Title: Bell inequalities from random variables

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Abstract: We derive a set of Bell inequalities using correlated random variables. Our inequalities are necessary conditions for the existence of a local realistic description of projective measurements on qubits. We analyze our inequalities for the case of two qubits and find that they are equivalent to the well known CHSH inequalities. We also discuss the sufficiency of our inequalities as well as their applicability to more than two qubits.

Bell inequalities from random variables

by Matthew B. Elliott
University of New Mexico

Presented at the Perimeter Institute
for Theoretical Physics
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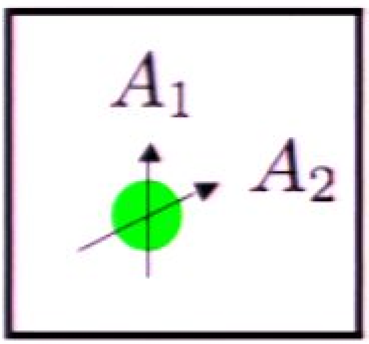
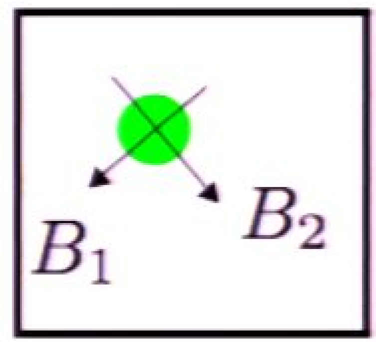
Local Realism



Alice

A	1	-1	$\cdots$
B	1	1	$\cdots$
AB	1	-1	$\cdots$



Bob

Repeat …

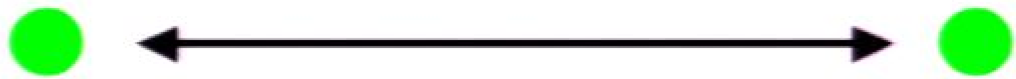
$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle$$

Bell Inequalities

J. Bell, Physics **1** 169 (1964).

Realism

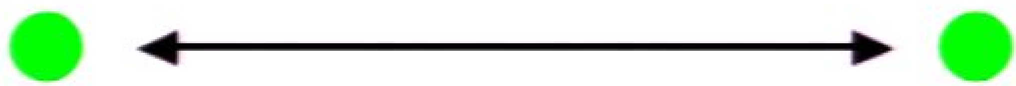
A_1	-1
A_2	1
$\vdots$	$\vdots$



B_1	1
B_2	-1
$\vdots$	$\vdots$

$\vdots$

A_1	-1
A_2	-1
$\vdots$	$\vdots$



B_1	-1
B_2	1
$\vdots$	$\vdots$

Locality $\Rightarrow$ Tables do not change unless particles interact

How to create the tables: standard picture

Probability distribution:

$$p(a_1, a_2, \ldots, b_1, b_2, \ldots)$$

A_1	-1
A_2	1
$\vdots$	$\vdots$

$$p(-1, 1, \ldots, 1, -1, \ldots)$$

B_1	1
B_2	-1
$\vdots$	$\vdots$

$\vdots$

A_1	-1
A_2	-1
$\vdots$	$\vdots$

$$p(-1, -1, \ldots, -1, 1, \ldots)$$

B_1	-1
B_2	1
$\vdots$	$\vdots$

$$Pr(A_1 B_1 \to +1) = \sum_{j \neq 1, k \neq 1} \sum_{a_j, b_k = 0}^{1} (p(1, a_2, \ldots, 1, b_2, \ldots) + p(-1, a_2, \ldots, -1, b_2, \ldots))$$

How to create the tables: random variables

A_1	$\mathcal{R}_1^A$
A_2	$\mathcal{R}_2^A$
$\vdots$	$\vdots$

B_1	$\mathcal{R}_1^B$
B_2	$\mathcal{R}_2^B$
$\vdots$	$\vdots$

A_1	-1
A_2	1
$\vdots$	$\vdots$

B_1	1
B_2	-1
$\vdots$	$\vdots$

$\vdots$

A_1	-1
A_2	-1
$\vdots$	$\vdots$

B_1	-1
B_2	1
$\vdots$	$\vdots$

$$\mathcal{R}_1^A \mathcal{R}_1^B = \mathcal{R}_{11}^{AB} \leftrightarrow A_1 B_1$$

How to create the tables: standard picture

Probability distribution:

$$p(a_1, a_2, \ldots, b_1, b_2, \ldots)$$

A_1	-1
A_2	1
$\vdots$	$\vdots$

$$p(-1, 1, \ldots, 1, -1, \ldots)$$

B_1	1
B_2	-1
$\vdots$	$\vdots$

$\vdots$

A_1	-1
A_2	-1
$\vdots$	$\vdots$

$$p(-1, -1, \ldots, -1, 1, \ldots)$$

B_1	-1
B_2	1
$\vdots$	$\vdots$

$$Pr(A_1 B_1 \to +1) = \sum_{j \neq 1, k \neq 1} \sum_{a_j, b_k = 0}^{1} \left(p(1, a_2, \ldots, 1, b_2, \ldots) + p(-1, a_2, \ldots, -1, b_2, \ldots) \right)$$

(Aside on)

Random Variables

How to create the tables: random variables

A_1	$\mathcal{R}_1^A$
A_2	$\mathcal{R}_2^A$
$\vdots$	$\vdots$

B_1	$\mathcal{R}_1^B$
B_2	$\mathcal{R}_2^B$
$\vdots$	$\vdots$

A_1	-1
A_2	1
$\vdots$	$\vdots$

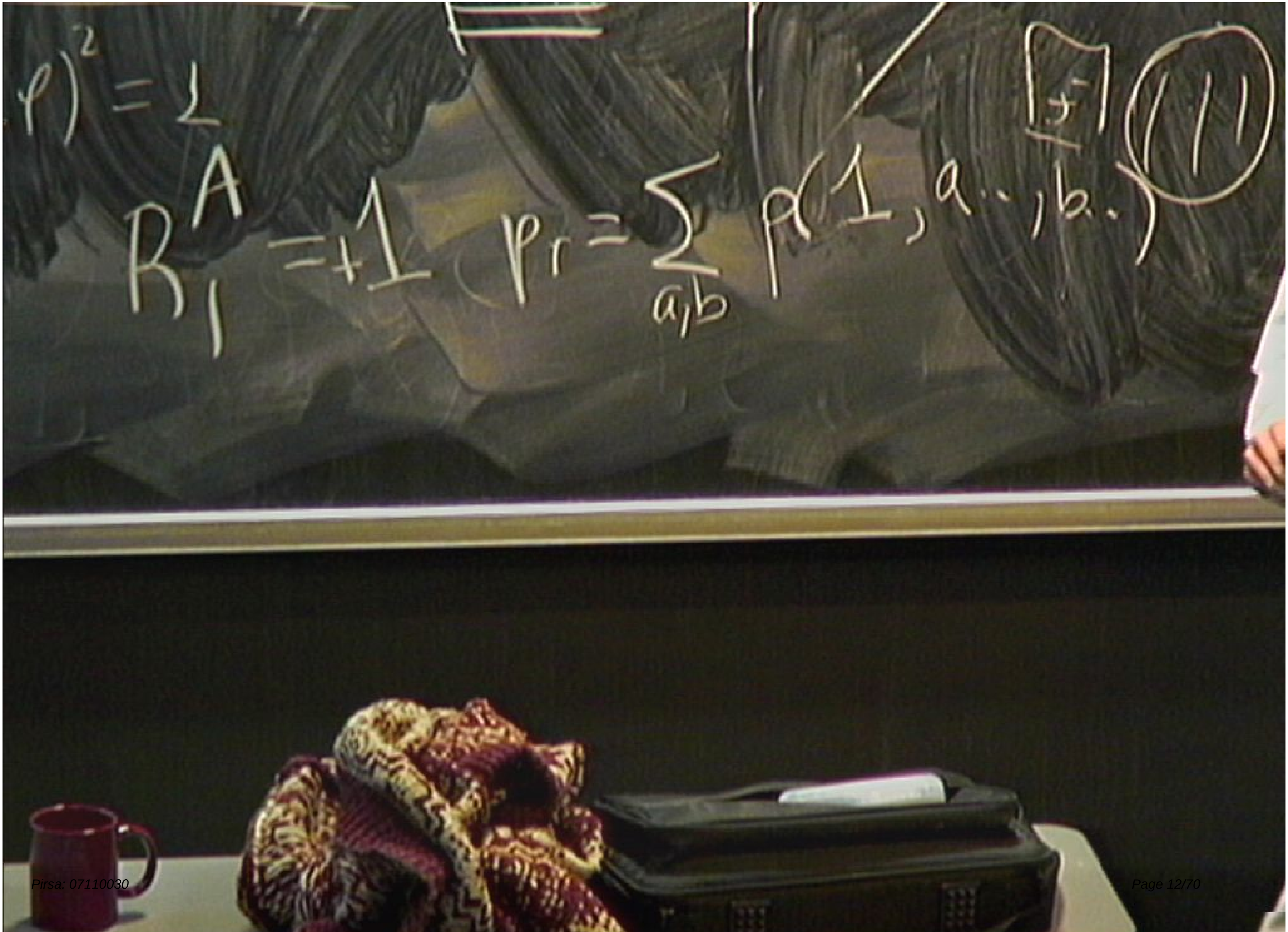
B_1	1
B_2	-1
$\vdots$	$\vdots$

$\vdots$

A_1	-1
A_2	-1
$\vdots$	$\vdots$

B_1	-1
B_2	1
$\vdots$	$\vdots$

$$\mathcal{R}_1^A \mathcal{R}_1^B = \mathcal{R}_{11}^{AB} \leftrightarrow A_1 B_1$$


$R_1^A = +1$
$\rho_1 = \sum_{a,b} \rho(1, a.., b..)$

(Aside on)

Random Variables

(Aside on)

Random Variables

Definition:

$$-1 \le a \le 1,\ \mathcal{R}(a) = \begin{cases} +1 & \text{with probability} \quad \frac{1}{2}(1+a) \\ -1 & \text{with probability} \quad \frac{1}{2}(1-a) \end{cases}$$

Average:

$$\overline{\mathcal{R}(a)} = (+1)\frac{1}{2}(1+a) + (-1)\frac{1}{2}(1-a) = a$$

Examples:

$\mathcal{R}(-1)$	$\mathcal{R}\left(-\frac{1}{2}\right)$	$\mathcal{R}(0)$	$\mathcal{R}\left(\frac{1}{2}\right)$	$\mathcal{R}(1)$
-1	-1	1	-1	1
-1	1	-1	1	1
-1	-1	-1	1	1
-1	-1	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p) \qquad \left(p = \overline{\mathcal{R}(a)\mathcal{R}(b)}\right)$$

Uncorrelated

$$\begin{aligned} p &= (+1)\left(\tfrac{1}{4}(1+a)(1+b) + \tfrac{1}{4}(1-a)(1-b)\right) \\ &\quad +(-1)\left(\tfrac{1}{4}(1+a)(1-b) + \tfrac{1}{4}(1-a)(1+b)\right) \\ &= (+1)\tfrac{1}{2}(1+ab) + (-1)\tfrac{1}{2}(1-ab) = ab \end{aligned}$$

Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p) \qquad \left(p = \overline{\mathcal{R}(a)\mathcal{R}(b)}\right)$$

Uncorrelated

$$\begin{aligned} p &= (+1)\left(\tfrac{1}{4}(1+a)(1+b) + \tfrac{1}{4}(1-a)(1-b)\right) \\ &\quad +(-1)\left(\tfrac{1}{4}(1+a)(1-b) + \tfrac{1}{4}(1-a)(1+b)\right) \\ &= (+1)\tfrac{1}{2}(1+ab) + (-1)\tfrac{1}{2}(1-ab) = ab \end{aligned}$$

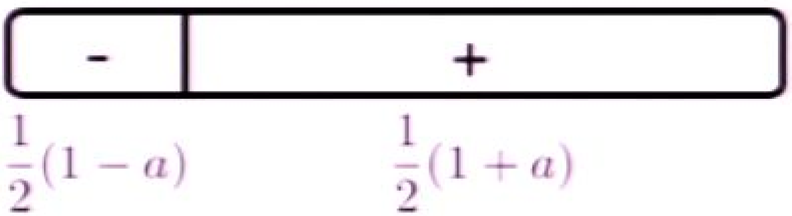
$$\boxed{\Rightarrow \mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(ab)}$$

Product of two random variables:

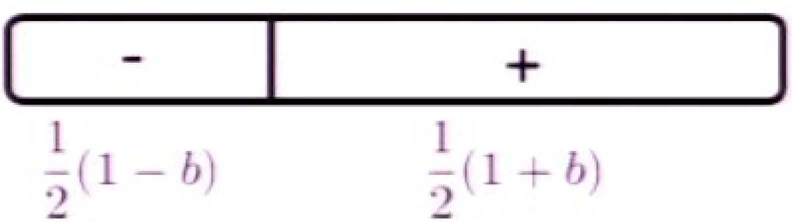
$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p)$$

Correlated

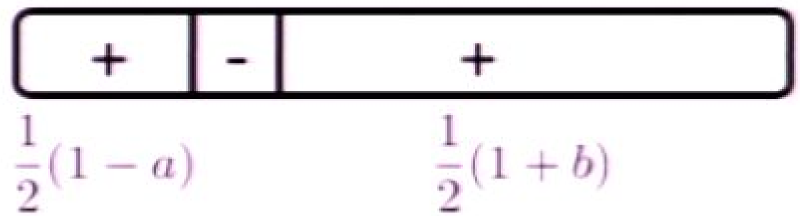
$\mathcal{R}(a)$



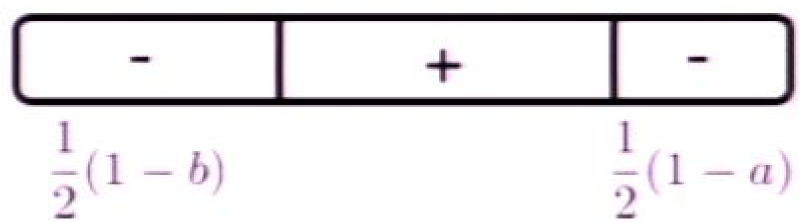
$\mathcal{R}(b)$



Correlated



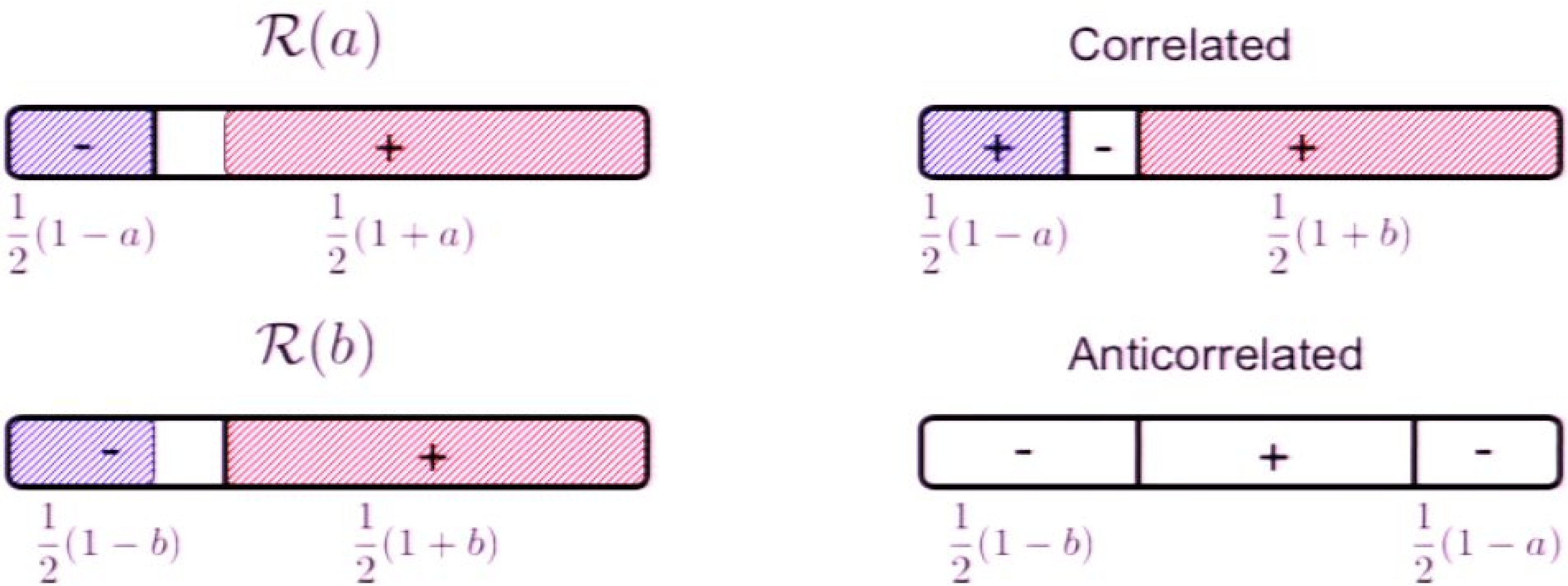
Anticorrelated



Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p)$$

Correlated

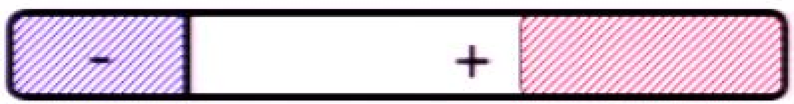


Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p)$$

Correlated

$\mathcal{R}(a)$

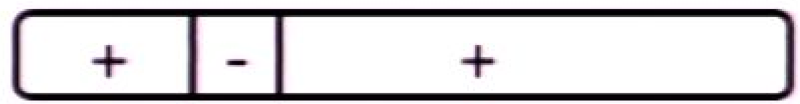


$\frac{1}{2}(1-a)$ $\frac{1}{2}(1+a)$

$\mathcal{R}(b)$

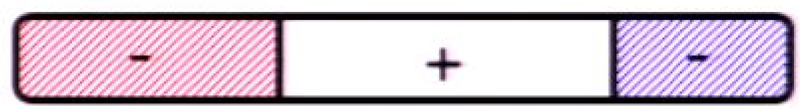
$\frac{1}{2}(1-b)$ $\frac{1}{2}(1+b)$

Correlated



$\frac{1}{2}(1-a)$ $\frac{1}{2}(1+b)$

Anticorrelated

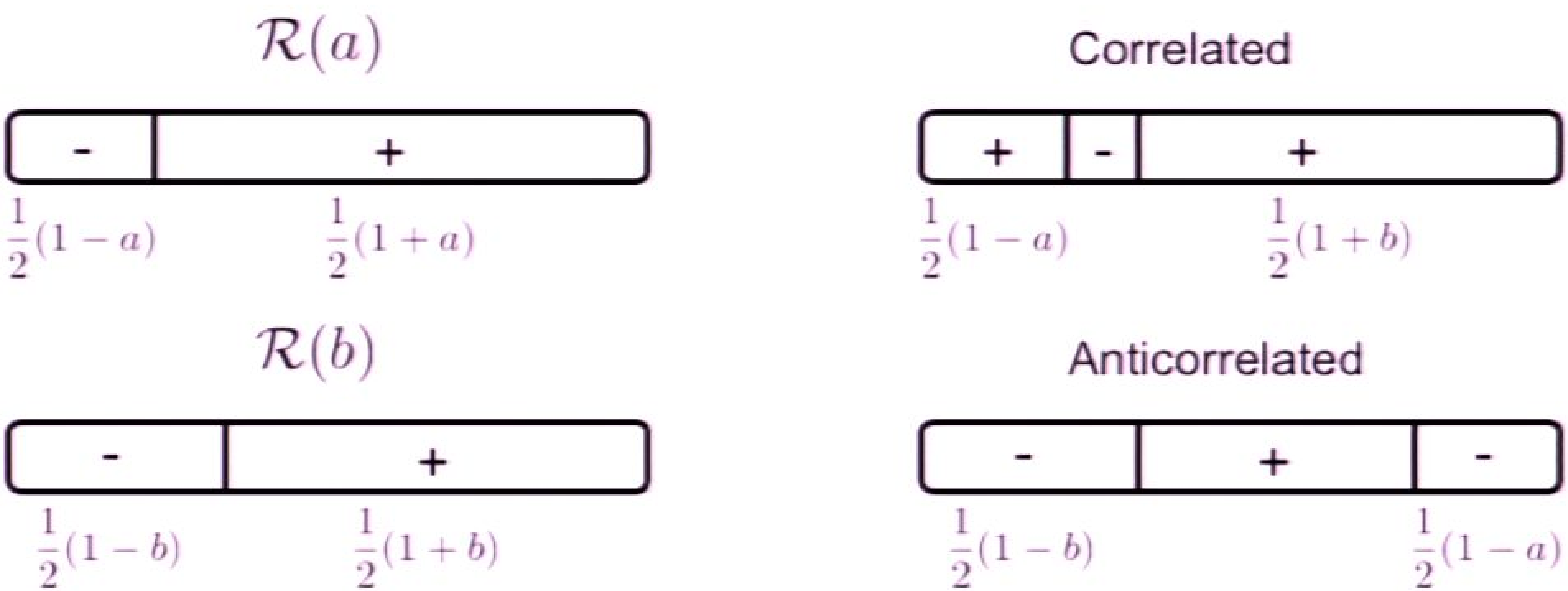


$\frac{1}{2}(1-b)$ $\frac{1}{2}(1-a)$

Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p)$$

Correlated



$$|a+b| - 1 \leq p \leq 1 - |a-b|$$

Product of many random variables:

Theorem:

Let $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$, with $a_1, a_2, \ldots, a_n \in [-1, 1]$.

Then $R(a_1) \cdots R(a_n) = R(p)$ is possible if and only if

$$\mathrm{p} \in [-1, 1] \text{ and}$$

$$(|a_1| + \cdots + |a_n|) - (n-1) \leq s_1 \cdots s_n p$$

$$\leq (n-1) - (|a_1| + \cdots + |a_{n-1}| - |a_n|),$$

where $s_j = a_j / |a_j|$ is the sign of $a_j \neq 0$.

Recall: $|a+b| - 1 \leq p \leq 1 - |a-b|$

(Back to)

Bell Inequalities

A_1	$\mathcal{R}(\langle A_1\rangle)$
A_2	$\mathcal{R}(\langle A_2\rangle)$
$\vdots$	$\vdots$

B_1	$\mathcal{R}(\langle B_1\rangle)$
B_2	$\mathcal{R}(\langle B_2\rangle)$
$\vdots$	$\vdots$

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$

$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$

$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_2 B_2\rangle)$$

$\Leftrightarrow$ Local realistic description exists

Nec. & Suff.

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_2 B_2\rangle)$$

$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(1)$$
$$\|$$
$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(p) = \mathcal{R}(a_1)\cdots\mathcal{R}(a_n) \Leftrightarrow$$

$$(|a_1| + \cdots + |a_n|) - (n-1) \leq s_1 \cdots s_n p$$

$$\leq (n-1) - (|a_1| + \cdots + |a_{n-1}| - |a_n|)$$

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(1)$$
$$\|$$
$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

$$\langle A_1 B_1\rangle| + |\langle A_1 B_2\rangle| + |\langle A_2 B_1\rangle| + |\langle A_2 B_2\rangle| - 3 \leq s_{11}s_{12}s_{21}s_{22}$$

$$\leq 3 - |\langle A_1 B_1\rangle| + |\langle A_1 B_2\rangle| + |\langle A_2 B_1\rangle| - |\langle A_2 B_2\rangle|$$

A	B	C	
			. . . . .
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(1)$$
$$\|$$
$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

$$\langle A_1 B_1\rangle| + |\langle A_1 B_2\rangle| + |\langle A_2 B_1\rangle| + |\langle A_2 B_2\rangle| - 3 \leq s_{11} s_{12} s_{21} s_{22}$$

$$\leq 3 - |\langle A_1 B_1\rangle| + |\langle A_1 B_2\rangle| + |\langle A_2 B_1\rangle| - |\langle A_2 B_2\rangle|$$

A	B	C	……
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

A B C ......

$$
\begin{array}{ccc}
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2 \\
\vdots & \vdots & \vdots \\
A_{\mathcal{M}_A} & B_{\mathcal{M}_B} & C_{\mathcal{M}_C}
\end{array}
$$

$$
\begin{array}{c}
\mathcal{R}(\langle A_j \rangle)\mathcal{R}(\langle B_k \rangle)\cdots = \mathcal{R}(\langle M_l \rangle) \\
\vdots
\end{array}
\Rightarrow \mathcal{R}(1) = \prod_{j=1}^{n} \mathcal{R}(M_j)
$$

Sufficient?

A	B	C	
●	●	●	. . . . . .
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

Necessary condition:

$$\mathcal{R}(\langle A_1 \rangle)\mathcal{R}(\langle B_1 \rangle) = \mathcal{R}(\langle A_1 B_1 \rangle)$$
$$\mathcal{R}(\langle A_1 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_1 B_2 \rangle)$$
$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_1 \rangle) = \mathcal{R}(\langle A_2 B_1 \rangle)$$
$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_2 B_2 \rangle)$$

$$\mathcal{R}(1)$$
$$\|$$
$$1 = \mathcal{R}(\langle A_1 B_1 \rangle)\mathcal{R}(\langle A_1 B_2 \rangle)\mathcal{R}(\langle A_2 B_1 \rangle)\mathcal{R}(\langle A_2 B_2 \rangle)$$

$$\langle A_1 B_1 \rangle| + |\langle A_1 B_2 \rangle| + |\langle A_2 B_1 \rangle| + |\langle A_2 B_2 \rangle| - 3 \leq s_{11} s_{12} s_{21} s_{22}$$

$$\leq 3 - |\langle A_1 B_1 \rangle| + |\langle A_1 B_2 \rangle| + |\langle A_2 B_1 \rangle| - |\langle A_2 B_2 \rangle|$$

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_2\rangle) = \mathcal{R}(\langle A_1 B_2\rangle)$$
$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
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$$\mathcal{R}(1)$$
$$\|$$
$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(p) = \mathcal{R}(a_1)\cdots\mathcal{R}(a_n) \Leftrightarrow$$

$$(|a_1| + \cdots + |a_n|) - (n-1) \leq s_1 \cdots s_n p$$

$$\leq (n-1) - (|a_1| + \cdots + |a_{n-1}| - |a_n|)$$

A_1	$\mathcal{R}(\langle A_1 \rangle)$
A_2	$\mathcal{R}(\langle A_2 \rangle)$
$\vdots$	$\vdots$

B_1	$\mathcal{R}(\langle B_1 \rangle)$
B_2	$\mathcal{R}(\langle B_2 \rangle)$
$\vdots$	$\vdots$

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$$\mathcal{R}(\langle A_1 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_1 B_2 \rangle)$$

$\Leftrightarrow$ Local realistic description exists

$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_1 \rangle) = \mathcal{R}(\langle A_2 B_1 \rangle)$$

$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_2 B_2 \rangle)$$

Nec. & Suff.

Necessary condition:

$$\mathcal{R}(\langle A_1 \rangle)\mathcal{R}(\langle B_1 \rangle) = \mathcal{R}(\langle A_1 B_1 \rangle)$$
$$\mathcal{R}(\langle A_1 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_1 B_2 \rangle)$$
$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_1 \rangle) = \mathcal{R}(\langle A_2 B_1 \rangle)$$
$$\mathcal{R}(\langle A_2 \rangle)\mathcal{R}(\langle B_2 \rangle) = \mathcal{R}(\langle A_2 B_2 \rangle)$$

$$1 = \mathcal{R}(\langle A_1 B_1 \rangle)\mathcal{R}(\langle A_1 B_2 \rangle)\mathcal{R}(\langle A_2 B_1 \rangle)\mathcal{R}(\langle A_2 B_2 \rangle)$$

Necessary condition:

$$\mathcal{R}(\langle A_1\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_1 B_1\rangle)$$
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$$\mathcal{R}(\langle A_2\rangle)\mathcal{R}(\langle B_1\rangle) = \mathcal{R}(\langle A_2 B_1\rangle)$$
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$$\mathcal{R}(1)$$
$$||$$
$$1 = \mathcal{R}(\langle A_1 B_1\rangle)\mathcal{R}(\langle A_1 B_2\rangle)\mathcal{R}(\langle A_2 B_1\rangle)\mathcal{R}(\langle A_2 B_2\rangle)$$

$$\mathcal{R}(p) = \mathcal{R}(a_1)\cdots\mathcal{R}(a_n) \Leftrightarrow$$

$$(|a_1| + \cdots + |a_n|) - (n-1) \leq s_1 \cdots s_n p$$

$$\leq (n-1) - (|a_1| + \cdots + |a_{n-1}| - |a_n|)$$

A	B	C	......
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

A	B	C	......
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

$$\mathcal{R}(\langle A_j \rangle)\mathcal{R}(\langle B_k \rangle)\cdots = \mathcal{R}(\langle M_l \rangle) \quad \vdots \quad \Rightarrow \mathcal{R}(1) = \prod_{j=1}^{n} \mathcal{R}(M_j)$$

Sufficient?

A B C . . . . . .

A_1 B_1 C_1

A_2 B_2 C_2

$\vdots$ $\vdots$ $\vdots$

$A_{\mathcal{M}_A}$ $B_{\mathcal{M}_B}$ $C_{\mathcal{M}_C}$

$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$

A B C ......

$$\begin{matrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ \vdots & \vdots & \vdots \\ A_{\mathcal{M}_A} & B_{\mathcal{M}_B} & C_{\mathcal{M}_C} \end{matrix}$$

$$\mathcal{R}(\langle A_j \rangle)\mathcal{R}(\langle B_k \rangle)\cdots = \mathcal{R}(\langle M_l \rangle) \\ \vdots$$

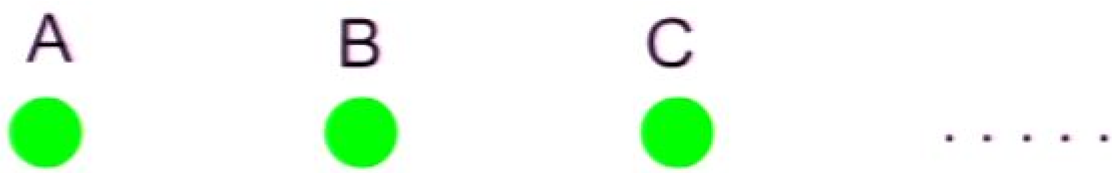
$$\Rightarrow \mathcal{R}(1) = \prod_{j=1}^{n} \mathcal{R}(M_j)$$

Sufficient?

A	B	C	......
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$



$$\begin{array}{ccc} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ \vdots & \vdots & \vdots \\ A_{\mathcal{M}_A} & B_{\mathcal{M}_B} & C_{\mathcal{M}_C} \end{array}$$

$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$

$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

A B C

$$\begin{array}{ccc} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ \vdots & \vdots & \vdots \\ A_{\mathcal{M}_A} & B_{\mathcal{M}_B} & C_{\mathcal{M}_C} \end{array}$$

$$\begin{array}{c} \mathcal{R}(\langle A_j \rangle)\mathcal{R}(\langle B_k \rangle)\cdots = \mathcal{R}(\langle M_l \rangle) \\ \vdots \end{array} \Rightarrow \mathcal{R}(1) = \prod_{j=1}^{n} \mathcal{R}(M_j)$$

Sufficient?

A	B	C	$\cdots\cdots$
A_1	B_1	C_1	
A_2	B_2	C_2	
$\vdots$	$\vdots$	$\vdots$	
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$	$C_{\mathcal{M}_C}$	

$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$

$$
\begin{array}{lll}
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2 \\
\vdots & \vdots & \vdots \\
A_{\mathcal{M}_A} & B_{\mathcal{M}_B} & C_{\mathcal{M}_C}
\end{array}
$$

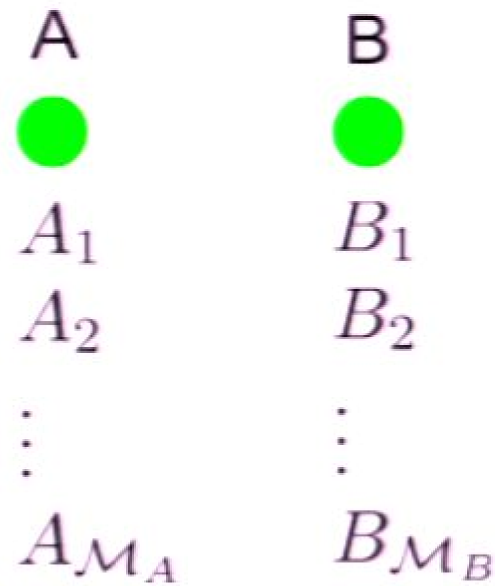
$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$

$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

Example 1: Two qubits.

$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$



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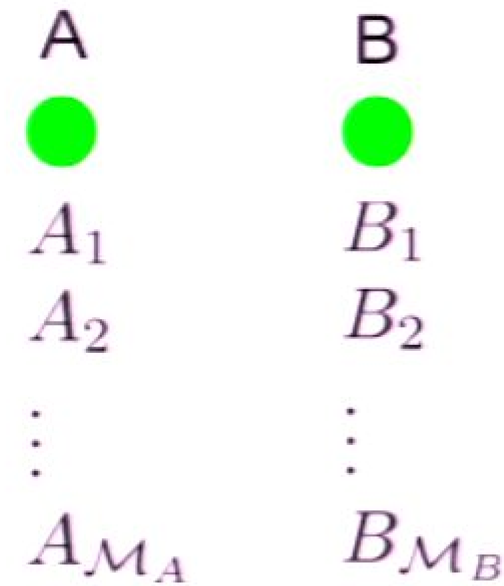

n=3:

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \le 1$$ ✔

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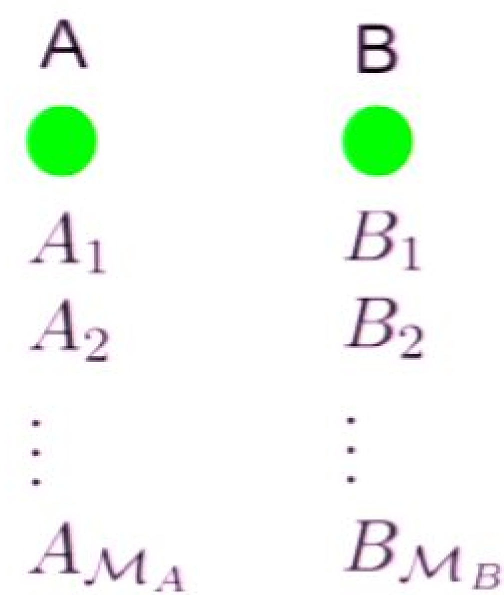
$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \le n-2$$



<u>n=3:</u>

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \le 1$$ ✔

<u>n=4:</u>

$$|\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \le 2$$ CHSH

Example 1: Two qubits.

$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

A B

$$A_1 \quad B_1$$
$$A_2 \quad B_2$$
$$\vdots \quad \vdots$$
$$A_{\mathcal{M}_A} \quad B_{\mathcal{M}_B}$$

<u>n=3:</u>

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \leq 1$$ ✔

<u>n=4:</u>

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<u>Easy:</u> $$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2 \Rightarrow \begin{array}{c} |\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| \\ - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2 \end{array}$$

<u>Hard:</u> $$\begin{array}{c} |\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| \\ - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2 \end{array} \Rightarrow \sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

Conjecture for k qubits: consider only up to $n=2^k$.

Example 1: Two qubits.

A B

A_1 B_1

A_2 B_2

$\vdots$ $\vdots$

$A_{\mathcal{M}_A}$ $B_{\mathcal{M}_B}$

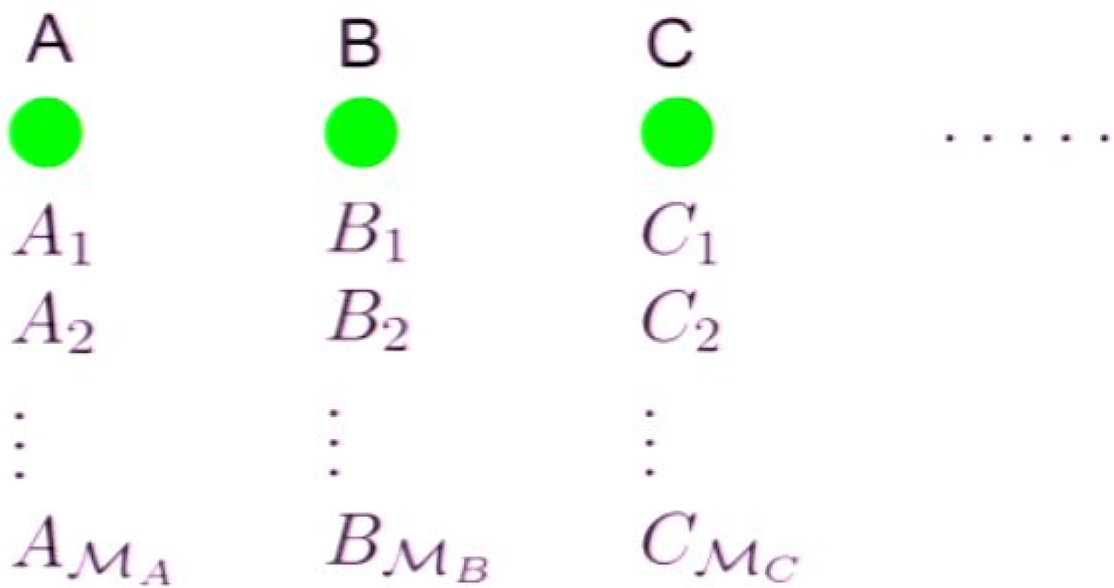
$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

n=3:

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \leq 1$$ ✔

n=4:

$$|\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2$$ CHSH



$$\sum_{j=1}^{n} |\langle M_j \rangle| - (n-1) \leq \prod_{j=1}^{n} s_j$$

$$\leq (n-1) - \sum_{j=1}^{n-1} |\langle M_j \rangle| + |\langle M_n \rangle|$$

$$\boxed{\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2}$$

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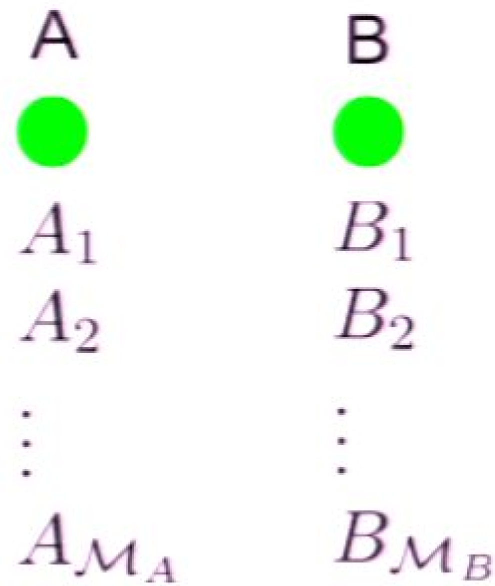

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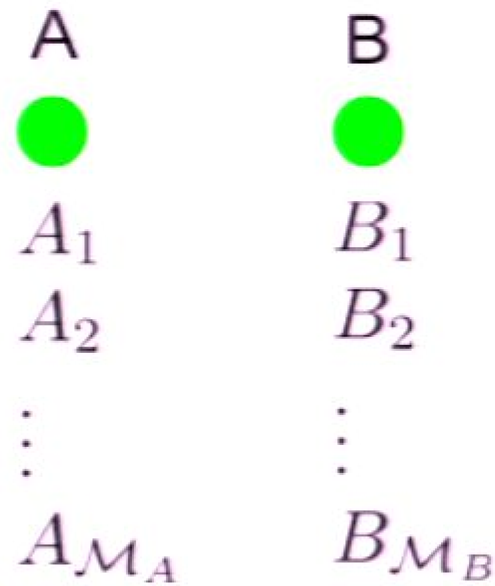

n=3:

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \leq 1$$ ✔

n=4:

$$|\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2$$ CHSH

Example 1: Two qubits.

$$\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$$

A	B
A_1	B_1
A_2	B_2
$\vdots$	$\vdots$
$A_{\mathcal{M}_A}$	$B_{\mathcal{M}_B}$

<u>n=3:</u>

$$|\langle A_j \rangle| + |\langle B_k \rangle| - s_{A_j} s_{B_k} s_{A_j B_k} |\langle A_j B_k \rangle| \leq 1$$ ✔

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<u>Easy:</u> $\sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2 \Rightarrow \begin{array}{c} |\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| \\ - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2 \end{array}$

<u>Hard:</u> $\begin{array}{c} |\langle A_i B_k \rangle| + |\langle A_i B_l \rangle| + |\langle A_j B_k \rangle| \\ - s_{ik} s_{il} s_{jk} s_{jl} |\langle A_j B_l \rangle| \leq 2 \end{array} \Rightarrow \sum_{j=1}^{n-1} |\langle M_j \rangle| - \left(\prod_{j=1}^{n} s_j \right) |\langle M_n \rangle| \leq n-2$

Conjecture for k qubits: consider only up to $n=2^k$.

Example 2: Insufficient for arbitrary correlations.

$$\mathcal{R}_1\left(-\tfrac{1}{3}\right)\mathcal{R}_2\left(-\tfrac{1}{3}\right)=\mathcal{R}_4\left(-\tfrac{1}{3}\right)$$

$$\mathcal{R}_1\left(-\tfrac{1}{3}\right)\mathcal{R}_3\left(-\tfrac{1}{3}\right)=\mathcal{R}_5\left(-\tfrac{1}{3}\right) \quad \Rightarrow \quad \mathcal{R}_1\left(-\frac{1}{3}\right)\mathcal{R}_2\left(-\frac{1}{3}\right)\mathcal{R}_3\left(-\frac{1}{3}\right)=\mathcal{R}\left(p\right)$$

$$\mathcal{R}_2\left(-\tfrac{1}{3}\right)\mathcal{R}_3\left(-\tfrac{1}{3}\right)=\mathcal{R}_6\left(-\tfrac{1}{3}\right)$$

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-2\leq -p\leq 2-\frac{1}{3}-\frac{1}{3}+\frac{1}{3}$$

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$\mathcal{R}_1\left(-\frac{1}{3}\right)$

$\mathcal{R}_2\left(-\frac{1}{3}\right)$

$\mathcal{R}_3\left(-\frac{1}{3}\right)$

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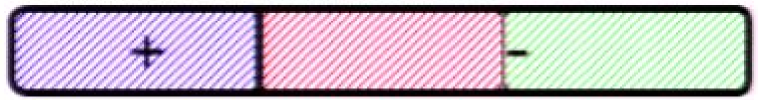

$$\mathcal{R}_1\left(-\frac{1}{3}\right)$$

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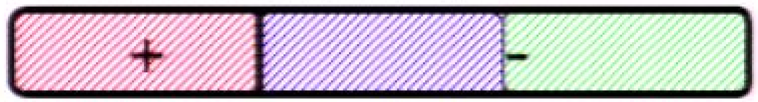
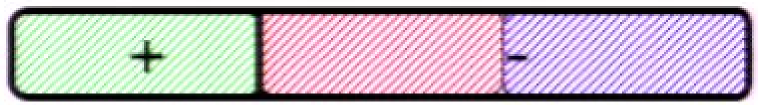


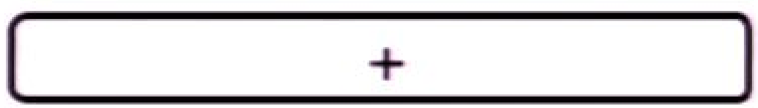
$$\mathcal{R}_2\left(-\frac{1}{3}\right)$$

$\mathcal{R}_2\left(-\frac{1}{3}\right)\mathcal{R}_3\left(-\frac{1}{3}\right) = \mathcal{R}_6\left(-\frac{1}{3}\right)$



$$\mathcal{R}_3\left(-\frac{1}{3}\right)$$

$$\mathcal{R}_1\left(-\frac{1}{3}\right)\mathcal{R}_2\left(-\frac{1}{3}\right)\mathcal{R}_3\left(-\frac{1}{3}\right)$$



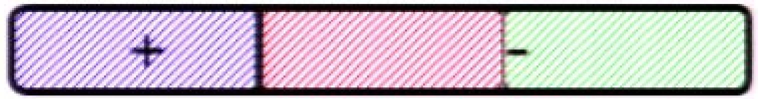
Comments:

- Arbitrary number of qubits
- Arbitrary number of measurements
- Two qubits: CHSH.

The big question:

- Sufficient for quantum correlations?

Example 2: Insufficient for arbitrary correlations.



$$\mathcal{R}_1\left(-\frac{1}{3}\right)$$

$$\mathcal{R}_1\left(-\tfrac{1}{3}\right)\mathcal{R}_2\left(-\tfrac{1}{3}\right)=\mathcal{R}_4\left(-\tfrac{1}{3}\right)$$

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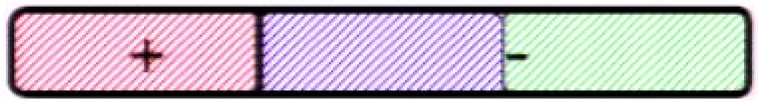

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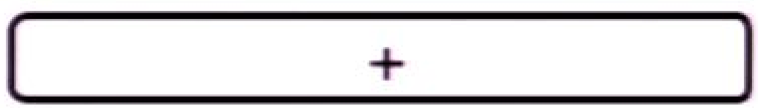



All
QM

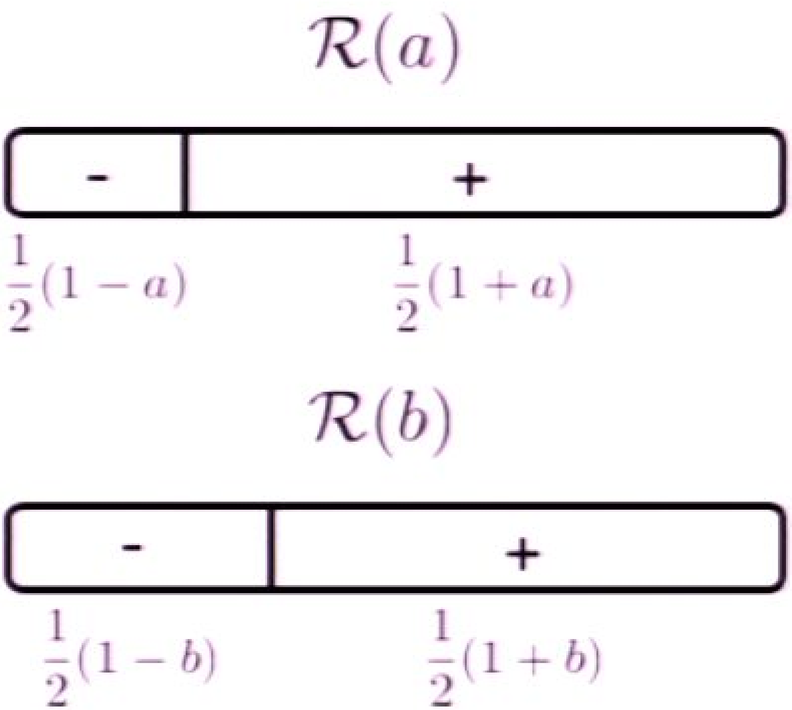

All
QM

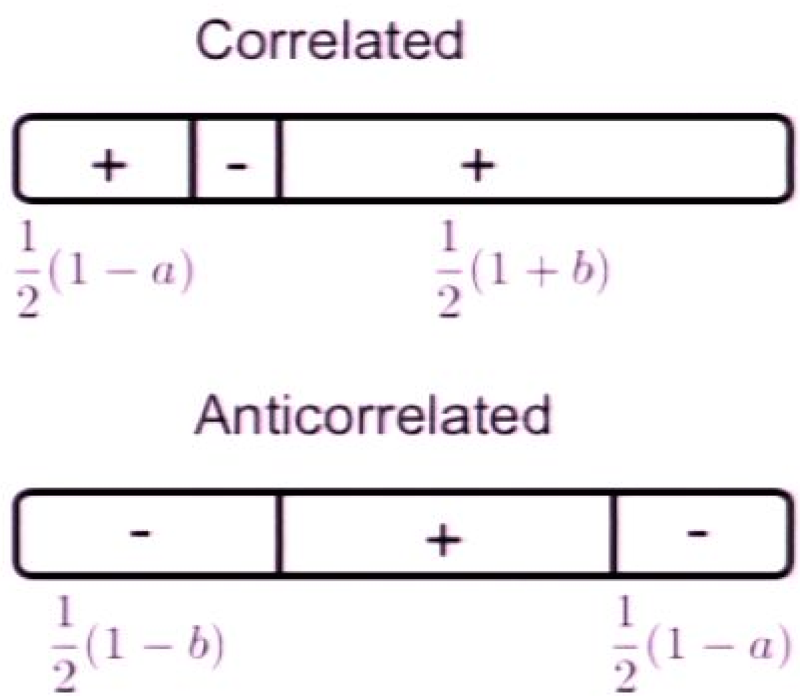

All
QM

Product of two random variables:

$$\mathcal{R}(a)\mathcal{R}(b) = \mathcal{R}(p)$$

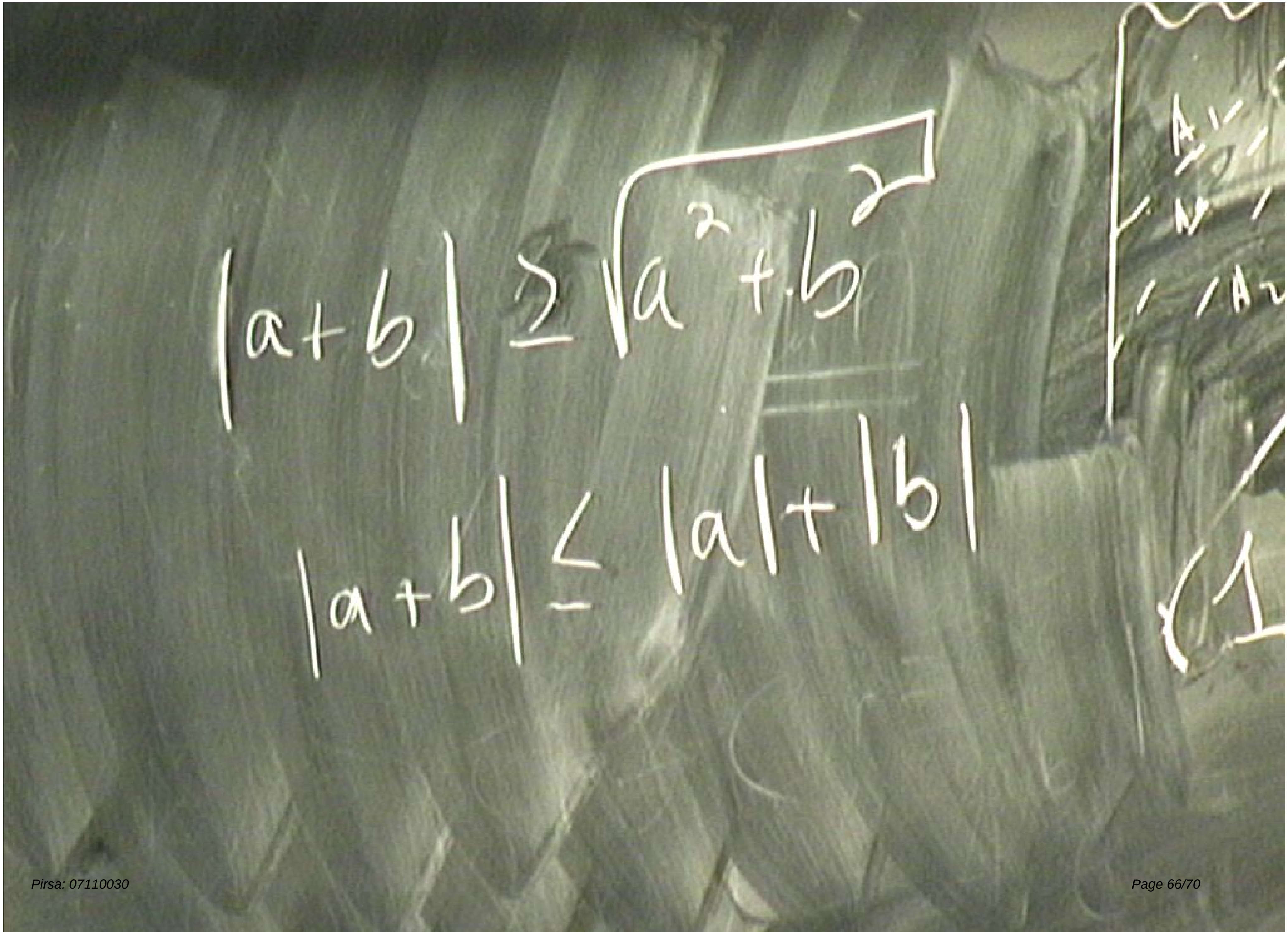
Correlated





$$|a+b| - 1 \leq p \leq 1 - |a-b|$$

$$|a+b| \leq a^2+b^2$$


$|a+b| \geq \sqrt{a^2+b^2}$
$|a+b| \leq |a|+|b|$

Product of many random variables:

Theorem:

Let $|a_1| \geq | a_2| \geq \cdots \geq |a_n|$, with $a_1, a_2, \ldots, a_n \in [-1, 1]$.

Then $R(a_1) \cdots R(a_n) = R(p)$ is possible if and only if

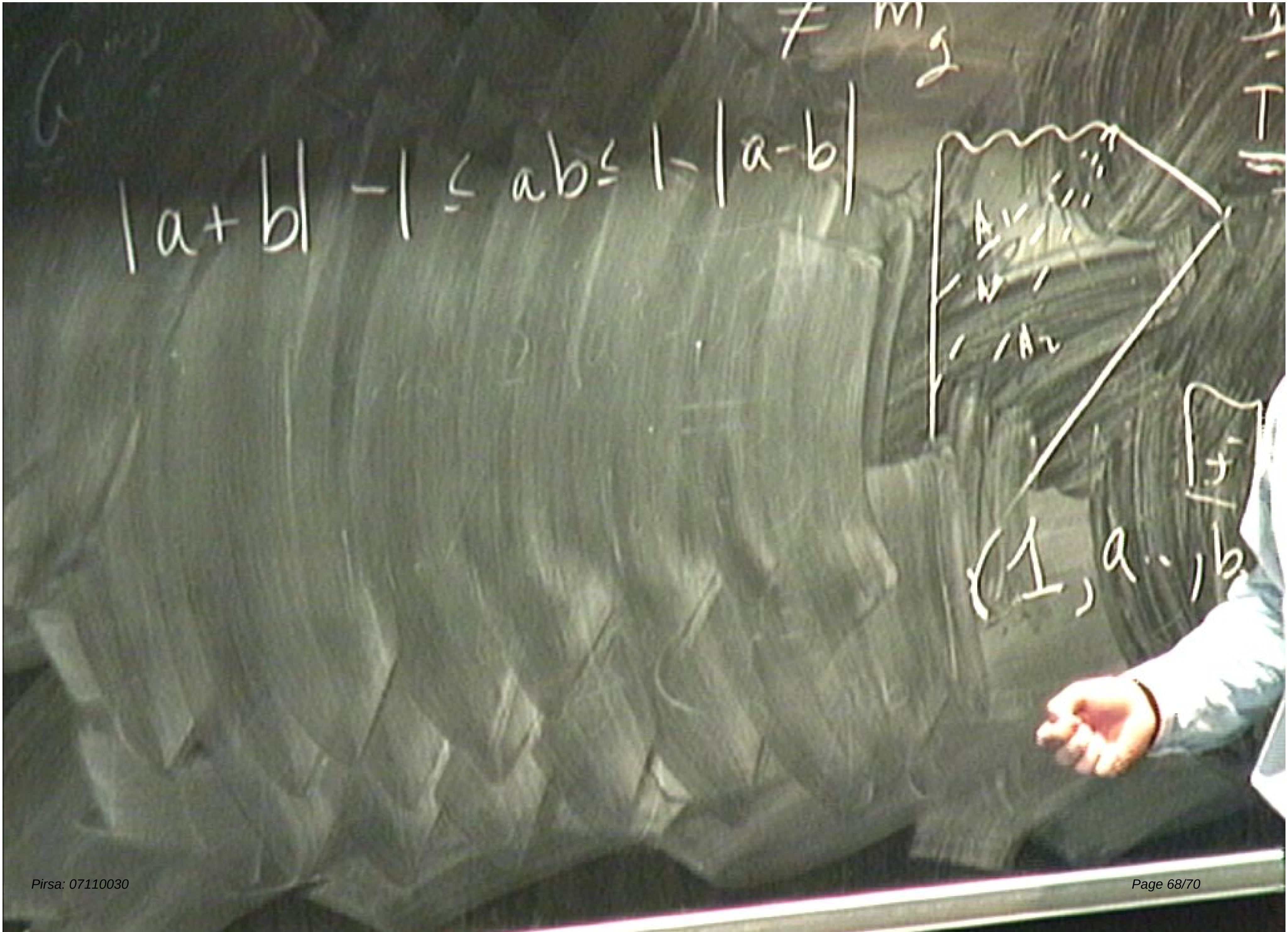
$$\mathrm{p} \in [-1, 1] \text{ and}$$

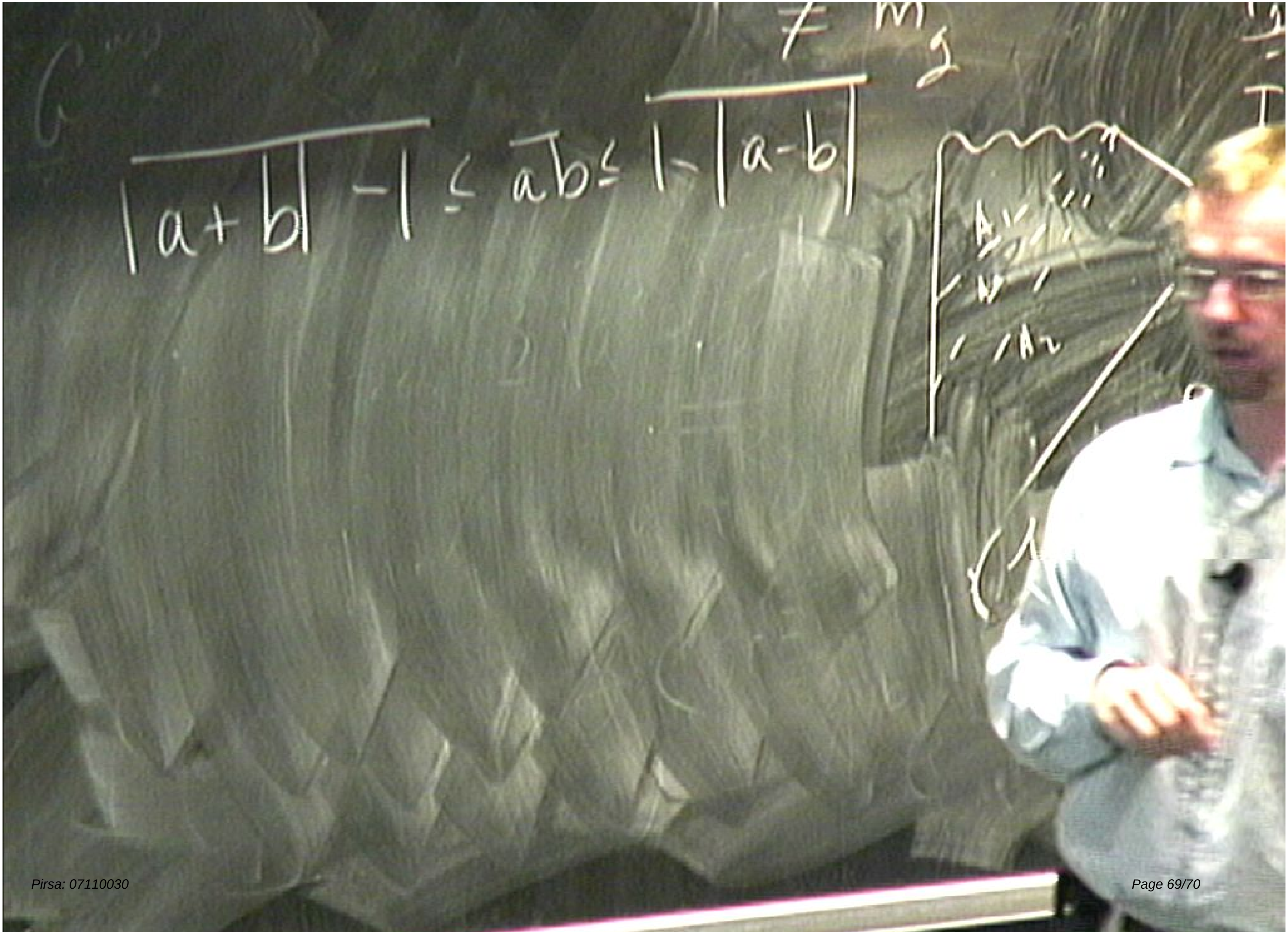
$$(|a_1| + \cdots + |a_n|) - (n-1) \leq s_1 \cdots s_n p$$

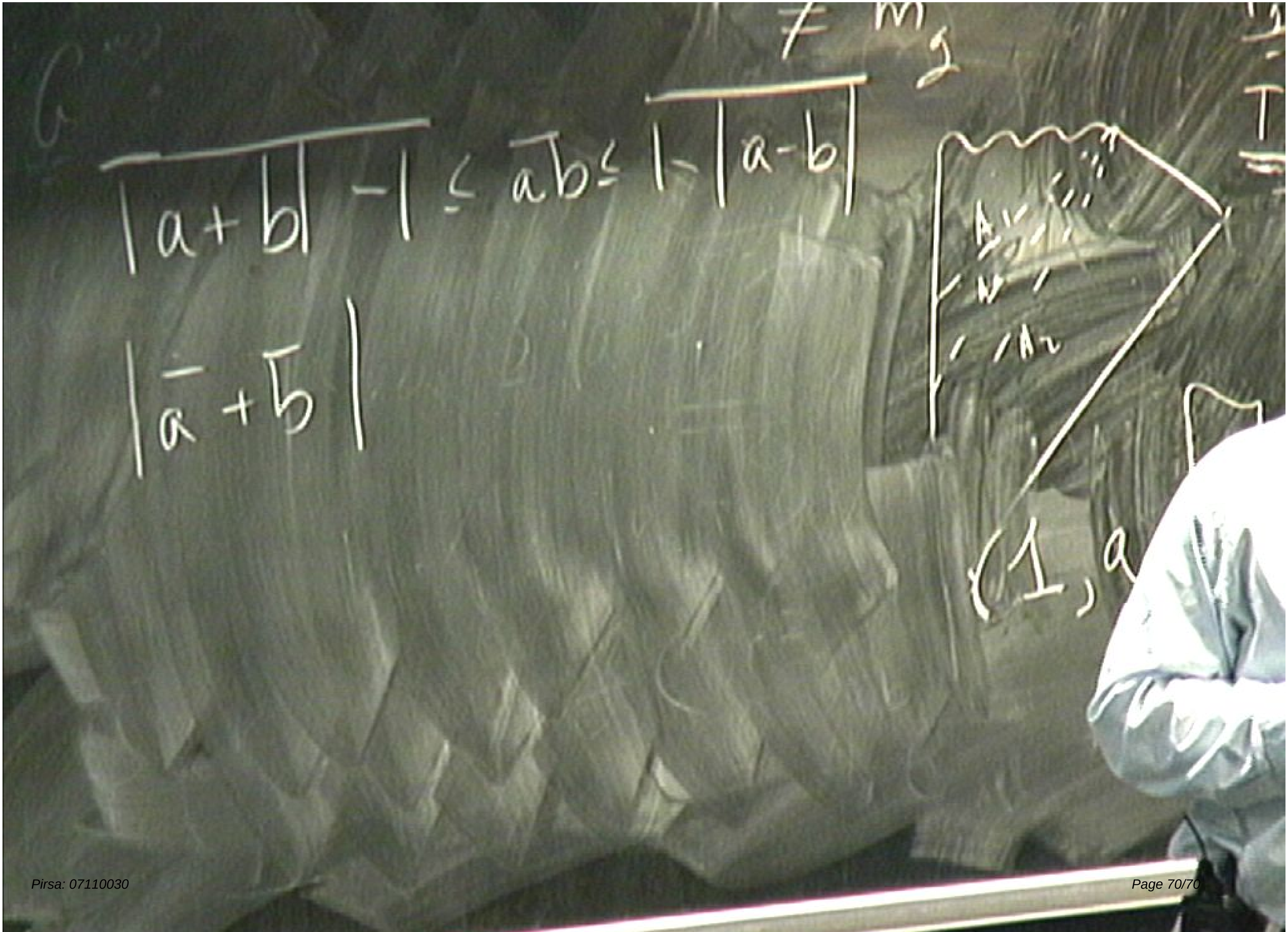
$$\leq (n-1) - (|a_1| + \cdots + |a_{n-1}| -| a_n|),$$

where $s_j = a_j / |a_j|$ is the sign of $a_j \neq 0$.

Recall: $|a+b| - 1 \leq p \leq 1 - |a-b|$


|a+b| −1 ≤ ab ≤ 1−|a−b|
(1, a, b


|a+b| − 1 ≤ ab ≤ 1 − |a−b|


|a+b| −1 ≤ āb ≤ 1 − |a−b|
|ā + b̄|
(1, a