Title: Multiloop Gluon Amplitudes and AdS/CFT

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Abstract: Some recent investigations into the structure of the AdS/CFT correspondence rely on input from increasingly complicated technical calculations. Two related examples in planar N=4 super Yang-Mills theory include testing consequences of integrability and exploring iteration relations amongst multiloop gluon scattering amplitudes. I will review the latest developments in these areas and the methods used to carry out relevant calculations through four loops.

Multiloop Gluon Amplitudes

and

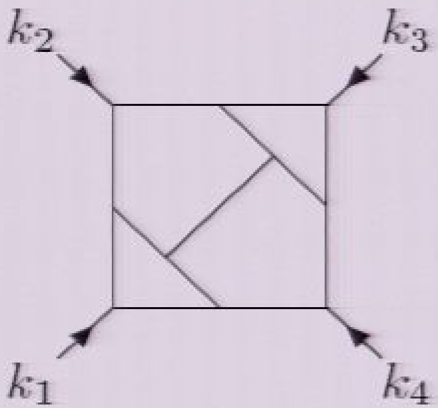
AdS/CFT

Marcus Spradlin

Brown University

In collaboration with F. Cachazo and A. Volovich

Snapshot



The meat of this work is a very efficient new algorithm for extracting certain quantities (the cusp anomalous dimension) from L-loop gluon amplitudes in $\mathcal{N}=4$ super-Yang Mills.

The method does have wider applicability, but our interest in these particular calculations stems from their important impact on studies of integrability and iteration relations in $\mathcal{N}=4$ super-Yang Mills. Concrete calculations are needed to test various conjectures and to shed light on hidden structure.

I will however begin at the beginning...

Introduction: Yang-Mills Theory

Of course there are many reasons to be interested in YM theory.

- The unique theory of interacting vector bosons
- A great deal of interesting mathematics
- And, of course, QCD and the 'Real World'!

The journey towards an analytic solution of this important and rich theory has been long and profitable.

Like in many areas of physics, if we can't solve the theory we're most interested in, we look for a simpler, similar model that we can solve!

This leads us to consider the $\mathcal{N} = 4$ supersymmetric version of the theory, which has even richer mathematical structure and is of course is of great independent interest since it is a theory of quantum gravity.

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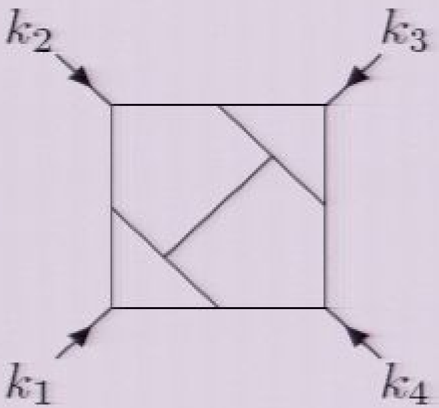

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Motivation

The motivation for our work was two-fold

- To unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and
- To exploit that structure to help simplify otherwise formidable computations.

Gluon Scattering Amplitudes in $\mathcal{N} = 4$ SYM

Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy, too many terms, hide structure.

Much interest in and progress on the calculation of tree-level amplitude calculations was stimulated by twistor sring theory. [Witten]

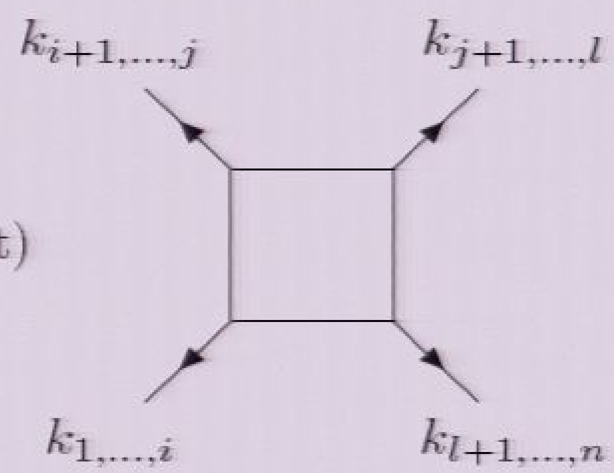
In fact, within a period of less than two years, the problem of calculating closed-form expressions for tree-level scattering amplitudes went from possible only in certain special cases to essentially completely solved.

[Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng, Witten] [Roiban, MS, Volovich] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Badger, Glover, Khoze]

One-Loop Amplitudes in $\mathcal{N} = 4$ SYM

In the $\mathcal{N} = 4$ theory, all one-loop integrals which appear in any Feynman diagram calculation can be reduced to a set of scalar box integrals using Passarino-Veltman reduction. Therefore scalar box integrals provide a complete basis for all one-loop amplitudes in $\mathcal{N} = 4$ [Bern, Dixon, Kosower].

$$\mathcal{A}^{1-\text{loop}} = \sum_{\text{boxes}} (\text{coefficient})$$

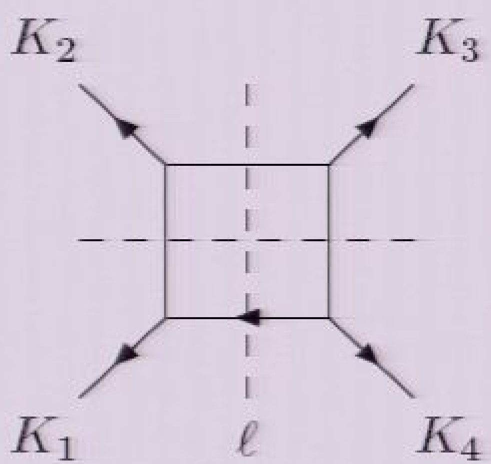


All one needs to calculate are the coefficients for a desired amplitude.

Unitarity-Based Methods

Any supersymmetric one-loop amplitude is completely determined by its branch cuts and discontinuities [Bern, Dixon, Dunbar, Kosower]. Therefore, it is natural to use unitarity cuts to compute these coefficients $\Longrightarrow$ 'unitarity based method' [Bern, Dixon, Kosower].

Each scalar box integral has has a unique leading singularity (though one has to use complex momenta to see it!), and the discontinuity of any desired amplitude across this singularity is given by a quadruple cut. [Britto, Cachazo, Feng]



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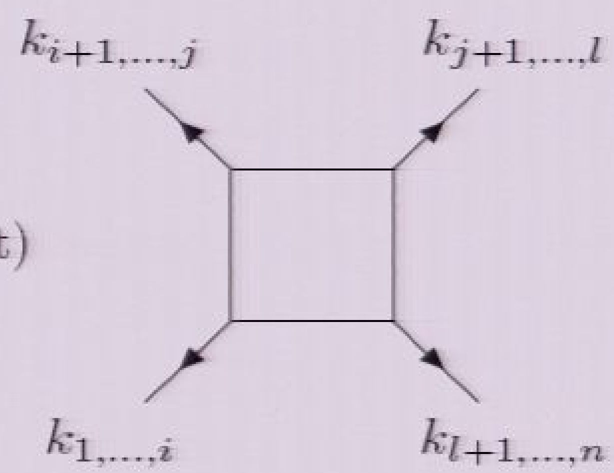

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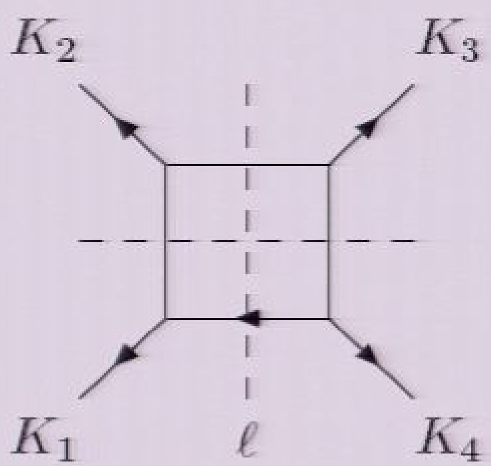

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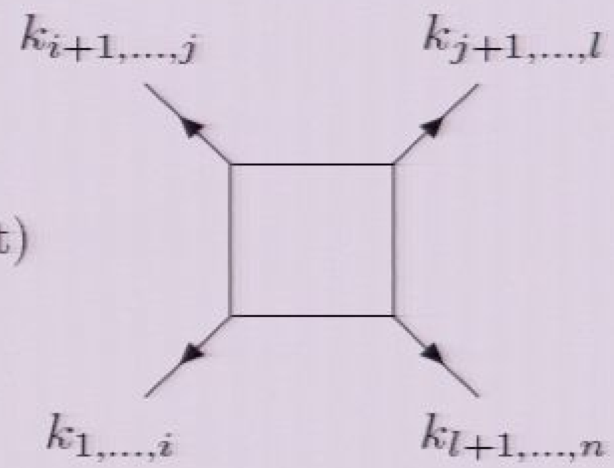

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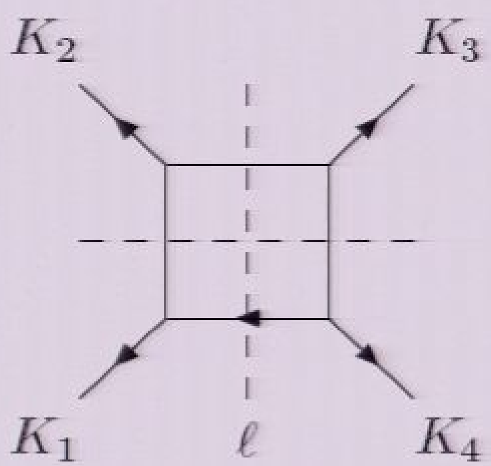

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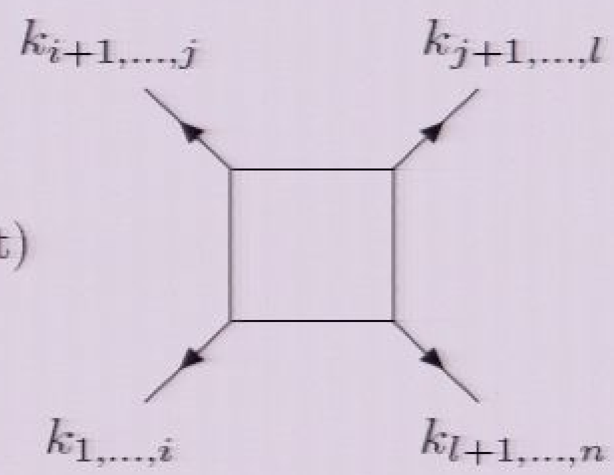

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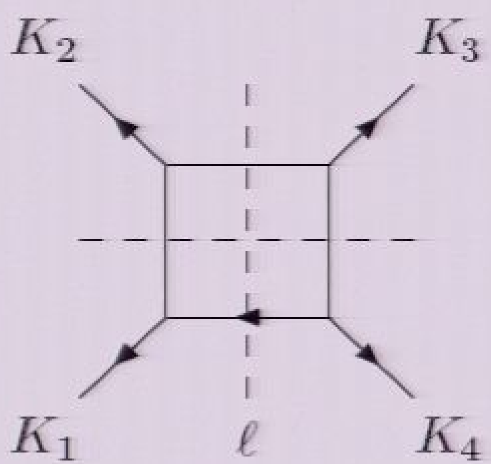

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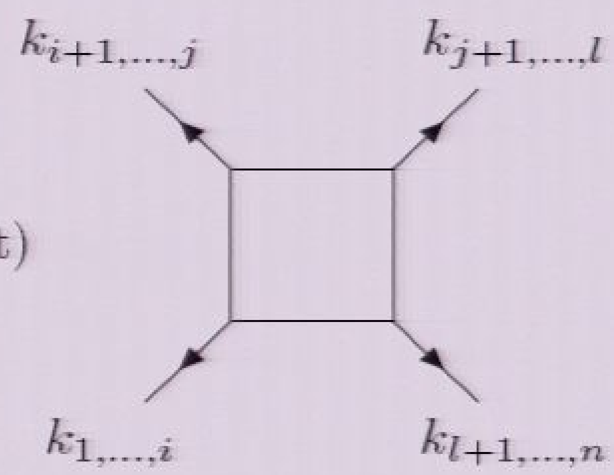

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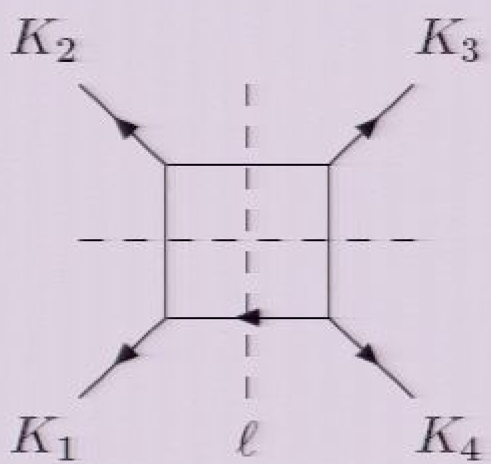

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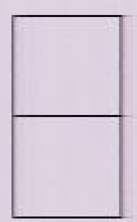
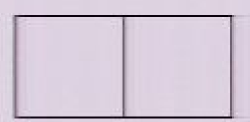
Unitarity-Based Methods at Higher Loops

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [Cachazo, Buchbinder] [Bern, Dixon, Smirnov] [Bern, Carrasco, Johansson, Kosower]

The first step in calculating an L-loop amplitude is to express it in terms of a (hopefully) small number of relatively simple scalar integrals. For example, the two-loop four-particle amplitude is given by [Bern, Rozowsky, Yan]

$\mathcal{A}/\mathcal{A}_{\text{tree}} \equiv M =$  $+$ 

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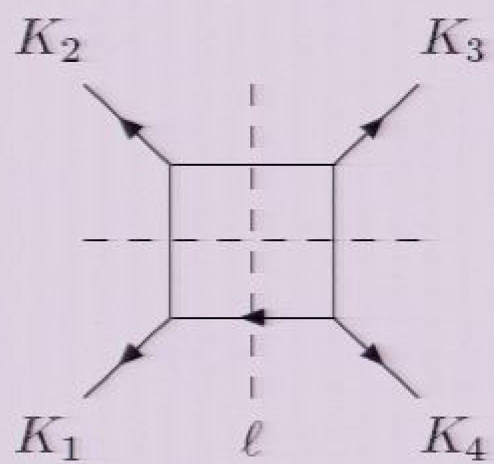

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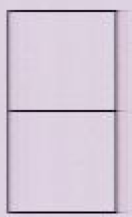
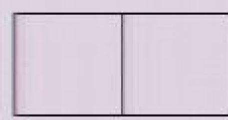
$\mathcal{A}/\mathcal{A}_{\text{tree}} \equiv M =$  $+$ 

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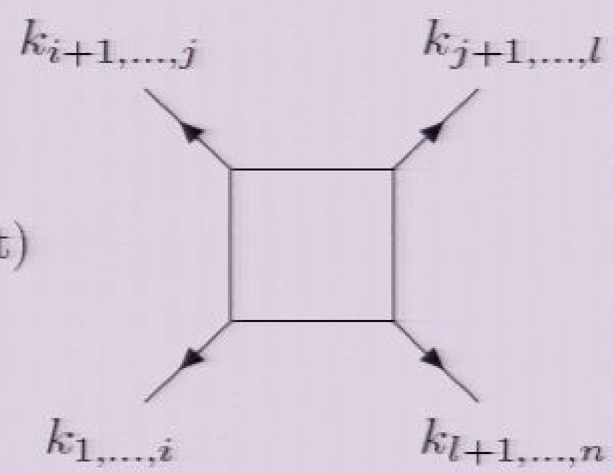

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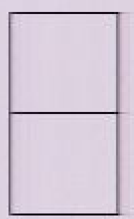
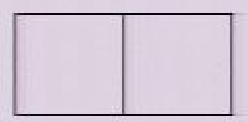
$\mathcal{A}/\mathcal{A}_{\text{tree}} \equiv M =$  $+$ 

Beyond one loop it is in general very difficult to determine which integrals contribute to any particular amplitude.

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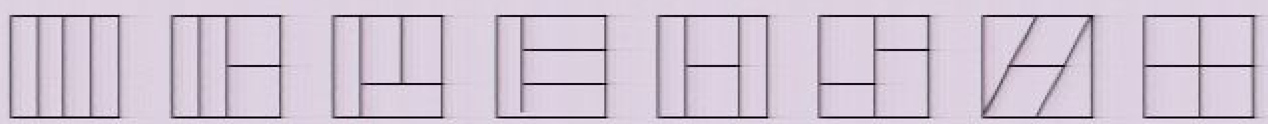
$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q+k_4)^2 (q+k_3+k_4)^2 (p-q)^2}.$$

The four-loop amplitude is equal to the sum of 8 integrals:



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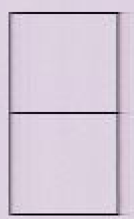

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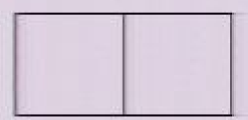

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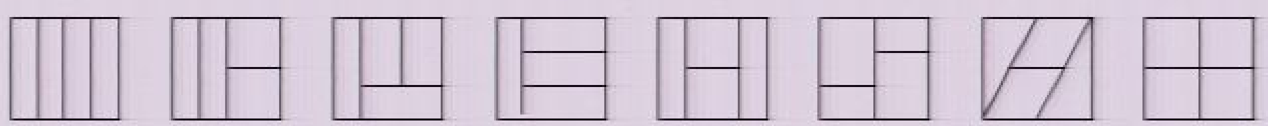

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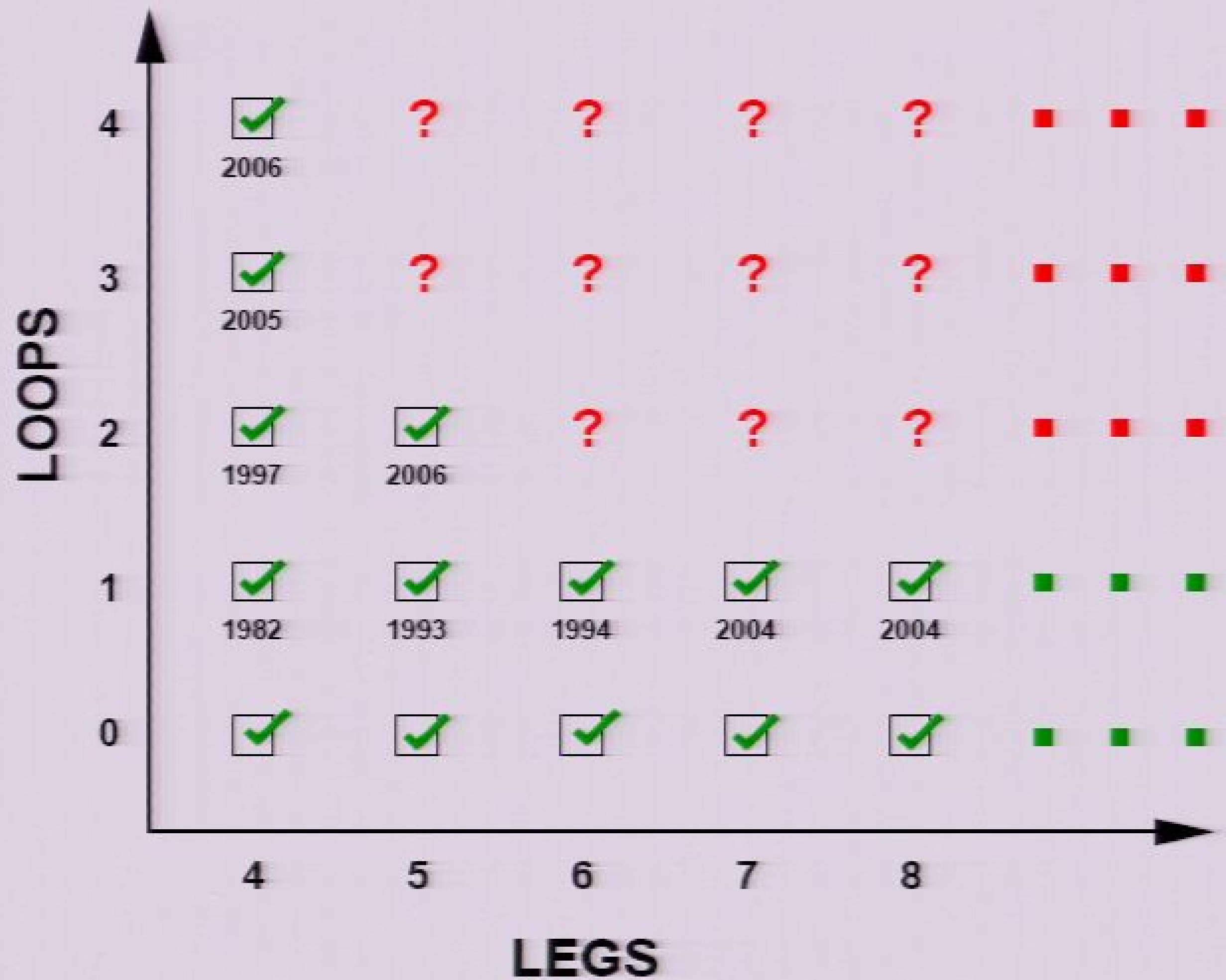

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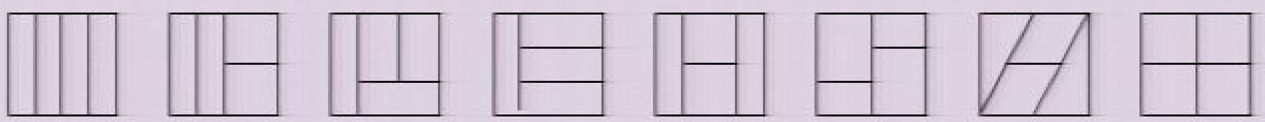

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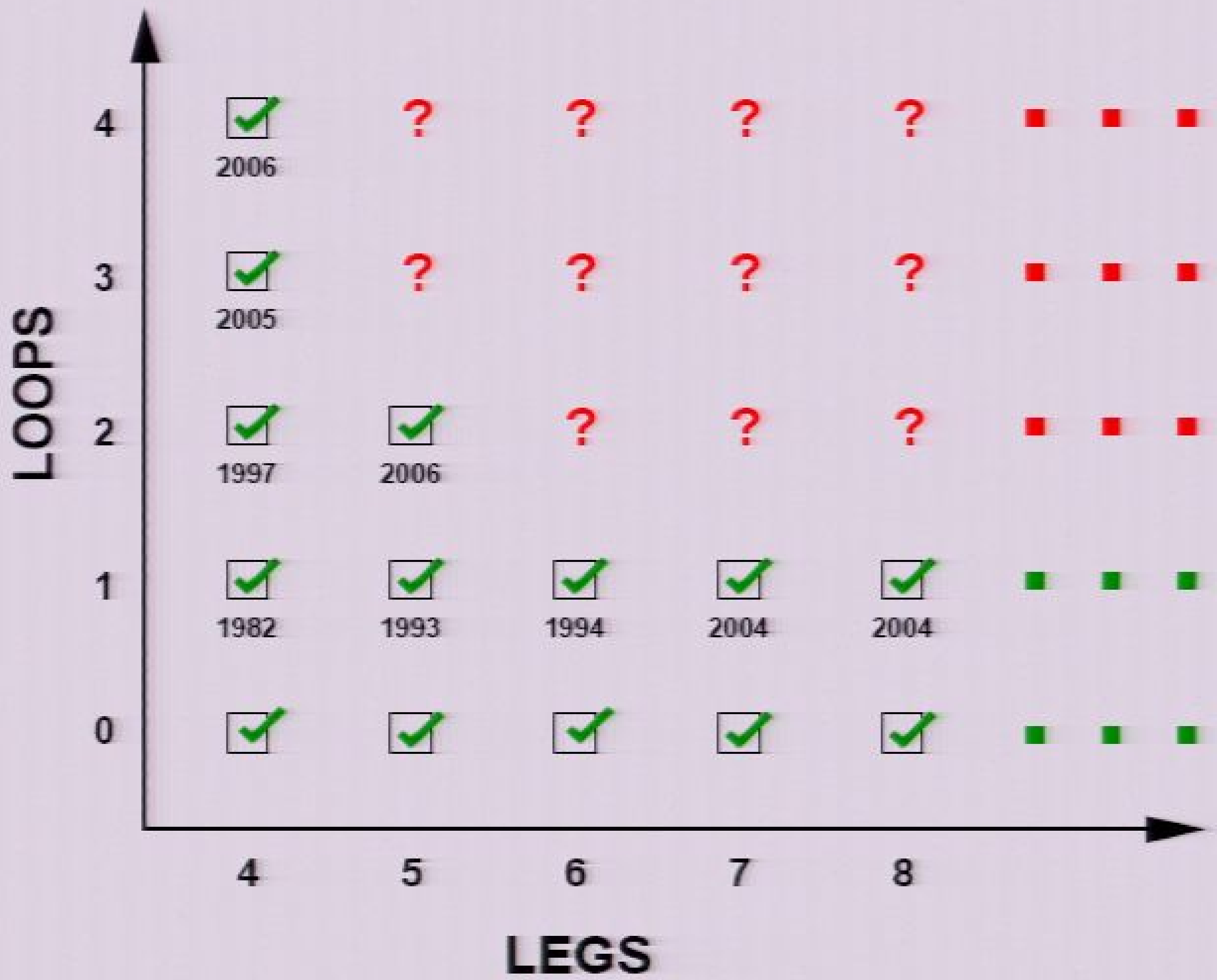

Universal Infrared Behavior of Loop Amplitudes

Resummation work by Sterman and Tejeda-Yeomans , and infrared singularity work by Catani, shows that in dimensional regularization to $D = 4 - 2\epsilon$, planar n-particle L loop MHV amplitudes satisfy iterative relations of the form

$$M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \ldots, M_n^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_n^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)\,,$$

where $P^{(L)}$ are some known polynomials.

The quantities $f^{(L)}$ and $g^{(L)}$ are the L-loop terms in the functions

$$f(\lambda) = \sum_{L=1}^{\infty} f^{(L)} \lambda^L, \qquad g(\lambda) = \sum_{L=1}^{\infty} g^{(L)} \lambda^L$$

respectively called the cusp anomalous dimension and collinear anomalous dimension. These two functions capture all information about the infrared singularities.

The Cusp Anomalous Dimension

The cusp anomalous dimension

$$f(\lambda) = 4\lambda - 4\zeta(2)\lambda^2 + (4\zeta(2)^2 + 12\zeta(4))\lambda^3 + \mathcal{O}(\lambda^4)$$

governs the behavior of twist-two operators in the limit of very large spin:

$$\Delta\left(\mathrm{Tr}[ZD^S Z]\right) = S + f(\lambda)\log S + \mathcal{O}(S^0), \qquad S \gg 1.$$

This quantity has long played an important role in quantitative checks of AdS/CFT: Gubser, Klebanov, Polyakov identified a certain string state in $AdS_5 \times S^5$ whose energy is $f(\lambda)$ thereby providing a prediction for the strong coupling behavior of this function.

Recently there has been much work on the apparent integrability of planar $\mathcal{N} = 4$ Yang-Mills, culminating in an exact prediction for $f(\lambda)$[Beisert, Eden, Staudacher].

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Targets: $f(\lambda)$ and $g(\lambda)$

Much less is known about $g(\lambda)$; I'll mention its AdS/CFT prediction later...

The one-loop four-particle amplitude takes the form

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{\log(st)}{\epsilon} - \log s \log t + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)$$

From the relation

$$M_4^{(L)}(\epsilon) = P^{(L)}(M_4^{(1)}(\epsilon), \ldots, M_4^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_4^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)$$

we see that we can read off the L loop contribution to $f(\lambda)$ and $g(\lambda)$ from the $1/\epsilon^2$ and $1/\epsilon$ singularities in the L loop amplitude.

Our interest in exploring the hidden structure in these amplitudes was partly motivated by the desire to develop an efficient algorithm for computing these quantities, which I will now briefly describe.

Preliminary Comments

- We consider the L-loop four-gluon amplitude in $D = 4 - 2\epsilon$.
- Supersymmetry determines the helicity structure of the amplitude to be the same as that of the tree-level amplitude.
- The ratio $M^{(L)} = A^{(L)}/A^{(0)}$ is therefore a function only of ϵ and the Mandelstam variables s, t.
- By crossing symmetry, the amplitude is symmetric under $s \leftrightarrow t$.
- The amplitude has dimensions of $M^{(L)} \sim [\mathrm{length}]^{2\epsilon L}$.
- Therefore, it can be written as

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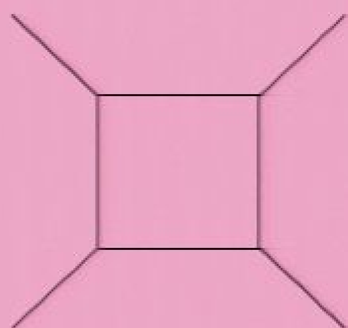

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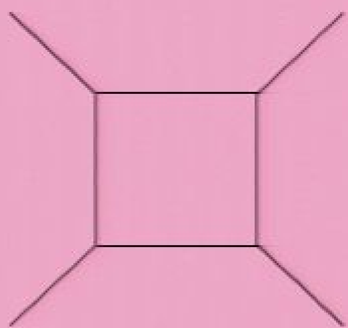

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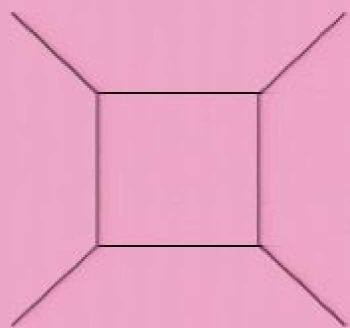

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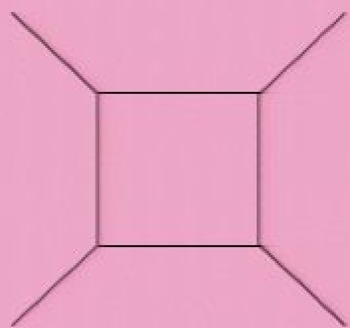

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Preliminary Comments

- We consider the L-loop four-gluon amplitude in $D = 4 - 2\epsilon$.
- Supersymmetry determines the helicity structure of the amplitude to be the same as that of the tree-level amplitude.
- The ratio $M^{(L)} = A^{(L)}/A^{(0)}$ is therefore a function only of ϵ and the Mandelstam variables s, t.
- By crossing symmetry, the amplitude is symmetric under $s \leftrightarrow t$.
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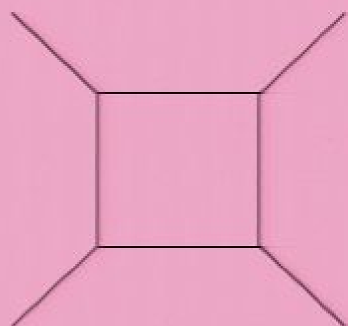

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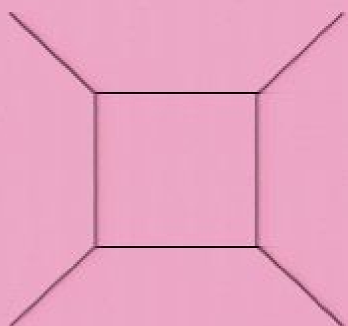

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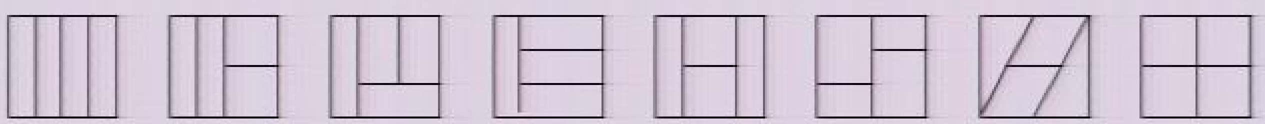
Beyond one loop it is in general very difficult to determine which integrals contribute to any particular amplitude.

We call this step ‘finding the integrand’. For example, the two-loop amplitude on the previous slide is

$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q+k_4)^2 (q+k_3+k_4)^2 (p-q)^2}.$$

The four-loop amplitude is equal to the sum of 8 integrals:



[Bern, Czakon, Dixon, Kosower, Smirnov]

But unitarity doesnt offer much help with evaluating these nasty integrals!

$\mathcal{N} = 4$ Yang-Mills Status Report

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The iterative relations imply that one has to sift all the way through to the ϵ^{-2} in order to find any 'new' information—the vast majority of the rational coefficients which specify the L-loop amplitude are completely determined in terms of lower loop amplitudes.

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The one unfixed number at order ϵ^{-2} is $f^{(L)}$. This quantity of particular interest to us; in fact it is the L-loop contribution to the cusp anomalous dimension.

Targets: $f(\lambda)$ and $g(\lambda)$

Much less is known about $g(\lambda)$; I'll mention its AdS/CFT prediction later...

The one-loop four-particle amplitude takes the form

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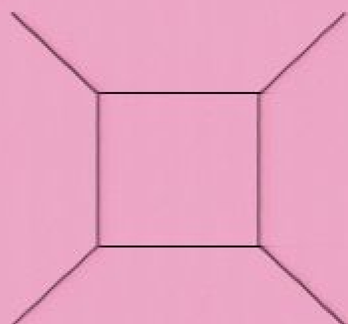

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For example, it would be great if there were a procedure to isolate those integrals which contribute to any particular coefficient of interest, say

$$\epsilon^{4-2L} H_{0101}(-x)$$

and that would enable us to calculate the rational number multiplying any given term without calculating everything else.

This is probably too much to hope for, but there is an efficient algorithm for reading off the coefficient of any term of the form

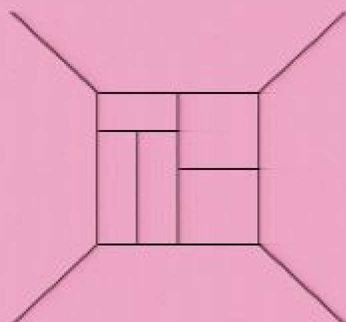
$$\epsilon^i \log^k x$$

These are precisely the terms touched by the cusp anoamlous dimension...

Some New Loop Technology

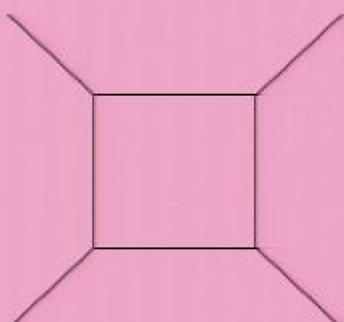
STEP 1. We observe that any dimensionally regulated L-loop four-particle Feynman integral can be written in the form (Mellin-Barnes representation)

[Smirnov, Tausk, Czakon, ...]



$$= \int_{-i\infty}^{+i\infty} dy\, x^y F(y,\epsilon), \qquad F(y,\epsilon) = F(-y,\epsilon)$$

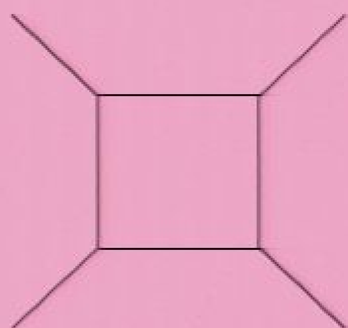
for some function $F(y,\epsilon)$, which is relatively easy to determine. As an example (not representative, because of its simplicity):



$$\Longrightarrow F(y,\epsilon) = \Gamma(1+\tfrac{1}{2}\epsilon+y)\Gamma^2(y-\tfrac{1}{2}\epsilon)\Gamma^2(-y-\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon-y).$$

The final integral over y is the really nasty one.

- Amplitudes are almost always studied in an expansion around $\epsilon = 0$.
- The leading singularity is ϵ^{-2L}, with higher order terms in the ϵ expansion becoming more and more complicated. For example at one-loop



$$= -\frac{2}{\epsilon^2} + \left[\frac{1}{4}L^2 + 4\zeta(2)\right] + \epsilon\Big[- H_{001}(-x) + LH_{01}(-x)$$
$$-\frac{1}{2}L^2H_1(-x) - 3\zeta(2)H_1(-x) - \frac{3}{2}\zeta(2)L - \frac{L^3}{12} + \frac{17\zeta(3)}{3}\Big]$$
$$+\epsilon^2\Big[H_{0001}(-x) + H_{0011}(-x) + H_{0101}(-x) + H_{1001}(-x) - \frac{1}{2}LH_{001}(-x)$$
$$-LH_{011}(-x) - LH_{101}(-x) + \frac{L^2}{2}H_{11}(-x) + 3\zeta(2)H_{11}(-x) + \frac{L^3}{12}H_1(-x)$$
$$-\zeta(3)H_1(-x) + \frac{3}{2}\zeta(2)LH_1(-x) + \frac{L^4}{64} + \frac{\zeta(2)}{24}L^2 - \frac{\zeta(3)}{2}L + \frac{41\pi^4}{720}\Big] + \cdots$$

where $L = \ln x$. Adapted from Bern,Dixon,Smirnov.

Some New Loop Technology

STEP 1. Any four-particle integral $= \int dy\, x^y F(y,\epsilon)$.

STEP 2. It we want to study some iterative equation, it is clearly tempting to try to collect all of the terms appearing in some relation inside one y integral, and then expand through $\mathcal{O}(\epsilon)$ under the y integral.

This is not possible, because $F(y,\epsilon)$ has poles which collide with the integration contour $\mathrm{Re}(y) = 0$ at $\epsilon = 0$, e.g.

$$F(y,\epsilon) = \Gamma(1 + \tfrac{1}{2}\epsilon + y)\Gamma^2(y - \tfrac{1}{2}\epsilon)\Gamma^2(-y - \tfrac{1}{2}\epsilon)\Gamma(1 - \tfrac{1}{2}\epsilon - y).$$

This signals that expanding in ϵ and performing the y integral do not commute—we are not allowed to expand in ϵ under the integral. We call these annoying poles obstructions because they obstruct our ability to collect everything under a single integral which is valid near $\epsilon = 0$.

Obstructions

The singularities which appear at $y = 0$ in the $\epsilon \to 0$ limit can be isolated by using the formula

$$\lim_{\epsilon \to 0} \frac{1}{y \pm \epsilon} = \mathcal{P}\frac{1}{y} \pm \pi\delta(y).$$

(and its derivatives). This leads to a simple, unique decomposition of any amplitude into

$$\int_{-i\infty}^{+i\infty} dy\, x^y \left[\mathcal{P}F(y) + G(y)\right],$$

where the first term is nonsginular and the obstruction terms $G(y)$ are given by a polynomial in derivatives acting on $\delta(y)$. If we note that

$$(\ln^2 x)^k = \int_{-i\infty}^{+i\infty} dy\, x^y \frac{d^{2k}}{dy^{2k}}\delta(y),$$

then we see that, in x space, obstructions are always polynomial in $\ln^2 x$ (they must be even in $\ln x$ because of the $x \to 1/x$ symmetry.

Product Algebra Structure

Thus, there is a canonical way to write any amplitude as a sum of two pieces:

$$\begin{aligned} \text{obstructions} &\leftrightarrow \text{singular part of the amplitude's Mellin transform} \\ &\leftrightarrow P(\ln^2 x) \text{ contributions to the amplitude} \\ \text{bulk term} &\leftrightarrow \text{smooth part of the Mellin transform at } y = 0 \\ &\leftrightarrow \text{harmonic polylogs in the amplitude.} \end{aligned}$$

One useful aspect of this decomposition is that

$$\begin{aligned} \text{obstruction} \times \text{obstruction} &= \text{obstruction} \\ \text{obstruction} \times \text{bulk term} &= \text{bulk term} \\ \text{bulk term} \times \text{bulk term} &= \text{bulk term} \end{aligned}$$

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Targets: $f(\lambda)$ and $g(\lambda)$

Much less is known about $g(\lambda)$; I'll mention its AdS/CFT prediction later...

The one-loop four-particle amplitude takes the form

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{\log(st)}{\epsilon} - \log s \log t + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)$$

From the relation

$$M_4^{(L)}(\epsilon) = P^{(L)}(M_4^{(1)}(\epsilon), \ldots, M_4^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_4^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)$$

we see that we can read off the L loop contribution to $f(\lambda)$ and $g(\lambda)$ from the $1/\epsilon^2$ and $1/\epsilon$ singularities in the L loop amplitude.

Our interest in exploring the hidden structure in these amplitudes was partly motivated by the desire to develop an efficient algorithm for computing these quantities, which I will now briefly describe.

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For example, the one, two and three loop obstructions are given by

$$
\begin{aligned}
M^{(1)} \sim\ & -\frac{2}{\epsilon^2} + \left[\frac{2\pi^2}{3} + \frac{\log^2 x}{4}\right] + \epsilon\left[\frac{17\zeta(3)}{3}\right] \\
& +\epsilon^2\left[\frac{41\pi^4}{720} + \frac{\pi^2\log^2 x}{24} + \frac{\log^4 x}{64}\right] \\
& +\epsilon^3\left[\frac{67\zeta(5)}{5} - \frac{59\pi^2\zeta(3)}{36} - \frac{11\zeta(3)\log^2 x}{24}\right] \\
& +\epsilon^4\left[-\frac{\pi^6}{4320} - \frac{70\zeta(3)^2}{9} - \frac{53\pi^4\log^2 x}{5760} + \frac{\log^6 x}{4608}\right] + \mathcal{O}(\epsilon^5),
\end{aligned}
$$

$$
\begin{aligned}
M^{(2)} \sim\ & \frac{2}{\epsilon^4} + \frac{1}{\epsilon^2}\left[-\frac{5\pi^2}{4} - \frac{\log^2 x}{2}\right] + \frac{1}{\epsilon}\left[-\frac{65\zeta(3)}{6}\right] + \left[-\frac{\pi^4}{90} + \frac{\pi^2\log^2 x}{24}\right] \\
& +\epsilon\left[-\frac{463\zeta(5)}{10} + \frac{77\pi^2\zeta(3)}{12} + \frac{25\zeta(3)\log^2 x}{12}\right] \\
& +\epsilon^2\left[-\frac{1999\pi^6}{30240} + \frac{95\zeta(3)^2}{18} + \frac{\pi^4\log^2 x}{42} + \frac{\pi^2\log^4 x}{32} + \frac{\log^6 x}{144}\right] + \mathcal{O}(\epsilon^3),
\end{aligned}
$$

$$
\begin{aligned}
M^{(3)} \quad \sim \quad & -\frac{4}{3\epsilon^6} + \frac{1}{\epsilon^4}\left[\frac{7\pi^2}{6} + \frac{\log^2 x}{2}\right] + \frac{1}{\epsilon^3}\left[\frac{31\zeta(3)}{3}\right] \\
& + \frac{1}{\epsilon^2}\left[-\frac{161\pi^4}{3240} - \frac{7\pi^2\log^2 x}{48} - \frac{\log^4 x}{32}\right] \\
& + \frac{1}{\epsilon}\left[\frac{967\zeta(5)}{15} - \frac{965\pi^2\zeta(3)}{108} - \frac{25\zeta(3)\log^2 x}{8}\right] \\
& + \left[\frac{244261\pi^6}{1632960} + \frac{107\zeta(3)^2}{18} - \frac{253\pi^4\log^2 x}{20160} - \frac{13\pi^2\log^4 x}{256} - \frac{3\log^6 x}{256}\right] \\
& + \mathcal{O}(\epsilon).
\end{aligned}
$$

It is straightforward to verify that these expressions satisfy the two-loop and three-loop iteration relations through order ϵ^0.

Summary

To summarize: In order to read off the L-loop cusp anomalous dimension from an L-loop four-gluon amplitude, we don't need to calculate the entire amplitude.

It is sufficient to start with the (relatively far simpler) expressions for the Mellin transform of the amplitude, and then just read off the coefficient of

$$\frac{\delta(y)}{\epsilon^2}$$

since only this particular coefficient contributes to the cusp anomalous dimension.

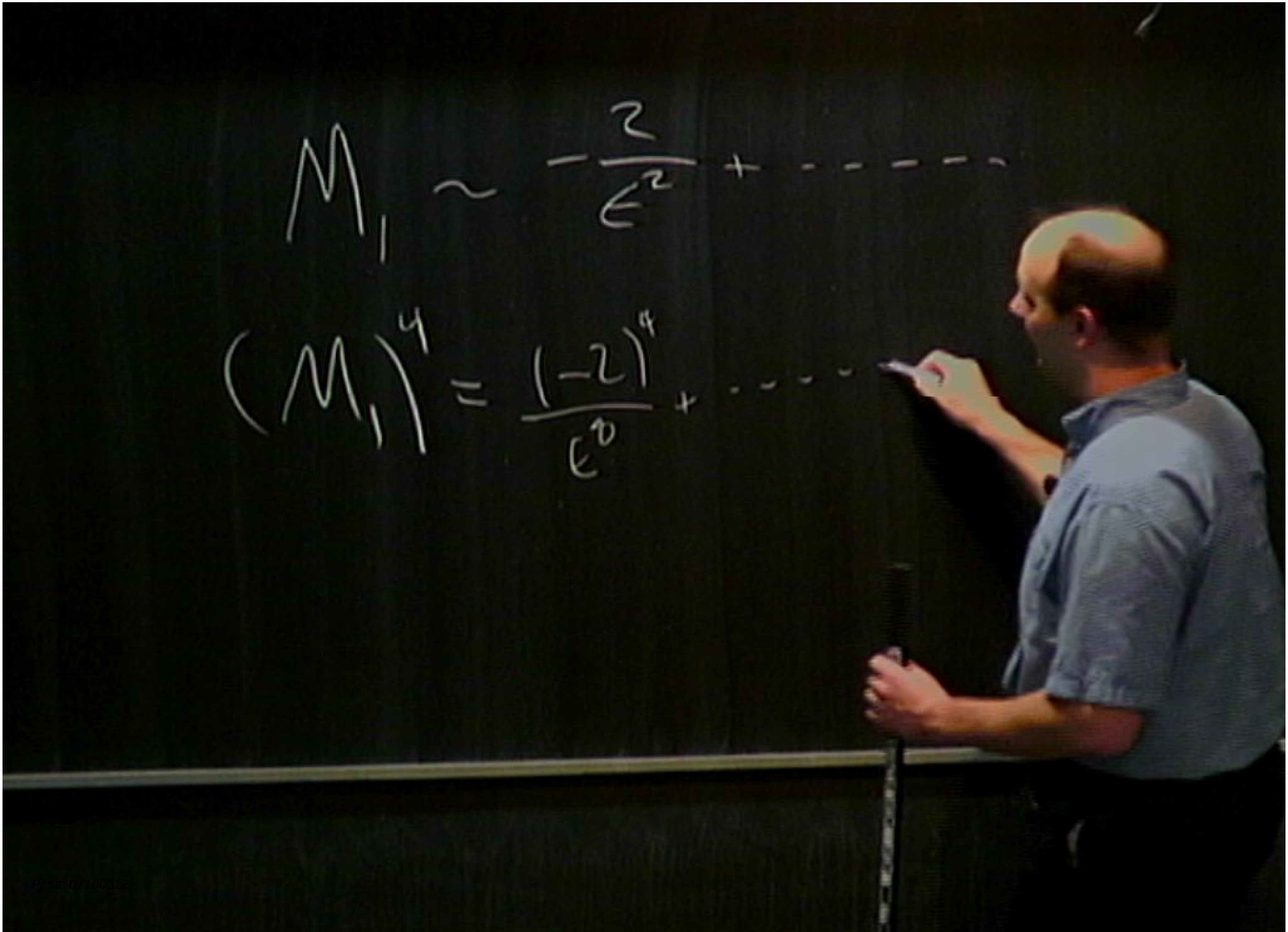
This algorithm is easily implemented in Mathematica (building on some code written by Czakon, and greatly optimizes the calculation of the cusp anomalous dimension.

Punchline

This method allows us direct access to the cusp anomalous dimension without having to first calculate both sides of the relation

$$M^{(4)} = \frac{1}{4}(M^{(1)})^4 - (M^{(1)})^2 M^{(2)} + M^{(1)} M^{(3)} + \frac{1}{2}(M^{(2)})^2 + \frac{1}{4} f^{(4)} M^{(1)}$$

as (complicated) functions of x, and then relying on delicate cancellations to expose the number $f^{(4)}$ that we are ultimately interested in.


$M_1 \sim -\frac{2}{\epsilon^2} + \cdots$
$(M_1)^4 = \frac{(-2)^4}{\epsilon^8} + \cdots$

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$$\frac{\quad}{\epsilon^2}$$

Results

As an application of our method we have obtained the four-loop cusp anomalous dimension

$$f^{(4)} = -117.1789 \pm 0.0002$$

in very good agreement with the BES conjecture

$$f^{(4)} = -(4\zeta(2)^3 + 24\zeta(2)\zeta(4) + 50\zeta(6) + 4\zeta(3)^2) = -117.1788285\ldots$$

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We also found the four-loop collinear anomalous dimension

$$g^{(4)} = -1240.9 \pm 0.3$$

[Thanks to SHARCNET!]

The ABDK Relations

Similar relations hold in any gauge theory.

However, it is has been conjectured that in $\mathcal{N} = 4$ Yang-Mills something special happens: the

$$+\mathcal{O}(\epsilon^0)$$

term in the iterative relation is believed to actually be

$$+C^{(L)} + \mathcal{O}(\epsilon^1)$$

where $C^{(L)}$ is a constant (independent of all of the gluon momenta)!

This conjecture has only been checked in three cases so far: 4 particles at 2 and 3 loops, and 5 particles at 2 loops [Anastasiou, Bern, Dixon, Kosower], [Bern, Dixon, Smirnov], [Cachazo, MS, Volovich], [Bern, Czakon, Kosower, Roiban, Smirnov]. It seems innocent but the consequence is profound...

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The BDS Ansatz

If true, then the all-loop, planar, four-particle amplitude sums up to

$$\log(\mathcal{A}/\mathcal{A}_{\rm tree}) = -\frac{f(\lambda)}{2\epsilon^2} - \frac{g(\lambda)}{\epsilon} + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \mathcal{O}(\epsilon^1)$$

where s, t are the usual Mandelstam invariants.

[A few inconsequential liberties have been taken in writing this equation.]

Recently, Alday and Maldacena have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling. For four particles, an explicit calculation yields precisely the structure shown above.

For more than four particles the calculation is more difficult but the singularity structure has been reproduced [Buchbinder] .

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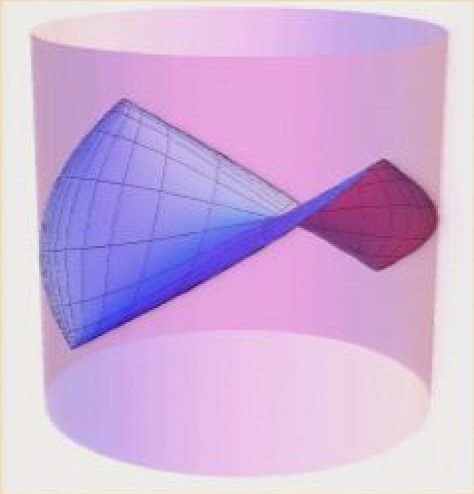
Their prescription is computationally equivalent to evaluating a certain Wilson loop composed of null line segments:



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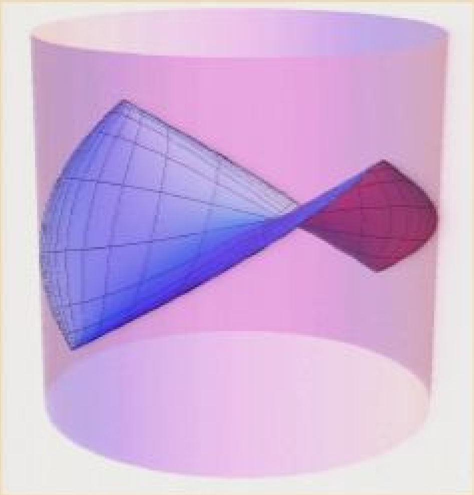

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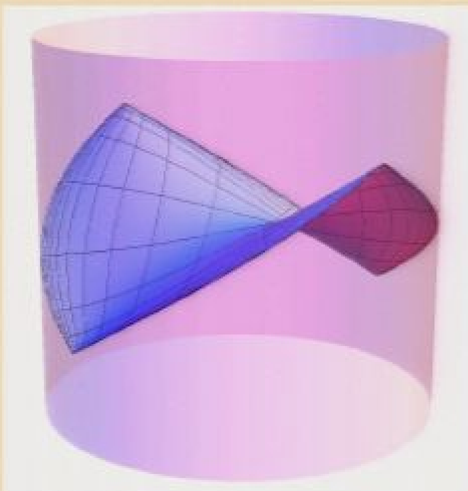

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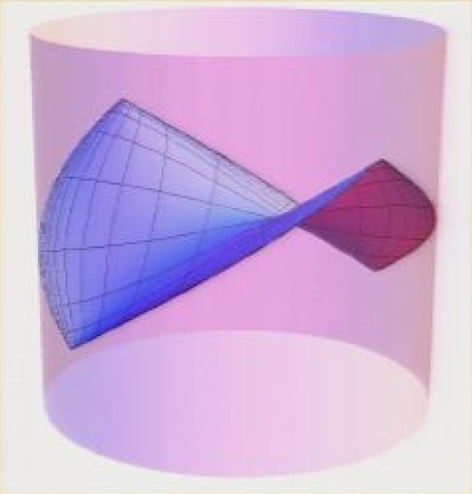

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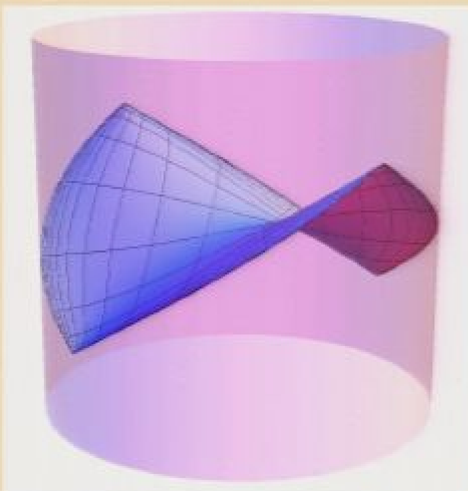

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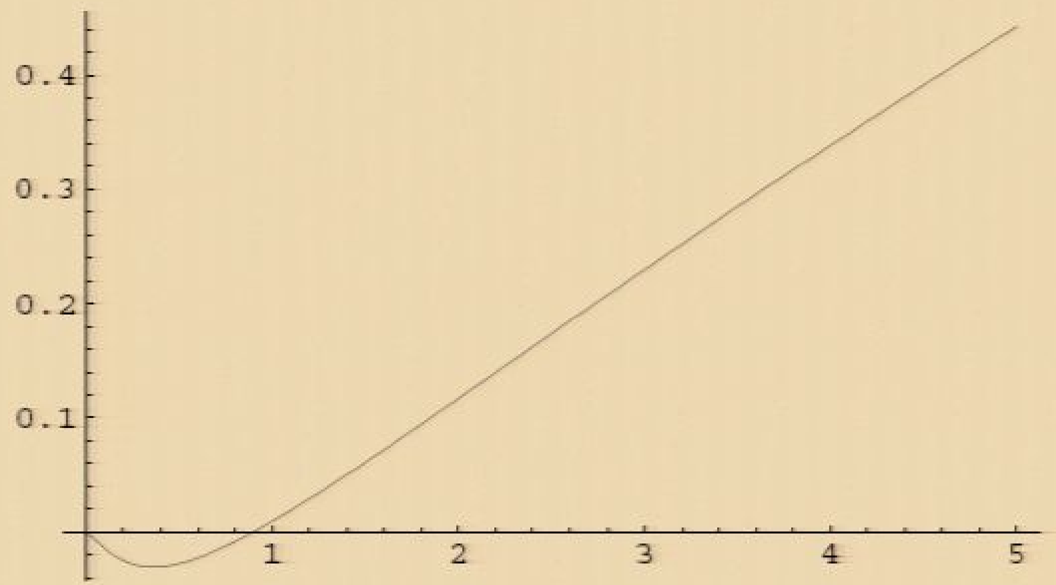
A Guess for What $g(\lambda)$ Might Look Like

A Padé approximant for $g(\lambda)$ based upon all available data (through four loops at weak coupling and leading order at strong coupling) looks like



It is an interesting open problem to relate $g(\lambda)$ to more familiar observables that one might compute from integrability, for example.

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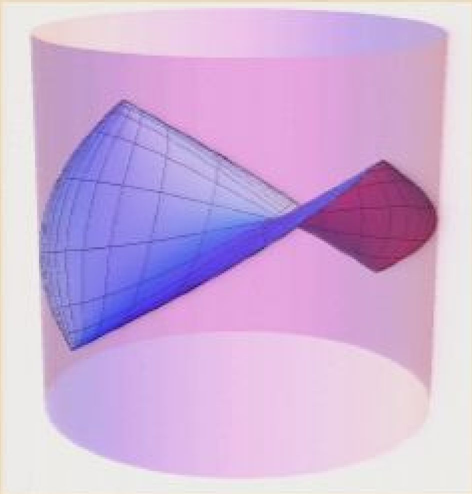

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If true, then the all-loop, planar, four-particle amplitude sums up to

$$\log(\mathcal{A}/\mathcal{A}_{\rm tree}) = -\frac{f(\lambda)}{2\epsilon^2} - \frac{g(\lambda)}{\epsilon} + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \mathcal{O}(\epsilon^1)$$

where s, t are the usual Mandelstam invariants.

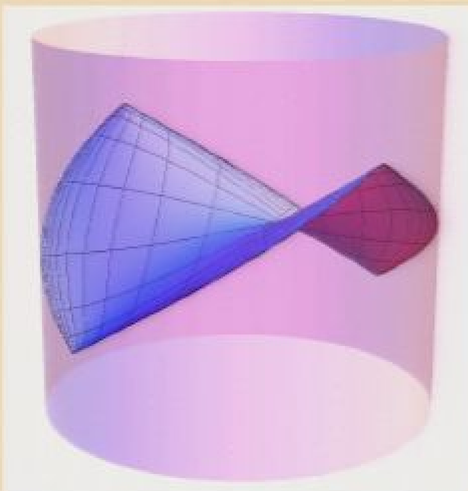
[A few inconsequential liberties have been taken in writing this equation.]

Recently, Alday and Maldacena have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling. For four particles, an explicit calculation yields precisely the structure shown above.

For more than four particles the calculation is more difficult but the singularity structure has been reproduced [Buchbinder] .

Alday-Maldacena

Their prescription is computationally equivalent to evaluating a certain Wilson loop composed of null line segments:

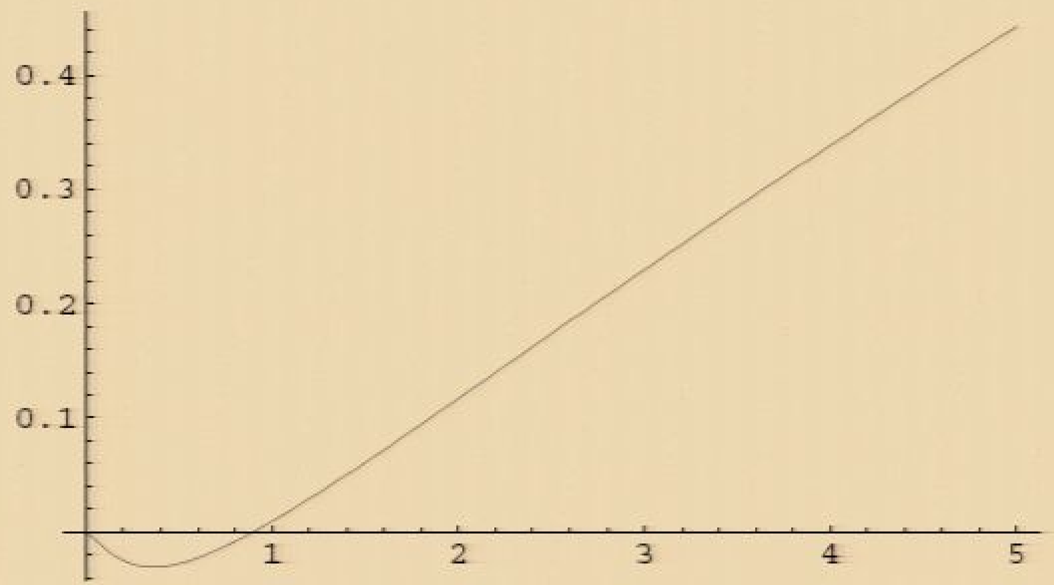


A byproduct of their calculation is the strong coupling prediction

$$g(\lambda) = 2(1 - \log 2)\sqrt{\frac{\lambda}{16\pi^2}} + \mathcal{O}(1)$$

A Guess for What $g(\lambda)$ Might Look Like

A Padé approximant for $g(\lambda)$ based upon all available data (through four loops at weak coupling and leading order at strong coupling) looks like



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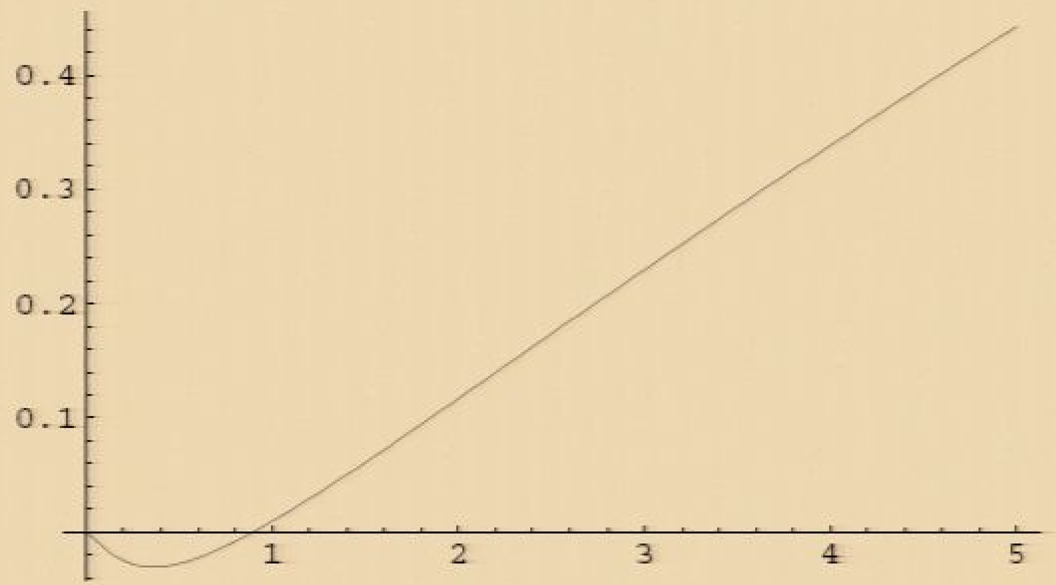

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$$M_1 \sim -\frac{2}{\epsilon^2} + \cdots\cdots$$

$$(M_1)^4 = \frac{(-2)^4}{\epsilon^8} + \cdots\cdots \frac{\quad}{\epsilon^2}$$

$$g \sim g + \partial f$$

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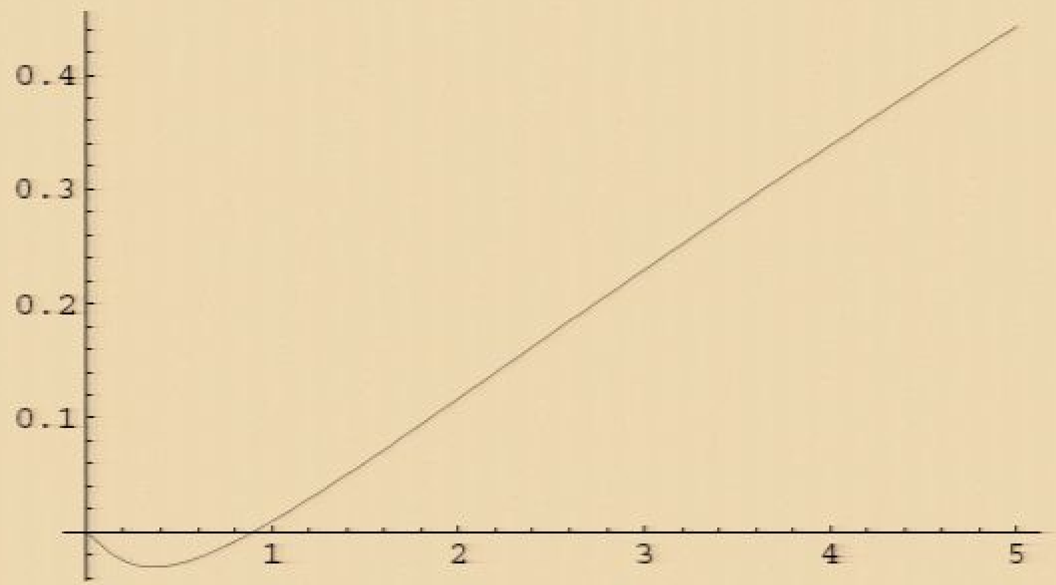

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Many Open Questions Remain

The Alday-Maldacena prescription reveals thatthe four-particle scattering amplitude is equal to a Wilson loop composed of null line segments, at least at strong coupling. Could this actually true at strong coupling? [Drummond et. al., Brandhuber et. al.]

An important role in this story is apparently played by a mysterious symmetry of planar scattering amplitudes called dual conformal symmetry. This symmetry is manifest in the Alday-Maldacena setup, but mysterious at weak coupling (though it has been shown to hold through five loops). It is partially responsible for fixing the form of the BDS ansatz.

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Obstructions

The singularities which appear at $y = 0$ in the $\epsilon \to 0$ limit can be isolated by using the formula

$$\lim_{\epsilon \to 0} \frac{1}{y \pm \epsilon} = \mathcal{P}\frac{1}{y} \pm \pi\delta(y).$$

(and its derivatives). This leads to a simple, unique decomposition of any amplitude into

$$\int_{-i\infty}^{+i\infty} dy\, x^y \left[\mathcal{P}F(y) + G(y)\right],$$

where the first term is nonsginular and the obstruction terms $G(y)$ are given by a polynomial in derivatives acting on $\delta(y)$. If we note that

$$(\ln^2 x)^k = \int_{-i\infty}^{+i\infty} dy\, x^y \frac{d^{2k}}{dy^{2k}}\delta(y),$$

then we see that, in x space, obstructions are always polynomial in $\ln^2 x$ (they must be even in $\ln x$ because of the $x \to 1/x$ symmetry.

Targets: $f(\lambda)$ and $g(\lambda)$

Much less is known about $g(\lambda)$; I'll mention its AdS/CFT prediction later...

The one-loop four-particle amplitude takes the form

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{\log(st)}{\epsilon} - \log s \log t + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)$$

From the relation

$$M_4^{(L)}(\epsilon) = P^{(L)}(M_4^{(1)}(\epsilon), \ldots, M_4^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_4^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)$$

we see that we can read off the L loop contribution to $f(\lambda)$ and $g(\lambda)$ from the $1/\epsilon^2$ and $1/\epsilon$ singularities in the L loop amplitude.

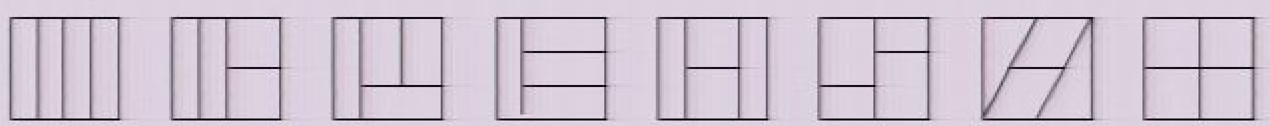
Our interest in exploring the hidden structure in these amplitudes was partly motivated by the desire to develop an efficient algorithm for computing these quantities, which I will now briefly describe.

Beyond one loop it is in general very difficult to determine which integrals contribute to any particular amplitude.

We call this step ‘finding the integrand’. For example, the two-loop amplitude on the previous slide is

$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q+k_4)^2 (q+k_3+k_4)^2 (p-q)^2}.$$

The four-loop amplitude is equal to the sum of 8 integrals:



[Bern, Czakon, Dixon, Kosower, Smirnov]

But unitarity doesnt offer much help with evaluating these nasty integrals!

Conclusion

We developed some techniques to aid in direct tests of the conjectured planar $\mathcal{N} = 4$ Yang-Mills S-matrix and multiloop iterative relations. As an application, we computed four-loop cusp and collinear anomalous dimensions.

The motivation behind this research is the desire to explore and uncover the rich mathematical structure underlying $\mathcal{N} = 4$ Yang-Mills theory.

Discovering such structures also has the pleasant side benefit of making previously difficult calculations much simpler.

Prospects are great for continued progress, both in supersymmetric gauge theories as well as QCD. There is definitely a lot more to learn and discover.

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