Title: Frame representations of quantum mechanics and the necessity of negativity in quasi-probability representations

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Abstract: Several finite dimensional quasi-probability representations of quantum states have been proposed to study various problems in quantum information theory and quantum foundations. These representations are often defined only on restricted dimensions and their physical significance in contexts such as drawing quantum-classical comparisons is limited by the non-uniqueness of the particular representation.
Here we show how the mathematical theory of frames provides a unified formalism which accommodates all known quasi-probability representations of finite dimensional quantum systems.

The frame representation of quantum mechanics and the necessity of negativity in quasi-probability representations

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Outline

1. Introduction
2. The frame representation
3. Negativity and non-classicality
4. Remarks

Discrete Wigner functions

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- Heiss and Weigert - d^2 points embedded in the sphere $\mathcal{S}^2$.
- Gibbons, Hoffman and Wootters - $d \times d$ lattice indexed by elements of a finite field of dimension d when d is a power of a prime number.

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- The application of any one of these quasi-probability representations in the context of determining criteria for the non-classicality of a given quantum task is limited in significance by the non-uniqueness of that particular representation.
- Ideally one would like to determine whether the task can be expressed as a classical process in any quasi-probability representation.

Quasi-probability representations

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Definition

Let Γ be a set with positive measure μ. A quasi-probability representation of $\mathrm{Herm}(\mathcal{H})$ is a map $\mathrm{Herm}(\mathcal{H}) \to L^2(\Gamma, \mu)$ which is linear and invertible.


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CAUTION


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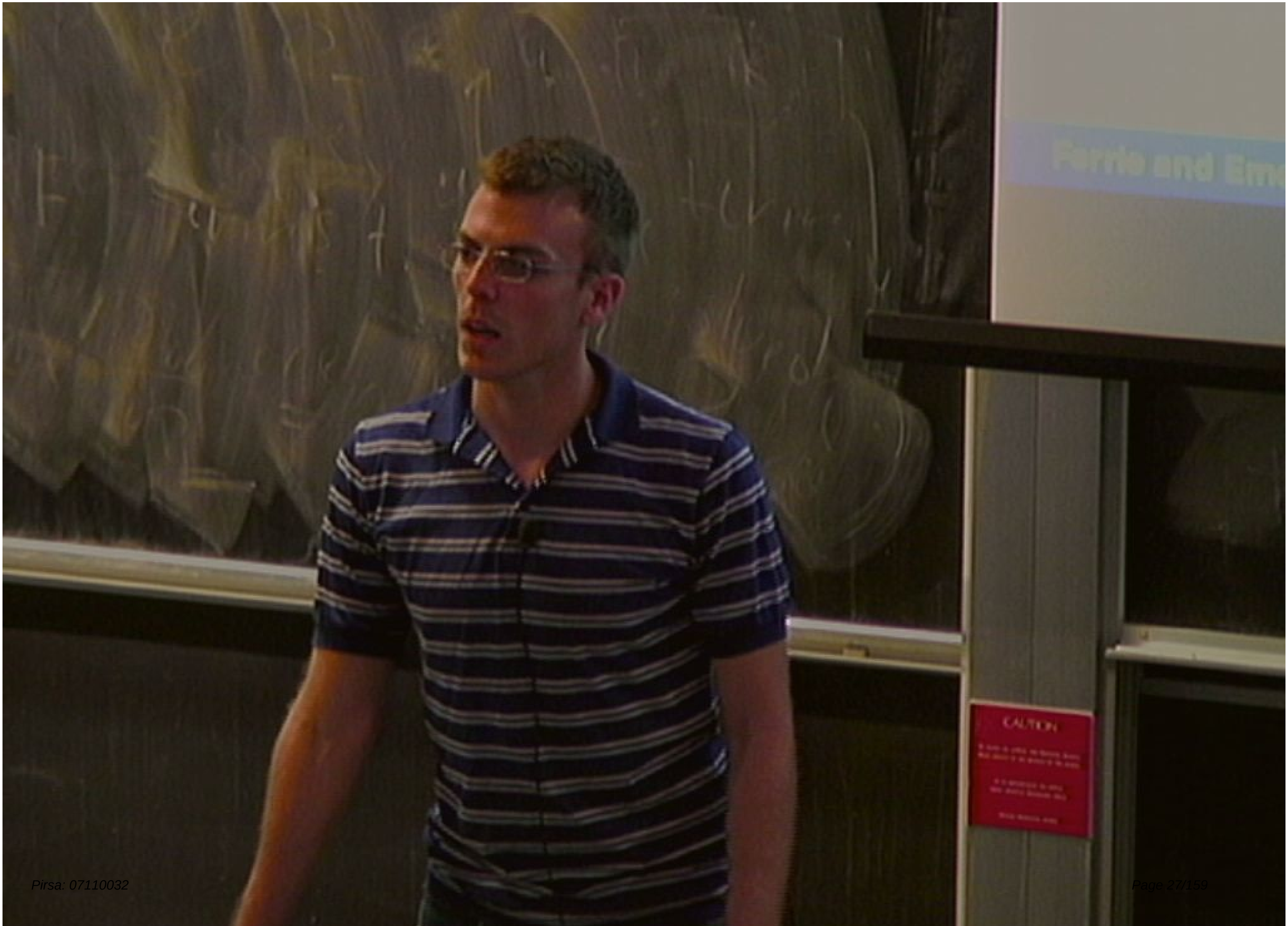
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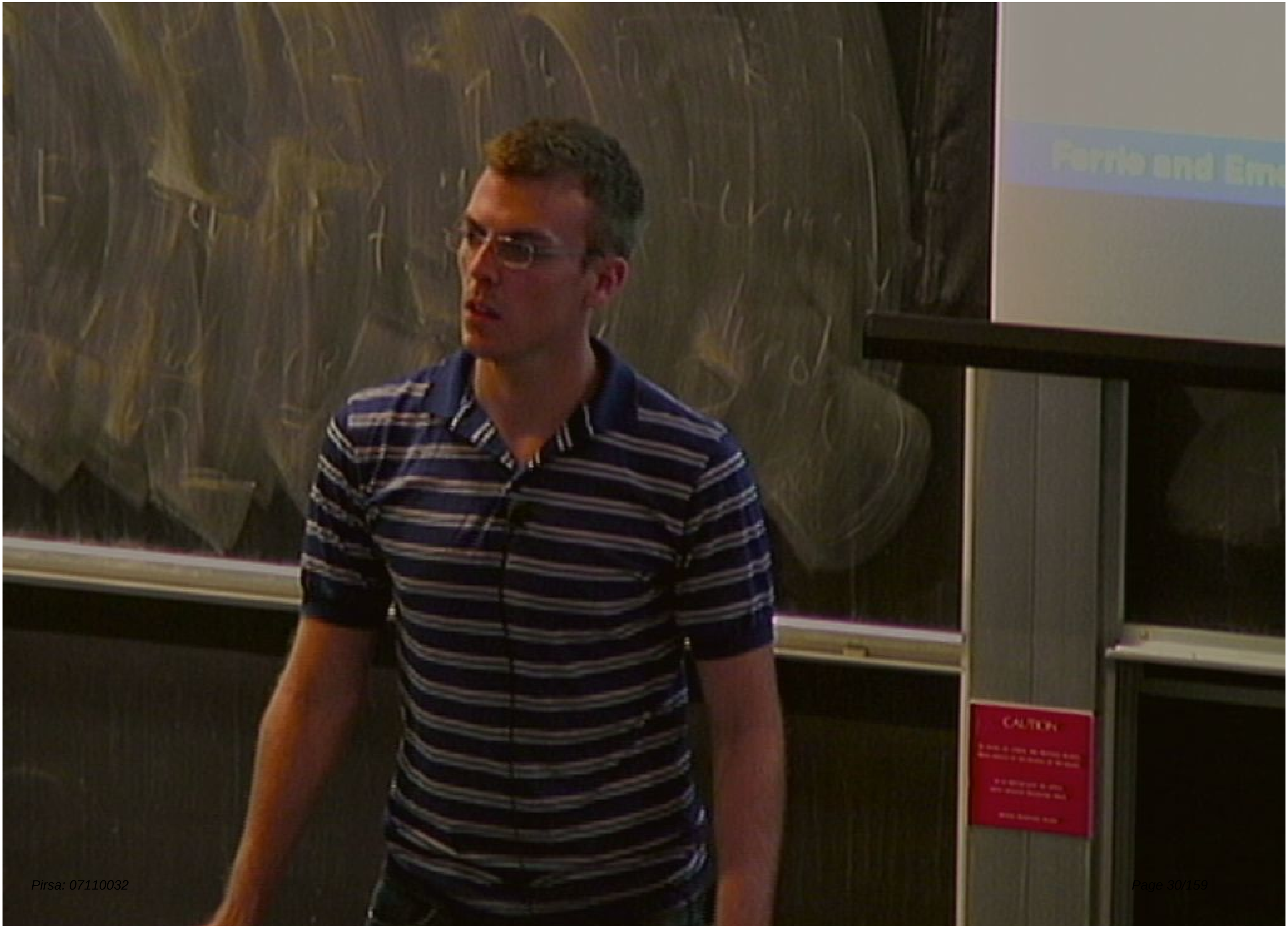
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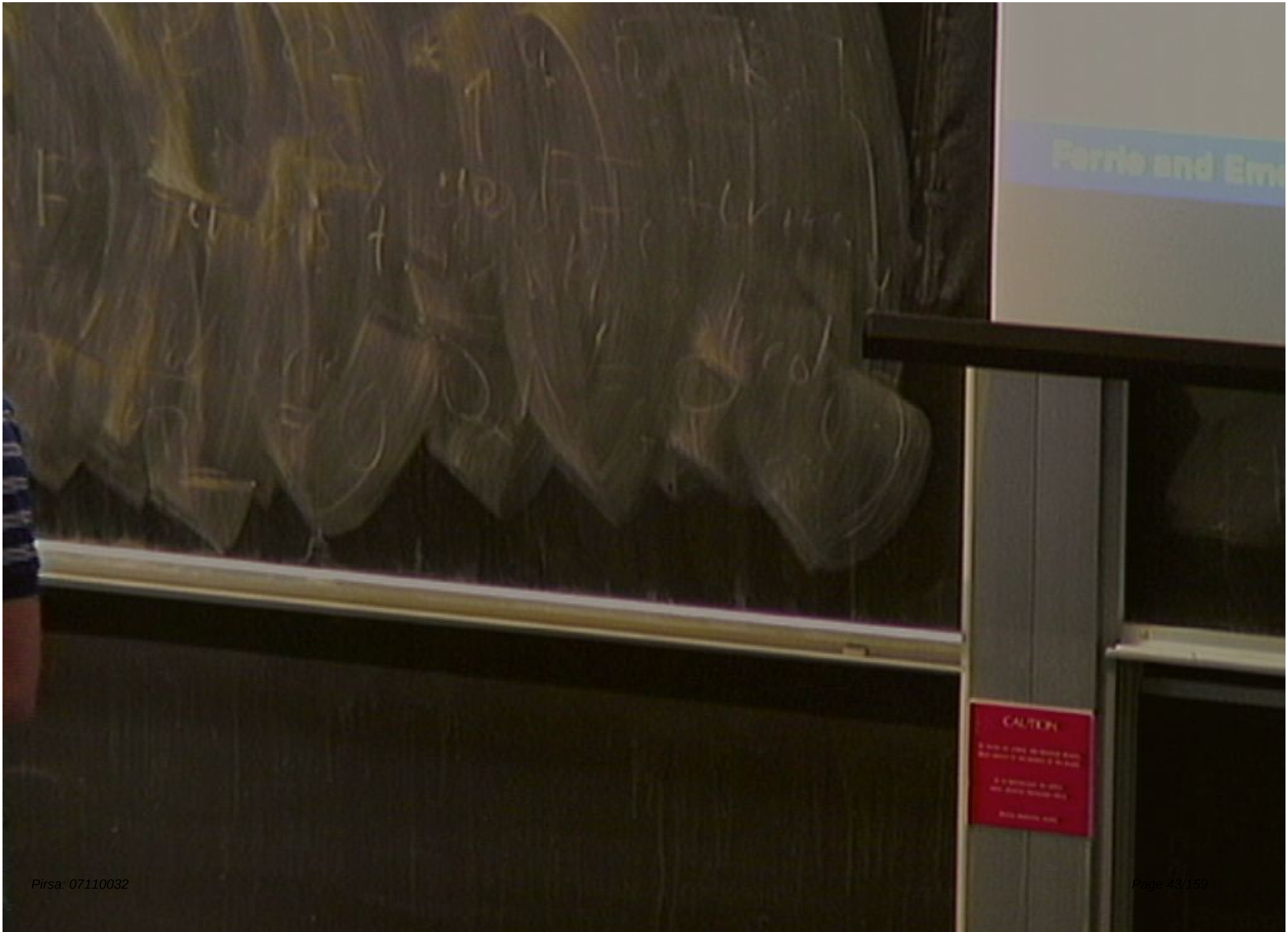
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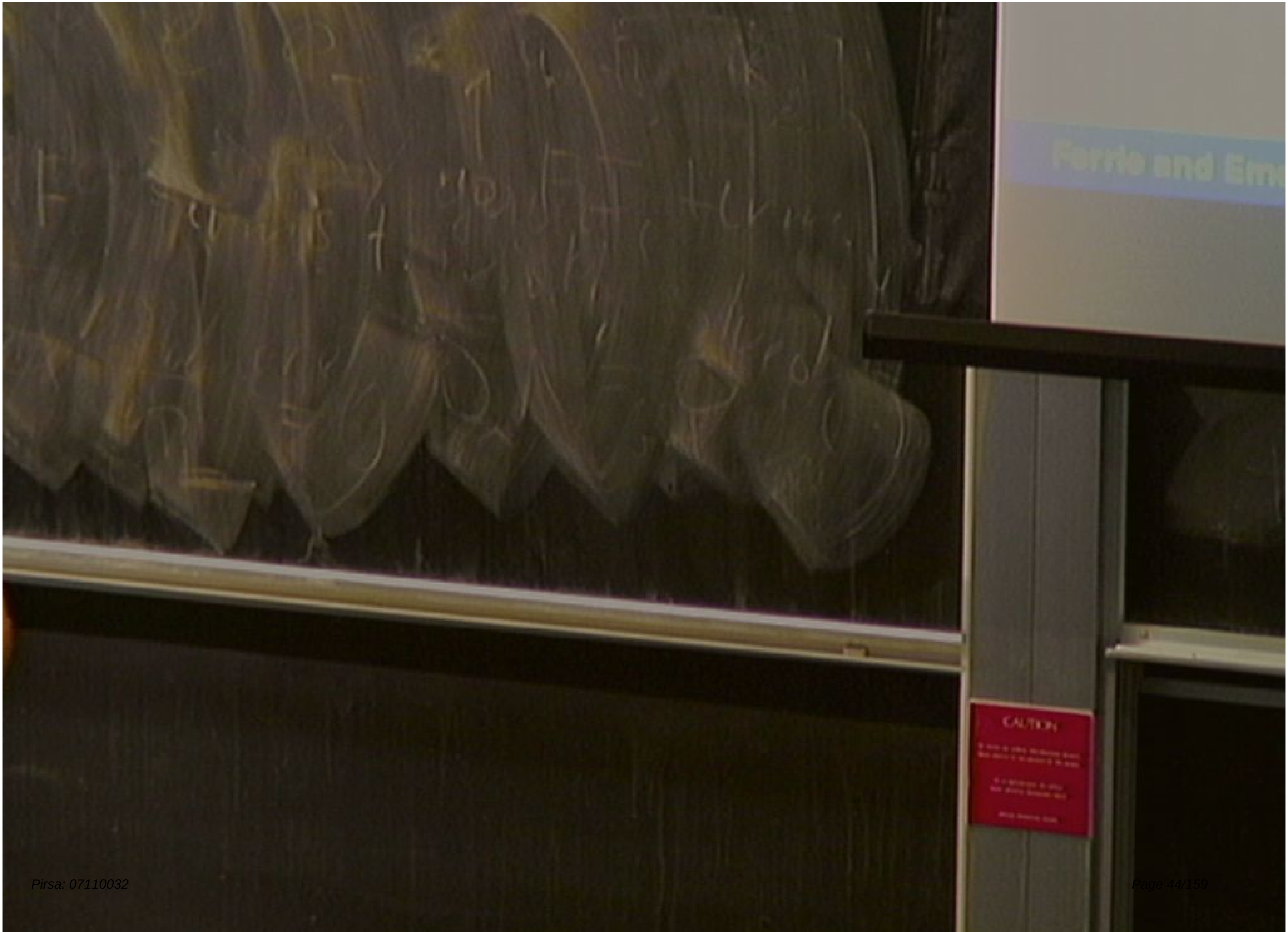
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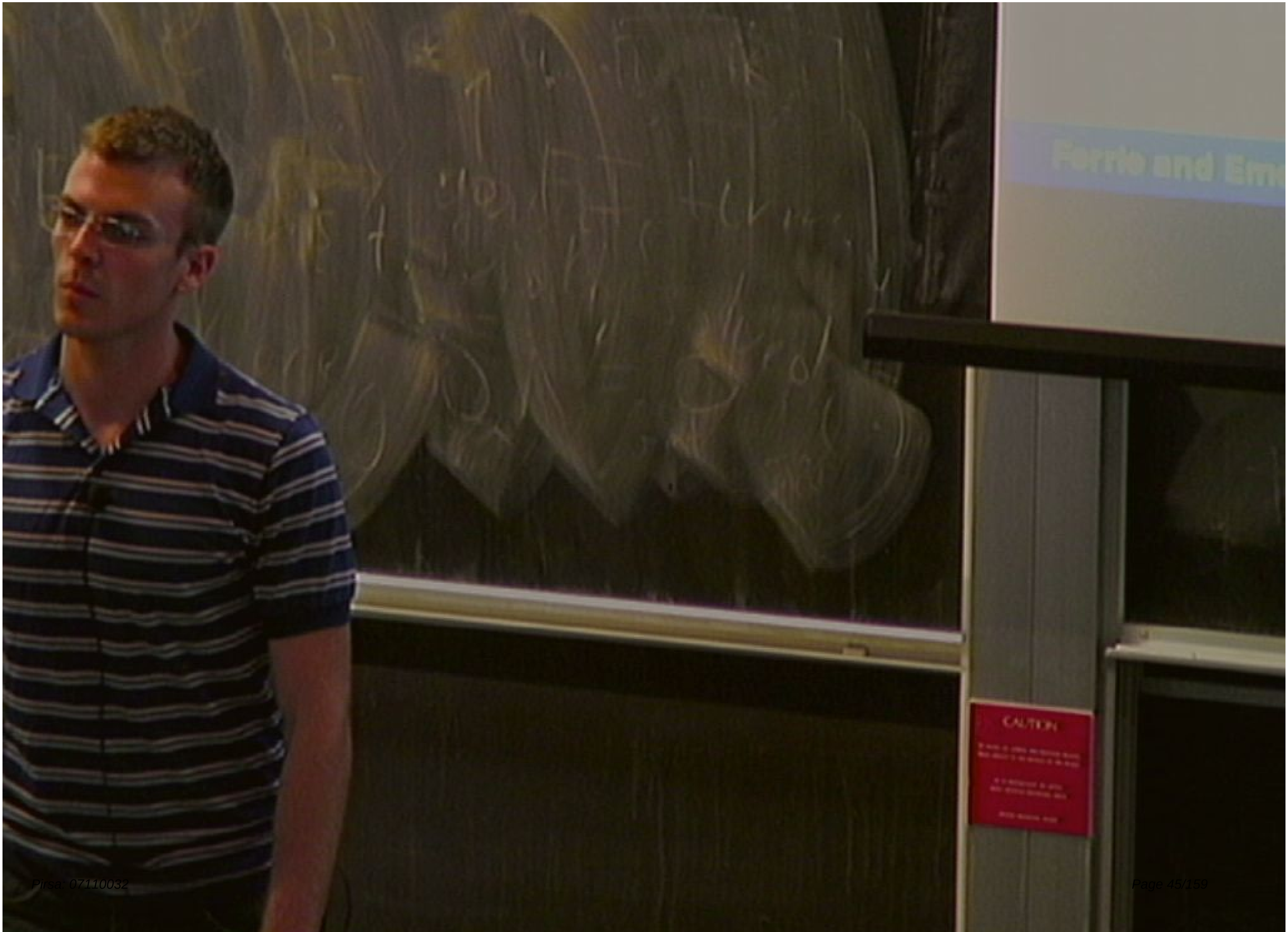
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Suppose there exists symmetry group on Γ, G, carrying a unitary representation $\hat{U} : G \to \mathrm{U}(\mathcal{H})$. A quasi-probability representation satisfying the covariance property $\hat{U}_g \hat{A} \hat{U}_g^\dagger \to \{A(g(\alpha))\}_{\alpha\in\Gamma}$ for all $\hat{A} \in \mathrm{Herm}(\mathcal{H})$ and $g \in G$ is a phase space representation.

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A mapping $\mathrm{Herm}(\mathcal{H}) \to L^2(\Gamma, \mu)$ of the form $\hat{A} \mapsto A(\alpha) := \langle \hat{E}(\alpha), \hat{A}\rangle$ is a frame representation of $\mathrm{Herm}(\mathcal{H})$.

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$$a := \min \left\{ \int_\Gamma d\mu(\alpha) |\langle \hat{A}, \hat{E}(\alpha) \rangle|^2 : \|\hat{A}\| = 1 \right\}.$$

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Since the space Γ is compact and, for each $\alpha \in \Gamma$, $\|\hat{E}(\alpha)\|$ is bounded, the integral (1) is bounded. Hence $\hat{E}$ is a frame. □

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- The frame representation provides two approaches lift any representation of states to a fully autonomous representation of quantum mechanics.

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- Define the algebraic multiplication as

$$(A \star_\Gamma B)(\alpha) := \int_{\Gamma^2} d\mu(\beta) d\mu(\gamma) A(\beta) B(\gamma) \mathfrak{F}(\alpha, \beta, \gamma),$$

  where $\mathfrak{F}(\alpha, \beta, \gamma) = \langle \hat{E}(\alpha), \hat{F}(\beta)\hat{F}(\gamma) \rangle$.

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4. For a system with probability distribution ρ, upon measurement $\{P_k\}_{k\in K}$, the probability of obtaining outcome k is

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- Then, the states and measurements cannot satisfy the criteria of a classical model: they must contain negativity.

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- We have also proven that the frame representation is equivalent to a linear invertible representation of quantum mechanics.
- We proved that such representations require either negativity or a deformation of the rule for calculating probabilities.
- We conjecture that these results continue to hold also for infinite dimensional separable Hilbert spaces.

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