Title: Quantum Phase Estimation

Date: Nov 14, 2007 04:00 PM

URL: http://pirsa.org/07110031

Abstract: We will compare quantum phase estimation from the point of view of quantum computation and quantum metrology. In the simplest cases, the former can be simplified to a sequential (unentangled) protocol, while the latter is parallel (entangled). We show that both protocols can be formally related with circuit identities and that they respond in exactly the same way to decoherence. We present sequential protocols for optimal estimation and frame synchronization in DQC1. Finally, we introduce new estimation protocols based on nonlinear Hamiltonians. We show that both optimal input states and product states with separable measurements improve the scaling of linear Hamiltonians. We will comment on the effect of decoherence in nonlinear protocols, and the role of entanglement in nonlinear protocols with product states.

Quantum phase estimation

Sergio Boixo

University of New Mexico
Los Alamos National Laboratory

November 14, 2007 / PIQuDos

SB, Caves, Datta and Shaji, Laser Physics, 16, 1525 (2006).
SB and Somma, arxiv:0708.1330.
SB, Flammia, Caves and Geremia, PRL 98, 090401 (2007).
SB, Datta, Flammia, Shaji, Bagan and Caves, accepted in PRA.

Quantum metrology

- Ramsey interferometry.

$$e^{i\phi/2(\sum^N z_j)}(|\uparrow\rangle + |\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + |\downarrow\rangle) \rightarrow$$
$$(|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) \,.$$

Gives "classical" uncertainty ($\phi = \omega t$)
$\delta\omega = \frac{1}{\sqrt{\nu N t}}$.

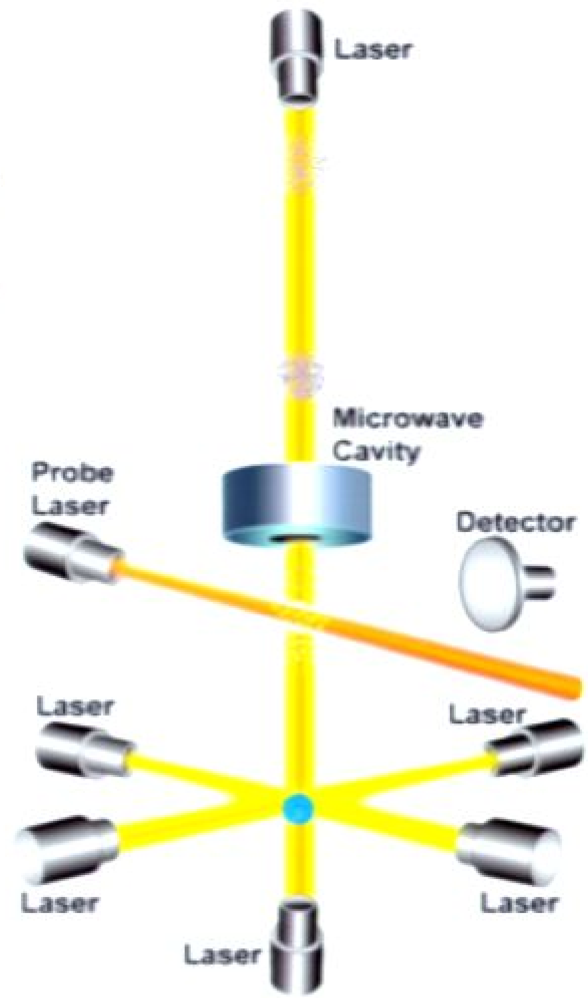


Figure: NIST F1.

Quantum metrology

- Ramsey interferometry.

$$e^{i\phi/2(\sum^N z_i)}(|\uparrow\rangle + |\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + |\downarrow\rangle) \rightarrow$$
$$(|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) .$$

Gives "classical" uncertainty ($\phi = \omega t$)
$\delta\omega = \frac{1}{\sqrt{\nu N t}}$.

- Entangled protocols.

$$e^{i\phi/2(\sum^N z_i)}(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle) \rightarrow$$
$$(|\uparrow\uparrow\cdots\uparrow\rangle + e^{-Ni\phi}|\downarrow\downarrow\cdots\downarrow\rangle) .$$

Gives $\delta\omega = \frac{1}{\sqrt{\nu} N t}$.

Atomic spin-squeezing, Mach-Zehnder interferometry with optical and NOON states.

BEC interferometry...

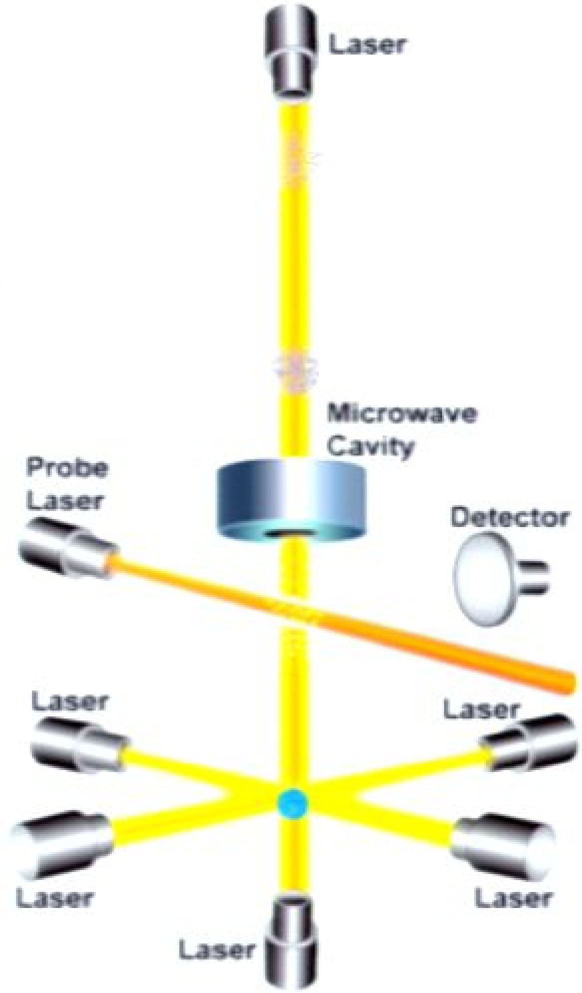


Figure: NIST F1.

Quantum metrology

- Ramsey interferometry.

$$e^{i\phi/2(\sum^N z_j)}(|\uparrow\rangle + |\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + |\downarrow\rangle) \rightarrow$$
$$(|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) \otimes \cdots \otimes (|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle) .$$

Gives "classical" uncertainty ($\phi = \omega t$)
$$\delta\omega = \frac{1}{\sqrt{\nu N t}} .$$

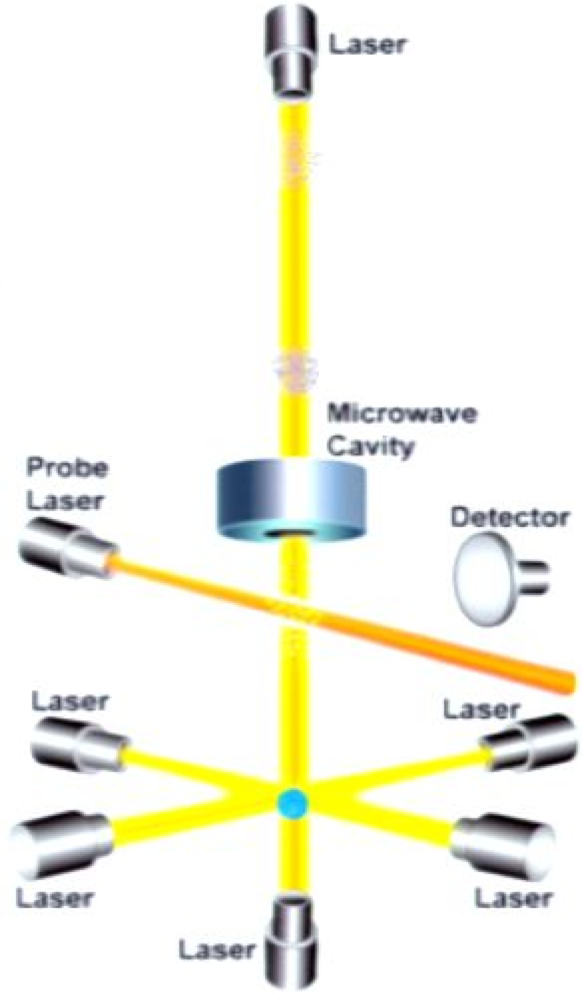


Figure: NIST F1.

Quantum metrology

- Ramsey interferometry.

$$e^{i\phi/2(\sum^N z_i)}(|\uparrow\rangle + |\downarrow\rangle)\otimes\cdots\otimes(|\uparrow\rangle + |\downarrow\rangle) \to (|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle)\otimes\cdots\otimes(|\uparrow\rangle + e^{-i\phi}|\downarrow\rangle).$$

Gives "classical" uncertainty ($\phi = \omega t$) $\delta\omega = \frac{1}{\sqrt{\nu N}t}$.

- Entangled protocols.

$$e^{i\phi/2(\sum^N z_i)}(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle) \to (|\uparrow\uparrow\cdots\uparrow\rangle + e^{-Ni\phi}|\downarrow\downarrow\cdots\downarrow\rangle).$$

Gives $\delta\omega = \frac{1}{\sqrt{\nu}Nt}$.

Atomic spin-squeezing, Mach-Zehnder interferometry with optical and NOON states.

BEC interferometry...

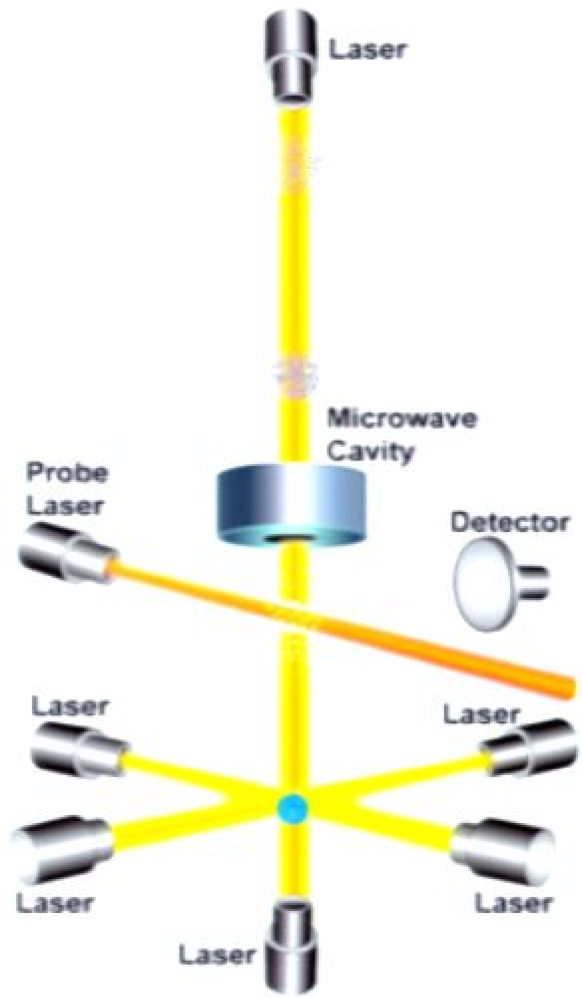


Figure: NIST F1.

Phase estimation in quantum computation

Phase estimation is used in many algorithms (i.e., Shor's), and it is actually BQP-Complete. The (simplest) optimal circuit is

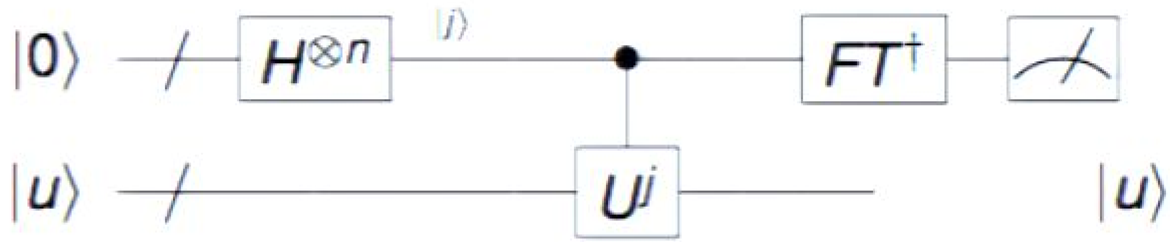


Figure: Kitaev's phase estimation

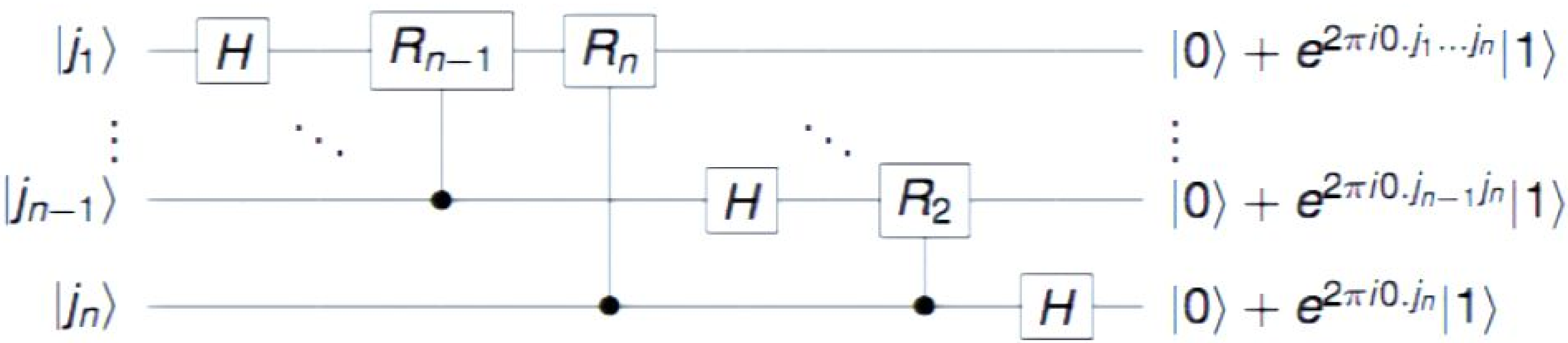
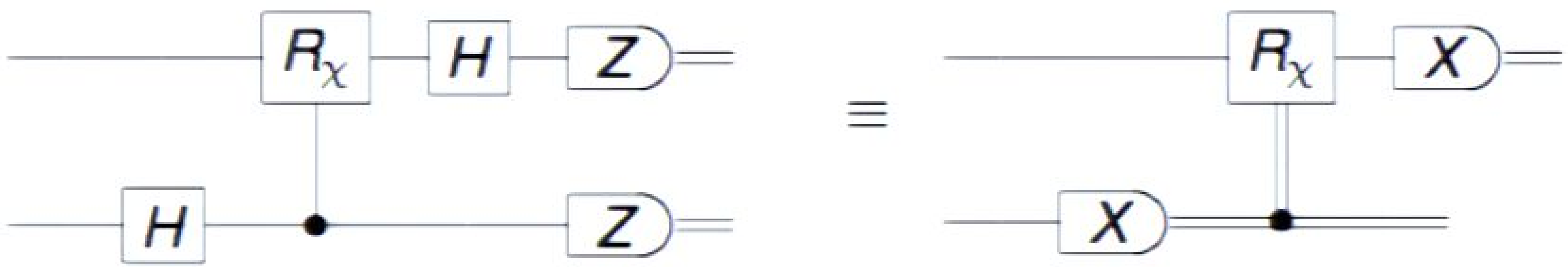


Figure: Quantum Fourier transform.

Cleve, Ekert, Macchiavello and Mosca, Proceedings of the Royal Society of London A 454, 339 (1998). Wocjan and Zhang, quant-ph/0606179. van Dam, D'Ariano, Ekert, Macchiavello and Mosca, PRL 98, 090501 (2007).

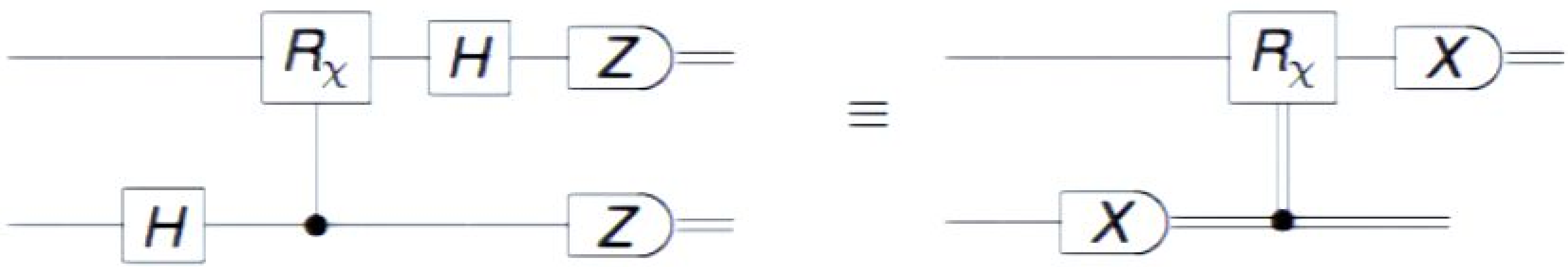
Inverse quantum Fourier transform

- Apply the principle of deferred measurement to $FT^\dagger$. Griffiths and Niu. PRL. 76, 3228 (1996).



Inverse quantum Fourier transform

- Apply the principle of deferred measurement to $FT^\dagger$. Griffiths and Niu. PRL. 76, 3228 (1996).



- In phase estimation, the classically controlled R_χ amounts to phase feedback which we ignore here. Knill, Ortiz and Somma, PRA 75, 012328 (2007), Higgins et al. arXiv:0709.2996.

Simplifying phase estimation

- Assuming $|u\rangle$ eigenstate of U (metrology), work with one ancilla at a time

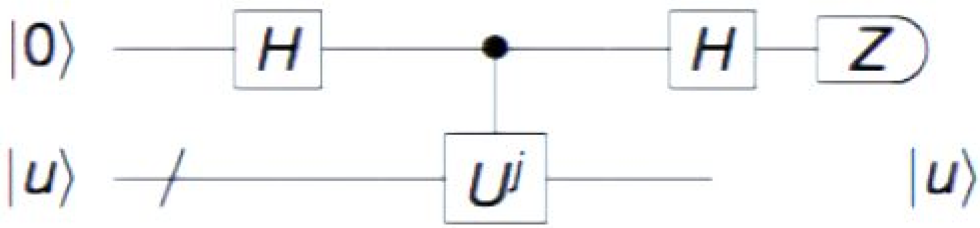
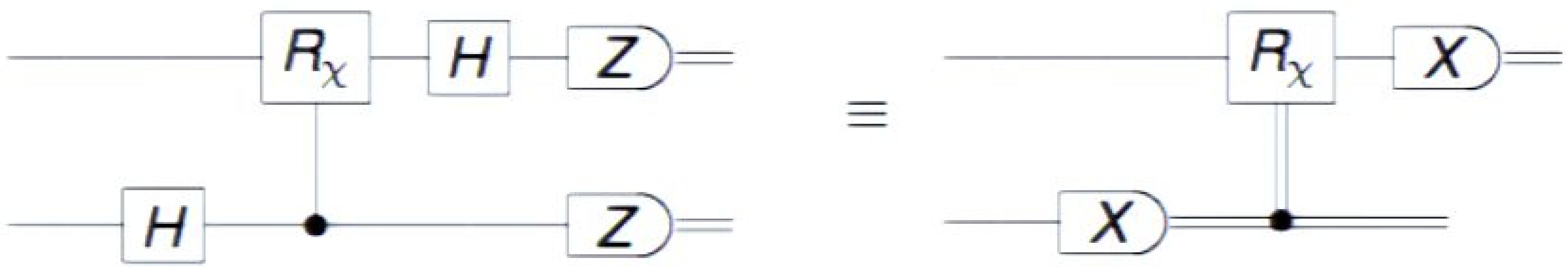


Figure: Single ancilla phase estimation

Inverse quantum Fourier transform

- Apply the principle of deferred measurement to $FT^\dagger$. Griffiths and Niu. PRL. 76, 3228 (1996).



- In phase estimation, the classically controlled R_χ amounts to phase feedback which we ignore here. Knill, Ortiz and Somma, PRA 75, 012328 (2007), Higgins et al. arXiv:0709.2996.

Simplifying phase estimation

- Assuming $|u\rangle$ eigenstate of U (metrology), work with one ancilla at a time

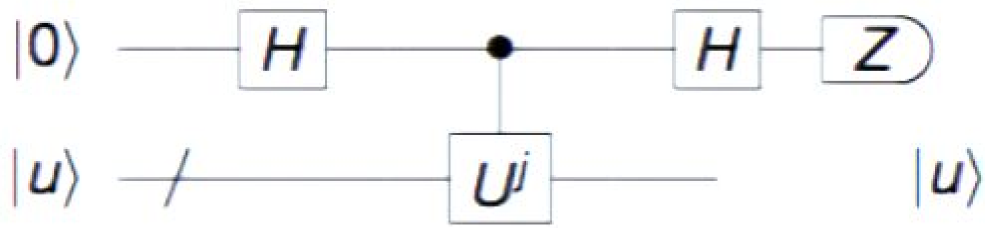


Figure: Single ancilla phase estimation

Simplifying phase estimation

- Assuming $|u\rangle$ eigenstate of U (metrology), work with one ancilla at a time

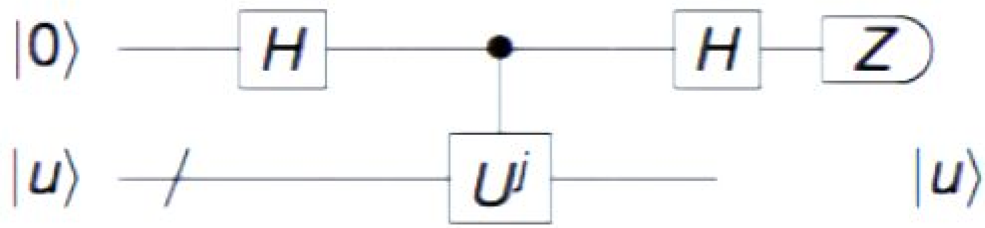
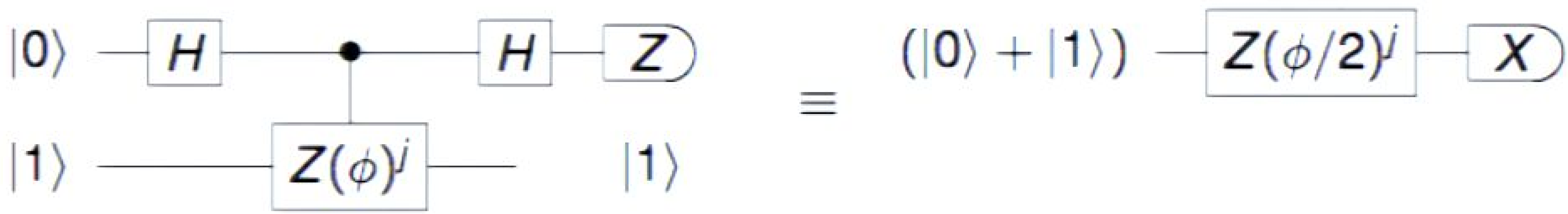


Figure: Single ancilla phase estimation

- Assume two eigenstates of U are accessible (metrology). Then $U \equiv Z(\phi)$. We are back with Ramsey.



From Ramsey to entanglement

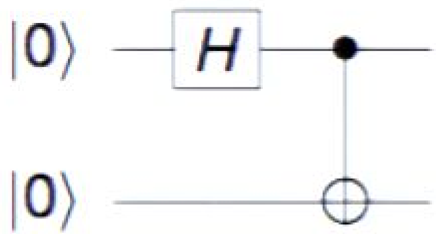
Formally, a sequential array of Z rotations is equivalent to a parallel array on a cat state.

From Ramsey to entanglement

Formally, a sequential array of Z rotations is equivalent to a parallel array on a cat state.

From Ramsey to entanglement

Formally, a sequential array of Z rotations is equivalent to a parallel array on a cat state.

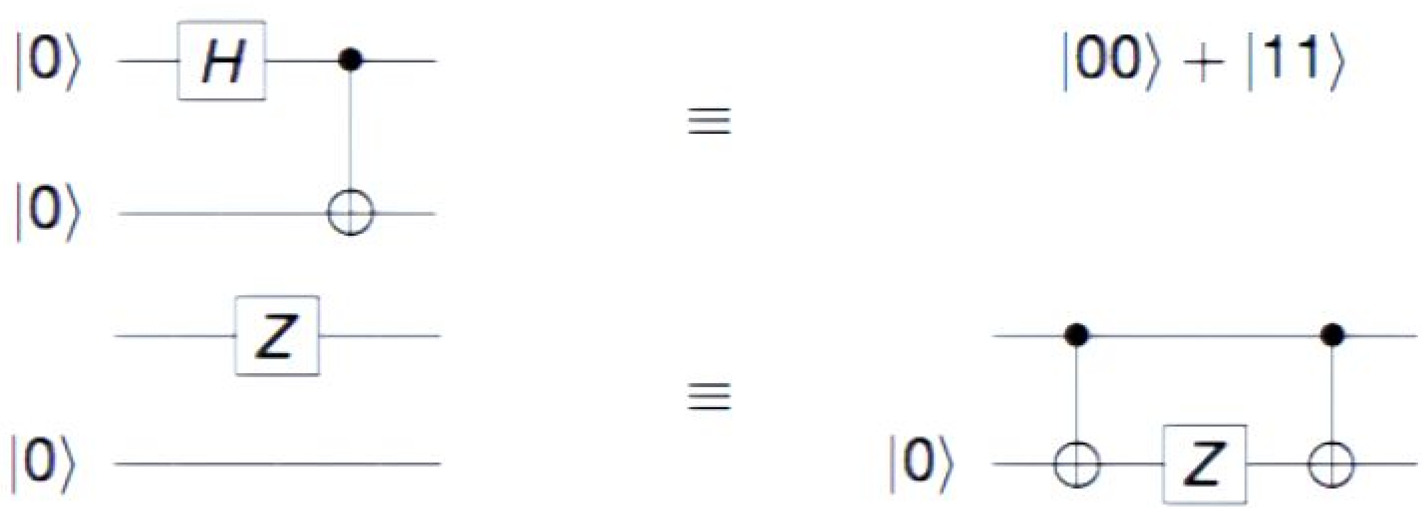


$\equiv$

$|00\rangle + |11\rangle$

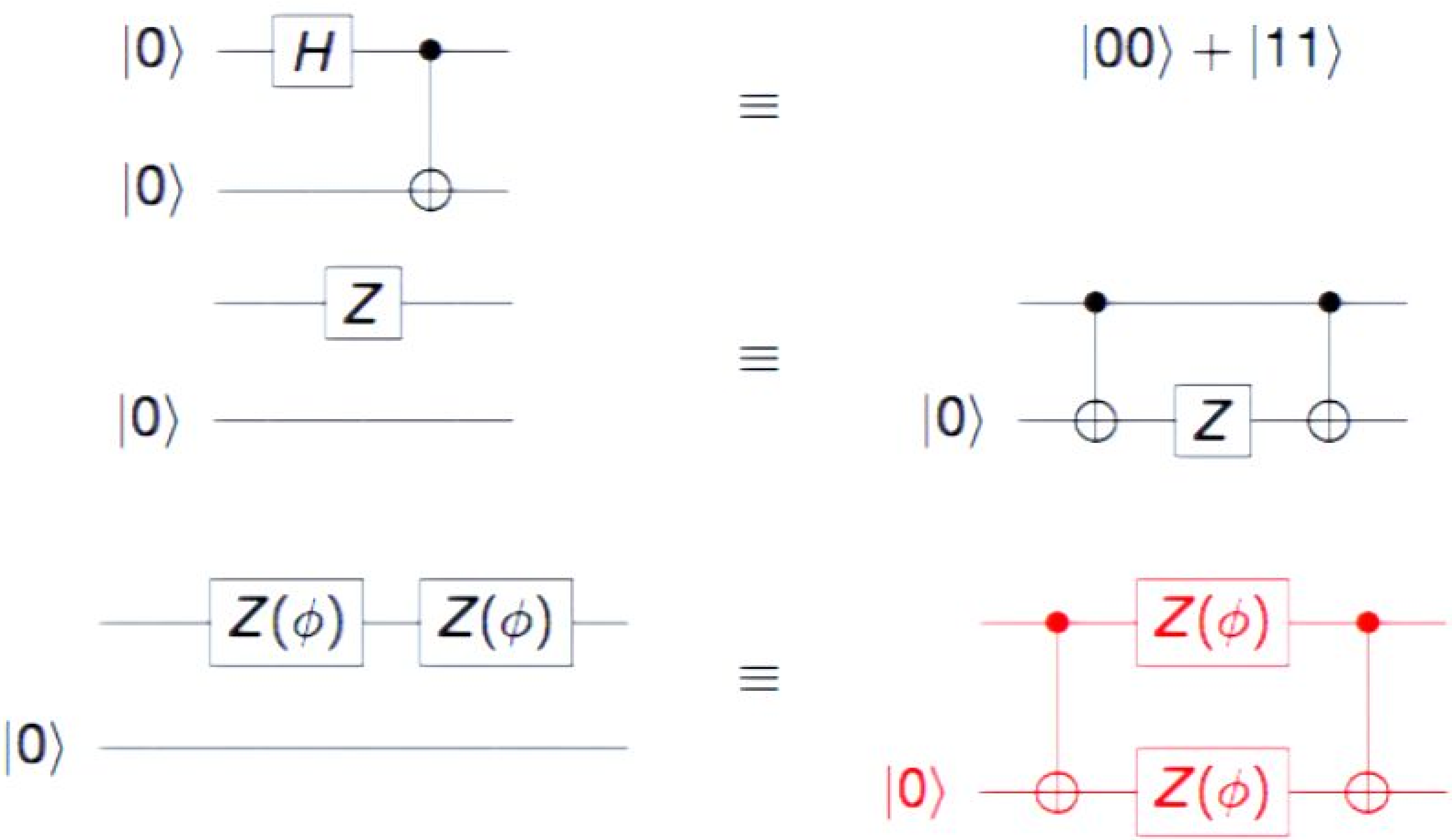
From Ramsey to entanglement

Formally, a sequential array of Z rotations is equivalent to a parallel array on a cat state.



From Ramsey to entanglement

Formally, a sequential array of Z rotations is equivalent to a parallel array on a cat state.



Decoherence

Is there any physics behind the formal relation between Ramsey and entangled interferometry? Are they equally robust? Let's introduce

1. Projection operator Π onto the $\{X, Y\}$ plane.
2. Uncorrelated decoherence channel $\mathcal{E}$ symmetric about the Z axis.

Decoherence in the $\{X, Y\}$ plane

$$\Pi \circ \mathcal{E}(\rho) \mapsto \lambda e^{-i\alpha Z}\, \Pi(\rho)\, e^{i\alpha Z} \quad \text{with} \quad \lambda \leq 1 \,.$$

Decoherence

Is there any physics behind the formal relation between Ramsey and entangled interferometry? Are they equally robust? Let's introduce

1. Projection operator Π onto the $\{X, Y\}$ plane.
2. Uncorrelated decoherence channel $\mathcal{E}$ symmetric about the Z axis.

Decoherence in the $\{X, Y\}$ plane

$$\Pi \circ \mathcal{E}(\rho) \mapsto \lambda e^{-i\alpha Z}\, \Pi(\rho)\, e^{i\alpha Z} \quad \text{with} \quad \lambda \leq 1\,.$$

Sequential protocol

$$\Pi \circ \mathcal{Z}(\phi) \circ \mathcal{E} \circ \cdots \circ \mathcal{Z}(\phi) \circ \mathcal{E}(\rho) = \lambda^N e^{-i\alpha NZ} \mathcal{Z}(N\phi) \circ \Pi(\rho) e^{i\alpha NZ}$$

Decoherence in the entanglement protocol

The cat state can be written as

$$\frac{1}{2^{n+1}}\left[\bigotimes_{j=1}^{N}(\mathbb{I}+Z_j)+\bigotimes_{j=1}^{N}(\mathbb{I}-Z_j)+\bigotimes_{j=1}^{N}(X_j+iY_j)+\bigotimes_{j=1}^{N}(X_j-iY_j)\right].$$

Entanglement protocol

$$\Pi^{\otimes N}\circ\mathcal{Z}(\phi)^{\otimes N}\circ\mathcal{E}^{\otimes N}(\rho_c)=\lambda^N e^{-i\alpha NZ}\mathcal{Z}(\phi)^{\otimes N}\circ\Pi^{\otimes N}(\rho_c)e^{i\alpha NZ}$$

Decoherence has the same effect on both protocols.

Huelga et. al. PRL 79, 3865 (1997).

SB, Caves, Datta and Shaji, Laser Physics, 16, 1525 (2006).

Mixed-state quantum computation

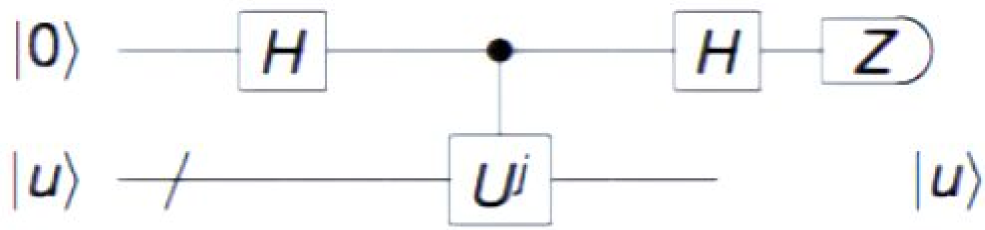


Figure: Single ancilla phase estimation

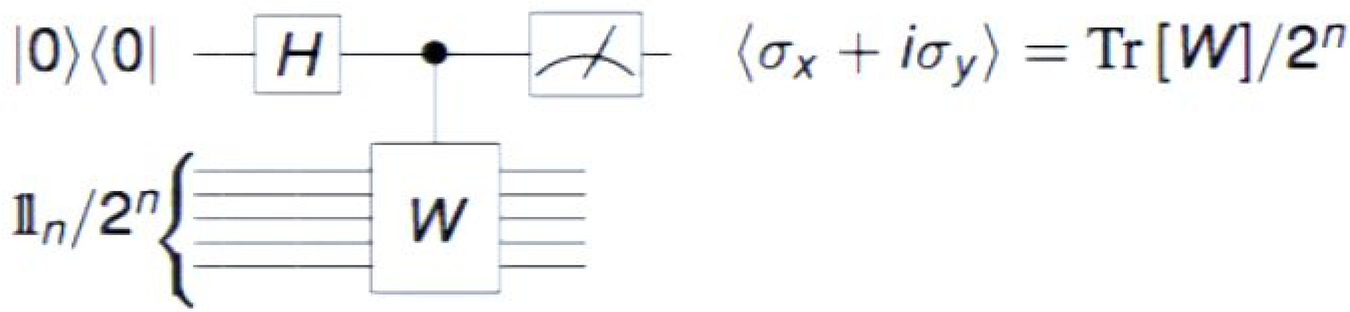


Figure: Mixed-state quantum computation

Knill and Laflamme, PRL 81, 5672 (1998).

Mixed-state quantum computation

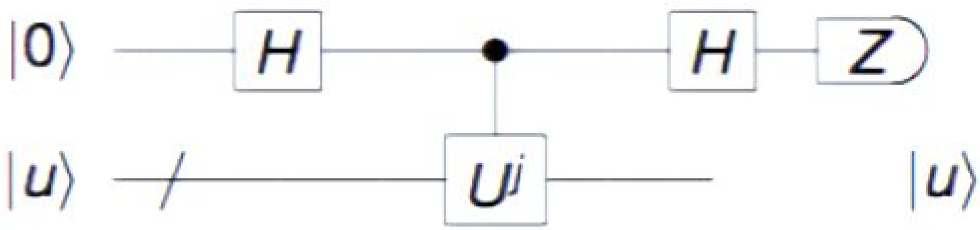


Figure: Single ancilla phase estimation

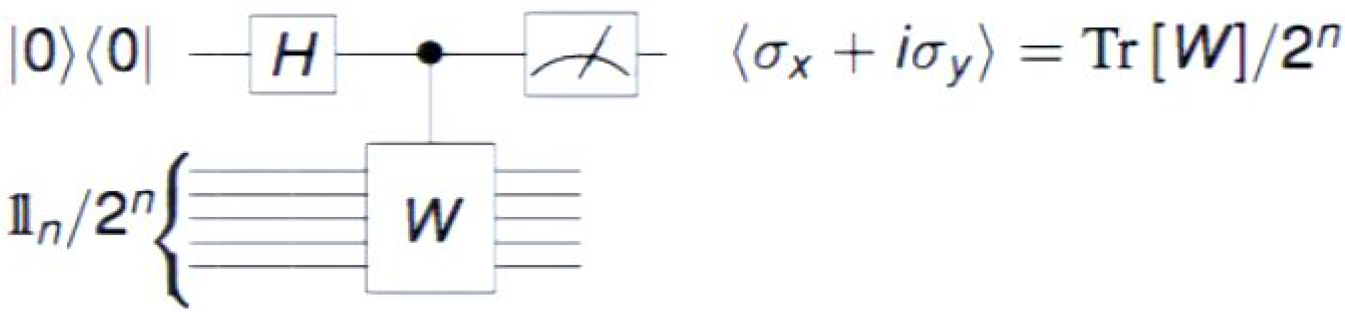


Figure: Mixed-state quantum computation

Knill and Laflamme, PRL 81, 5672 (1998).

Heisenberg picture

Mixed-state QC: inner product for the adjoint action of U.

$$\mathrm{Tr}\,[W] = \mathrm{Tr}\,[U^\dagger VUO]$$

Mixed-state estimation

$U(T) = e^{-i\omega T H_0}$. Assume pseudo-orthogonal $\{H_0, H_1, H_2\}$ spanning a $\mathfrak{su}(2)$ Lie algebra:

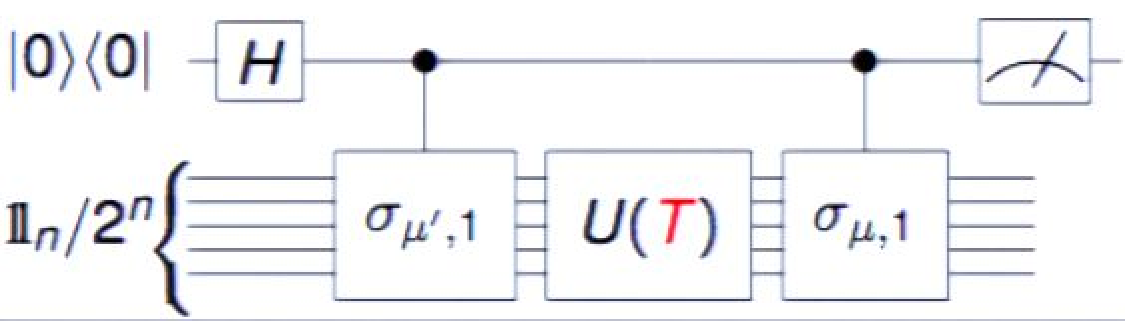
$$k \cos(2\omega T) = \mathrm{Tr}\left[U^\dagger(T) H_1 U(T) H_1\right] .$$

Let $H_j = \sum_{\mu=1}^{L} e^{\mu,j}\sigma_{\mu,j}$, with $\sigma_{\mu,j}$ Pauli product.

$$k \cos(2\omega T) = \sum_{\mu,\mu'} e^{\mu,1} e^{\mu',1} \mathrm{Tr}\left[U^\dagger(T)\sigma_{\mu,1} U(T)\sigma_{\mu',1}\right] .$$

Mixed-state estimation

Run L^2 times the circuit



SB and Somma, arxiv:0708.1330.

Mixed-state quantum computation

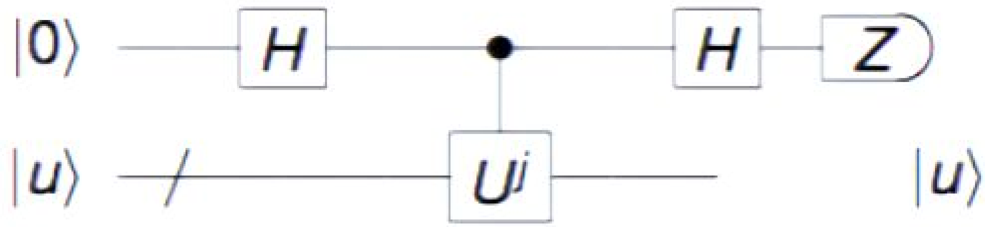
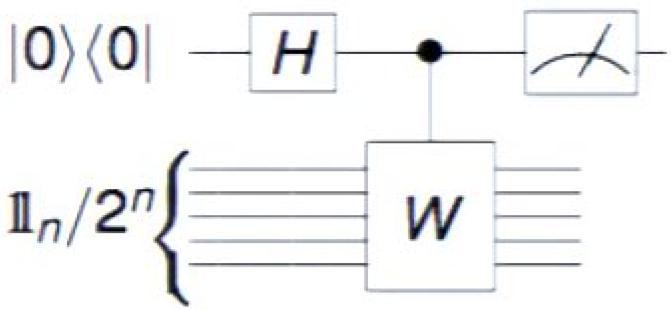


Figure: Single ancilla phase estimation



$$\langle \sigma_x + i\sigma_y \rangle = \mathrm{Tr}\,[W]/2^n$$

Figure: Mixed-state quantum computation

Knill and Laflamme, PRL 81, 5672 (1998).

Heisenberg picture

Mixed-state QC: inner product for the adjoint action of U.

$$\mathrm{Tr}\,[W] = \mathrm{Tr}\,[U^\dagger VUO]$$

Mixed-state estimation

$U(T) = e^{-i\omega T H_0}$. Assume pseudo-orthogonal $\{H_0, H_1, H_2\}$ spanning a $\mathfrak{su}(2)$ Lie algebra:

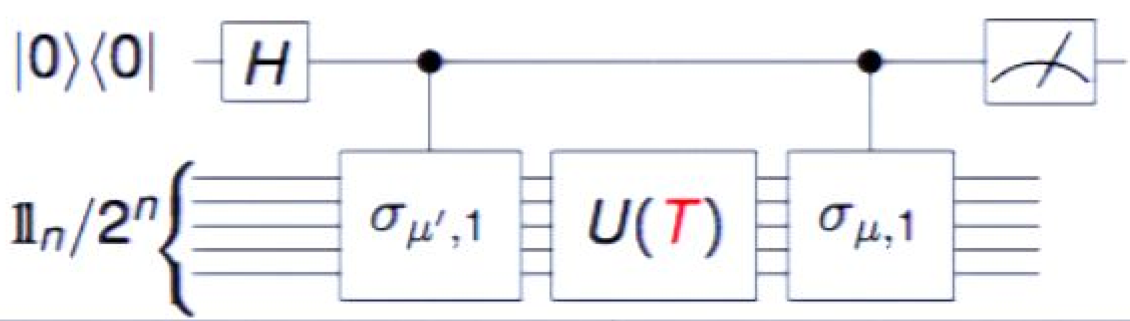
$$k \cos(2\omega T) = \mathrm{Tr}\left[U^\dagger(T) H_1 U(T) H_1\right] .$$

Let $H_j = \sum_{\mu=1}^{L} e^{\mu,j} \sigma_{\mu,j}$, with $\sigma_{\mu,j}$ Pauli product.

$$k \cos(2\omega T) = \sum_{\mu,\mu'} e^{\mu,1} e^{\mu',1} \mathrm{Tr}\left[U^\dagger(T) \sigma_{\mu,1} U(T) \sigma_{\mu',1}\right] .$$

Mixed-state estimation

Run L^2 times the circuit



SB and Somma, arxiv:0708.1330.

More mixed-state estimation

1. Sometimes $L = 1$: $H_1 = \sigma_1$ and

$$\begin{aligned} \left[\sigma_{\mu,0}, \sigma_{\mu',0}\right] &= 0 \quad \forall\, \mu, \mu' \,, \\ \left\{\sigma_{\mu,0}, \sigma_1\right\} &= 0 \,, \\ \left[\sigma_{\mu',0}, \sigma_1\right] &= 0 \quad \forall\, \mu' \neq \mu \,. \end{aligned}$$

Mixed-state estimation

$U(T) = e^{-i\omega T H_0}$. Assume pseudo-orthogonal $\{H_0, H_1, H_2\}$ spanning a $\mathfrak{su}(2)$ Lie algebra:

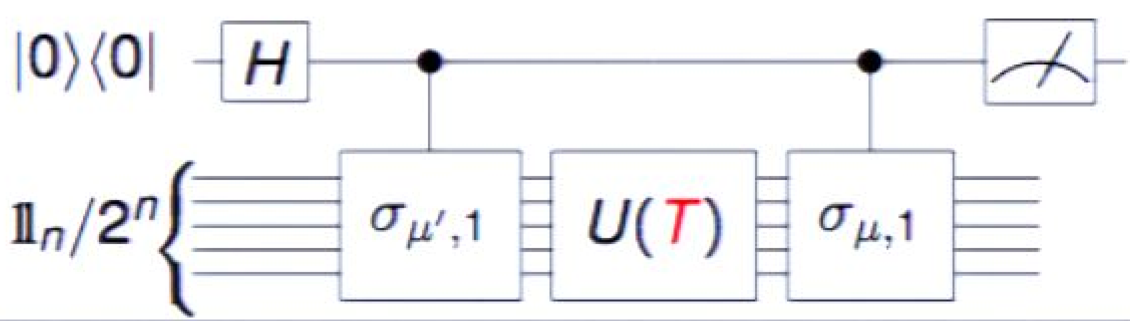
$$k\cos(2\omega T) = \mathrm{Tr}\left[U^\dagger(T) H_1 U(T) H_1\right].$$

Let $H_j = \sum_{\mu=1}^{L} e^{\mu,j}\sigma_{\mu,j}$, with $\sigma_{\mu,j}$ Pauli product.

$$k\cos(2\omega T) = \sum_{\mu,\mu'} e^{\mu,1} e^{\mu',1}\mathrm{Tr}\left[U^\dagger(T)\sigma_{\mu,1} U(T)\sigma_{\mu',1}\right].$$

Mixed-state estimation

Run L^2 times the circuit



SB and Somma, arxiv:0708.1330.

More mixed-state estimation

- Sometimes $L = 1$: $H_1 = \sigma_1$ and

$$\begin{aligned} \left[\sigma_{\mu,0}, \sigma_{\mu',0}\right] &= 0 \quad \forall\, \mu, \mu' \,, \\ \left\{\sigma_{\mu,0}, \sigma_1\right\} &= 0 \,, \\ \left[\sigma_{\mu',0}, \sigma_1\right] &= 0 \quad \forall\, \mu' \neq \mu \,. \end{aligned}$$

Mixed-state estimation

$U(T) = e^{-i\omega T H_0}$. Assume pseudo-orthogonal $\{H_0, H_1, H_2\}$ spanning a $\mathfrak{su}(2)$ Lie algebra:

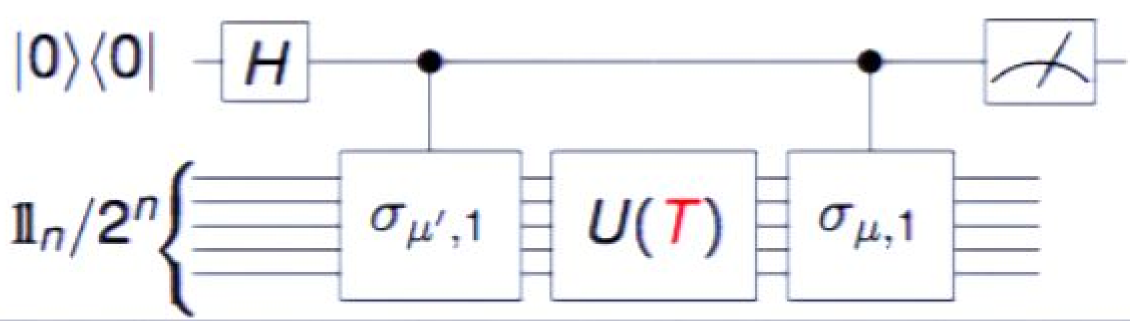
$$k\cos(2\omega T) = \mathrm{Tr}\left[U^\dagger(T) H_1 U(T) H_1\right] .$$

Let $H_j = \sum_{\mu=1}^{L} e^{\mu,j} \sigma_{\mu,j}$, with $\sigma_{\mu,j}$ Pauli product.

$$k\cos(2\omega T) = \sum_{\mu,\mu'} e^{\mu,1} e^{\mu',1} \mathrm{Tr}\left[U^\dagger(T)\sigma_{\mu,1} U(T)\sigma_{\mu',1}\right] .$$

Mixed-state estimation

Run L^2 times the circuit



SB and Somma, arxiv:0708.1330.

More mixed-state estimation

- Sometimes $L = 1$: $H_1 = \sigma_1$ and

$$\begin{aligned} [\sigma_{\mu,0}, \sigma_{\mu',0}] &= 0 \quad \forall\, \mu, \mu' \,, \\ \{\sigma_{\mu,0}, \sigma_1\} &= 0 \,, \\ [\sigma_{\mu',0}, \sigma_1] &= 0 \quad \forall\, \mu' \neq \mu \,. \end{aligned}$$

More mixed-state estimation

1. Sometimes $L = 1$: $H_1 = \sigma_1$ and

$$\begin{aligned} \left[\sigma_{\mu,0}, \sigma_{\mu',0}\right] &= 0 \quad \forall\, \mu, \mu' \,, \\ \left\{\sigma_{\mu,0}, \sigma_1\right\} &= 0 \,, \\ \left[\sigma_{\mu',0}, \sigma_1\right] &= 0 \quad \forall\, \mu' \neq \mu \,. \end{aligned}$$

2. Black box: Include phase feedback.

More mixed-state estimation

1. Sometimes $L = 1$: $H_1 = \sigma_1$ and

$$\begin{aligned}[\sigma_{\mu,0}, \sigma_{\mu',0}] &= 0 \quad \forall\, \mu, \mu' \,, \\ \{\sigma_{\mu,0}, \sigma_1\} &= 0 \,, \\ [\sigma_{\mu',0}, \sigma_1] &= 0 \quad \forall\, \mu' \neq \mu \,.\end{aligned}$$

2. Black box: Include phase feedback.
3. Multi-parameter: $H_0 = \sum_{\nu=1}^{P} \omega^\nu \sigma_\nu$. Solve Euler angles equations or find σ

$$\begin{aligned}[\sigma_\nu, \sigma] &= 0 \,, \\ \{\sigma_{\nu'}, \sigma\} &= 0 \; \forall\, \nu' \neq \nu \,.\end{aligned}$$

$\bar{S}_\nu(\epsilon T) \equiv \mathrm{Trotter}(\epsilon T(H_0 + \sigma H_0 \sigma), p)$. Then

Multi-parameter with Trotter

$$\bar{S}_\nu(T) = e^{-i\omega^\nu T \sigma_\nu} + \mathcal{O}(\|H_0\| \epsilon^{p-1} T^p) \,.$$

Mixed-state frame synchronization

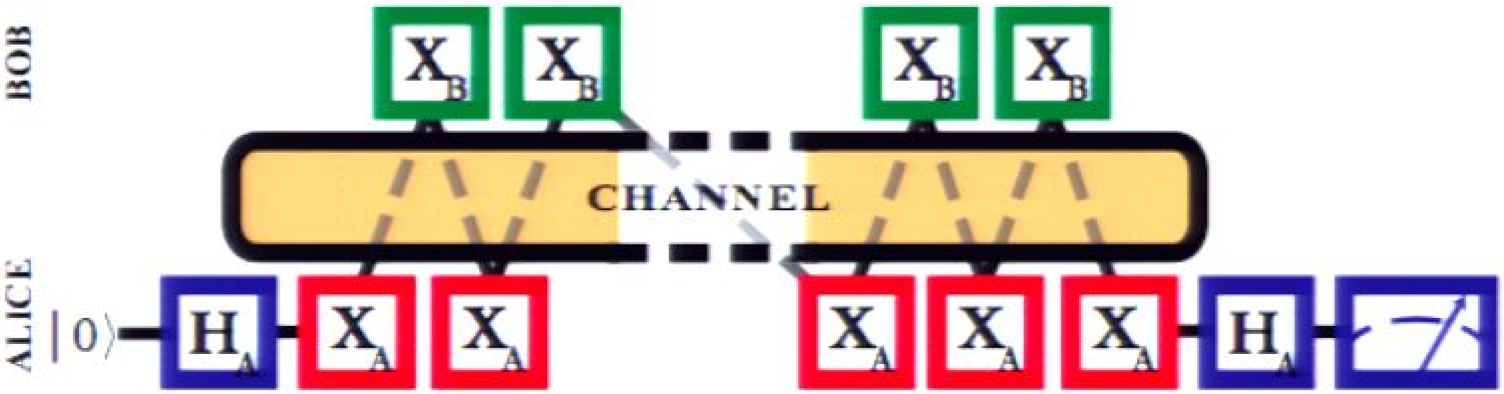


Figure: Sequential frame synchronization. Alice's X_A and Bob's X_B give $X_A X_B = e^{-2i\omega Z}$. de Burgh and Bartlett, PRA 72, 042301 (2005).

Mixed-state frame synchronization

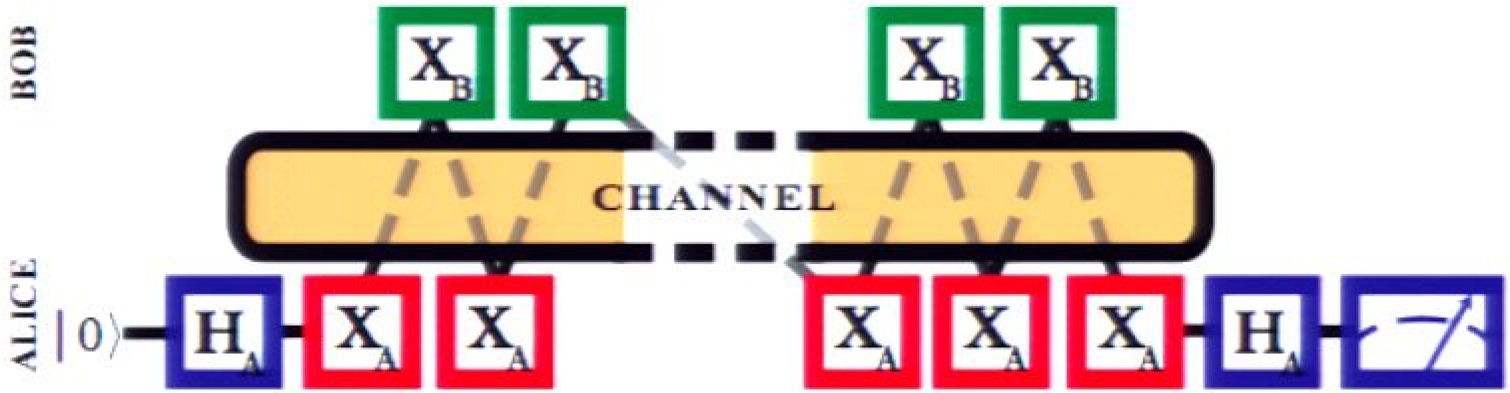


Figure: Sequential frame synchronization. Alice's X_A and Bob's X_B give $X_A X_B = e^{-2i\omega Z}$. de Burgh and Bartlett, PRA 72, 042301 (2005).

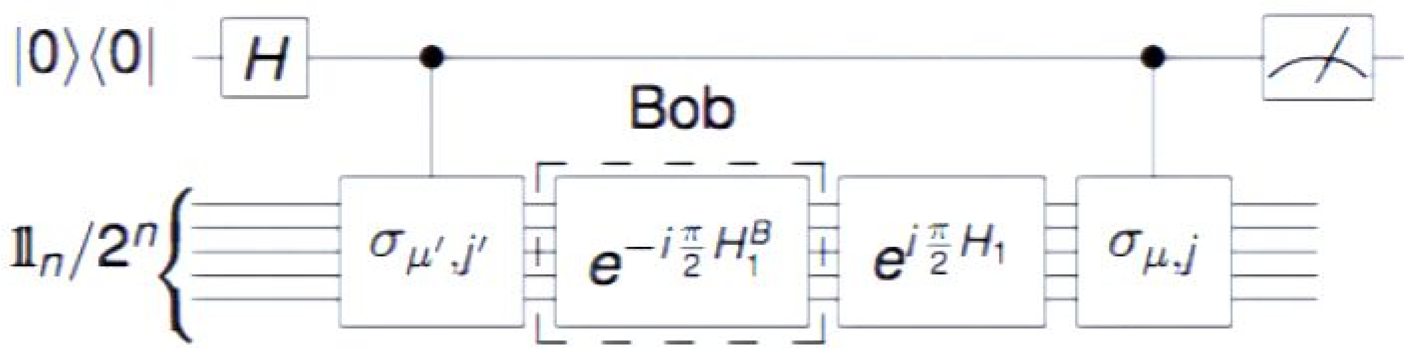


Figure: Frame synchronization with mixed-states. $H_1^B = e^{i\omega H_0} H_1 e^{-i\omega H_0} \Rightarrow e^{i\pi H_1/2} e^{-i\pi H_1^B/2} = e^{-2i\omega H_0}$. SB and Somma, arxiv:0708.1330.

Adaptive Bayesian protocol

- Measurement outcome $\mathcal{N}(\cos(\omega T), \Delta)$.
- Previous estimator $\hat{\omega}_{l-1}$, $\Delta_{l-1} \approx \Delta$ and time T_{l-1}.
- Find $T_l = a_l T_{l-1}$ s. t. $2\hat{\omega}_{l-1} T_l = \pi/2 + 2p_l\pi$. For simplicity, assume $(a_l\Delta)^3 \ll 1$, so the cosine goes linear. Measurement $+$ Bayes' rule

$$\begin{aligned} \hat{\omega}_l &\approx \frac{1}{2T_l}\left((\pi/2 + 2p_l\pi) - \frac{(a_l)^2}{1+(a_l)^2} x_l\right) , \\ \Delta_l &\approx \Delta \, . \end{aligned}$$

95% credible interval

$$\hat{\omega}_l - 1.96\Delta_l/(2T_l) \leq \omega \leq \hat{\omega}_l + 1.96\Delta_l/(2T_l) \, .$$

What about Grover?

$U_G = e^{i\theta|S\rangle\langle S|}$, with $\theta \neq 0$, or $U_G = \mathbb{1}$.
After Q calls

$$\langle \sigma_z \rangle = \mathrm{Tr}\,[W_Q U_G \cdots U_G W_0 \sigma_z W_0^\dagger U_G^\dagger \cdots U_G^\dagger W_Q^\dagger \sigma_z]/2^{n+1} \ .$$

Following Knill & Laflamme:

No Grover

$$\left| \langle \sigma_z \rangle|_{U_B = \mathbf{1}} - \langle \sigma_z \rangle|_{U_B = e^{i\theta|S\rangle\langle S|}} \right| \leq 4Q/2^{n+1} \ .$$

So Q needs to be $\mathcal{O}(2^n)$.

Generalized phase estimation

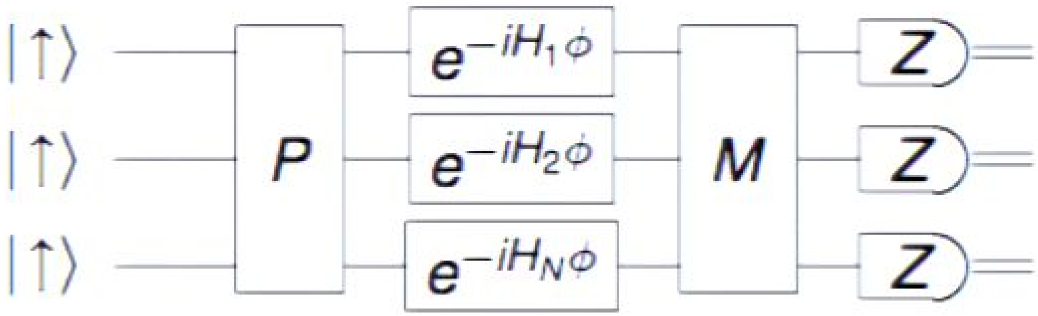


Figure: Standard phase estimation.

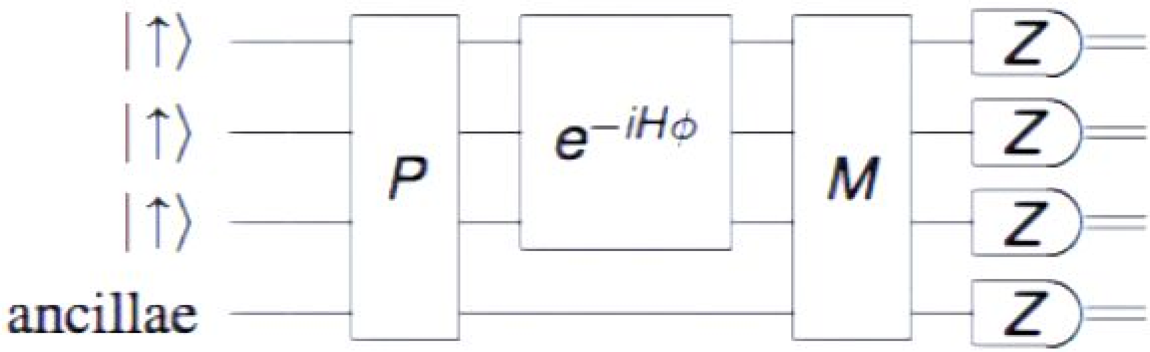


Figure: Generalized phase estimation.

What about Grover?

$U_G = e^{i\theta|S\rangle\langle S|}$, with $\theta \neq 0$, or $U_G = \mathbb{1}$.
After Q calls

$$\langle \sigma_Z \rangle = \mathrm{Tr}\,[W_Q U_G \cdots U_G W_0 \sigma_Z W_0^\dagger U_G^\dagger \cdots U_G^\dagger W_Q^\dagger \sigma_Z]/2^{n+1} \ .$$

Following Knill & Laflamme:

No Grover

$$\left| \langle \sigma_Z \rangle|_{U_B = \mathbf{1}} - \langle \sigma_Z \rangle|_{U_B = e^{i\theta|S\rangle\langle S|}} \right| \leq 4Q/2^{n+1} \ .$$

So Q needs to be $\mathcal{O}(2^n)$.

Generalized phase estimation

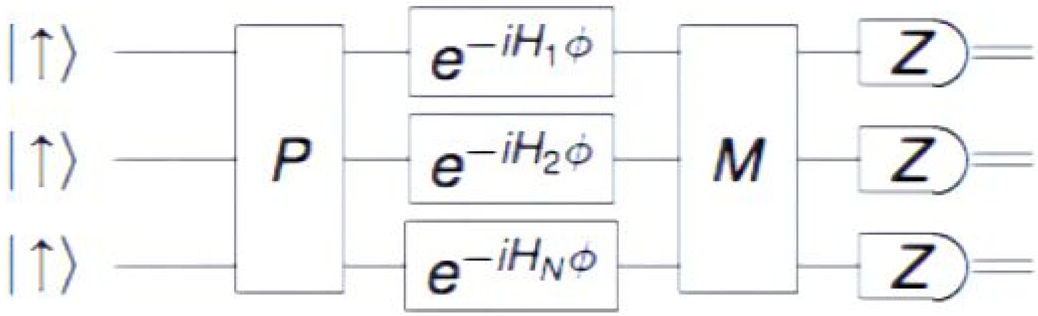


Figure: Standard phase estimation.

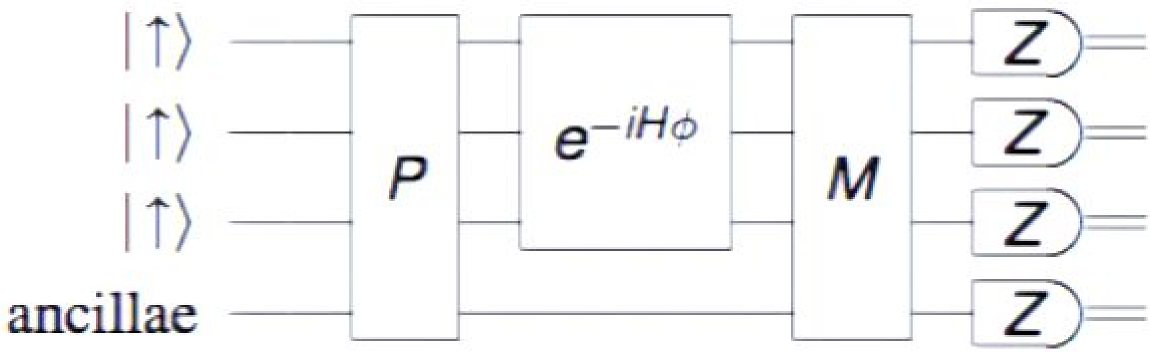


Figure: Generalized phase estimation.

Generalized phase estimation

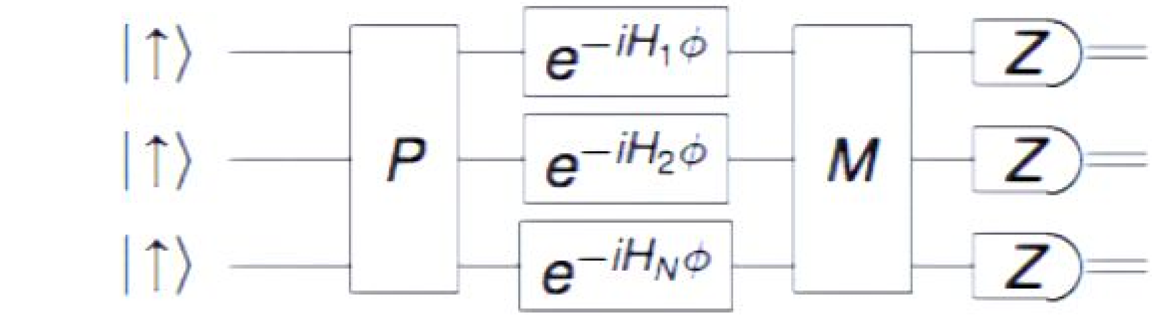


Figure: Standard phase estimation.

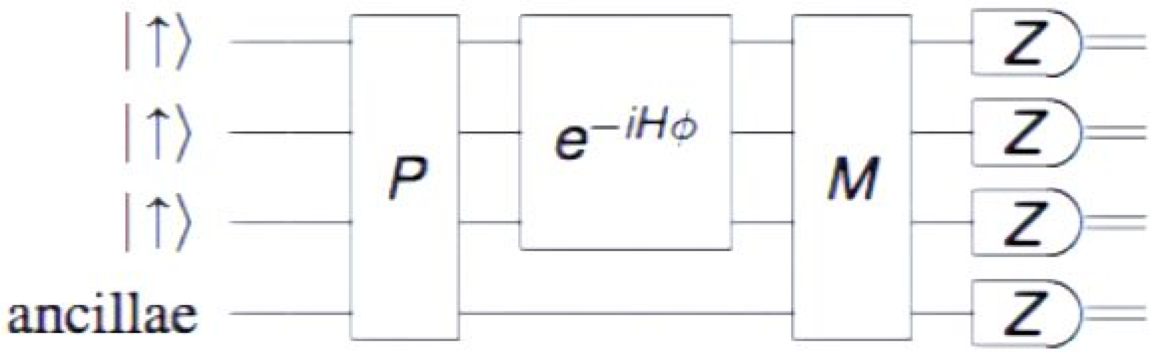
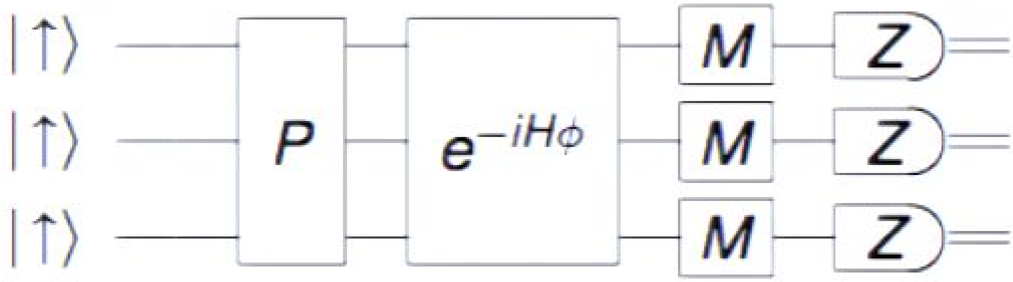


Figure: Generalized phase estimation.

Ancillae do not help

Using Fisher information, $e^{i(\phi H + H_A(t))} \simeq e^{i\phi H}$.

Entangled generalized phase estimation



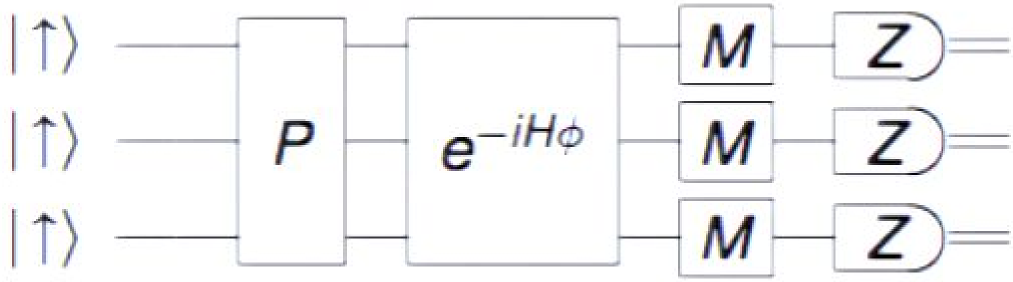
SB, Flammia, Caves and Geremia, PRL 98, 090401 (2007).

Figure: Entangled generalized phase estimation.

Use k-body couplings

$$H = \left(\sum_{j=1}^{N} H_j \right)^k = \sum_{j_1, j_2, \cdots, j_k} H_{j_1} H_{j_2} \cdots H_{j_k} \quad \text{has } N^k \text{ terms} .$$

Entangled generalized phase estimation



SB, Flammia, Caves and Geremia, PRL 98, 090401 (2007).

Figure: Entangled generalized phase estimation.

Use k-body couplings

$$H = \left(\sum_{j=1}^{N} H_j\right)^k = \sum_{j_1, j_2, \cdots, j_k} H_{j_1} H_{j_2} \cdots H_{j_k} \quad \text{has } N^k \text{ terms} .$$

Generalized scalings (QCR)

$$\delta\omega \geq \frac{1}{\sqrt{\nu}} \frac{1}{t\Delta H} \geq \frac{1}{\sqrt{\nu}} \frac{1}{t\|H\|_s} \text{ where } \|H\|_s = \Lambda_{\max} - \Lambda_{\min} = \mathcal{O}(N^k) .$$

Entangled states and separable measurements.

Product state generalized estimation

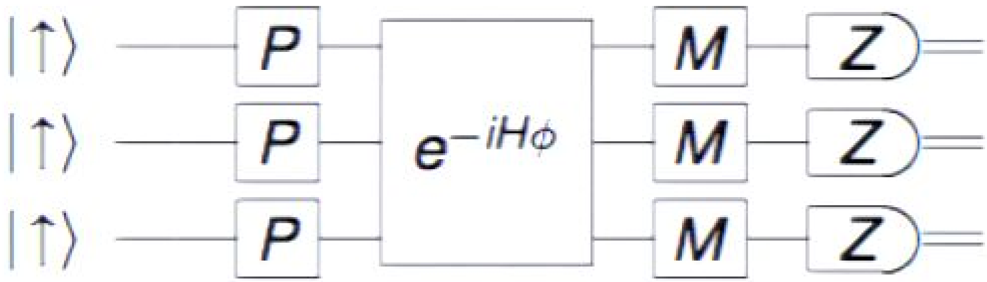


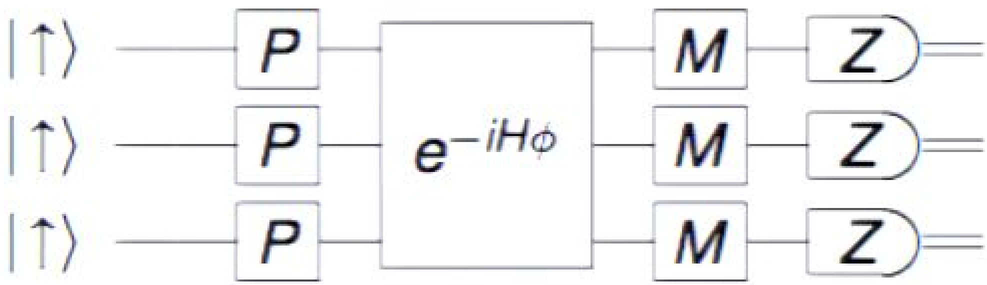
Figure: Product state generalized estimation.

SB, Datta, Flammia, Shaji, Bagan and Caves, arxiv:0710.0285.

Example: $H = (\sum Z_j/2)^k = J_z^k$, measurement J_y, and $\langle J_y \rangle = 0$,

$$\left.\begin{array}{ll} \langle J_y \rangle_t & \approx \phi k \langle J_x \rangle \langle J_z \rangle^{k-1} + \mathcal{O}(\phi^2) \\ (\Delta J_y)_t & \approx \sqrt{N}/2 + \mathcal{O}(\phi^2) \end{array}\right\} \delta\phi \approx \frac{(\Delta J_y)_t}{|d\langle J_y \rangle_t / d\phi|} \approx \frac{1}{N^{k-1/2}} + \mathcal{O}(\phi) \, .$$

Product state generalized estimation



SB, Datta, Flammia, Shaji, Bagan and Caves, arxiv:0710.0285.

Figure: Product state generalized estimation.

Example: $H = (\sum Z_j/2)^k = J_z^k$, measurement J_y, and $\langle J_y \rangle = 0$,

$$\left.\begin{aligned} \langle J_y \rangle_t &\approx \phi k \langle J_x \rangle \langle J_z \rangle^{k-1} + \mathcal{O}(\phi^2) \\ (\Delta J_y)_t &\approx \sqrt{N}/2 + \mathcal{O}(\phi^2) \end{aligned}\right\} \delta\phi \approx \frac{(\Delta J_y)_t}{|d\langle J_y \rangle_t / d\phi|} \approx \frac{1}{N^{k-1/2}} + \mathcal{O}(\phi) \, .$$

Product state nonlinear protocol

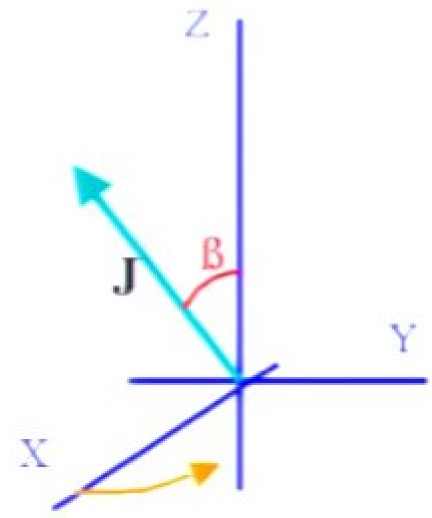
Cramér-Rao gives

$$\delta\omega \approx \frac{1}{tN^{k-1/2}} \, .$$

The initial entanglement always yields an improvement $1/\sqrt{N}$.

Product state quadratic (J_z^2) protocol

- Hamiltonian $J_z^2 = (\sum Z/2)^2$.
- Initial state $|\Psi_\beta\rangle = e^{-i\beta J_y}|J, J\rangle$.
- Measurements $J_{x,y}$.



$$|\Psi_\beta(t)\rangle = \sum_{m=-J}^{J} d^J_{mJ}(\beta) e^{-i\phi m^2} |J, m\rangle$$

$$\langle J_y \rangle_\phi = J \sin\beta\, r^{2J-1} \sin[(2J-1)\theta] \simeq J \sin\beta \sin(2J\phi\cos\beta)$$

$$\langle J_y^2 \rangle_\phi = \simeq \frac{J}{2}[1 - \sin^2\beta \sin^2(2J\phi\cos\beta)]$$

with

$$r = (1 - \sin^2\phi \sin^2\beta)^{1/2}$$

$$\theta = \tan^{-1}(\tan\phi\cos\beta)$$

Sensitivity

$$\delta\phi_{x,y} \simeq \frac{1}{\sqrt{2} J^{3/2} |\sin 2\beta|} .$$

SB, Datta, Flammia, Shaji, Bagan and Caves, arxiv:0710.0285. Partner, Black, and Geremia, arxiv:0708.2730. Rey, Jiang, and Lukin, arxiv:0706.3376. Choi and Sundaram, arxiv:0709.3842

Product state quadratic (J_z^2) protocol

Figure: Sensitivity (solid red lines) vs. ϕ for $\beta = \pi/4$: (a) J_x measurements; (b) J_y measurements. $J = 200$. Green line is $1/\sqrt{2}J^{3/2}$. Dashed blue is J_x measurements when $\beta = \pi/2$.

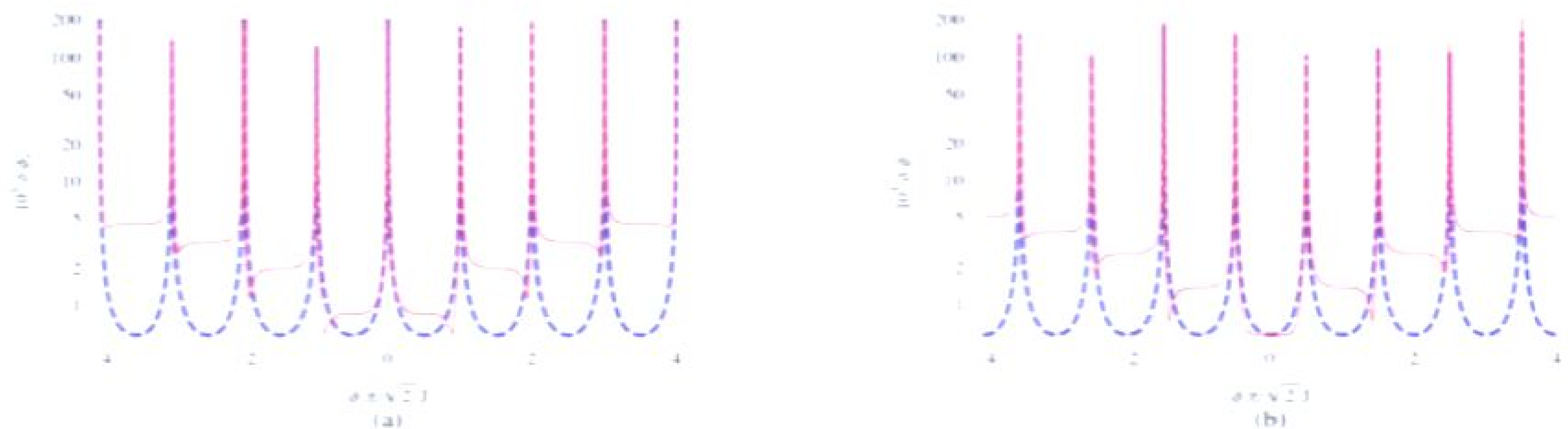


Figure: Central fringes for $J = 2\,500$. Dashed blue lines are coherent state approximations.

Nonlinear protocol decoherence

Uncorrelated dephasing decoherence $\dot{\rho} = \Gamma/2(Z\rho Z - \rho)$.

$$\langle J_{x,y} \rangle_\Gamma = e^{-\Gamma t} \langle J_{x,y} \rangle_0 \ ,$$

$$(\Delta J_{x,y})^2_\Gamma = e^{-2\Gamma t} (\Delta J_{x,y})^2_0 + \frac{J}{2}(1 - e^{-2\Gamma t}) \ ,$$

$$\delta\gamma^2_\Gamma = \delta\gamma^2 \left(1 + \frac{J(e^{2\Gamma t} - 1)}{2(\Delta J_{x,y})^2_0} \right) \ .$$

$T = \nu t$ total time available for measurements.
Optimal $t = (2\Gamma)^{-1}$, gives

Scaling with decoherence

$$\delta\gamma_\Gamma = \sqrt{\frac{e\Gamma}{T}} \frac{1}{J^{3/2} |\sin 2\beta|} \ .$$

Same as with entanglement.

SB, Datta, Flammia, Shaji, Bagan and Caves, arxiv:0710.0285. Huelga et. al. PRL 79, 3865 (1997).

What about generated entanglement?

Linear entropy $L(\rho) = 1 - \mathrm{Tr}\left(\rho^2\right)$.

$$L(\rho) = 1 - \sum_{a=0}^{N_A} \binom{N_A}{a} p^a (1-p)^{N_A - a} \sum_{b=0}^{N_A} \binom{N_A}{b} p^b (1-p)^{N_A - b} R^{N_B}(b-a)$$

with $p = \cos^2(\beta/2)$,

$$R^{N_B}(b-a) = \left(1 - \sin^2(\beta)\sin^2(\phi(b-a))\right)^{N_B} \approx 1 - N_B \sin^2(\beta)\phi^2(b-a)^2$$

Linear entropy

To first order

$$L(\rho) = \frac{1}{2} N_A N_B \phi^2 \sin^4(\beta).$$

Small in the region of estimation. $(\phi \lesssim 1/J)$

Conclusions

1. Sequential and entangled protocols are formally related, and have identical decoherence. What's the use of entanglement?
2. The sequential protocol can (often) be implemented with DQC1.
3. Nonlinear Hamiltonians give better metrology.
4. Nonlinear separable metrology is robust, and doesn't seemingly need entanglement.

Product state generalized estimation

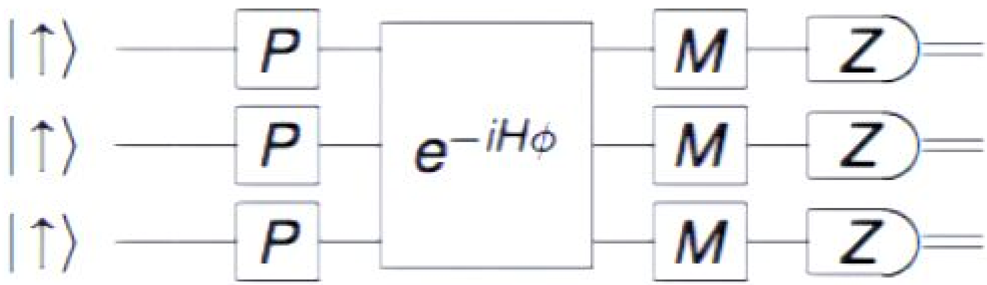


Figure: Product state generalized estimation.

SB, Datta, Flammia, Shaji, Bagan and Caves, arxiv:0710.0285.

Example: $H = (\sum Z_j/2)^k = J_z^k$, measurement J_y, and $\langle J_y \rangle = 0$,

$$\left.\begin{array}{ll} \langle J_y \rangle_t & \approx \phi k \langle J_x \rangle \langle J_z \rangle^{k-1} + \mathcal{O}(\phi^2) \\ (\Delta J_y)_t & \approx \sqrt{N}/2 + \mathcal{O}(\phi^2) \end{array}\right\} \delta\phi \approx \frac{(\Delta J_y)_t}{|d\langle J_y \rangle_t / d\phi|} \approx \frac{1}{N^{k-1/2}} + \mathcal{O}(\phi) \, .$$

Conclusions

1. Sequential and entangled protocols are formally related, and have identical decoherence. What's the use of entanglement?
2. The sequential protocol can (often) be implemented with DQC1.
3. Nonlinear Hamiltonians give better metrology.
4. Nonlinear separable metrology is robust, and doesn't seemingly need entanglement.