Title: Sicmubbery and the Geometry of Quantum State Space

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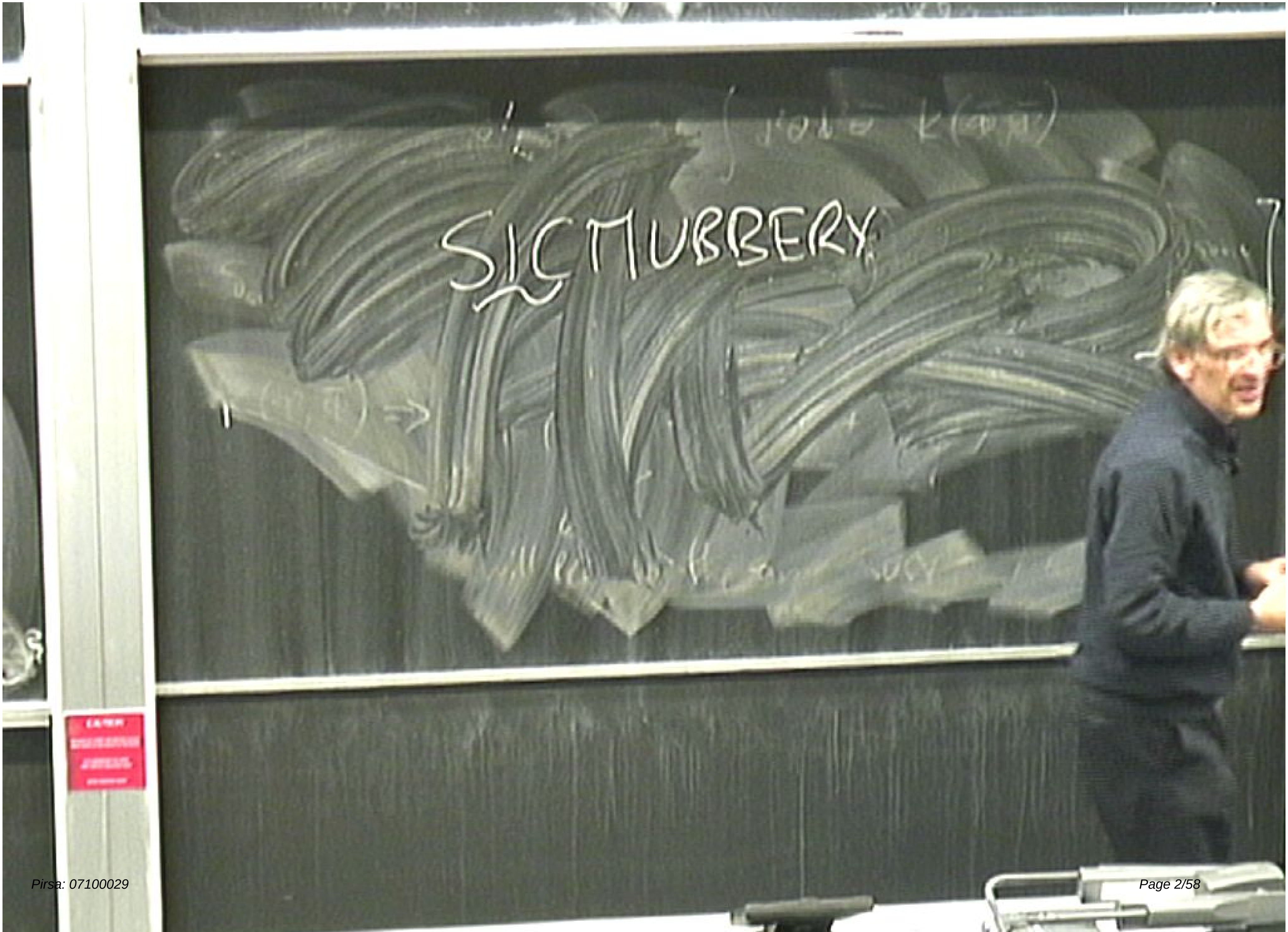
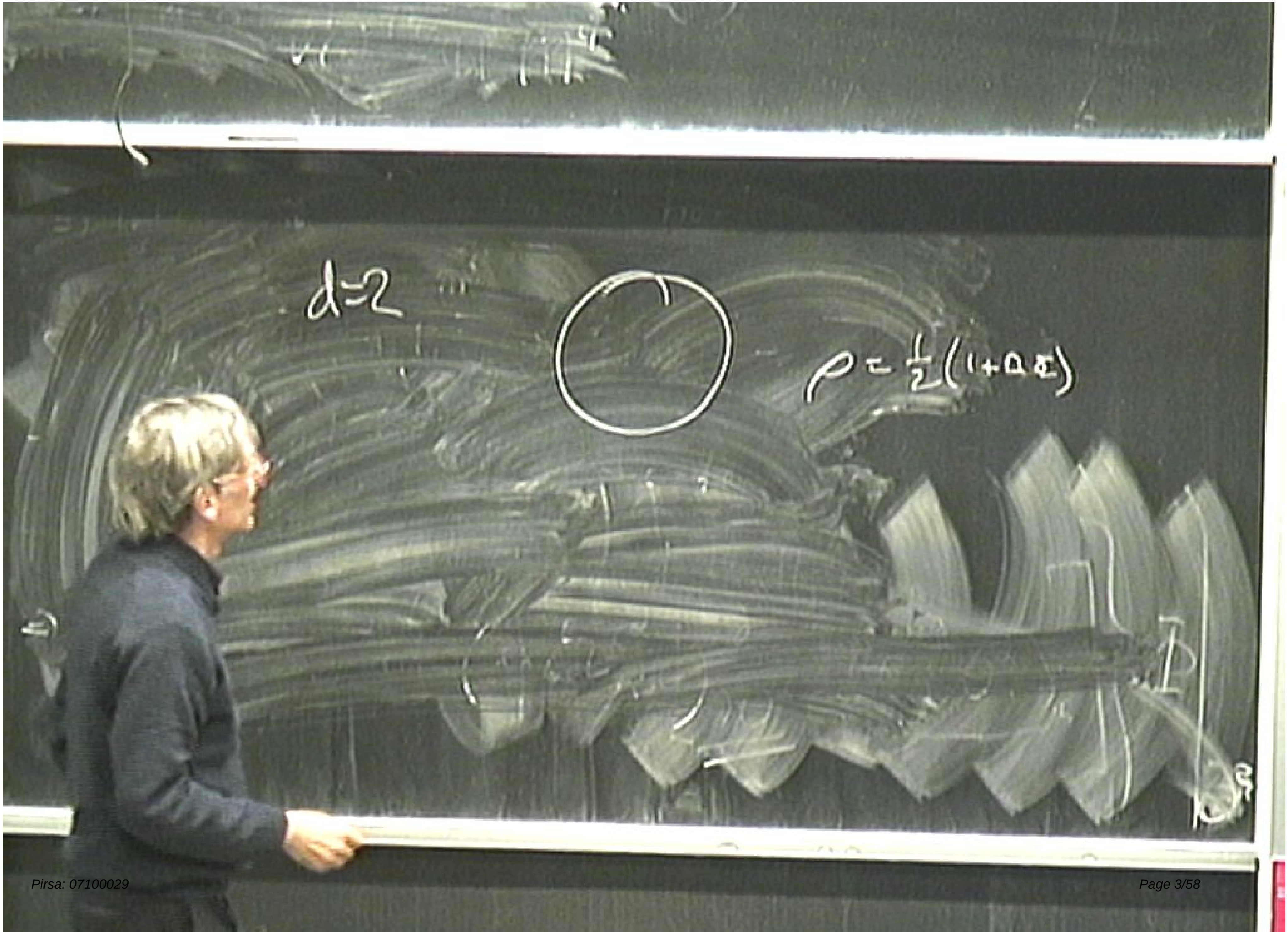
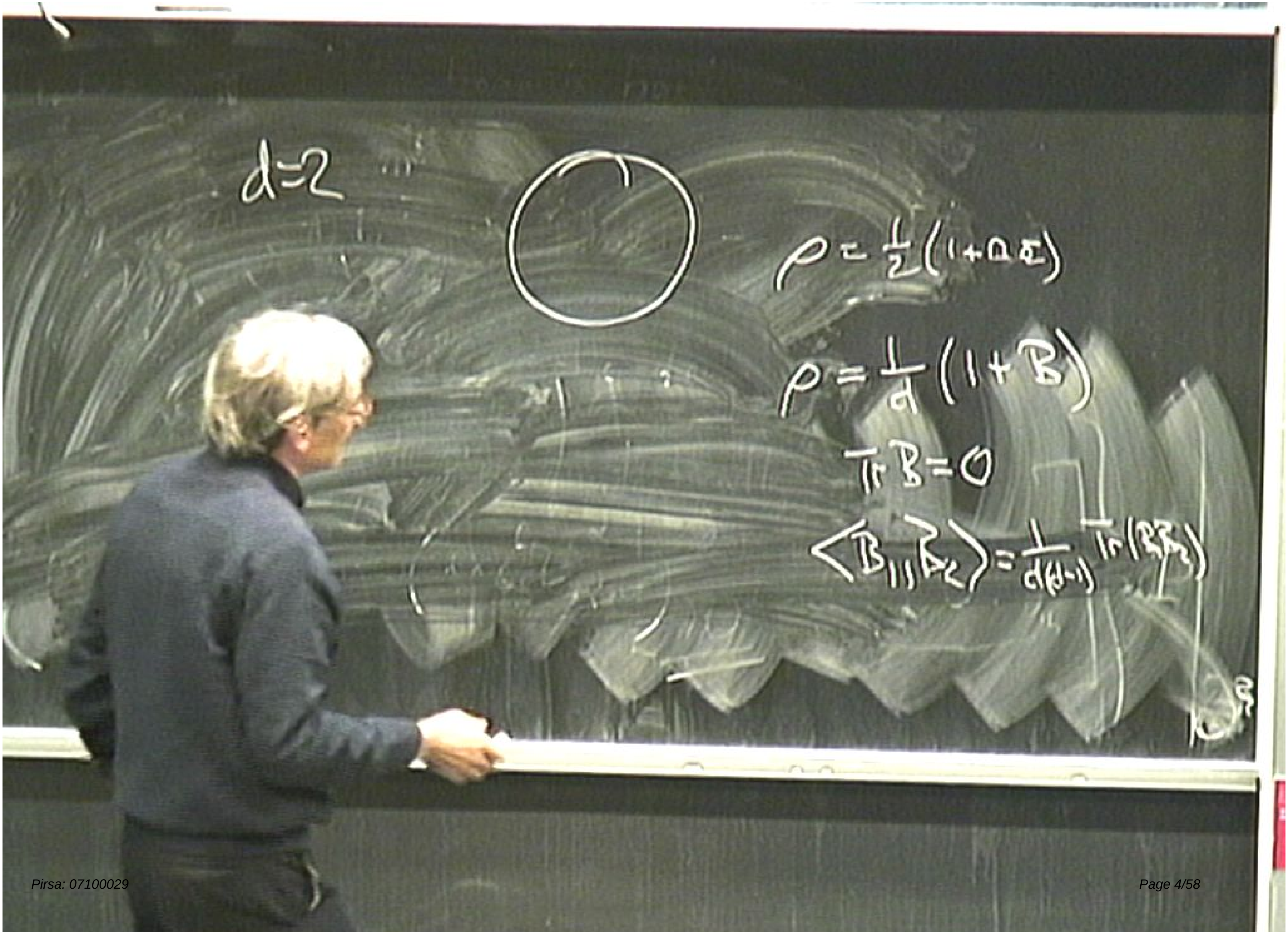
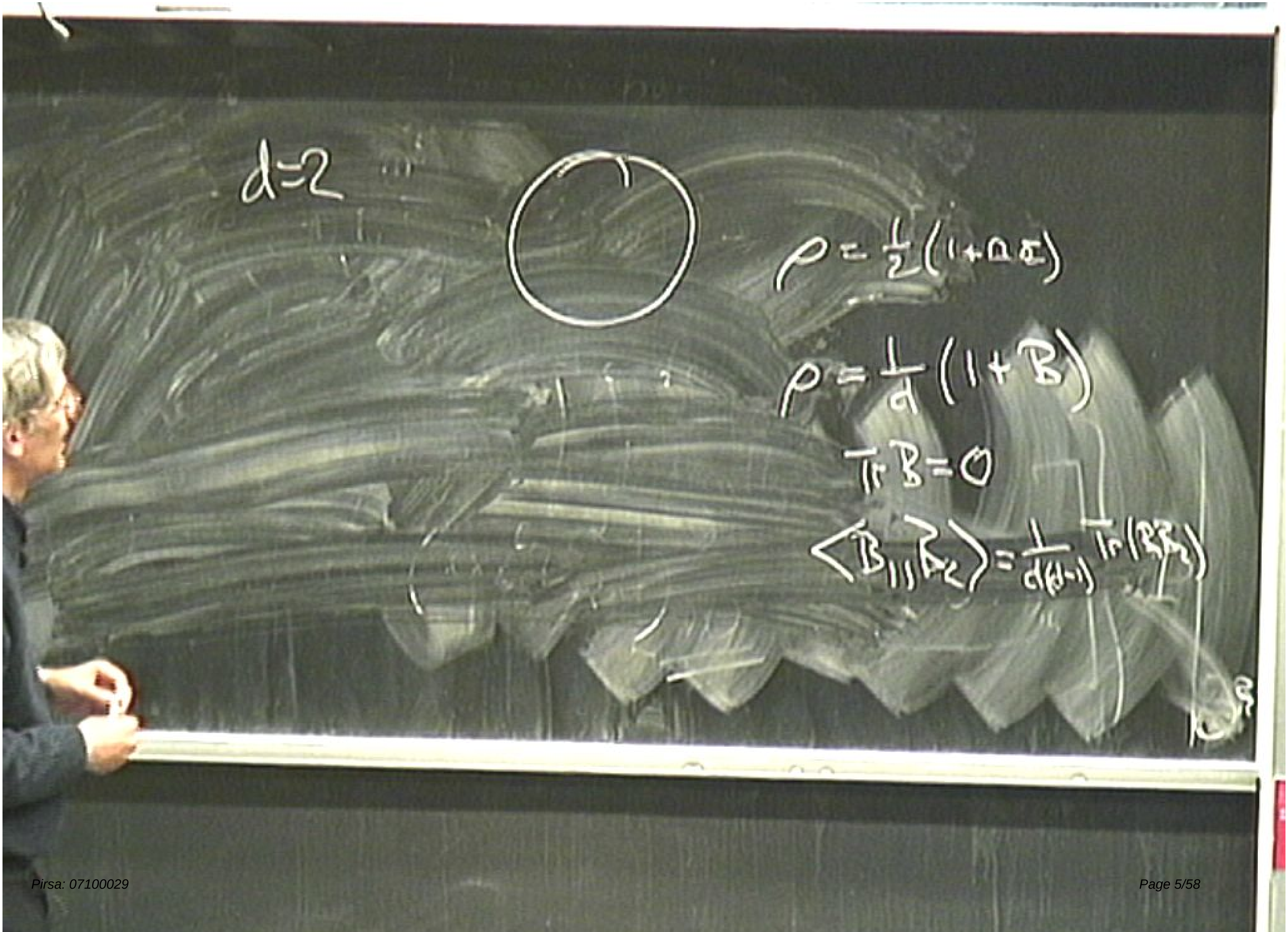
Abstract: The solution of many problems in quantum information is critically dependent on the geometry of the space of density matrices. For a Hilbert space of dimension 2 this geometry is very simple: it is simply a sphere. However for Hilbert spaces of dimension greater than 2 the geometry is much more interesting as the bounding hypersurface is both highly symmetric (it has a d^2 real parameter symmetry group, where d is the dimension) and highly convoluted. The problem of getting a better understanding of this hypersurface is difficult (it is hard even in the case of a single qutrit). It is also, we believe, both physically important and mathematically deep. In this talk we relate the problem to MUBs (mutually unbiased bases) and SIC-POVMs (symmetric informationally complete positive operator valued measures). These structures were originally introduced in connection with tomography. However, that by no means exhausts their importance. In particular their existence (non-existence???) in a given dimension is a source of significant insight into the state space geometry in that dimension. SIC-POVMs are especially important in this regard as they provide a a natural set of coordinates for state space. In this talk we give an overview of the problem. We then go on to describe some recent results obtained in collaboration with Chris Fuchs and Hoan Dang (also see recent work by Wootters and Sussman). In particular we describe the connection with minimum uncertainty states. These states, besides being interesting in themselves (they are a kind of discrete analogue of coherent states with important cryptographic applications), suggest a potentially fruitful line of attack on the still outstanding SIC existence problem.




$d=2$
$\rho = \frac{1}{2}(1 + a\cdot\sigma)$


d=2
ρ = 1/2 (1 + n·σ)
ρ = 1/d (1 + B)
Tr B = 0
⟨B_1, B_2⟩ = 1/(d(d-1)) Tr(B_1 B_2)



$d=2$
$\rho = \frac{1}{2}(1 + \vec{n}\cdot\vec{\sigma})$
$\rho = \frac{1}{d}(1 + B)$
$\mathrm{Tr}\, B = 0$
$\langle B_1, B_2 \rangle = \frac{1}{d(d-1)} \mathrm{Tr}(B_1 B_2)$

$d=2$

$$\rho = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma})$$

$$\rho = \frac{1}{d}(1 + B)$$

$$\mathrm{Tr}\, B = 0$$

$$\langle B_1, B_2 \rangle = \frac{1}{d(d-1)} \mathrm{Tr}(B_1 B_2)$$

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$d=2$

$$\rho = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma})$$

$\frac{1}{d+1}$

$$\rho = \frac{1}{d}(1 + B)$$

$$\mathrm{Tr}\, B = 0$$

$$\langle B_1, B_2 \rangle = \frac{1}{d(d-1)} \mathrm{Tr}(B_1 B_2)$$

$d=2$

$$\rho = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma})$$

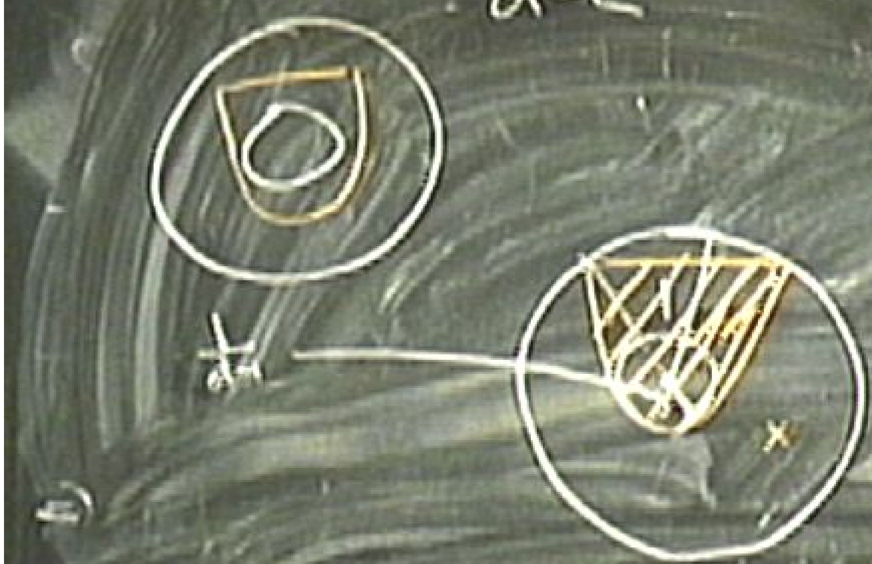
$$\rho = \frac{1}{d}(1 + B)$$

$$\mathrm{Tr}\, B = 0$$

$$\langle B, B\rangle = 1$$

$$\langle B_1, B_2\rangle = \frac{1}{d(d-1)}\mathrm{Tr}(B_1 B_2)$$

$d=2$



$$\rho = \frac{1}{2}(1+\vec{a}\cdot\vec{\sigma})$$

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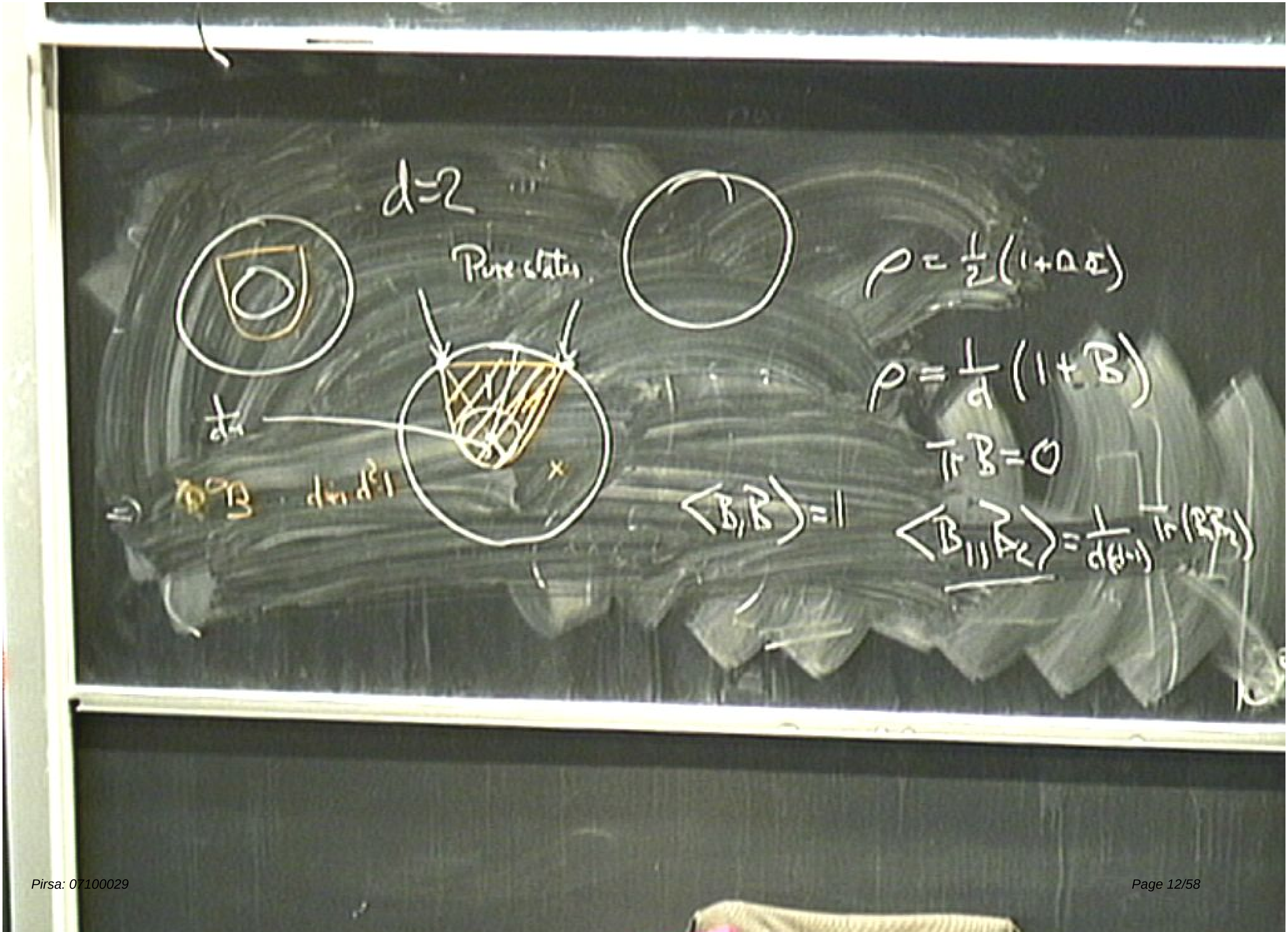
d=2
Pure states
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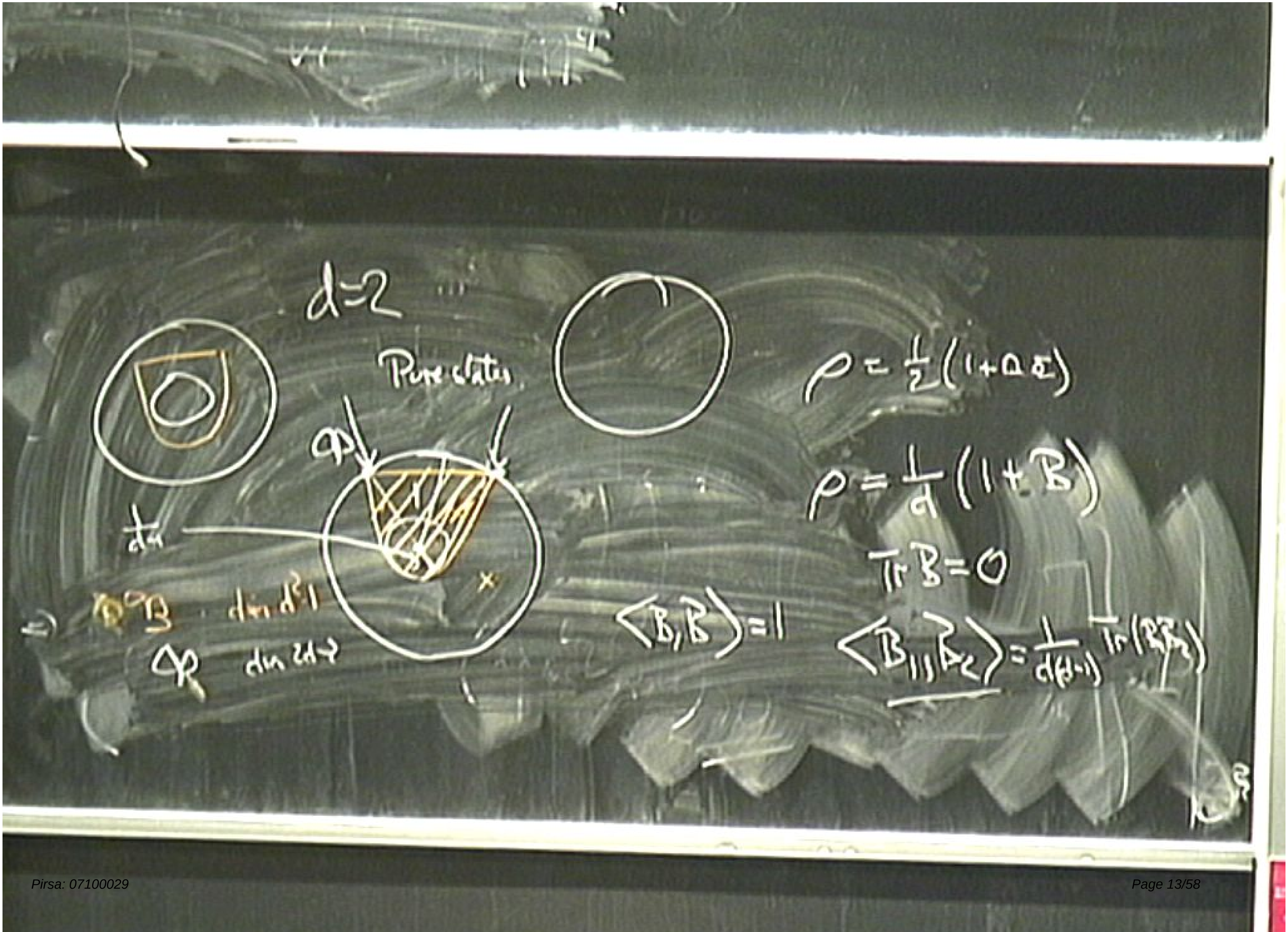

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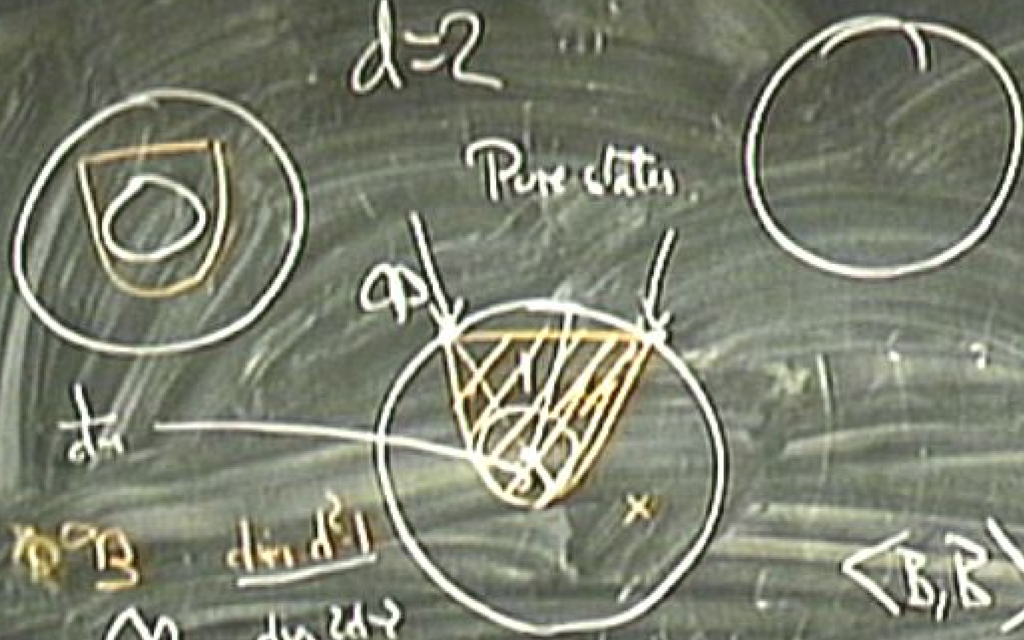
Tr B = 0
⟨B,B⟩=1
⟨B1,B2⟩ = 1/(d(d-1)) Tr(B1B2)
dim d²-1
dim 2d-2

$d=2$



$\Rightarrow\ \mathcal{R}^{\circ}\mathcal{B}\quad \dim d^2-1$

$\mathcal{R}\quad \dim 2d-2$

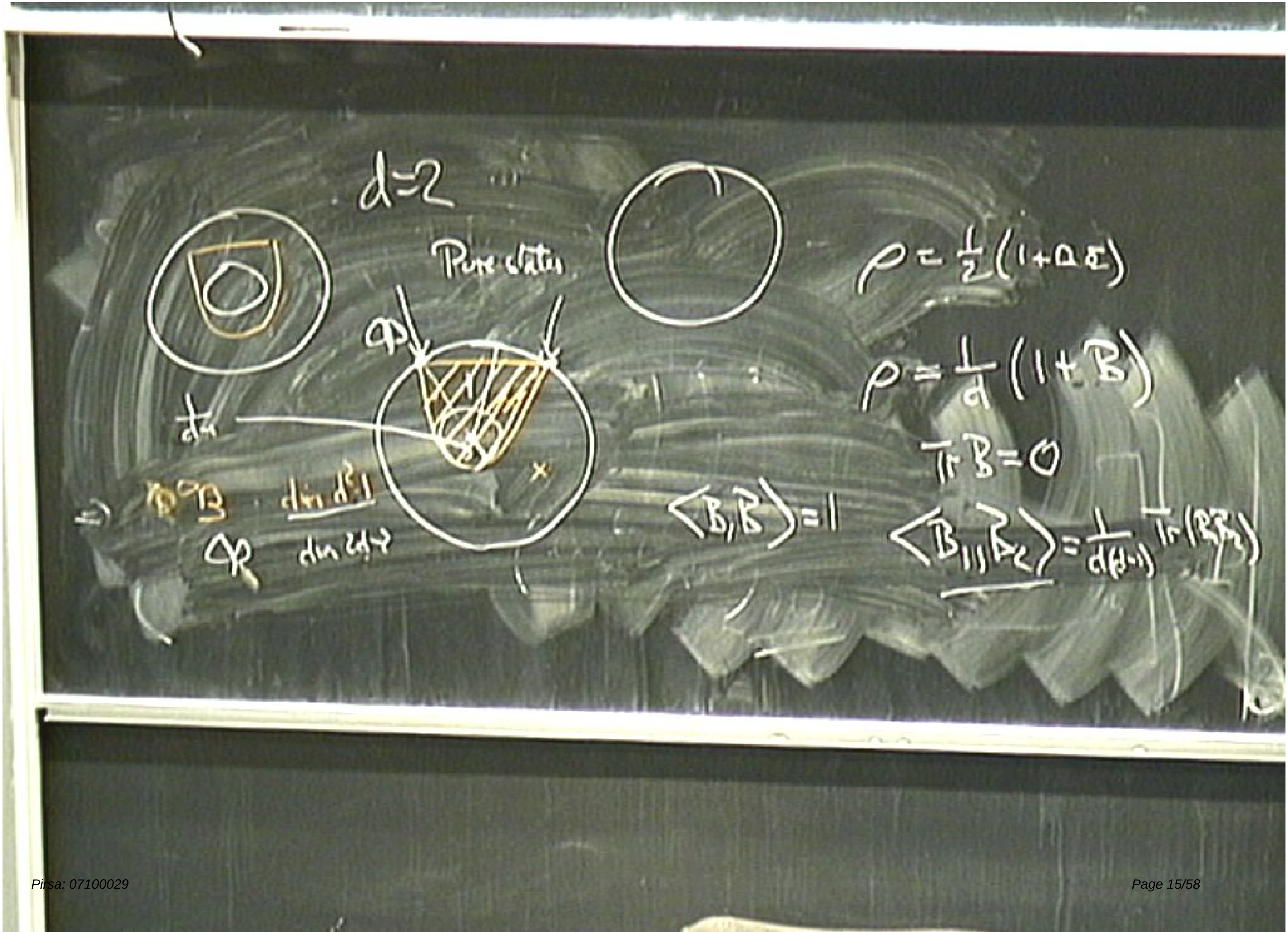
$$\rho = \frac{1}{2}(1+\vec{a}\cdot\vec{\sigma})$$

$$\rho = \frac{1}{d}(1+B)$$

$$\mathrm{Tr}\,B = 0$$

$$\langle B, B\rangle = 1$$

$$\langle B_1, B_2\rangle = \frac{1}{d(d-1)}\mathrm{Tr}(B_1 B_2)$$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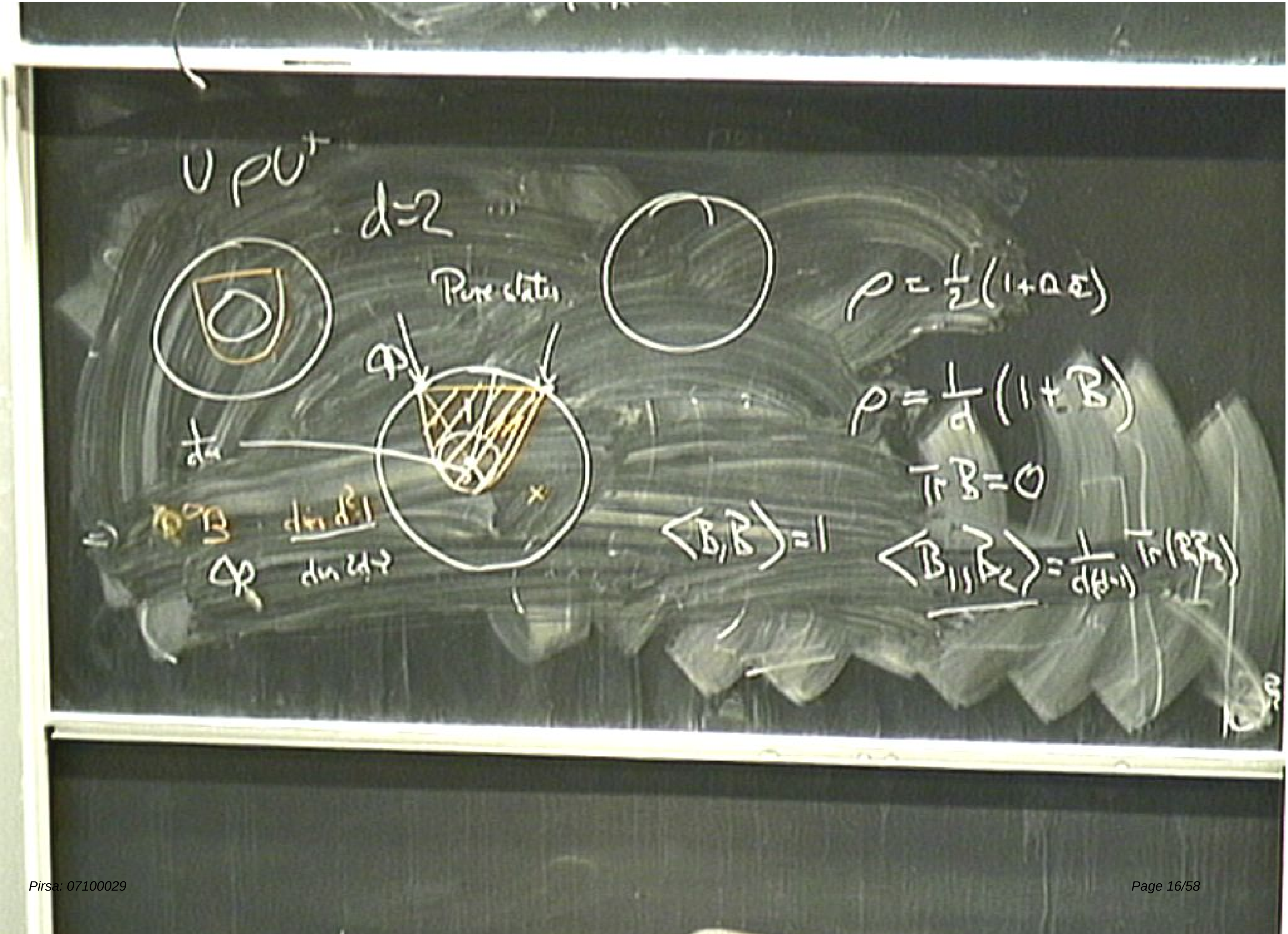
dim 2d-2


U ρ U†
d=2
Pure states
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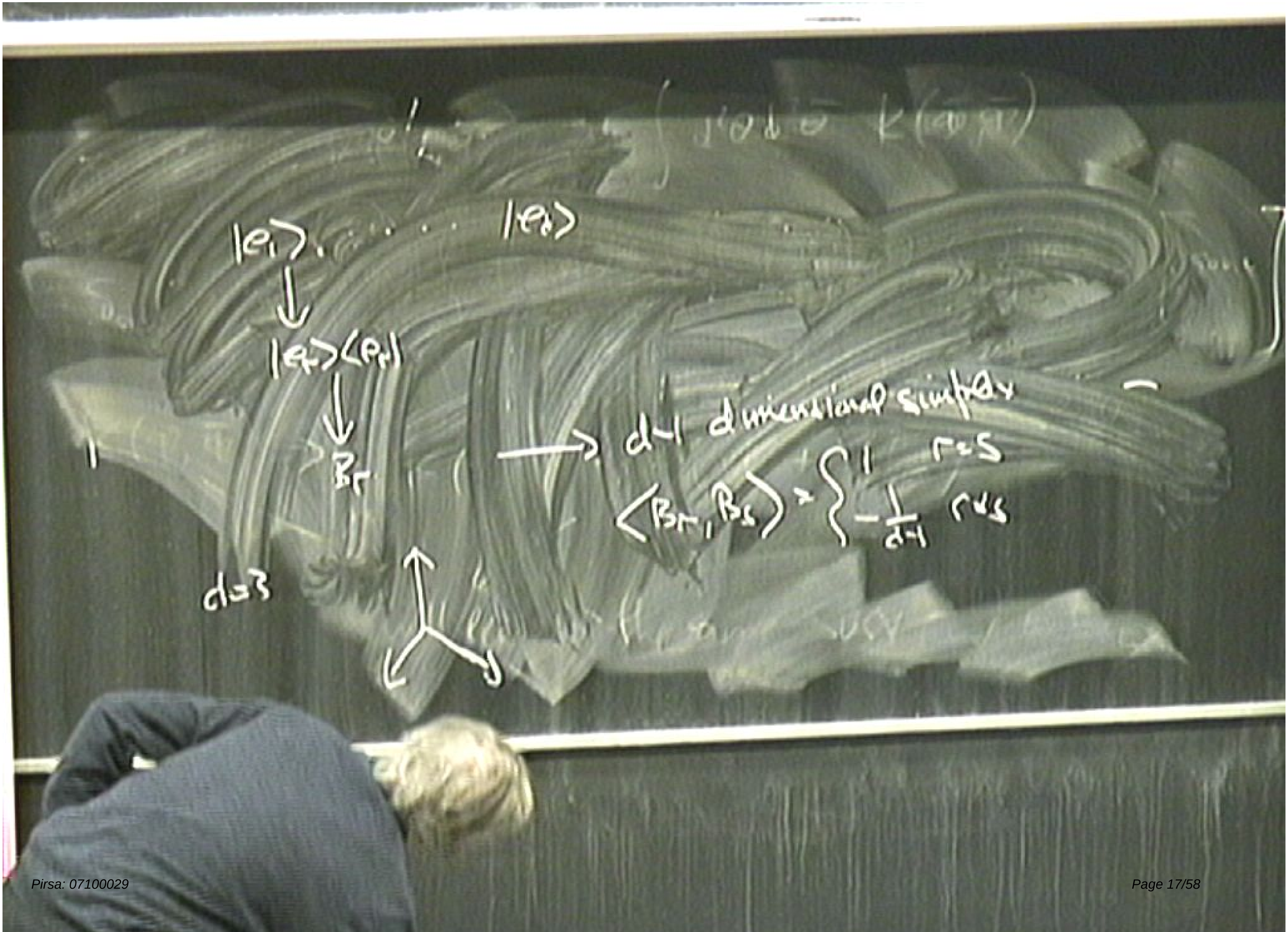
ρ = 1/d (1 + B)
Tr B = 0
⟨B, B⟩ = 1
⟨B_1, B_2⟩ = 1/(d(d-1)) Tr(B_1 B_2)


|e1⟩ . . . . |ed⟩
|er⟩⟨er|
Br
d-1 dimensional simplex
⟨Br, Bs⟩ = { 1 r=s ; -1/(d-1) r≠s
d=3

$|e_1\rangle, \ldots\ldots |e_d\rangle$

$\downarrow$

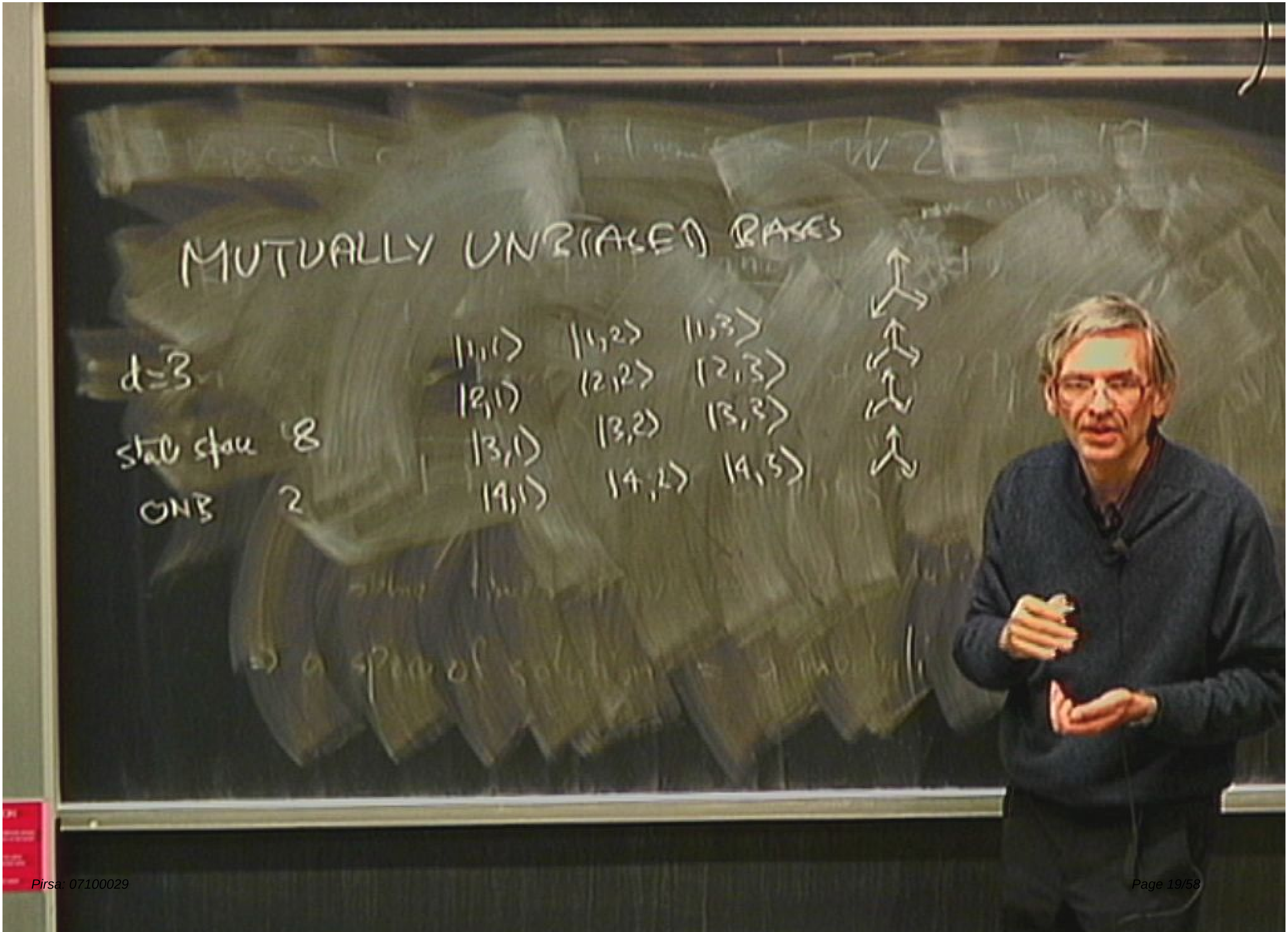
$|e_r\rangle\langle e_r|$

$\downarrow$

$B_r \longrightarrow$ d−1 dimensional simplex

$$\langle B_r, B_s\rangle = \begin{cases} 1 & r=s \\ -\frac{1}{d-1} & r\neq s \end{cases}$$

d=3


MUTUALLY UNBIASED BASES
d=3
stat space 8
ONB 2
|1,1⟩ |1,2⟩ |1,3⟩
|2,1⟩ |2,2⟩ |2,3⟩
|3,1⟩ |3,2⟩ |3,3⟩
|4,1⟩ |4,2⟩ |4,3⟩

MUTUALLY UNBIASED BASES

$d=3$

state space 8

ONB 2

$\|1,1\rangle$	$\|1,2\rangle$	$\|1,3\rangle$
$\|2,1\rangle$	$\|2,2\rangle$	$\|2,3\rangle$
$\|3,1\rangle$	$\|3,2\rangle$	$\|3,3\rangle$
$\|4,1\rangle$	$\|4,2\rangle$	$\|4,3\rangle$

$|e_1\rangle, \ldots, |e_{d^2}\rangle$

$\downarrow$

$|e_r\rangle\langle e_r|$

$\downarrow$

B_r

$\longrightarrow$ d^2-1 dimensional simplex

$$\langle B_r, B_s\rangle = \begin{cases} 1 & r=s \\ -\frac{1}{d^2-1} & r\neq s \end{cases}$$

$|e_1\rangle, \ldots, |e_{d}\rangle$

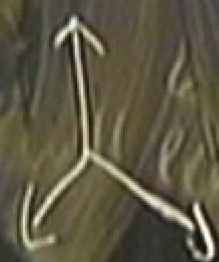
$|e_r\rangle\langle e_r|$

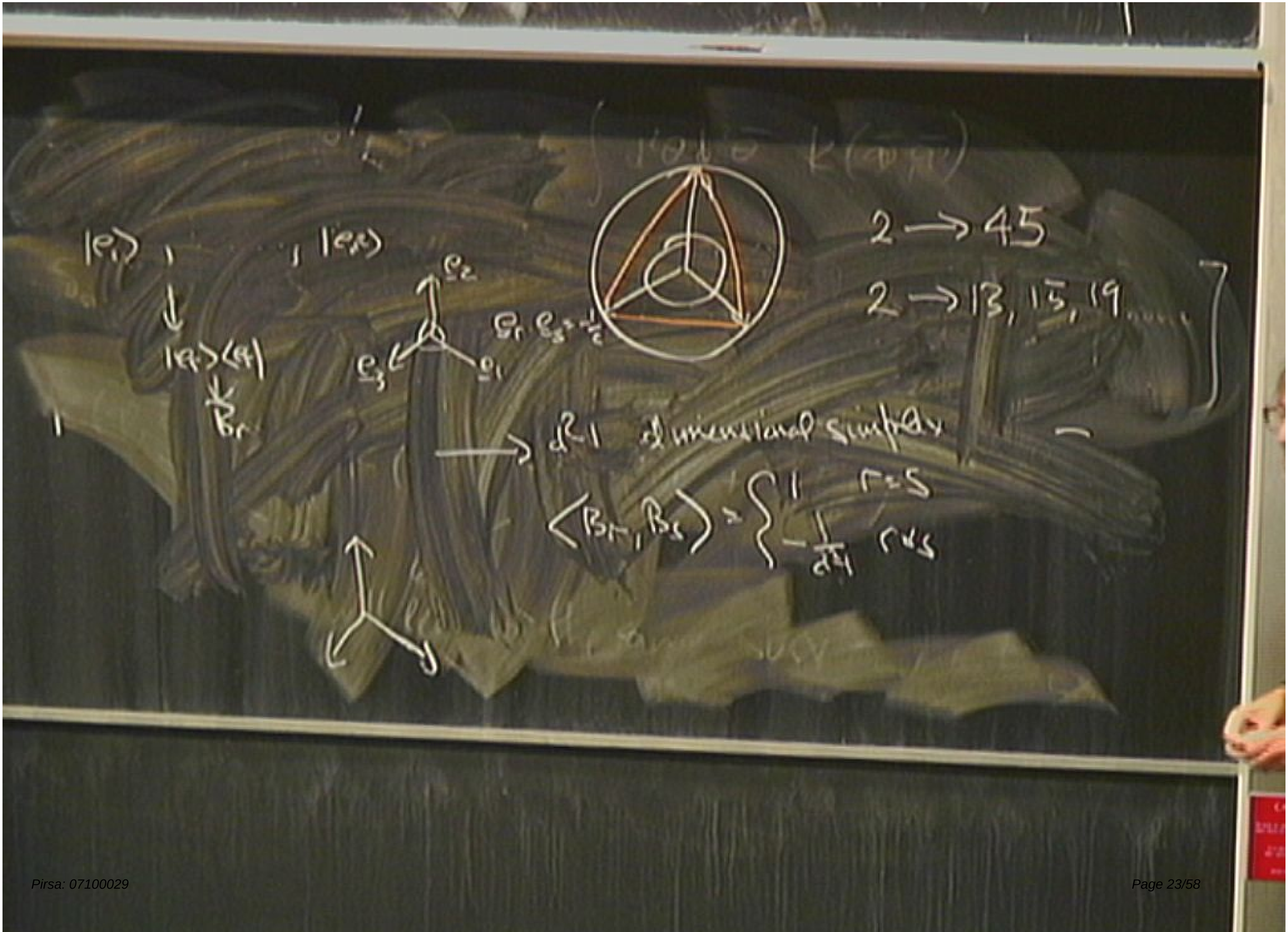
B_r

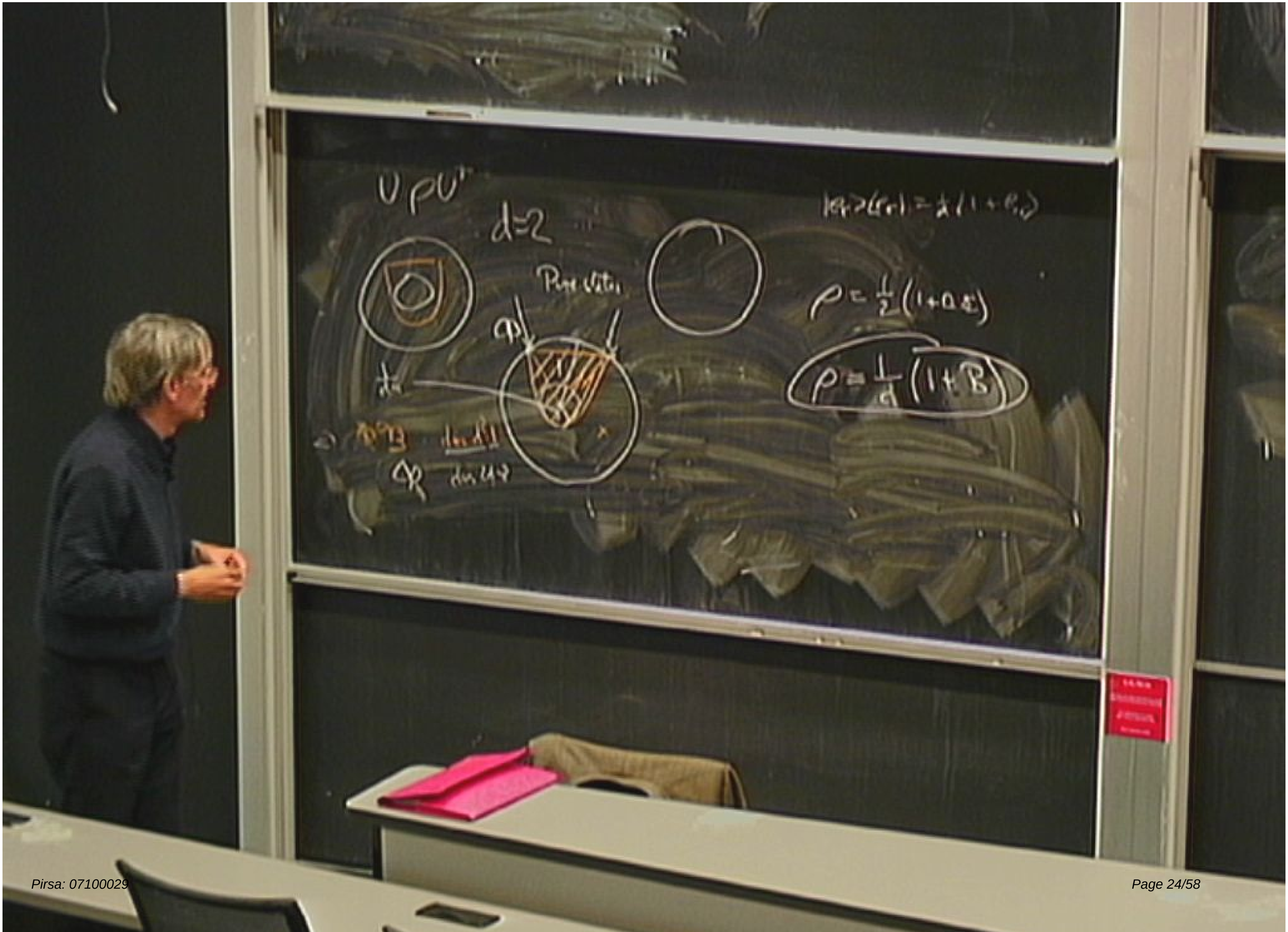
$e_r \cdot e_s = -\frac{1}{2}$

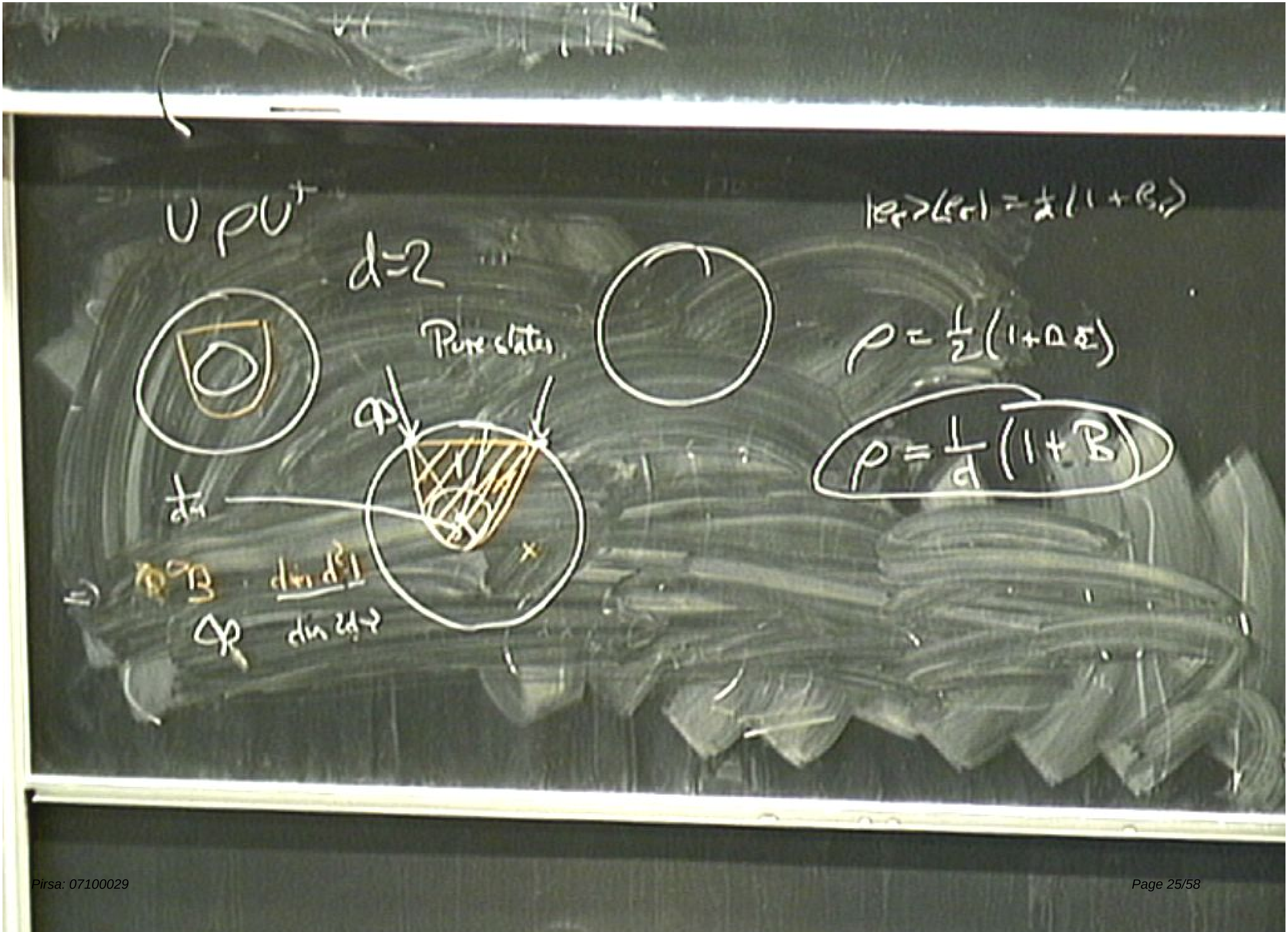
$\longrightarrow d^2-1$ dimensional simplex

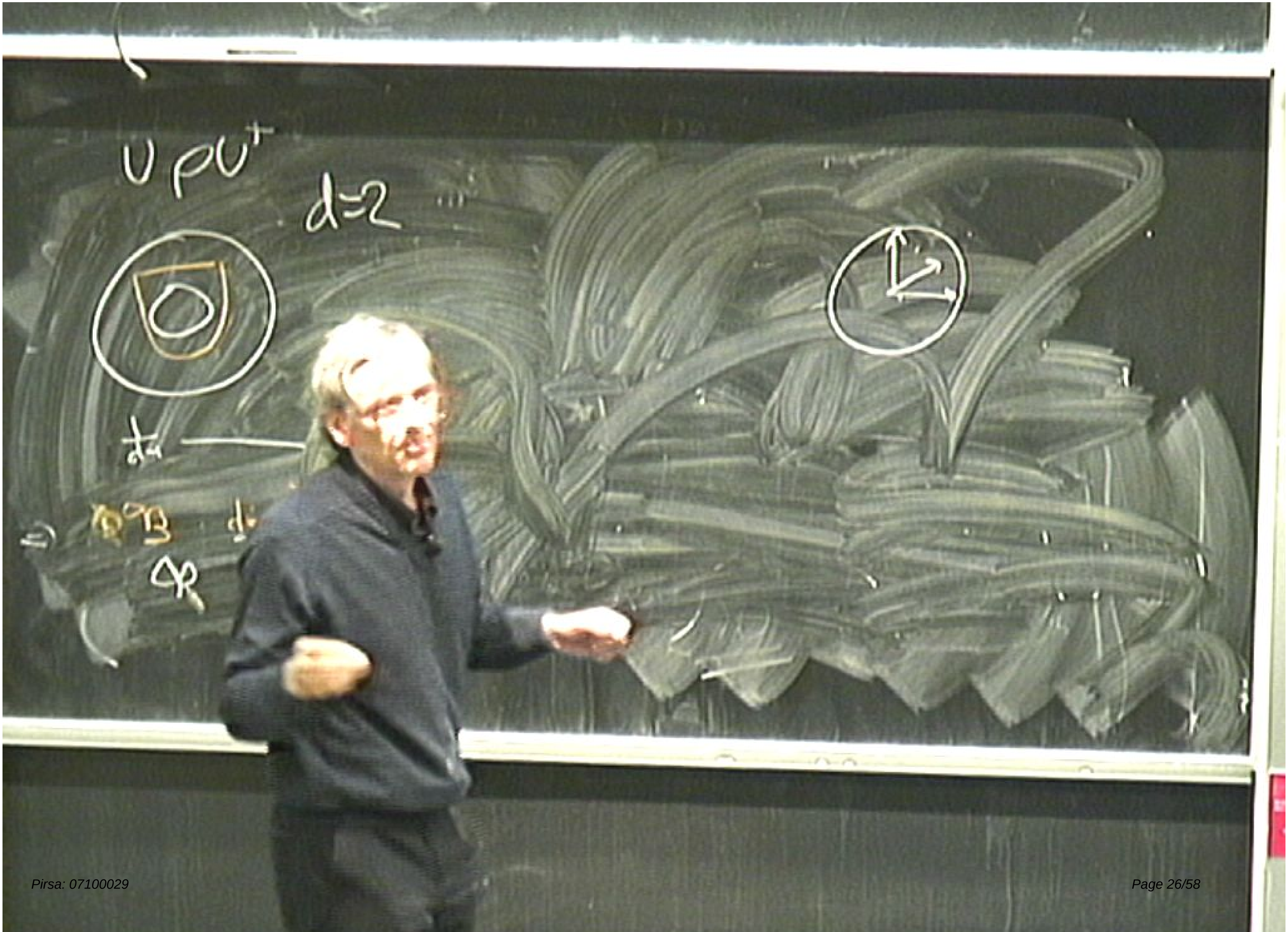
$$\langle B_r, B_s\rangle = \begin{cases} 1 & r=s \\ -\frac{1}{d^2-1} & r\neq s \end{cases}$$




|e1⟩ , |e2⟩
|e1⟩⟨e1|
Br
e2
e3
e1
e1 · e3 = -1/2
2 → 45
2 → 13, 15, 19, ...
→ d²-1 dimensional simplex
⟨Br , Bs⟩ = { 1 r=s ; -1/(d+1) r≠s


d=2


U ρ U†
d=2
Pure states
ρ = 1/2 (1 + a·σ)
ρ = 1/d (1 + B)


UρU+
d=2

$U\rho U^{\dagger}$

$d=2$

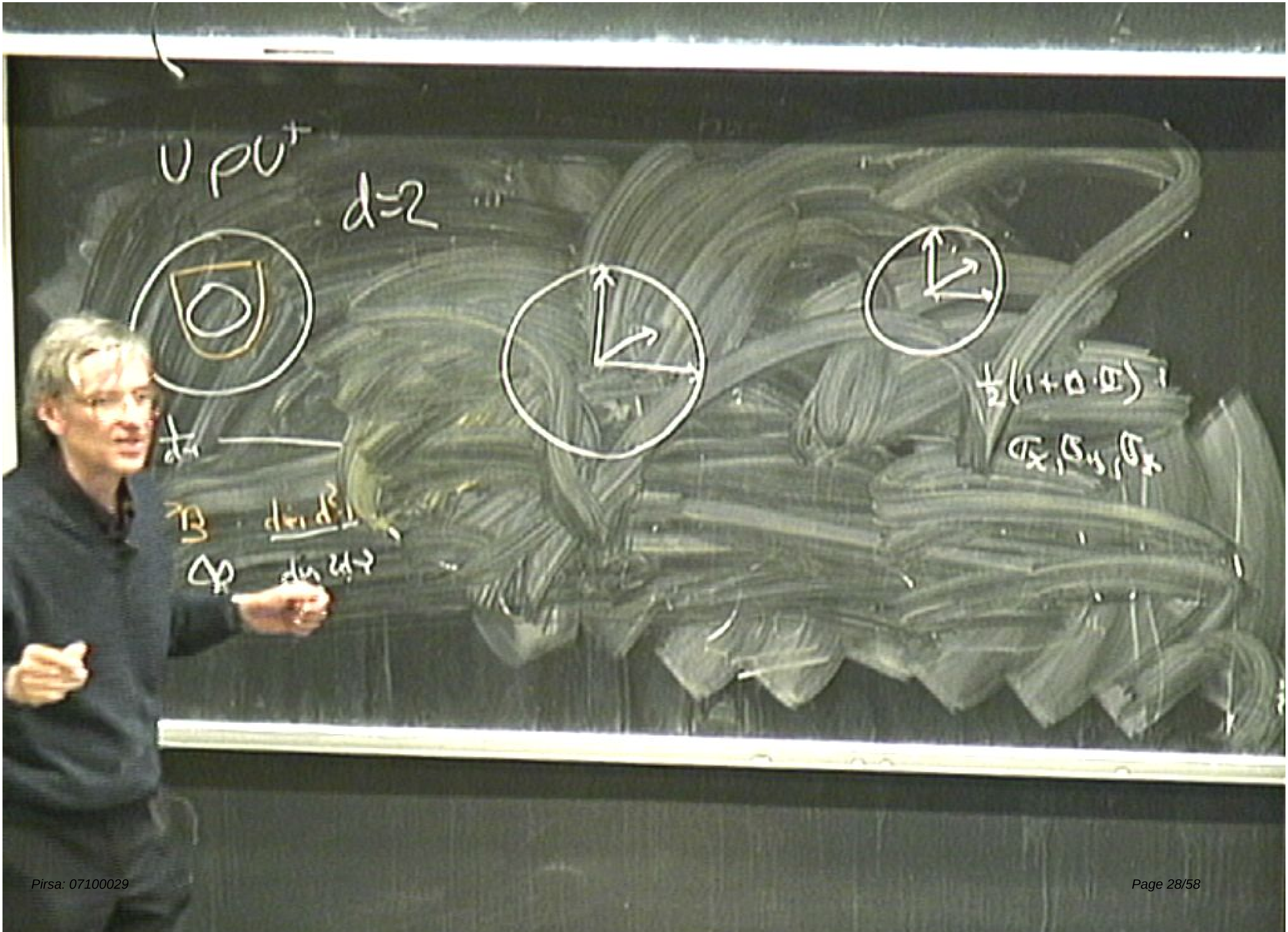
$\frac{1}{2}(1+\vec{n}\cdot\vec{\sigma})$

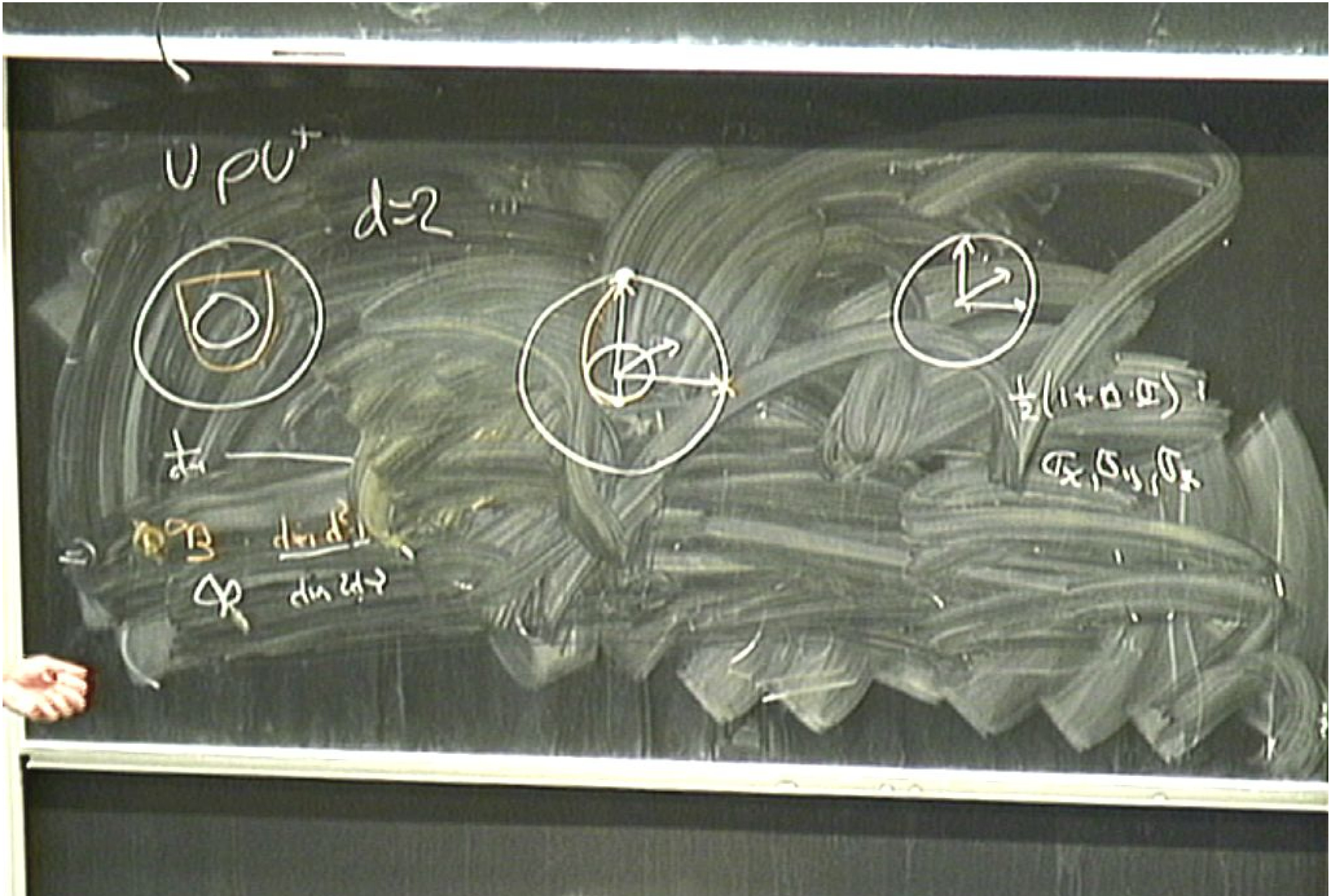
$\sigma_x, \sigma_y, \sigma_z$

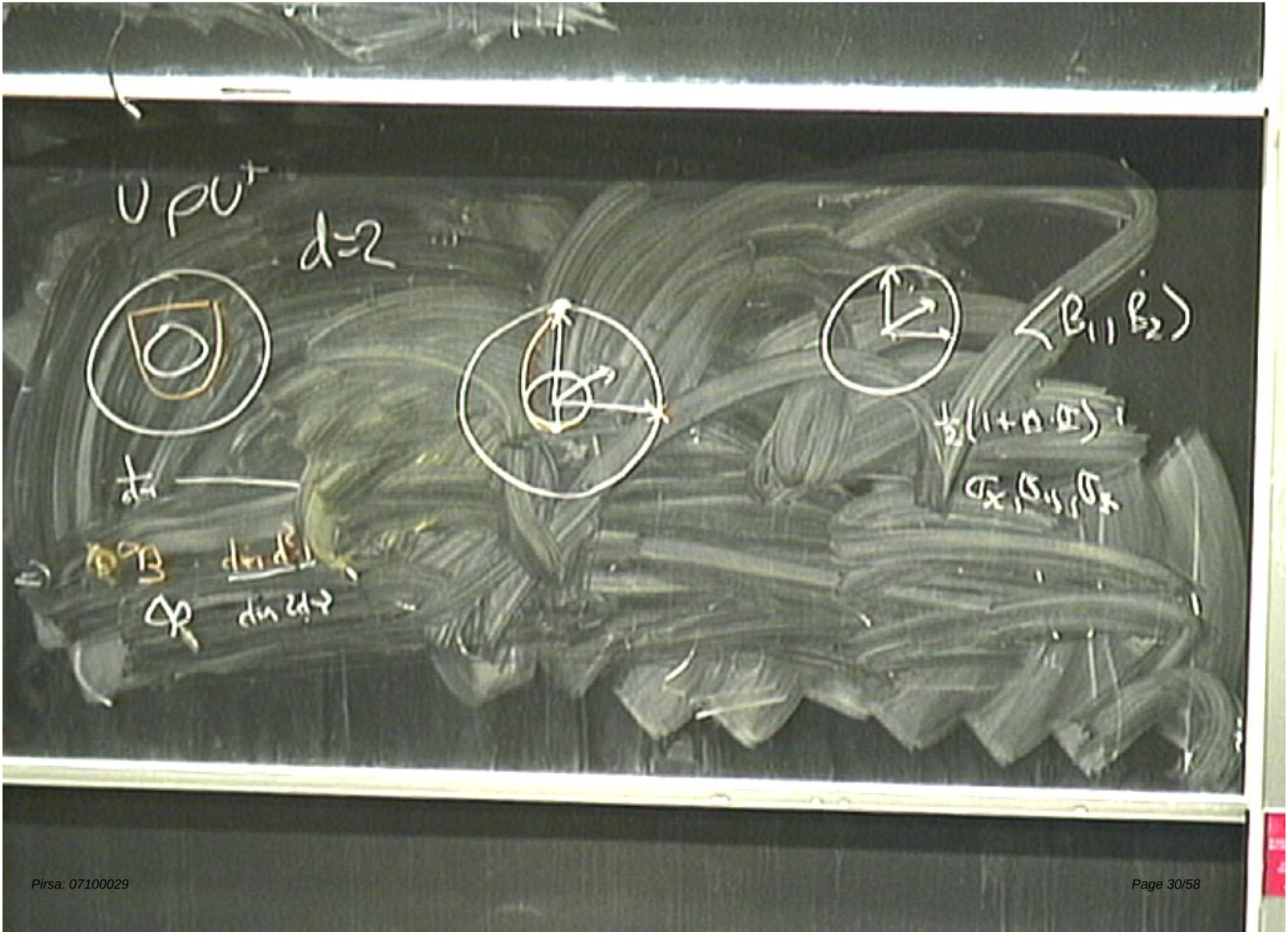
$\frac{1}{d_A}$ ———

$\Rightarrow$ ρ^{AB} $\dim d^2-1$

ρ_R $\dim 2d-2$


d=2
σx, σy, σz


U ρ U†
d=2
½(1+a·σ)
σx, σy, σz


U ρ U†
d=2
⟨B₁, B₂⟩
½(1+n·σ)
σx, σy, σz
dim d²-1
dim 2d-2

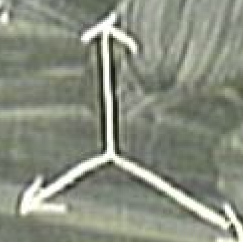
$U \rho U^\dagger$

$d=2$

$\langle B_1, B_2 \rangle$

$\frac{1}{2}(1 + n \cdot \sigma)$

$\sigma_x, \sigma_y, \sigma_z$



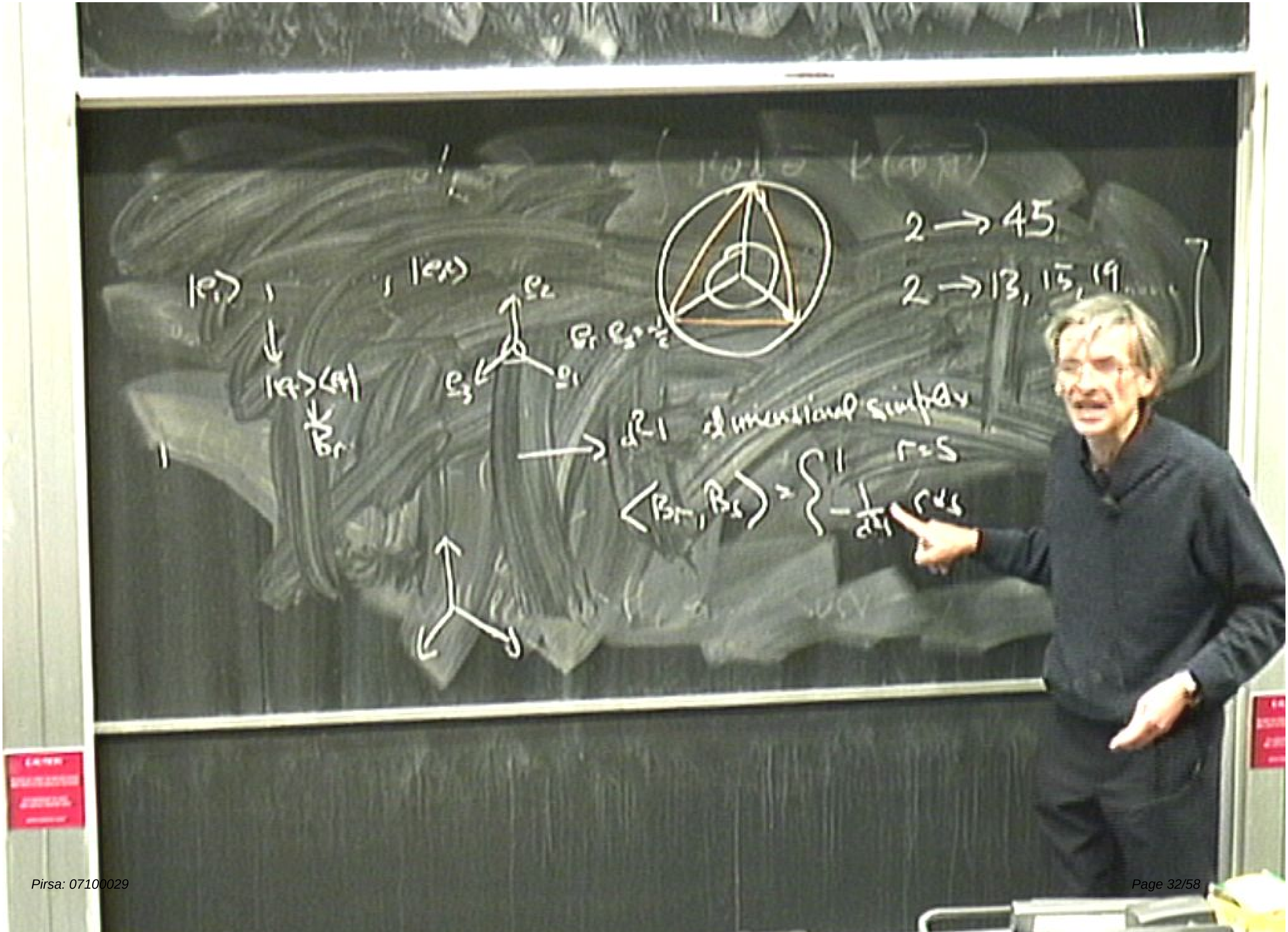
$d=4$

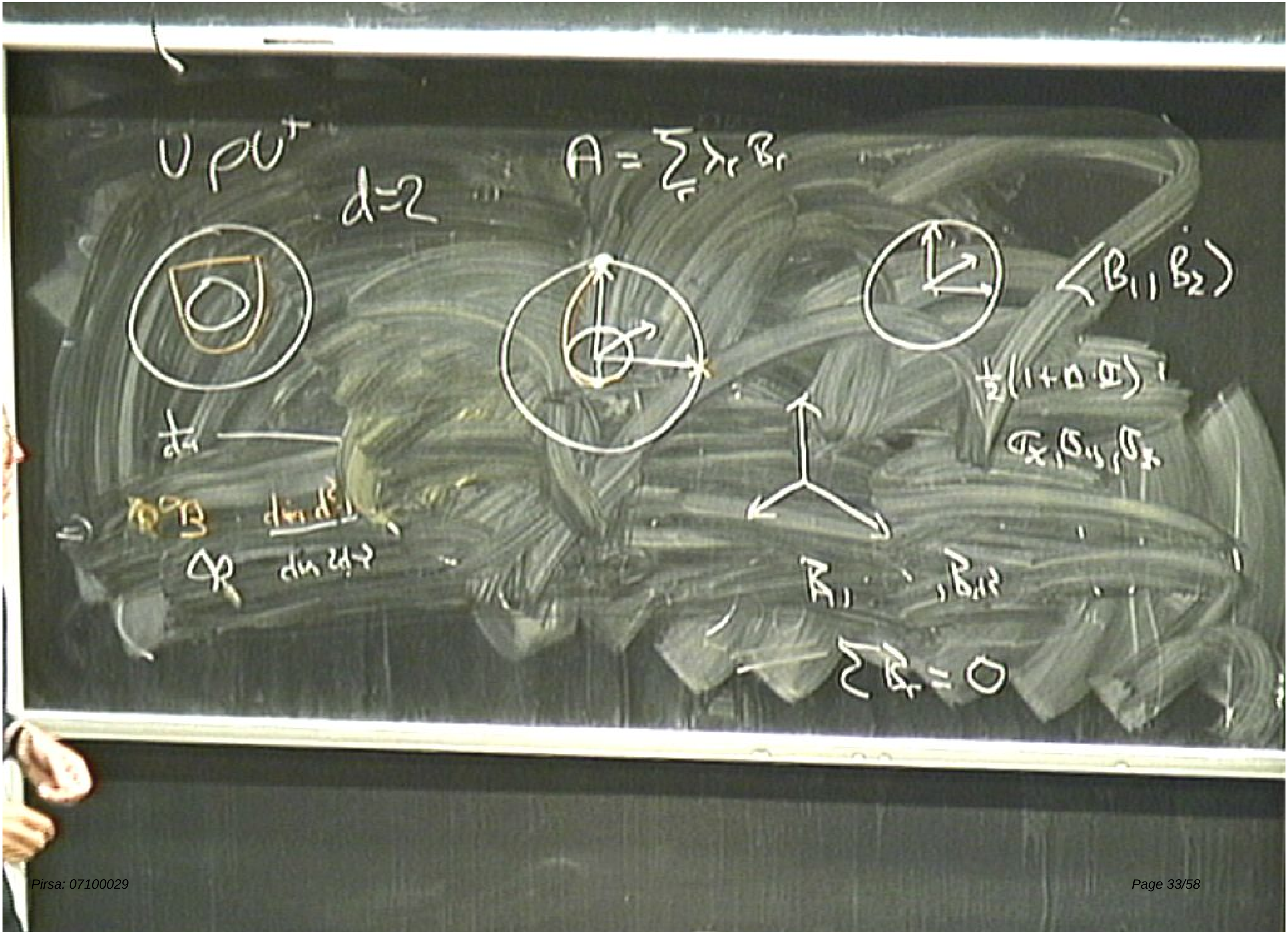
$\otimes B \quad \dim d^2-1$

$\mathbb{C}P \quad \dim 2d-2$

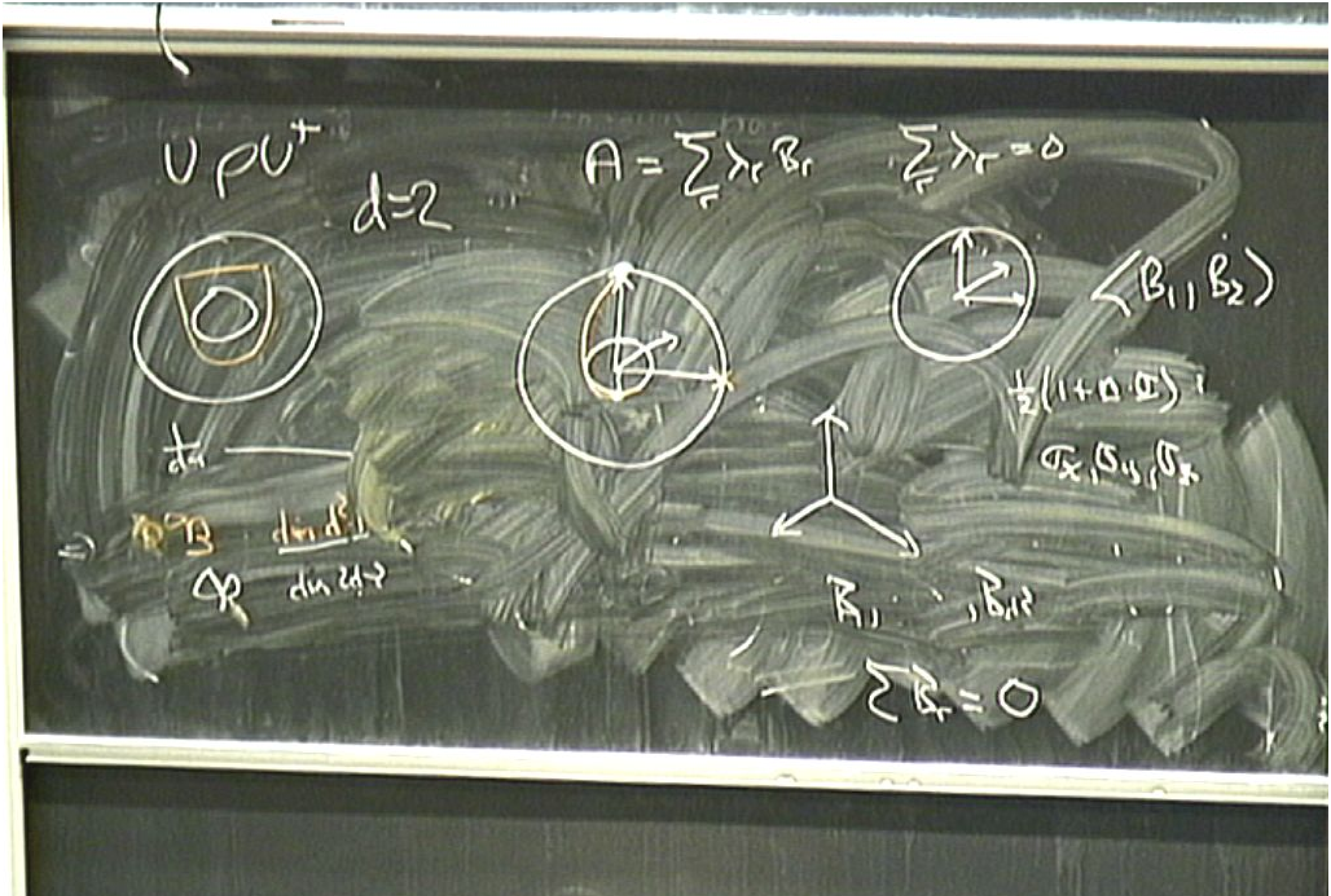
$B_1, \ldots, B_{d^2}$

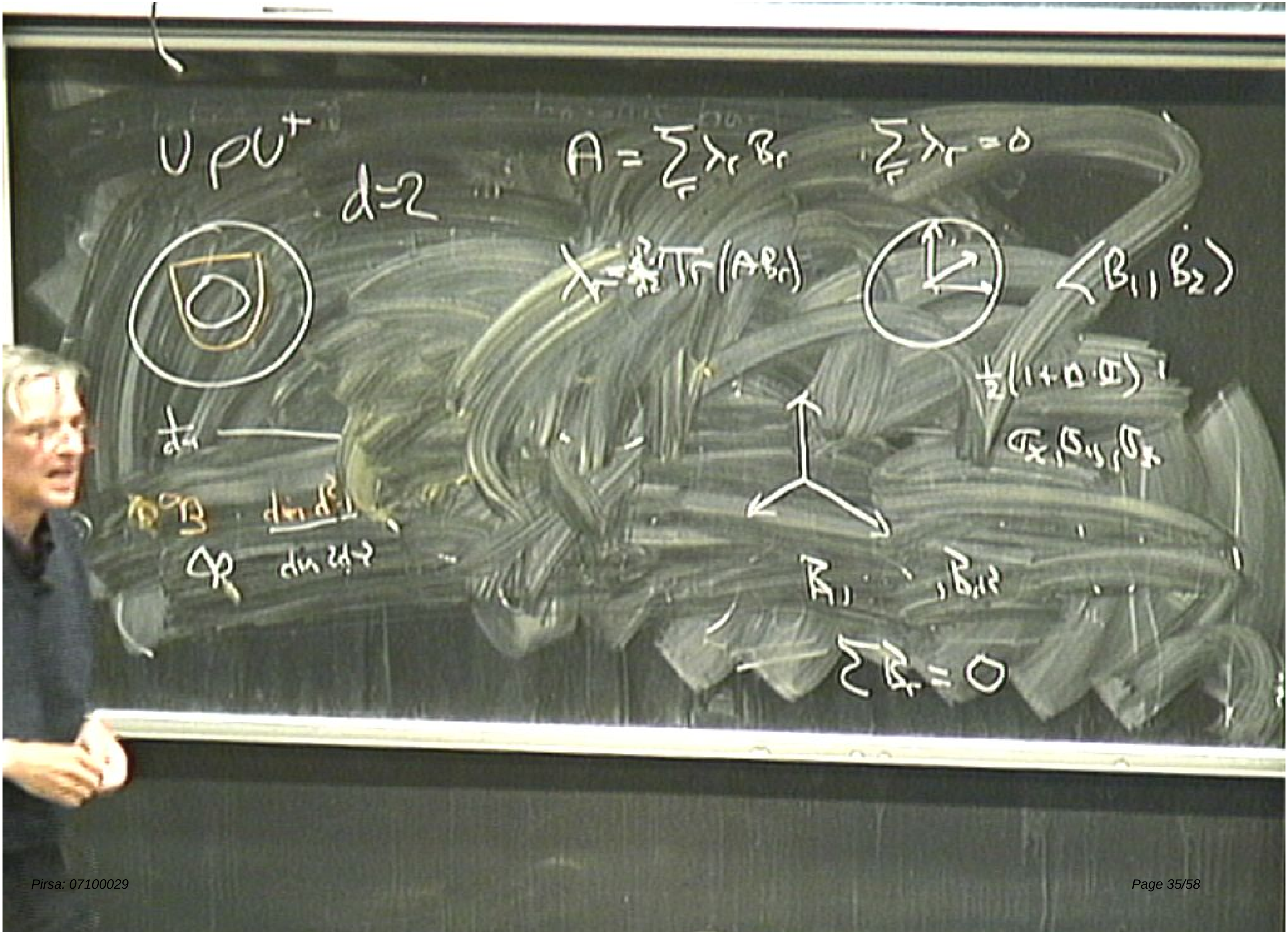
$\sum B_i = 0$


2 → 45
2 → 13, 15, 19
dimensional simplex
r=s



U ρ U†
d=2
A = Σ λ B
(B1, B2)
½(1 + n·σ)
σx, σy, σz
ΣB = 0


$U\rho U^{\dagger}$
$d=2$
$A = \sum_r \lambda_r B_r$
$\sum_r \lambda_r = 0$
(B_1, B_2)
$\sigma_x, \sigma_y, \sigma_z$
$\sum B_r = 0$



U ρ U†
d=2
A = Σ λr Br
Σ λr = 0
⟨B1, B2⟩
σx, σy, σz
Σ Br = 0

$U \rho U^\dagger$

$d=2$

$A = \sum_r \lambda_r B_r \quad \sum_r \lambda_r = 0$

$\lambda_r = \frac{?}{?} Tr(A B_r)$

$\langle B_1, B_2 \rangle$

$\langle \psi, \phi \rangle = 0$

$\frac{1}{2}(1 + n \cdot \sigma)$

$\frac{1}{d+1}$

$P = |\psi\rangle\langle\psi| \quad P' = |\phi\rangle\langle\phi|$

$\sigma_x, \sigma_y, \sigma_z$

$\Rightarrow \; \mathbb{R}^{?} B \quad \frac{d_n d^2}{d_n 2d+?}$

$\mathbb{CP}$

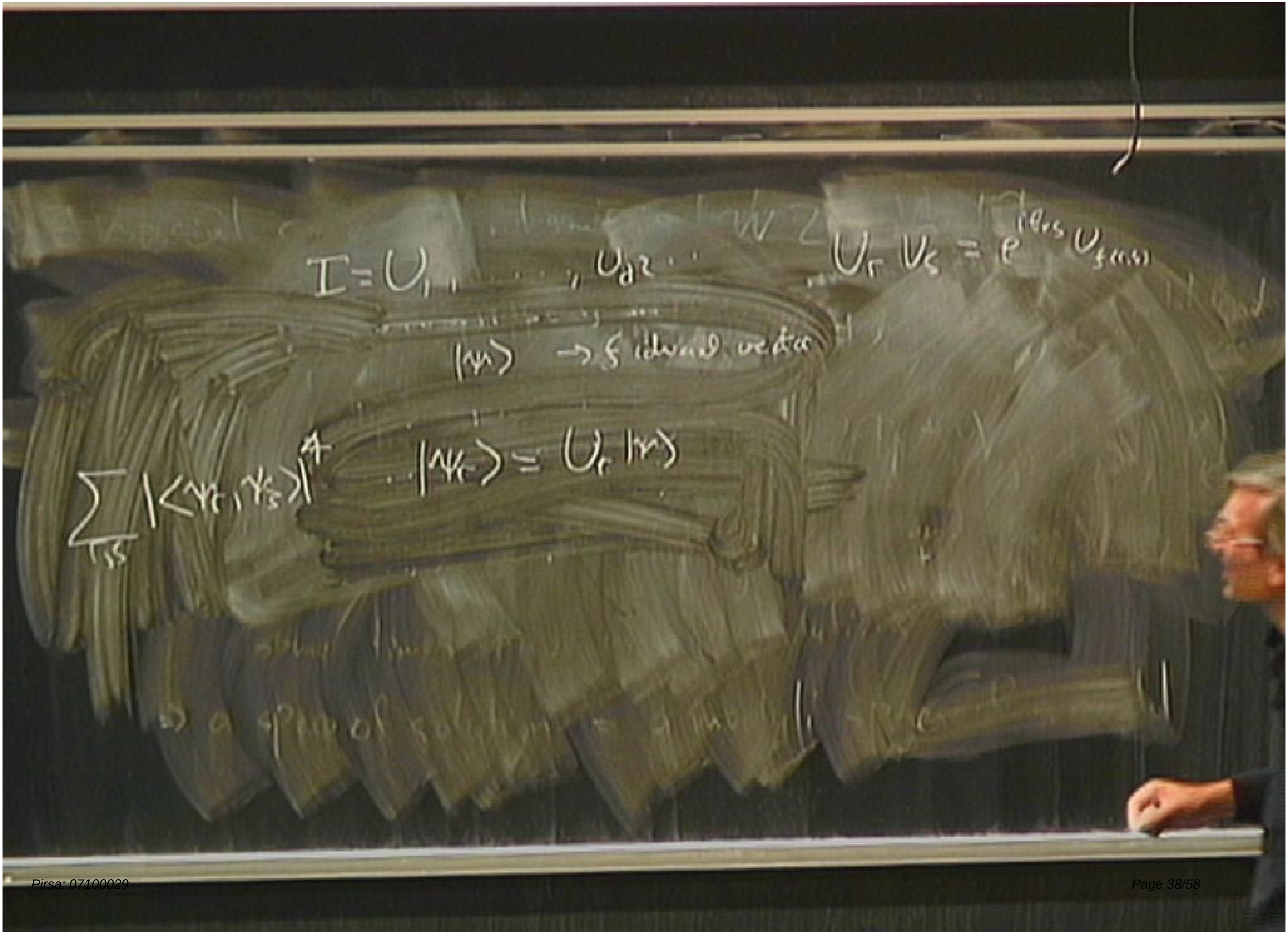
$B \qquad B'$

$\langle B, B' \rangle = -\frac{1}{d+1}$

$$I = U_1, \ldots, U_{d^2} \cdots \qquad U_r U_s = e^{i\theta_{rs}} U_{f(r,s)}$$

$|\psi_0\rangle \to$ fiducial vector

$$|\psi_r\rangle = U_r |\psi_0\rangle$$


$I = U_1, \ldots, U_{d^2}$
$U_r U_s = e^{i\theta_{rs}} U_{f(r,s)}$
$|\psi\rangle \to$ fiducial vector
$|\psi_r\rangle = U_r |\psi\rangle$
$\sum_{r,s} |\langle \psi_r, \psi_s \rangle|^4$

$I = U_1, \ldots, U_{d^2}$ $\quad U_r U_s = e^{i\theta_{rs}} U_{f(r,s)}$

$|\psi\rangle \to$ fiducial vector

$|\psi_r\rangle = U_r |\psi\rangle$

$$\sum_{r,s} |\langle \psi_r | \psi_s \rangle|^4$$

$$\sum_r |\langle \psi | U_r | \psi \rangle|^4$$

$I = U_1, \ldots, U_{d^2}$ $U_r U_s = e^{i\theta_{r,s}} U_{f(r,s)}$

$|\psi\rangle \to$ fiducial vector

Weyl-Heisenberg group: $|0\rangle, |1\rangle, \ldots, |d-1\rangle$

$Z|r\rangle = \omega^r |r\rangle$ $\omega = e^{\frac{2\pi i}{d}}$

$X|r\rangle = |r+1\rangle$

$D_{rs} = X^r Z^s$

$I = U_1, \ldots, U_{d^2} \ldots \qquad U_r U_s = e^{i\theta_{rs}} U_{f(r,s)}$

$|\psi\rangle \rightarrow$ fiducial vector

Weyl–Heisenberg group

$|0\rangle, |1\rangle, \ldots, |d-1\rangle$

$p = (p_1, p_2)$

$\omega = e^{\frac{2\pi i}{d}}$

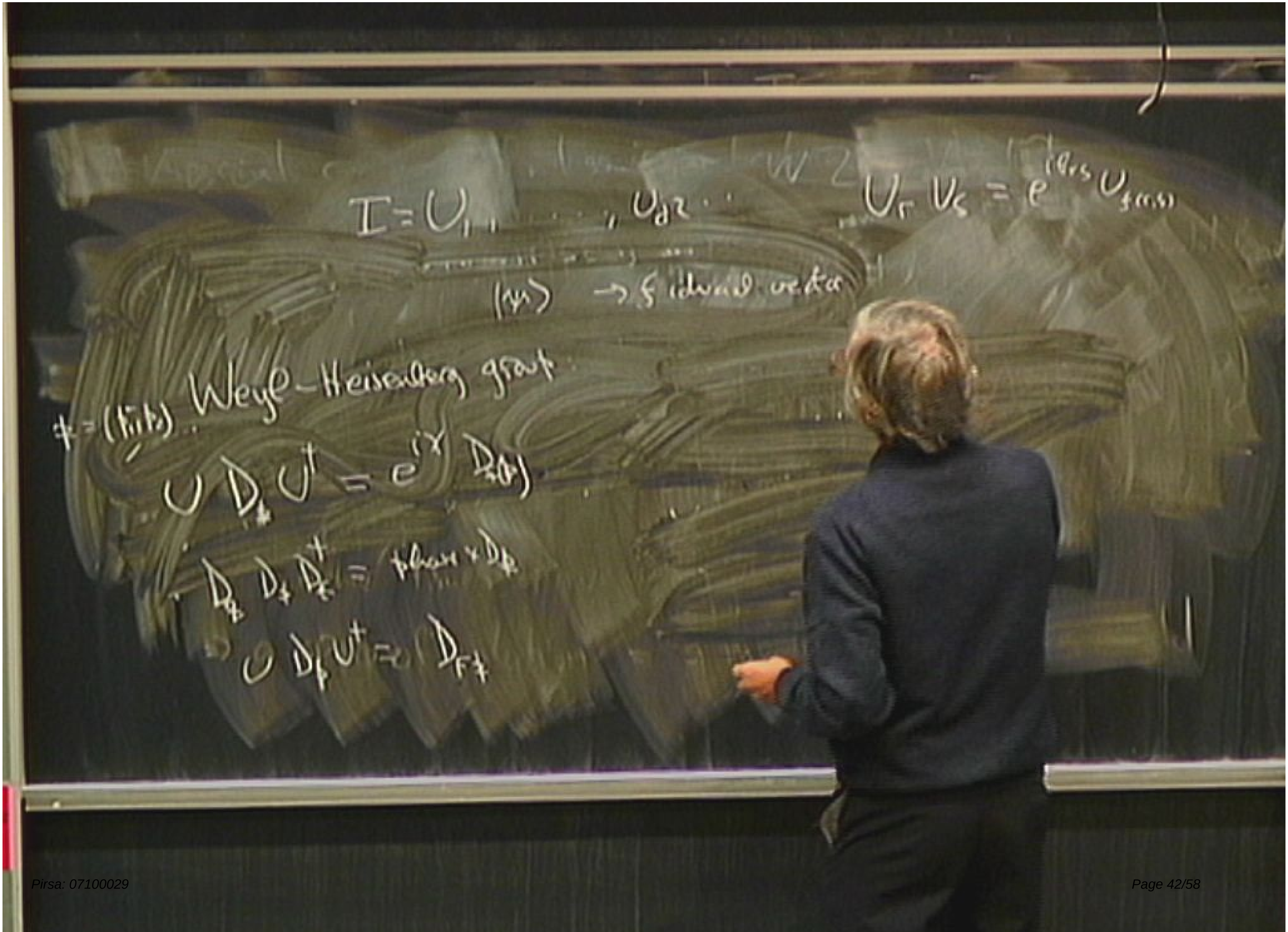
$Z|r\rangle = \omega^r |r\rangle$

$U D_p U^\dagger = e^{i\chi} D_{f(p)}$

$X|r\rangle = |r+1\rangle$

$D_p \quad D_p D_q^\dagger = \text{phase} \times D_{p}$

$D_{rs} = X^r Z^s$


$I = U_1, \ldots, U_{d^2}$
$U_r V_s = e^{i\theta_{r,s}} U_{f(r,s)}$
$|\psi\rangle$ → fiducial vector
Weyl-Heisenberg group
$U D_p U^\dagger = e^{i\chi} D_{F(p)}$
$D_q D_p D_q^\dagger$ = phase × D_p
$U D_p U^\dagger = D_{Fp}$

Weyl–Heisenberg group

$\vec{p} = (p_1, p_2)$

$$U D_{\vec p} U^\dagger = e^{i\chi} D_{F(\vec p)}$$

$$D_{\vec q} D_{\vec p} D_{\vec q}^\dagger = \text{phase} \times D_{\vec p}$$

$$\Rightarrow U D_{\vec p} U^\dagger = D_{F\vec p}$$

$I = U_1, \ldots, U_{d^2}$ $U_r U_s = e^{i\theta_{r,s}} U_{f(r,s)}$

$F p = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$\alpha, \beta, \gamma, \delta \in \mathbb{Z}_d$

Weyl–Heisenberg group:

$p = (p_1, p_2)$

$\det F = 1$

$U D_p U^\dagger = e^{i\chi} D_{F(p)}$

$D_p D_q D_p^\dagger = \text{phase} \times D_q$

$U D_p U^\dagger = D_{Fp}$

$I = U_1, \ldots, U_{d^2}$ $\quad U_r U_s = e^{i\theta_{r,s}} U_{f(r,s)}$

Weyl–Heisenberg group

$p = (p_1, p_2)$

$U D_p U^\dagger = e^{i\chi} D_{f(p)}$

$D_q D_p D_q^\dagger = \text{phase} \times D_p$

$\Rightarrow U D_p U^\dagger = D_{Fp}$

$F p = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$\alpha, \beta, \gamma, \delta \in \mathbb{Z}_d$

$\det F = 1 \longrightarrow$ symplectic unitary

anti-unitary $\det F = -1$

$U_{F_1} U_{F_2} = U_{F_1 F_2}$

$I = U_1, \ldots, U_{d^2} \cdots$ $\quad U_r U_s = e^{i\theta_{r,s}} U_{f(r,s)}$

Weyl–Heisenberg group

$p = (p_1, p_2)$

$U D_p U^\dagger = e^{i\chi} D_{f(p)}$

$D_q D_p D_q^\dagger = \text{phase} \times D_p$

$U D_p U^\dagger = D_{Fp}$

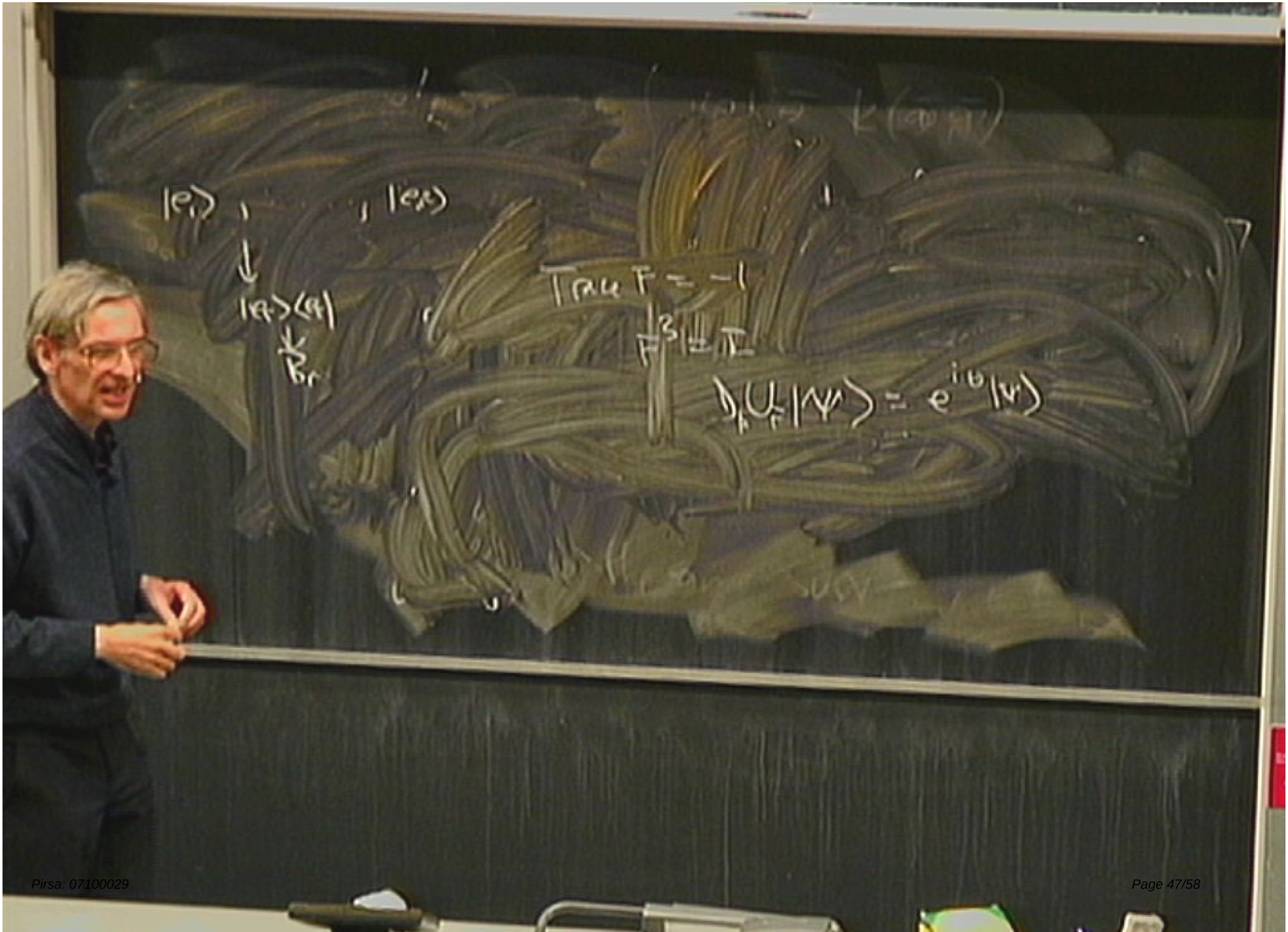
$F p = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$\alpha, \beta, \gamma, \delta \in \mathbb{Z}_d$

$\det F = 1 \longrightarrow$ symplectic unitary

anti-unitary $\quad \det F = -1$

$U_{F_1} U_{F_2} = U_{F_1 F_2}$


|e1⟩
|e2⟩
|e1⟩⟨e1|
Trace F = -1
F^3 = I
U_F|ψ⟩ = e^{iθ}|ψ⟩

SICS & MINIMUM UNCERTAINTY STATES.

dimension = prime

SICS & MINIMUM UNCERTAINTY STATES.

dimension = prime

$|0,0\rangle_1 \quad \ldots \quad |0,d-1\rangle$ Basis 0

$|1,0\rangle_1 \quad \ldots \quad |1,d-1\rangle$ 1

$|d,0\rangle \quad \ldots \quad |d,d-1\rangle$ d

SICS & MINIMUM UNCERTAINTY STATES.

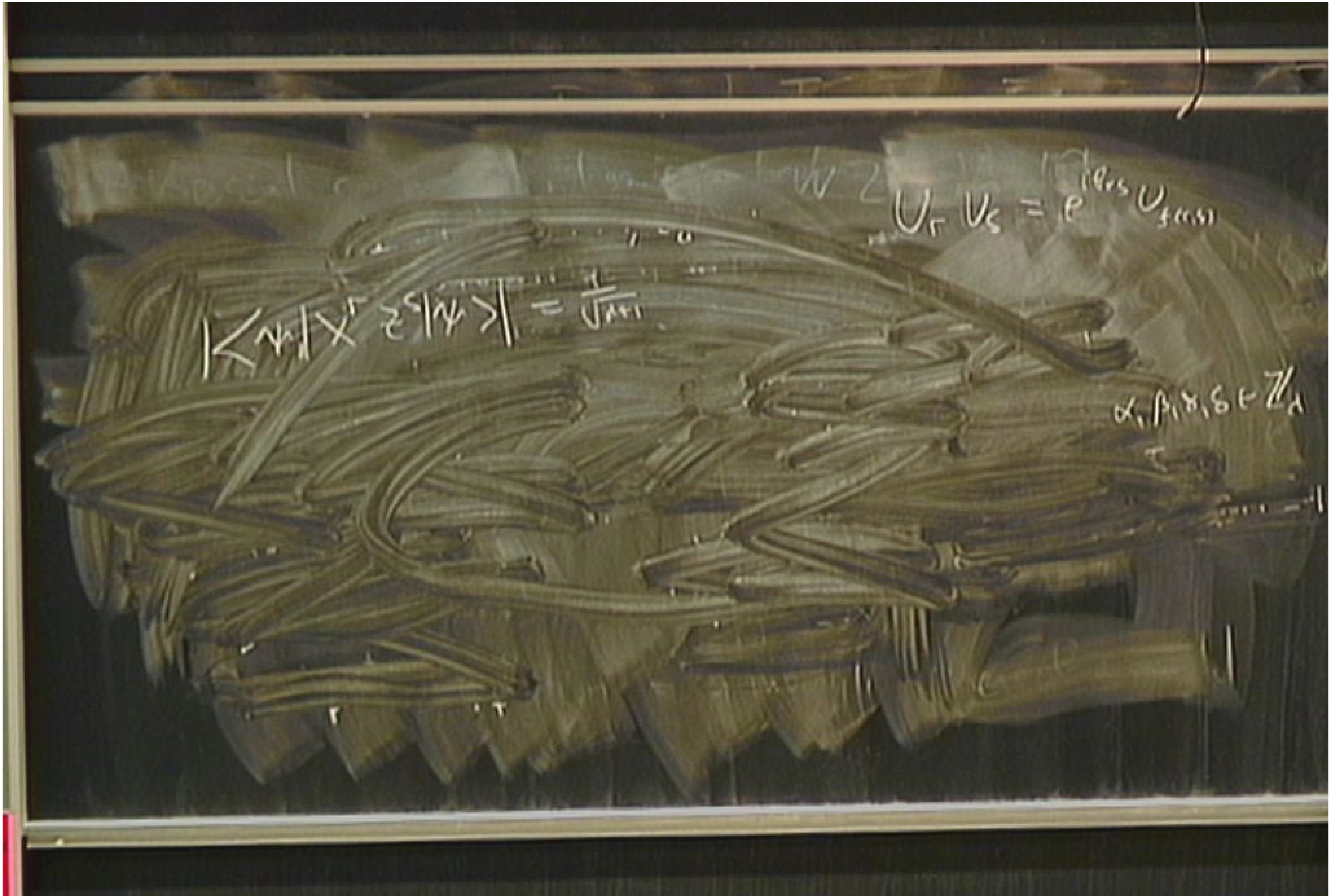
dimension = prime

$|0,0\rangle_1 \quad \ldots \quad |0,d-1\rangle$ Basis 0

$|1,0\rangle_1 \quad \ldots \quad |1,d-1\rangle$ 1

$\vdots$

$|d,0\rangle \quad \ldots \quad |d,d-1\rangle$ d.



$U_r U_s = e^{i\theta_{r,s}} U_{f(r,s)}$
$|\langle\psi| X_r Z_s |\psi\rangle| = \frac{1}{\sqrt{d+1}}$
$\alpha, \beta, \gamma, \delta \in \mathbb{Z}_d$

$$\left|\langle\psi_1|X^r Z^s|\psi\rangle\right| = \frac{1}{\sqrt{d+1}}$$

SICS & MINIMUM UNCERTAINTY STATES.

dimension = prime

	$\vert 0,0\rangle_1$	$\vert 0,d-1\rangle$	Basis 0
	$\vert 1,0\rangle_1$	$\vert 1,d-1\rangle$	1
		$\vert d,d-1\rangle$	d.

$$|\langle\psi|m,r\rangle|^2 = p_{mr}$$

$$\sum_r p_{mr}\, p_{m,(r+s)} = \begin{cases} \frac{2}{d+1} & s=0 \\ \frac{1}{d+1} & s\neq 0 \end{cases}$$

$$\sum_{r=0}^{d-1} p_{mr}^2 = \frac{2}{d+1}$$

$$\sum_{r=0}^{d-1} p_{mr}^2 = \frac{2}{d+1}$$

$$S_m = -\log_2\left(\sum_{r=0}^{d-1} p_{mr}^2\right)$$

$$-\sum_r p_{mr} \log p_{mr}$$

$$\sum_{r=0}^{d-1} p_{mr}^2 = \frac{2}{d+1}$$

$$S_m = -\log_2\left(\sum_{r=0}^{d-1} p_{mr}^2\right) \qquad -\sum_r p_{mr} \log_2 p_{mr}$$

$$S = \sum_m S_m \qquad S \geq (d+1)\left(\log_2 \frac{d+1}{2}\right)$$

$$= \text{ if and only if } \quad S_0 = S_1 = S_2 = \cdots = S_d = \log_2 \frac{d+1}{2}$$

$$\sum_{r=0}^{d-1} p_{mr}^2 = \frac{2}{d+1}$$

$$S_m = -\log_2\left(\sum_{r=0}^{d-1} p_{mr}^2\right) \qquad -\sum_r p_{mr}\log_2 p_{mr}$$

$$S = \sum_m S_m \qquad S \geq (d+1)\left(\log_2 \frac{d+1}{2}\right)$$

$=$ if and only if $S_0 = S_1 = S_2 = \cdots = S_d = \log_2 \frac{d+1}{2}$

$\lambda_r B_r \qquad \sum_r \lambda_r = 0$

$\mathrm{Tr}(A B_r)$

$\langle B_1, B_2 \rangle$

$$\mathrm{Tr}(A_r A_s) = \begin{cases} \kappa_1 & r = s \\ \kappa_2 & r \neq s \end{cases}$$