Title: Quantum Reference Frames and the Classification of Rotationally-Invariant Maps

Date: Oct 24, 2007 04:00 PM

URL: http://pirsa.org/07100033

Abstract: We give a convenient representation for any map which is covariant with respect to an irreducible representation of SU(2), and use this representation to analyze the evolution of a quantum directional reference frame when it is exploited as a resource for performing quantum operations. We introduce the moments of a quantum reference frame, which serve as a complete description of its properties as a frame, and investigate how many times a quantum directional reference frame represented by a spin-j system can be used to perform a certain quantum operation with a given probability of success. We provide a considerable generalization of previous results on degradation of reference frame, from which follows a classification of the dynamics of spin-j system under the repeated action of any covariant map with respect to SU(2).
Joint work with Lana Sheridan, Martin Laforest and Stephen Bartlett

Quantum Reference Frames and the Classification of Rotationally-Invariant Maps

Jean Christian Boileau (University of Toronto)
Lana Sheridan (University of Waterloo)
Martin Laforest (University of Waterloo)
Stephen Bartlett (University of Sydney)

Reference : arXiv:0709.0142

NSERC
CRSNG

Presented at the Perimeter Institute
October 24th 2007

Topics:

- Introduce a representation for any map which is covariant with respect to an irreducible representation of SU(2).
  - Proof
- Use this representation to study the dynamics of quantum directional reference frame
  - What is a quantum directional reference frame (QDRF)?
  - We introduce the *moments* of a QDRF.
  - We generalize the concept of *quality* and *longevity* of a QDRF, and analyze them in function of the evolution of the moments.
  - Examples:
    - Measurement of a spin-1 particle.
    - One qubit Rotation
- Conclusion and Open Questions

Part I:

The classification of Rotationally-Invariant Maps

Background about SU(2)

- SU(2): set of 2 by 2 unitary complex matrices with determinant one
  - ✓ SU(2) is isomorphic (up to a sign) to the group of space rotations SO(3)
- SU(2) is a (simply connected) Lie group, and the corresponding Lie Algebra is spanned by the three Pauli matrices
- Consider an irreducible representation of SU(2) acting on a (2*j*+1)-dimensional Hilbert space

A representation *R* is a homomorphism

$$R : SU(2) \rightarrow U(H_{2j+1})$$

It is irreducible if there exist no none-trivial subspace $S \in H_{2j+1}$ such that $R(\Omega)s \in S$ for all $\Omega \in SU(2)$ and $s \in S$.

Background about SU(2)

- There exists operators J_x, J_y and J_z such that

$$[J_x, J_y] = iJ_z \qquad [J_y, J_z] = iJ_x \qquad [J_z, J_x] = iJ_y$$

- In physics, J_x, J_y and J_z represented the angular momentum operator of a spin-j.
- Every element R of the irrep of SU(2) can be written as $e^{iv\hat{J}}$ for some vector v and where $\hat{J} := (J_x, J_y, J_z)$.

Covariance With Respect to SU(2)

- We say that a map χ is covariant with respect to SU(2) iff

$$\chi[R(\Omega)(\cdot)R(\Omega)^\dagger] = R(\Omega)\chi(\cdot)R(\Omega)^\dagger$$

for all $\Omega \in SU(2)$.

i.e. R is the irrep of SU(2) for some $(2j+1)$-dimensional Hilbert space.

Representation for maps which are covariant with respect to an irreducible representation of SU(2)

- Define

$$\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k\in\{x,y,z\}} J_k \rho_j J_k$$

where ρ_j is a state associated to the Hilbert space of dimension *2j+1*.

Theorem 1: Any map ξ which is covariant with respect to a spin-*j* irreducible representation of SU(2) has the form

$$\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$$

for some real coefficients q_n.

Proof

- First show that $\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k \rho_j J_k$ is covariant with respect to SU(2):

$$R(\Omega)^{-1}\zeta(R(\Omega)\rho_j R(\Omega)^{-1})R(\Omega) = \zeta(\rho_j) \qquad \forall\, \Omega \in SU(2).$$

Any SU(2) element can be decomposed into rotations around the Y and Z axes:

$$R(\Omega) = R_z(\theta)R_y(\psi)R_z(\phi)$$

Consider: $R_z(\theta)$

$$\begin{cases} J_x \to \cos\theta J_x + \sin\theta J_y \\ J_y \to -\sin\theta J_x + \cos\theta J_y \\ J_z \to J_z \end{cases}$$

- Therefore any map of the form $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$ must also be covariant.
- To prove that every covariant map can be written as above, we use the Liouville representation:

$$\rho_j \to |\rho_j\rangle\rangle \quad \text{vector of dimension } d^2$$

Proof

- First show that $\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k \rho_j J_k$ is covariant with respect to SU(2):

$$R(\Omega)^{-1}\zeta(R(\Omega)\rho_j R(\Omega)^{-1})R(\Omega) = \zeta(\rho_j) \qquad \forall\, \Omega \in SU(2).$$

Any SU(2) element can be decomposed into rotations around the Y and Z axes:

$$R(\Omega) = R_z(\theta)R_y(\psi)R_z(\phi)$$

Consider: $R_z(\theta)$

$$\left\{ \begin{array}{l} J_x \to \cos\theta J_x + \sin\theta J_y \\ J_y \to -\sin\theta J_x + \cos\theta J_y \\ J_z \to J_z \end{array} \right.$$

- Therefore any map of the form $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$ must also be covariant.
- To prove that every covariant map can be written as above, we use the Liouville representation:

$$\rho_j \rightarrow |\rho_j\rangle\rangle \quad \text{vector of dimension } d^2$$

Liouville Representation of a superoperator

Kraus Representation: $$\xi(\cdot) = \sum_k E_k \cdot E_k^\dagger$$

$$\updownarrow$$

Liouville Representation: $$\mathcal{K}(\xi) = \sum_k E_k^* \otimes E_k$$

Covariance in the Liouville Representation

$$(R^*(\Omega) \otimes R(\Omega))\mathcal{K}(\xi) = \mathcal{K}(\xi)(R^*(\Omega) \otimes R(\Omega))$$

for all $\Omega \in SU(2)$.

For an irrep of SU(2): $R^*(\Omega) = e^{-i\pi J_y} R(\Omega) e^{i\pi J_y}$

So $(R(\Omega) \otimes R(\Omega))\mathcal{K}'(\xi) = \mathcal{K}'(\xi)(R(\Omega) \otimes R(\Omega))$

where $\mathcal{K}'(\xi) = (e^{i\pi J_y} \otimes I)\mathcal{K}(\xi)(e^{-i\pi J_y} \otimes I)$

- For $\Omega \in SU(2)$, all irreps of the group generated by $R(\Omega) \otimes R(\Omega)$ have multiplicity one. Each irrep correspond to an integer from *0* to *2j*.
- By Schur's Lemma,

$$\mathcal{K}'(\xi) = \sum_{k=0}^{2j} c_k \Pi_k$$

  for some complex parameter c_k and orthogonal projectors Π_k.

- Therefore, there is exactly *2j+1* linearly independent matrix that can be used as a basis to write -- in the Liouville representation -- any covariant map $\mathcal{K}(\xi)$ with respect to SU(2).
  - ✓ We need to find the right set of linearly independent matrices.

Covariance in the Liouville Representation

$$(R^*(\Omega) \otimes R(\Omega))\mathcal{K}(\xi) = \mathcal{K}(\xi)(R^*(\Omega) \otimes R(\Omega))$$

for all $\Omega \in SU(2)$.

For an irrep of SU(2): $R^*(\Omega) = e^{-i\pi J_y} R(\Omega) e^{i\pi J_y}$

So $(R(\Omega) \otimes R_j(\Omega))\mathcal{K}'(\xi) = \mathcal{K}'(\xi)(R(\Omega) \otimes R(\Omega))$

where $\mathcal{K}'(\xi) = (e^{i\pi J_y} \otimes I)\mathcal{K}(\xi)(e^{-i\pi J_y} \otimes I)$

- Therefore any map of the form $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$ must also be covariant.
- To prove that every covariant map can be written as above, we use the Liouville representation:

$$\rho_j \to |\rho_j\rangle\rangle \quad \text{vector of dimension } d^2$$

Covariance in the Liouville Representation

$$(R^*(\Omega) \otimes R(\Omega))\mathcal{K}(\xi) = \mathcal{K}(\xi)(R^*(\Omega) \otimes R(\Omega))$$

for all $\Omega \in SU(2)$.

For an irrep of SU(2): $R^*(\Omega) = e^{-i\pi J_y} R(\Omega) e^{i\pi J_y}$

So $(R(\Omega) \otimes R_j(\Omega))\mathcal{K}'(\xi) = \mathcal{K}'(\xi)(R(\Omega) \otimes R(\Omega))$

where $\mathcal{K}'(\xi) = (e^{i\pi J_y} \otimes I)\mathcal{K}(\xi)(e^{-i\pi J_y} \otimes I)$

- For $\Omega \in SU(2)$, all irreps of the group generated by $R(\Omega) \otimes R(\Omega)$ have multiplicity one. Each irrep correspond to an integer from *0* to *2j*.
- By Schur's Lemma, $$\mathcal{K}'(\xi) = \sum_{k=0}^{2j} c_k \Pi_k$$

  for some complex parameter c_k and orthogonal projectors Π_k.
- Therefore, there is exactly *2j+1* linearly independent matrix that can be used as a basis to write -- in the Liouville representation -- any covariant map $\mathcal{K}(\xi)$ with respect to SU(2).
  - ✓ We need to find the right set of linearly independent matrices.

Kraus representation:

$$\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k\in\{x,y,z\}} J_k \rho_j J_k$$

$$\downarrow$$

Liouville representation:

$$\mathcal{K}(\zeta) = \frac{1}{j(j+1)} \sum_{k\in\{x,y,z\}} J_k^* \otimes J_k$$

$$\zeta^{\circ n}(\rho_j)$$

$$\downarrow$$

Liouville representation:

$$\mathcal{K}(\zeta^{\circ n}) = (\frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k^* \otimes J_k)^n$$

Need to show that the matrices $(\sum_{k \in \{x,y,z\}} J_k^* \otimes J_k)^n$

are linearly independent for $0 \leq n \leq 2j$.

It is not to hard to show:

(a) $$\sum_i J_i^* \otimes J_i = -e^{-i\pi J_y} \otimes I \left(\sum_i J_i \otimes J_i \right) e^{i\pi J_y} \otimes I$$

(b) $$\sum_i J_i \otimes J_i = \frac{1}{2}(\mathcal{J}^2 - 2j(j+1)I)$$

where $$\mathcal{J}^2 = \sum_i (J_i \otimes I + I \otimes J_i)^2$$

(total angular momentum)

(a) implies $\sum_i J_i^* \otimes J_i$ and $\sum_i J_i \otimes J_i$ share the same number of distinct eigenvalues.

(b) implies that $\sum_i J_i \otimes J_i$ has 2j+1 distinct eigenvalues.

- By the fundamental theorem of algebra, this implies that there exist no polynomial of degree 2j that has the 2j+1 distinct eigenvalues as roots.
- This implies the matrices $\left(\sum_{k\in\{x,y,z\}} J_k^* \otimes J_k\right)^n$ for $0 \leq n \leq 2j$ are linearly independent.

- We showed that $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$ where the q_n are complex.
- The q_n are in fact real.
  - Proof by induction based on the fact that ξ is a positive map.

Open Question

- What about covariance with respect to other Lie Group? Is there any interesting representation in function of the associated Lie Algebra generators?
- What are the restrictions on the q_n parameters?

Part II:

Dynamics of Quantum Directional Reference Frame

What is a Quantum Directional Reference Frame?

- Consider the initial state of a spin-j: $\rho_j^{(0)}$
- Suppose that $\rho_j^{(0)}$ depends only on some "classical" direction $\hat{n}$.
  - If R is the rotation that transforms $\hat{n}$ to $\hat{n}'$, then
$$R\rho_j^{(0)}(\hat{n})R^{-1} = \rho_j^{(0)}(\hat{n}')$$
  - The state $\rho_j^{(0)}$ is also covariant under rotations about the $\hat{n}$-axis.
- Therefore, $\rho_j^{(0)}$ is diagonal in the basis consisting of the eigenvectors of $J_{\hat{n}}$.

Scenario

Quantum Reference Frame

Reservoir: contains many identical subsystems of dimension *d*.

$\rho_j^{(0)}$

$\textcircled{1}$ $\textcircled{2}$ $\textcircled{3}$ ...

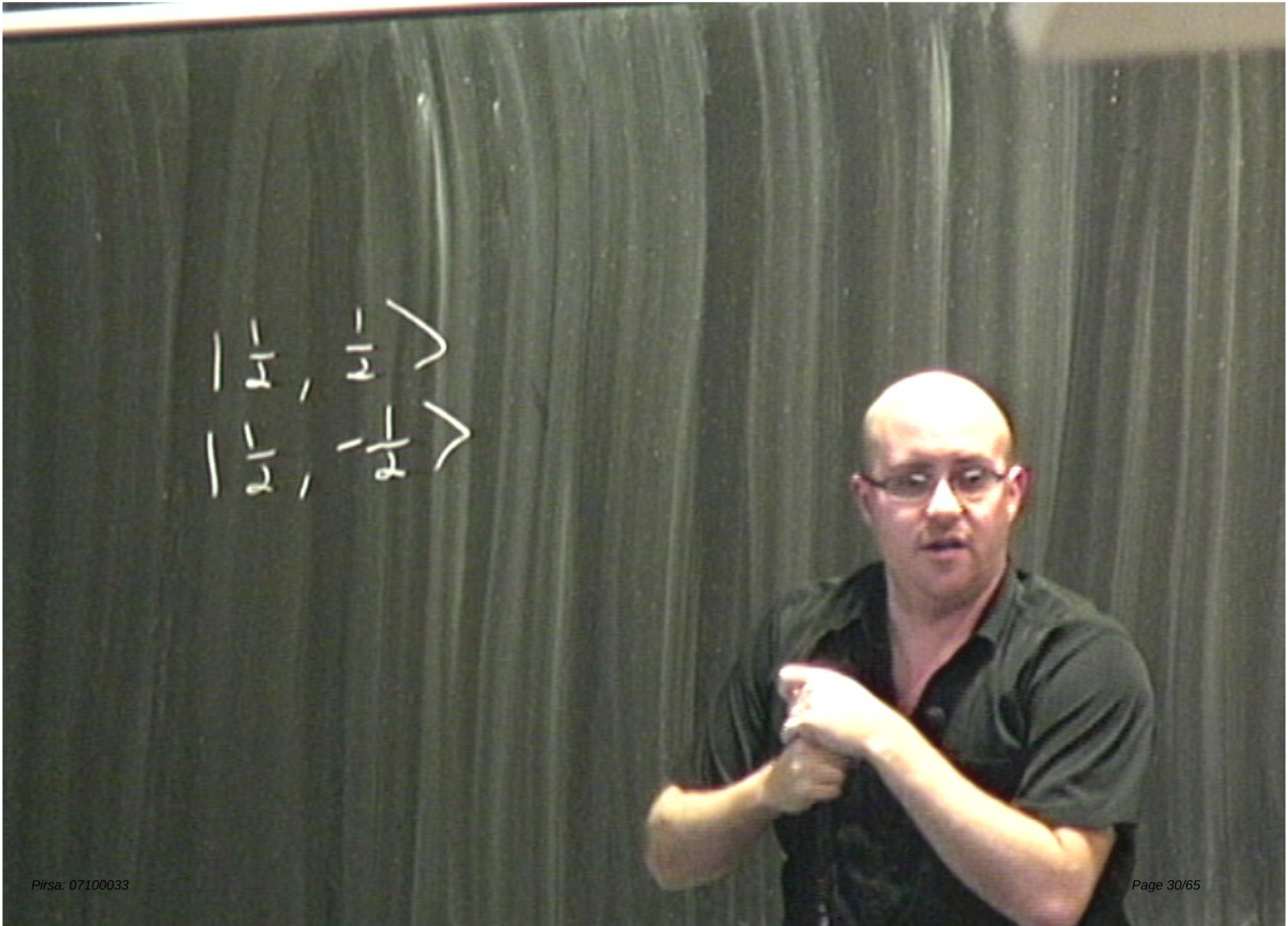
apply the map

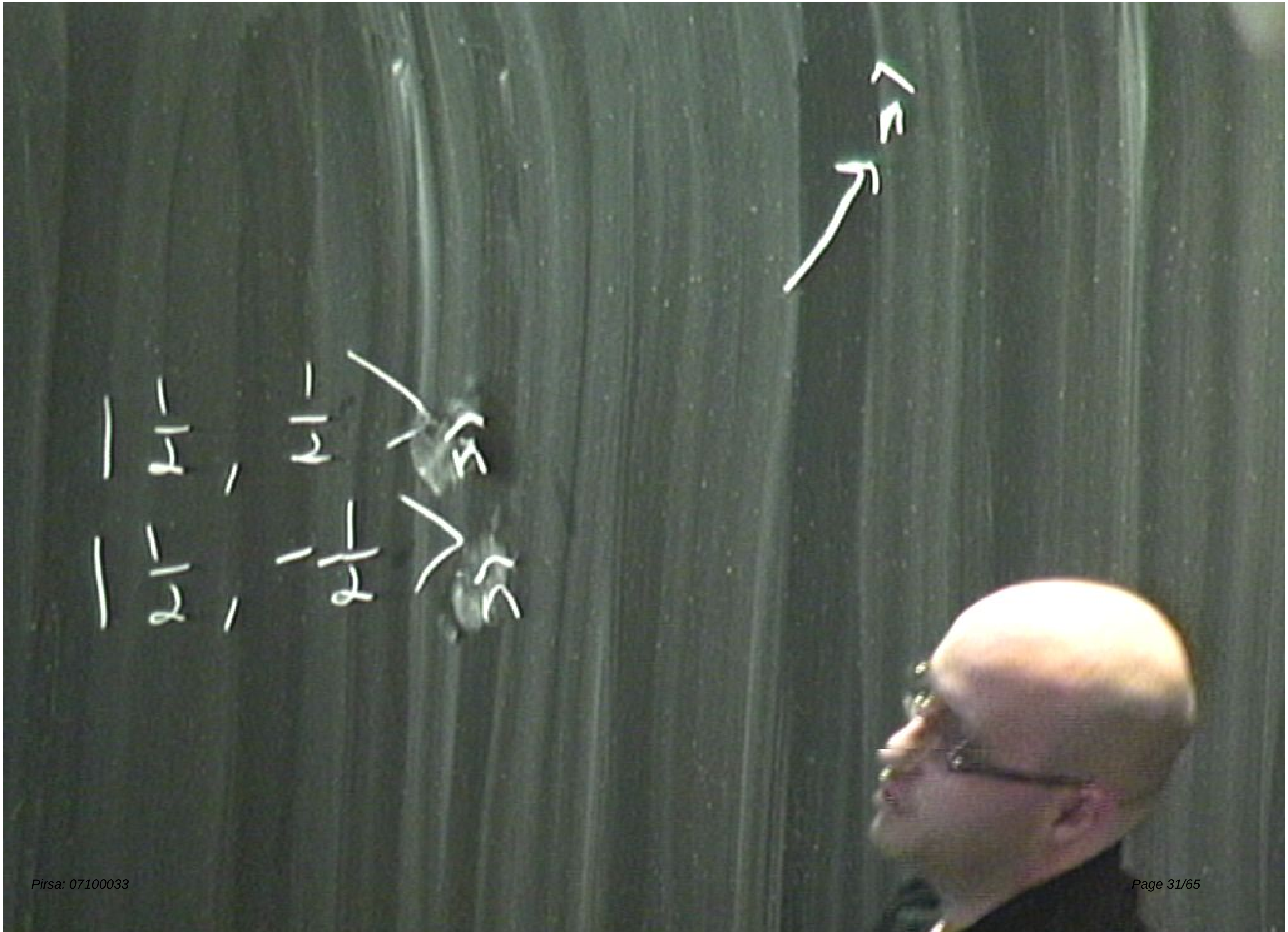
$$\chi(\rho_j^{(0)} \otimes \textcircled{1})$$

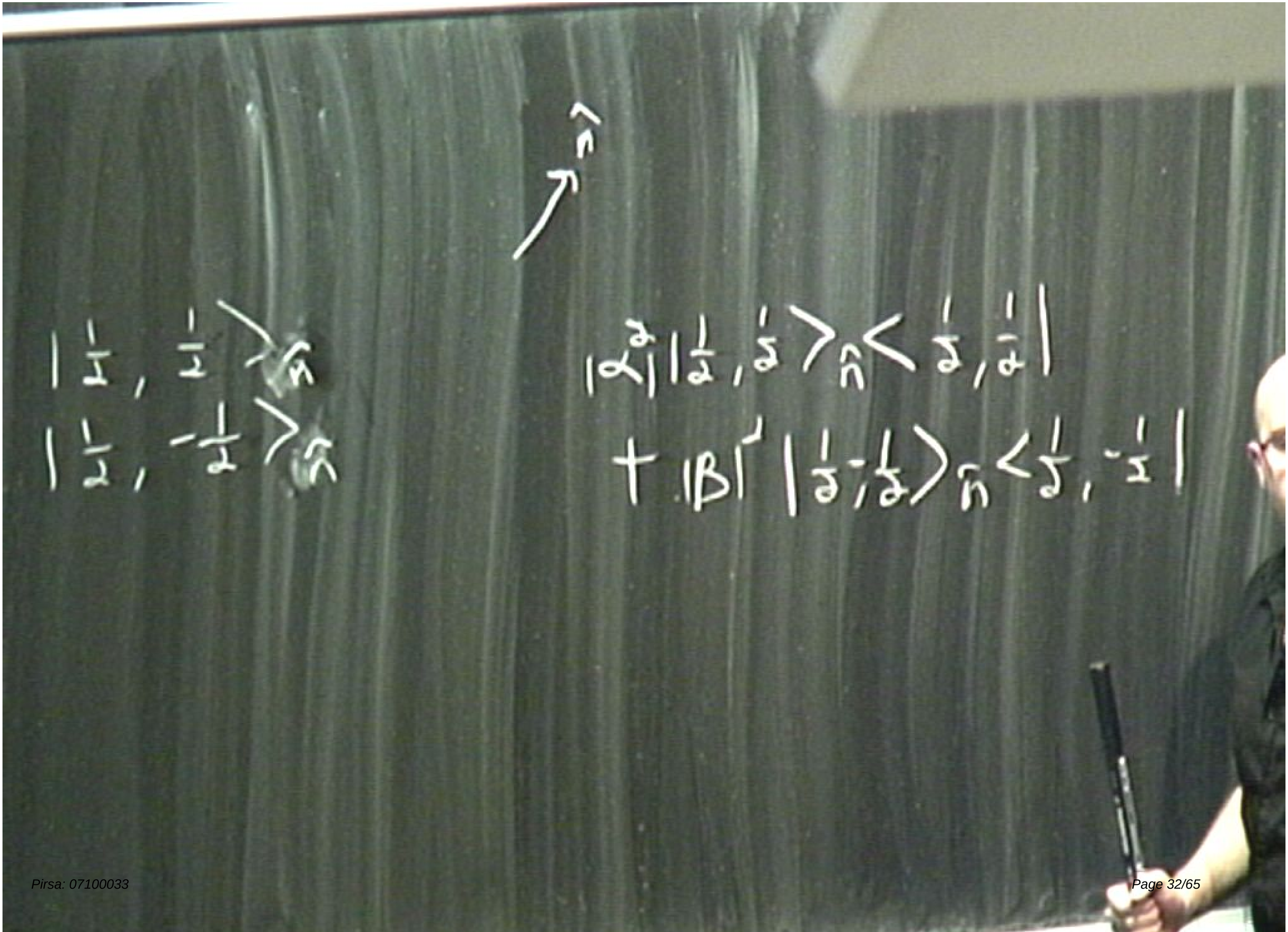
$$\rho_j^{(1)} = \xi(\rho_j^{(0)}) = Tr_{\textcircled{1}}[\chi(\rho_j^{(0)} \otimes \textcircled{1})]$$

What is a Quantum Directional Reference Frame?

- Consider the initial state of a spin-j: $\rho_j^{(0)}$
- Suppose that $\rho_j^{(0)}$ depends only on some "classical" direction $\hat{n}$.
  - If R is the rotation that transforms $\hat{n}$ to $\hat{n}'$, then
  $$R\rho_j^{(0)}(\hat{n})R^{-1} = \rho_j^{(0)}(\hat{n}')$$
  - The state $\rho_j^{(0)}$ is also covariant under rotations about the $\hat{n}$-axis.
- Therefore, $\rho_j^{(0)}$ is diagonal in the basis consisting of the eigenvectors of $J_{\hat{n}}$.


|1/2, 1/2⟩
|1/2, -1/2⟩


$\hat{n}$
$|\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}}$
$|\frac{1}{2}, -\frac{1}{2}\rangle_{\hat{n}}$


n̂
|1/2, 1/2⟩_n̂
|1/2, −1/2⟩_n̂
|α|² |1/2, 1/2⟩_n̂ ⟨1/2, 1/2|
+ |β|² |1/2, −1/2⟩_n̂ ⟨1/2, −1/2|

Scenario

Quantum Reference Frame

Reservoir: contains many identical subsystems of dimension *d*.

$\rho_j^{(0)}$ $\quad$ ① ② ③...

apply the map

$$\chi(\rho_j^{(0)} \otimes \textcircled{1})$$

$$\rho_j^{(1)} = \xi(\rho_j^{(0)}) = Tr_{\textcircled{1}}[\chi(\rho_j^{(0)} \otimes \textcircled{1})]$$

Quantum Reference Frame

$\rho_j^{(n-1)}$

Reservoir: contains many identical subsystems of dimension *d*.

apply the map

$\chi(\rho_j^{(n-1)} \otimes \textcircled{n})$

X: discarded

$$\rho_j^{(n)} = \xi(\rho_j^{(n-1)}) = Tr_{\textcircled{n}}[\chi(\rho_j^{(n-1)} \otimes \textcircled{n})]$$

Extra Assumptions:

- The joint map χ is rotationally-invariant.
- The state of the subsystems in the reservoir are invariant under space-rotations.

✓ *This implies that* the back-action map ξ on the quantum reference frame is rotationally-invariant.

  ✓ Which implies that $\rho_j^{(k)}$ for all k is diagonal in the basis given by the eigenvectors of $J_{\hat{n}}$

Quantum Reference Frame

$\rho_j^{(n-1)}$

Reservoir: contains many identical subsystems of dimension *d*.

apply the map

$\chi(\rho_j^{(n-1)} \otimes n)$

X: discarded

$$\rho_j^{(n)} = \xi(\rho_j^{(n-1)}) = Tr_{n}\,[\chi(\rho_j^{(n-1)} \otimes n)]$$

Extra Assumptions:

- The joint map χ is rotationally-invariant.
- The state of the subsystems in the reservoir are invariant under space-rotations.

✓ *This implies that* the back-action map ξ on the quantum reference frame is rotationally-invariant.

  ✓ Which implies that $\rho_j^{(k)}$ for all k is diagonal in the basis given by the eigenvectors of $J_{\hat{n}}$

Previous Related Works

1) S. Bartlett, T. Rudolph, R. Spekkens, and P. Turner, **Degradation of a Quantum Reference Frame,** *New J. Phys.* **8**, 58 (2006).
2) D. Poulin and J. Yard, **Dynamics of a Quantum Reference Frame**, *New J. Phys.* **9**, 156 (2007).
   a) The joint operator (χ) considered is restricted to measurements
   b) d=2
   c) the *quality function* is fixed (somewhat arbitrarily).

In 2), the states of the subsystems of the reservoir are not necessarily rotationally-invariant.

We use the term **quality function** for any function ***F*** that is meant to quantify the ability of the reference frame to perform a particular task.

We use the term **quality function** for any function ***F*** that is meant to quantify the ability of the reference frame to perform a particular task.

The quality measure should not be biased such that it favors a quantum reference frame that is pointed in any particular direction relative to some external frame. All directions must be equally valid. Therefore, ***F*** does not depend on the direction of $\hat{n}$, but only on the eigenvalues of ρ_j.

Moments

- An equivalent set of parameters to the eigenvalues of ρ_j are the moments of ρ_j:

$$\{Tr[\rho_j J_{\hat{n}}^{\ell}] \mid 1 \le \ell \le 2j\}$$

- Any quality function ***F*** depends only on those moments.
- To analyze the behavior of ***F***, it is sufficient to study the evolution of the moments.

General Recursion Formula for the Moments

$$Tr[\rho_j^{(k)} J_{\hat{n}}^{\ell}] = \sum_{i=0}^{2j} A_i^{(\ell)} Tr[\rho_j^{(k-1)} J_{\hat{n}}^{i}]$$

where the $A_i^{(\ell)}$'s are real coefficients.

Get ride of some of the coefficients

Theorem 2:

If ℓ is even, then

$$\mathrm{Tr}[\rho_j^{(k)} J_{\hat{n}}^{\ell}] = \sum_{i=0}^{\ell/2} A_{2i}^{(\ell)} \mathrm{Tr}[\rho_j^{(k-1)} J_{\hat{n}}^{2i}]$$

and if ℓ is odd, then

$$\mathrm{Tr}[\rho_j^{(k)} J_{\hat{n}}^{\ell}] = \sum_{i=1}^{(\ell+1)/2} A_{2i-1}^{(\ell)} \mathrm{Tr}[\rho_j^{(k-1)} J_{\hat{n}}^{2k-1}].$$

Proof: Corollary of Theorem 1 (by induction using commutator relations).

Longevity

- We are interested in the scaling, with respect to Hilbert space dimension, of how many times a quantum reference frame can be used before the value of its quality function ***F*** falls below a certain threshold.

BRST06: Longevity scales as $O(j^2)$. (specific ***F***)

- Because we consider any quality function ***F***, the longevity of the reference frame can be arbitrary (in general).
- But we can study the scaling of the moments.

Theorem 3: Consider a quantum reference frame with initial state $\rho_j^{(0)}$, which is used for performing a rotationally-invariant joint operation χ_j. If this operation induces a disturbance map

$$\xi = \sum_{n=0}^{2j} q_n(j)\zeta^{\circ n}$$

that satisfies the following assumptions:

- there exists some n_{max} such that $q_n=0$ for all $n \geq n_{max}$,
- $q_n \leq O(1)$ and
- $Tr[\rho_j^{(0)} J_{\hat{n}}^{\ell}] = O(j^{\ell})$.

then the number of times that such a quantum reference frame can be used before its ℓ^{th} moment falls below a certain threshold value scales as j^2.

The proof is based on Theorem 2.

Example A.1: Measurements of spin-1/2

- Suppose that the reservoir consists of spin-1/2 systems. Each spin is either parallel or anti-parallel to $\hat{n}$ (with the same probability).
- The goal is to use the quantum reference frame to guess the direction of each spin-1/2.
- The optimal joint measurement χ is a projection onto the subspaces corresponding to different values of the total angular momentum.
- $$F_{\frac{1}{2}}^{(k)} := Tr\left[\frac{1}{2}\sum_{\mu\in\{-\frac{1}{2},\frac{1}{2}\}}\Pi_{j+\mu}(\rho_j^{(k)}\otimes|\mu\rangle\langle\mu|)\right]$$

Measurement of spin-1/2

- In term of the first moment, we can rewrite the quality function:

$$F_{\frac{1}{2}}^{(k)} = \frac{1}{2} + \frac{1}{2j+1} Tr[\rho_j^{(k)} J_z]$$

- Theorem 2 tells us

$$Tr[\rho_j^{(k)} J_z] = Tr[\rho_j^{(0)} J_z] \left(A_0^{(1)} \right)^k .$$

- Simple calculation give us

$$A_0^{(1)} = 1 - \frac{2}{(2j+1)^2}$$

Example A.2: Measurements of spin-1

- Suppose that the reservoir consists of spin-1 systems. Each spin has either 1, 0 or -1 angular momentum in the $\hat{n}$ direction (with the same probability).
- The goal is to use the quantum reference frame to guess the angular momentum in the $\hat{n}$ direction of each spin-1.
- The optimal joint measurement χ is a projection onto the subspaces corresponding to different values of the total angular momentum.
- $$F_1^{(k)} = Tr\left[\frac{1}{3}\sum_{\mu\in\{-1,0,1\}} \Pi_{j+\mu}(\rho_j^{(k)} \otimes |\mu\rangle\langle\mu|)\right]$$

Measurements of spin-1

- In terms of moments:

$$F_1^{(k)} = \frac{1}{6} + \frac{[(2j+1)^2 - 2]}{6j(j+1)(2j+1)} Tr[\rho_j^{(k)} J_z] + \frac{1}{2j(j+1)} Tr[\rho_j^{(k)} J_z^2].$$

- We can use again Theorem 2 to evaluate the moments. Simple calculations give us the values of the three *A*'s coefficients.

Example B: Pauli Operator

- Suppose, we want to implement a Pauli Z operation on a qubit:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

using a quantum reference frame to define the z-axis.

- Cannot be implemented without errors if the reference frame is quantum and restricted to a finite space.

Gate Fidelity

- We can pick the quality function to be

$$F_{\text{gate}}(Z,\tau) \equiv \frac{\int_\Omega d\mu_\Omega \left| Tr[E(\Omega)^\dagger Z] \right|^2 + d}{d^2 + d}$$

where $E(\Omega)$ are the Kraus operators of the approximate gate.

Different Methods To Implement the Gate

1. Projective Measurement

$$\{\Lambda(\Omega) = (2j+1)R(\Omega)|e\rangle\langle e|R(\Omega)^\dagger,\ \Omega \in SU(2)\}$$

where $|e\rangle := |j, m_z = j\rangle$

Gate Fidelity

- We can pick the quality function to be

$$F_{\text{gate}}(Z,\tau) \equiv \frac{\int_\Omega d\mu_\Omega \left|Tr[E(\Omega)^\dagger Z]\right|^2 + d}{d^2 + d}$$

where $E(\Omega)$ are the Kraus operators of the approximate gate.

Different Methods To Implement the Gate

1. Projective Measurement

$$\{\Lambda(\Omega) = (2j+1)R(\Omega)|e\rangle\langle e|R(\Omega)^{\dagger},\ \Omega \in SU(2)\}$$

where $|e\rangle := |j, m_z = j\rangle$

Different Methods To Implement the Gate

2. Filtering operation:

$$(2j+1)\int_{\Omega} d\mu_{\Omega} R_j(\Omega)\otimes R_{1/2}(\Omega)\Big[|j,j\rangle_z\langle j,j|\otimes Z\Big] R_j(\Omega)^{-1}\otimes R_{1/2}(\Omega)^{-1}$$

$$=$$

$$\frac{2j}{2j+2}\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}$$ (not unitary)

where Π_k is the projector into the subspace of total angular momentum k.

Different Methods To Implement the Gate

3. Use coupling between the spins of the quantum reference frame and of the reservoir :

Use $$H = w\left(J_x^{(\frac{1}{2})}J_x^{(j)} + J_y^{(\frac{1}{2})}J_y^{(j)} + J_z^{(\frac{1}{2})}J_z^{(j)}\right)$$

to implement $$\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}$$ (unitary)

Different Methods To Implement the Gate

2. Filtering operation:

$$(2j+1)\int_{\Omega} d\mu_{\Omega} R_j(\Omega)\otimes R_{1/2}(\Omega)\Big[|j,j\rangle_z\langle j,j|\otimes Z\Big] R_j(\Omega)^{-1}\otimes R_{1/2}(\Omega)^{-1}$$

$$=$$

$$\frac{2j}{2j+2}\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}$$ (not unitary)

where Π_k is the projector into the subspace of total angular momentum *k*.

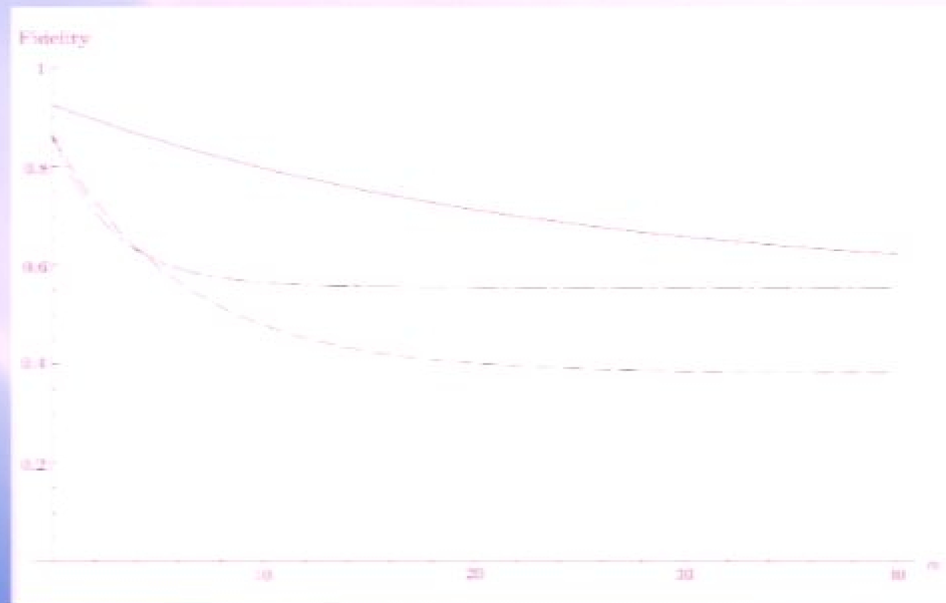
Different Methods To Implement the Gate

3. Use coupling between the spins of the quantum reference frame and of the reservoir :

Use $$H = w\left(J_x^{(\frac{1}{2})}J_x^{(j)} + J_y^{(\frac{1}{2})}J_y^{(j)} + J_z^{(\frac{1}{2})}J_z^{(j)}\right)$$

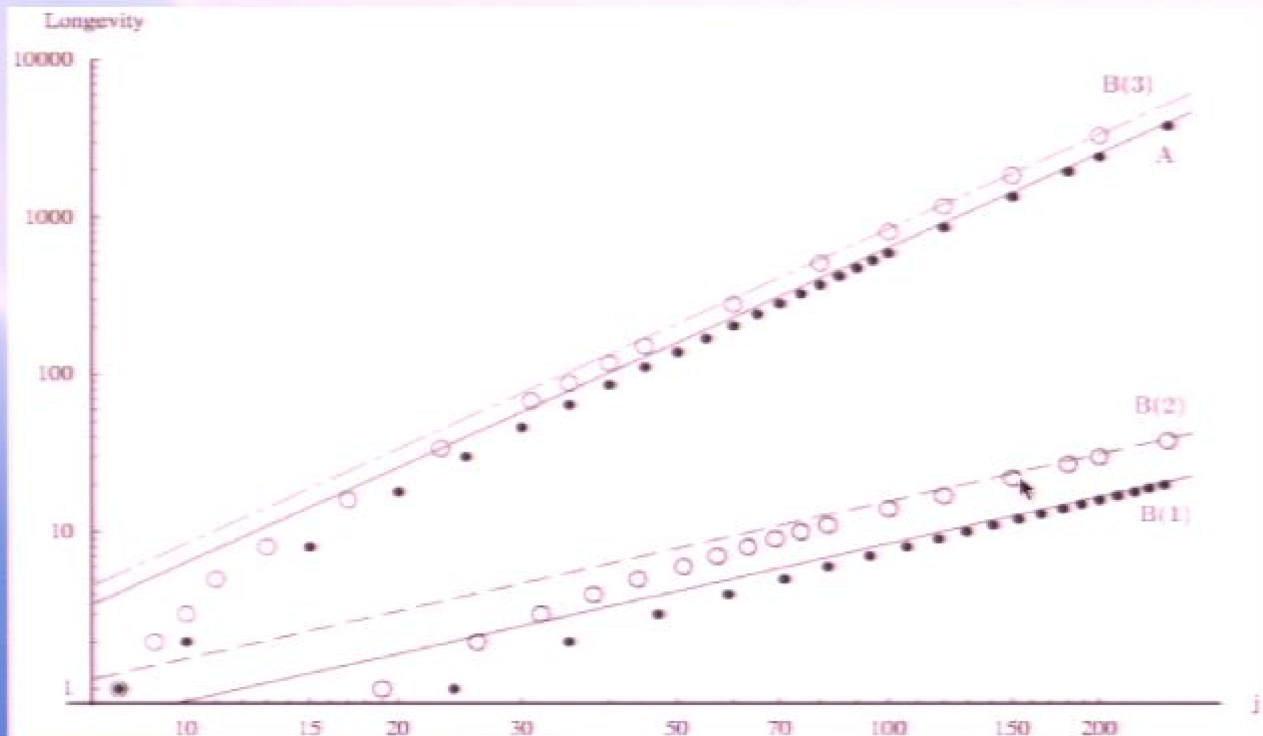
to implement $$\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}$$ (unitary)

Different Results



A plot of the gate fidelity with number of repetitions, *n*, for *j=8* for the three methods, (2.1) (dot-dashed line), (2.2) (dashed line), and 2.3) (solid line). This behavior of this value of *j* is representative

Longevity

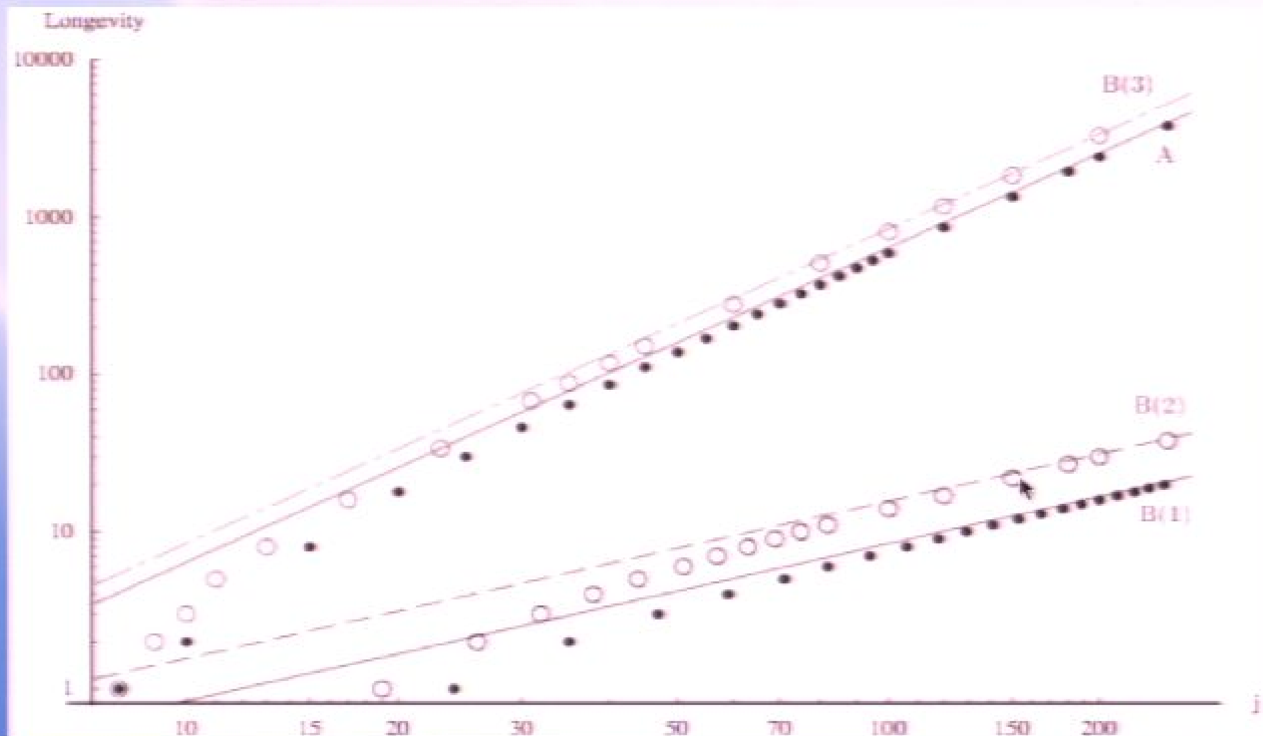


Longevity
10000
1000
100
10
1
B(3)
A
B(2)
B(1)
j
10
15
20
30
50
70
100
150
200

Conclusion

- We generalize the concept of quality function and introduce the moments of a quantum reference frame.
- We give recursive equations (Theorem 2) for how the moments evolve with the number of uses of the quantum reference frame.
- We derive sufficient conditions (Theorem 3) for the longevity of a quantum reference frame to scale by a factor proportional to square the dimension of the quantum reference frame.
- Finally, we applied our results to different examples such as the use of a quantum directional reference frame to measure a spin-1 particle or to implement an Pauli operator on a qubit. The tools that we developed can be use to compare different methods to perform some operation using a quantum reference frame as we showed in our last example.

Longevity



Longevity
10000
1000
100
10
1
B(3)
A
B(2)
B(1)
j
10
15
20
30
50
70
100
150
200

Conclusion

- We generalize the concept of quality function and introduce the moments of a quantum reference frame.
- We give recursive equations (Theorem 2) for how the moments evolve with the number of uses of the quantum reference frame.
- We derive sufficient conditions (Theorem 3) for the longevity of a quantum reference frame to scale by a factor proportional to square the dimension of the quantum reference frame.
- Finally, we applied our results to different examples such as the use of a quantum directional reference frame to measure a spin-1 particle or to implement an Pauli operator on a qubit. The tools that we developed can be use to compare different methods to perform some operation using a quantum reference frame as we showed in our last example.

Acknowledgements

- Perimeter Institute
- University of Sydney
- Rob Spekkens
- Terry Rudolph

NSERC
CRSNG