Title: Entanglement is an important resource ??!!

Date: Nov 28, 2007 04:00 PM

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Abstract: We attempt at characterizing the correlations present in the quantum computational model DQC1, introduced by Knill and Laflamme [Phys. Rev. Lett. 81, 5672 (1998)]. The model involves a collection of qubits in the completely mixed state coupled to a single control qubit that has nonzero purity. Although there is little or no entanglement between two parts of this system, it provides an exponential speedup in certain problems. On the contrary, we find that the quantum discord across the most natural split is nonzero for typical instances of the DQC1 ciruit. Nonzero values of discord indicate the presence of nonclassical correlations. We propose quantum discord as figure of merit for characterizing the resources present in this computational model. This
might be a complementary measure for counting resources in quantum
information science.

Entanglement is an important resource ??!!

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November 28, 2007

Entanglement is an important resource...

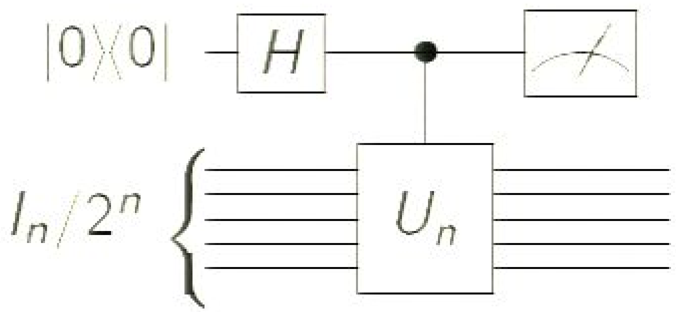
- Teleportation and Superdense Coding
- Quantum Algorithms
  - Deutsch-Jozsa, Simon etc
  - Grover
  - Shor
- Quantum Communications and Cryptography
  - Channel Capacity
  - Security
- Quantum Metrology
  - Heisenberg Limit & beyond
- Rich behavior of condensed matter systems

But actually...

- *For pure state quantum computation ... the presence of multi-partite entanglement, with a number of parties that increases unboundedly with input size, is necessary if the quantum algorithm is to offer an exponential speed-up over classical computation.* (Jozsa/Linden,02)
- *An algorithm (quantum system) can be classically efficiently simulated to within a prescribed tolerance η if a suitably small amount of global entanglement (depending on η) is present.* (Vidal,03)

Power of one qubit

$$\rho_{n+1} = \frac{1}{2^{n+1}} \begin{pmatrix} I_n & U_n \\ U_n^\dagger & I_n \end{pmatrix}$$



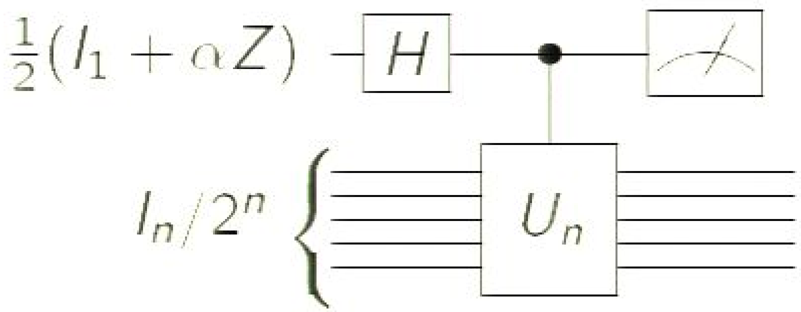
- Measure the top qubit
- $\langle X \rangle = \mathrm{Re}[\mathrm{Tr}(U_n)]/2^n$ and $\langle Y \rangle = \mathrm{Im}[\mathrm{Tr}(U_n)]/2^n$
- Can evaluate $\mathrm{Tr}(U_n)/2^n$ up to a fixed accuracy efficiently
- True only for unitaries that can be represented efficiently by a quantum circuit
- The classical problem is apparently HARD !

Knill & Laflamme, PRL, **81**, 5672, (1998).

sub-unity polarization

If the top qubit is not pure, then

$$\rho_{n+1}(\alpha) = \frac{1}{2^{n+1}} \begin{pmatrix} I_n & \alpha U_n \\ \alpha U_n^\dagger & I_n \end{pmatrix}$$



- $\alpha < 1$ makes the evaluation more difficult
- The number of runs required to estimate the normalized trace goes as $L \sim 1/\alpha^2$
- Power of even the tiniest fraction of a qubit

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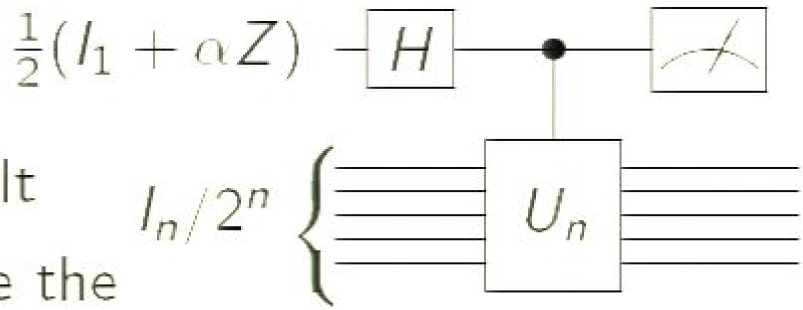

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- The number of runs required to estimate the normalized trace goes as $L \sim 1/\alpha^2$
- Power of even the tiniest fraction of a qubit
- Tracing out the top qubit leaves the remaining qubits in the completely mixed state
- The top qubit is always separable from the remaining qubits.

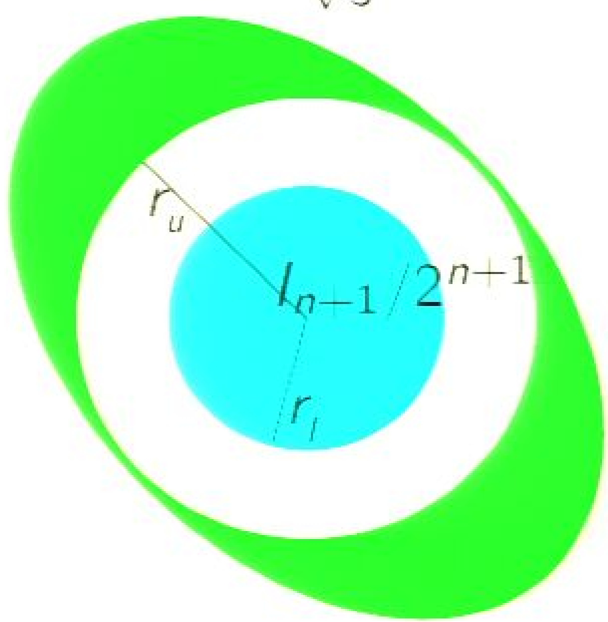
One more fact...

- The distance from the completely mixed state (in trace norm) at all times is

$$\sqrt{\mathrm{Tr}(\rho_{n+1}(\alpha) - I_{n+1}/2^{n+1})^2} = \alpha 2^{-(n+1)/2}$$

The largest separable ball:

$$r_u = 3 \times 6^{-(n+1)/2}, \; r_l = \frac{1}{\sqrt{3}} 6^{-(n+1)/2}$$



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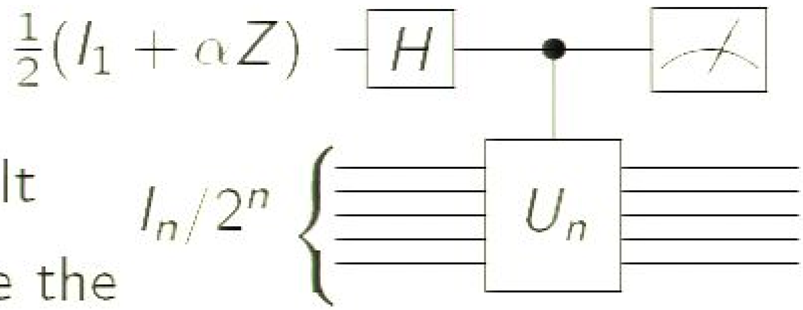

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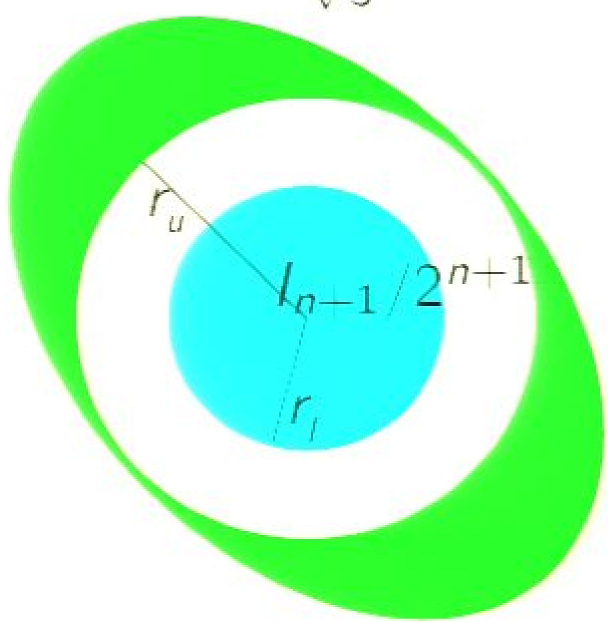

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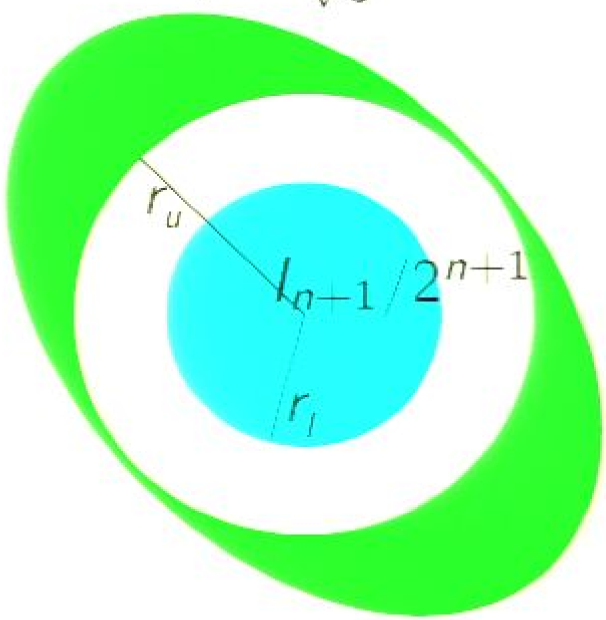


- $3 \times 6^{-(n+1)/2} \leq \alpha 2^{-(n+1)/2} \leq \frac{1}{\sqrt{3}} 6^{-(n+1)/2}$

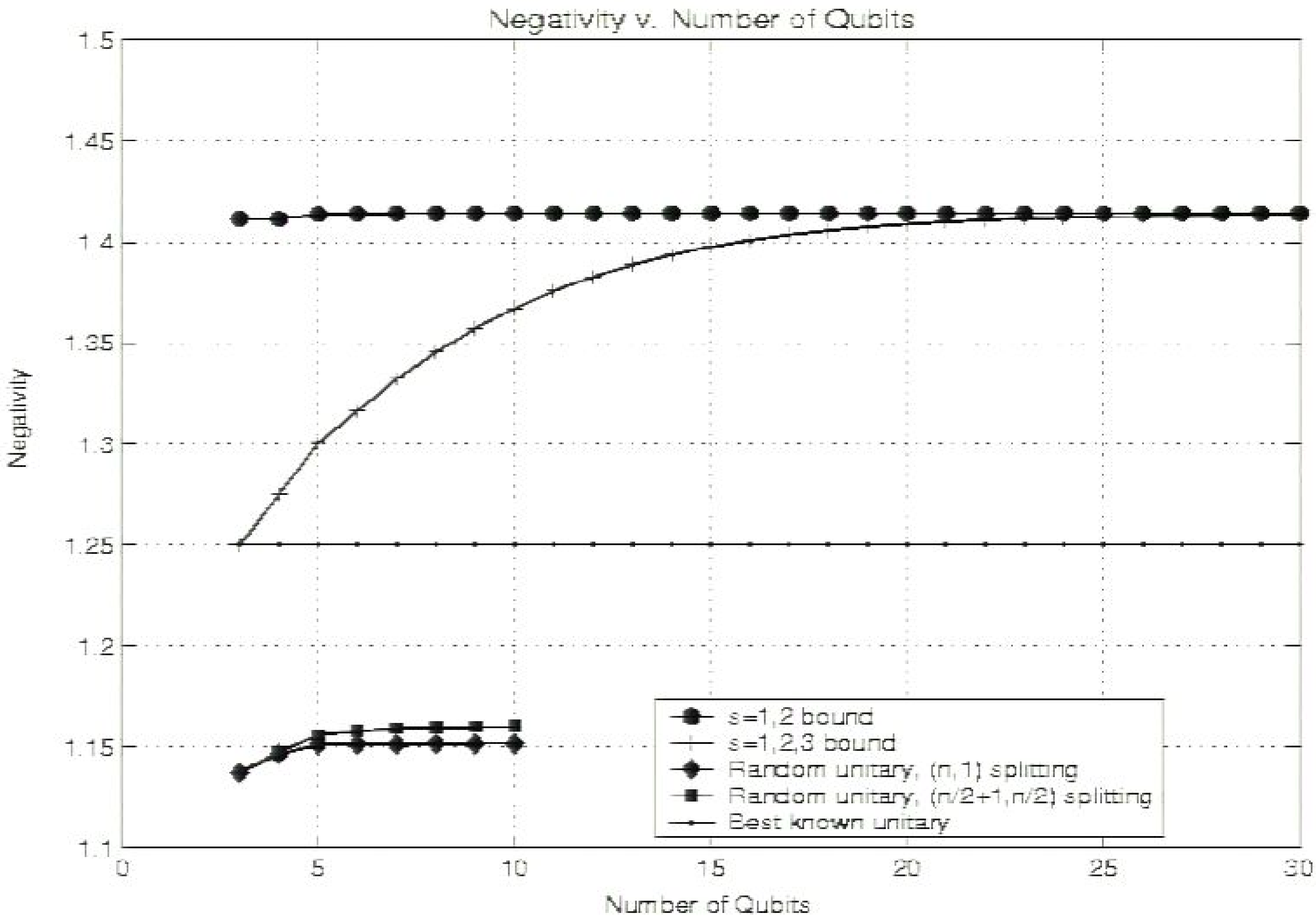
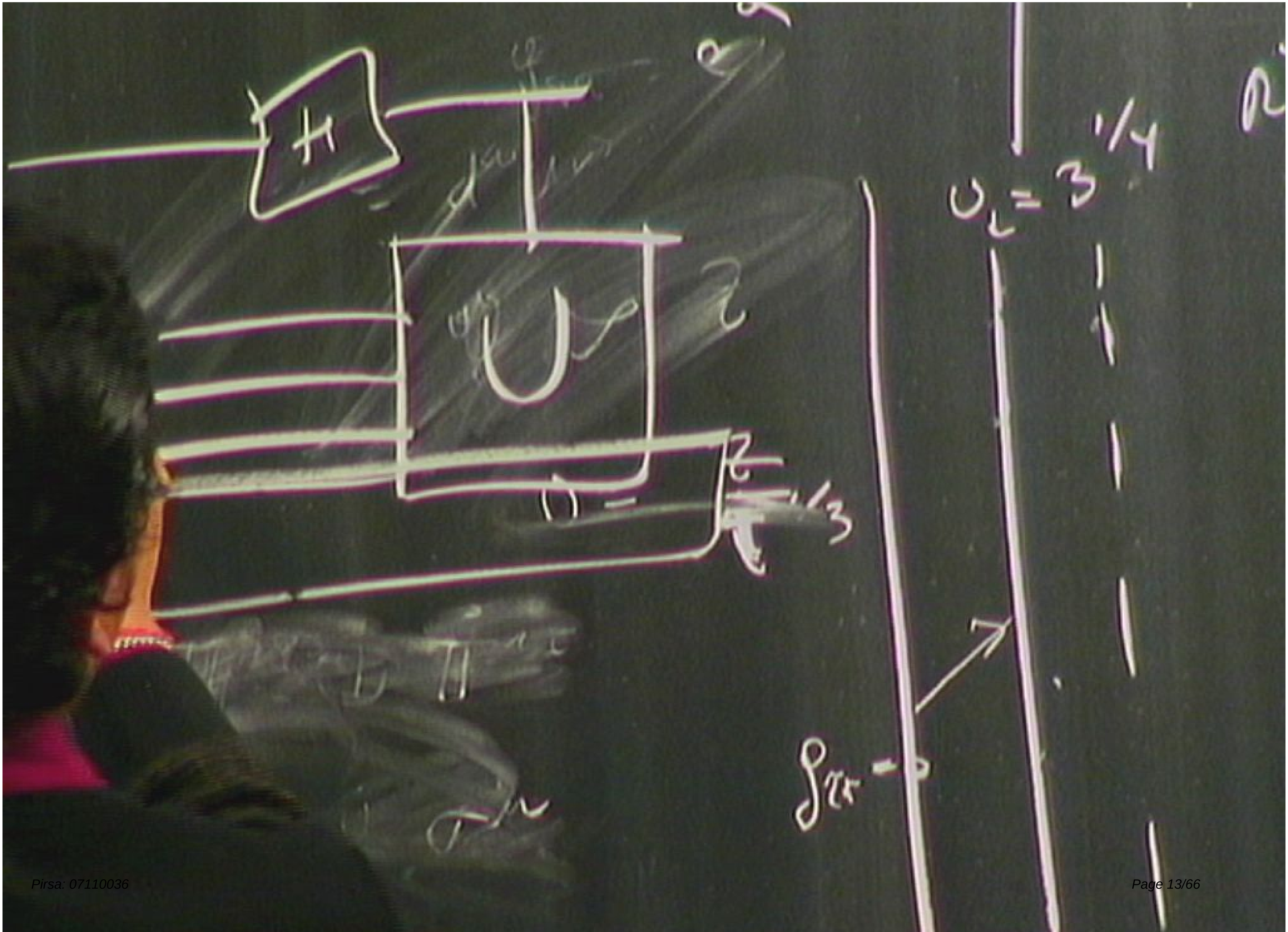
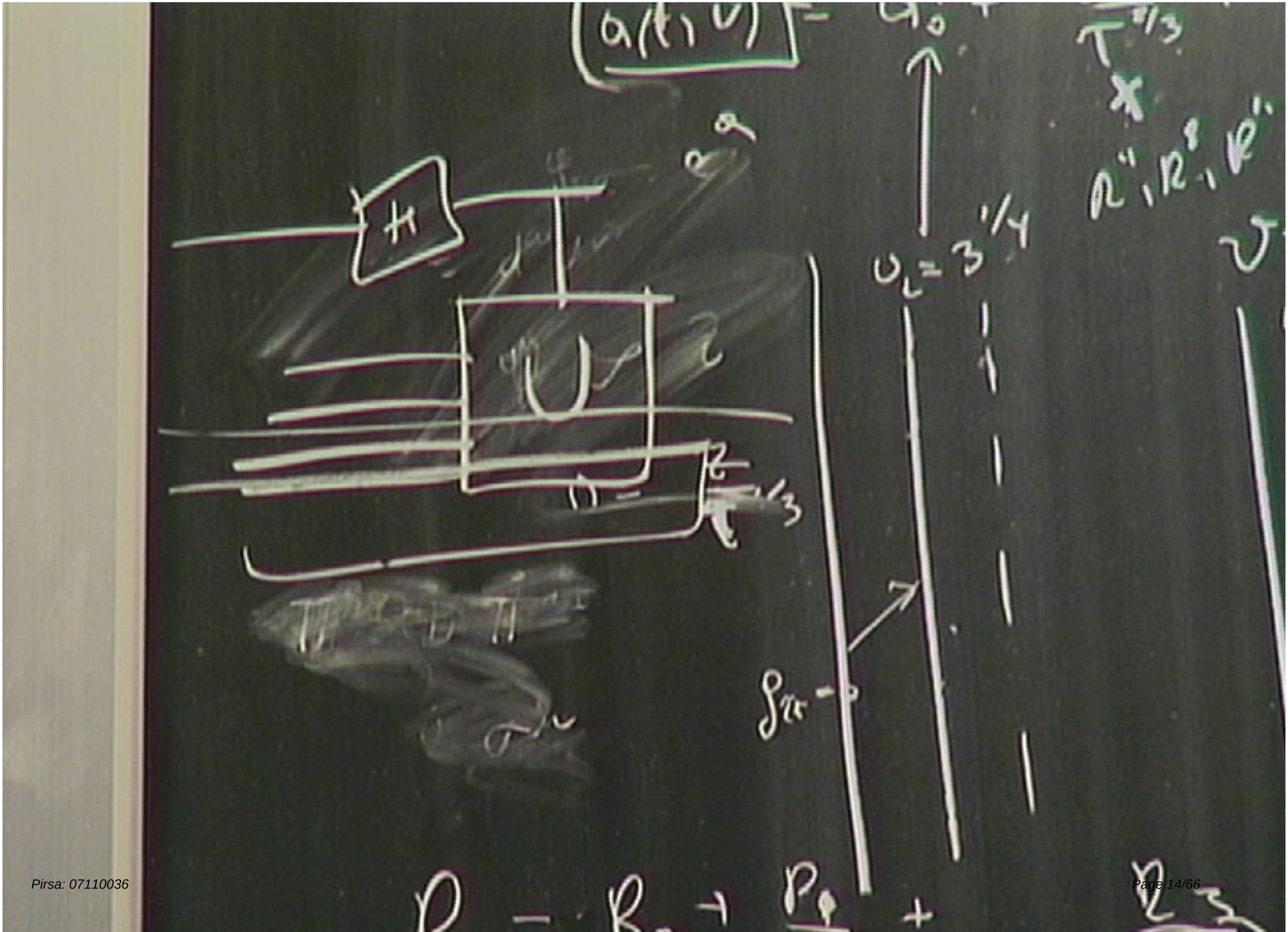


Figure: $\mathcal{N}_{1,2} = \sqrt{1+\alpha^2}$, $\mathcal{N}_{1,2,3|_{\alpha=1}} = \sqrt{2} - \frac{1}{2^{7/6}2^{n/3}} + \mathcal{O}(\frac{1}{2^{2n/3}})$

AD, S. T. Flammia, C. M. Caves, PRA, **72**, 042316, (2005)

Entanglement is an important resource ??!!


H
U


a(t, v)
H
U
v_c = 3^{1/4}

To obtain the bounds...

- Write

$$\breve{\rho}_\alpha = \frac{I_n + \alpha \breve{C}}{2^{n+1}}$$

where

$$\breve{C} \equiv \begin{pmatrix} 0 & \breve{U}_n^\dagger \\ \breve{U}_n & 0 \end{pmatrix}$$

- Using the binomial theorem, we can expand $\mathrm{Tr}(\breve{\rho}_\alpha^s)$ in terms of $\mathrm{Tr}(\breve{C}^k)$:

$$\mathrm{Tr}(\breve{\rho}_\alpha^s) = \left(\frac{1}{2^{n+1}}\right)^s \sum_{k=0}^{s} \binom{s}{k} \alpha^k \mathrm{Tr}(\breve{C}^k)$$

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- For odd k, $\mathrm{Tr}(\breve{C}^k) = 0$
- For $k = 2$, $\mathrm{Tr}(\breve{C}^2) = 2\mathrm{Tr}(\breve{U}_n \breve{U}_n^\dagger) = 2\mathrm{Tr}(U_n U_n^\dagger) = 2^{n+1}$
- $$\sum_{j=1}^{2^{n+1}} \lambda_j^s = \mathrm{Tr}(\breve{\rho}_\alpha^s) = \frac{1}{2^s 2^{(n+1)(s-1)}}[(1+\alpha)^s + (1-\alpha)^s], s = 1, 2, 3$$
- The task then is to maximize $\mathcal{N}(\rho_\alpha) = \sum_j |\lambda_j|$ w.r.t the above constraints
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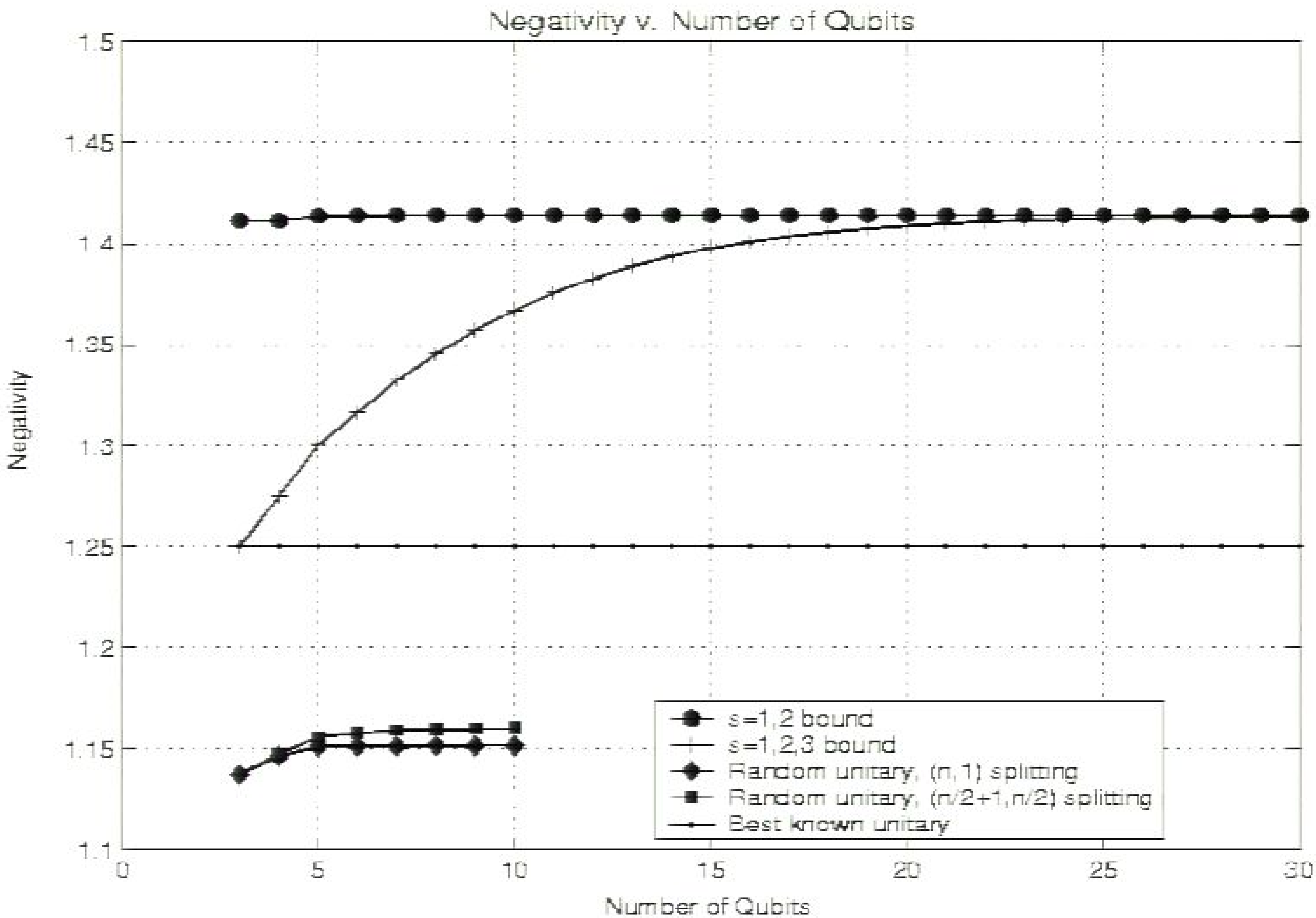

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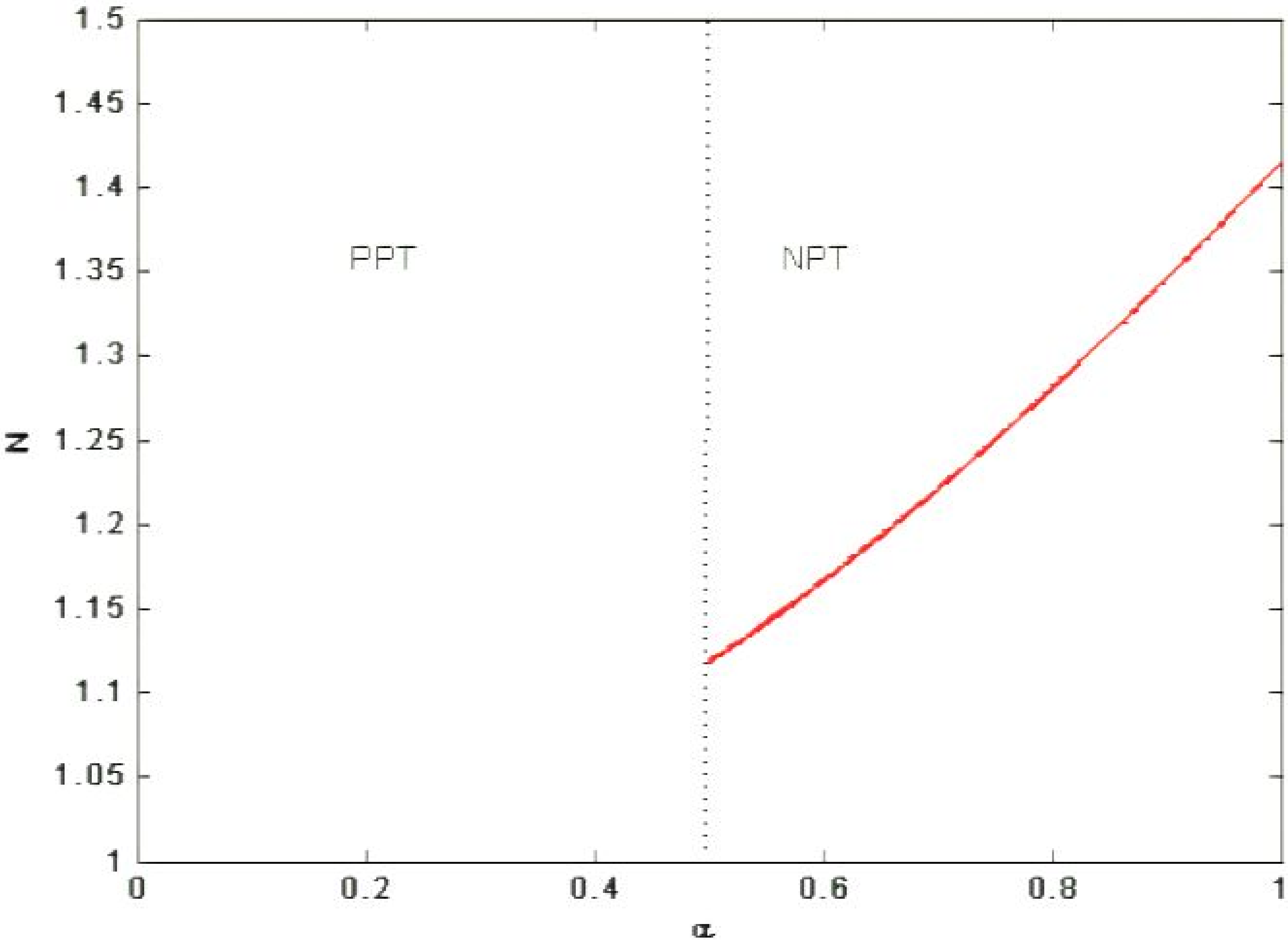
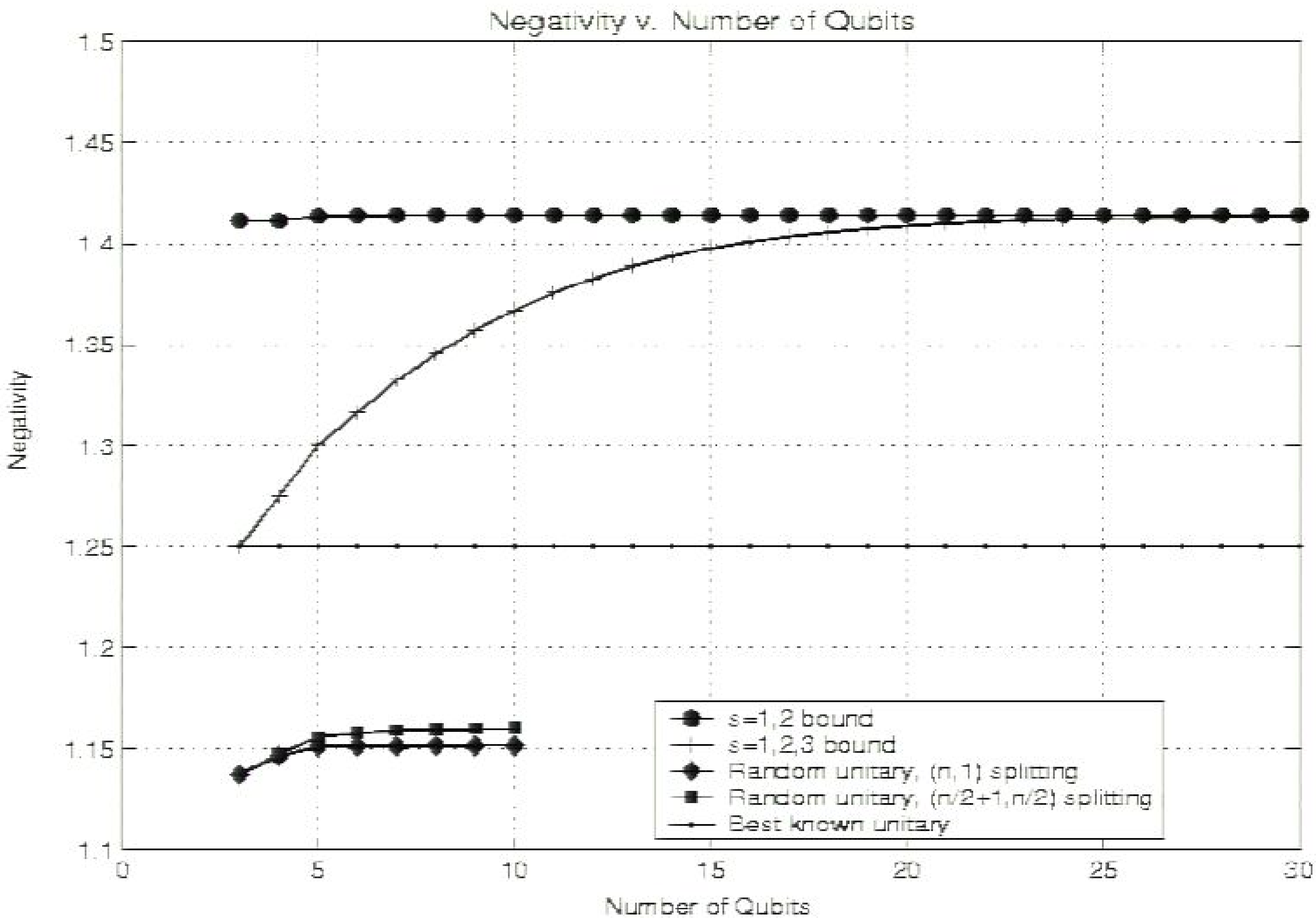


Figure: $\mathcal{N}_{DQC} = \sqrt{2}$. For an n qubit pure state, $\mathcal{N}_{max} \sim 2^{n/2}$



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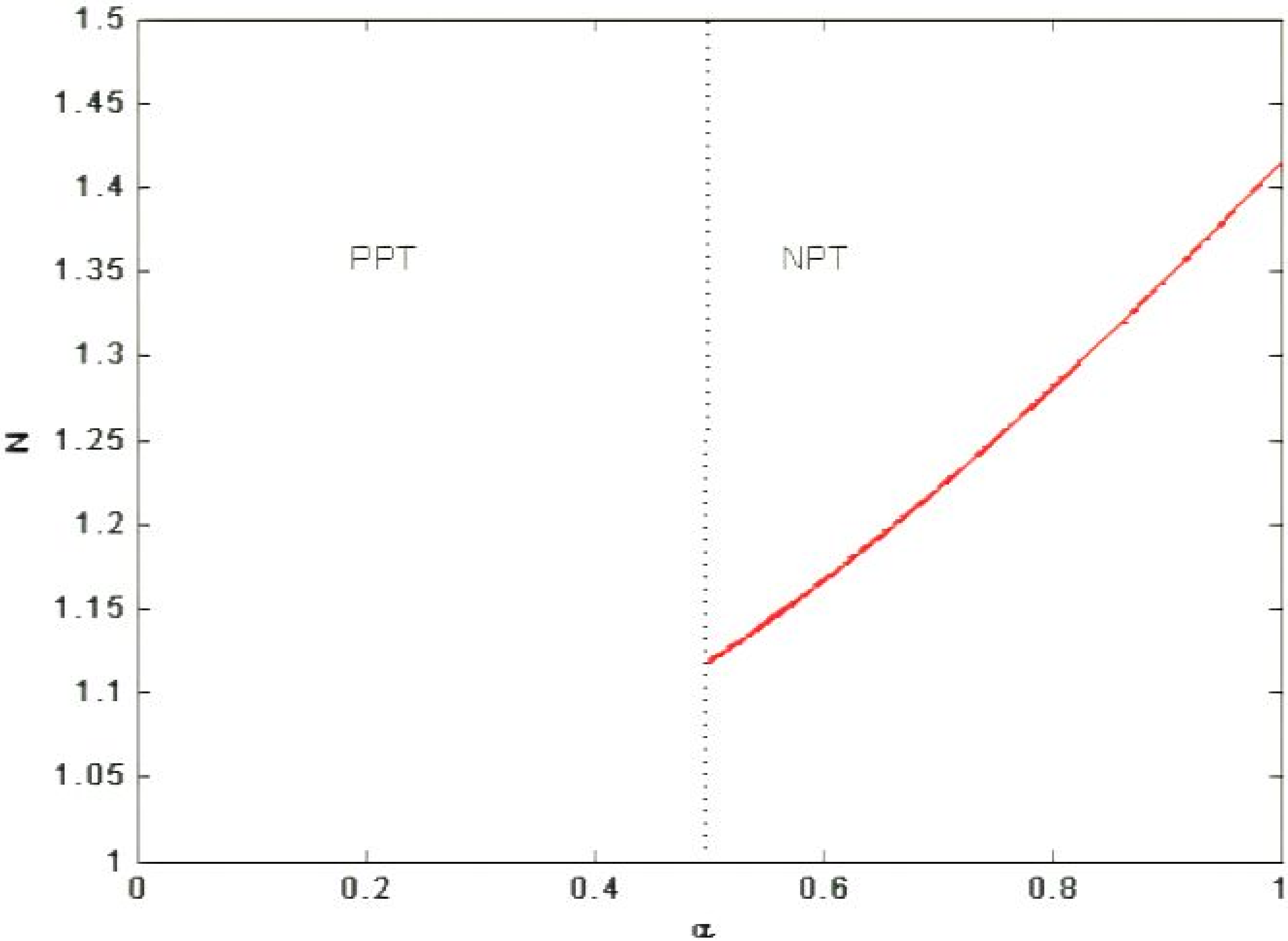

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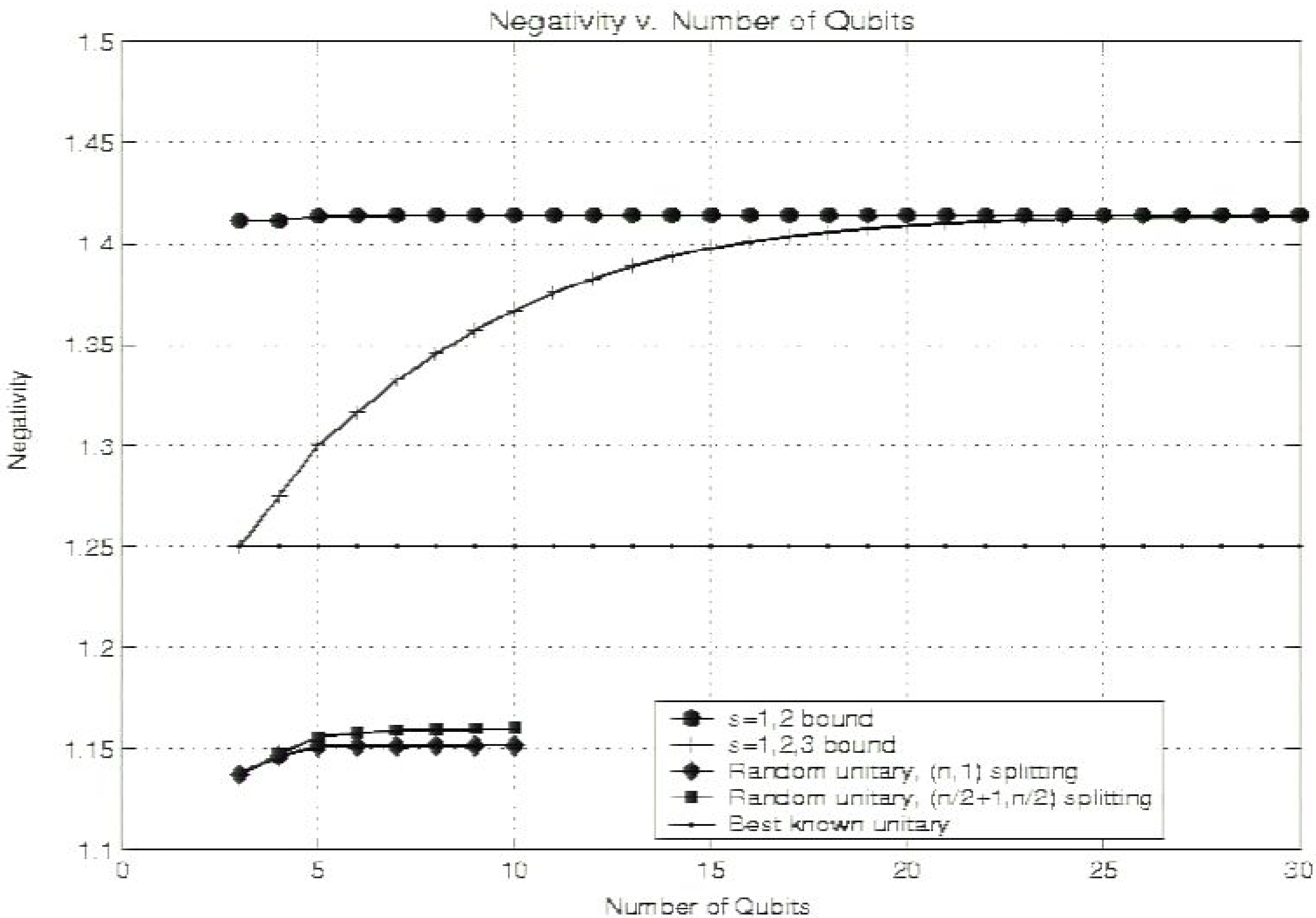

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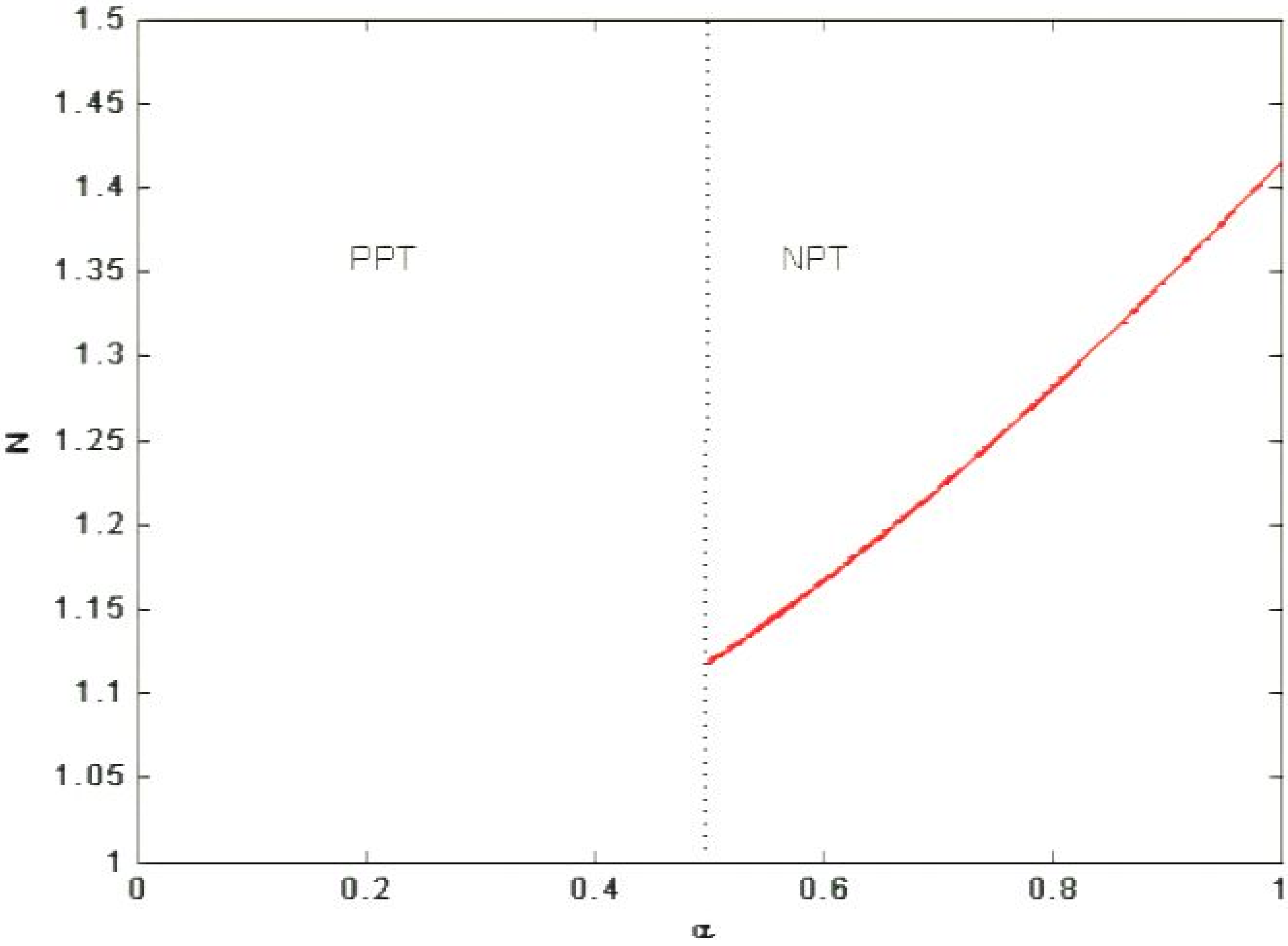

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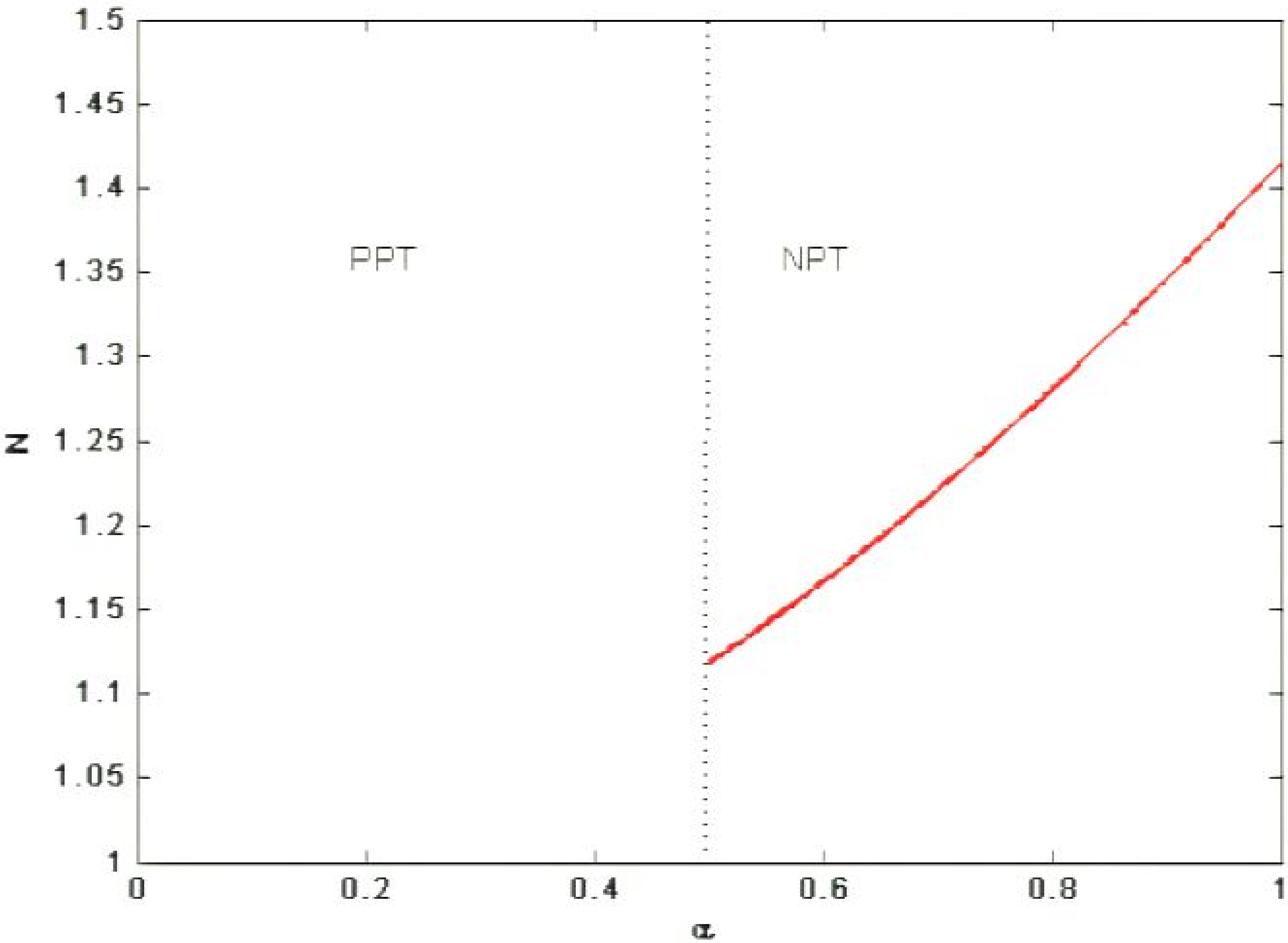

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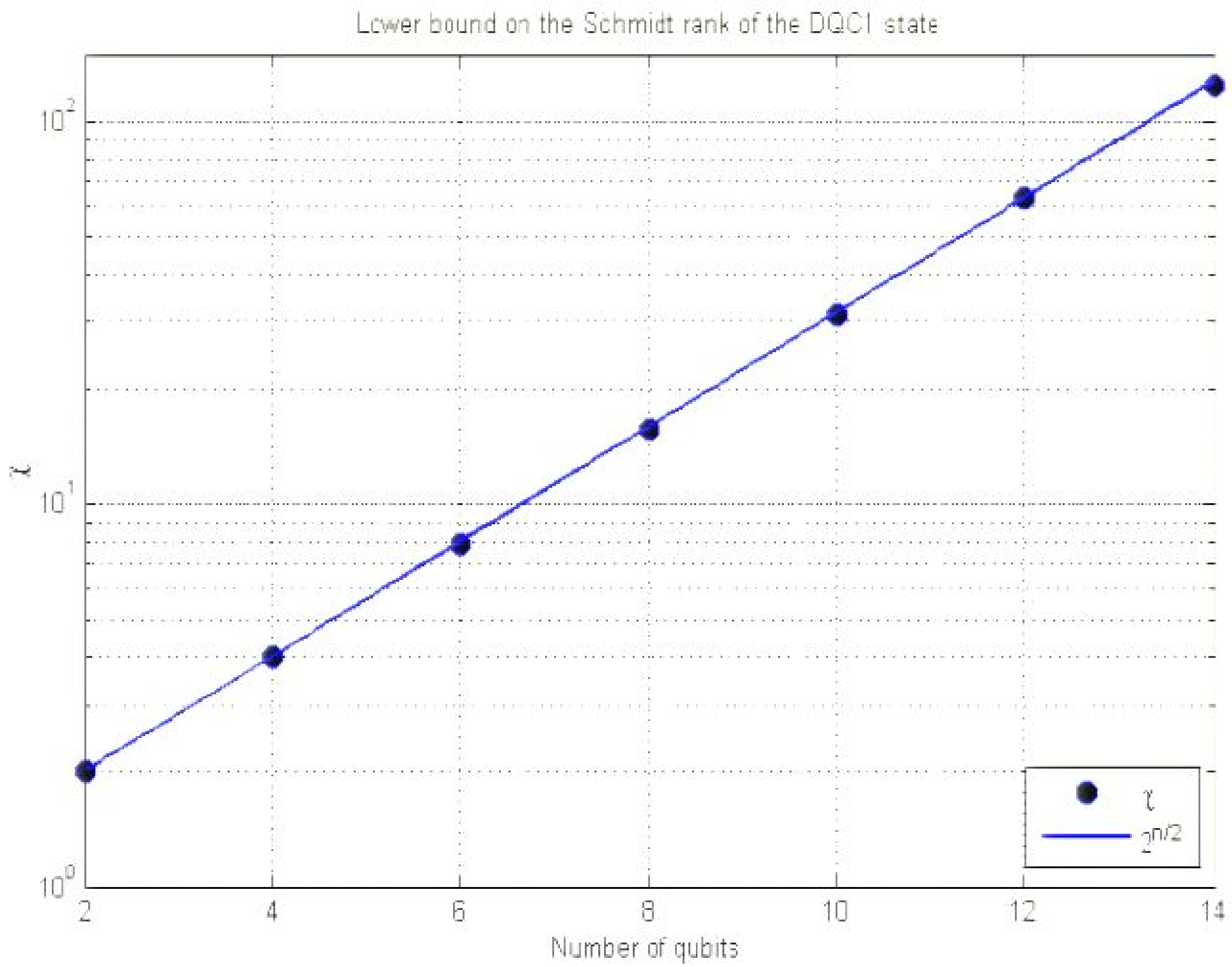
Exponential speedup with bounded bi-partite entanglement !!!!

Classical Simulatability

- In pure state quantum computation, systems with little entanglement can be simulated efficiently using MPS techniques.
- For these studies, entanglement is measured in terms of the Schmidt rank, which is a monotone for pure states.
- For mixed states, the Schmidt rank is *not* an entanglement monotone.

$$\rho_{n+1} = \sum_{i=1}^{\chi^\sharp} \lambda_i^\sharp \, O_{iA} \otimes O_{iB}.$$

where $\mathrm{Tr}(O_{iA}^\dagger O_{jA}) = \mathrm{Tr}(O_{iB}^\dagger O_{jB}) = \delta_{ij}$. The Schmidt rank $\chi^\sharp$ is a measure of correlations between parts A and B.



In addition to there being an exponential number of Schmidt coefficients, they are all arbitrarily close to $2^{-n/2}$.

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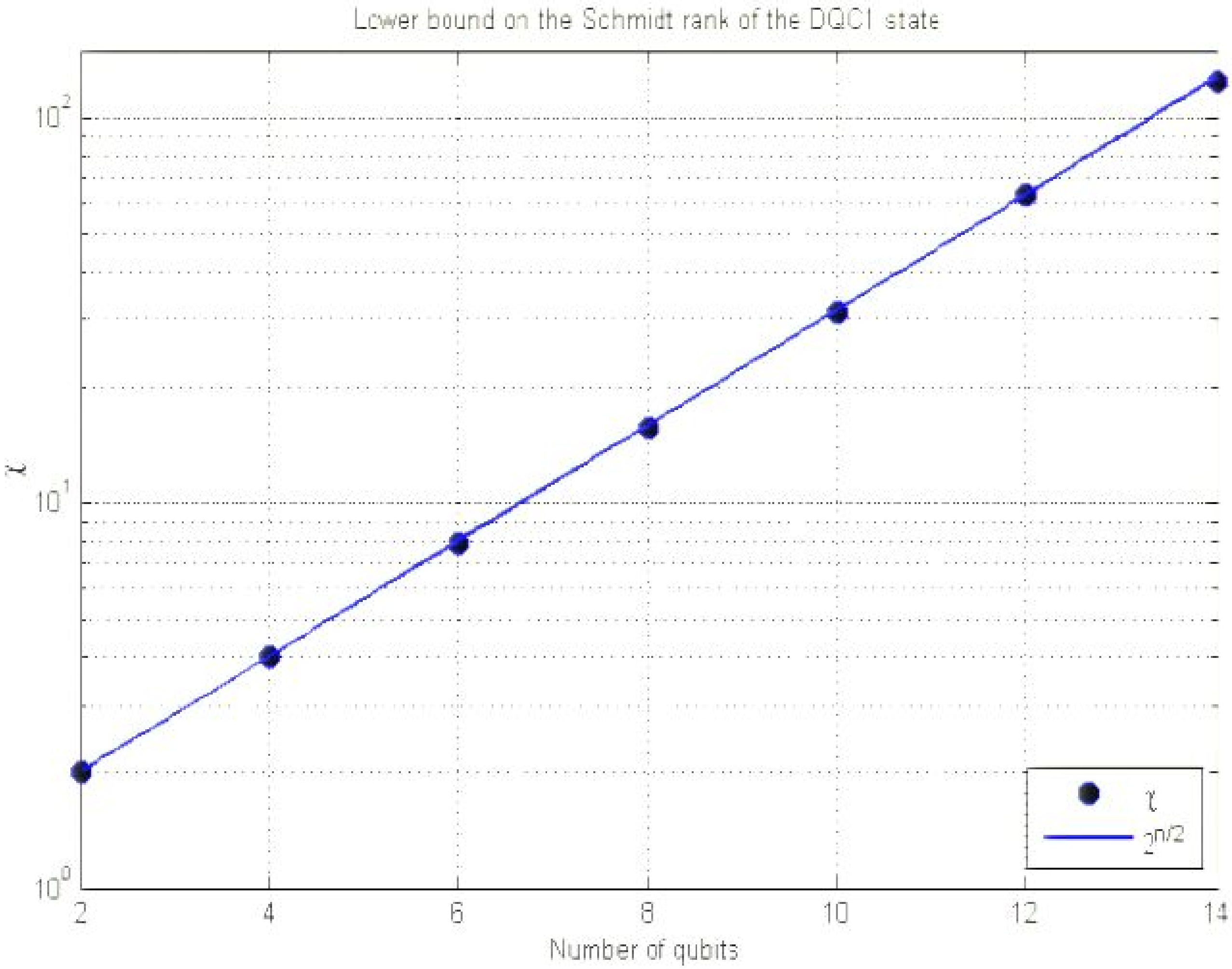

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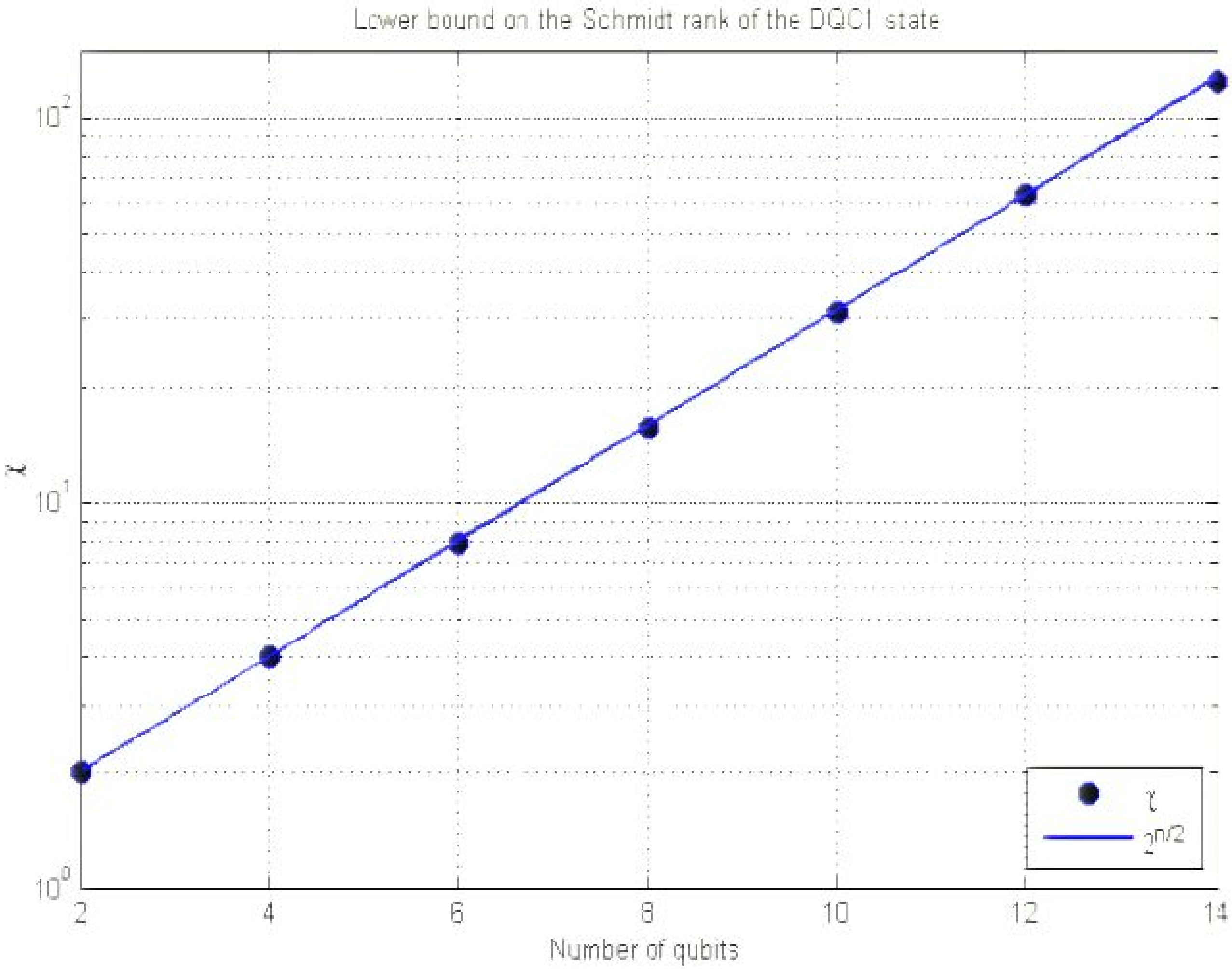

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So we have...

- Evaluation of the normalized trace of an unitary efficiently... this is an exponential speedup over the best known classical algorithm
- This circuit generates very little entanglement
- Yet it cannot be simulated classically in an efficient manner
- Correlations seem to be the key to speed-up
- The vital question of the reason behind the power of quantum computation remains unanswered

AD, G. Vidal, PRA, **75**, 042310 (2007).

Let's step back and recall...

- Measure of Ignorance: $H(X)$
- Measure of Correlations : $J(X:Y) = H(Y) - H(Y|X)$.

$$p_x = \sum_y p_{x.y} \qquad p_y = \sum_x p_{x.y}$$

$$p_{x|y=Y} = \frac{p_{x.y=Y}}{p_{y=Y}}$$

Since $H(Y|X) = H(X.Y) - H(X)$.

$$I(X:Y) = H(X) + H(Y) - H(X.Y)$$

- $I(X:Y) = J(X:Y)$ for a classical distribution

H is the Shannon entropy

For quantum systems

- With a joint density matrix, $\rho_{X,Y}$

$$\mathcal{I}(X:Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{X,Y})$$

- To get the conditional entropy

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For quantum systems

- With a joint density matrix, $\rho_{X.Y}$

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- To get the conditional entropy
  Make projective measurements on a subsystem $\{\Pi_j^X\}$
  Then

$$\rho_{Y|\Pi_j^X} = \Pi_j^X \rho_{X.Y} \Pi_j^X / p_j$$

$$p_j = \mathrm{Tr}[\Pi_j^X \rho_{X.Y} \Pi_j^X]$$

- $J(X:Y) = S(\rho_Y) - \sum_j p_j S(\rho_{Y|\Pi_j^X})$.
  Then

$$\begin{aligned} D &= \mathcal{I}(X:Y) - J(X:Y) \\ &= S(\rho_X) - S(\rho_{X.Y}) + \sum_j p_j S(\rho_{Y|\Pi_j^X}) \end{aligned}$$

Dependance on measurements

- Maximum extractable classical information: $\mathcal{J}(X:Y) = \max_{\{\Pi_j^Y\}} J(X:Y)$
- Purely quantum correlations:

$$\begin{aligned} \mathcal{D} &= \mathcal{I}(X:Y) - \mathcal{J}(X:Y) \\ &= S(\rho_X) - S(\rho_{X.Y}) + \min_{\{\Pi_j^X\}} \sum_j p_j S(\rho_{Y|\Pi_j^X}) \end{aligned}$$

$\mathcal{D}$ is a measure of 'purely quantum' correlations (Ollivier/Zurek,02)

- $0 \le \mathcal{D} \le S(\rho_X)$
- $\mathcal{D} = 0 \Leftrightarrow \rho_{X.Y} = \sum_j \Pi_j^X \rho_{X.Y} \Pi_j^X$

For quantum systems

- With a joint density matrix, $\rho_{X.Y}$

$$\mathcal{I}(X:Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{X.Y})$$

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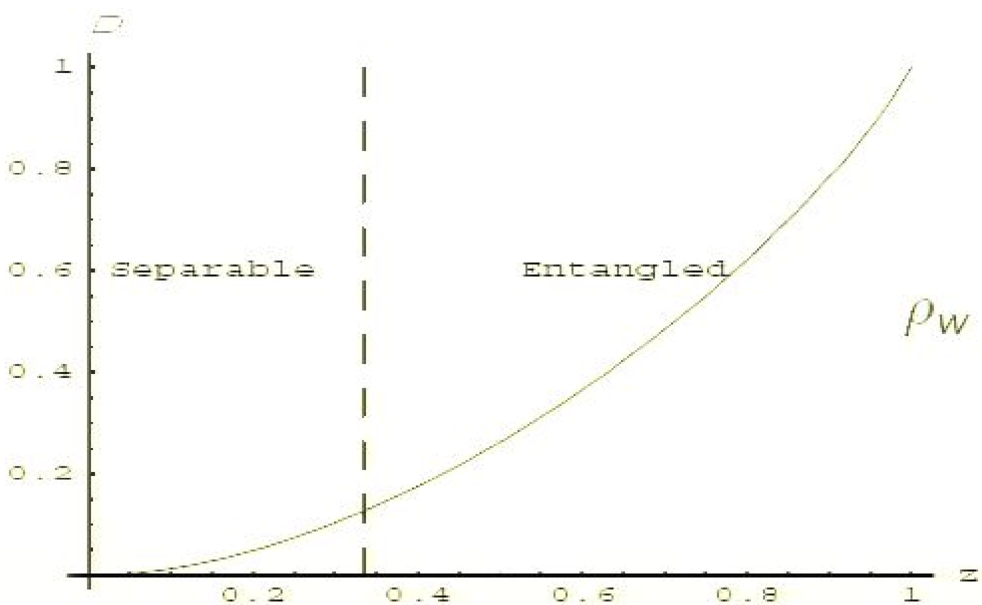
Further Properties of Discord

- For pure states

$$
\begin{aligned}
\mathcal{D} &= S(\rho_X) - \cancelto{0}{S(\rho_{X,Y})} + \min_{\{\Pi_j^X\}} \sum_j p_j \cancelto{0}{S(\rho_{Y|\Pi_j^X})} \\
&= S(\rho_X)
\end{aligned}
$$

- $\mathcal{D}$ is an entanglement monotone on pure states
- Fundamentally different from Werner's notion of classicality



$$\rho_W = \frac{1-z}{4}\mathbb{I} + z|\Phi^+\rangle\langle\Phi^+|$$

Dependance on measurements

- Maximum extractable classical information:
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- Purely quantum correlations:

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\mathcal{D} &= \mathcal{I}(X:Y) - \mathcal{J}(X:Y) \\
&= S(\rho_X) - S(\rho_{X,Y}) + \min_{\{\Pi_j^X\}} \sum_j p_j S(\rho_{Y|\Pi_j^X})
\end{aligned}
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$\mathcal{D}$ is a measure of 'purely quantum' correlations (Ollivier/Zurek,02)

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Further Properties of Discord

- For pure states

$$\begin{aligned}\mathcal{D} &= S(\rho_X) - \cancelto{0}{S(\rho_{X,Y})} + \min_{\{\Pi_j^X\}} \sum_j p_j \cancelto{0}{S(\rho_{Y|\Pi_j^X})} \\ &= S(\rho_X)\end{aligned}$$

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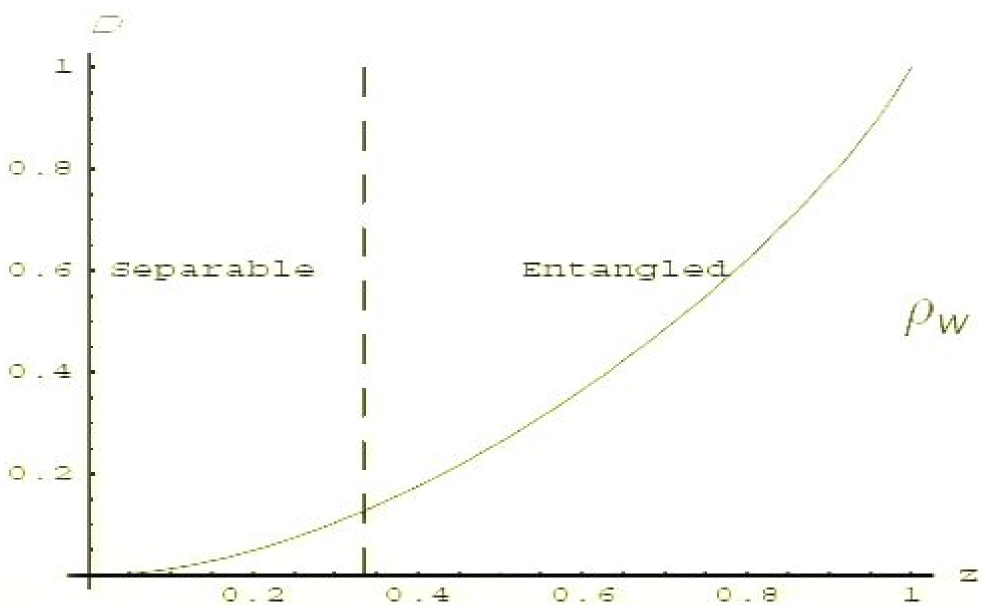

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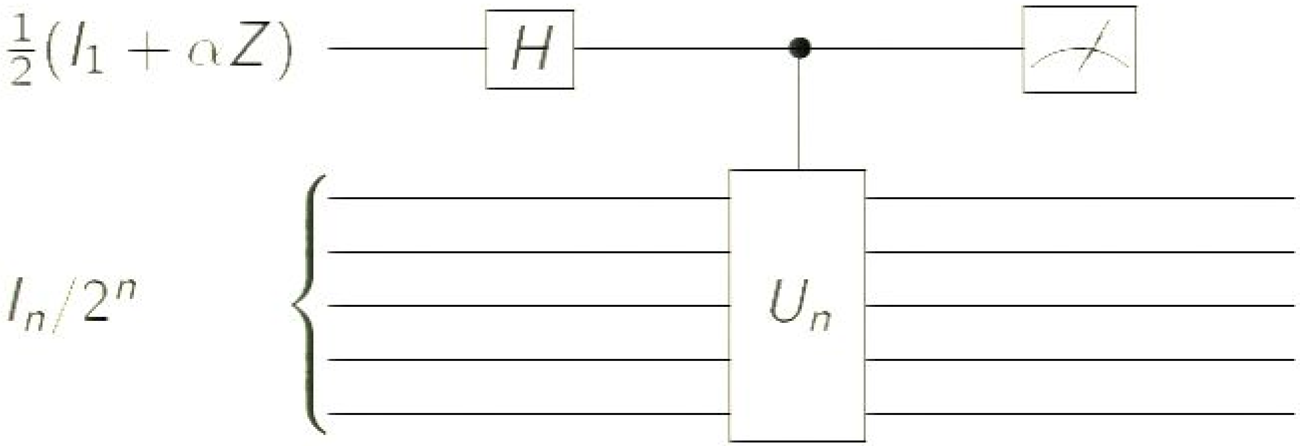
$$\begin{aligned}\rho &= \frac{1}{4}\Big(|+\rangle\langle+| \otimes |0\rangle\langle 0| \; + \; |-\rangle\langle-| \otimes |1\rangle\langle 1| \; + \\ &\qquad |0\rangle\langle 0| \otimes |-\rangle\langle-| \; + \; |1\rangle\langle 1| \otimes |+\rangle\langle+| \Big)\end{aligned}$$

DQC1



Density matrix just prior to the measurement is

$$\rho_{n+1}(\alpha) = \frac{1}{2^{n+1}} \begin{pmatrix} I_n & \alpha U_n \\ \alpha U_n^\dagger & I_n \end{pmatrix}$$

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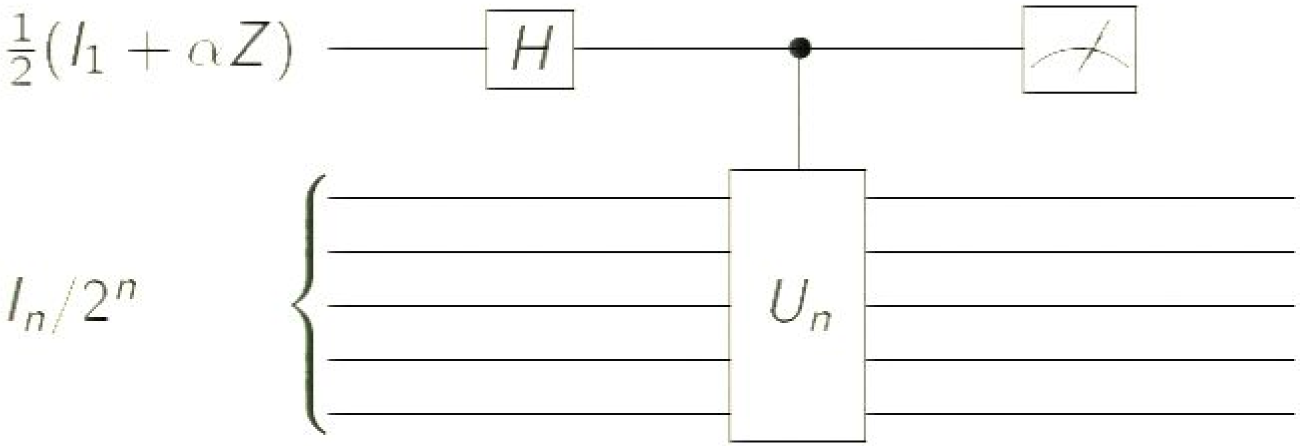

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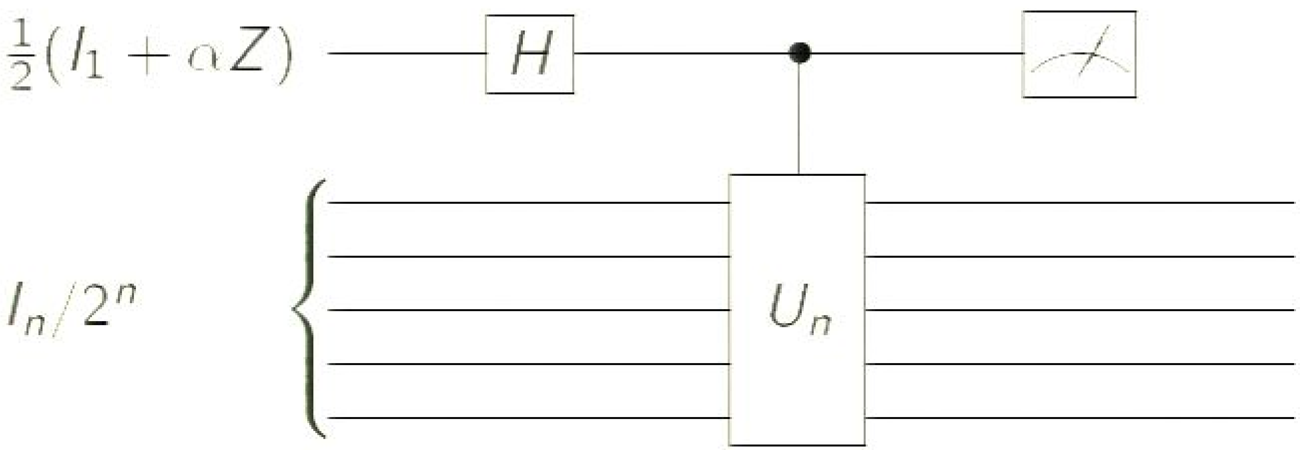

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Discord in DQC1

Measurements:

- $\Pi_{\pm} = \frac{1}{2}(I_1 + \mathbf{a}.\boldsymbol{\sigma})$
- Then, for a random U,

$$\sum_j p_j S(\rho_{Y|\Pi_j^X}) = \frac{1}{2}[H(\mathbf{q}_+) + H(\mathbf{q}_-)] + \frac{\alpha}{2}(\tau_R \cos\phi + \tau_I \sin\phi)[H(\mathbf{q}_+) - H(\mathbf{q}_-)]$$

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Entanglement is an important resource ??!!

Putting the pieces together...

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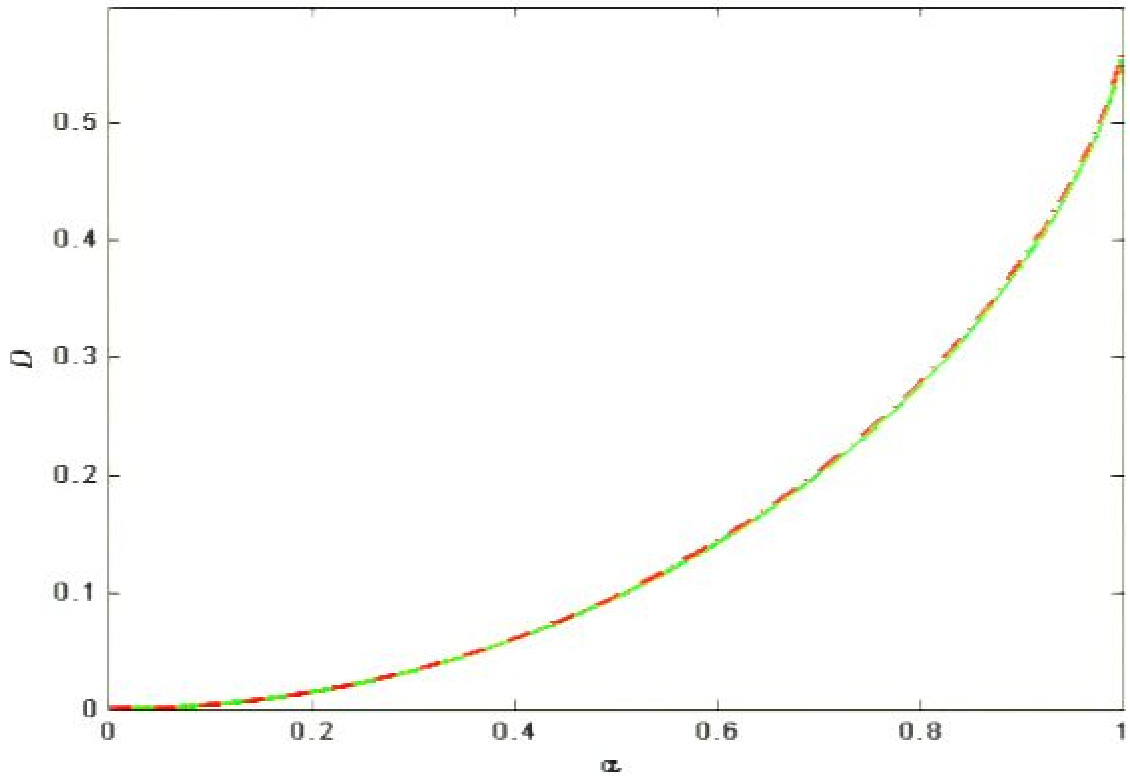

Figure: Discord in an n qubit DQC circuit with a typical unitary

$$
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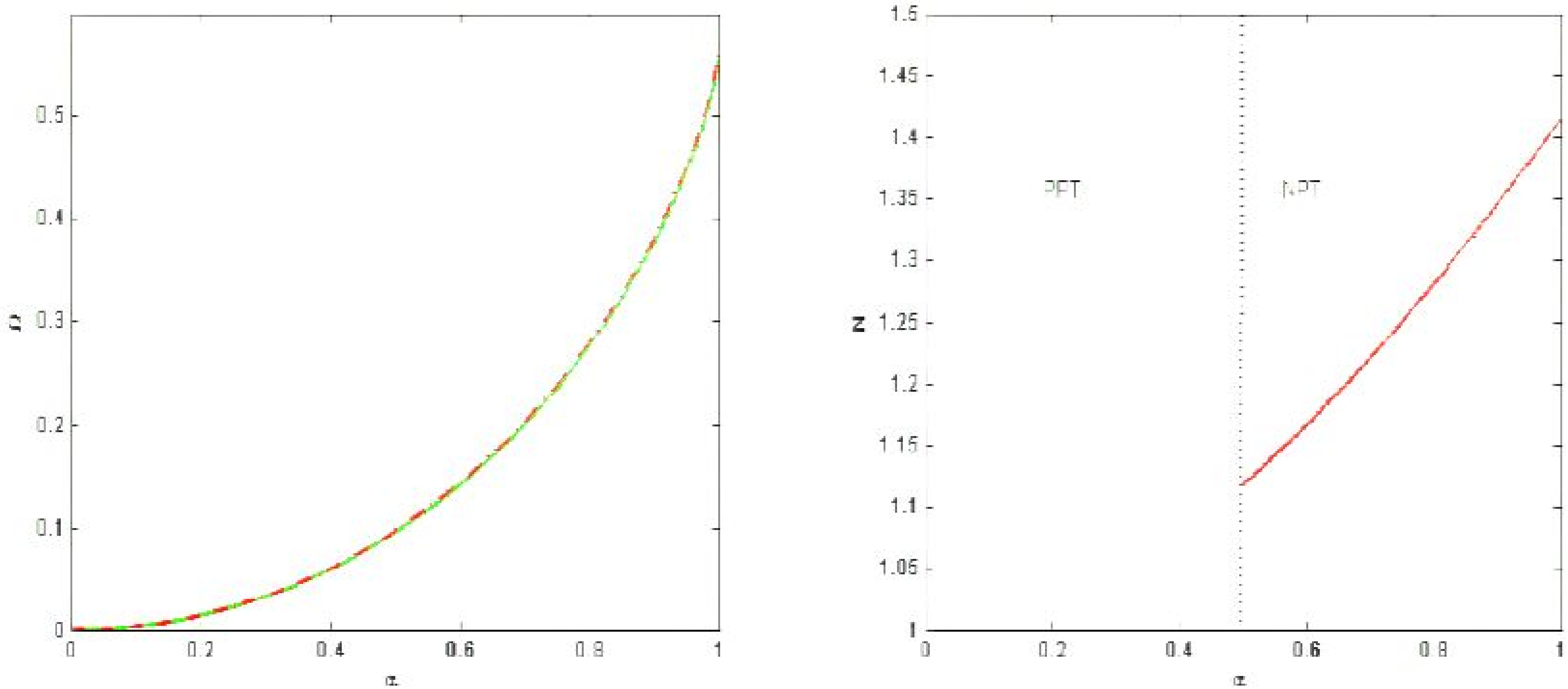
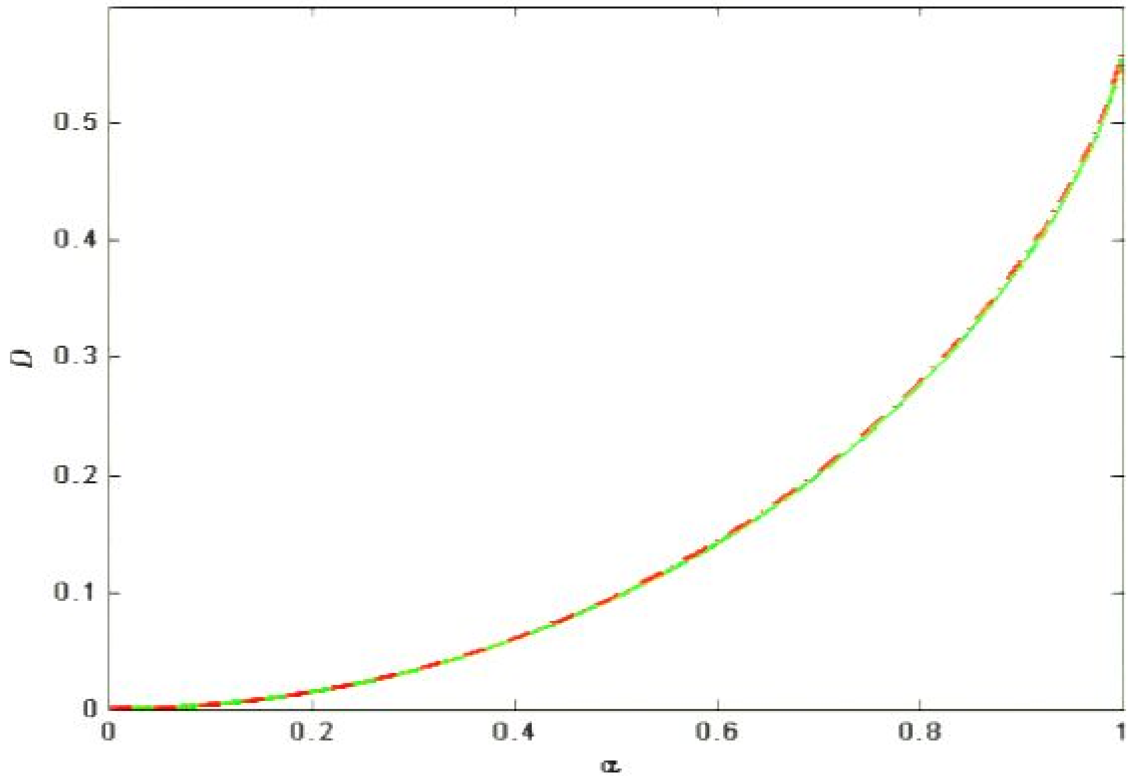


For $\alpha = 1$, $\mathcal{D}_{DQC} = 2 - \log e = 0.5573$, $\mathcal{N}_{DQC} = \sqrt{2}$.
For an n qubit pure state, $\mathcal{D}_{max} = 1$, $\mathcal{N}_{max} \sim 2^{n/2}$



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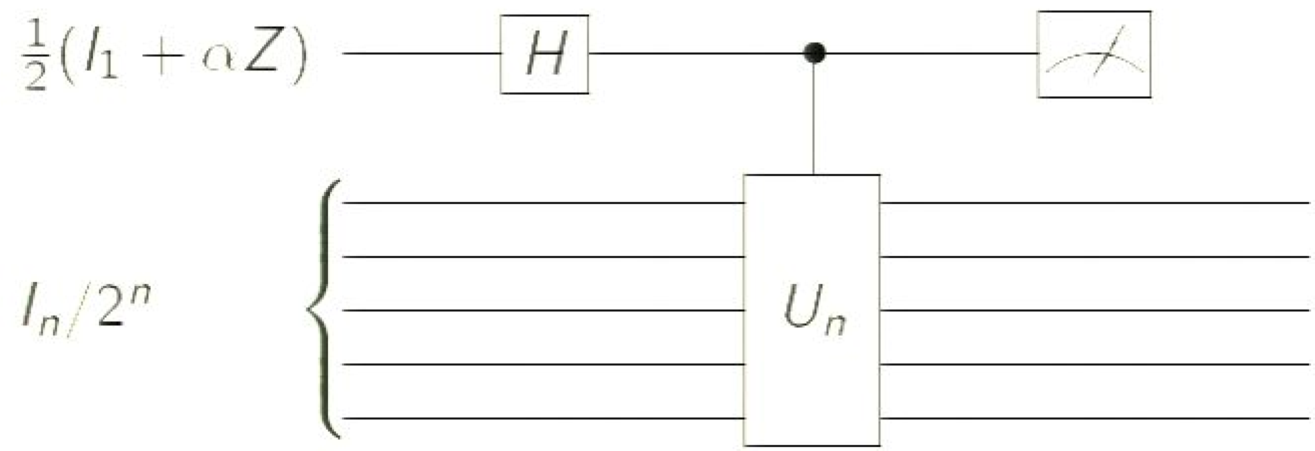
DQC1



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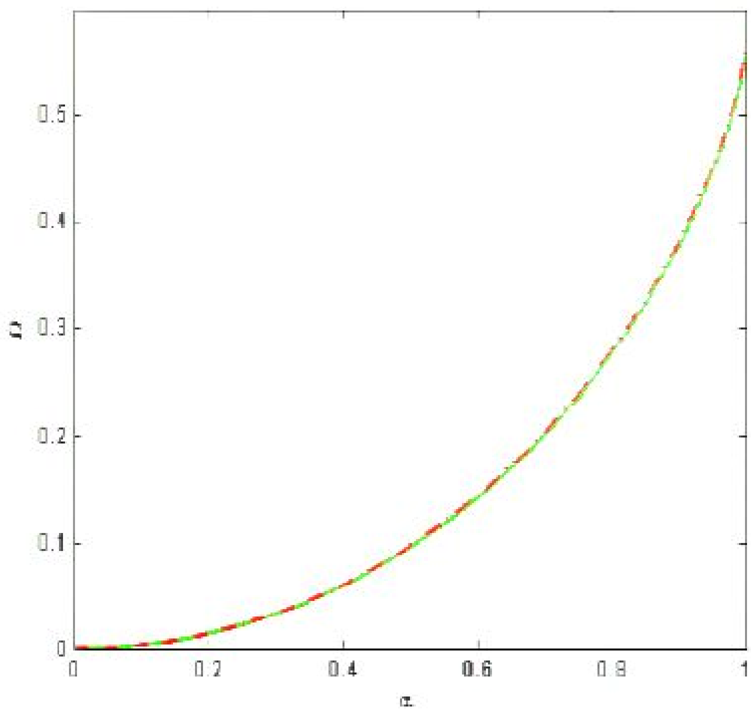

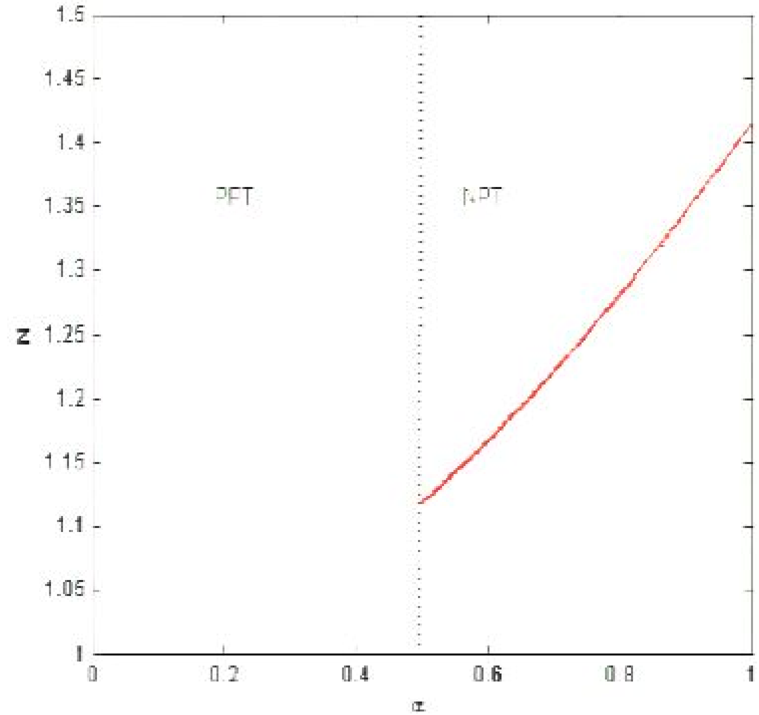

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To end with...

- There's more to quantum information science than entanglement
- Maybe discord taps that unknown
- It reduces to entanglement on pure states
- Discord and entanglement complement each other

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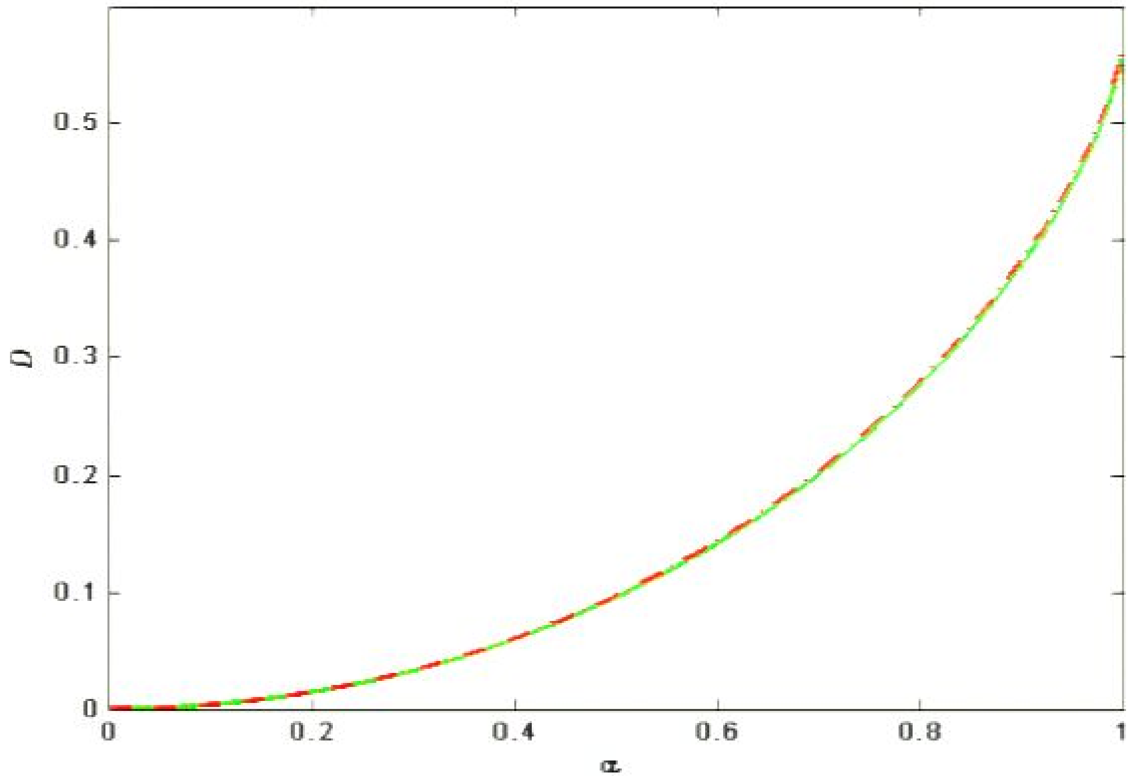

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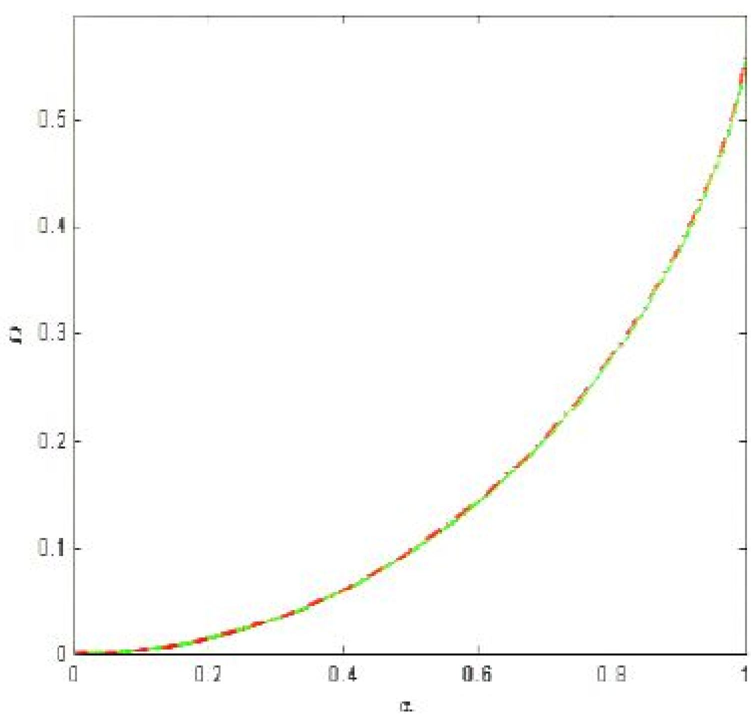

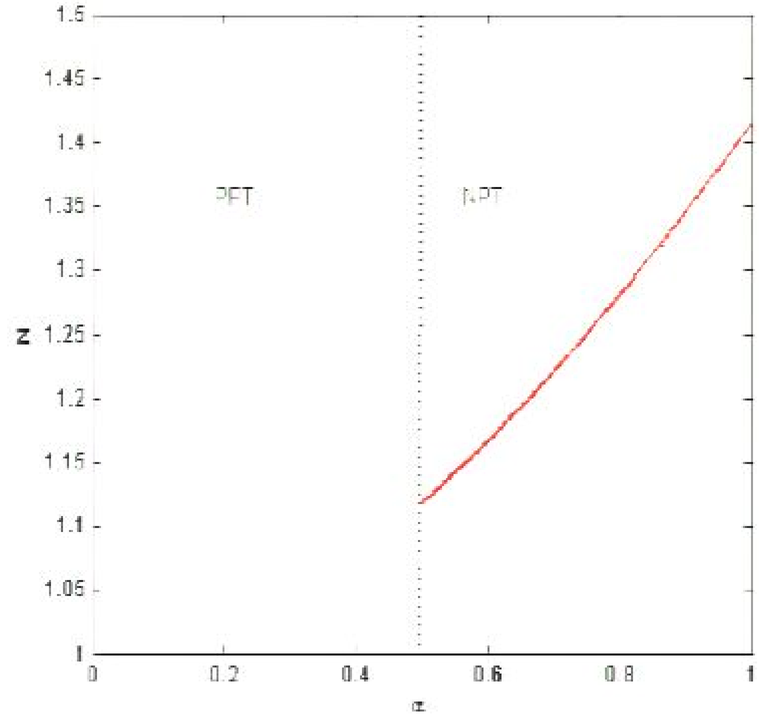

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