Title: Geometric Aspects of Quantum State Spaces

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Abstract: The manifold of pure quantum states can be regarded as a complex projective space endowed with the unitary-invariant Fubini-Study metric.
The physical characteristics of a given quantum system can then be represented by a variety of geometrical structures that can be identified in this manifold.
This talk will review a number of examples of such structures as they arise in the state spaces of spin-1/2, spin-1, spin-3/2, and spin-2 systems, and various types of entangled systems, all of which have fascinating and beautiful geometries associated with them.
The geometric approach offers interesting insights into the nature of quantum systems, and is also useful in the consideration of foundational issues such as those related to the measurement problem.

1. Hilbert space H and projective Hilbert space PH

Let us review briefly how quantum mechanics is ordinarily formulated.

A physical system is represented by a wave function $\psi(\mathbf{x}, t)$, which for each time t belongs to a complex Hilbert space $\mathcal{H}$.

We also require a set of linear operators on $\mathcal{H}$ corresponding to observables.

The wave function characterises the 'state' of the system at time t.

In the case of a single particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential $\phi(\mathbf{x})$, the evolution of the system is given by the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) \;=\; \left(-\frac{1}{2m}\nabla^2 + \phi(\mathbf{x})\right)\psi(\mathbf{x}, t) \;.$$

Given an initial condition $\psi(\mathbf{x}, 0)$, the Schrödinger equation determines $\psi(\mathbf{x}, 0)$ for $t > 0$, in terms of which we can work out the probabilities of the outcomes of measurements at future times.

The results of such calculations depend on the wave function only up to an overall complex factor.

The state of the system is not given by $\psi(\mathbf{x})$ itself, but rather by an equivalence class modulo transformations of the form

$$\psi(\mathbf{x},t) \;\rightarrow\; \Lambda(t)\psi(\mathbf{x},t),$$

that is to say, by a 'ray' through the origin in $\mathcal{H}$.

The space of such rays is called projective Hilbert space, denoted $\mathcal{P}H$.

The standard operations of quantum mechanics can be referred to $\mathcal{P}H$ directly, without consideration of $\mathcal{H}$ itself.

For example, the Schrödinger equation is not invariant under a change of phase and scale for $\psi(\mathbf{x})$, whereas the *projective* Schrödinger equation

$$\begin{aligned} \mathrm{i}\hbar\left[\psi(\mathbf{y})\frac{\partial\psi(\mathbf{x})}{\partial t}-\psi(\mathbf{x})\frac{\partial\psi(\mathbf{y})}{\partial t}\right] \;=\; & -\frac{1}{2m}\left[\psi(\mathbf{y})\nabla^2\psi(\mathbf{x})-\psi(\mathbf{x})\nabla^2\psi(\mathbf{y})\right] \\ & +\left[\phi(\mathbf{x})-\phi(\mathbf{y})\right]\psi(\mathbf{x})\psi(\mathbf{y}) \end{aligned}$$

is invariant under such transformations.

Had Schrödinger elected to present this relation as his wave equation, none of the physical consequences would have differed.

2. Pure States

There is a beautiful geometry associated with the projective Hilbert space $\mathcal{P}H$ which is so compelling in its richness that, in my opinion, all physicists should become acquainted with it.

The basic idea can be sketched as follows.

For simplicity we use an index notation for the Hilbert space $\mathcal{H}$. Instead of $\psi(\mathbf{x})$ we write ψ^α, where the Greek index α labels components of the Hilbert-space vector with respect to a basis.

The highly effective use of the index notation for Hilbert space was popularised by Geroch (1970).

For the complex conjugate of ψ^α we write $\bar{\psi}_\alpha$.

The 'downstairs' index reminds us that $\bar{\psi}_\alpha$ is a 'bra' vector, i.e., it belongs to the dual of the vector space to which ψ^α belongs.

The usual inner product between ψ^α and $\bar{\psi}_\alpha$ can be written $\bar{\psi}_\alpha\psi^\alpha$, with an implied summation over the repeated index.

In the case of a wave function, this is equivalent to $\int_{\mathbb{R}^3} \bar{\psi}(\mathbf{x})\psi(\mathbf{x})d^3x$.

In the Dirac notation this is $\langle\bar{\psi}|\psi\rangle$.

By use of the index notation the Schrödinger equation can be represented in the compact form $\mathrm{i}\hbar\partial_t\psi^\alpha = H^\alpha_\beta\psi^\beta$, where H^α_β is the Hamiltonian operator, $\partial_t = \partial/\partial t$, and for the projective Schrödinger equation we have

$$\mathrm{i}\hbar\psi^{[\alpha}\partial_t\psi^{\beta]} \;=\; \psi^{[\alpha}H^{\beta]}_\gamma\psi^\gamma,$$

where the skew brackets indicate antisymmetrisation.

A Hilbert space vector ξ^α can also represent homogeneous coordinates for the corresponding point in the projective Hilbert space $\mathcal{P}H$.

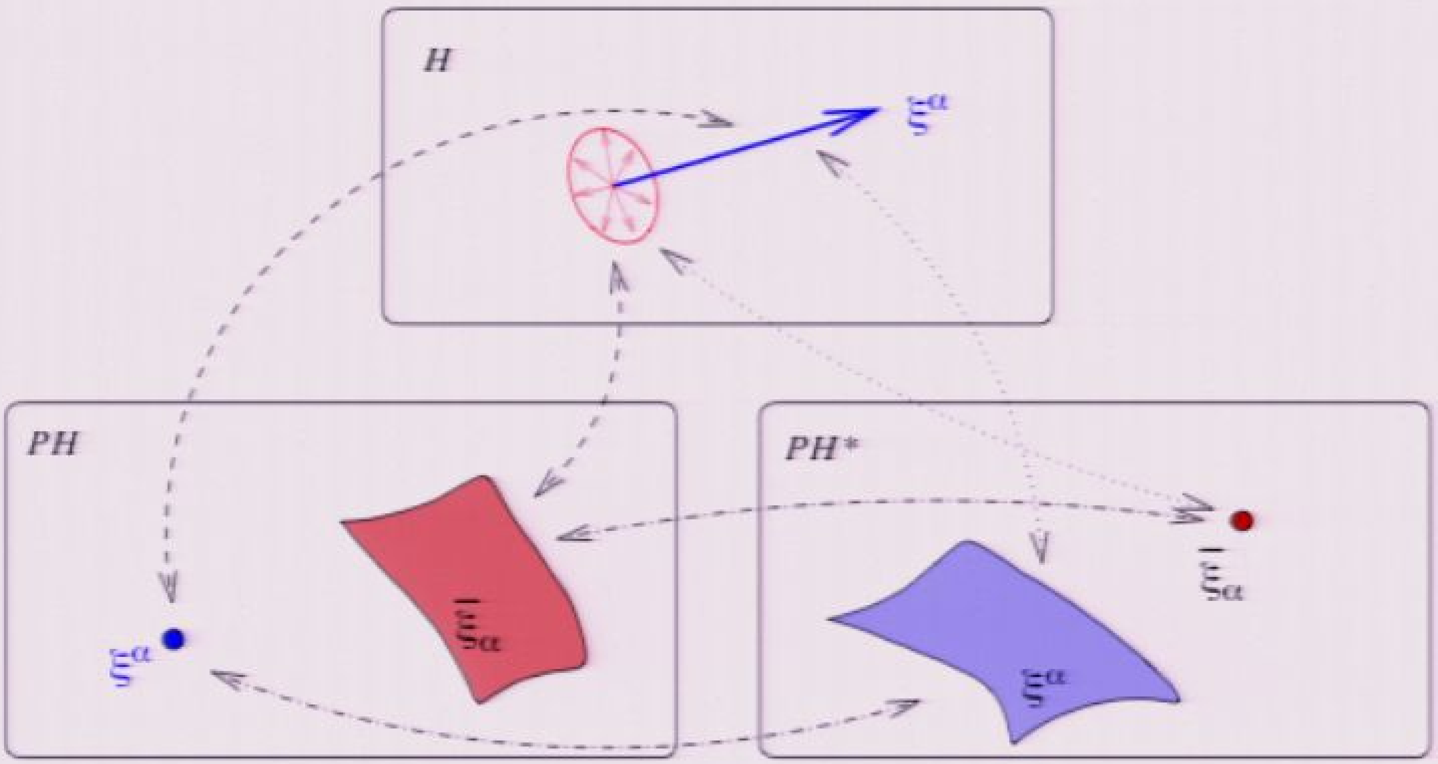
This is valid when we consider relations homogeneous in ξ^α, for which the scale is irrelevant.

For example the complex conjugate $\bar{\xi}_\alpha$ of a 'point' in $\mathcal{P}H$ can be represented by the linear subspace (hyperplane) of points ψ^α in $\mathcal{P}H$ satisfying $\bar{\xi}_\alpha\psi^\alpha = 0$.

The set of all such hyperplanes constitutes the dual space $\mathcal{P}H^*$.

The points of $\mathcal{P}H^*$ correspond to hyperplanes in $\mathcal{P}H$.

Conversely, the points of $\mathcal{P}H$ correspond to hyperplanes in $\mathcal{P}H^*$.



One of the advantages of the use of projective geometry in quantum mechanics is that it allows us to represent states (points) and dual states (hyperplanes) as geometrical objects coexisting in the same space $\mathcal{P}H$.

The complex conjugation operation, in particular, determines a so-called Hermitian correspondence between points hyperplanes.

3. Superposition of states

The join of two distinct points ξ^α and η^α in $\mathcal{P}H$ is a complex projective line, represented by points in $\mathcal{P}H$ of the form

$$\psi^\alpha = A\xi^\alpha + B\eta^\alpha ,$$

where A and B are complex numbers, not both zero.

A neat way of characterising this line is in terms of the tensor $L^{\alpha\beta} = \xi^{[\alpha}\eta^{\beta]}$.

Physically, $L^{\alpha\beta}$ represents the system of all possible quantum mechanical superpositions of the states ξ^α and η^α.

If $\mathcal{P}H = CP^n$, the n-dimensional complex projective space, then $L^{\alpha\beta}$ it has $\frac{1}{2}n(n+1)$ complex components, which can be viewed as the coordinates of the given line.

The fundamental property of these line coordinates is that their ratios are independent of the choice of the two points ξ^α and η^α, in such a way that all points on the given line are treated on an equal footing.

4. Transition probability

The simplest situation in which a probabilistic idea arises in quantum theory is also the simplest situation in which the concept of 'distance' arises.

The transition probability for the states ξ^α and η^α determines an angle θ as follows:

$$\cos^2 \frac{1}{2}\theta \; = \; \frac{\xi^\alpha \bar{\eta}_\alpha \eta^\beta \bar{\xi}_\beta}{\xi^\gamma \bar{\xi}_\gamma \eta^\delta \bar{\eta}_\delta} \, .$$

Clearly, θ is independent of the scale and phase of ξ^α and η^α.

This angle defines a distance between the states ξ^α and η^α in $\mathcal{P}H$.

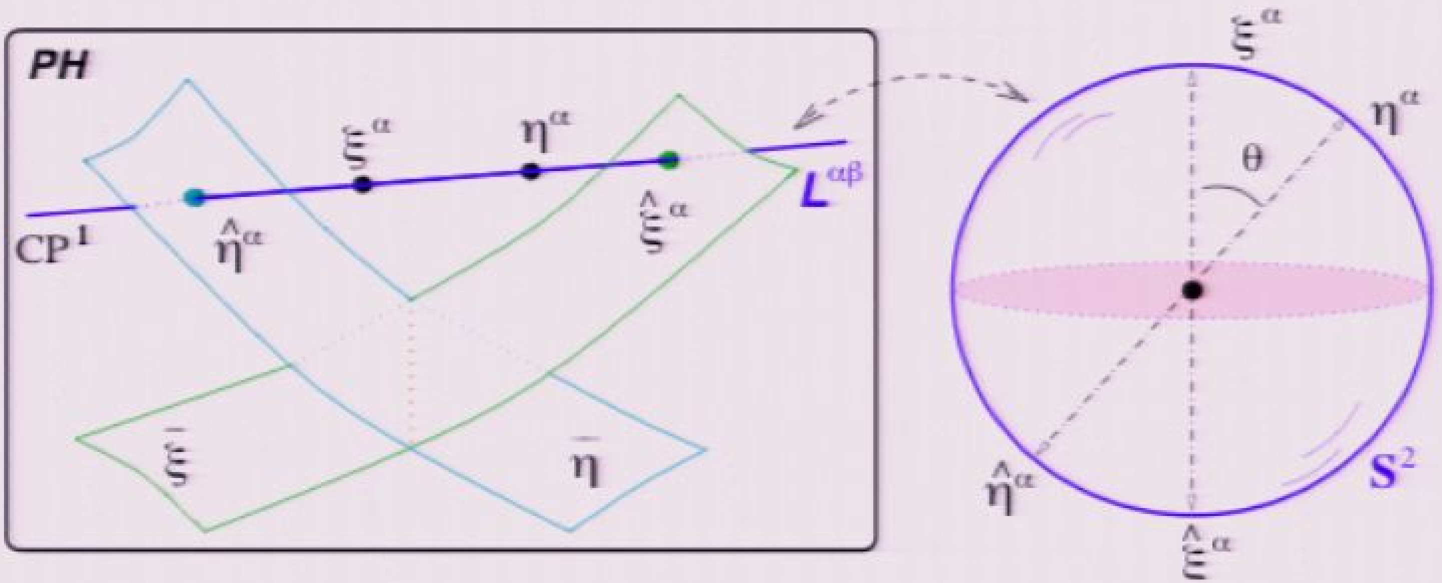
If the states coincide, then $\theta = 0$; for orthogonal states we have $\theta = \pi$.

Suppose we set $\theta = ds$ and $\xi^\alpha = \psi^\alpha$, $\eta^\alpha = \psi^\alpha + d\psi^\alpha$.

By use of the expression for the transition probability, expanded to second order, we find that the infinitesimal distance ds between two neighbouring states is

$$\mathrm{d}s^2 \; = \; 8 \frac{\psi^{[\alpha} d\psi^{\beta]} \bar{\psi}_{[\alpha} d\bar{\psi}_{\beta]}}{(\bar{\psi}_\gamma \psi^\gamma)^2} \, ,$$

an expression known to geometers as the Fubini-Study metric.



The introduction of the Fubini-Study metric illustrates how the notions of probability and distance become interlinked, once quantum theory is formulated in a geometric manner.

The *geodesic distance* with respect to the Fubini-Study metric determines the transition probability between two states.

The nontrivial metrical geometry of the Fubini-Study manifold is responsible for much of the surprising richness of the quantum world, and we will see various examples of this as we examine the state spaces of various quantum systems.

5. Spin $\frac{1}{2}$ systems

We consider first the state space of a single spin-$\frac{1}{2}$ particle. The Hilbert space is $\mathbb{C}^2$, spanned by a pair of spin eigenstates with eigenvalues, say, $S_z = \pm\frac{1}{2}$.

The spin-z eigenstates can be written $|\uparrow\rangle$ and $|\downarrow\rangle$, and a generic state $|\psi\rangle$ is thus $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$, $a, b \in \mathbb{C}$. Projectively, the state space is a CP^1.

In real terms this is a two-sphere of radius $\frac{1}{2}$. To see this we recall that a general mixed state of the spin-$\frac{1}{2}$ particle is represented by the density matrix:

$$\hat{\rho} = \begin{pmatrix} t-z & x-\mathrm{i}y \\ x+\mathrm{i}y & t+z \end{pmatrix}, \tag{1}$$

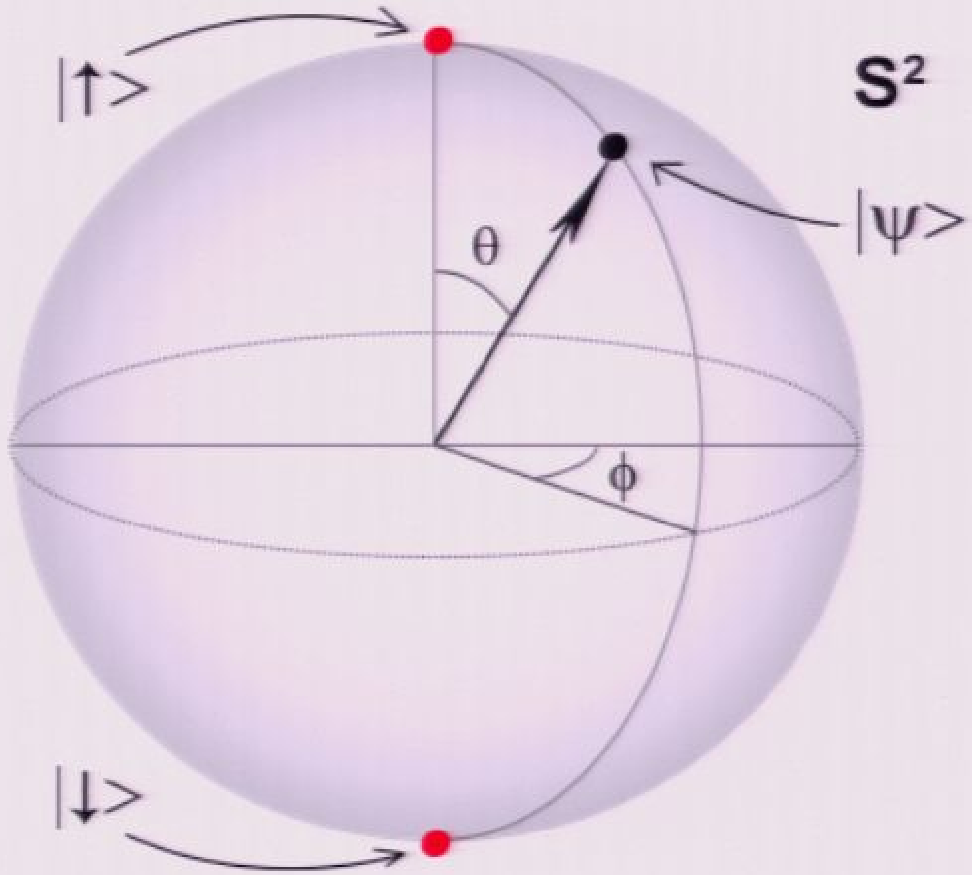
where the trace condition $tr\hat{\rho} = 1$ implies $t = \frac{1}{2}$.

Writing $r^2 = x^2 + y^2 + z^2$, we find that the eigenvalues of $\hat{\rho}$ are $\lambda_\pm = t \pm r$.

Since $\hat{\rho}$ is nonnegative, the eigenvalues are nonnegative: $t - r \geq 0$.

The trace condition then says that $x^2 + y^2 + z^2 \leq (\frac{1}{2})^2$. In other words, the space of 2×2 density matrices is a ball of radius $\frac{1}{2}$.

For pure states, the density matrix is degenerate with $\lambda_- = 0$; that is, $x^2 + y^2 + z^2 = (\frac{1}{2})^2$. Hence the pure state space is the surface of the ball.



State space of a spin-$\frac{1}{2}$ particle.

A general pure state can be represented by spherical coordinates on $\mathbb{S}^2$.

Letting the two-component spinor ψ^A, $A = 0, 1$, represent a point on $\mathbb{P}^1$, we relate this to the corresponding point on $\mathbb{S}^2$ by writing

$$\psi^A = \left(\cos\tfrac{1}{2}\theta\,,\ \ \sin\tfrac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}\phi}\right) \qquad (0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi). \tag{2}$$

We let $\alpha^A = (1,\, 0)$ represent the spin-up state.

This corresponds to the north pole on $\mathbb{S}^2$, for which $\theta = 0$.

Then $\bar{\alpha}^A = \epsilon^{AB}\bar{\alpha}_B = (0,\, 1)$ represents the spin down state, where ϵ^{AB} is the anti-symmetric spinor and $\bar{\alpha}_B$ is the complex conjugate of α^B.

We use α^A and $\bar{\alpha}^A$ as our basis in Hilbert space and express the general pure state as $\psi^A = u\alpha^A + v\bar{\alpha}^A$, where (u, v) are the homogeneous coordinates of that point.

6. Spin-1 systems

The specification of a physical system implies geometrical structure on the state space.

Our point of view is that *all* of the physical details of a quantum system can be represented by additional projective geometrical features.

We shall illustrate this point with several examples.

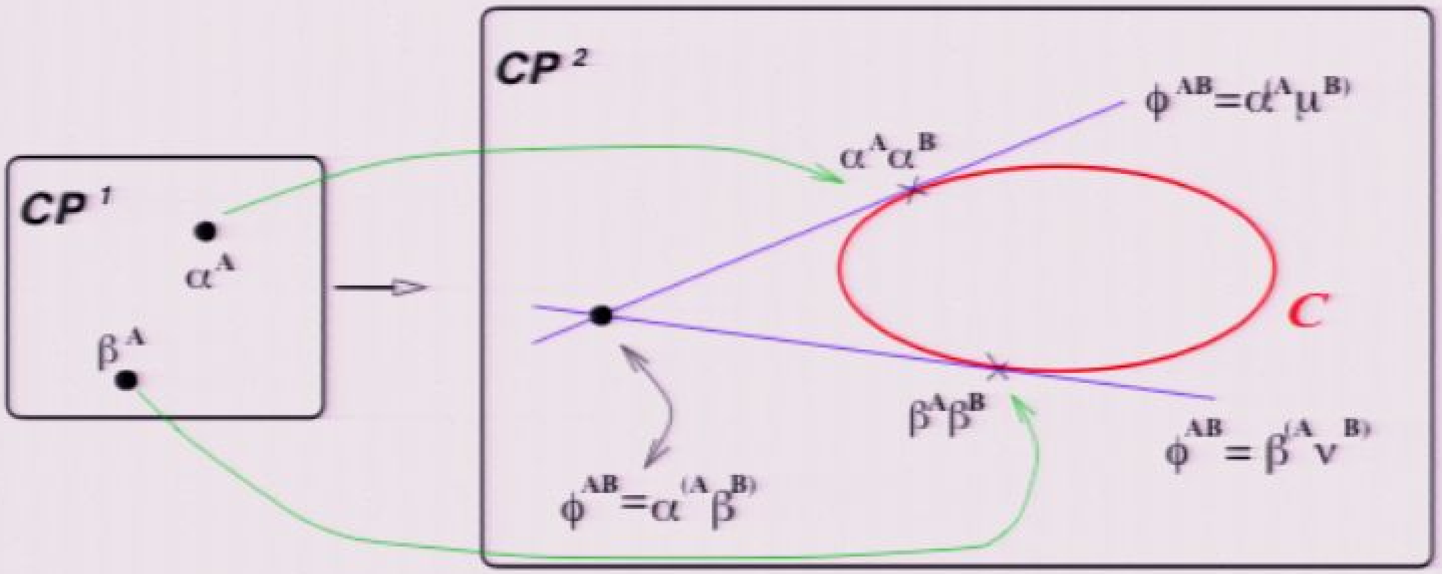
Let us now consider the spin degrees of freedom of a nonrelativistic spin 1 particle, as represented by a symmetric spinor ϕ^{AB} $(A, B = 0, 1)$.

The Hilbert space has three dimensions, and the corresponding projective Hilbert space is CP^2.

A symmetric spinor has a natural decomposition $\phi^{AB} = \alpha^{(A}\beta^{B)}$, where α^A and β^A are called 'principal spinors'.

There is a special conic $\mathcal{C}$, corresponding to degenerate spinors of the form $\phi^{AB} = \psi^A\psi^B$ for some repeated principal spinor ψ^A.

The identification of $\mathcal{C}$ is sufficient to induce the structure of a spin 1 system on the state space.



In particular, through any generic point in CP^2 there are two lines tangent to $\mathcal{C}$, and the corresponding tangent points determine the principal spinors, up to scale.

The conic $\mathcal{C}$ in CP^2 can be represented by a map from CP^1 to CP^2 such that if (t, u) are homogeneous coordinates on CP^1, we have the Veronese embedding

$$\mathcal{C}: \ (t, u) \rightarrow (t^2, tu, u^2) \ ,$$

where (t^2, tu, u^2) represent homogeneous coordinates on CP^2.

7. Measurement of S_z in the case of a spin 1 system

Now we look at measurements.

Since a complex projective line represents a sphere S^2, the specification of the spin direction in $\mathbb{R}^3$ determines a point on S^2, and hence on $\mathcal{C}$.

For quantum mechanics the conic is required to be compatible with the complex conjugation operation on the state space in the sense that if we conjugate a point of $\mathcal{C}$, then the resulting line is tangent to $\mathcal{C}$.

If we take the complex conjugate of a state on $\mathcal{C}$, the resulting line is tangent to the conic at a point, which we call the conjugate of the original point on $\mathcal{C}$.

This establishes a Hermitian correspondence between pairs of points on $\mathcal{C}$.

For a state $\phi^{AB} = \psi^A\psi^B$ the conjugate line consists of states of the form $\lambda^{(A}\bar{\psi}^{B)}$ for arbitrary λ^A.

This line touches the conic $\mathcal{C}$ at the point $\bar{\psi}^A\bar{\psi}^B$.

For any choice of spin axis there are three possible spin states, with eigenvalues $1, -1$ and 0.

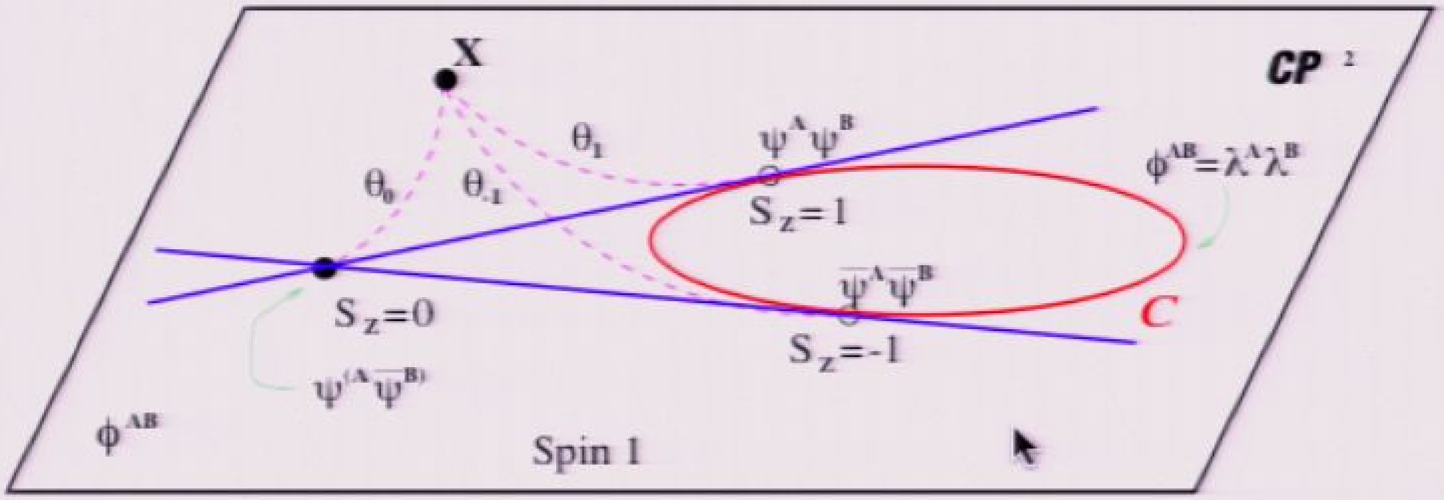
We need to see how these are represented in the geometry.

The spin eigenstates are the points $\psi^A\psi^B$ and $\bar{\psi}^A\bar{\psi}^B$ on $\mathcal{C}$, having the eigenvalues 1 and -1, together with a third point $\psi^{(A}\bar{\psi}^{B)}$ obtained by intersecting the lines tangent to the conic $\mathcal{C}$ at the other two points, corresponding to eigenvalue 0.

When a spin measurement is made, the initial state corresponds to a generic point X in CP^2, and the measurement is defined by a spin axis.

The state then 'jumps' from its initial point to one of the three spin eigenstates associated with the choice of axis.

Quantum theory, as such, states nothing about the 'mechanism' whereby this jump is achieved.



We can, however, compute the probabilities, and describe the result in geometrical terms.

First we calculate the distance from X to each of the three spin eigenstates, by use of the Fubini-Study metric.

This gives us three angles θ_1, θ_{-1}, and θ_0. For each angle we compute $P(\theta) = \frac{1}{2}(1 + \cos\theta)$, which gives the transition probability to that state.

It is not obvious that the three probabilities computed in this way sum up to one, given any initial state in which the measurement is performed, but they do: this is a 'miracle' of the Fubini-Study geometry.

8. Spin-$\frac{3}{2}$ systems and the twisted cubic curve

In the case of a projective plane, there is a conic $\mathcal{C}$, corresponding to degenerate spinors obtained by a special map from a projective line to a plane.

In three-dimensional projective space CP^3 there are two different kinds of locus to be considered, each of which is in some respects a proper analogue of the conic, namely, the quadric surface Q^2 and the twisted cubic curve $\mathcal{T}$.

When viewed as the state space of a quantum mechanical system, the quadric surface in CP^3 characterises the disentangled states of a pair of spin $\frac{1}{2}$ particles, the geometry of which we shall come to shortly

The twisted cubic, the simplest nonplanar curve in projective geometry.

It plays an essential role in the geometry of the state space of a spin $\frac{3}{2}$ particle.

The twisted cubic can be represented by a map from CP^1 to CP^3 of the form

$$\mathcal{T}: \ (t,u) \rightarrow (t^3, t^2u, tu^2, u^3) \ ,$$

where (t^3, t^2u, tu^2, u^3) represents the homogeneous coordinates of a point on $\mathcal{T}$ in CP^3.

In order to proceed further, we introduce a spinorial notation and let the symmetric spinor $\psi^{ABC} = \psi^{(ABC)}$ denote homogeneous coordinates on CP^3.

Then, the twisted cubic curve is determined by the relation $\tau_{AB} = 0$, where

$$\tau_{AB} \triangleq \psi_{CD(A}\psi_{B)}{}^{CD} .$$

As a consequence we see that $\mathcal{T}$ is given by the common intersection of two-dimensional net of quadric surfaces in CP^3.

Here the indices on ψ^{ABC} are raised and lowered according to the standard conventions, so for example, $\psi_B{}^{CD} = \epsilon_{AB}\psi^{ACD}$.

The general solution to the algebraic relations given by $\tau_{AB} = 0$ takes the form $\psi^{ABC} = \xi^A\xi^B\xi^C$ for arbitrary ξ^A.

Then if we parametrise a point $\xi^A \in CP^1$ according to the scheme $\xi^A = (t, u)$, we recover the map $\tau : \ CP^1 \to CP^3$ noted above.

The specification of a twisted cubic $\mathcal{T}$ in CP^3 induces a *null polarity* on the state space, i.e., a natural correspondence between points and planes such that the polar plane of a given point includes that point.

The null polarity is given by the map

$$\psi^{ABC} \rightarrow \psi_{ABC} = \epsilon_{AP}\epsilon_{BQ}\epsilon_{CR}\psi^{PQR} .$$

It follows as an elementary spinor identity that $\psi^{ABC}\psi_{ABC} = 0$ for any choice of ψ^{ABC}.

In the case of a point $\psi^{ABC} = \xi^A\xi^B\xi^C$ on $\mathcal{T}$, the corresponding polar plane intersects $\mathcal{T}$ solely at that point, with a three-fold degeneracy, and is called the *osculating plane* at that point.

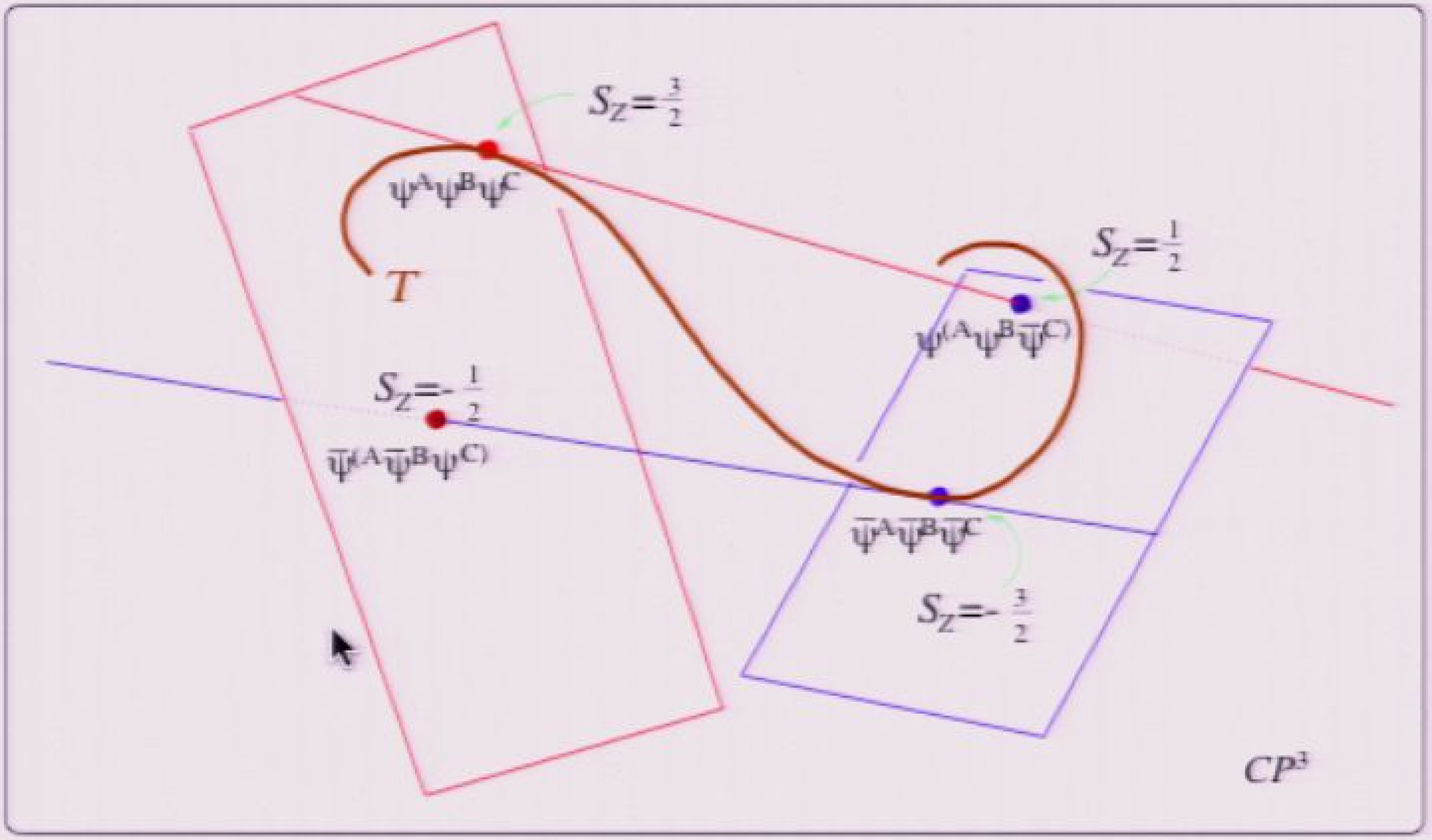
Through a given point $\xi^A\xi^B\xi^C \in \mathcal{T}$, the associated tangent line is given by points of the form $\xi^{(A}\xi^B\eta^{C)}$, with η^A arbitrary.

We say that a generic point of CP^3, with three distinct principal spinors, is of type $\{1, 1, 1\}$.

The point that lie on tangents to $\mathcal{T}$ are of type $\{2, 1\}$, whereas the points of $\mathcal{T}$ are of type $\{3\}$.

A necessary and sufficient condition for a point to be of type $\{2, 1\}$ is the vanishing of the invariant $\mathcal{G} = \tau_{AB}\tau^{AB}$.

Hence we see that the tangent lines to $\mathcal{T}$ generates a quartic surface $\mathfrak{G}$ in CP^3.



For quantum mechanics the twisted cubic has to be self-conjugate in the sense that the complex conjugate plane of any point on $\mathcal{T}$ has to be the osculating plane of another point on $\mathcal{T}$.

The choice of a point on $\mathcal{T}$ determines a spin axis.

For each spin axis, there are four possible spin eigenstates, with eigenvalues $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$.

Two of the spin states, corresponding to the eigenvalues $\pm\frac{3}{2}$, lie on $\mathcal{T}$ itself.

These two states can be written $\psi^A\psi^B\psi^C$ and $\bar{\psi}^A\bar{\psi}^B\bar{\psi}^C$, where $\bar{\psi}^A = \epsilon^{AB}\bar{\psi}_B$ and $\bar{\psi}_B = \mathbf{c}(\psi^B)$.

The complex conjugate of the state $\psi^{ABC} = \psi^A\psi^B\psi^C$ on the twisted cubic $\mathcal{T}$ is the plane $\bar{\psi}_{ABC} = \bar{\psi}_A\bar{\psi}_B\bar{\psi}_C$ in CP^3, and this plane osculates $\mathcal{T}$ at the point $\bar{\psi}^A\bar{\psi}^B\bar{\psi}^C$.

Through the point $\psi^A\psi^B\psi^C$ there is a unique line tangent to $\mathcal{T}$, and this line intersects the plane $\bar{\psi}_A\bar{\psi}_B\bar{\psi}_C$ at the point $\psi^{(A}\psi^B\bar{\psi}^{C)}$.

This point is the spin $\frac{1}{2}$ eigenstate with respect to that choice of axis.

Conversely, the tangent line to $\mathcal{T}$ at the spin $-\frac{3}{2}$ state $\bar{\psi}^A\bar{\psi}^B\bar{\psi}^C$ intersects the osculating plane of $\mathcal{T}$ at $\psi^A\psi^B\psi^C$ at the point $\bar{\psi}^{(A}\bar{\psi}^B\psi^{C)}$, which is the spin $-\frac{1}{2}$ state.

9. Spin-2 systems and the rational quartic curve

A similar analysis can be pursued in connection with the geometry of a spin-2 system, for which the state space is CP^4, endowed with a rational quartic curve.

The geometry of this curve is closely related to the famous Petrov classification of gravitational fields.

There are two levels of specialisation in the description of the state space. If we take CP^4 with the Fubini-Study metric, but without the specification of a rational quartic curve, then we have the state space appropriate for a generic five-state system.

If we take CP^4 with the rational quartic without the Fubini-Study geometry, then we have the set-up appropriate for the geometry of gravitational fields, but without bringing quantum mechanics into play.

That is the situation where the Petrov scheme arises.

Finally, when we bring both the rational quartic and the Fubini-Study metric into the picture, we have the state space geometry for a spin 2 quantum system.

Let us consider first the case when we have a rational quartic curve R in CP^4, but without consideration of the metric.

For each spin axis, there are four possible spin eigenstates, with eigenvalues $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$.

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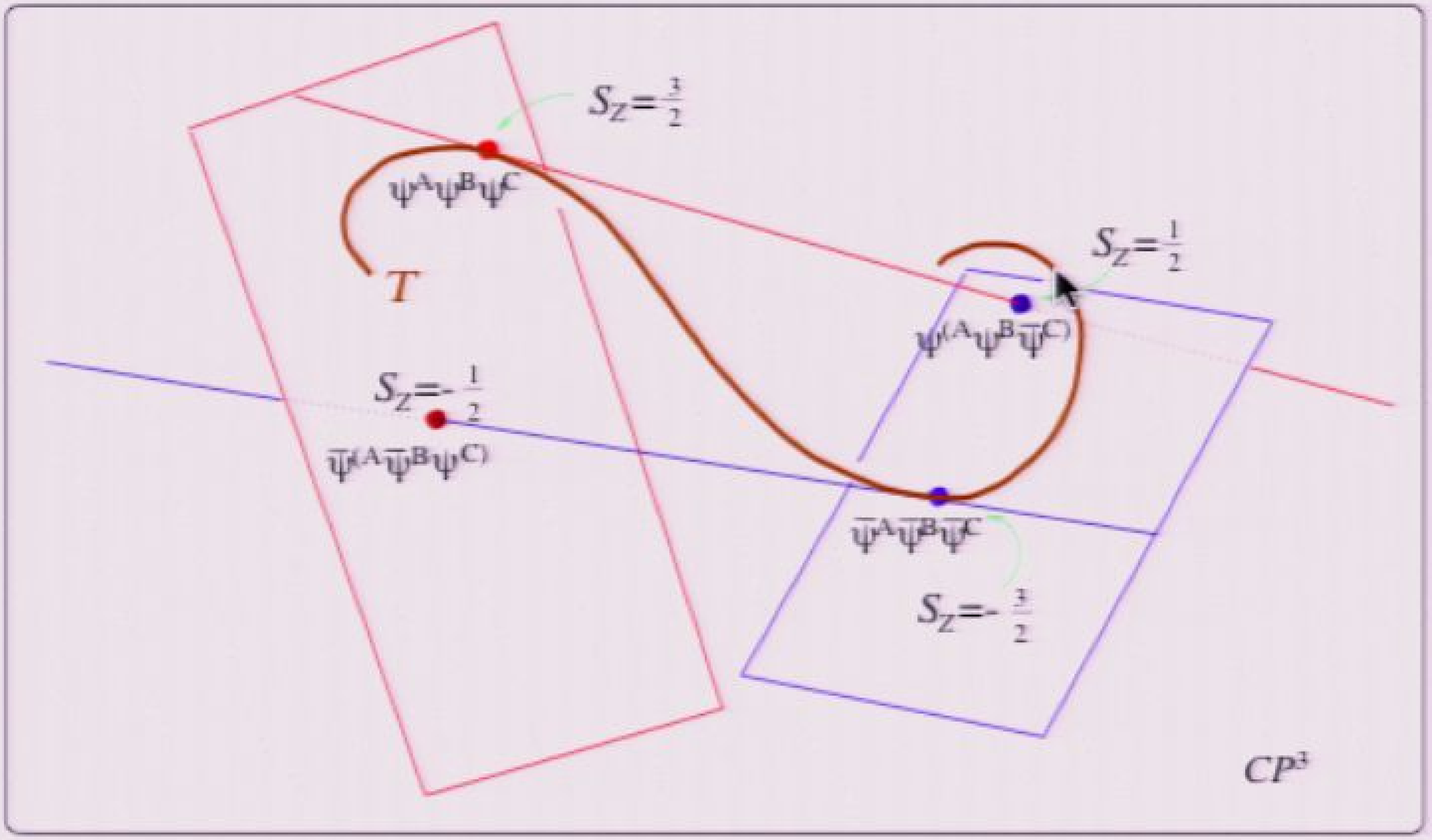

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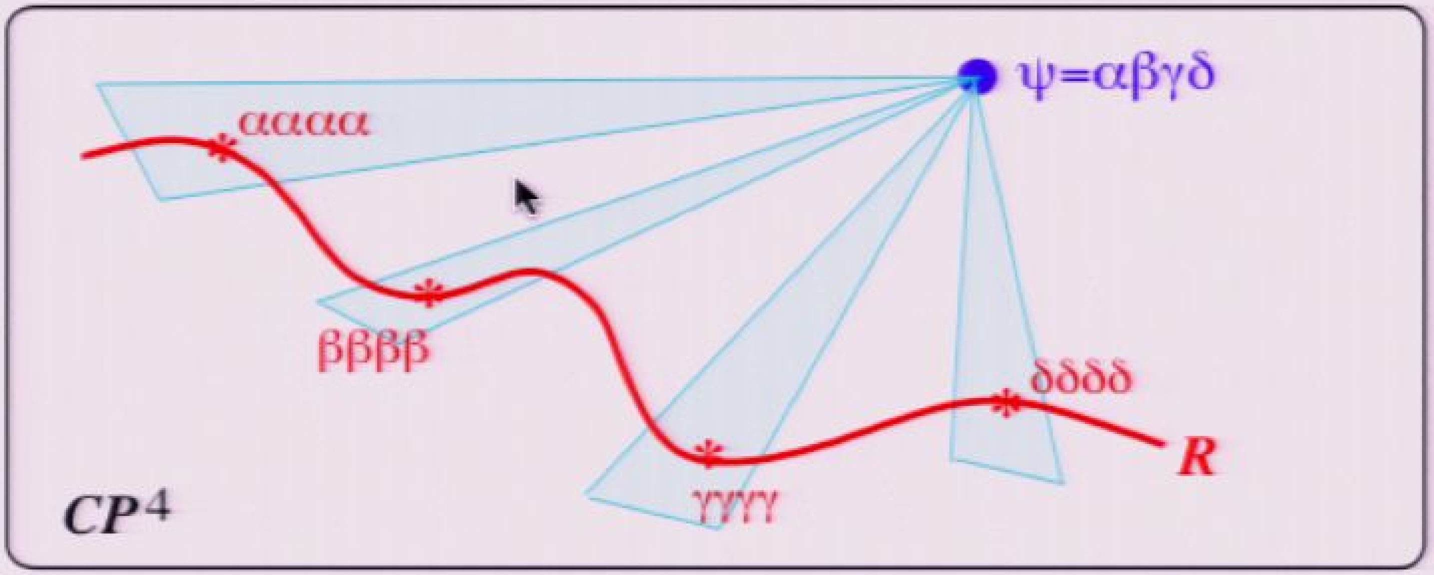


The various types of degeneracies that can arise characterised in terms of the geometry of the rational quartic curve.

The states of type $\{3,1\}$ or type $\{4\}$ constitute together a sextic 2-surface $\mathfrak{M} \in CP^4$ ruled by the tangent lines of $\mathcal{R}$.

A necessary and sufficient condition for a spinor to lie on $\mathfrak{M}$ is the vanishing of the following invariants:

$$\mathcal{I} = \psi_{ABCD}\psi^{ABCD}, \quad \mathcal{J} = \psi_{AB}{}^{CD}\psi_{CD}{}^{EF}\psi_{EF}{}^{AB} .$$

In particular, the chosen state on $\mathcal{R}$ has $S_z = 2$ with respect to the corresponding z-axis, and the complex conjugate states has $S_z = -2$.

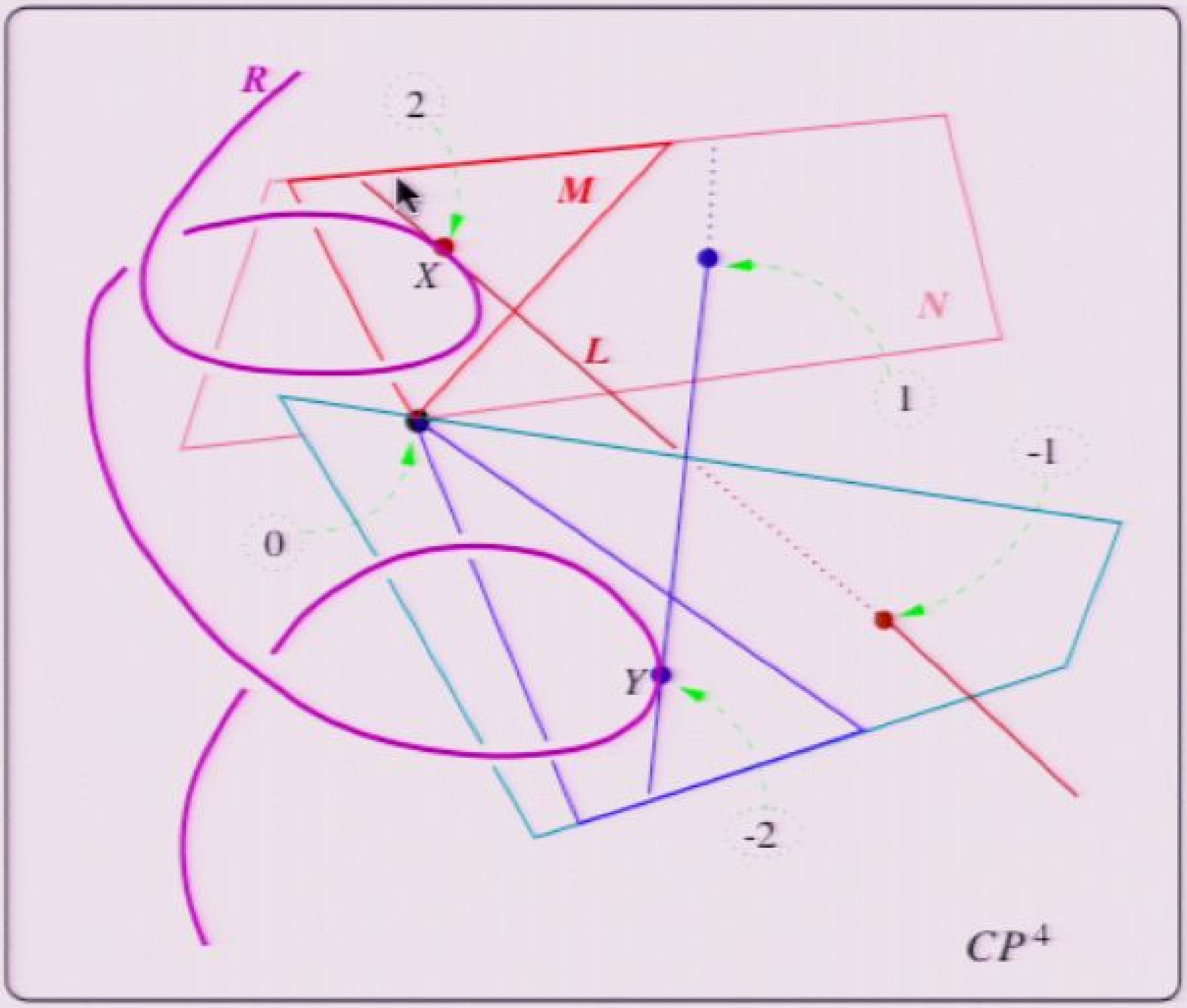
The $S_z = 1$ state obtained by intersecting the tangent line of an $S_z = 2$ state with the osculating solid of the corresponding $S_z = -2$ state.

The $S_z = -1$ state is obtained by intersecting the osculating solid of the $S_z = 2$ state with the tangent line of the $S_z = -2$ state.

Finally, the $S_z = 0$ state is the point obtained by intersecting the osculating planes at $S_z = 2$ and $S_z = -2$ states.

The $S_z = 0$ states are 'real' in the sense that $\psi^{ABCD} \propto \bar{\psi}^{ABCD}$ for these points.

Thus the $S_z = 0$ states are given by the real points of the surface $\mathfrak{K}$.



Geometry of the quantum state space of a spin-2 system.

10. Two spin $\frac{1}{2}$ particles

Now we consider the spin degrees of freedom of an entangled pair of spin $\frac{1}{2}$ particles.

The generic two-particle state ψ^{AB} for a pair of such particles (e.g., an electron and a positron) has a 4-dimensional Hilbert space, and the state space is CP^3.

There is a preferred point Z in CP^3, corresponding to the singlet state of total spin 0, for which $\psi^{AB} = \psi^{[AB]}$.

The conjugate plane $\bar{Z}$ contains the triplet states of total spin 1, for which $\psi^{AB} = \psi^{(AB)}$.

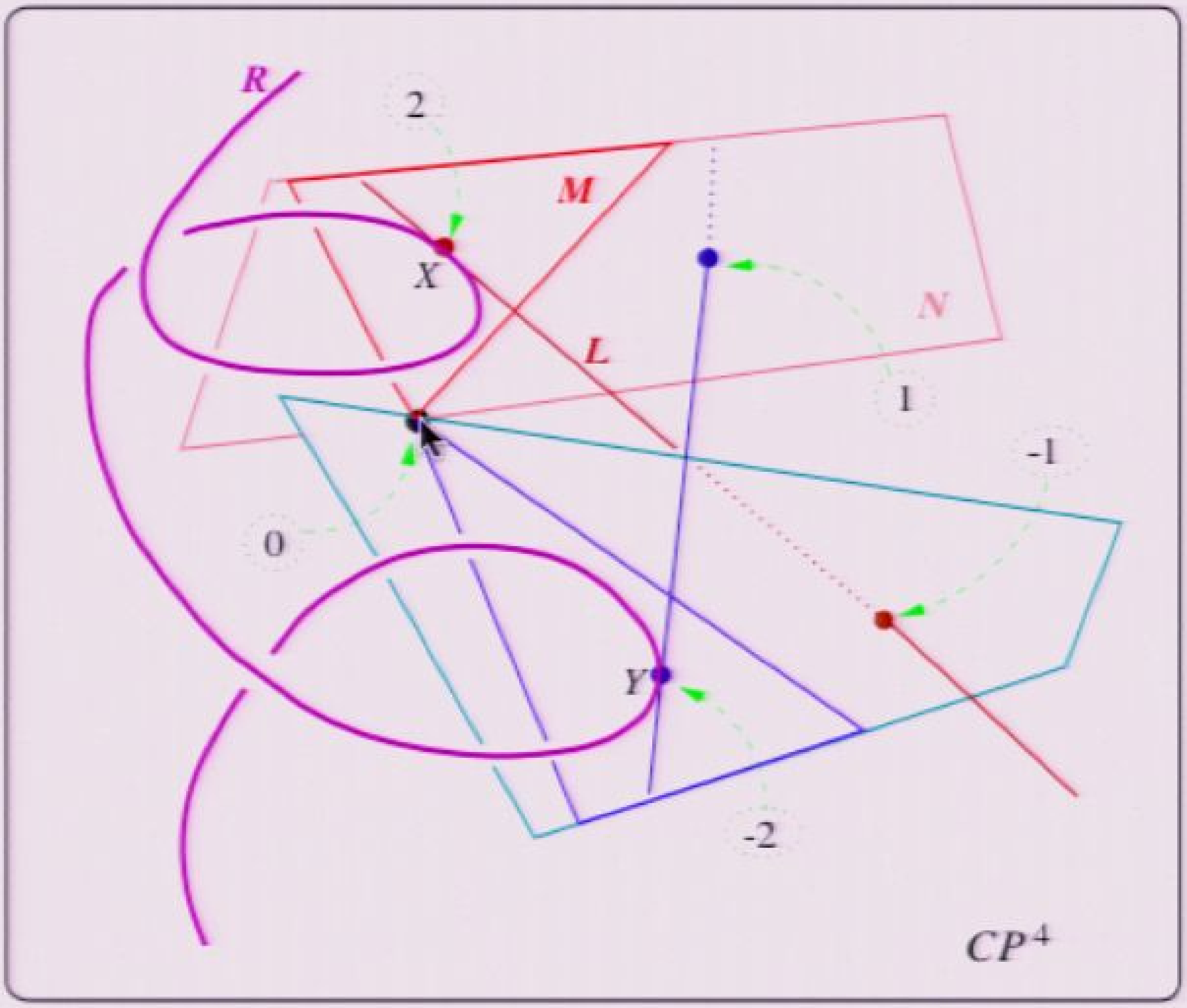
We note that $\bar{Z}$ is endowed with a self-conjugate conic $\mathcal{C}$, each point of which defines a spin axis.

There is also a surface $Q \in CP^3$, given by the quadratic equation

$$\epsilon_{AC}\epsilon_{BD}\psi^{AB}\psi^{CD} = 0 .$$

Q consists of states of the $disentangled$ form $\psi^{AB} = \xi^A\eta^B$, representing an embedding of the product of the state spaces of the individual spin $\frac{1}{2}$ particles.

The pure states off the quadric are the $entangled$ states.



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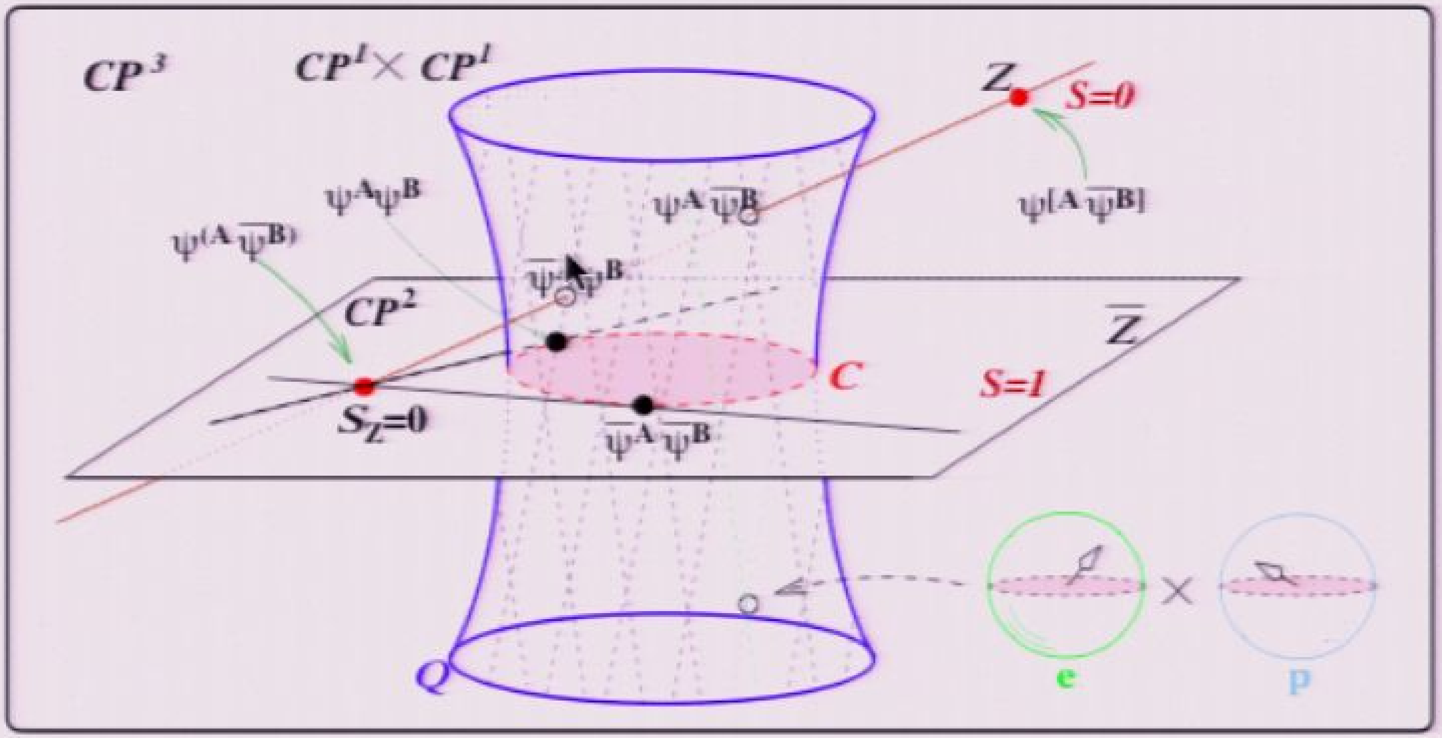


Quantum entanglement.

Suppose we start with a combined state of total spin 0 for the two particles, and we measure the spin of the first particle (say, the electron) relative to a given choice of axis.

This will disentangle the state, so the result lies on Q.

11. Three spin $\frac{1}{2}$ particles

The state space $\mathbb{P}^7$ of the three-qubit system is obtained by projecting the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

There are several different types of entanglement that can result.

First, we have the completely disentangled states.

These constitute a triply-ruled three-surface $\mathcal{D} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Next, we have the partly entangled states, where one of the particles is disentangled from the other two.

There are three such systems of partly entangled states, each of which constitutes a four-dimensional variety $Q_i \subset \mathbb{P}^7$ $(i = 1, 2, 3)$ with the structure of $\mathbb{P}^1 \times \mathbb{P}^3$.

The $\mathbb{P}^1$ in each case represents the state space of the disentangled particle, and the $\mathbb{P}^3$ represents the state space of the remaining entangled pair.

It should be evident that $\mathcal{D} = Q_1 \cap Q_2 \cap Q_3$.

Finally, we have the states for which all three particles are entangled.

The states of total spin $S = \frac{3}{2}$ form a hyperplane $\mathbb{P}^3_{\rm sym} \subset \mathbb{P}^7$.

These states are represented by totally symmetric spinors, i.e. those satisfying $\psi^{ABC} = \psi^{(ABC)}$, where the round brackets denote symmetrisation.

The states of total spin $\frac{1}{2}$ also constitute a hyperplane of dimension three, which we call $\mathbb{P}^3_{\rm asym}$. The 'asymmetric' states are those that are of the form

$$\psi^{ABC} = \alpha^A \epsilon^{BC} + \beta^B \epsilon^{CA} + \gamma^C \epsilon^{AB} \tag{3}$$

for some $\alpha^A, \beta^A, \gamma^A$.

The symmetric states and the asymmetric states are orthogonal.

The hyperplane $\mathbb{P}^3_{\rm sym}$ intersects $\mathcal{D}$ in a twisted cubic curve $\mathcal{T} = \mathbb{P}^3_{\rm sym} \cap \mathcal{D}$.

The space $\mathbb{P}^3_{\rm asym}$ intersects each of the varieties $\{Q_i\}_{i=1,2,3}$ in a line $\mathcal{L}_i = Q_i \cap \mathbb{P}^3_{\rm asym}$.

These lines represent the singlet states of the entangled pairs in the $\{Q_i\}$.

This follows because the states of $\mathbb{P}^3_{\rm asym}$ can be expressed in the form $\alpha^A \epsilon^{BC} + \beta^B \epsilon^{CA} + \gamma^C \epsilon^{AB}$.

The intersection of $\mathbb{P}^3_{\rm asym}$ with Q_1 thus takes the form $\alpha^A \epsilon^{BC}$, where $\alpha^A \in \mathbb{P}^1$ and $\epsilon^{BC} \in \mathbb{P}^3$.

Since ϵ^{BC} is antisymmetric, it corresponds to the singlet state in $\mathbb{P}^3$. Then as α^A varies, we obtain the 'singlet' line $\mathcal{L}_1$.

An interesting feature of 3-qubit entanglement is that there are six different natural classes of entanglement.

These the: totally disentangled states, the three configurations of partly entangled states, and two different classes of totally entangled states:

We consider the space $\mathcal{D}$ of totally disentangled states, and let $\mathcal{H}$ denote the six-dimensional variety generated by the system of 3-hyperplanes tangent to $\mathcal{D}$.

Then $\mathcal{H}$ turns out to be a quartic surface in $\mathbb{P}^7$, consisting of those points for which the so-called Cayley invariant vanishes:

$$\psi_A{}^{BC}\psi_{BCD}\psi_P{}^{RS}\psi_{QRS}\epsilon^{AP}\epsilon^{BQ} = 0. \quad (4)$$

A necessary and sufficient condition for ψ^{ABC} to satisfy this relation is that

$$\psi^{ABC} = x^A\beta^B\gamma^C + \alpha^A y^B\gamma^C + \alpha^A\beta^B z^C \quad (5)$$

for some $\alpha^A, \beta^B, \gamma^C, x^A, y^B, z^C$.

If $\alpha^A\beta^B\gamma^C$ is a point in $\mathcal{D}$, then the tangent plane to $\mathcal{D}$ at that point consists of states of the form (5) for some choice of x^A, y^B, z^C.

12. Schrödinger evolution

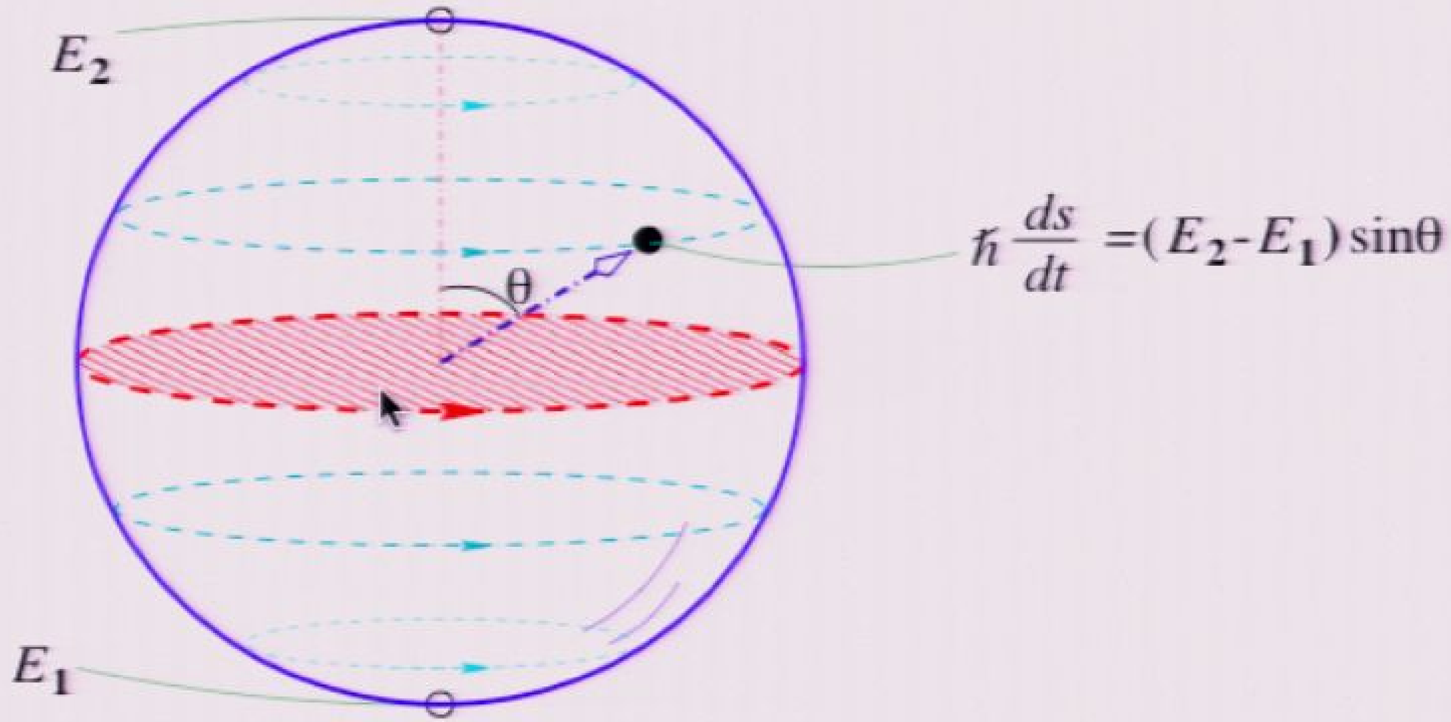
The geometry of quantum mechanics is very rich, once specific physical systems are brought into play, even when there are only a few degrees of freedom.

This picture can be further developed by consideration of the dynamics of a quantum system, which can be pictured as a vector field on the state manifold.

Such a vector field generates a symmetry of the Fubini-Study geometry, i.e., an action of the projective unitary group.

In the case of an $(n+1)$-dimensional Hilbert space, the state space is CP^n, which can be viewed as a real manifold Γ of dimension $2n$, with a symmetry group generated by a family of $n(n+2)$ Killing vector fields.

The generic Killing field on Γ has $n+1$ fixed points, corresponding to the eigenstates of a nondegenerate Hamiltonian.



In the case of a 2-dimensional Hilbert space, the state space is CP^1, and the specification of a Killing field selects out a pair of polar points on S^2, corresponding to energy eigenstates E_0 and E_1.

The relevant symmetry is then given by a rigid rotational flow about this axis, the angular frequency being determined by Planck's formula $E_1 - E_0 = \hbar\omega$.

13. Manifesto

When one is doing mathematical work, there are essentially two different ways of thinking about the subject: the algebraic way, and the geometric way.

With the algebraic way, one is all the time writing down equations and following rules of deduction, and interpreting these equations to get more equations.

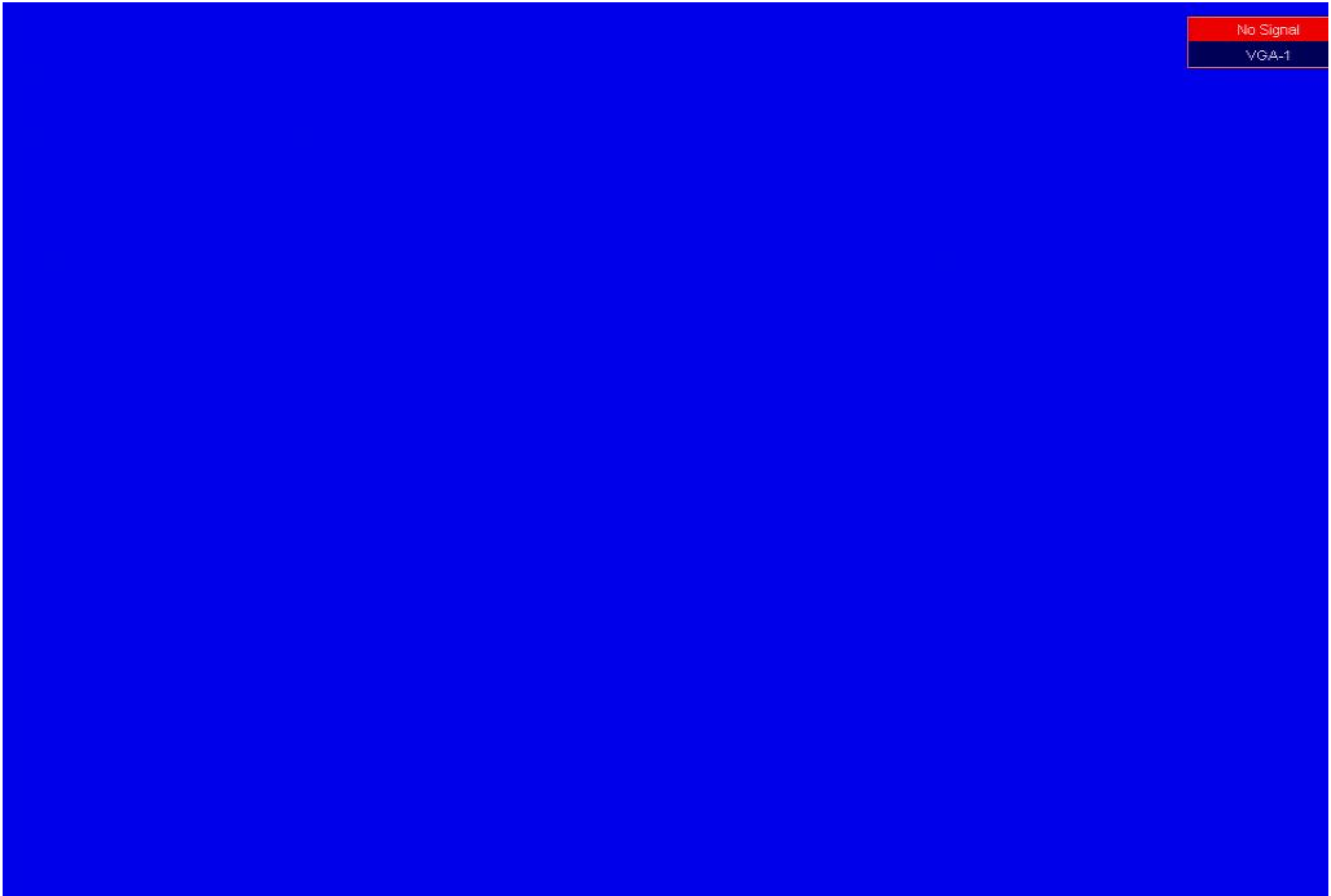
With the geometric way, one is thinking in terms of pictures; pictures which one imagines in space in some way, and one just tries to get a feeling for the relationships between the quantities occurring in those pictures.

Now, a good mathematician has to be a master of both of those ways of thinking, but even so they will have a preference for one way or the other; I don't think they can avoid it.

In my own case, my preference is especially for the geometrical way.

. . .

The methods of projective geometry are the most powerful ones.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