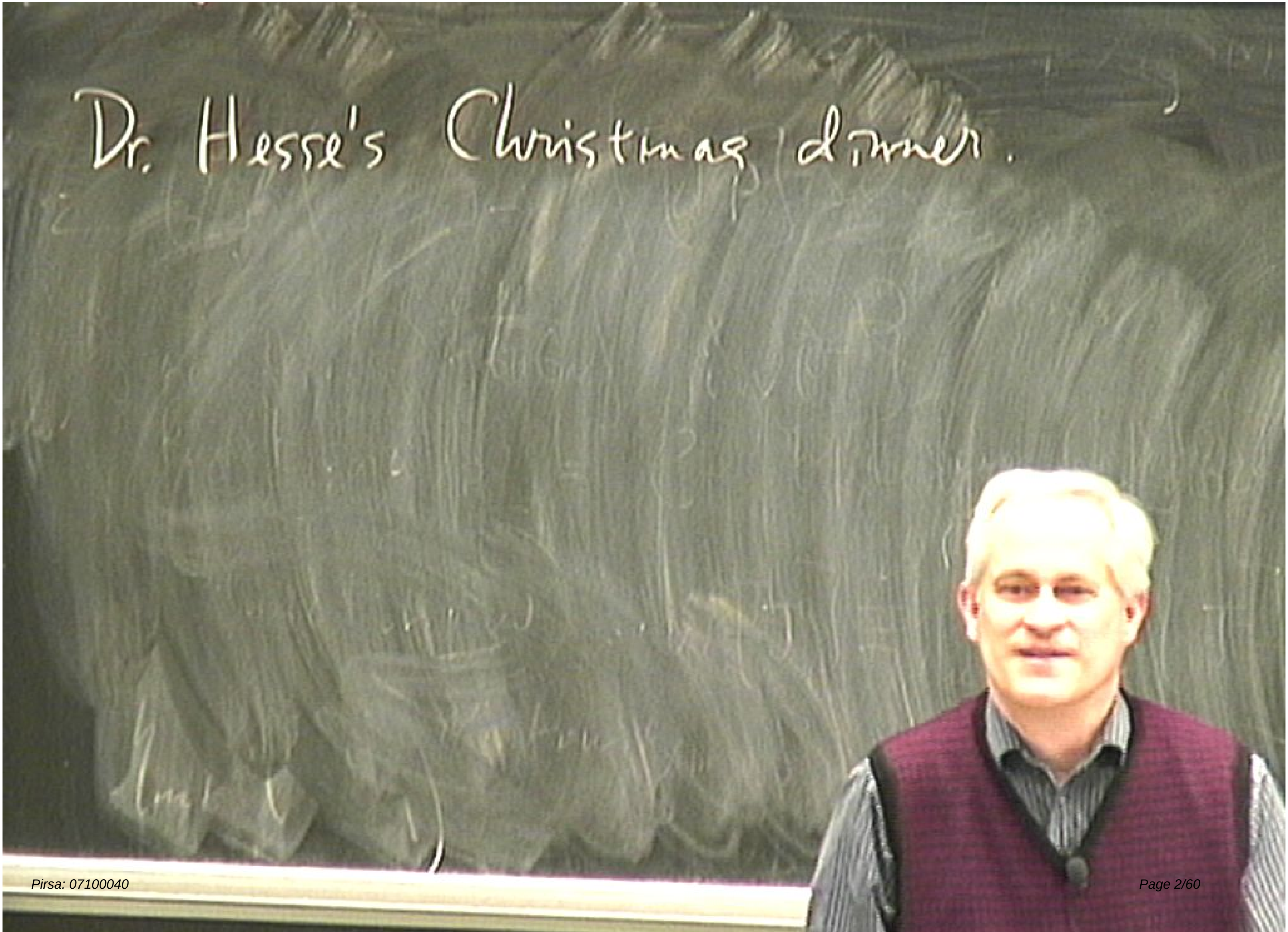
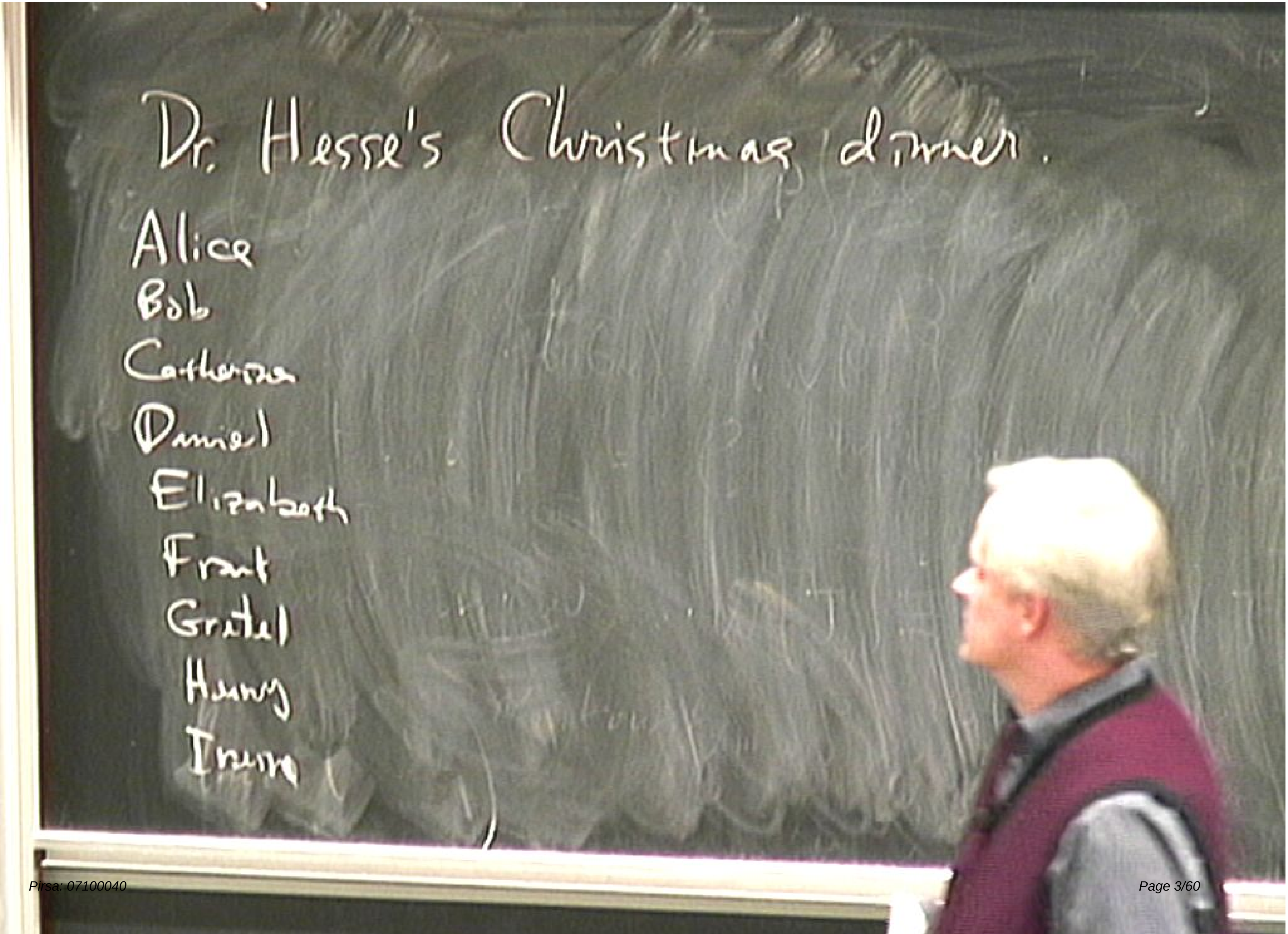
Title: d=3 SIC POVMs and Elliptic Curves

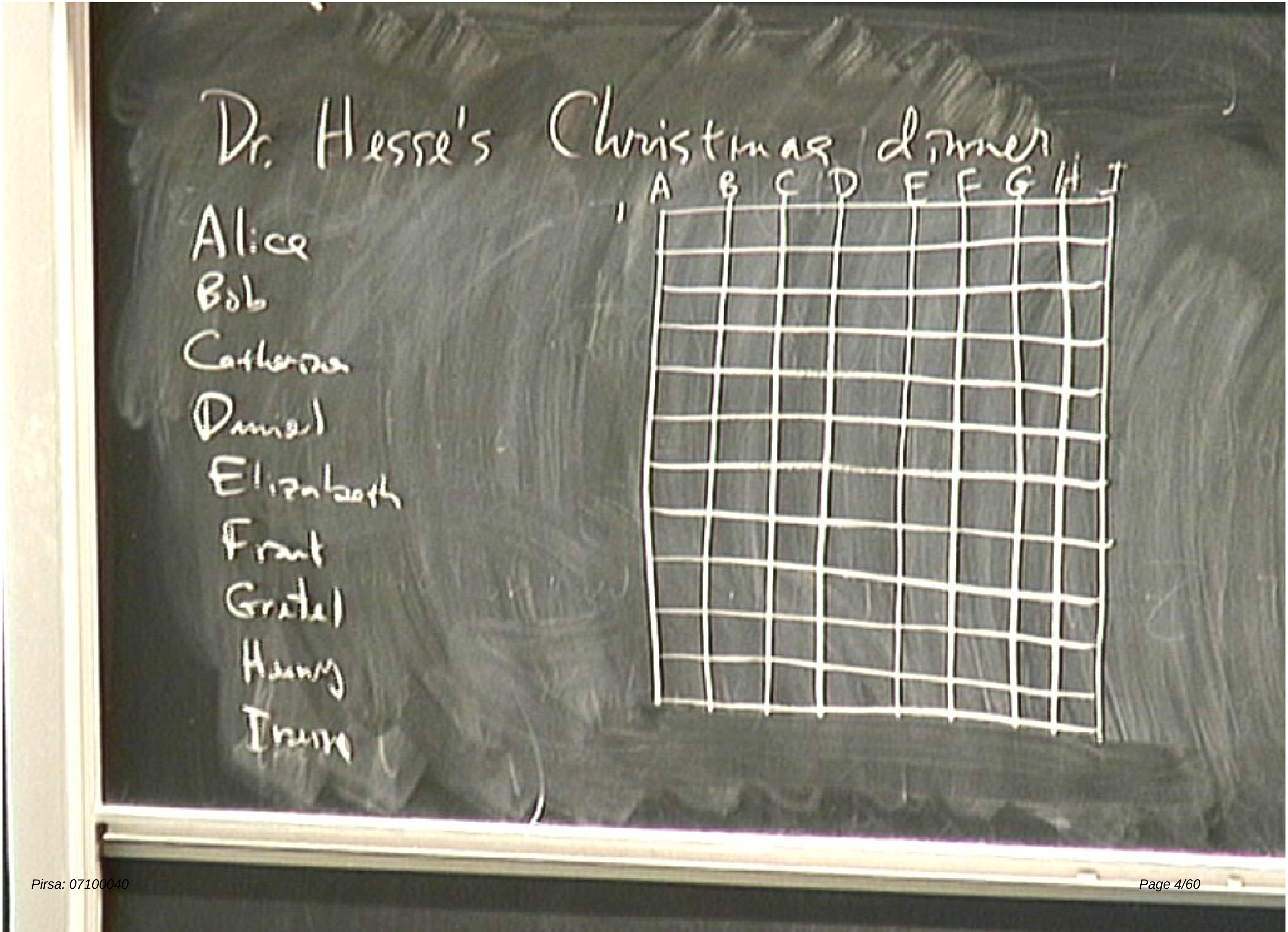
Date: Oct 30, 2007 04:00 PM

URL: http://pirsa.org/07100040

Abstract: The simplest algebraic curves of genus one are the nonsingular cubics in two-dimensional complex projective space. Interpreting CP^2 as the space of pure quantum states associated with a Hilbert space of dimension three, I will show how various properties of d=3 symmetric informationally complete positive operator valued measures can be understood in terms of the geometry of such curves. The resulting structure, although of considerable complexity, is very beautiful from a mathematical perspective.


Dr. Hesse's Christmas dinner.


Dr. Hesse's Christmas dinner.
Alice
Bob
Catherine
Daniel
Elizabeth
Frank
Gretel
Henry
Irene


Dr. Hesse's Christmas dinner
A B C D E F G H I
Alice
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Catherine
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Henry
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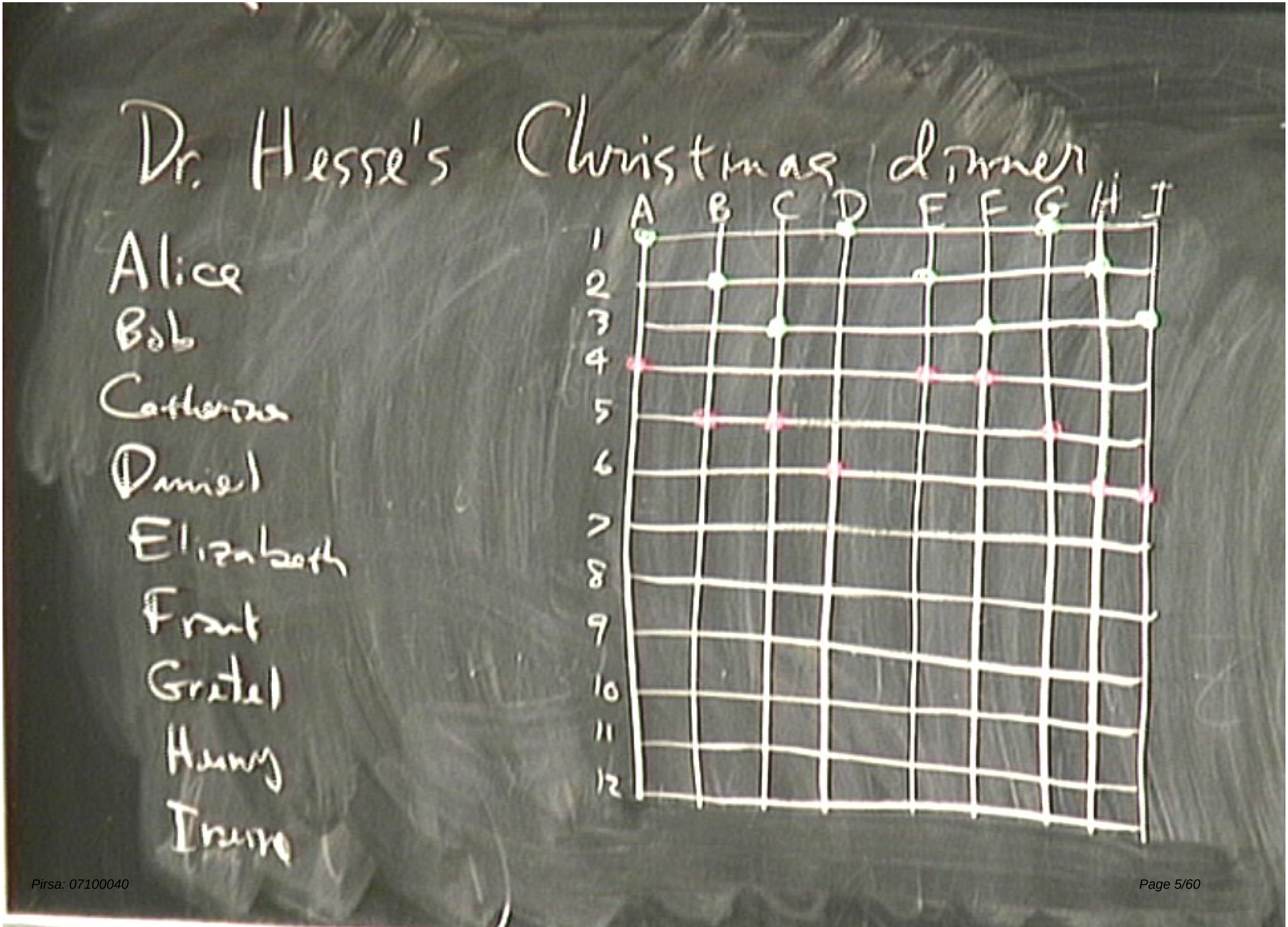
Dr. Hesse's Christmas dinner
Alice
Bob
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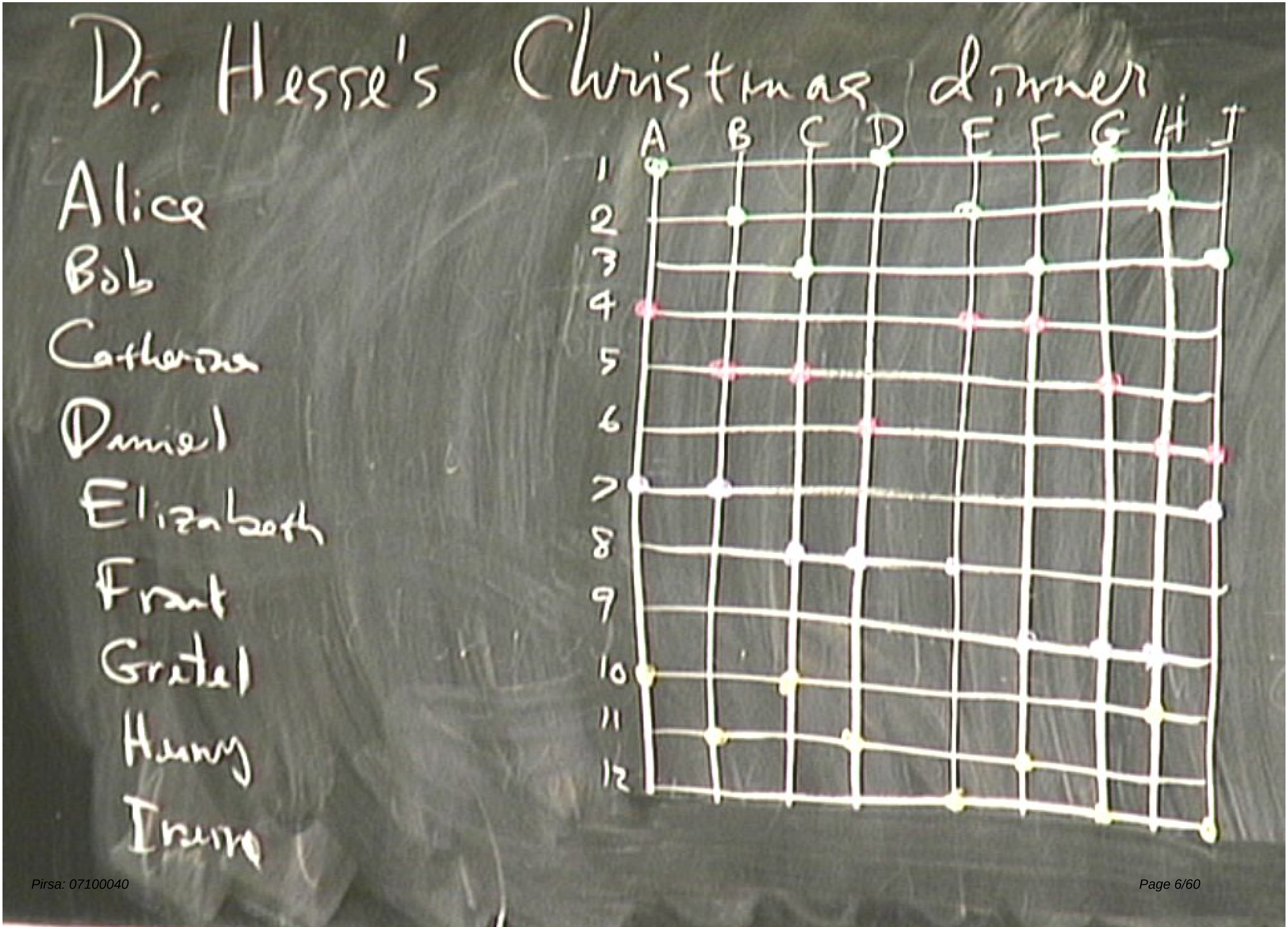
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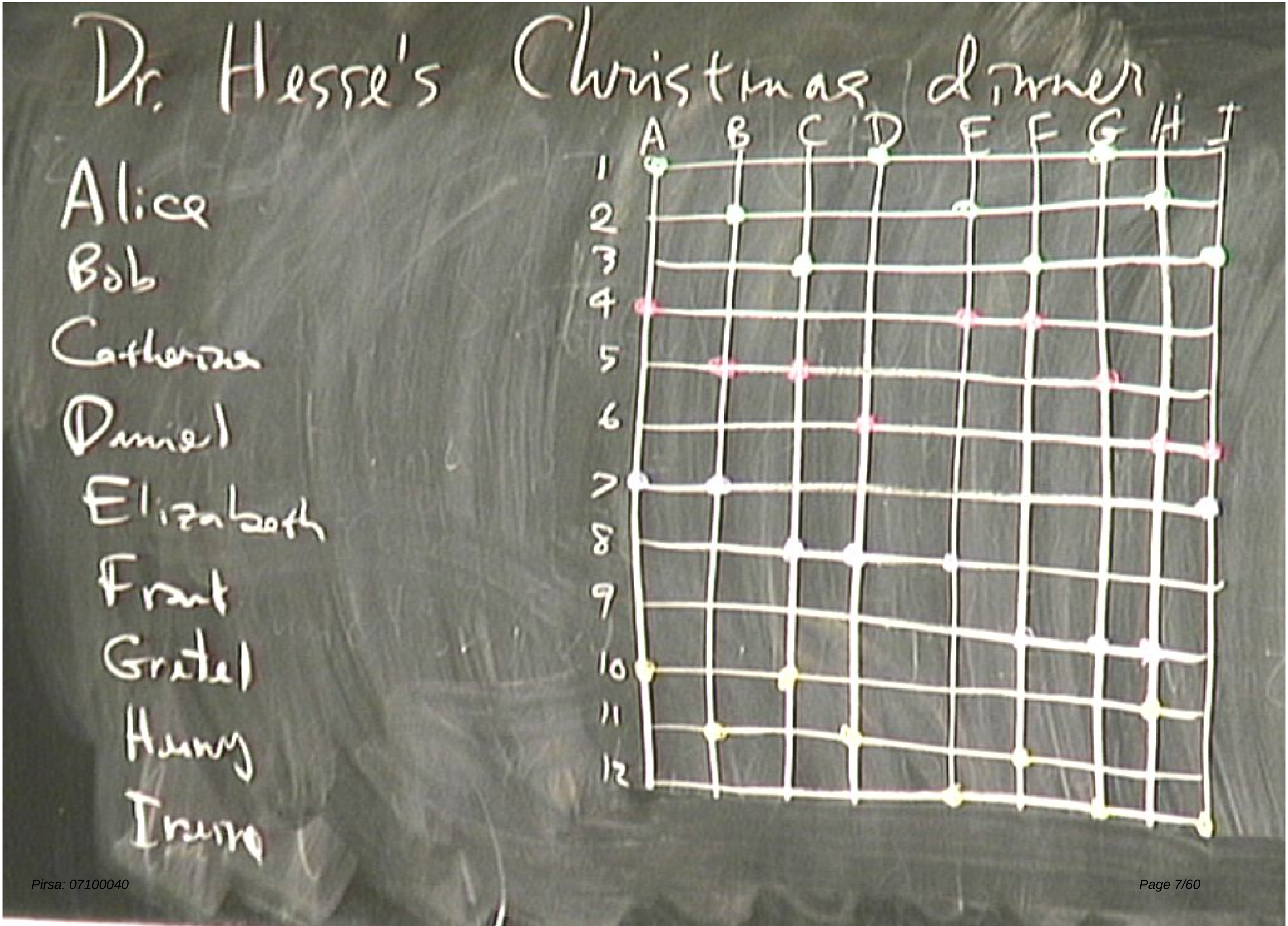
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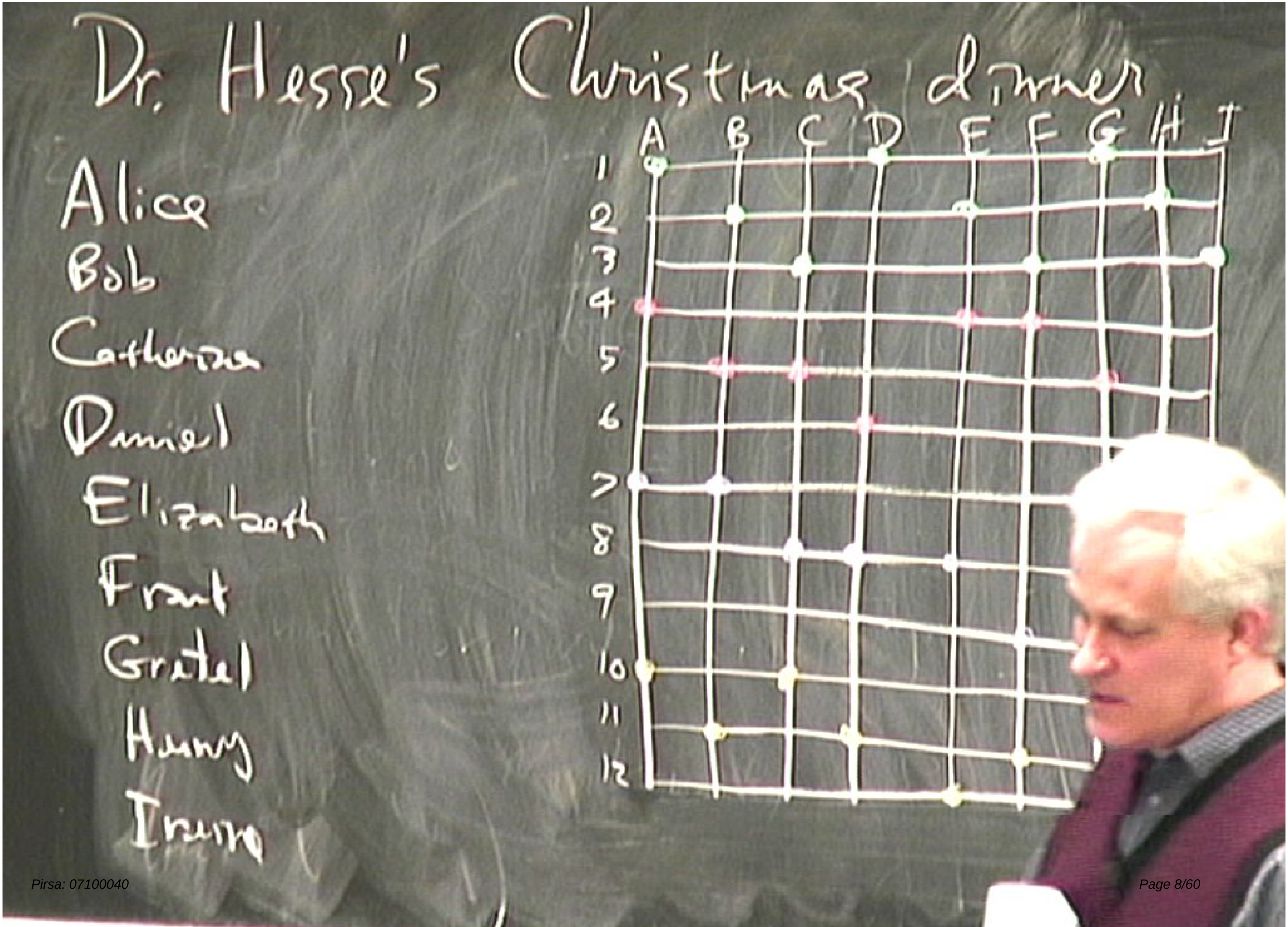
A B C D E F G H I
1 2 3 4 5 6 7 8 9 10 11 12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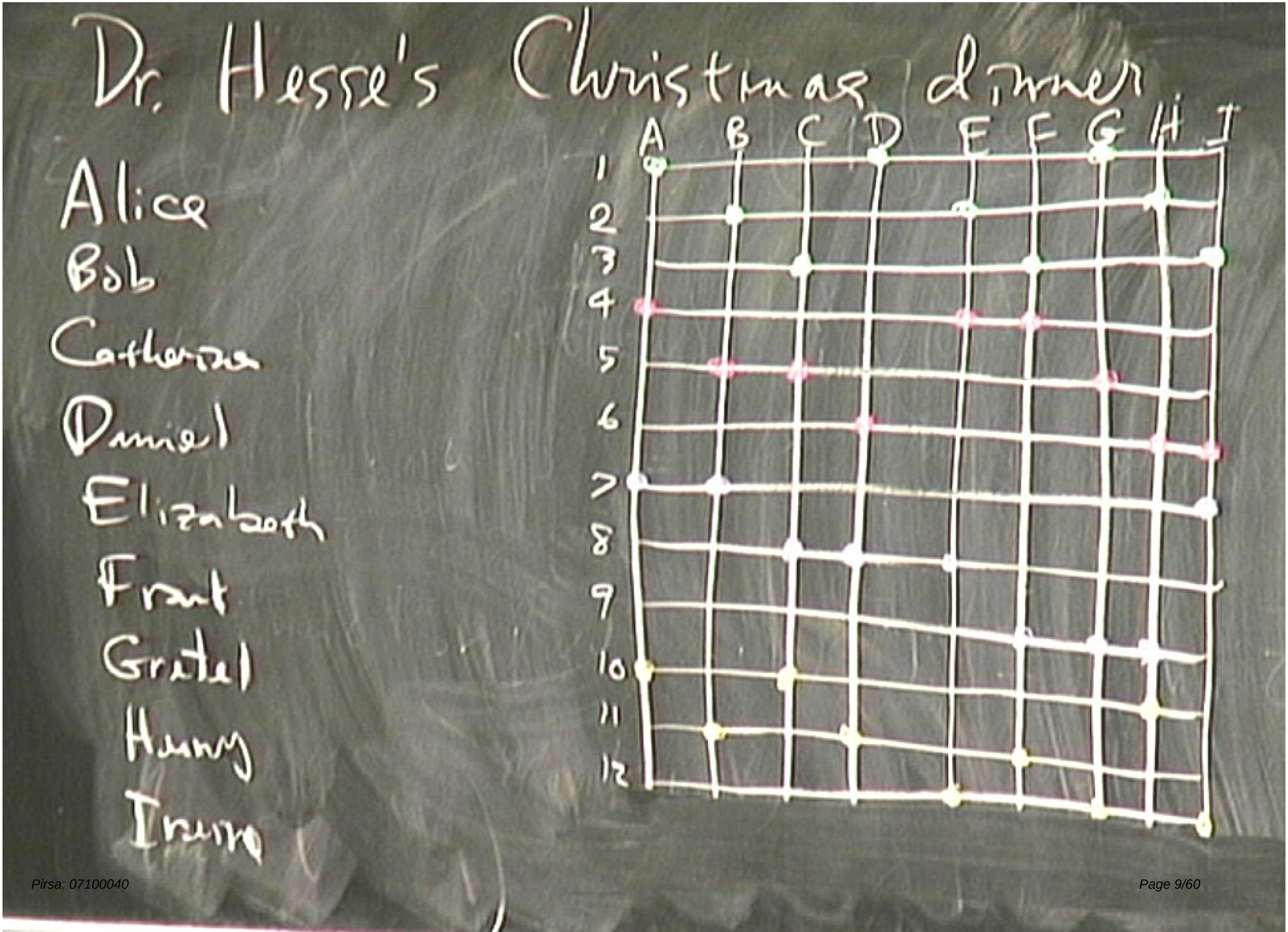

Dr. Hesse's Christmas dinner
A B C D E F G H I
1 2 3 4 5 6 7 8 9 10 11 12
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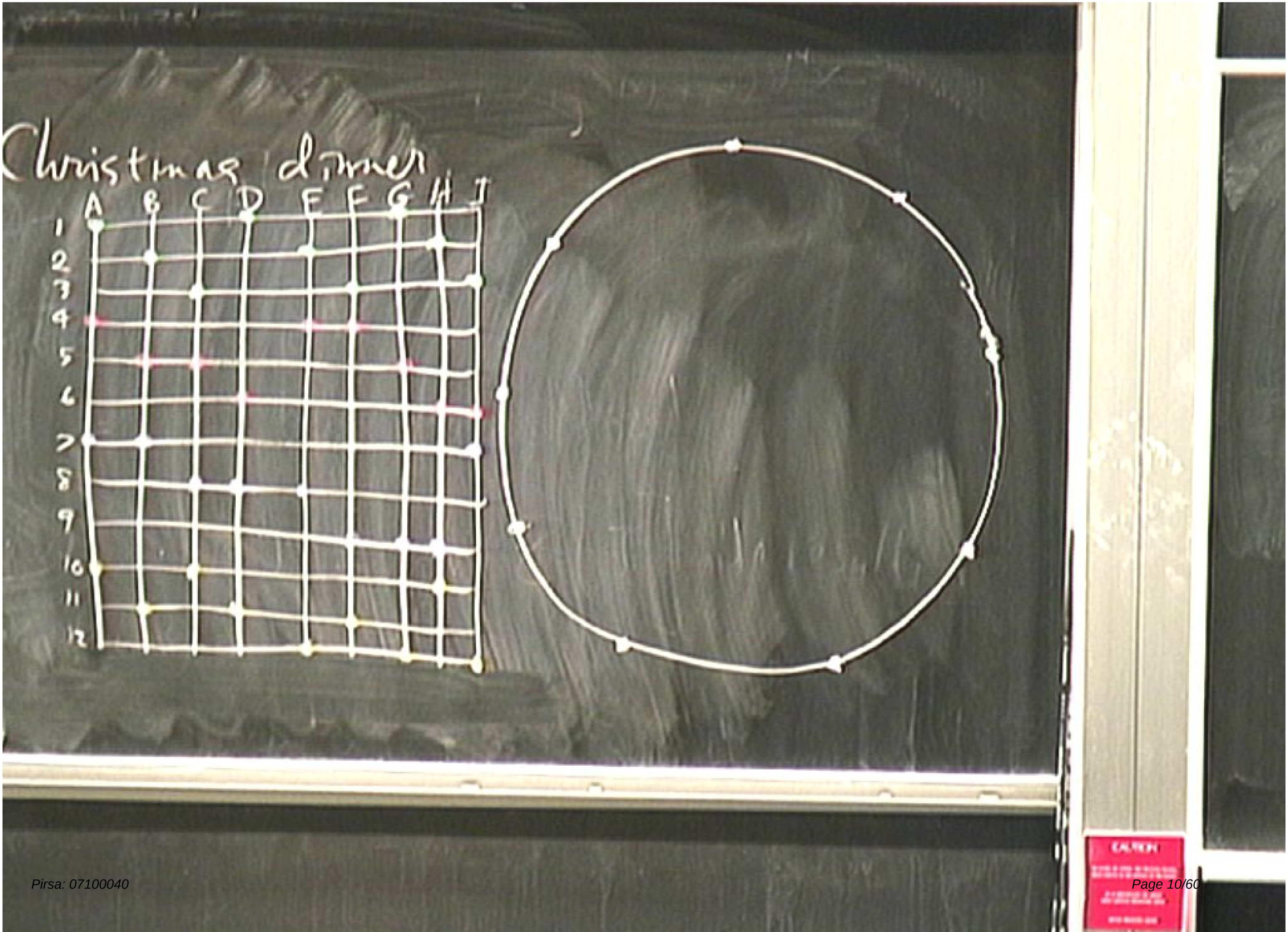
Christmas dinner
A B C D E F G H I
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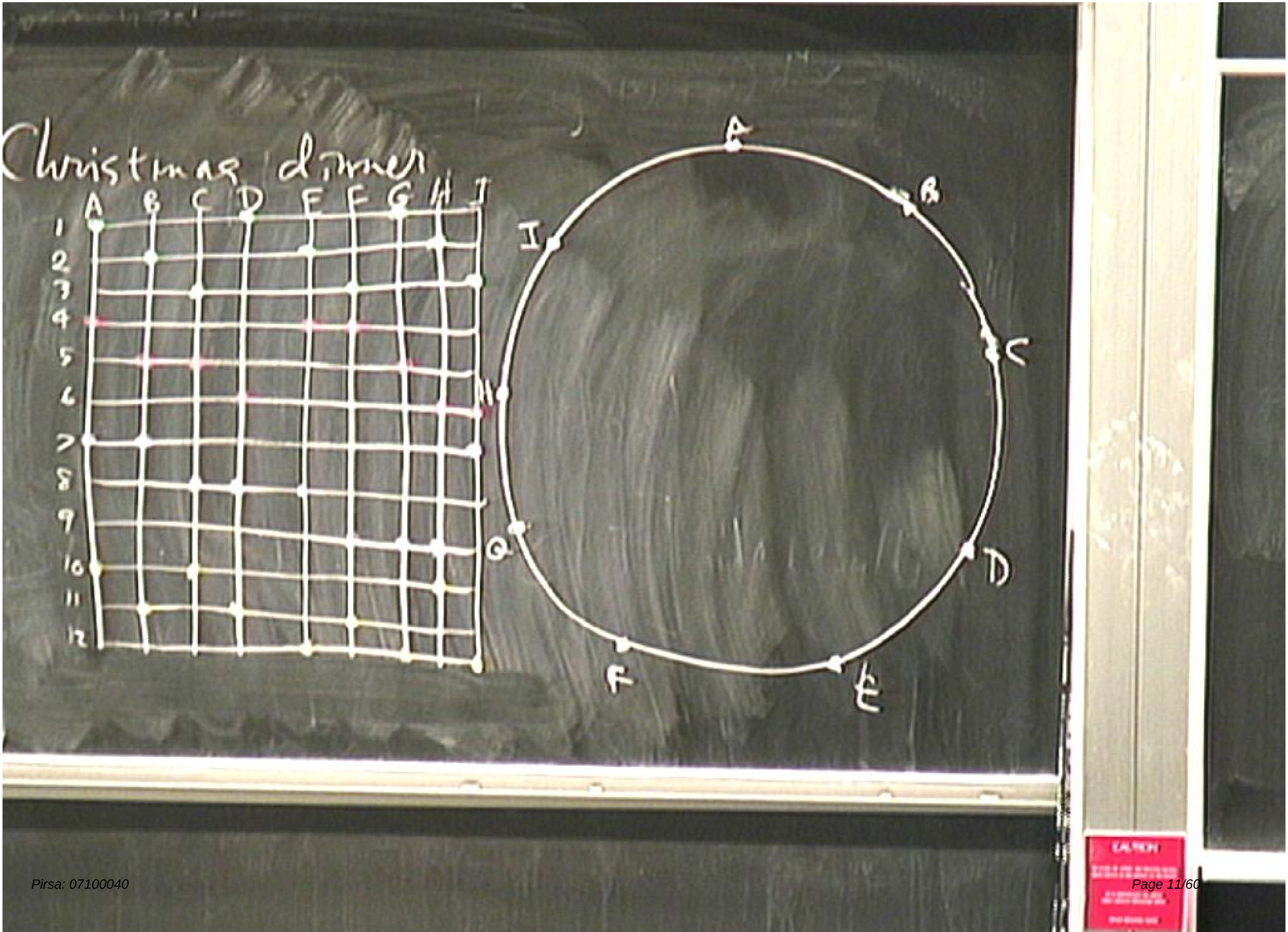
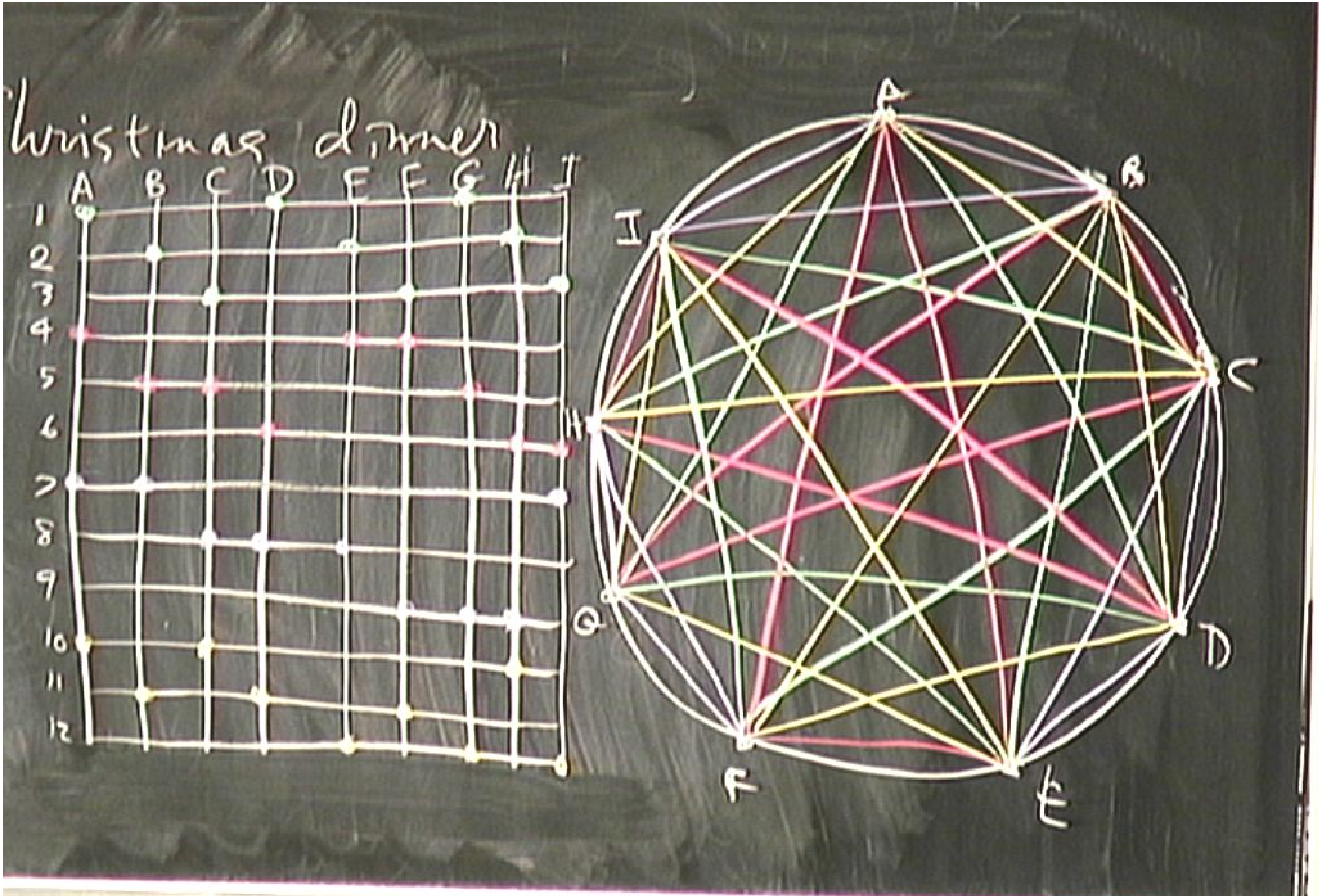
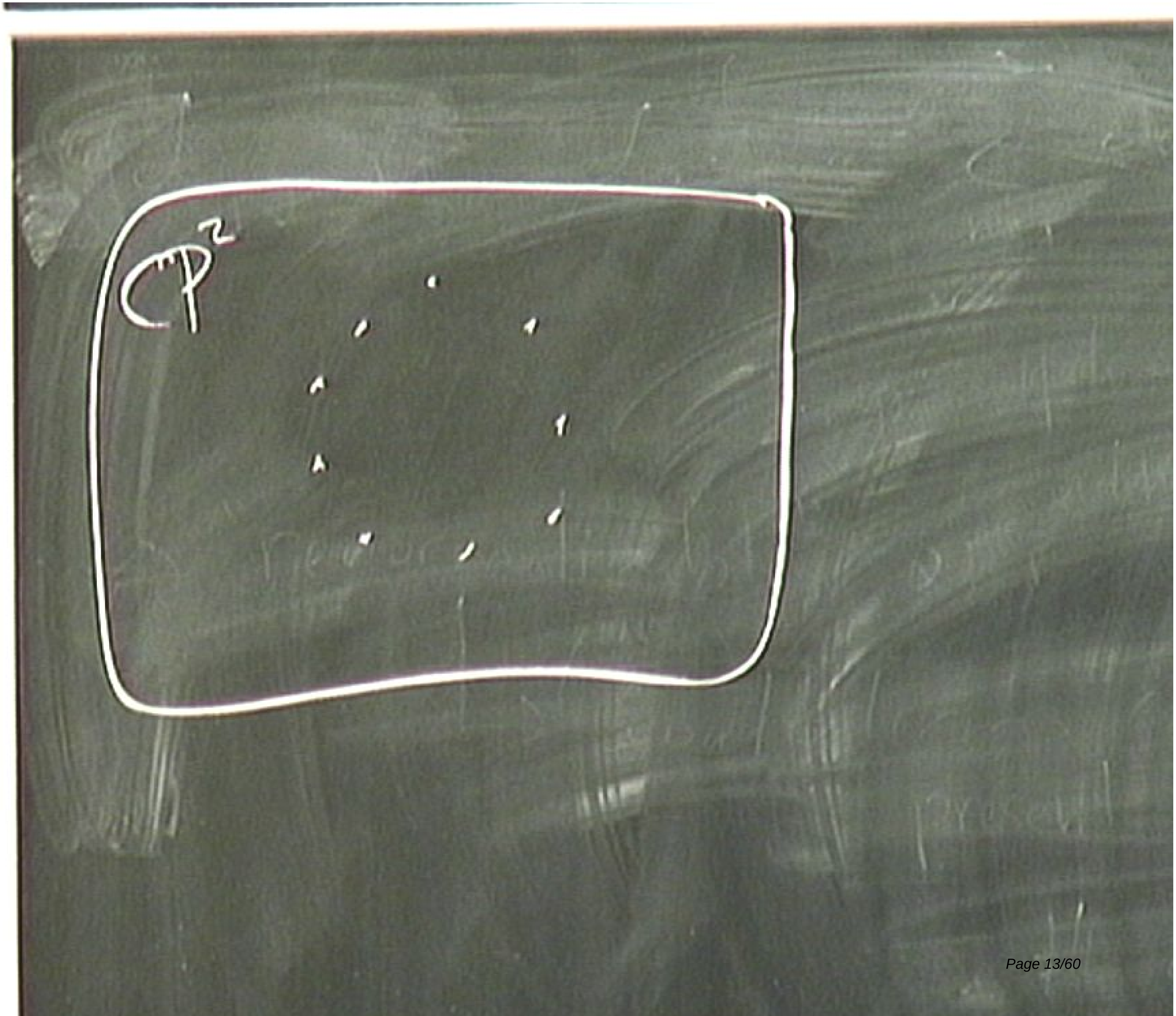
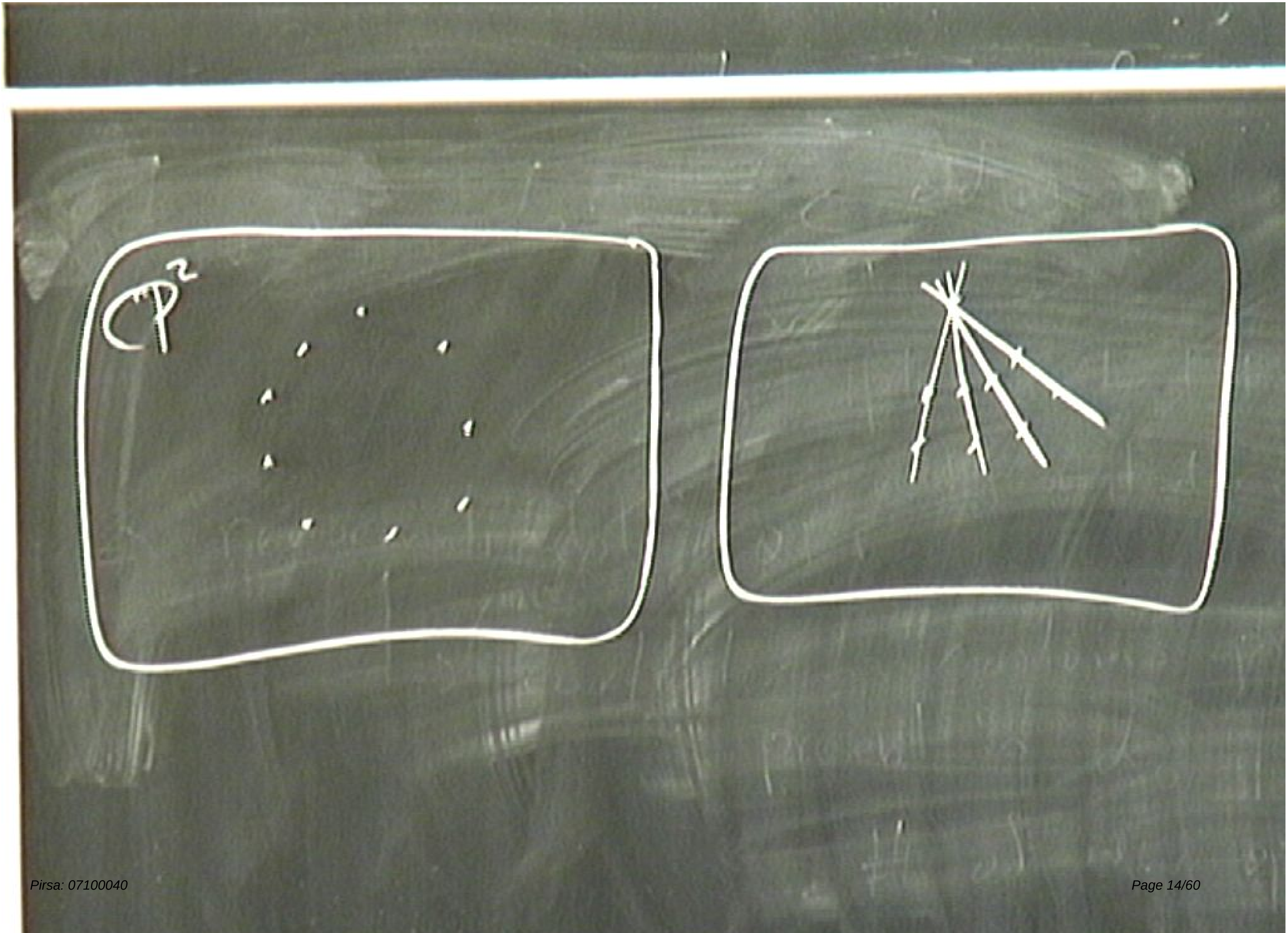
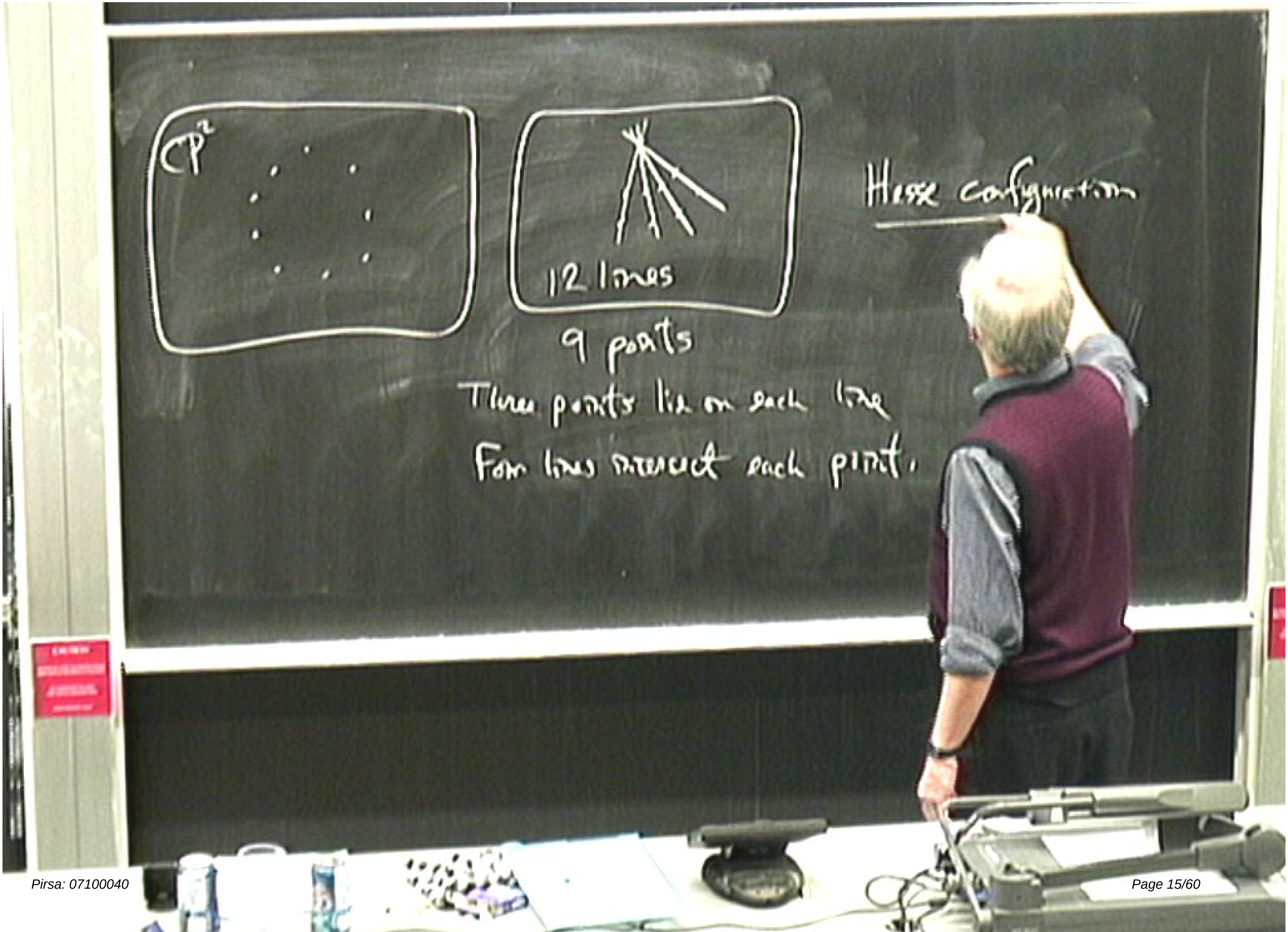

Christmas dinner

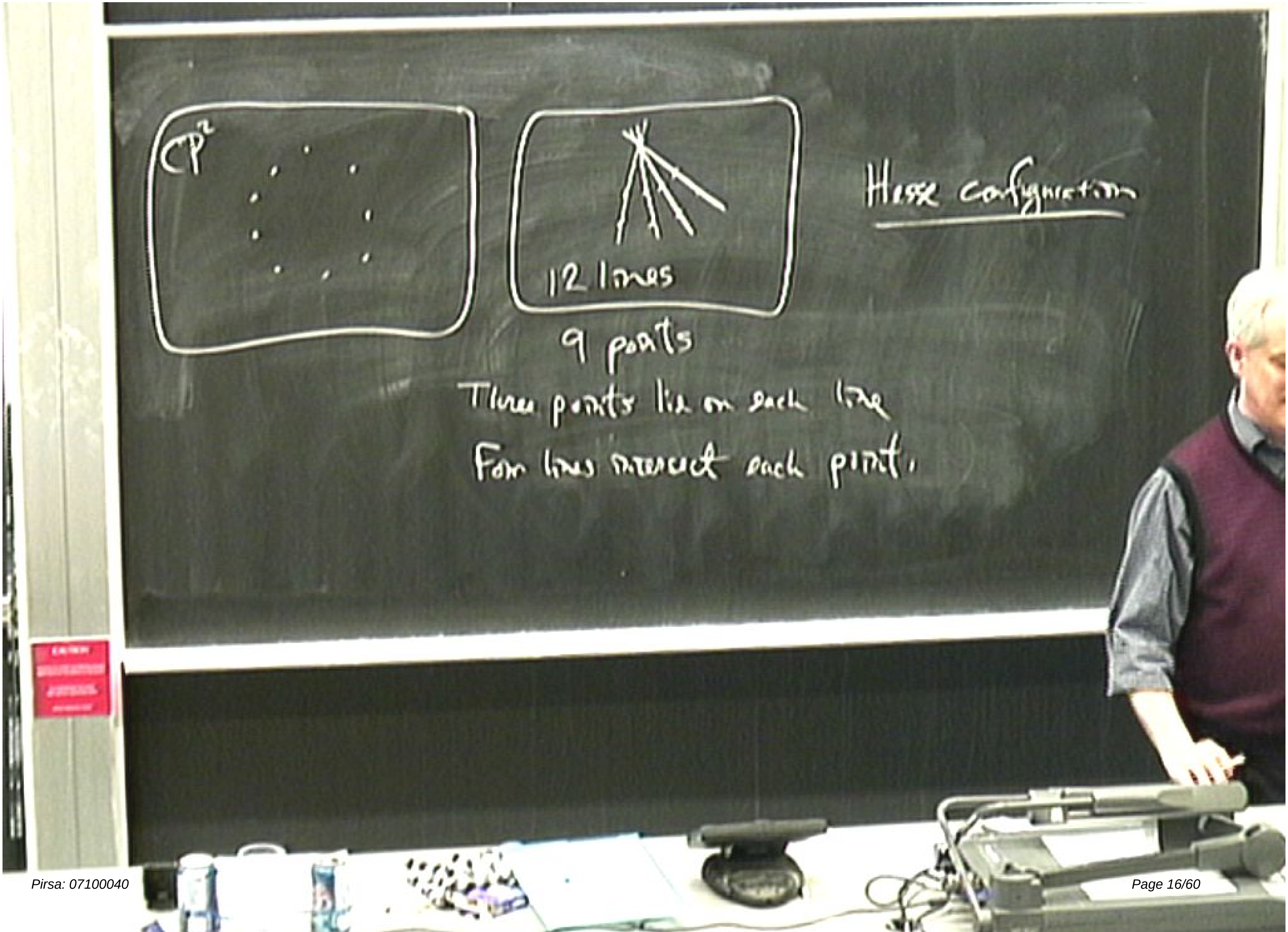

hristmas dinner
A B C D E F G H I
1 2 3 4 5 6 7 8 9 10 11 12
A B C D E F G H I



$\mathbb{C}P^2$


$\mathbb{CP}^2$


CP²
12 lines
9 points
Hesse configuration
Three points lie on each line
Four lines intersect each point.


CP2
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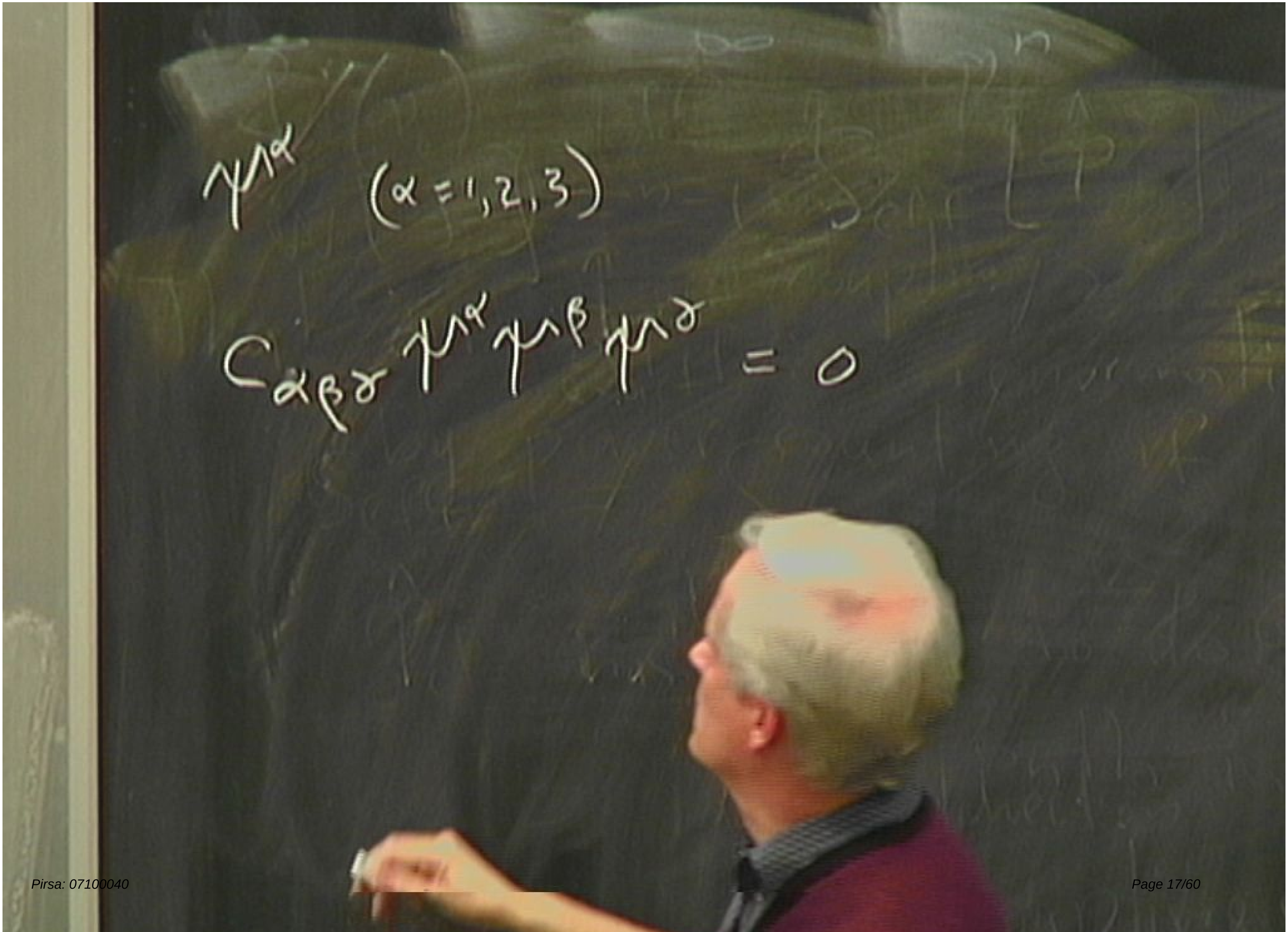
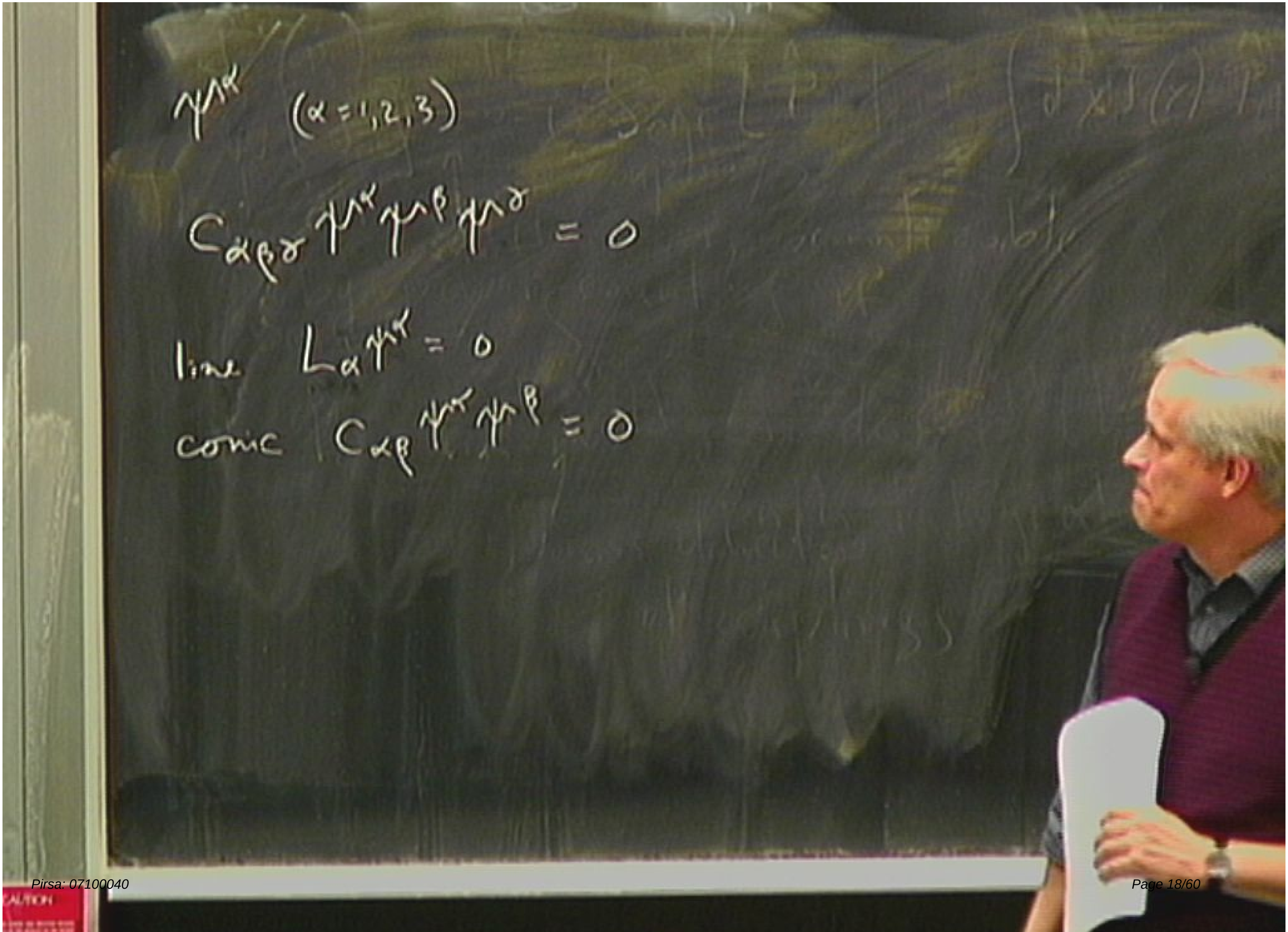

$\psi^{1\alpha}$ $(\alpha = 1, 2, 3)$
$C_{\alpha\beta\gamma}\,\psi^{1\alpha}\psi^{1\beta}\psi^{1\gamma} = 0$

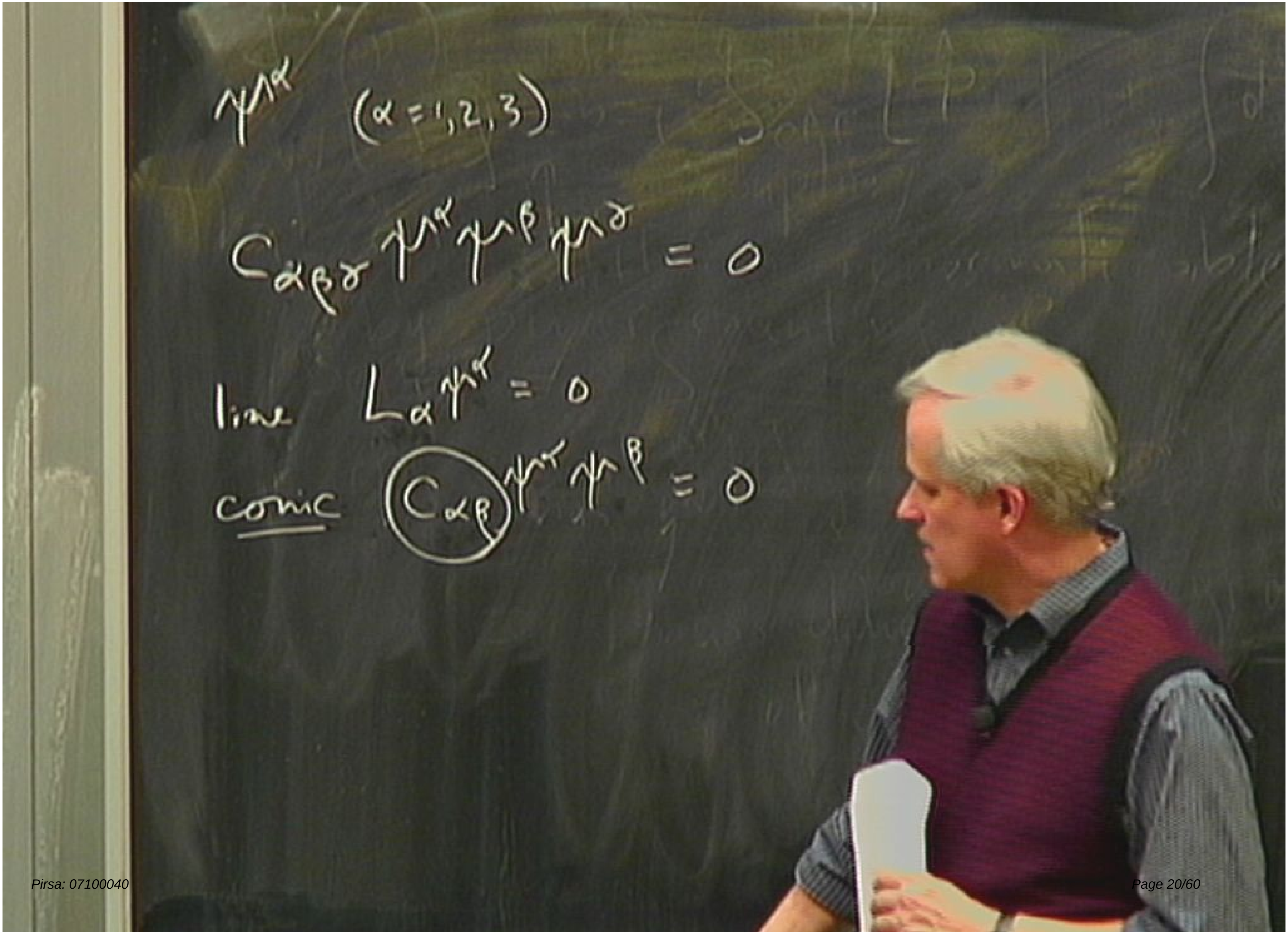

$\psi^{A\alpha}$ $(\alpha = 1, 2, 3)$
$C_{\alpha\beta\gamma}\psi^{A\alpha}\psi^{A\beta}\psi^{A\gamma} = 0$
line $L_\alpha \psi^{A\alpha} = 0$
conic $C_{\alpha\beta}\psi^{A\alpha}\psi^{A\beta} = 0$

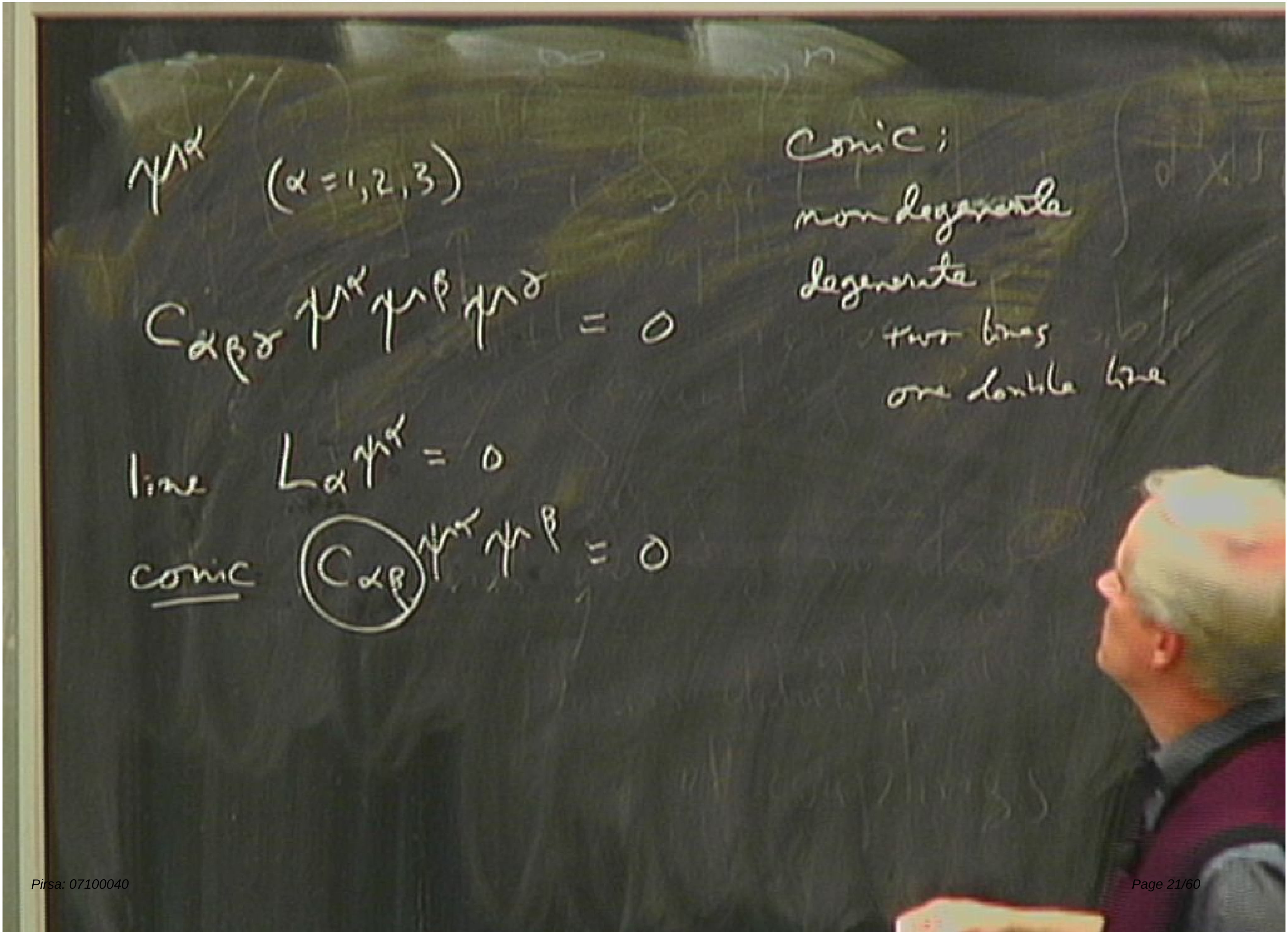
$\psi^{A\alpha}$ $\quad (\alpha = 1, 2, 3)$

$$C_{\alpha\beta\gamma}\psi^{A\alpha}\psi^{A\beta}\psi^{A\gamma} = 0$$

line $\quad L_\alpha \psi^{A\alpha} = 0$

conic $\quad C_{\alpha\beta}\psi^{A\alpha}\psi^{A\beta} = 0$


ψ^{α} $(\alpha = 1, 2, 3)$
$C_{\alpha\beta\gamma}\psi^{\alpha}\psi^{\beta}\psi^{\gamma} = 0$
line $L_{\alpha}\psi^{\alpha} = 0$
conic $C_{\alpha\beta}\psi^{\alpha}\psi^{\beta} = 0$


ψ^{α} $(\alpha = 1, 2, 3)$
$C_{\alpha\beta\gamma}\psi^{\alpha}\psi^{\beta}\psi^{\gamma} = 0$
line $L_{\alpha}\psi^{\alpha} = 0$
conic $C_{\alpha\beta}\psi^{\alpha}\psi^{\beta} = 0$
conic:
non degenerate
degenerate
two lines
one double line

conic ;

non degenerate

degenerate

two lines

one double line

cubic

cubic

nondegenerate

nonsingular $C_{\alpha\beta\gamma}\psi^{\alpha}\psi^{\beta}\psi^{\gamma} \neq 0 \quad \forall \psi$

singular

conic:

nondegenerate

degenerate

two lines

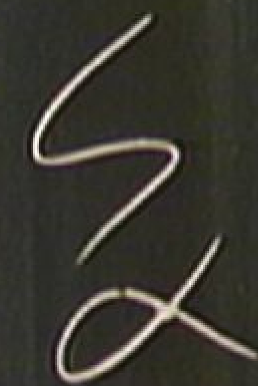
one double line

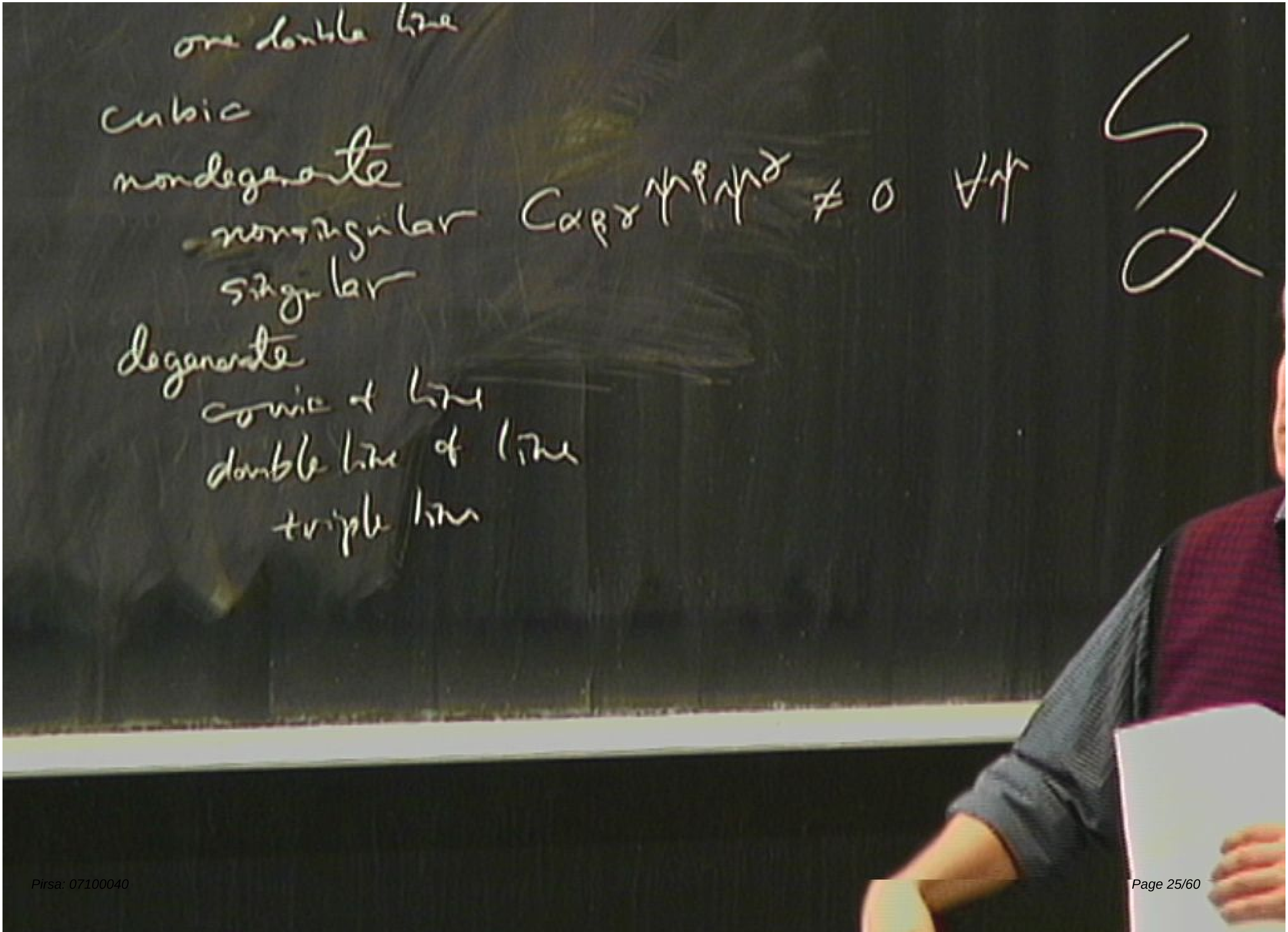
cubic

nondegenerate

nonsingular $C_{\alpha\beta\gamma}\eta^{\alpha}\eta^{\beta}\eta^{\gamma} \neq 0 \quad \forall \eta$

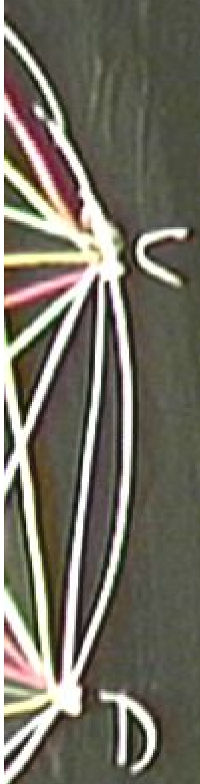
singular




one double line
cubic
nondegenerate
nonsingular $C_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma \neq 0 \ \forall \psi$
singular
degenerate
conic & line
double line & line
triple line

$$C_{\alpha\beta\gamma}\, \psi^{\alpha} \psi^{\beta} \psi^{\gamma}$$

$$C_{\underline{\alpha\beta}\gamma}\, \psi^{\gamma}$$



$$C_{\alpha\beta\gamma}\,\psi^{\mu\alpha}\,\psi^{\rho}\,\psi^{\mu\gamma}$$

$$C_{\underline{\alpha}\beta\gamma}\,\psi^{\mu\gamma}$$

$$\Delta_H =$$

$$C_{\alpha\beta\gamma}\,\psi^{\alpha}\psi^{\beta}\psi^{\gamma}$$

$$C_{\underline{\alpha\beta}\gamma}\,\psi^{\gamma} = M_{\alpha\beta}$$

$$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\sigma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}\,M_{\beta\gamma}\,M_{\gamma\tau}$$

$$C_{\alpha\beta\gamma}\,\psi^{\alpha}\psi^{\beta}\psi^{\gamma}$$

$$\psi^{\alpha} \longrightarrow \zeta^{\alpha}_{\beta}\,\psi^{\beta}$$

$$C_{\underline{\alpha\beta}\gamma}\,\psi^{\gamma} = M_{\alpha\beta}$$

$$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\sigma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}\,M_{\beta\gamma}\,M_{\gamma\tau}$$

$C_{\alpha\beta\gamma}\psi^{\mu\alpha}\psi^{\nu\beta}\psi^{\lambda\gamma}$

$\psi^{\mu\alpha} \longrightarrow \zeta^{\alpha}_{\beta}\psi^{\mu\beta}$

$C_{\underline{\alpha\beta}\gamma}\psi^{\mu\gamma} = M_{\alpha\beta}$

$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\gamma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}\,M_{\beta\sigma}\,M_{\gamma\tau}$

$$C_{\alpha\beta\gamma}\,\psi^\alpha\psi^\beta\psi^\gamma$$

$$\psi^\alpha \longrightarrow L^\alpha_\beta\,\psi^\beta$$

$$C_{\underline{\alpha\beta}\gamma}\,\psi^\gamma = M_{\alpha\beta}$$

$$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\gamma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}M_{\beta\sigma}M_{\gamma\tau}$$

$$\Delta_H = 0$$ another cubic curve!

$\rightarrow L_{\rho}^{\alpha} \psi^{\beta}$

$= M_{\alpha\beta} M_{\beta\gamma} M_{\gamma\tau}$

$C_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma$

$\psi^\alpha \longrightarrow L^\alpha_\beta \psi^\beta$

$C_{\alpha\beta\gamma}\psi^\gamma = M_{\alpha\beta}$

$\Delta_H = \det(M) = \frac{1}{6}\,\epsilon^{\alpha\beta\pi}\,\epsilon^{\rho\sigma\tau}\, M_{\alpha\rho} M_{\beta\sigma} M_{\pi\tau}$

$\Delta_H = 0$ another cubic curve!

C

$C_{\alpha\beta\gamma}\,\psi^{\alpha}\psi^{\beta}\psi^{\gamma}$

$\psi^{\alpha} \longrightarrow L^{\alpha}_{\beta}\,\psi^{\beta}$

$C_{\alpha\beta\gamma}\,\psi^{\gamma} = M_{\alpha\beta}$

$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\gamma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}\,M_{\beta\sigma}\,M_{\gamma\tau} \leftarrow \Delta_H$

$\Delta_H = 0$ another cubic curve!

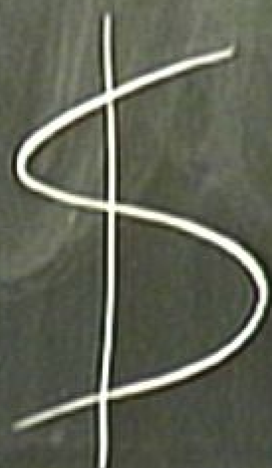
$C_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma$

$\psi^\alpha \longrightarrow L^\alpha_\beta\,\psi^\beta$

$C_{\alpha\beta\gamma}\psi^\gamma = M_{\alpha\beta}$

$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha\beta\gamma}\,\varepsilon^{\rho\sigma\tau}\,M_{\alpha\rho}\,M_{\beta\sigma}\,M_{\gamma\tau}$

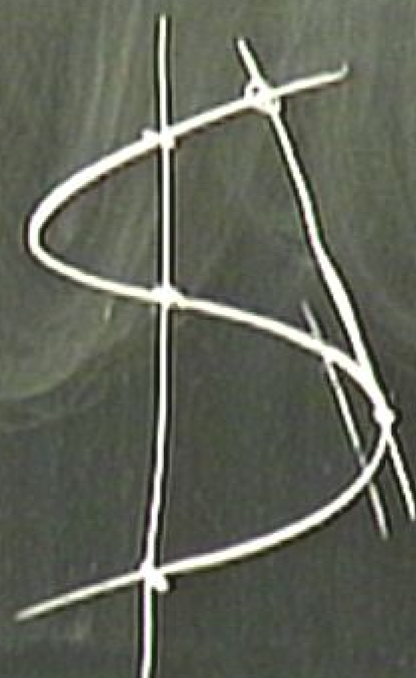
$\Delta_H = 0$ another cubic curve!

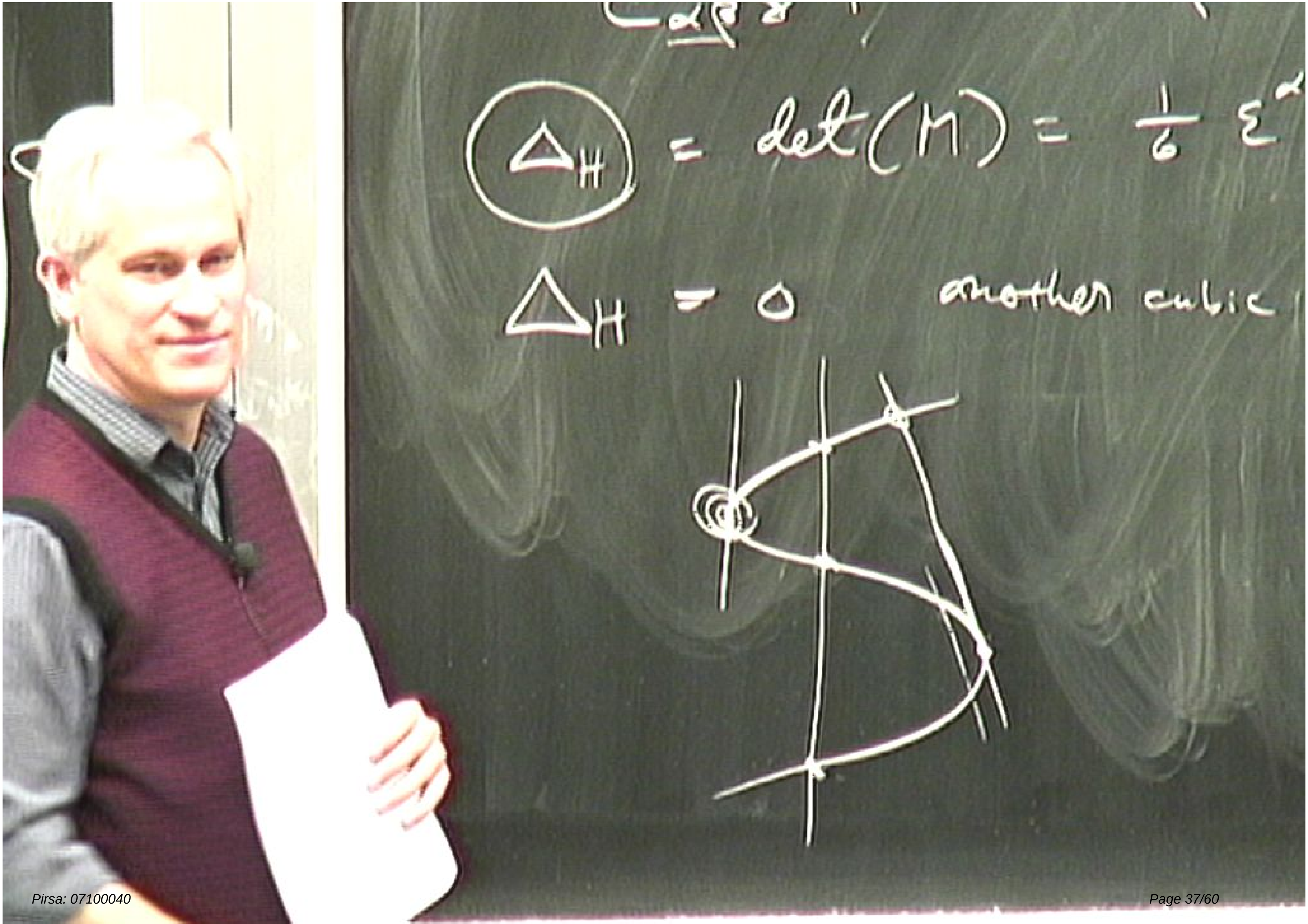


$C_{\alpha\beta\gamma}\, \psi \ldots = T_{\alpha\beta}$

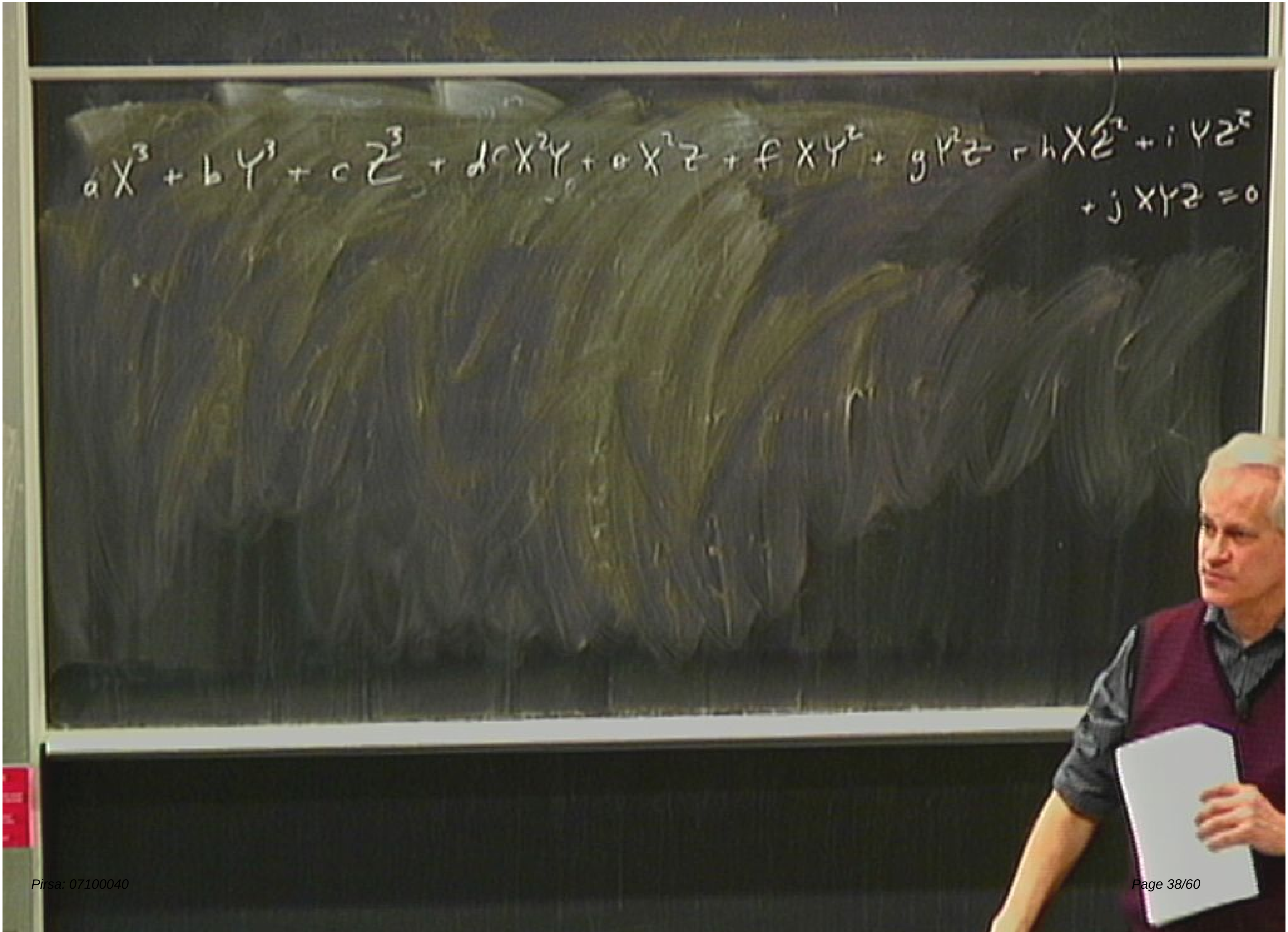
$$\Delta_H = \det(M) = \frac{1}{6}\, \varepsilon^{\alpha\beta\delta}\, \varepsilon^{\rho\sigma\epsilon} M \ldots$$

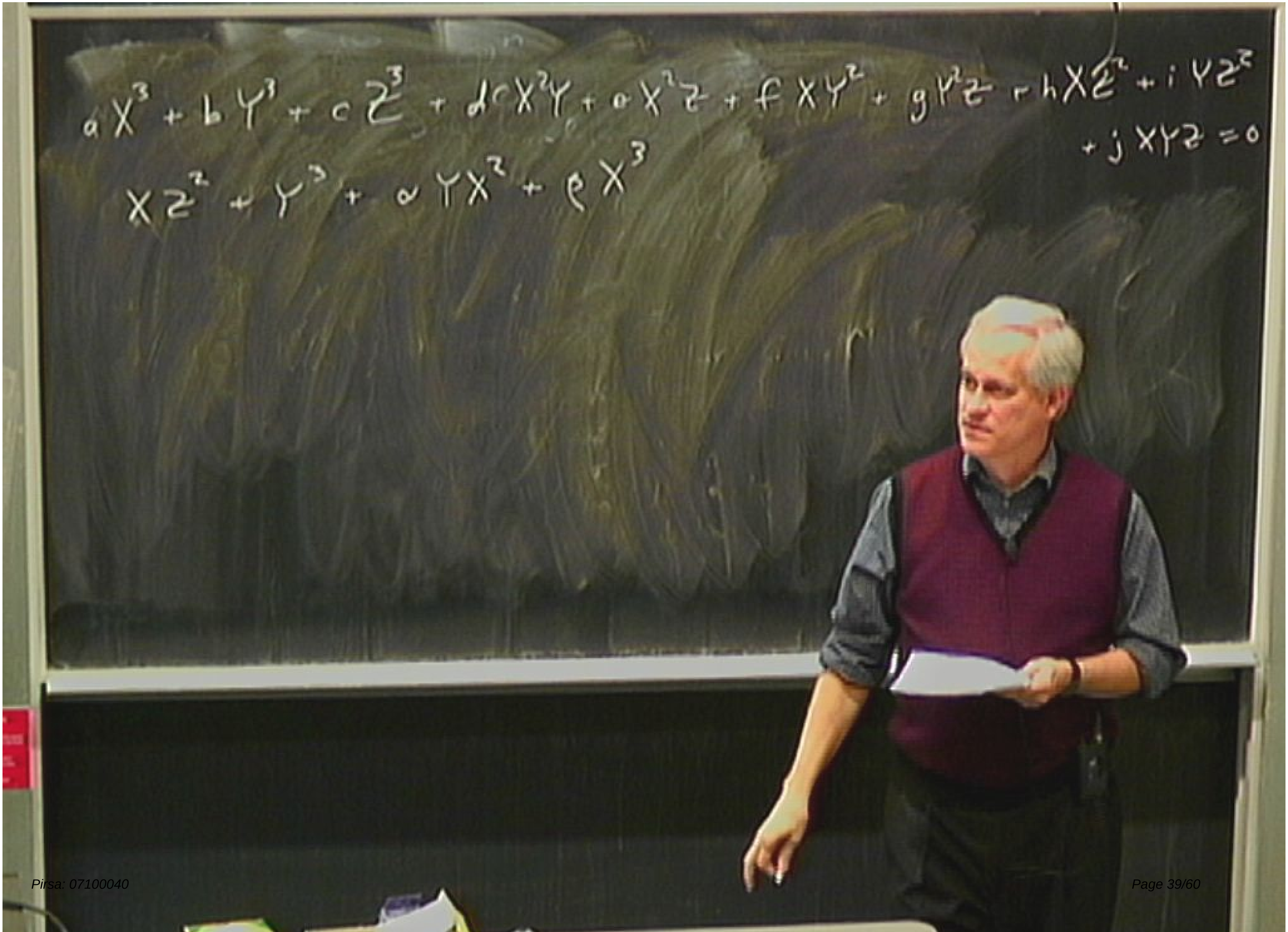
$\Delta_H = 0$ another cubic curve!

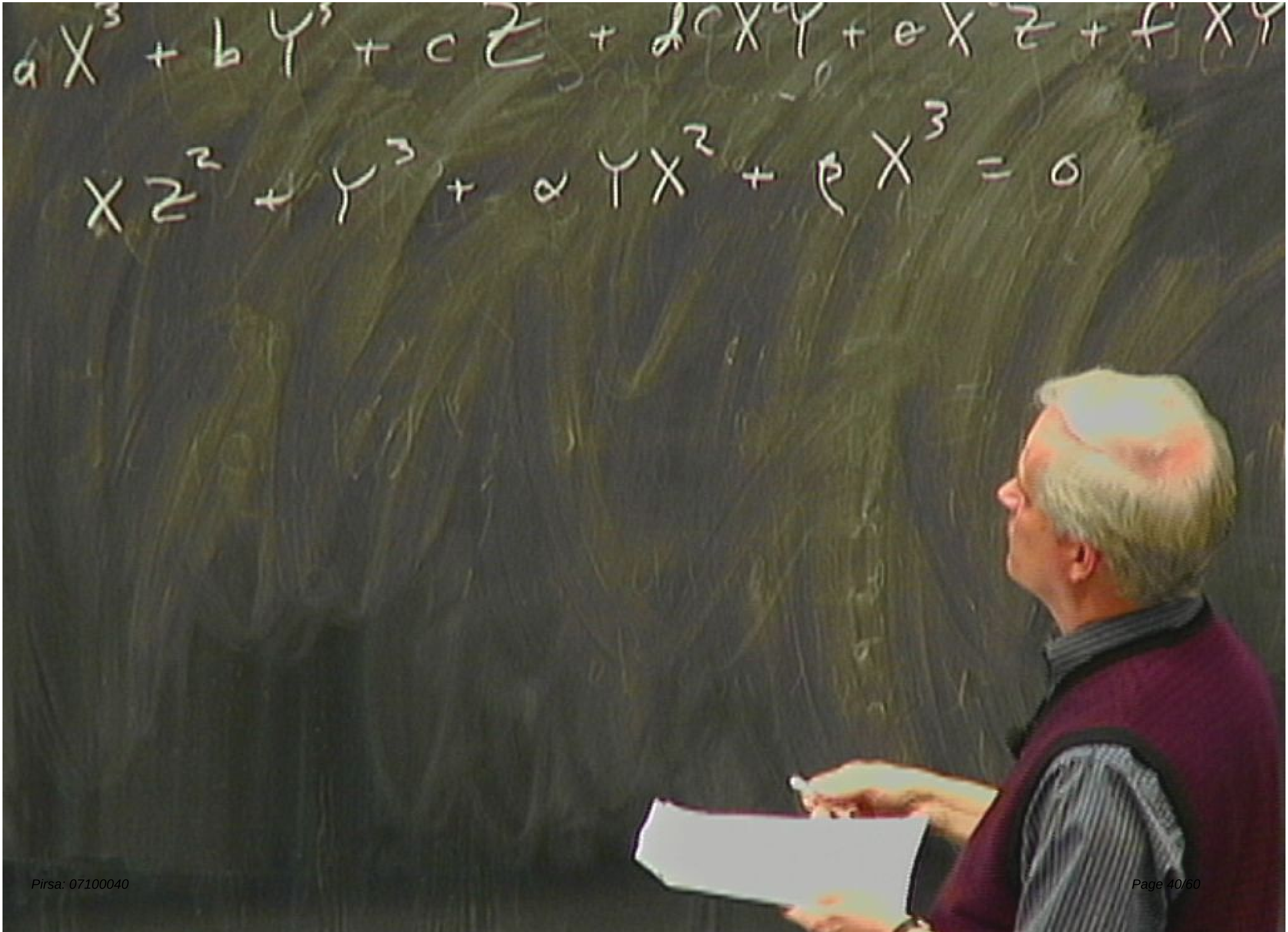




$\Delta_H = \det(M) = \frac{1}{6}\,\varepsilon^{\alpha}$
$\Delta_H = 0$ another cubic


$aX^3 + bY^3 + cZ^3 + dX^2Y + eX^2Z + fXY^2 + gY^2Z + hXZ^2 + iYZ^2$
$+ jXYZ = 0$


$aX^3 + bY^3 + cZ^3 + dX^2Y + eX^2Z + fXY^2 + gY^2Z + hXZ^2 + iYZ^2$
$+ jXYZ = 0$
$XZ^2 + Y^3 + \alpha YX^2 + \beta X^3$


$aX^3 + bY^3 + cZ + dXY + eXZ + fXY$
$XZ^2 + Y^3 + \alpha YX^2 + \beta X^3 = 0$

$$XZ^2 + Y^3 + \alpha Y X^2 + \beta X^3 = 0$$

$$\tilde{Z}^2 + \tilde{Y}^3 + \alpha \tilde{Y} + \beta = 0$$

$$aX^3 + bY^3 + cZ^3 + dX^2Y + eX^2Z + fXY^2 + gY^2Z + hX$$

$$XZ^2 + Y^3 + \alpha YX^2 + \beta X^3 = 0$$

$$\tilde{Z}^2 + \tilde{Y}^3 + \alpha\tilde{Y} + \beta = 0 \qquad \tilde{Y} = \frac{Y}{X} \qquad \tilde{Z} = \frac{Z}{X}$$

$$X^3 + Y^3 + Z^3 + a\,XYZ = 0$$

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

Hessian cubic

$$X^2 + Y^2 + Z^2 + k\,XYZ = 0$$

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

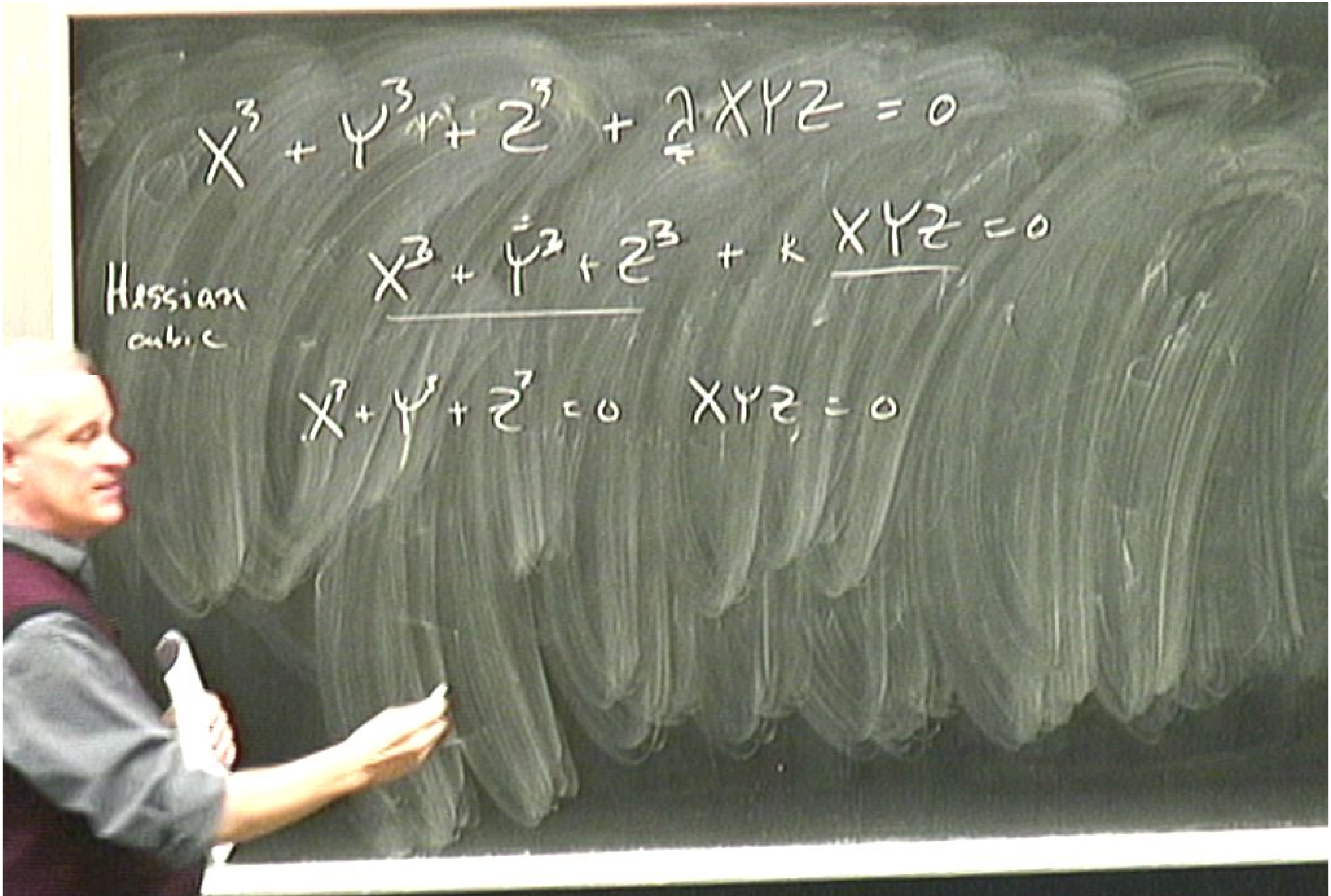
Hessian cubic

$$X^3 + Y^3 + Z^3 + k\, XYZ = 0$$

$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$

Hessian cubic

$\underline{X^3 + Y^3 + Z^3} + k\,\underline{XYZ} = 0$


$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$
Hessian cubic
$X^3 + Y^3 + Z^3 + k XYZ = 0$
$X^3 + Y^3 + Z^3 = 0$
$XYZ = 0$

$$X^3 + Y^3 + Z^3 + 2XYZ = 0$$

Hessian cubic

$$\underline{X^3 + Y^3 + Z^3} + k\,\underline{XYZ} = 0$$

$$X^3 + Y^3 + Z^3 = 0 \qquad XYZ = 0$$

$$Z = 0 \Longrightarrow X = -(1)^{1/3}\,Y$$

$= 0$

$XYZ = 0$ $\quad \omega = \frac{-1 + i\sqrt{3}}{2}, \quad \omega^3 = 1$

0

$$(X, Y, Z) =$$

$$(0\ 1\ -1) \quad (0\ 1\ -\omega) \quad (0\ 1\ -\omega^2)$$

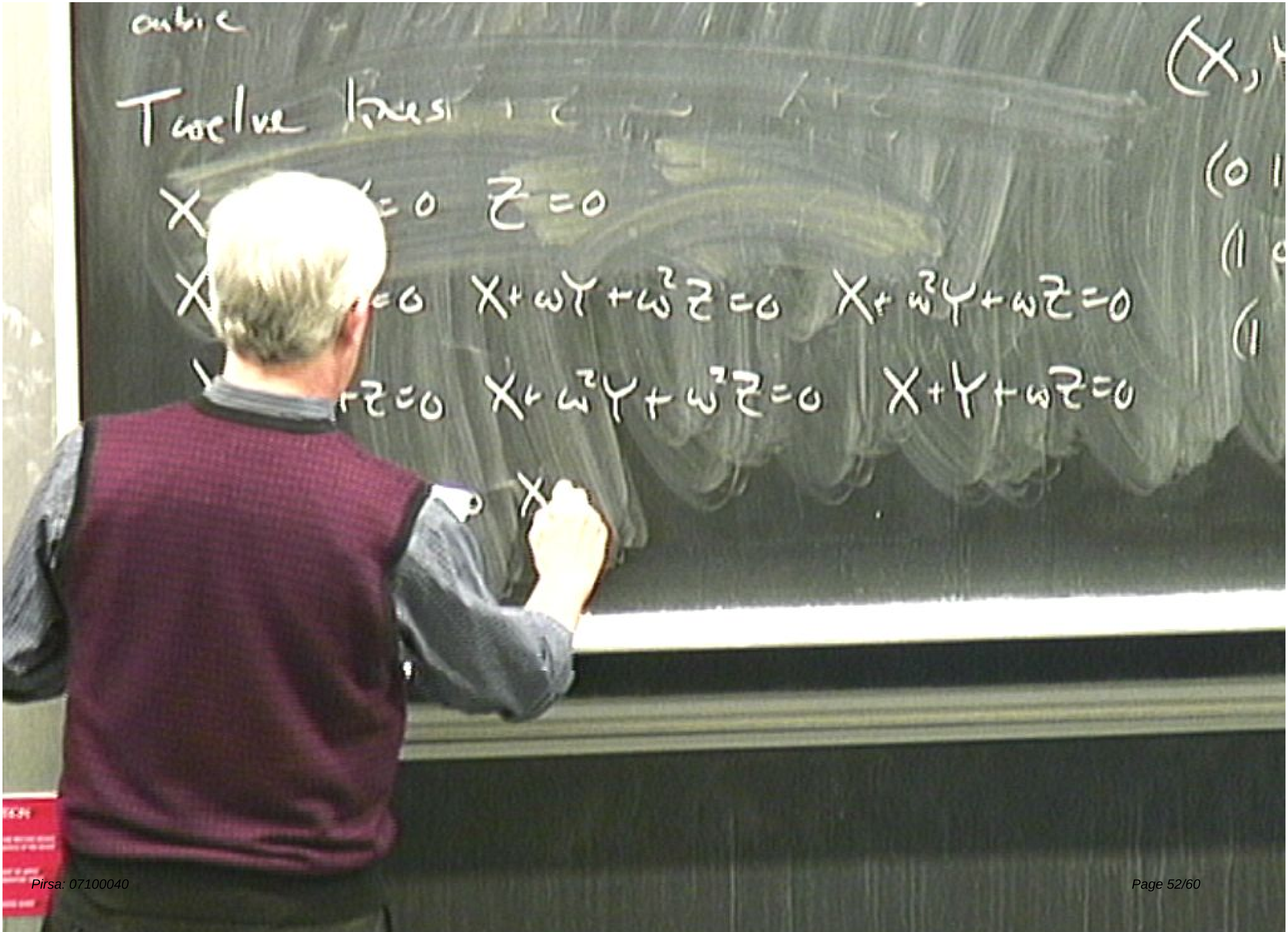
$$(1\ 0\ -1) \quad (1\ 0\ -\omega) \quad (1\ 0\ -\omega^2)$$

$$(1\ -1\ 0) \quad (1\ -\omega\ 0) \quad (1\ -\omega^2\ 0)$$

$$X Z^2 + Y^3 + \alpha Y X^2 + \beta X^3 = 0$$

$$\tilde{Z}^2 + \tilde{Y}^3 + \alpha \tilde{Y} + \beta = 0 \qquad \tilde{Y} = \frac{Y}{X} \qquad \tilde{Z} = \frac{Z}{X}$$

$$C_{\alpha\beta\gamma} = \bar{X}_\alpha \bar{X}_\beta \bar{X}_\gamma + \bar{Y}_\alpha \bar{Y}_\beta \bar{Y}_\gamma + \bar{Z}_\alpha \bar{Z}_\beta \bar{Z}_\gamma + \lambda \bar{X}_\alpha \bar{Y}_\beta \bar{Z}_\gamma$$


cubic
Twelve lines
Z = 0
X + ωY + ω²Z = 0   X + ω²Y + ωZ = 0
X + ω²Y + ω²Z = 0   X + Y + ωZ = 0

cubic

Twelve lines:

$X=0 \quad Y=0 \quad Z=0$

$X+Y+Z=0 \quad X+\omega Y+\omega^2 Z=0 \quad X+\omega^2 Y+\omega Z=0$

$X+\omega Y+Z=0 \quad X+\omega^2 Y+\omega^2 Z=0 \quad X+Y+\omega Z=0$

$X+\omega^2 Y+Z=0 \quad X+\omega Y+\omega Z=0 \quad X+Y+\omega^2 Z=0$

$(X, Y$

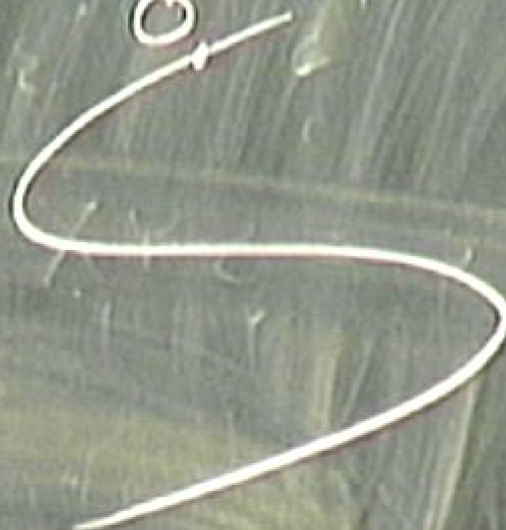
$(0 \; 1$

$(1 \; 0$

$(1$

$$\ldots^3 + Y^3 + Z^3 + k\,XYZ = 0 \qquad \omega = \frac{-1 + \ldots}{2}$$

O



$$(X, Y, Z) =$$

$$(0\ 1\ -1)\quad (0\ 1\ -\omega \ldots$$

$$(1\ 0\ -1)\quad (1\ 0\ -\omega \ldots$$

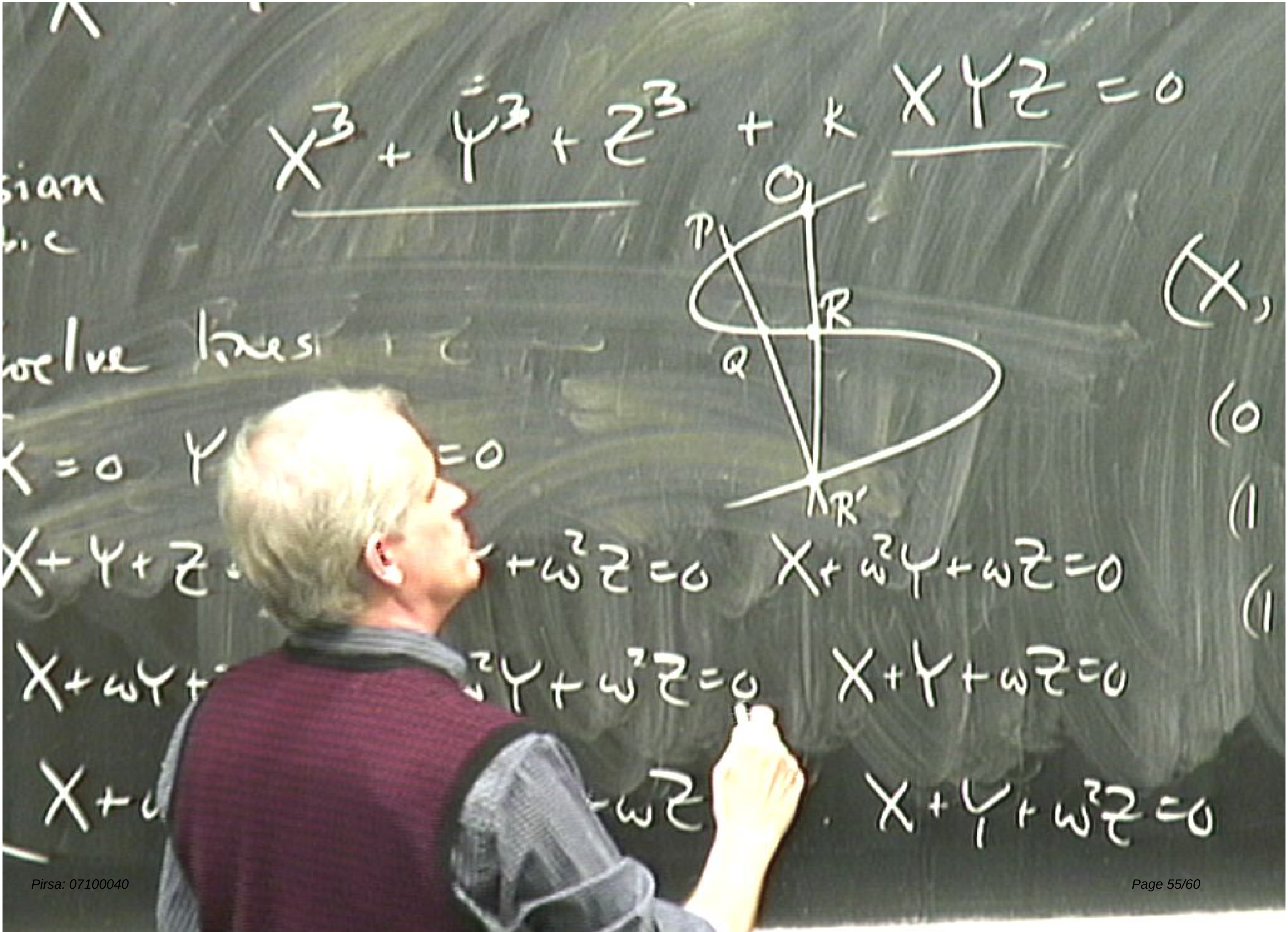
$$(1\ -1\ 0)\quad (1\ -\omega\ \ldots$$

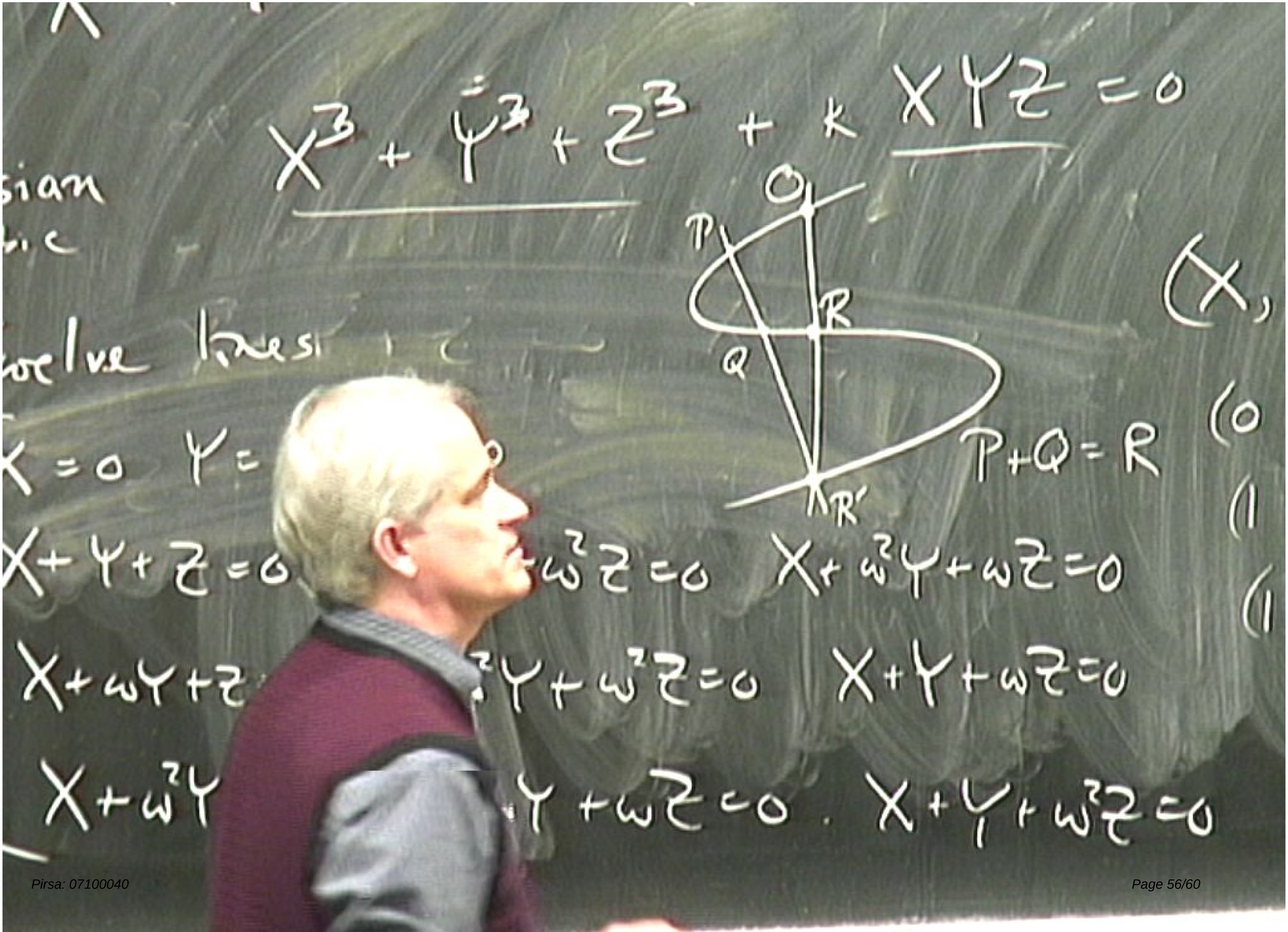
$$Z = 0$$

$$X + \omega Y + \omega^2 Z = 0 \qquad X + \omega^2 Y + \omega Z = 0$$

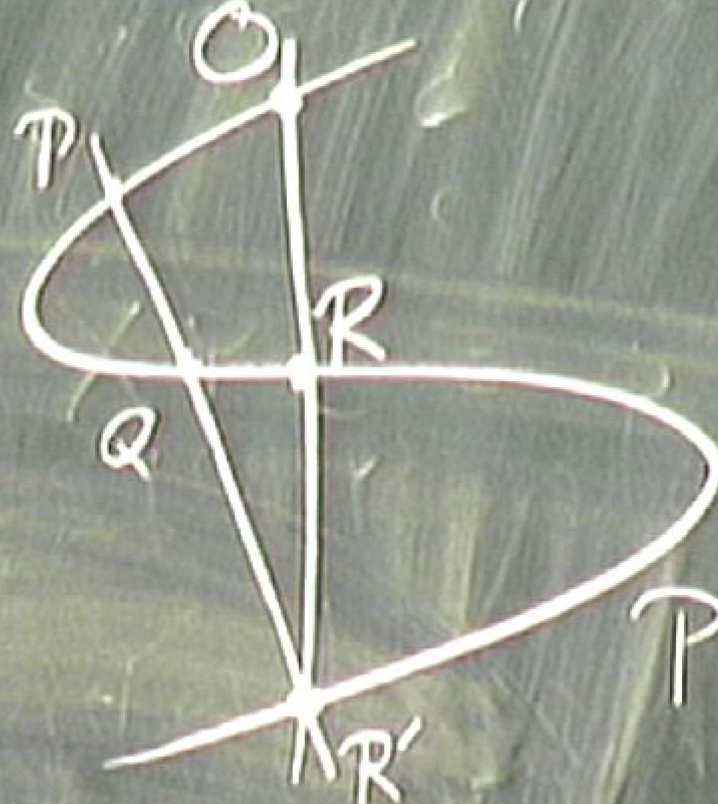
$$X + \omega^2 Y + \omega^2 Z = 0 \qquad X + Y + \omega Z = 0$$

$$\ldots = 0 \qquad X + \omega Y + \omega Z = 0 \qquad X + Y + \omega^2 Z = 0$$


X³ + Y³ + Z³ + k XYZ = 0
sian
tic
twelve lines
X = 0 Y =0
X + Y + Z +ω²Z = 0
X + ω²Y + ωZ = 0
X + ωY + ²Y + ω²Z = 0
X + Y + ωZ = 0
ωZ
X + Y + ω²Z = 0
O
P
Q
R
R′


X^3 + Y^3 + Z^3 + k XYZ = 0
sian
tic
elve lines
X = 0 Y =
X+Y+Z=0
w^2Z=0
X+w^2Y+wZ=0
X+wY+Z
Y+w^2Z=0
X+Y+wZ=0
X+w^2Y
Y+wZ=0
X+Y+w^2Z=0
O
P
Q
R
R'
P+Q=R
(X,
(0
(1
(1

$+ Y^3 + Z^3 + k\,XYZ = 0$



$Z = 0$

$P + Q = R$

$(X, Y$

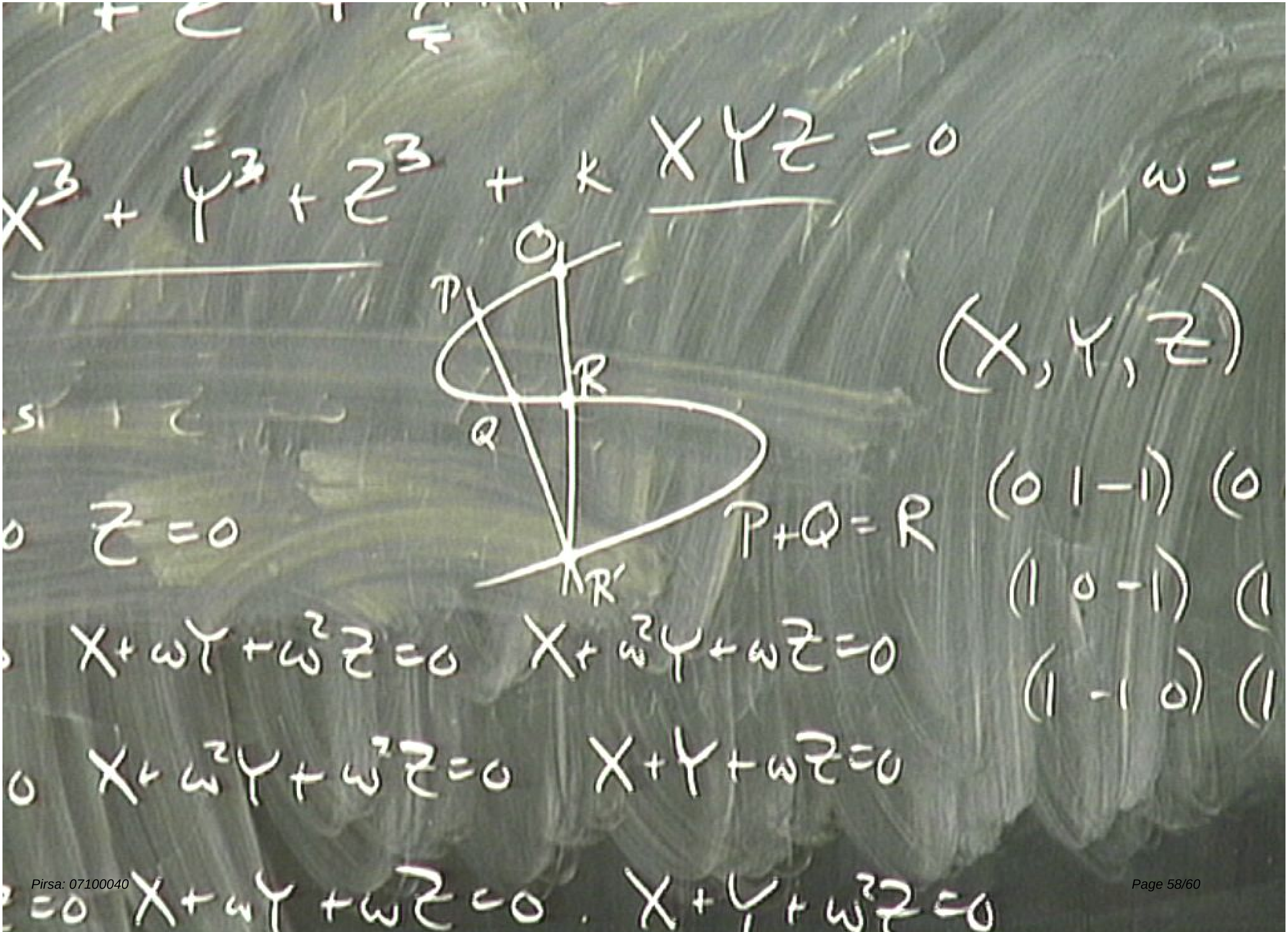
$(0\ 1$

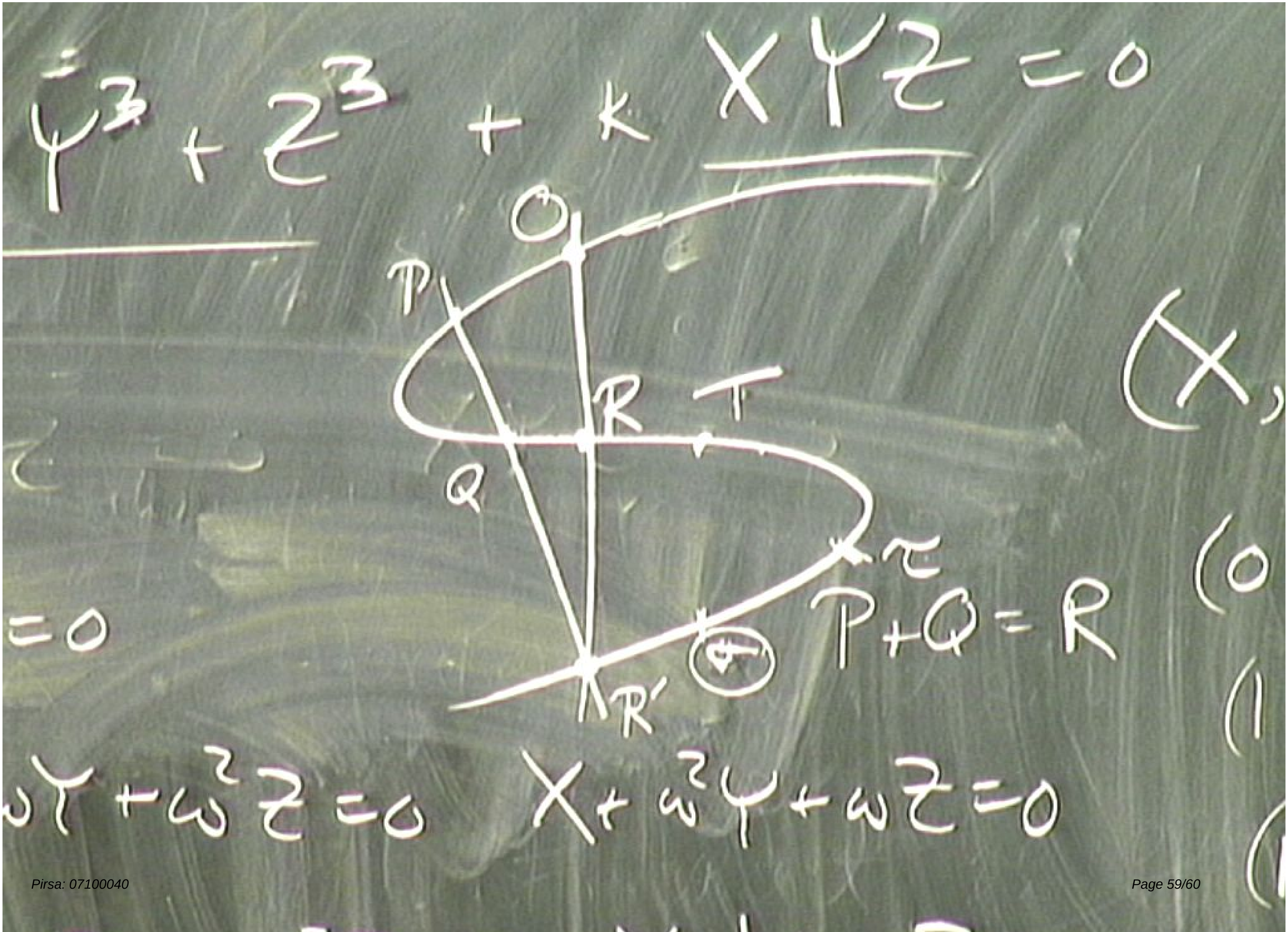
$(1$

$(1$

$+ \omega Y + \omega^2 Z = 0 \qquad X + \omega^2 Y + \omega Z = 0$

$X + \omega^2 Y + \omega^2 Z = 0 \qquad X + Y + \omega Z = 0$


X³ + Y³ + Z³ + k XYZ = 0
ω =
O
P
R
Q
R'
(X, Y, Z)
P+Q=R
(0 1 −1) (0
(1 0 −1) (1
(1 −1 0) (1
Z = 0
X+ωY+ω²Z=0
X+ω²Y+ωZ=0
X+ω²Y+ω²Z=0
X+Y+ωZ=0
X+ωY+ωZ=0
X+Y+ω²Z=0


Y³ + Z³ + k XYZ = 0
O
P
R
T
Q
τ
=0
P+Q=R
R′
(X,
(0
(1
ωY+ω²Z=0
X+ω²Y+ωZ=0

$$\omega = \frac{-1 + i\sqrt{3}}{2} \qquad \omega^3 = 1$$

$$(X, Y, Z) = \left\{ \begin{matrix} (0\ 1\ {-1}) & (0\ 1\ {-\omega}) & (0\ 1\ {-\omega^2}) \\ (1\ 0\ {-1}) & (1\ 0\ {-\omega}) & (1\ 0\ {-\omega^2}) \\ (1\ {-1}\ 0) & (1\ {-\omega}\ 0) & (1\ {-\omega^2}\ 0) \end{matrix} \right\}$$

$P + Q = R$

$+\omega Z = 0$

$+\omega Z = 0$

$+Y + \omega^2 Z = 0$