

Title: The resource theory of quantum reference frames: manipulations and monotones

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Abstract: Every restriction on quantum operations defines a resource theory, determining how quantum states that cannot be prepared under the restriction may be manipulated and used to circumvent the restriction. A superselection rule is a restriction that arises through the lack of a classical reference frame. The states that circumvent it (the resource) are quantum reference frames. We consider the resource theories that arise from three types of superselection rule, associated respectively with lacking: (i) a phase reference, (ii) a frame for chirality, and (iii) a frame for spatial orientation. Focussing on pure unipartite quantum states, we identify the necessary and sufficient conditions for a deterministic transformation between two resource states to be possible and, when these conditions are not met, the maximum probability with which the transformation can be achieved. We also determine when a particular transformation can be achieved reversibly in the limit of arbitrarily many copies and find the maximum rate of conversion. (joint work with Gilad Gour)

$(\hat{n})$  assume RF for  $\hat{n}$

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lack of Cartesian frame  $\rightarrow$  rot. inv. states

$$T(\Omega)\rho T(\Omega)^\dagger = \rho \quad \forall \Omega \in \text{SU}(2)$$

$$T(\Omega) = e^{i\Omega \cdot \mathbf{J}}$$

$(+\hat{n})$  assume RF for  $\hat{n}$

lack of Cartesian frame  $\rightarrow$  rot. inv. states

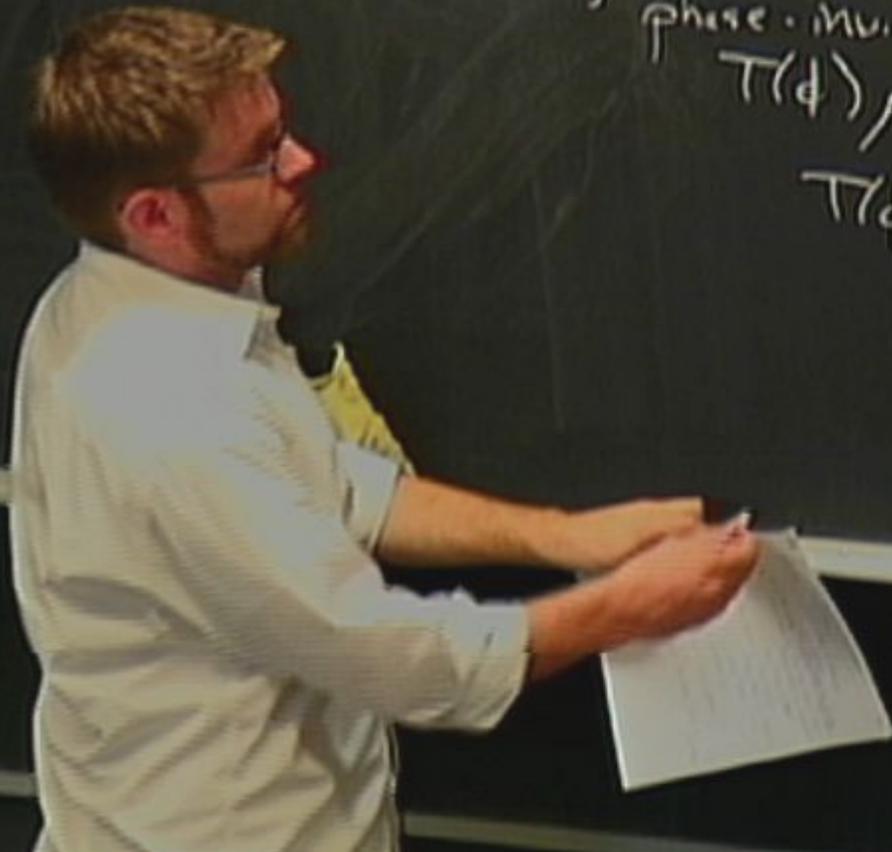
$$T(\Omega)\rho T(\Omega)^\dagger = \rho \quad \forall \Omega \in \text{SU}(2)$$

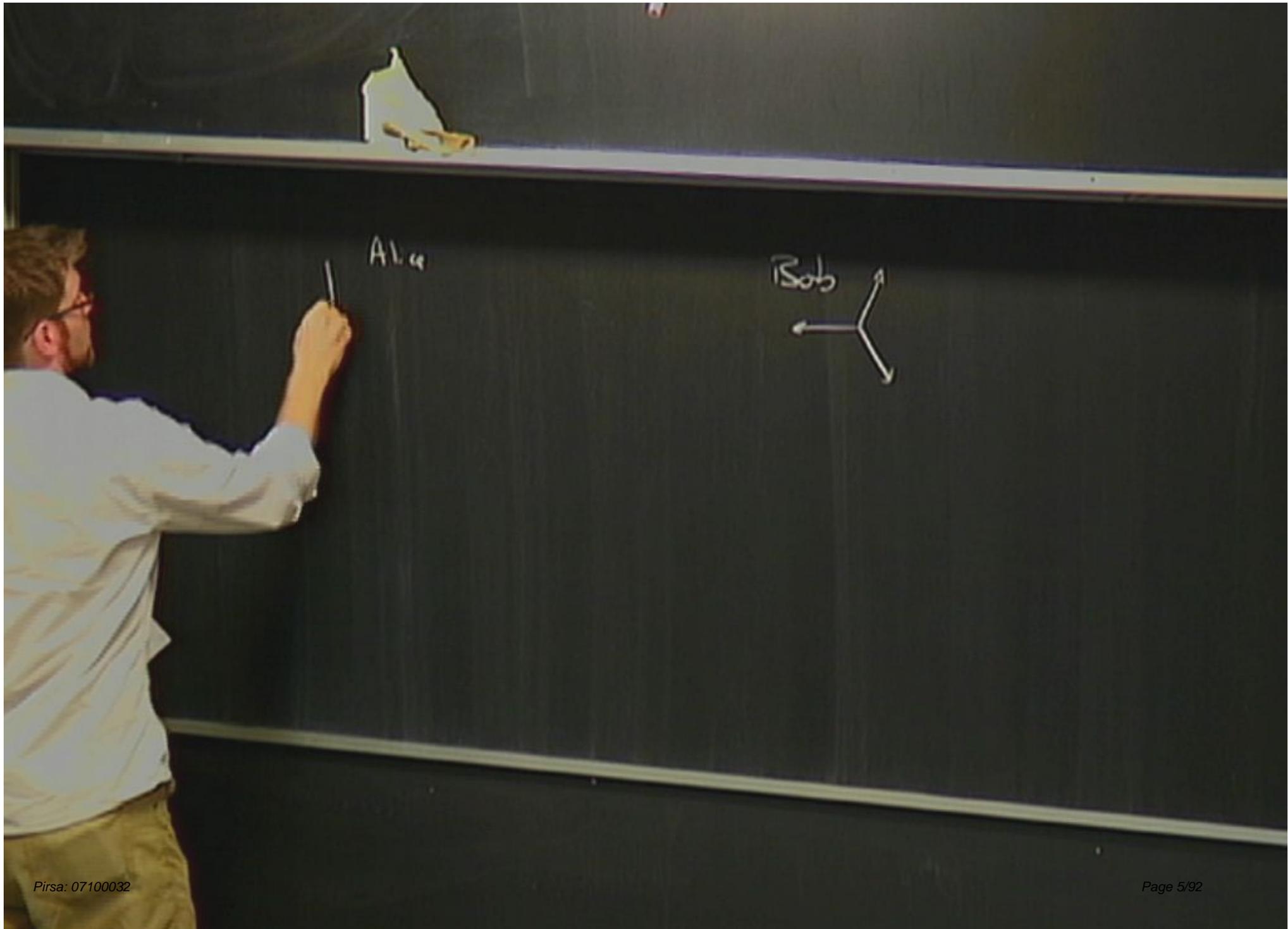
$$T(\Omega) = e^{i\Omega \cdot \mathbf{J}}$$

lack of phase reference  $\rightarrow$  phase-inv. states

$$T(\phi)\rho T(\phi)^\dagger = \rho \quad \forall \phi \in \text{U}(1)$$

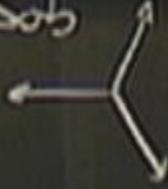
$$T(\phi) = e^{i\phi N}$$

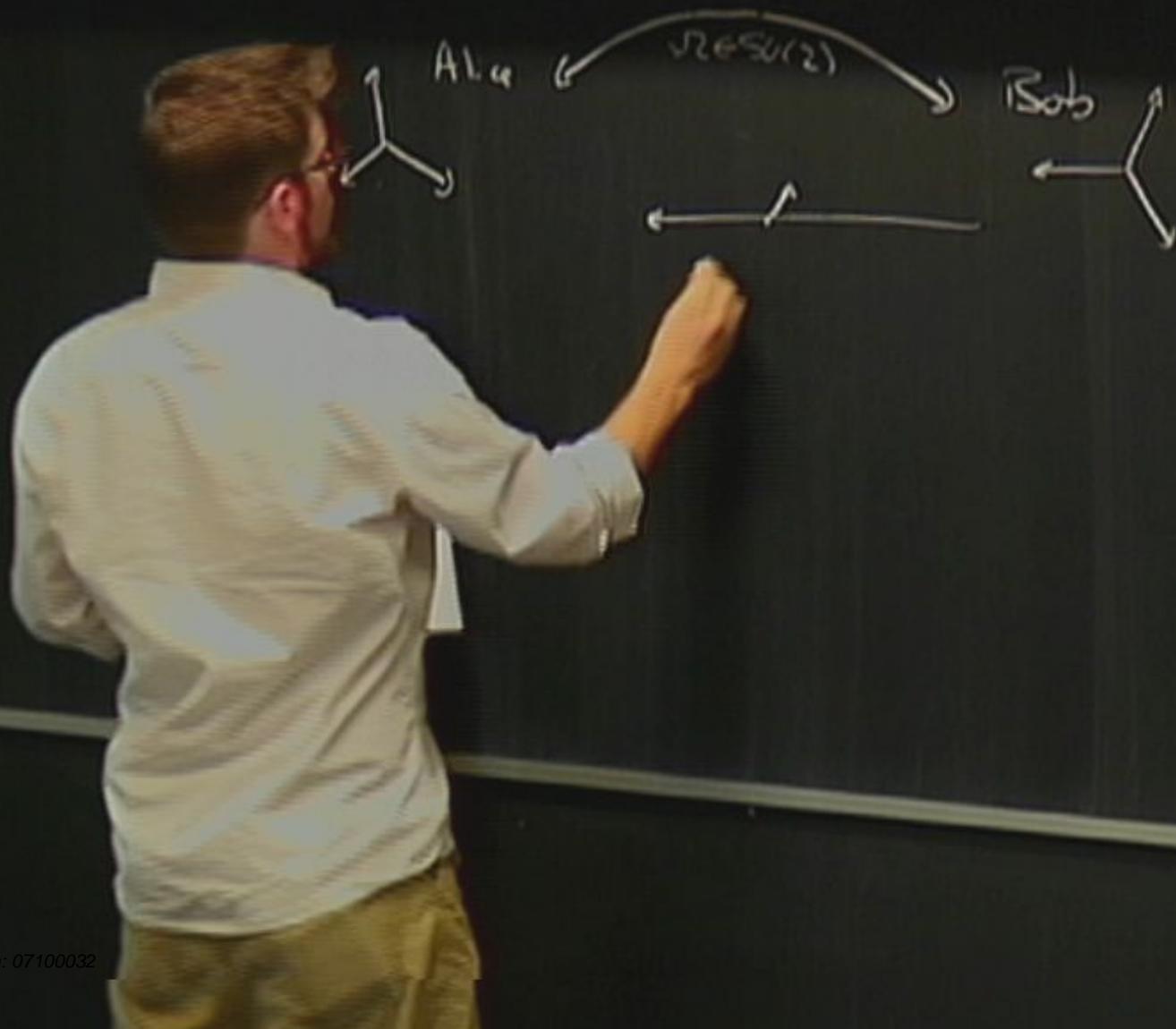


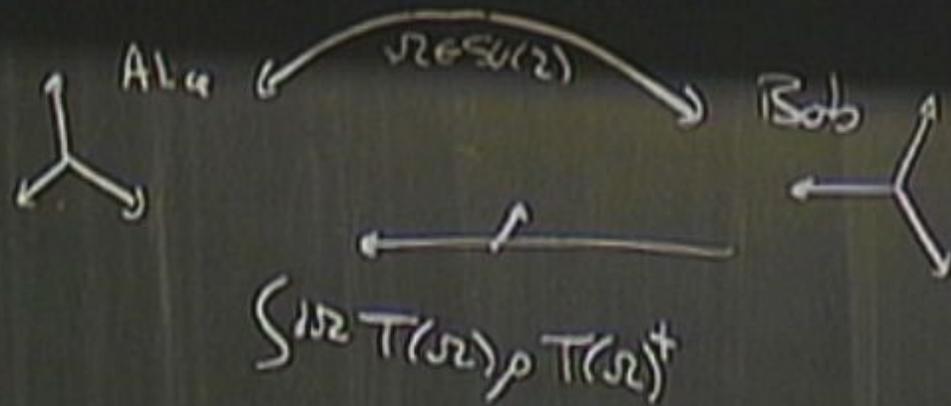


Alice

Bob







lack of Cartesian frame  $\rightarrow$  rot. inv. states

$$T(\Omega) \rho T(\Omega)^\dagger = \rho \quad \forall \Omega \in SU(2)$$

$$T(\Omega) = e^{i\Omega \cdot \mathbf{J}}$$

lack of phase reference  $\rightarrow$  phase-inv. states

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In general, RF is associated w/  $T(\phi) = e^{i\phi N}$

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$$T(\phi) \rho T(\phi)^\dagger = \rho \quad \forall \phi \in \text{U}(1)$$

In general, RF is associated w/

$$T(\phi) = e^{i\phi N}$$

- group  $G$

- unitary rep'n of  $G$   $T: G \rightarrow \mathcal{U}(\mathcal{H})$

$$\int \Omega T(\Omega) \rho T(\Omega)^\dagger$$

$(\hat{n})$  assume RF for  $\hat{n}$

lack of Cartesian frame  $\rightarrow$  rot. inv. states

$$T(\Omega)\rho T(\Omega)^\dagger = \rho \quad \forall \Omega \in SU(2)$$

lack of phase reference  $\rightarrow$  phase-inv. states

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• unitary rep'n of  $G$   $T: G \rightarrow \mathcal{U}(\mathcal{H})$

$$T(\Omega) \rho T(\Omega)^\dagger = \rho \quad \forall \Omega \in \text{SO}(2)$$

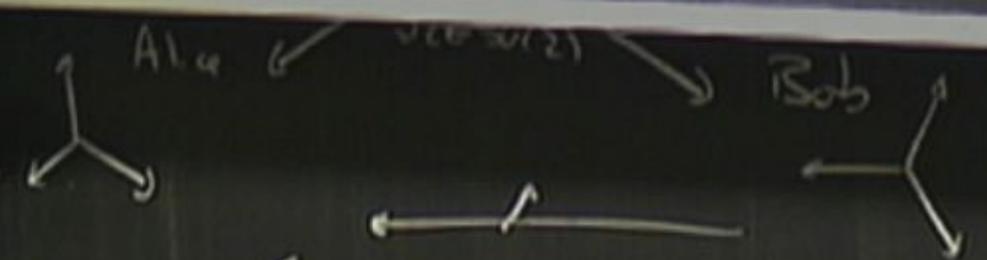
lack of phase reference  $\rightarrow$  phase inv. states  
 $T(\Omega) = e^{i\Omega J}$

$$T(\phi) \rho T(\phi)^\dagger = \rho \quad \forall \phi \in U(1)$$

In general, RF is associated w/  $T(\phi) = e^{i\phi R}$

- group  $G$
- unitary rep'n of  $G$   $T: G \rightarrow \mathcal{U}(\mathcal{H})$

lack of RF for  $G$  implies  $T(g) \rho T(g)^\dagger = \rho \quad \forall g \in G$   
 only preparable states



$$\int \Omega T(\Omega) \rho T(\Omega)^\dagger$$

$$T(\Omega) = e^{i\Omega J}$$

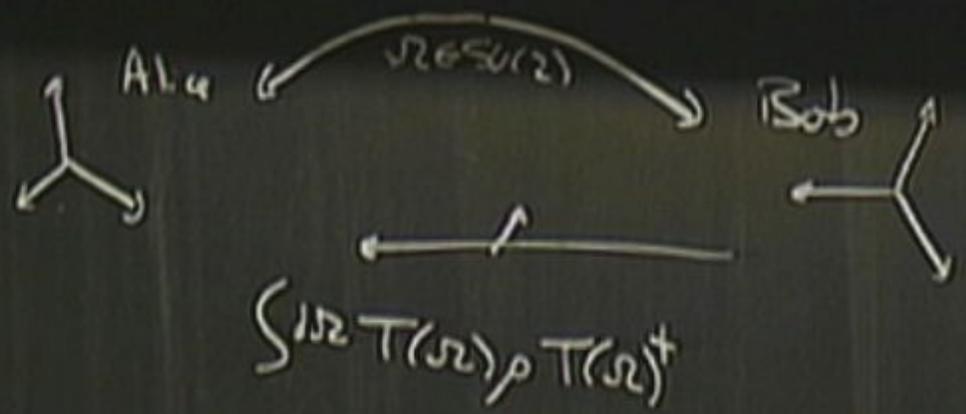
phase inv. states

$$T(\phi)\rho T(\phi)^\dagger = \rho \quad \forall \phi \in U(1)$$

In general, RF is associated w/

- group  $G$
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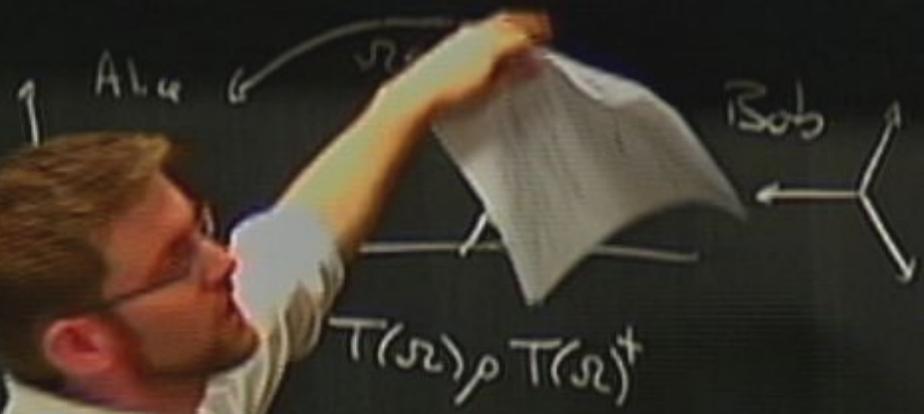


phase mu. states  
 $T(\phi) \rho T(\phi)^\dagger = \rho \quad \forall \phi \in U(1)$   
 In general, RF is associated w/

- group  $G$
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lack of RF for  $G$  implies  $T(g) \rho T(g)^\dagger = \rho \quad \forall g \in G$   
 only preparable states

lack of RF for  $G \equiv G$ -SSR



Only implementable op's under G-S/R

$$\tau(g) \circ \mathcal{E} \circ \tau^+(g) = \mathcal{E} \quad \forall g \in G$$

Only implementable op's under G-SSR

$$\mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^+(g) = \mathcal{E} \quad \forall g \in G$$

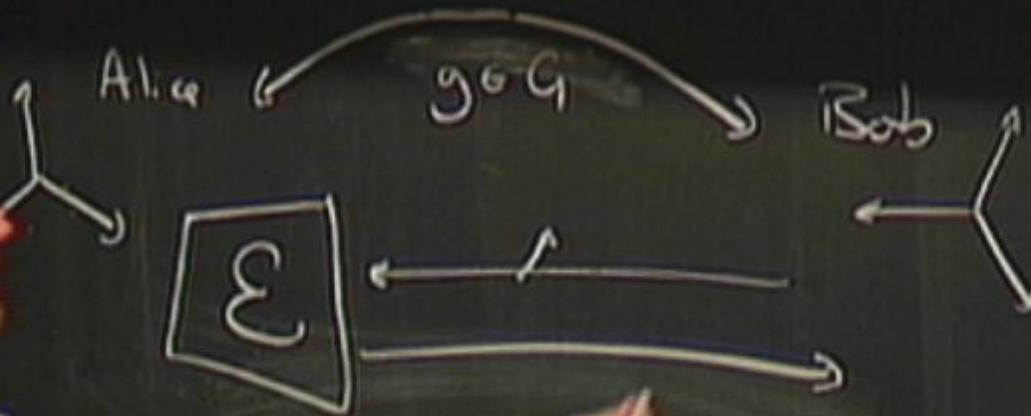
$$\text{where } \mathcal{T}(g)[X] = \mathcal{T}(g)X\mathcal{T}^+(g)$$

only preparable states!



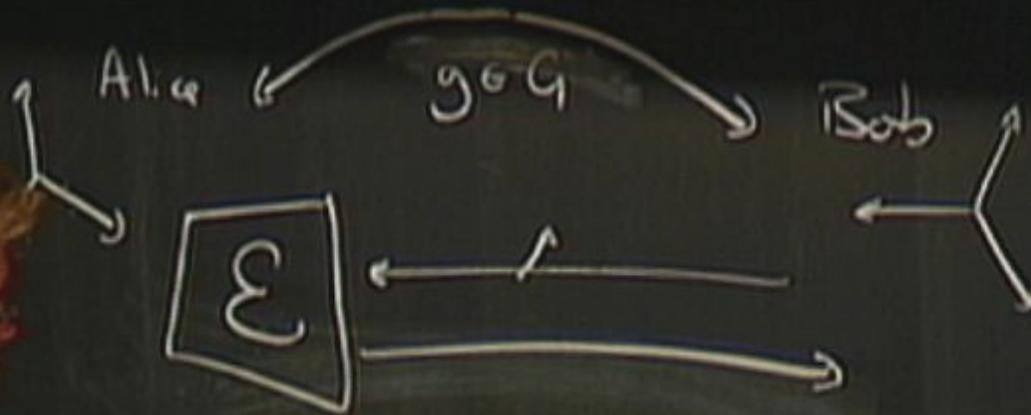
$$\int d\Omega T(\Omega) \rho T(\Omega)^\dagger$$

only preparable states



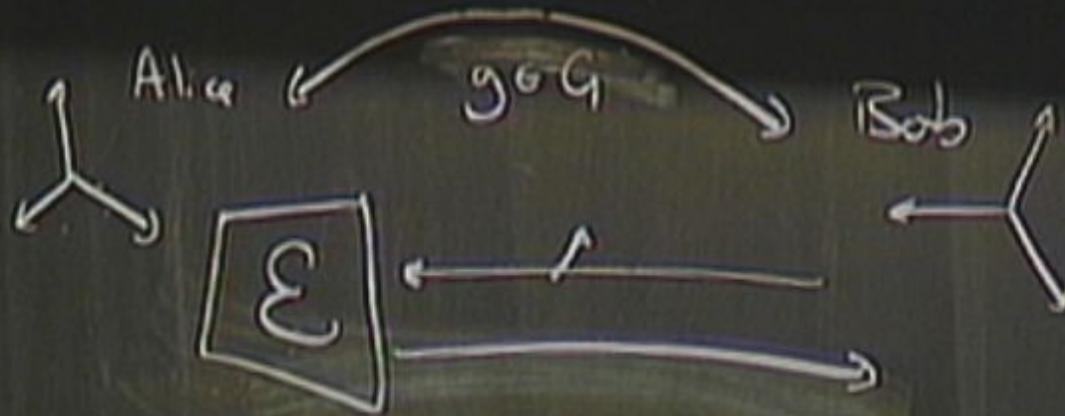
$$\left[ \begin{array}{c} \mathcal{E} \\ \mathcal{E} \end{array} \right]$$

only preparable states!



$$\mathcal{E}[\tau(\rho) \otimes \tau^+(\rho)]$$

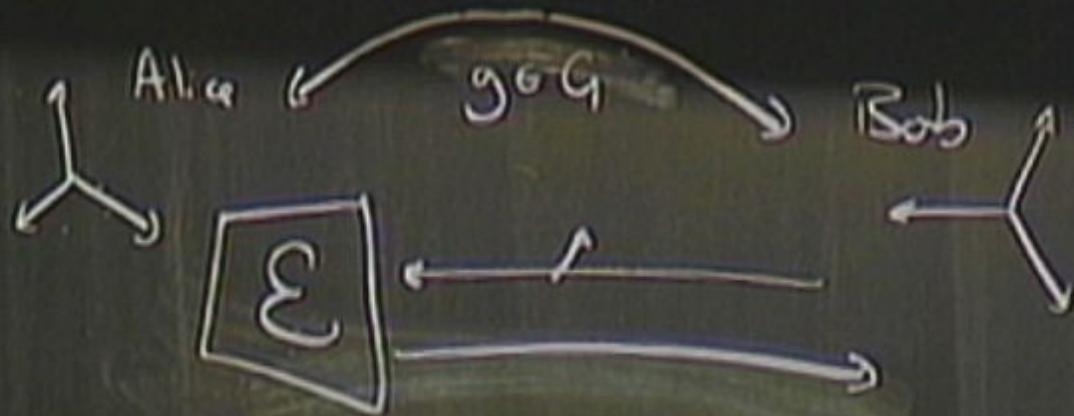
only preparable states



$$S_{dg} \tau(\rho) \varepsilon [ \tau(\rho) \tau(\rho) ] \tau(\rho)$$

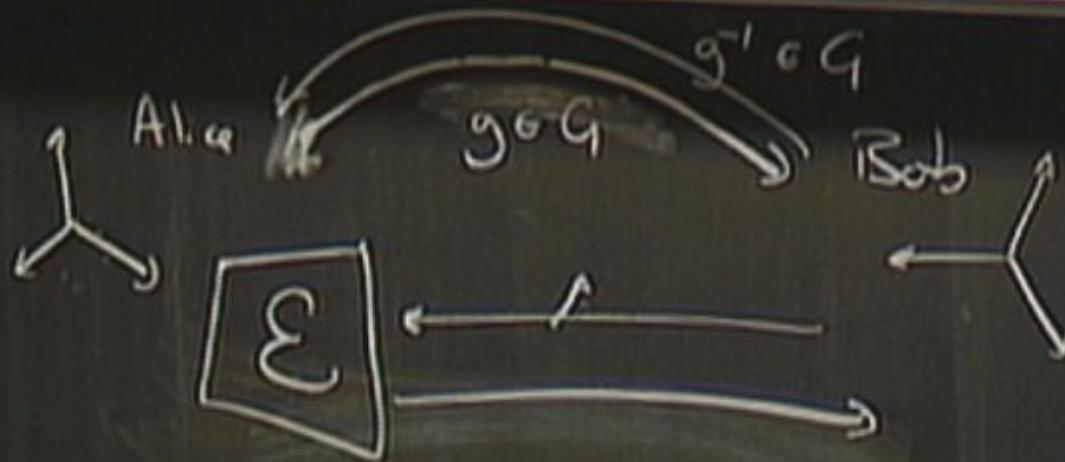
$$= S_{dg} \tau(\rho) \cdot \varepsilon \cdot \tau(\rho)$$

only preparable states



$$\mathcal{E} = \sum_{dg} \mathcal{T}(g) \cdot \mathcal{E} \cdot \mathcal{T}^\dagger(g)$$

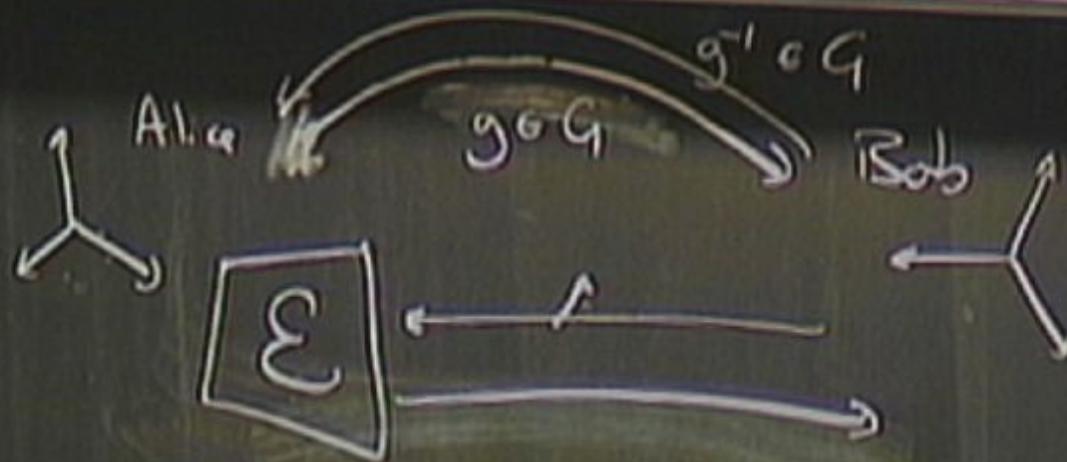
only preparable states



$$\mathcal{E} = S \circ g \circ \mathcal{T}(g) \circ \mathcal{E} \circ [\mathcal{T}(g) \circ \mathcal{T}(g)] \circ \mathcal{T}(g)$$

$$\mathcal{E} = S \circ g \circ \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g)$$

only preparable states



$$\mathcal{E} = Sdg \tau(g) \cdot \mathcal{E} \cdot \tau(g)$$

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Only implementable op's under G-SR

$$\mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^+(g) = \mathcal{E} \quad \forall g \in G$$

$$\text{where } \mathcal{T}(g)[x] = \mathcal{T}(g)x\mathcal{T}^+(g)$$

Only implementable op's under  $G$ -SSR

$$\mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) = \mathcal{E} \quad \forall g \in G$$

$$\text{where } \mathcal{T}(g)[X] = \mathcal{T}(g)X\mathcal{T}^\dagger(g)$$

Kraus decomposition for  $G$ -inv. op's

Only implementable op's under G-SIR

$$\begin{aligned} \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &= \mathcal{E} \quad \forall g \in G \\ \text{where } \mathcal{T}(g)[X] &= \mathcal{T}(g)X\mathcal{T}^\dagger(g) \end{aligned}$$

Kraus decomposition for G-inv. op's

$$\mathcal{E} \leftrightarrow \{K_\mu\}$$

$$\mathcal{E} = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$$

Only implementable op's under  $G$ -SSR

$$\begin{aligned} \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &= \mathcal{E} \quad \forall g \in G \\ \text{where } \mathcal{T}(g)[X] &= \mathcal{T}(g)X\mathcal{T}^\dagger(g) \end{aligned}$$

Kraus decomposition for  $G$ -inv. op's

$$\begin{aligned} \mathcal{E} &\leftrightarrow \{K_H\} & \mathcal{E} &= \sum_H K_H(\cdot)K_H^\dagger \\ \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &\leftrightarrow \{L_H(g) = \mathcal{T}(g)K_H\mathcal{T}^\dagger(g)\} \end{aligned}$$



Only implementable

$$\boxed{\begin{aligned} \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &= \mathcal{E} \quad \forall g \in G \\ \text{where } \mathcal{T}(g)[x] &= \mathcal{T}(g)x\mathcal{T}^\dagger(g) \end{aligned}}$$

Kraus decomposition for G-inv. op's

$$\begin{aligned} \mathcal{E} &\leftrightarrow \{K_H\} & \mathcal{E} &= \sum_H K_H(\cdot)K_H^\dagger \\ \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &\leftrightarrow \{L_H(g) = \mathcal{T}(g)K_H\mathcal{T}^\dagger(g)\} \\ \mathcal{T}(g)K_H\mathcal{T}^\dagger(g) &= \sum_{H'} u_{HH'} \end{aligned}$$

Only implementable

$$\tau(g) \circ \mathcal{E} \circ \tau^\dagger(g) = \mathcal{E} \quad \forall g \in G$$

where  $\tau(g)[X] = \tau(g) X \tau^\dagger(g)$

Kraus decomposition for G-inv. op's

$$\mathcal{E} \leftrightarrow \{K_\mu\}_{\mu=1}^n \quad \mathcal{E} = \sum_{\mu} K_\mu(\cdot) K_\mu^\dagger$$
$$\tau(g) \circ \mathcal{E} \circ \tau^\dagger(g) \leftrightarrow \{L_{\mu}(g) = \tau(g) K_\mu \tau^\dagger(g)\}$$

$$\tau(g) K_\mu \tau^\dagger(g) = \sum_{\mu'} u_{\mu\mu'}(g) K_{\mu'}$$

Only implementable

$$\begin{aligned} \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &= \mathcal{E} \quad \forall g \in G \\ \text{where } \mathcal{T}(g)[x] &= \mathcal{T}(g)x\mathcal{T}^\dagger(g) \end{aligned}$$

Kraus decomposition for G-inv. op's

$$\begin{aligned} \mathcal{E} &\leftrightarrow \{K_H\}_{H \in \mathcal{H}} \quad \mathcal{E} = \sum_H K_H(\cdot)K_H^\dagger \\ \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &\leftrightarrow \{L_H(g) = \mathcal{T}(g)K_H\mathcal{T}^\dagger(g)\} \end{aligned}$$

$$\mathcal{T}(g)K_H\mathcal{T}^\dagger(g) = \sum_{H'} u_{HH'}(g)K_{H'}$$

$$\vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$
$$v(g') \vec{K}(g) = v(g') v(g) \vec{K}$$

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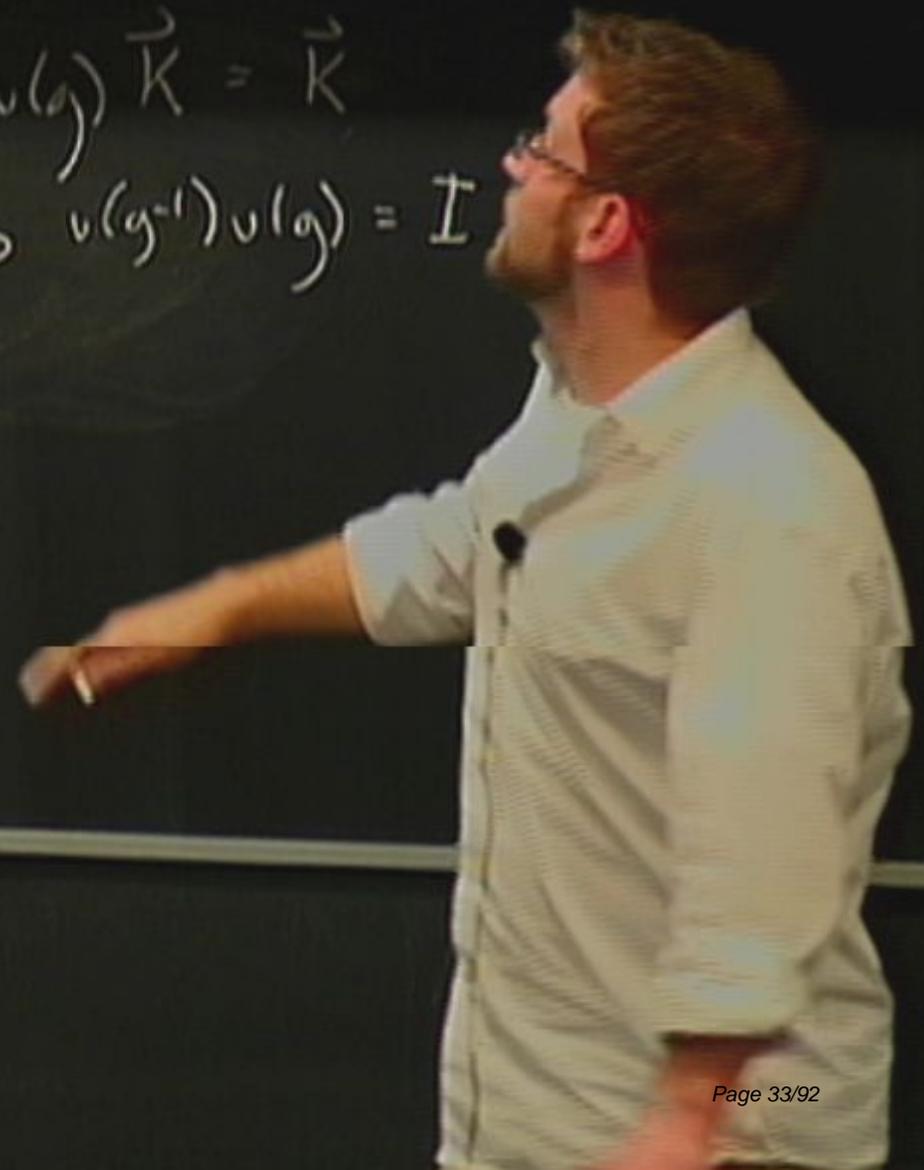
→

$\mathbb{R}^n \xrightarrow{M} \mathbb{R}^n$

$$K(g) = v(g) K$$

$$v(g^{-1}) \vec{K}(g) = v(g^{-1}) v(g) \vec{K} = \vec{K}$$

$$\rightarrow v(g^{-1}) v(g) = I$$



$$\vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$

$$v(g') \vec{K}(g) = v(g') v(g) \vec{K} = \vec{K}$$

$$\rightarrow v(g^{-1}) v(g) = I$$

$$v(g^{-1}) = v^{-1}(g)$$

$$T(g \cdot g') \vec{K} = \sum_{\mu} u_{\mu\mu'}(g) T(g') \vec{K}_{\mu'} T(g')^{\dagger}$$

$$\vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$

$$v(g') \vec{K}(g) = v(g') v(g) \vec{K} = \vec{K}$$

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$$T(g \cdot g') K_H T(g \cdot g')^\dagger = \sum_{\mu} u_{\mu\mu}(g) T(g') K_H T(g')^\dagger$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$\vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$

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$$T(g \cdot g') K_H T(g \cdot g')^\dagger = \sum_{\mu} u_{\mu\mu}(g) T(g') K_H T(g')^\dagger$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$v(g \cdot g') \vec{K} = v(g) v(g') \vec{K}$$

$$v(g \cdot g') = v(g) v(g')$$

$$\vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$

$$v(g') \vec{K}(g) = v(g') v(g) \vec{K} = \vec{K}'$$

$$\rightarrow v(g^{-1}) v(g) = I$$

$$v(g^{-1}) = v^{-1}(g)$$

$$T(g \cdot g') K_H T(g \cdot g')^\dagger = \sum_{\mu} u_{\mu H}(\mu, g) T(\mu, g') K_H T(\mu, g')^\dagger$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$v(g \cdot g') \vec{K} = v(g) v(g') \vec{K}$$

$$v(g \cdot g') = v(g) v(g')$$

$$\textcircled{2} \quad \vec{K}(g) = v(g) \vec{K} \quad (\vec{K})_H = K_H$$

$$v(g') \vec{K}(g) = v(g') v(g) \vec{K} = \vec{K}'$$

$$\rightarrow v(g^{-1}) v(g) = I$$

$$v(g^{-1}) = v^{-1}(g)$$

$$\textcircled{1} \quad T(g \cdot g') K_H T(g \cdot g')^\dagger = \sum_{\mu} u_{H\mu}(g) T(g') K_H T(g')^\dagger$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$v(g \cdot g') \vec{K} = v(g) v(g') \vec{K}$$

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$$\rightarrow v(g') v(g) = I$$

$$v(g^{-1}) = v^{-1}(g)$$

$$\textcircled{1} \quad T(g \cdot g') K_H T(g \cdot g')^\dagger = \sum_H U_{HH'}(g) T(g') K_H T(g')^\dagger$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$v(g \cdot g') \vec{K} = v(g) v(g') \vec{K}$$

$$v(g \cdot g') = v(g) v(g')$$

$w$  takes  $u(g)$  to block diag. form

$$\tilde{K}(g) = u(g) \tilde{K}$$

$$w \tilde{K}(g) = w u(g) w^t w \tilde{K}$$

$$\tilde{K}'(g) = u'(g) \tilde{K}'$$



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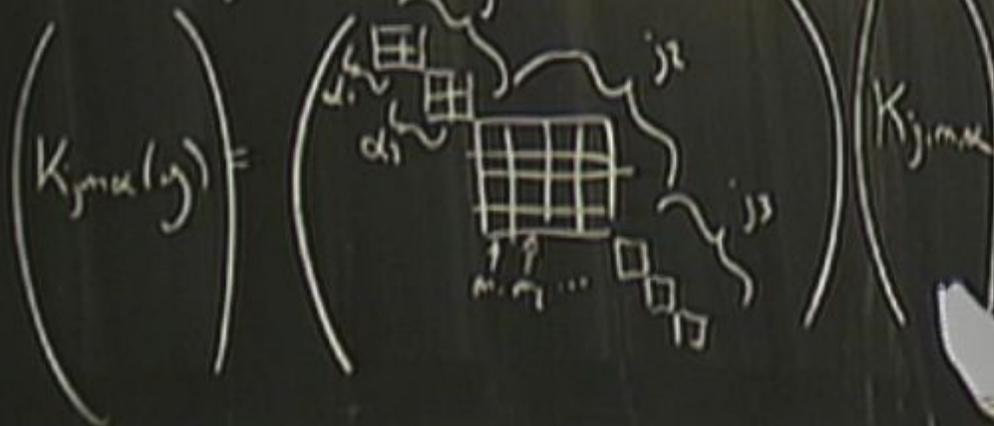


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$$T(g) K_{j,m,\alpha} T^+(g) = \sum_{\alpha} u_{\alpha\alpha}^{(j)}(g) K_{j,m,\alpha} \quad \forall g \in G$$

Diagram labels:  $\alpha$  (row),  $\alpha$  (column),  $\alpha$  (multi),  $\alpha$  (basis)

w takes  $u(g)$  to Wack ding, form

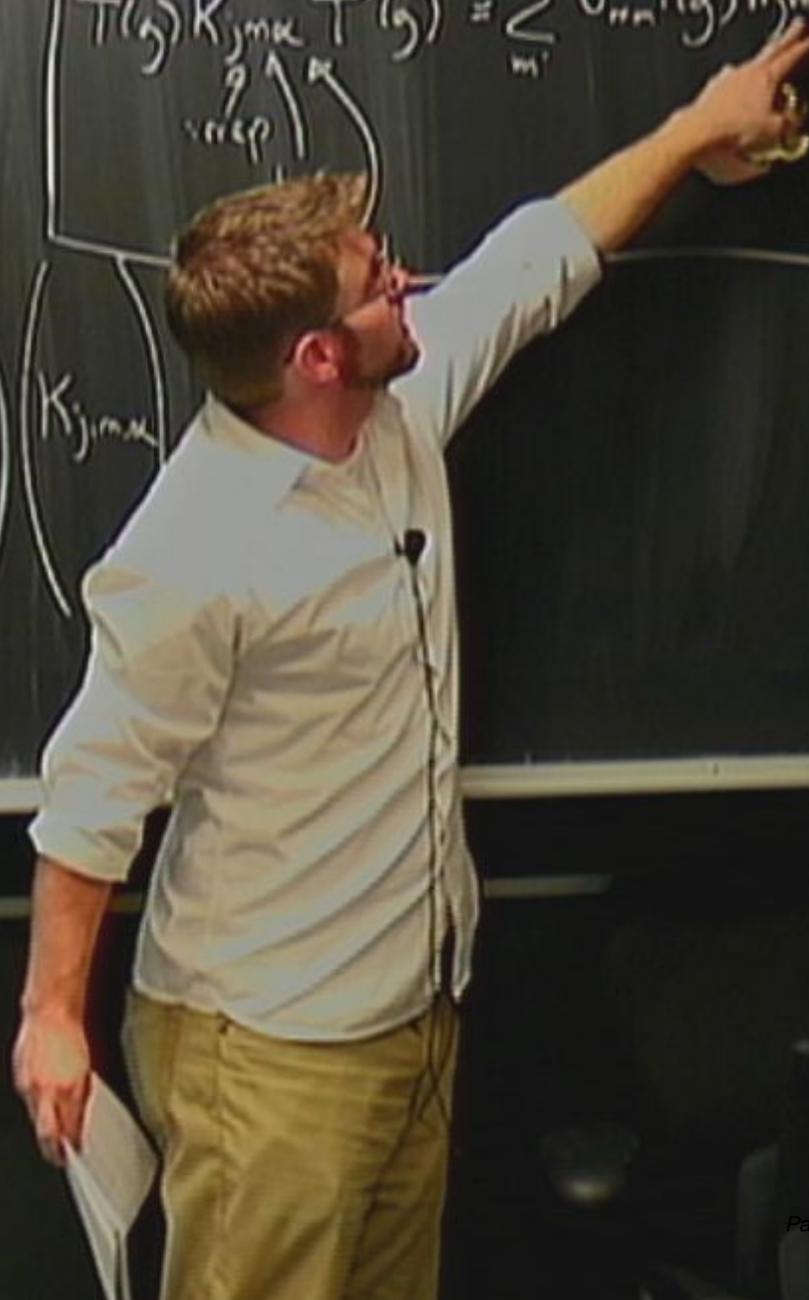
$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) w^t w K$$

$$\tilde{K}'(g) = u'(g) K'$$



$$T(g) K_{j,m,x} T^+(g) = \sum_{m'} u_{m'm}(g) K_{j,m',x} \quad \forall g \in G$$

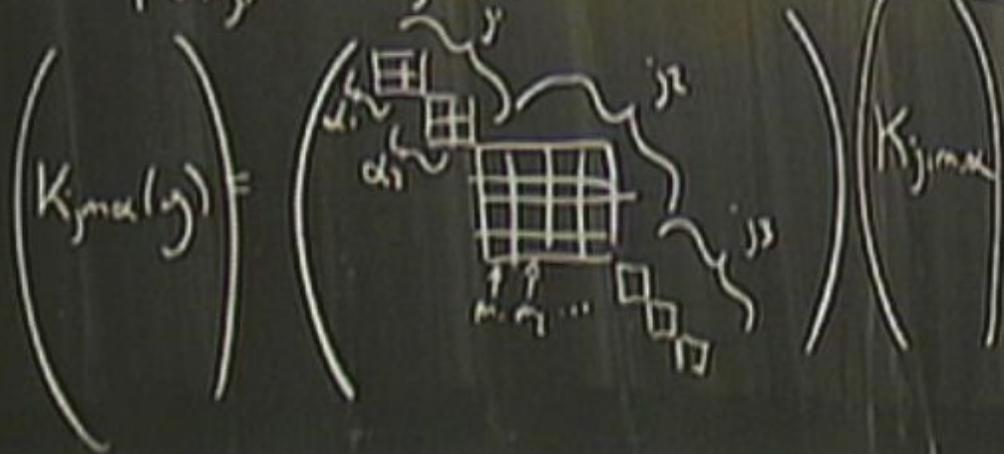


w takes  $u(g)$  to block diag. form

$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) w^t w K$$

$$\tilde{K}'(g) = u'(g) K'$$



$$T(g) K_{j,m \times k} T^+(g) = \sum_{m'} u_{mm'}^{(j)}(g) K_{j,m' \times k} \quad \forall g \in G$$

Annotations:  $g$  (pointing to  $T(g)$ ),  $u_{mm'}$  (pointing to the sum),  $basis$  (pointing to  $K_{j,m' \times k}$ ),  $mult.$  (pointing to the sum).

w takes  $u(g)$  to Wock diag. form

$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) w^t w K$$

$$\tilde{K}'(g) = u'(g) K'$$

$$T(g) K_{j,m,\alpha} T^+(g) = \sum_{m'} u_{mm'}^{(j)}(g) K_{j,m',\alpha} \quad \forall g \in G$$

$\uparrow$   $\uparrow$   $\uparrow$   
 rep basis mult.

$$K_{j,m,\alpha}(g) =$$



$$T(g) [K_{j,m,\alpha}] = \sum_{m'} u_{mm'}^{(j)}(g) K_{j,m',\alpha}$$



w takes  $u(g)$  to Wokk ding, form

$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) w^\dagger w K$$

$$\tilde{K}'(g) = u(g) K'$$

$$T(g) K_{j,m,\alpha} T^\dagger(g) = \sum_{m'} U_{mm'}^{(j)}(g) K_{j,m',\alpha} \quad \forall g \in G$$

$\begin{matrix} \nearrow & \text{rep} \\ \uparrow & \text{basis} \\ \searrow & \text{mult.} \end{matrix}$



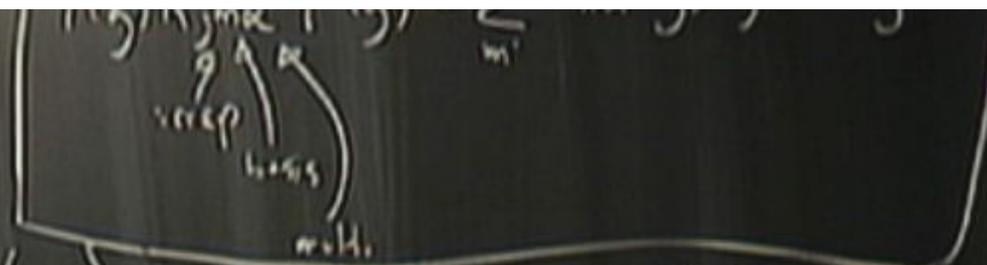
$$T(g) [K_{j,m,\alpha}] = \sum_{m'} U_{mm'}^{(j)}(g) K_{j,m',\alpha}$$

$$T(g) |j, m, \alpha\rangle = \sum_{m'} U_{mm'}^{(j)}(g) |j, m', \alpha\rangle$$

$$K(g) = u(g) K$$

$$w K(g) = w u(g) w^\dagger w K$$

$$K'(g) = u'(g) K'$$



$$U(g) [K_{j,m,\alpha}] = \sum_{m'} U_{mm'}^{(j)}(g) K_{j,m',\alpha}$$

$$T(g) |j, m, \alpha\rangle = \sum_{m'} U_{mm'}^{(j)}(g) |j, m', \alpha\rangle$$

$$\{K_{j,m,\alpha} | m\}$$

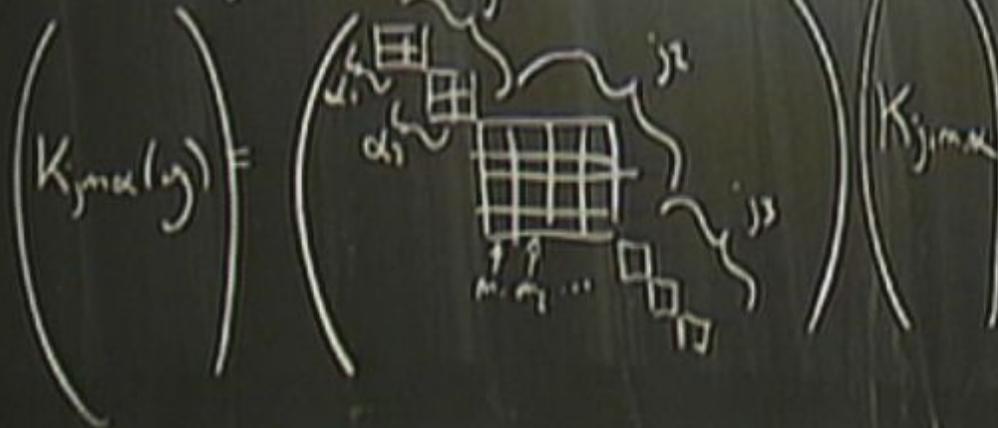


w takes  $u(g)$  to Wokk ching form

$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) w^+ w K$$

$$\tilde{K}'(g) = u(g) K'$$



$$T(g) K_{j,m,\alpha} T^+(g) = \sum_{n=1}^{\infty} u_{nn}^{(j)}(g) K_{j,m,\alpha} \quad \forall g \in G$$

$\begin{matrix} \nearrow \text{diag} \\ \text{basis} \\ \searrow \text{multi} \end{matrix}$

$$T(g) [K_{j,m,\alpha}] = \sum_{n=1}^{\infty} u_{nn}^{(j)}(g) K_{j,m,\alpha}$$

$$T(g) |j, m, \alpha\rangle = \sum_{n=1}^{\infty} u_{nn}^{(j)}(g) |j, m, \alpha\rangle$$

$$\{K_{j,m,\alpha} | m\rangle\}$$

Only implementable op's under G-S/R

$$\boxed{\begin{aligned} \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &= \mathcal{E} \quad \{g \in G \\ \text{where } \mathcal{T}(g)[X] &= \mathcal{T}(g) X \mathcal{T}^\dagger(g) \end{aligned}}$$

Kraus decomposition for G-inv. op's

$$\begin{aligned} \mathcal{E} &\leftrightarrow \{K_H\}_{H \in \mathcal{H}} \quad \mathcal{E} = \sum_H K_H(\cdot) K_H^\dagger \\ \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) &\leftrightarrow \{L_H(g) = \mathcal{T}(g) K_H \mathcal{T}^\dagger(g)\} \end{aligned}$$

$$\boxed{\mathcal{T}(g) K_H \mathcal{T}^\dagger(g) = \sum_{H'} u_{HH'}(g) K_{H'}}$$

$$\textcircled{1} \mathcal{T}(g \cdot g') K_H \mathcal{T}^\dagger(g \cdot g') = \sum_{H'} u_{HH'}(g) \mathcal{T}(g') K_{H'} \mathcal{T}^\dagger(g')$$

$$\vec{K}(g \cdot g') = v(g) \vec{K}(g')$$

$$v(g \cdot g') \vec{K} = v(g) v(g') \vec{K}$$

$$\boxed{v(g \cdot g') = v(g) v(g')}$$

Resource Theory for  $U(1)$ -SSR

$$T(\phi) = e^{i\phi N} \quad \forall \phi \in U(1)$$

Representation Theory For  $U(n)$ -SSR

$$T(\phi) = e^{i\phi \hat{N}} \quad \forall \phi \in U(1)$$

$U(1)$ -invariant state,  
 $e^{i\phi \hat{N}} \rho_0 e^{-i\phi \hat{N}} = \rho$

$$[\rho, \hat{N}] = 0$$

$$U(\phi) = e^{i\phi N} \quad \forall \phi \in U(1)$$

$U(1)$ -invariant states,  
 $\rho_0 - i\phi \hat{N} = \rho$   
 $[\rho, \hat{N}] = 0$   
 $U(1)$ -inv. operations



$w$  takes  $u(g)$  to block diag. form

$$\bar{K}(g) = u(g) K$$

$$w \bar{K}(g) = w u(g) w^t w K$$

$$\bar{K}'(g) = u'(g) K'$$

$$T(g) K_{j m \alpha} T^+(g) = \sum_{m'} u_{m m'}^{(j)}(g) K_{j m' \alpha} \quad \forall g \in G$$

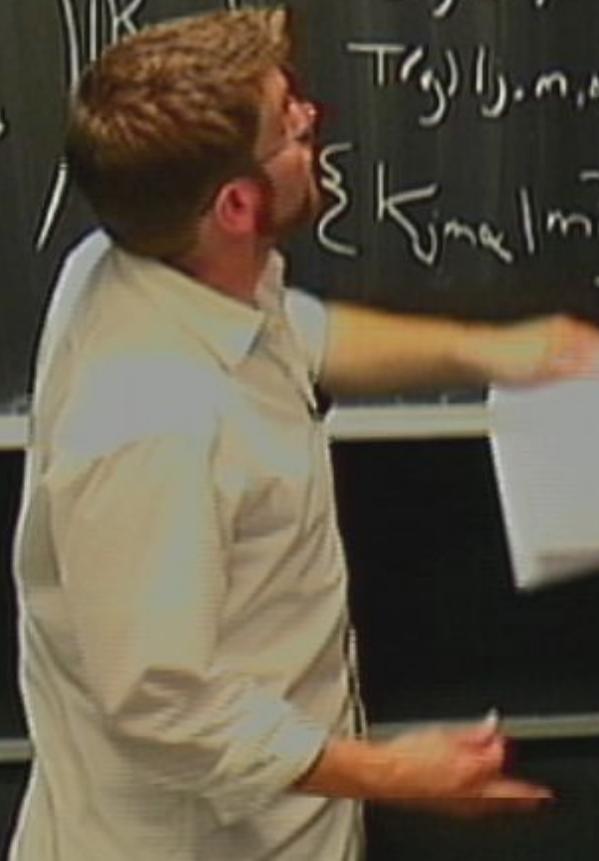
$\begin{matrix} \nearrow & \text{diag} \\ \text{diag} & \nearrow \\ \text{mult.} & \end{matrix}$



$$T(g) [K_{j m \alpha}] = \sum_{m'} u_{m m'}^{(j)}(g) K_{j m' \alpha}$$

$$T(g) |j, m, \alpha\rangle = \sum_{m'} u_{m m'}^{(j)}(g) |j, m', \alpha\rangle$$

$$\{ |K_{j m \alpha} | m \rangle \}$$



# Resource Theory For $U(1)$ -SSR

$$T(\phi) = e^{i\phi\hat{N}} \quad \forall \phi \in U(1)$$

$U(1)$ -invariant states

$$\rho e^{-i\phi\hat{N}} = \rho$$

$$[\rho, \hat{N}] = 0$$

$U(1)$ -inv. operations

inreps labelled by  $k \in \mathbb{Z}$

$$U^{(k)}(\phi) = e^{ik\phi}$$

$$e^{i\phi\hat{N}}$$

# (1) - SSR

$$T(\phi) = e^{i\phi\hat{N}} \quad \forall \phi \in U(1)$$

$U(1)$ -invariant states,

$$e^{i\phi\hat{N}} \rho_0 e^{-i\phi\hat{N}} = \rho_0$$

$$[\rho_0, \hat{N}] = 0$$

$U(1)$ -inv. operations

irreps labelled by  $k \in \mathbb{Z}$

$$U^{(k)}(\phi) = e^{ik\phi}$$

$$e^{i\phi\hat{N}} K_{k,\kappa} e^{-i\phi\hat{N}} = e^{ik\phi} K_{k,\kappa}$$

$$e^{i\phi N} K_{k,\alpha} e^{-i\phi N} = e^{i\phi k} K_{k,\alpha}$$

$$K_{k,\alpha} = \sum_{n=0}^{\infty} \frac{e^{(k,\alpha)n}}{n!} |n\rangle \langle n|$$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)} |n\rangle \langle n'|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)} e^{i\phi(n-n')} |n\rangle \langle n'| = \binom{\quad}{\quad} e^{ik\phi}$$

$n' = n - k$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)}$$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)} |n\rangle \langle n'|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)} e^{i\phi(n-n')} |n\rangle \langle n'| = \binom{n}{n'} e^{ik\phi}$$

$n' = n - k$



$$\sum_{n'} c_n^{(k,\alpha)} |n\rangle \langle n-k|$$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)} |n\rangle \langle n'|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)} e^{i\phi(n-n')} |n\rangle \langle n'| = \binom{n}{n'} e^{ik\phi}$$

$n' = n - k$

$$K_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n-k|$$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)} |n\rangle \langle n'|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)} e^{i\phi(n-n')} |n\rangle \langle n'| = \left( \begin{matrix} \text{---} \\ \text{---} \end{matrix} \right) e^{ik\phi}$$

$n' = n - k$

$$K_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n-k|$$

$$= S_k \tilde{K}_{k,\alpha}$$

$$\tilde{K}_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n|$$

$$S_k = \sum_n |n+k\rangle \langle n|$$

$$K_{k,\alpha} = \sum_{n,n'} c_{n,n'}^{(k,\alpha)} |n\rangle \langle n'|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)} e^{i\phi(n-n')} |n\rangle \langle n'| = \left( \begin{matrix} \text{---} \\ \text{---} \end{matrix} \right) e^{ik\phi}$$

$n' = n - k$

$$K_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n-k|$$

$$= S_k K_{k,\alpha}$$

$$\tilde{K}_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n|$$

$$S_k = \sum_n |n+k\rangle \langle n|$$

$$|1\rangle + |2\rangle \rightarrow |0\rangle + |1\rangle$$

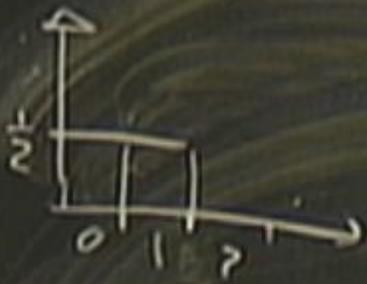
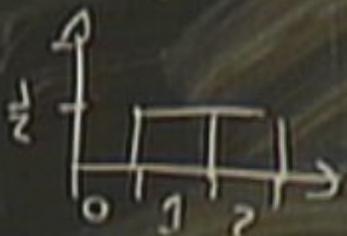
inv. op's

$$\mathcal{E} \rightarrow \{K_H\}_{H=1}^n \quad \mathcal{E} = \sum_H K_H(\cdot) K_H^\dagger$$

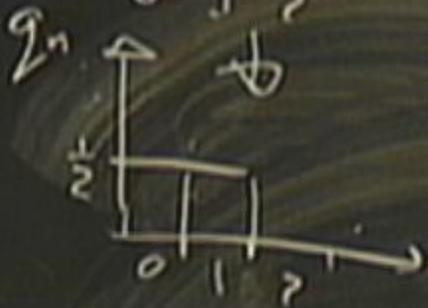
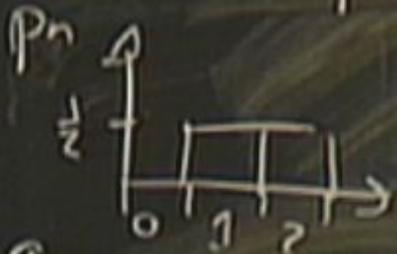
$$T(g) \cdot \mathcal{E} \cdot T(g)^\dagger \leftrightarrow \{L_H(g) = T(g) K_H T(g)^\dagger\}$$

$$T(g) K_H T(g)^\dagger = \sum_{H'} u_{HH'}(g) K_{H'}$$

$$|1\rangle + |2\rangle \rightarrow |0\rangle + |1\rangle$$



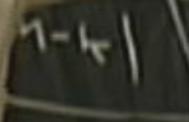
$$|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$K_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n|$$

$$\sum_{n,n'} c_{n,n'}^{(k,\alpha)}$$

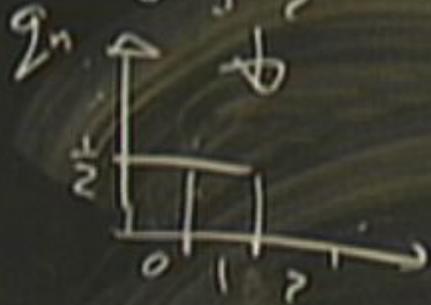
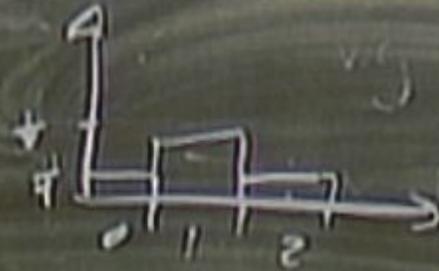
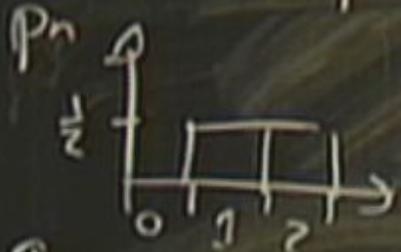
$$|n\rangle \langle n| = \left( \dots \right) e^{ikf}$$



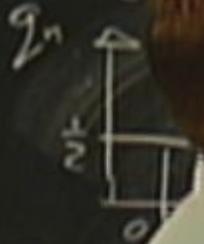
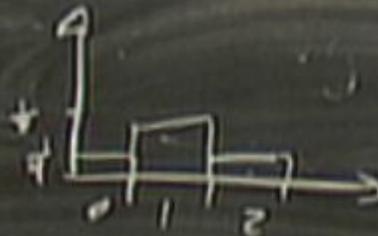
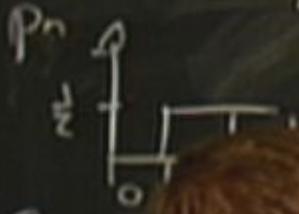
$$K_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle \langle n|$$

$$S_k = \sum_n |n+k\rangle \langle n|$$

$$|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$|\psi\rangle = \sum_n p_n |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$E_0 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|$$

$$|\psi\rangle = \sum_n p_n |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$

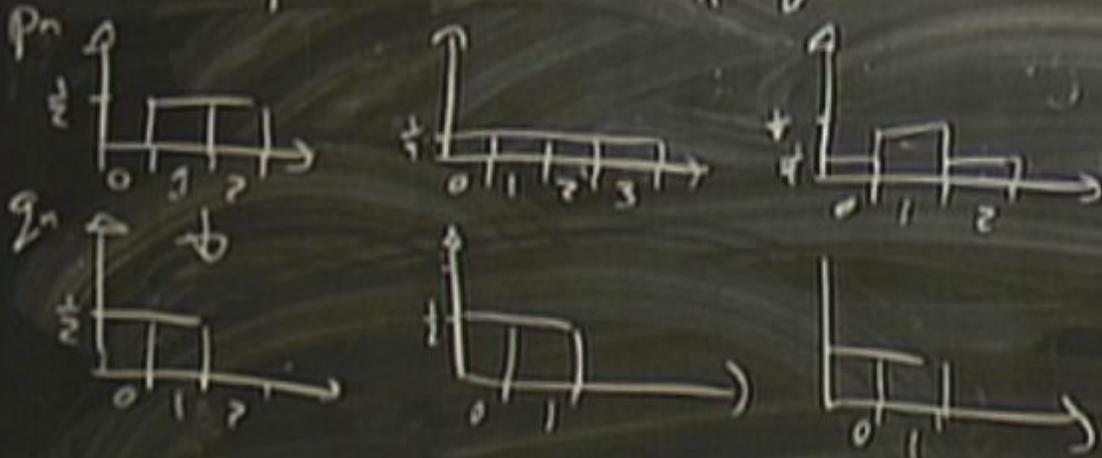


$$E_0 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|$$

Thm. (Pirsa)

$$|\psi\rangle = \sum_n p_n |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$E_0 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|$$

Thm: Traced. possible iff

$$\vec{p} = \sum_{k=0}^{\infty} w_k$$

$$|\psi\rangle = \sum_n p_n |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



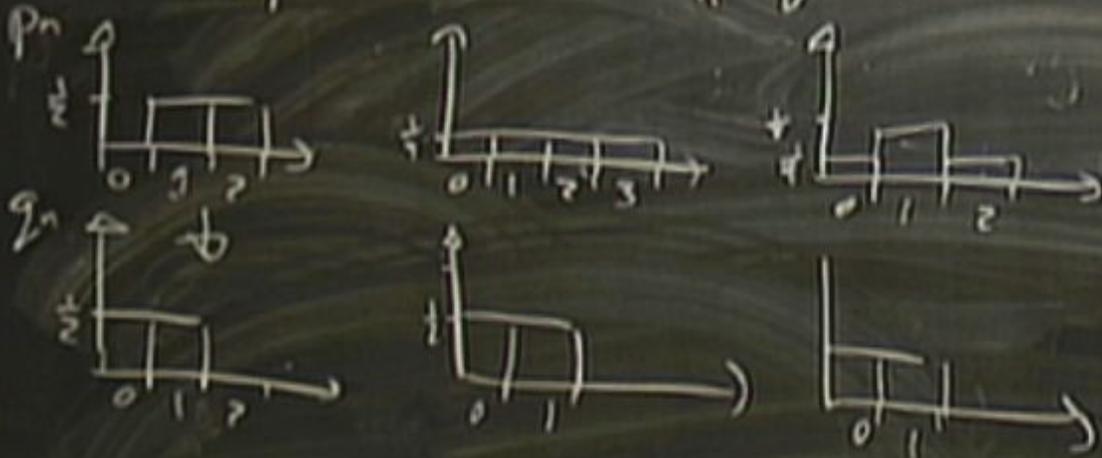
$$E_0 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|$$

Thm: Trunc. possible iff

$$\vec{p} = \sum_{k=0}^{\infty} w_k \vec{v}_k \vec{q}_k$$

$$|\psi\rangle = \sum_n |p_n\rangle |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$E_0 = |10\rangle\langle 01| + \frac{1}{2}|11\rangle\langle 11|$$

$$E_1 = \frac{1}{2}|11\rangle\langle 11| + |2\rangle\langle 2|$$

Thm: Trunc. possible if

$$\vec{p} = \sum_{k=0}^{\infty} w_k \vec{\gamma}_k$$

$$|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle \rightarrow |\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$$



$$E_0 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$E_1 = \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|$$

Thm: Trivial possible SF

$$\vec{\beta} = \left( \sum_{k=0}^{\infty} w_k \gamma_k \right) \vec{q}$$

w takes  $u(g)$  to block diag. form

$$\tilde{K}(g) = u(g) K$$

$$w \tilde{K}(g) = w u(g) u^\dagger w K$$

$$\tilde{K}'(g) = u(g) K'$$



$$T(g) K_{j,m,\alpha} T^\dagger(g) = \sum_{n'} U_{nn'}^{(j)}(g) K_{j,m',\alpha} \quad \forall g \in G$$

Diagram with arrows:  $g$  (top),  $U_{nn'}$  (left),  $U_{nn'}$  (right),  $mult.$  (bottom)

$$T(g) [K_{j,m,\alpha}] = \sum_{n'} U_{nn'}^{(j)}(g) K_{j,m',\alpha}$$

$$T(g) |j, m, \alpha\rangle = \sum_{n'} U_{nn'}^{(j)}(g) |j, m', \alpha\rangle$$

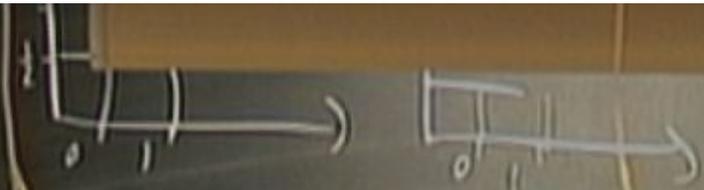
$$\{ |j, m, \alpha\rangle \}$$

G-frames, monotonies



$G$ -freeness monotonies

his  $F$  nonincreasing under  $G$ -invariant op's.



$$\vec{\mu} = \left( \sum_{k=1}^K w_k \gamma_k \right) \vec{e}$$

Deterministic single-copy transfer

G-freeness monotonies

nonincreasing under G-invariant op's.

Stochastic

$G$ -firmness monotonies

his  $F$  nonincreasing under  $G$ -invariant op's.

Stochastic single-copy trans.

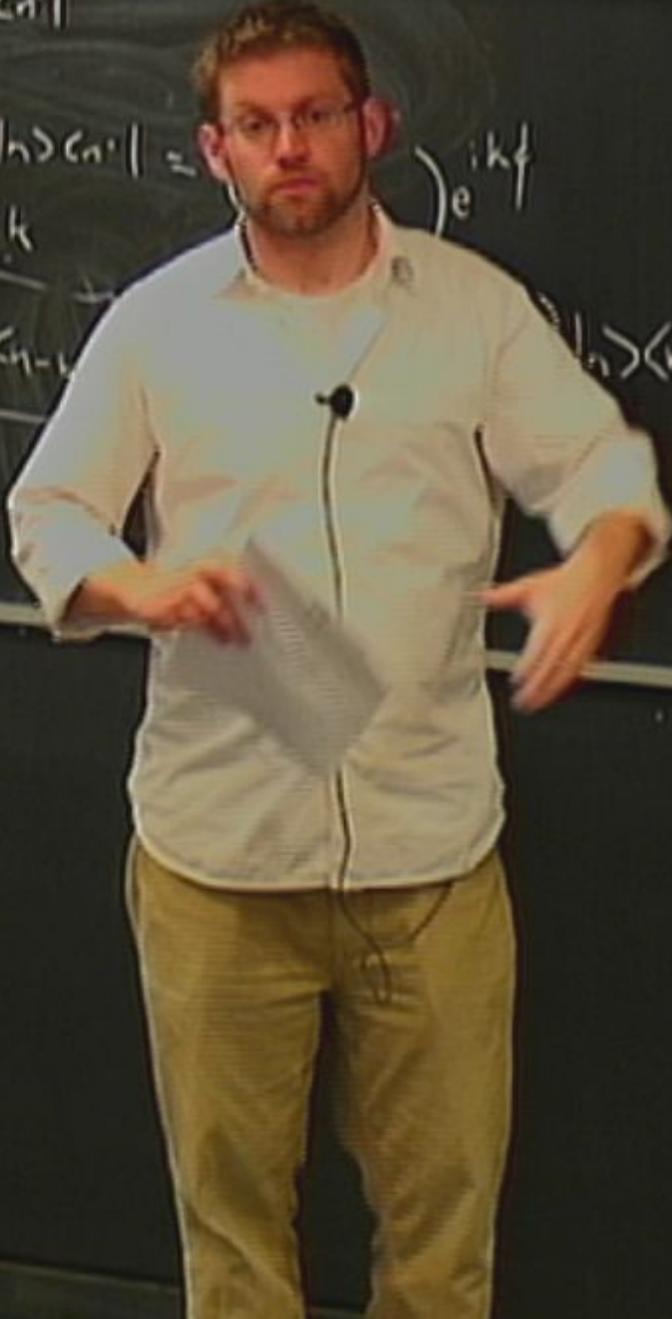
$$K_{k,k} = \sum_{n=0}^{\infty} \binom{k+k}{n} |n\rangle \langle n|$$

$$\sum_{n=0}^{\infty} \binom{k+k}{n} \frac{1}{e} e^{i(k-k)n} |n\rangle \langle n| = e^{ikf}$$

$n \leq n-k$

$$K_{k,k} = \sum_{n=0}^{\infty} \binom{k+k}{n} |n\rangle \langle n-k|$$

$$= \sum_k K_{k,k}$$



E.A. 10/10

$G$ -framesness monotones

$h$ 's  $F$  nonincreasing under  $G$ -invariant op's.

Stochastic single copy trans

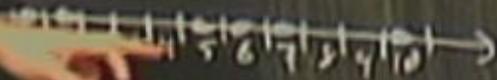
$$n\text{-spec}(\Psi) = \{n \mid \rho_n \neq 0\}$$

$G$ -framesness monotones

$h$ 's  $F$  nonincreasing under  $G$ -invariant op's.

Stochastic single copy trans

$$n\text{-spec}(\Psi) = \{n \mid \rho_n \neq 0\}$$

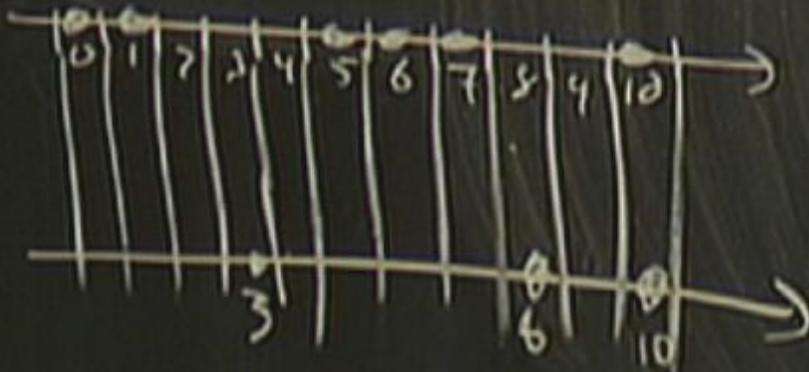


q-transmissives monotones

his F nonincreasing under q-invariant op'

Stochastic single-copy trans

$$n\text{-spec}(\psi) = \{n \mid p_n \neq 0\}$$

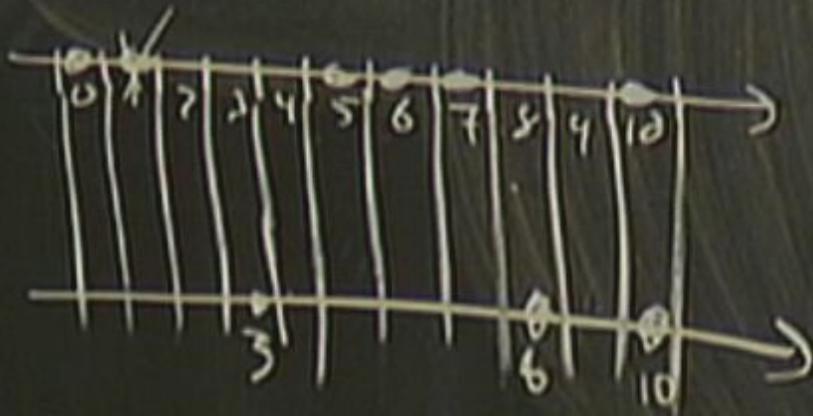


G-traneness monotonies

his F nonincreasing under G-invariant op's.

Stochastic single copy trans.

$$n\text{-spec}(\psi) = \{n \mid p_n \neq 0\}$$



G-freeness monotonies

his F nonincreasing under G-invariant op's.

Stochastic single rep.

$$n\text{-spec}(\psi) = \sum n_i$$

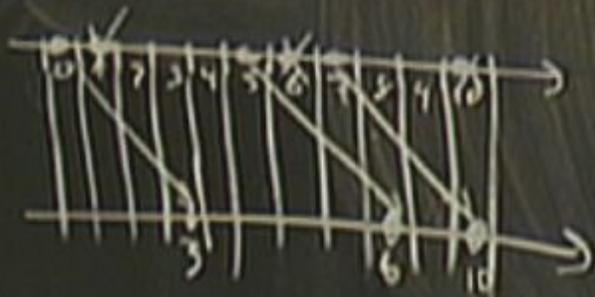
Thm: stochastic transf.s possible  
 $n\text{-spec}(\phi) \leq n\text{-spec}(\psi)$

$G$ -framedness monotones

his  $F$  nonincreasing under  $G$ -invariant op's.

Stochastic single-copy trans

$$n\text{-spec}(\psi) = \{ \sum_n |p_n| \}$$

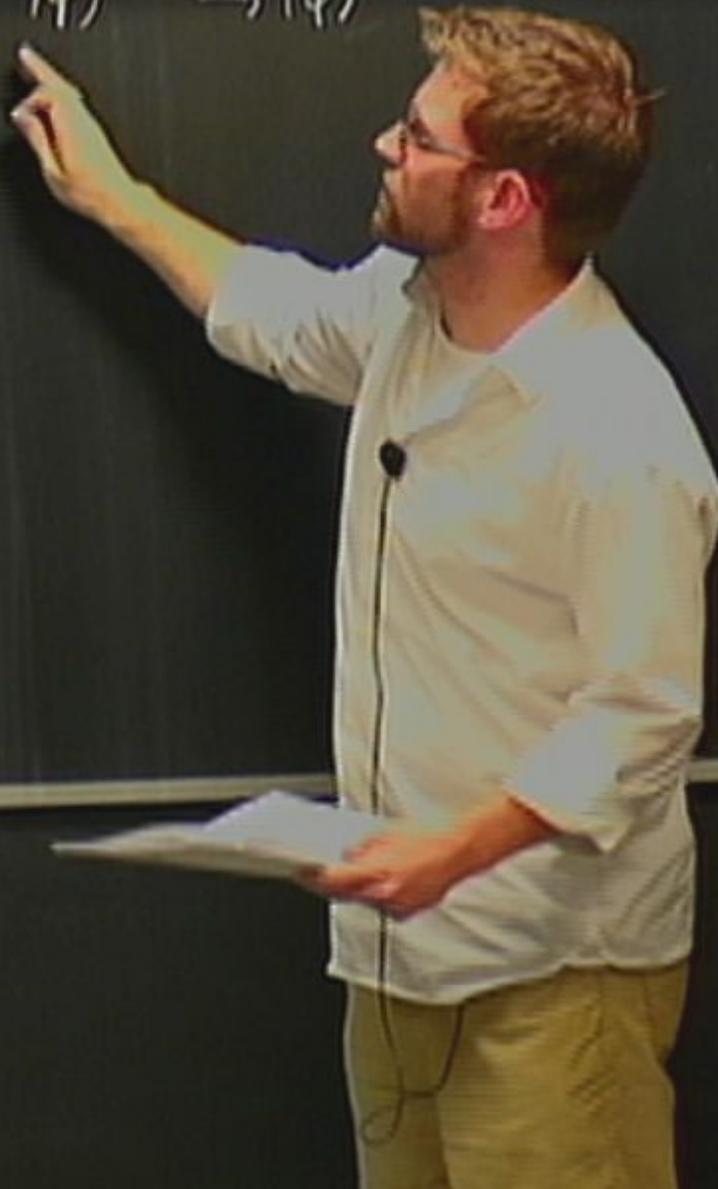


Thm: stochastic trans possible  
iff  $\exists k \sum_n |p_n| \leq n\text{-spec}(\psi) + k$

$$Pr(\psi \rightarrow \phi) = \min_n \left( \frac{p_n}{p_n + k} \right)$$

Asympt. transf.  
 $|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{p_n} |n\rangle$

$|\psi\rangle^{\otimes N} \rightarrow |\psi\rangle^{\otimes M}$



Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\psi\rangle^{\otimes M}$

$$|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{p_n} |n\rangle$$

$$|\psi\rangle^{\otimes N} = \sum_{n_1, \dots, n_N} \sqrt{p_{n_1} \dots p_{n_N}} |n_1, \dots, n_N\rangle$$



Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$

$$|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{p_n} |n\rangle$$

$$|\psi\rangle^{\otimes N} = \sum_n \sqrt{r_n} |n\rangle$$

$$V(|\psi\rangle^{\otimes N}) = V(|\psi\rangle^{\otimes M})$$

$$N V(\psi) = M V(\phi)$$

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{V(\psi)}{V(\phi)}$$

Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$

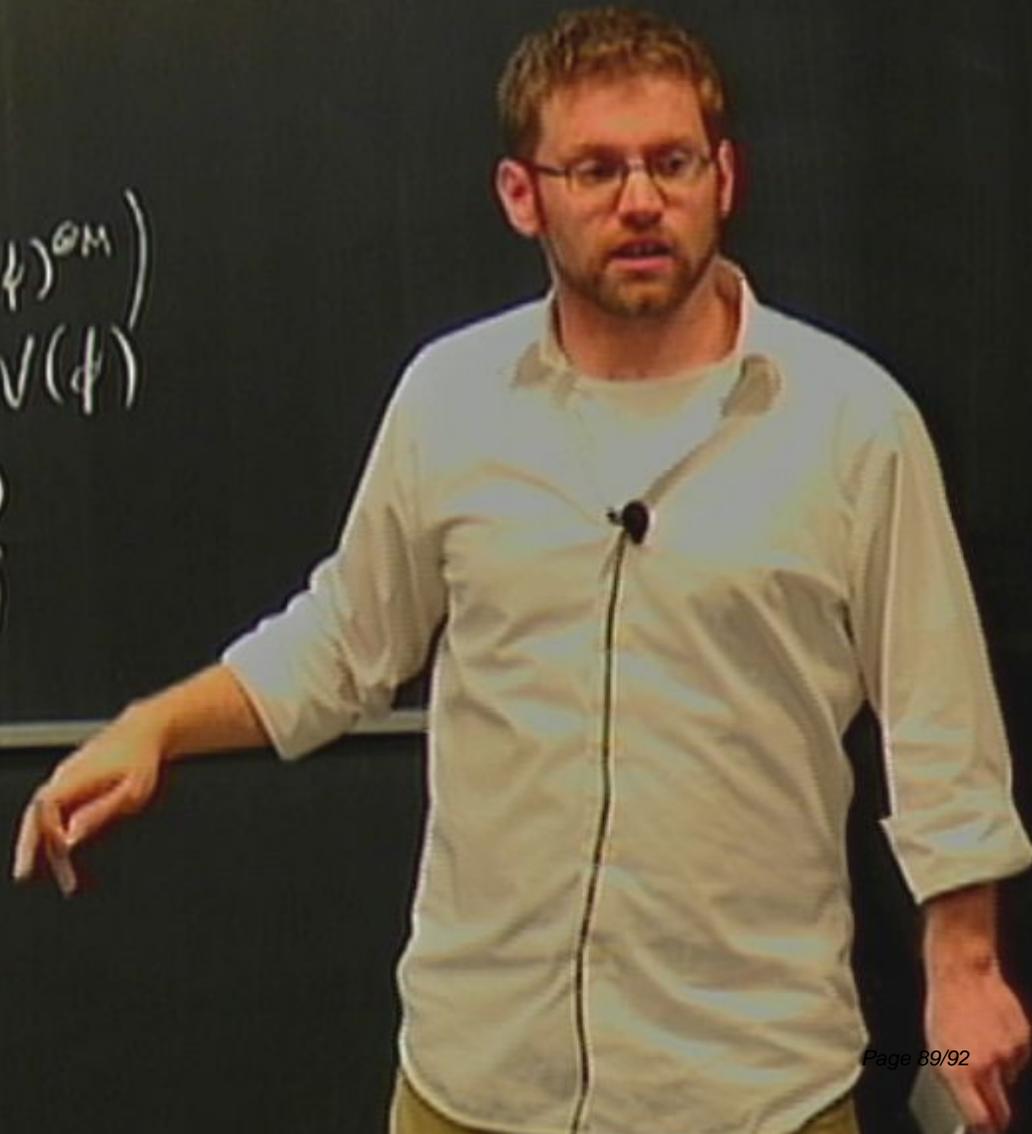
$$|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{p_n} |n\rangle$$

$$|\psi\rangle^{\otimes N} = \sum_n \sqrt{r_n} |n\rangle$$

$$V(|\psi\rangle^{\otimes N}) = V(|\psi\rangle^{\otimes M})$$

$$N V(\psi) = M V(\phi)$$

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{V(\psi)}{V(\phi)}$$



Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$   
 $|\psi\rangle = \sum_{n=0}^d \sqrt{p_n} |n\rangle \quad \forall n \in \{0, \dots, d\}, p_n \neq 0$   
 $|\psi\rangle^{\otimes N} = \sum_n \sqrt{r_n} |n\rangle$

$$V(|\psi\rangle^{\otimes N}) = V(|\psi\rangle^{\otimes M})$$

$$N V(\psi) = M V(\phi)$$

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{V(\psi)}{V(\phi)}$$

Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$   
 $|\psi\rangle = \sum_{n=0}^d \sqrt{p_n} |n\rangle \quad \forall n \in \{0, \dots, d\}, p_n \neq 0$  "gapless number spectrum"  
 $|\psi\rangle^{\otimes N} = \sum_n \sqrt{r_n} |n\rangle$

$$V(|\psi\rangle^{\otimes N}) = V(|\psi\rangle^{\otimes M})$$

$$N V(\psi) = M V(\phi)$$

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{V(\psi)}{V(\phi)}$$

Asympt. transf.  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$   
 $|\psi\rangle = \sum_{n=0}^d \sqrt{p_n} |n\rangle \quad \forall n \in \{0, \dots, d\}, p_n \neq 0$  "gapless number spectrum"  
 $|\psi\rangle^{\otimes N} = \sum_n \sqrt{r_n} |n\rangle$

$$V(|\psi\rangle^{\otimes N}) = V(|\psi\rangle^{\otimes M}) \quad (|10\rangle + |11\rangle)^{\otimes N}$$

$$N V(\psi) = M V(\phi) \quad \downarrow \mathcal{P}$$

$$(|10\rangle + |12\rangle)^{\otimes M}$$

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{V(\psi)}{V(\phi)}$$