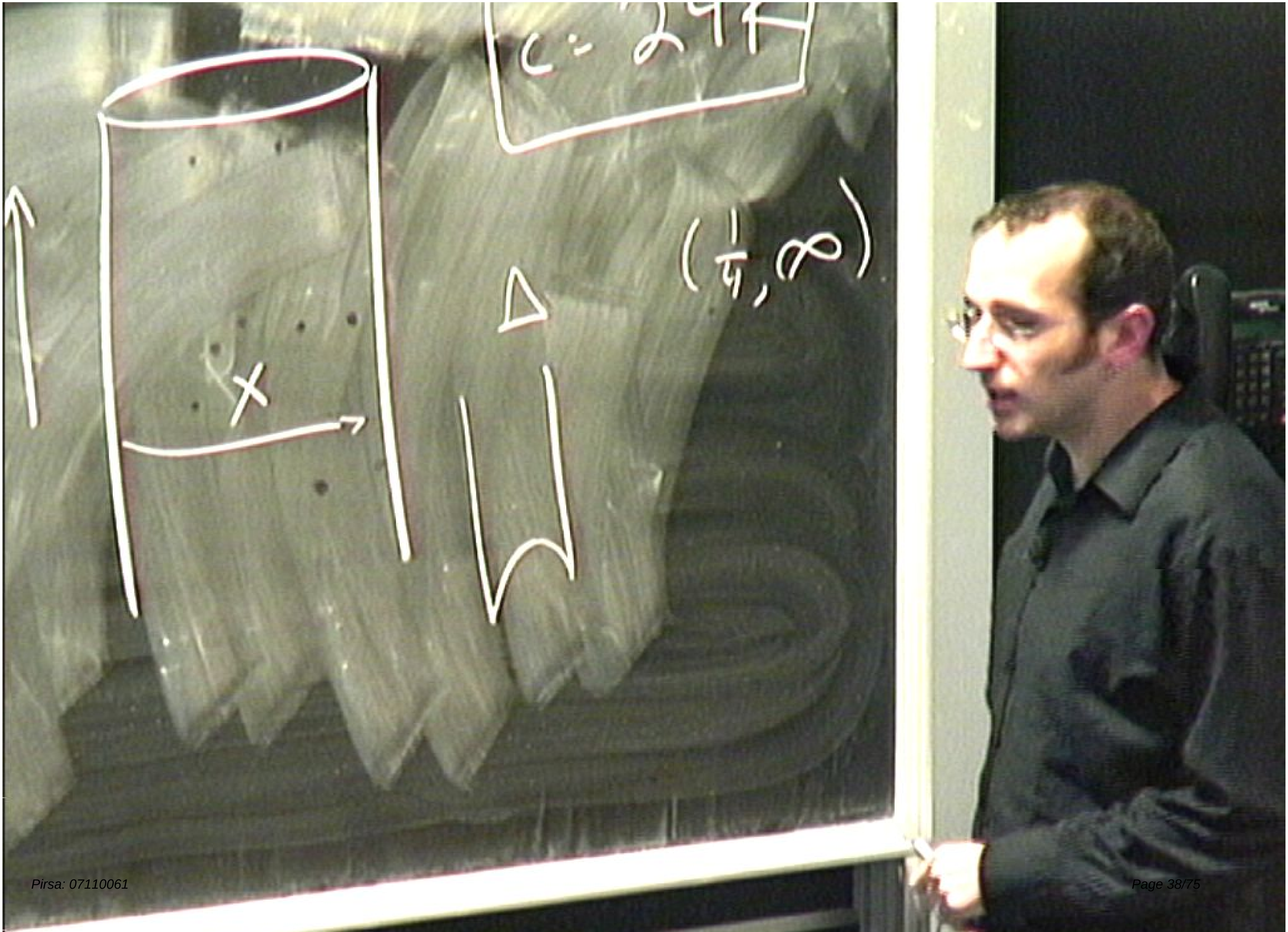
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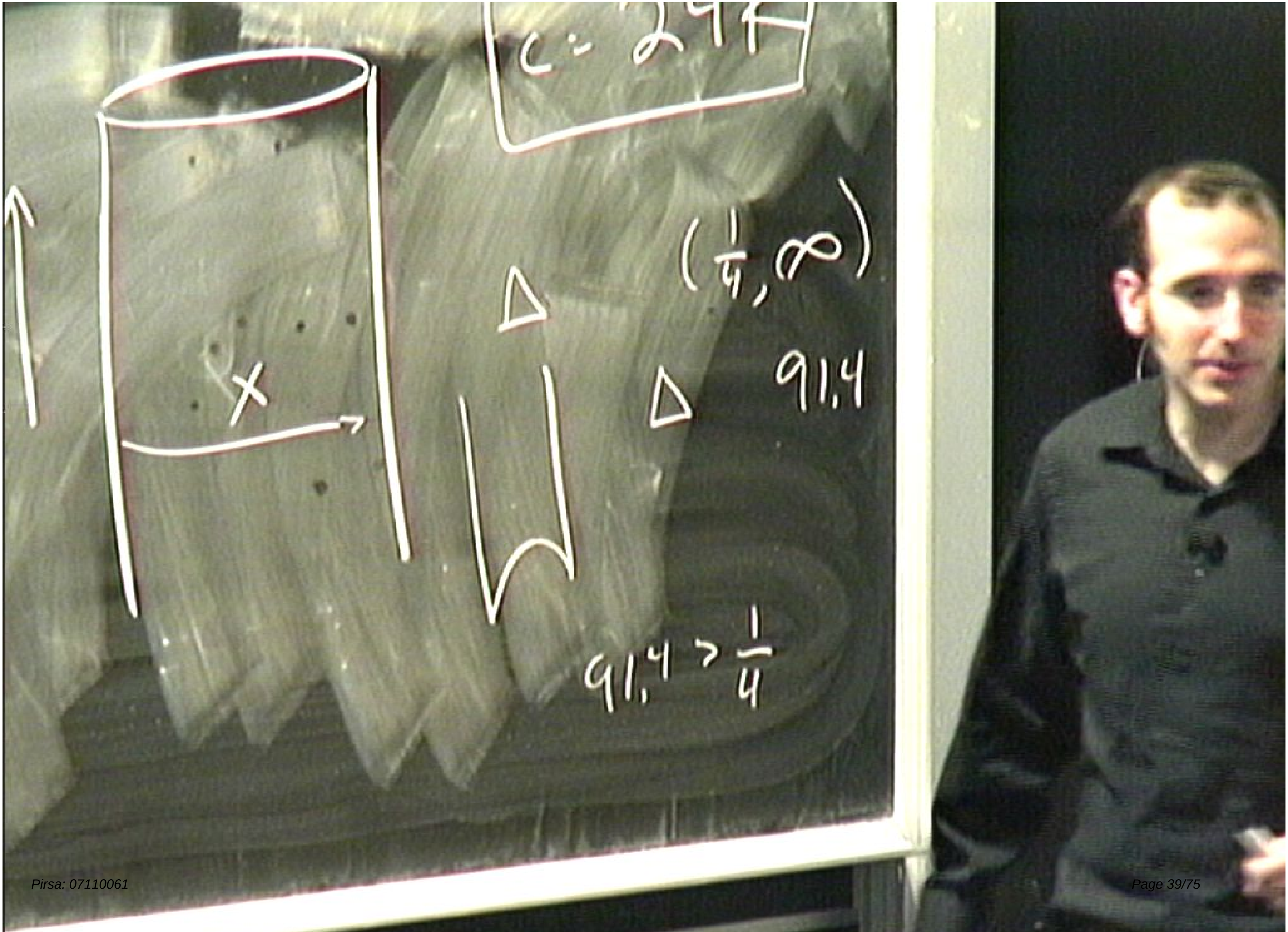
Δ
x


x
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Δ 91.4
91.4 > 1/4

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The Regularized Partition Function

We must prove our regularization scheme works:

- Our $Z(\tau, s)$ is related to the Selberg Zeta function for the fundamental domain $H_2/SL(2,\mathbb{Z})$
- The limit $s \to 0$ is finite because the Laplacian on $H_2/SL(2,\mathbb{Z})$ has no discrete eigenvalue at $\lambda = 1/4$

There is a straightforward procedure for computing terms in $Z(\tau)$. Letting $\tau = \theta + i\beta$,

$$Z(\tau) = Z_{0,1}(\tau) + |\eta(\tau)|^{-2}\left(c_0 + c_1\beta^{-1} + c_2\beta^{-2} + \ldots\right) + \mathcal{O}(e^{\beta})$$

where the c_i are computable:

$$c_0 = -6, \qquad c_1 = \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)}, \qquad c_2 = \ldots$$

Physical Implications

Physical Interpretation

Does this answer have a sensible physical interpretation?

We are trying to compute the thermal partition function of a quantum mechanical system. This should be of the form

$$\mathrm{Tr}\left(e^{-\beta H - i\theta J}\right) = \sum_{\Delta,\bar{\Delta}} F(\Delta,\bar{\Delta})\, q^{\Delta}\, \bar{q}^{\bar{\Delta}}$$

where $q = e^{2\pi i\tau}$ and $\tau = \theta + i\beta$. In order for this to converge, the set of Δ and $\bar{\Delta}$ appearing in this sum should be discrete. The $F(\Delta,\bar{\Delta})$ should be positive integers.

Our answer is not of this form, because of the polynomial terms in β. Even worse, some of our coefficients are negative!

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There is a straightforward procedure for computing terms in $Z(\tau)$. Letting $\tau = \theta + i\beta$,

$$Z(\tau) = Z_{0,1}(\tau) + |\eta(\tau)|^{-2}\left(c_0 + c_1\beta^{-1} + c_2\beta^{-2} + \dots\right) + \mathcal{O}(e^{\beta})$$

where the c_i are computable:

$$c_0 = -6, \qquad c_1 = \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)}, \qquad c_2 = \dots$$

The Regularized Partition Function

We must prove our regularization scheme works:

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Physical Implications

Physical Interpretation

Does this answer have a sensible physical interpretation?

We are trying to compute the thermal partition function of a quantum mechanical system. This should be of the form

$$\mathrm{Tr}\left(e^{-\beta H - i\theta J}\right) = \sum_{\Delta,\bar{\Delta}} F(\Delta,\bar{\Delta})\, q^{\Delta}\, \bar{q}^{\bar{\Delta}}$$

where $q = e^{2\pi i\tau}$ and $\tau = \theta + i\beta$. In order for this to converge, the set of Δ and $\bar{\Delta}$ appearing in this sum should be discrete. The $F(\Delta,\bar{\Delta})$ should be positive integers.

Our answer is not of this form, because of the polynomial terms in β. Even worse, some of our coefficients are negative!

So $Z(\tau)$ does not have a sensible quantum mechanical interpretation. One possible implication is that three dimensional gravity simply does not make sense. Are there other possibilities?

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Complex Geometries

Another possibility is that we were too restrictive in our partition sum.

What if we include complex geometries? Complex geometries have arisen before in AdS/CFT, in the computation of geodesics and Wilson loops.

Classically, 3D gravity can be written as an $SL(2, \mathbf{C})$ gauge theory

$$A = e + i\omega$$

The A and $\bar{A}$ sectors do not interact, so we will allow them to live on different 3-manifolds with the same conformal boundary. The corresponding metric is no longer real.

The full partition function is then the square $|Z(\tau)|^2$ of a holomorphic partition function

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau), \qquad Z_{0,1}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}$$

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Complex Geometries II

We now sum over two copies (holomorphic and anti-holomorphic) of $SL(2,\mathbb{Z})/\mathbb{Z}$. The sum is divergent, but can be regularized as before.

Since this is a holomorphic sum, none of the pesky polynomial terms appear. The resulting partition function now has a proper quantum mechanical interpretation as a trace over a Hilbert space.

The answer is Witten's extremal partition function:

$$Z(\tau) = \sum_{n=0}^{k} a_n T_n J = \sum_{\Delta \geq -k}^{\infty} F(\Delta) q^{\Delta}$$

where J is Klein's modular invariant function and T_k is a Hecke operator. The coefficients $F(\Delta)$ are all positive integers.

Black Hole Entropy

Examples:

What do these partition functions look like?

For $k = 1$,

$$Z(\tau) = J(\tau) = q^{-1} + 196884q + \dots$$

The conformal field theory with partition function $J(\tau)$ is known. It is the famous "Monster CFT" of Frenkel, Lepowski & Meurman, with central charge $c = 24$.

For $k > 1$ it is not known whether analogous CFTs exist, but one can still construct the associated partition functions. E.g.

$$Z_{k=3}(\tau) = q^{-3} + q^{-1} + 1 + 2593096794q + \dots$$

The conjecture that such CFTs exist has survived several consistency checks, although questions have been raised.

Witten, Gaiotto & Yin, Gaberdiel, ...

Black Holes

The coefficients of the modular function $Z(\tau)$ grow exponentially at large Δ

$$F(\Delta) \sim e^{4\pi\sqrt{k\Delta}} + \dots$$

What are all these states?

AdS_3 gravity contains black holes, discovered by Banados, Teitelboim & Zanelli. Their entropy

$$S_{BH} = \frac{A}{4G} = 4\pi\sqrt{k\Delta}$$

agrees precisely with this asymptotic growth. For example, the entropy of the $k = 1$, $\Delta = 1$ states is

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Black Holes II

The subleading terms in the asymptotic growth can computed and compared with corrections to the Bekenstein-Hawking formula for black hole entropy

$$F(\Delta) \sim \frac{k^{1/4}}{\sqrt{2}\Delta^{3/4}} e^{4\pi\sqrt{k\Delta}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\Delta k}}\right)\right) \left(1 + \mathcal{O}\left(\frac{k}{\Delta}\right)\right)$$

There are two types of corrections:

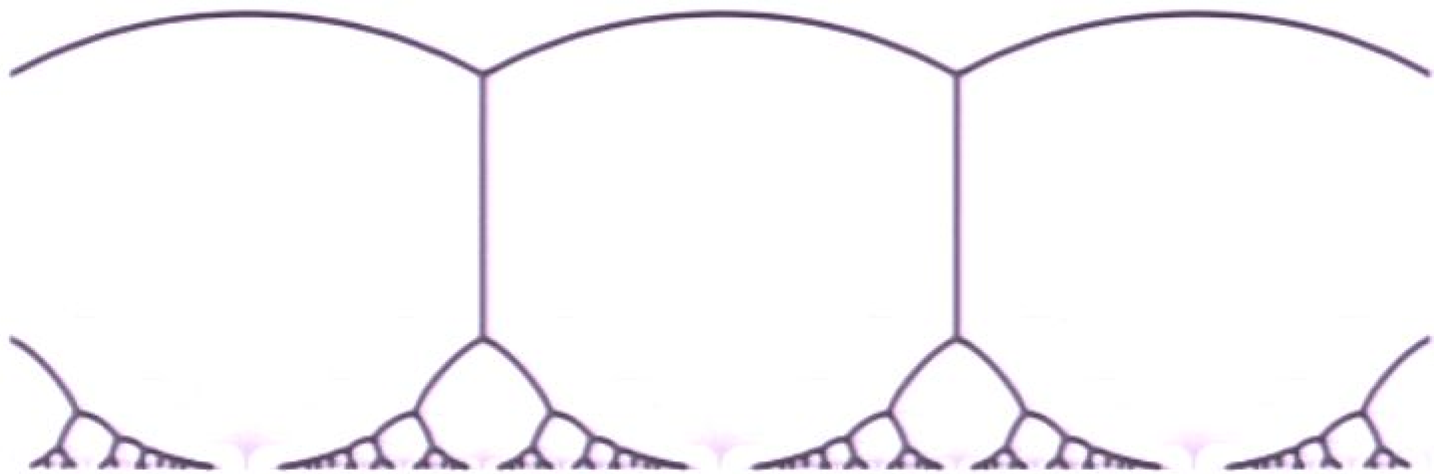
- Subleading in Δk: come from the fact that S_{BH} is computed at fixed temperature, whereas $F(\Delta)$ is computed at fixed energy
- Subleading in Δ/k: one-loop effects, which come from nearby (Brown & Henneaux) geometries

In both cases, an infinite series of subleading terms can be computed and matched term by term with corrections to $F(\Delta)$

Phase Transitions

Phase Diagram

For different values of τ, different saddles $M_{c,d}$ dominate the partition function.



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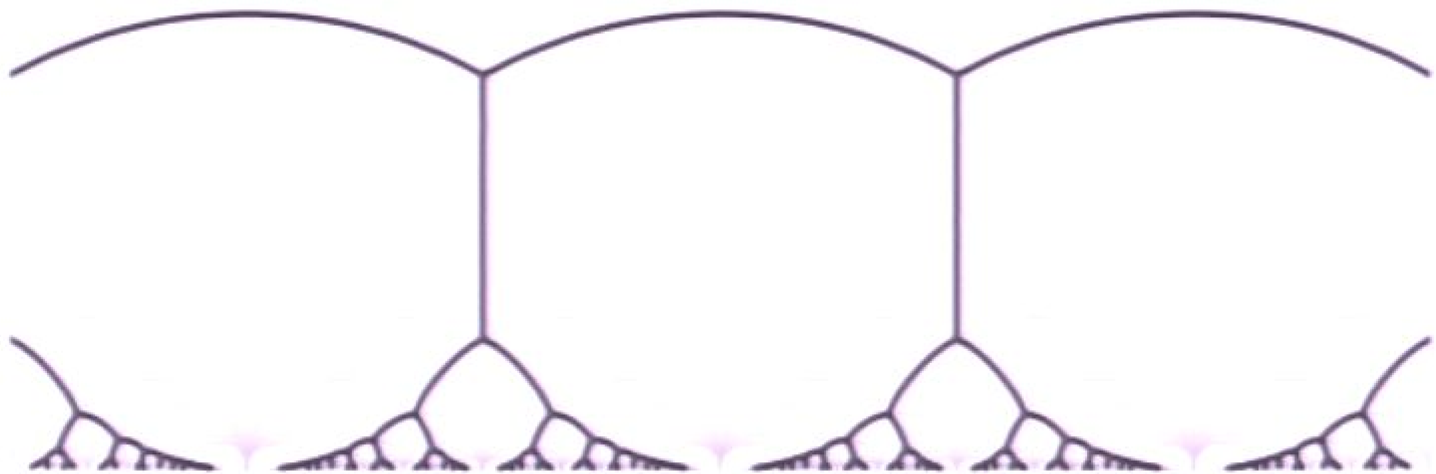

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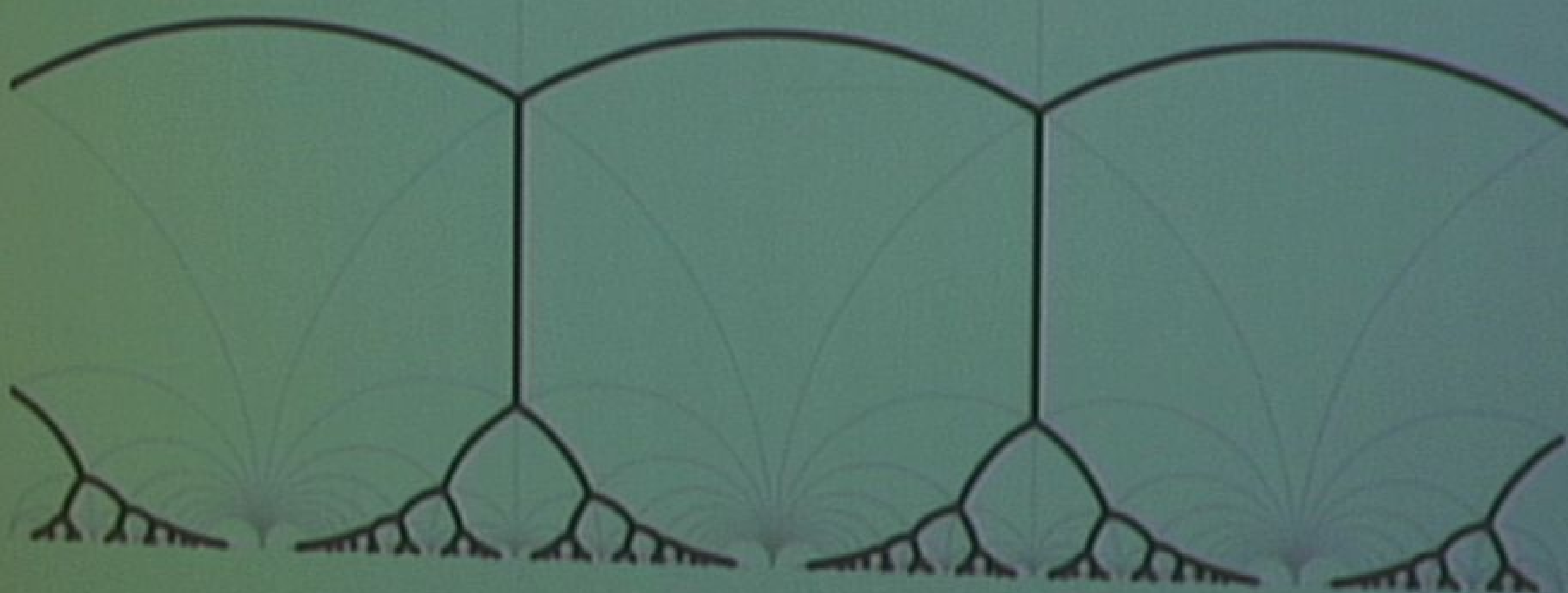

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Phase Transitions

As τ moves from one region to another we expect a phase transition.

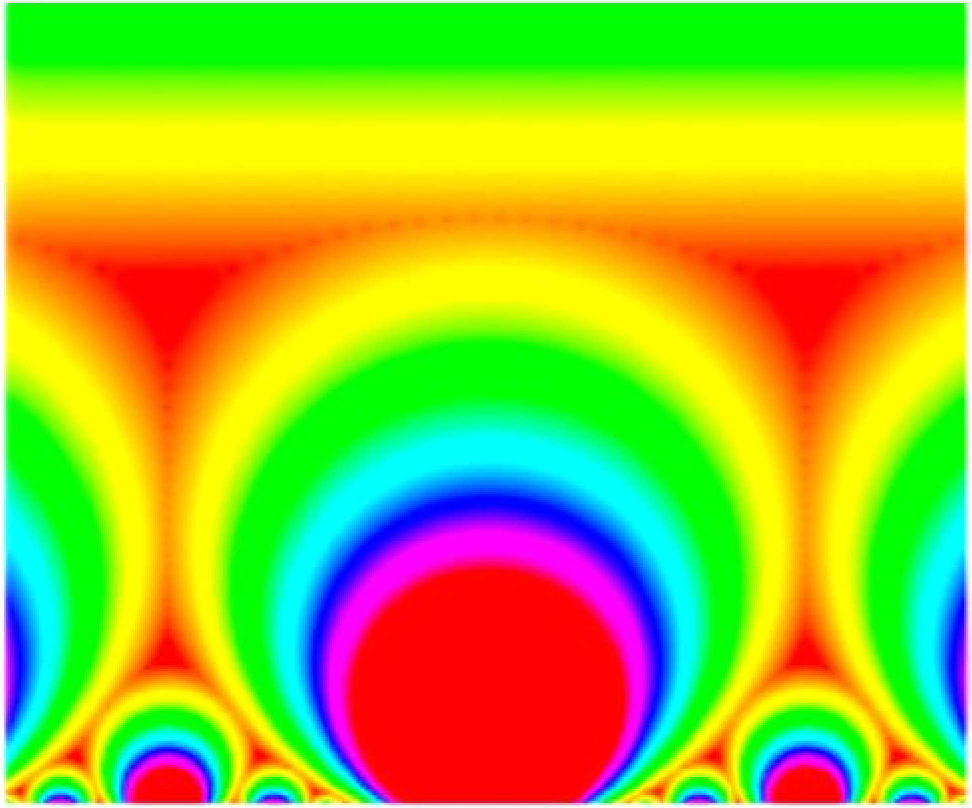
Puzzle: Phase transitions are associated with non-analyticity in τ, but our $Z(\tau)$ are analytic functions on the upper half τ-plane.

Resolution: The theory of phase transitions developed by Lee & Yang and Fischer.

- In the classical ($k \to \infty$) limit, the partition function $Z(\tau)$ becomes non-analytic along phase boundaries.
- This non-analyticity arises because at finite k, $Z(\tau)$ has k zeroes on the phase boundary. The zeroes become dense on this curve in the $k \to \infty$ limit.

Phase Transitions II

For example, it can be proven that all zeroes of $T_n J$ lie along phase boundaries in the τ plane, and that these zeroes become dense in the $n \to \infty$.



Asai, Kaneko & Ninomiya

Phase Transitions III

The extremal partition functions have zeroes along the phase boundaries as well.

In fact, *any* modular invariant, holomorphic partition function $Z(\tau)$ will have this property, provided that it does not have too many additional primary states at low energy:

$$F(\Delta) \leq e^{2\pi(0.77)(\Delta+k)}, \qquad \Delta \leq 0$$

This bound is physically sensible:

- $\mathcal{O}(e^k)$ additional primary states $\approx \mathcal{O}(e^k)$ additional light fields
- This many additional states would lead to species problems at large k, making black holes unstable

Conclusion

Three dimensional quantum gravity can teach us a lot:

- The sum over real geometries is not consistent with a quantum mechanical Hilbert space interpretation
- One resolution is to sum over complex geometries
- Understand black hole entropy & its corrections
- Hawking-Page transition as a Lee–Yang phase transition

But many puzzles remain:

- Do higher k CFTs exist?
- Do they have Monster symmetry?
- Higher genus partition functions?
- Cosmic censorship and chronology protection?
- Quasinormal modes and black hole quantization?