

Title: Quantum Reference Frames and the Classification of Rotationally-Invariant Maps

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Abstract: We give a convenient representation for any map which is covariant with respect to an irreducible representation of $SU(2)$, and use this representation to analyze the evolution of a quantum directional reference frame when it is exploited as a resource for performing quantum operations. We introduce the moments of a quantum reference frame, which serve as a complete description of its properties as a frame, and investigate how many times a quantum directional reference frame represented by a spin- j system can be used to perform a certain quantum operation with a given probability of success. We provide a considerable generalization of previous results on degradation of reference frame, from which follows a classification of the dynamics of spin- j system under the repeated action of any covariant map with respect to $SU(2)$.

Joint work with Lana Sheridan, Martin Laforest and Stephen Bartlett

Quantum Reference Frames and the Classification of Rotationally-Invariant Maps

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Topics:

- Introduce a representation for any map which is covariant with respect to an irreducible representation of $SU(2)$.
 - Proof
- Use this representation to study the dynamics of quantum directional reference frame
 - What is a quantum directional reference frame (QDRF)?
 - We introduce the *moments* of a QDRF.
 - We generalize the concept of *quality* and *longevity* of a QDRF, and analyze them in function of the evolution of the moments.
 - Examples:
 - Measurement of a spin-1 particle.
 - One qubit Rotation
- Conclusion and Open Questions

Part I:

The classification of
Rotationally-Invariant Maps

Background about SU(2)

- SU(2): set of 2 by 2 unitary complex matrices with determinant one
 - ✓ SU(2) is isomorphic (up to a sign) to the group of space rotations SO(3)
- SU(2) is a (simply connected) Lie group, and the corresponding Lie Algebra is spanned by the three Pauli matrices
- Consider an irreducible representation of SU(2) acting on a $(2j+1)$ -dimensional Hilbert space

A representation R is a homomorphism

$$R : SU(2) \rightarrow U(H_{2j+1})$$

It is irreducible if there exist no non-trivial subspace $S \in H_{2j+1}$ such that $R(\Omega)s \in S$ for all $\Omega \in SU(2)$ and $s \in S$.

Background about SU(2)

- There exists operators J_x , J_y and J_z such that
$$[J_x, J_y] = iJ_z \quad [J_y, J_z] = iJ_x \quad [J_z, J_x] = iJ_y$$
- In physics, J_x , J_y and J_z represented the angular momentum operator of a spin- j .
- Every element R of the irrep of SU(2) can be written as $e^{i\mathbf{v}\hat{J}}$ for some vector \mathbf{v} and where $\hat{J} := (J_x, J_y, J_z)$.

Covariance With Respect to $SU(2)$

- We say that a map χ is covariant with respect to $SU(2)$ iff

$$\chi[R(\Omega)(\cdot)R(\Omega)^\dagger] = R(\Omega)\chi(\cdot)R(\Omega)^\dagger$$

for all $\Omega \in SU(2)$.

i.e. R is the irrep of $SU(2)$ for some $(2j+1)$ -dimensional Hilbert space.

Representation for maps which are covariant with respect to an irreducible representation of SU(2)

- Define

$$\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k \rho_j J_k$$

where ρ_j is a state associated to the Hilbert space of dimension $2j+1$.

Theorem 1: Any map ξ which is covariant with respect to a spin- j irreducible representation of SU(2) has the form

$$\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j) \quad \text{for some real coefficients } q_n.$$

Proof

- First show that $\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k \rho_j J_k$ is covariant with respect to $SU(2)$:

$$R(\Omega)^{-1} \zeta(R(\Omega) \rho_j R(\Omega)^{-1}) R(\Omega) = \zeta(\rho_j) \quad \forall \Omega \in SU(2).$$

Any $SU(2)$ element can be decomposed into rotations around the Y and Z axes:

$$R(\Omega) = R_z(\theta) R_y(\psi) R_z(\phi)$$

Consider: $R_z(\theta)$

$$\left\{ \begin{array}{l} J_x \rightarrow \cos \theta J_x + \sin \theta J_y \\ J_y \rightarrow -\sin \theta J_x + \cos \theta J_y \\ J_z \rightarrow J_z \end{array} \right.$$

- Therefore any map of the form $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{on}(\rho_j)$ must also be covariant.
- To prove that every covariant map can be written as above, we use the Liouville representation:

$$\rho_j \rightarrow |\rho_j\rangle\rangle \quad \text{vector of dimension } d^2$$

Proof

- First show that $\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k \rho_j J_k$ is covariant with respect to $SU(2)$:

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Liouville Representation of a superoperator

Kraus
Representation:

$$\xi(\cdot) = \sum_k E_k \cdot E_k^\dagger$$



Liouville
Representation:

$$\mathcal{K}(\xi) = \sum_k E_k^* \otimes E_k$$

Covariance in the Liouville Representation

$$(R^*(\Omega) \otimes R(\Omega))\mathcal{K}(\xi) = \mathcal{K}(\xi)(R^*(\Omega) \otimes R(\Omega))$$

for all $\Omega \in SU(2)$.

For an irrep of $SU(2)$: $R^*(\Omega) = e^{-i\pi J_y} R(\Omega) e^{i\pi J_y}$

So $(R(\Omega) \otimes R(\Omega))\mathcal{K}'(\xi) = \mathcal{K}'(\xi)(R(\Omega) \otimes R(\Omega))$

where $\mathcal{K}'(\xi) = (e^{i\pi J_y} \otimes I)\mathcal{K}(\xi)(e^{-i\pi J_y} \otimes I)$

- For $\Omega \in SU(2)$, all irreps of the group generated by $R(\Omega) \otimes R(\Omega)$ have multiplicity one. Each irrep correspond to an integer from 0 to $2j$.

- By Schur's Lemma,
$$\mathcal{K}'(\xi) = \sum_{k=0}^{2j} c_k \Pi_k$$

for some complex parameter c_k and orthogonal projectors Π_k .

- Therefore, there is exactly $2j+1$ linearly independent matrix that can be used as a basis to write -- in the Liouville representation -- any covariant map $\mathcal{K}(\xi)$ with respect to $SU(2)$.

✓ We need to find the right set of linearly independent matrices.

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Kraus representation:

$$\zeta(\rho_j) = \frac{1}{j(j+1)} \sum_{k \in \{x, y, z\}} J_k \rho_j J_k$$



Liouville
representation:

$$\mathcal{K}(\zeta) = \frac{1}{j(j+1)} \sum_{k \in \{x, y, z\}} J_k^* \otimes J_k$$

$$\zeta^{\circ n}(\rho_j)$$



Liouville
representation:

$$\mathcal{K}(\zeta^{\circ n}) = \left(\frac{1}{j(j+1)} \sum_{k \in \{x,y,z\}} J_k^* \otimes J_k \right)^n$$

Need to show that the matrices $\left(\sum_{k \in \{x,y,z\}} J_k^* \otimes J_k \right)^n$

are linearly independent for $0 \leq n \leq 2j$.

It is not too hard to

show:

$$(a) \quad \sum_i J_i^* \otimes J_i = -e^{-i\pi J_y} \otimes I \left(\sum_i J_i \otimes J_i \right) e^{i\pi J_y} \otimes I$$

$$(b) \quad \sum_i J_i \otimes J_i = \frac{1}{2}(\mathcal{J}^2 - 2j(j+1)I)$$

$$\text{where } \mathcal{J}^2 = \sum_i (J_i \otimes I + I \otimes J_i)^2$$

(total angular momentum)

(a) implies $\sum_i J_i^* \otimes J_i$ and $\sum_i J_i \otimes J_i$ share the same number of distinct eigenvalues.

(b) implies that $\sum_i J_i \otimes J_i$ has $2j+1$ distinct eigenvalues.

- By the fundamental theorem of algebra, this implies that there exist no polynomial of degree $2j$ that has the $2j+1$ distinct eigenvalues as roots.
- This implies the matrices $(\sum_{k \in \{x,y,z\}} J_k^* \otimes J_k)^n$ for $0 \leq n \leq 2j$ are linearly independent.

- We showed that $\xi(\rho_j) = \sum_{n=0}^{2j} q_n \zeta^{\circ n}(\rho_j)$ where the q_n are complex.
- The q_n are in fact real.
 - Proof by induction based on the fact that ξ is a positive map.

Open Question

- What about covariance with respect to other Lie Group? Is there any interesting representation in function of the associated Lie Algebra generators?
- What are the restrictions on the q_n parameters?

Part II:

Dynamics of Quantum Directional Reference Frame

What is a Quantum Directional Reference Frame?

- Consider the initial state of a spin- j : $\rho_j^{(0)}$
- Suppose that $\rho_j^{(0)}$ depends only on some “classical” direction \hat{n} .
 - If R is the rotation that transforms \hat{n} to \hat{n}' , then
$$R\rho_j^{(0)}(\hat{n})R^{-1} = \rho_j^{(0)}(\hat{n}')$$
 - The state $\rho_j^{(0)}$ is also covariant under rotations about the \hat{n} -axis.
- Therefore, $\rho_j^{(0)}$ is diagonal in the basis consisting of the eigenvectors of $J_{\hat{n}}$.

Scenario

**Quantum
Reference Frame**

Reservoir: contains many
identical subsystems of
dimension d .

$$\rho_j^{(0)}$$



apply the map

$$\chi(\rho_j^{(0)} \otimes \textcircled{1})$$

$$\rho_j^{(1)} = \xi(\rho_j^{(0)}) = \text{Tr}_{\textcircled{1}} [\chi(\rho_j^{(0)} \otimes \textcircled{1})]$$

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$\frac{1}{H}, \frac{1}{H} \rangle$
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$\frac{1}{2}, \frac{1}{2}$
 $\frac{1}{2}, \frac{1}{2}$

$\frac{1}{2}$



$$\begin{aligned} & \left| \frac{1}{2}, \frac{1}{2} \right\rangle_n \\ & \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_n \end{aligned}$$

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Quantum Reference Frame

$$\rho_j^{(n-1)}$$

Reservoir: contains many identical subsystems of dimension d .



apply the map

$$\chi(\rho_j^{(n-1)} \otimes \textcircled{n})$$

X: discarded

$$\rho_j^{(n)} = \xi(\rho_j^{(n-1)}) = \text{Tr}_{\textcircled{n}} [\chi(\rho_j^{(n-1)} \otimes \textcircled{n})]$$

Extra Assumptions:

- The joint map χ is rotationally-invariant.
 - The state of the subsystems in the reservoir are invariant under space-rotations.
- ✓ *This implies that the back-action map ξ on the quantum reference frame is rotationally-invariant.*
- ✓ Which implies that $\rho_j^{(k)}$ for all k is diagonal in the basis given by the eigenvectors of $J_{\hat{n}}$

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Previous Related Works

- 1) S. Bartlett, T. Rudolph, R. Spekkens, and P. Turner, **Degradation of a Quantum Reference Frame**, *New J. Phys.* **8**, 58 (2006).
- 2) D. Poulin and J. Yard, **Dynamics of a Quantum Reference Frame**, *New J. Phys.* **9**, 156 (2007).
 - a) The joint operator (χ) considered is restricted to measurements
 - b) $d=2$
 - c) the *quality function* is fixed (somewhat arbitrarily).

In 2), the states of the subsystems of the reservoir are not necessarily rotationally-invariant.

We use the term **quality function** for any function F that is meant to quantify the ability of the reference frame to perform a particular task.

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The quality measure should not be biased such that it favors a quantum reference frame that is pointed in any particular direction relative to some external frame. All directions must be equally valid. Therefore, F does not depend on the direction of \hat{n} , but only on the eigenvalues of ρ_j .

Moments

- An equivalent set of parameters to the eigenvalues of ρ_j are the moments of ρ_j :

$$\{Tr[\rho_j J_{\hat{n}}^\ell] \mid 1 \leq \ell \leq 2j\}$$

- Any quality function F depends only on those moments.
- To analyze the behavior of F , it is sufficient to study the evolution of the moments.

General Recursion Formula for the Moments

$$\text{Tr}[\rho_j^{(k)} J_{\hat{n}}^\ell] = \sum_{i=0}^{2j} A_i^{(\ell)} \text{Tr}[\rho_j^{(k-1)} J_{\hat{n}}^i]$$

where the $A_i^{(\ell)}$'s are real coefficients.

Get ride of some of the coefficients

Theorem 2:

If ℓ is even, then

$$\text{Tr}[\rho_j^{(k)} J_{\hat{n}}^\ell] = \sum_{i=0}^{\ell/2} A_{2i}^{(\ell)} \text{Tr}[\rho_j^{(k-1)} J_{\hat{n}}^{2i}]$$

and if ℓ is odd, then

$$\text{Tr}[\rho_j^{(k)} J_{\hat{n}}^\ell] = \sum_{i=1}^{(\ell+1)/2} A_{2i-1}^{(\ell)} \text{Tr}[\rho_j^{(k-1)} J_{\hat{n}}^{2i-1}].$$

Proof: Corollary of Theorem 1 (by induction using commutator relations).

Longevity

- We are interested in the scaling, with respect to Hilbert space dimension, of how many times a quantum reference frame can be used before the value of its quality function F falls below a certain threshold.

BRST06: Longevity scales as $O(j^2)$. (specific F)

- Because we consider any quality function F , the longevity of the reference frame can be arbitrary (in general).
- But we can study the scaling of the **moments**.

Theorem 3: Consider a quantum reference frame with initial state $\rho_j^{(0)}$, which is used for performing a rotationally-invariant joint operation χ_j . If this operation induces a disturbance map

$$\xi = \sum_{n=0}^{2j} q_n(j) \zeta^{\circ n}$$

that satisfies the following assumptions:

- there exists some n_{max} such that $q_n = 0$ for all $n \geq n_{max}$,
- $q_n \leq O(1)$ and
- $\text{Tr}[\rho_j^{(0)} J_{\hat{n}}^\ell] = O(j^\ell)$.

then the number of times that such a quantum reference frame can be used before its ℓ^{th} moment falls below a certain threshold value scales as j^2 .

The proof is based on Theorem 2.

Example A.1: Measurements of spin-1/2

- Suppose that the reservoir consists of spin-1/2 systems. Each spin is either parallel or anti-parallel to \hat{n} (with the same probability).
- The goal is to use the quantum reference frame to guess the direction of each spin-1/2.
- The optimal joint measurement χ is a projection onto the subspaces corresponding to different values of the total angular momentum.

$$F_{\frac{1}{2}}^{(k)} := \text{Tr} \left[\frac{1}{2} \sum_{\mu \in \{-\frac{1}{2}, \frac{1}{2}\}} \Pi_{j+\mu}(\rho_j^{(k)}) \otimes |\mu\rangle\langle\mu| \right]$$

Measurement of spin-1/2

- In term of the first moment, we can rewrite the quality function:

$$F_{\frac{1}{2}}^{(k)} = \frac{1}{2} + \frac{1}{2j+1} \text{Tr}[\rho_j^{(k)} J_z]$$

- Theorem 2 tells us

$$\text{Tr}[\rho_j^{(k)} J_z] = \text{Tr}[\rho_j^{(0)} J_z] \left(A_0^{(1)} \right)^k .$$

- Simple calculation give us

$$A_0^{(1)} = 1 - \frac{2}{(2j+1)^2}$$

Example A.2: Measurements of spin-1

- Suppose that the reservoir consists of spin-1 systems. Each spin has either 1, 0 or -1 angular momentum in the \hat{n} direction (with the same probability).
- The goal is to use the quantum reference frame to guess the angular momentum in the \hat{n} direction of each spin-1.
- The optimal joint measurement χ is a projection onto the subspaces corresponding to different values of the total angular momentum.

$$F_1^{(k)} = \text{Tr} \left[\frac{1}{3} \sum_{\mu \in \{-1, 0, 1\}} \Pi_{j+\mu}(\rho_j^{(k)}) \otimes |\mu\rangle\langle\mu| \right]$$

Measurements of spin-1

- In terms of moments:

$$F_1^{(k)} = \frac{1}{6} + \frac{[(2j+1)^2 - 2]}{6j(j+1)(2j+1)} \text{Tr}[\rho_j^{(k)} J_z] + \frac{1}{2j(j+1)} \text{Tr}[\rho_j^{(k)} J_z^2].$$

- We can use again Theorem 2 to evaluate the moments. Simple calculations give us the values of the three A 's coefficients.

Example B: Pauli Operator

- Suppose, we want to implement a Pauli Z operation on a qubit:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

using a quantum reference frame to define the z -axis.

- Cannot be implemented without errors if the reference frame is quantum and restricted to a finite space.

Gate Fidelity

- We can pick the quality function to be

$$F_{\text{gate}}(Z, \tau) \equiv \frac{\int_{\Omega} d\mu_{\Omega} |\text{Tr}[E(\Omega)^{\dagger} Z]|^2 + d}{d^2 + d}$$

where $E(\Omega)$ are the Kraus operators of the approximate gate.

Different Methods To Implement the Gate

1. Projective Measurement

$$\{\Lambda(\Omega) = (2j + 1)R(\Omega)|e\rangle\langle e|R(\Omega)^\dagger, \Omega \in SU(2)\}$$

where $|e\rangle := |j, m_z = j\rangle$

Gate Fidelity

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Different Methods To Implement the Gate

2. Filtering operation:

$$\begin{aligned}
 (2j+1) \int_{\Omega} d\mu_{\Omega} R_j(\Omega) \otimes R_{1/2}(\Omega) \left[|j, j\rangle_z \langle j, j| \otimes Z \right] R_j(\Omega)^{-1} \otimes R_{1/2}(\Omega)^{-1} \\
 = \\
 \frac{2j}{2j+2} \Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}} \quad \text{(not unitary)}
 \end{aligned}$$

where Π_k is the projector into the subspace of total angular momentum k .

Different Methods To Implement the Gate

3. Use coupling between the spins of the quantum reference frame and of the reservoir :

Use
$$H = w \left(J_x^{(\frac{1}{2})} J_x^{(j)} + J_y^{(\frac{1}{2})} J_y^{(j)} + J_z^{(\frac{1}{2})} J_z^{(j)} \right)$$

to implement
$$\prod_{j+\frac{1}{2}} - \prod_{j-\frac{1}{2}} \quad (\text{unitary})$$

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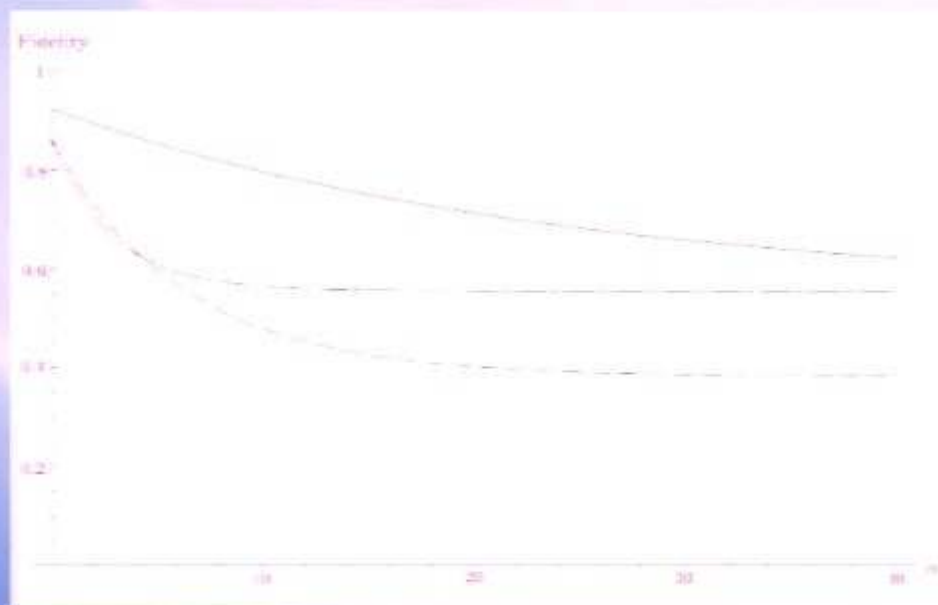
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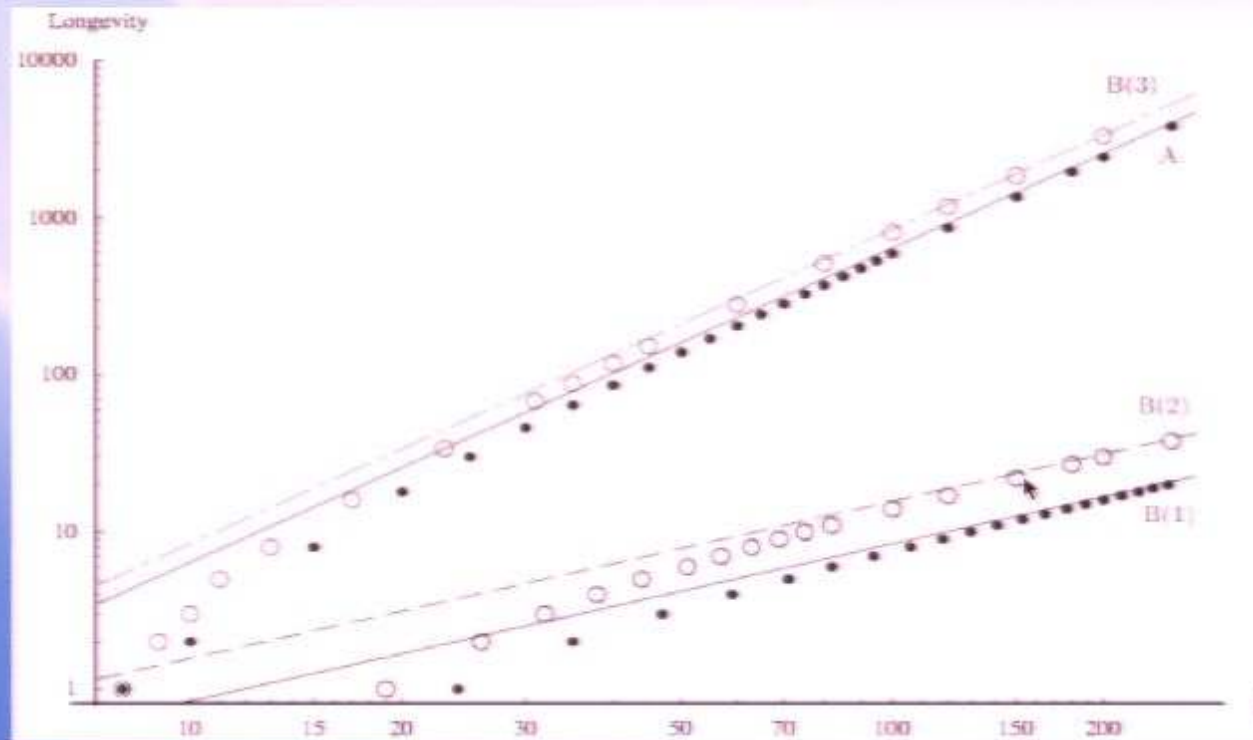
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$$\prod_{j+\frac{1}{2}} - \prod_{j-\frac{1}{2}} \quad (\text{unitary})$$

Different Results



A plot of the gate fidelity with number of repetitions, n , for $j=8$ for the three methods, (2.1) (dot-dashed line), (2.2) (dashed line), and 2.3) (solid line). This behavior of this value of j is representative

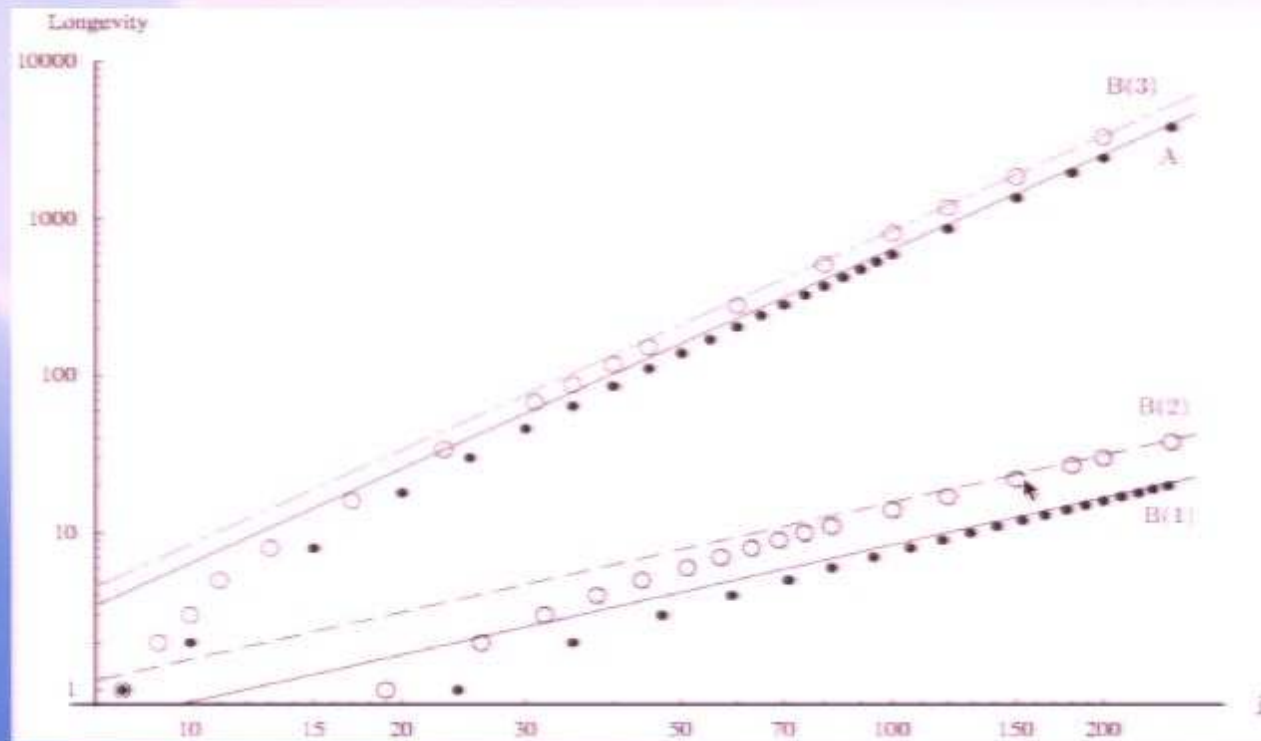
Longevity



Conclusion

- We generalize the concept of quality function and introduce the moments of a quantum reference frame.
- We give recursive equations (Theorem 2) for how the moments evolve with the number of uses of the quantum reference frame.
- We derive sufficient conditions (Theorem 3) for the longevity of a quantum reference frame to scale by a factor proportional to square the dimension of the quantum reference frame.
- Finally, we applied our results to different examples such as the use of a quantum directional reference frame to measure a spin-1 particle or to implement an Pauli operator on a qubit. The tools that we developed can be use to compare different methods to perform some operation using a quantum reference frame as we showed in our last example.

Longevity



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Acknowledgements

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