Title: Nonperturbative effects in matrix models and topological strings

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Abstract: It has been known for a long time that instanton effects control the large order behavior of the perturbation series in quantum mechanics and gauge theories. I present a study of this connection in the context of matrix models in 1/N-expansion and topological strings.
I will show how to compute the one-instanton corrections for a generic matrix model. Due to a recent matrix model inspired formalism for the topological string amplitudes on local Calabi-Yau manifolds, this can be used to compute nonperturbative effects in topological string theory and make predictions about the asymptotics of the string perturbation series. I discuss various cases where our predictions can be tested, yielding spectacular agreement with the asymptotics extracted by standard numerical methods.

Nonperturbative effects in Matrix Models and Topological Strings

Marlene Weiss

CERN & ETH Zurich

collaboration with M.Mariño, R. Schiappa
arXiv:0711.xxxx

16.10.2007

Outline

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Introduction

Topological strings

Consider the A-model on a Calabi-Yau X

$$F(Q, g_s) = \sum_{d,g} N_{d,g} Q^d g_s^{2g-2}, \quad Q = \mathrm{e}^{-t}$$

count worldsheet instantons

perturbative in Q, g_s

B-model on X_{mirror} → compute $F_g(Q)$ exactly in Q

...but can we go beyond perturbation theory in g_s?

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Large Order behavior & Nonperturbative effects

- QM, QFT: Standard relation between instanton effects and large-order behavior of the perturbation series
- Asymptotics of $\frac{1}{N}$-expansion of gauge theories controlled by nonperturbative corrections $\sim e^{-N}$

  $\updownarrow$

- D-brane instanton effects in string dual

[Alexandrov Kazakov Kutasov]

Applications to Topological String Theory

If the gauge theory has a string dual:

instanton effect in gauge theory $\leftrightarrow$ asymptotics of string amplitudes

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instanton effect in gauge theory ↔ asymptotics of string amplitudes

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  ☞ can be tested with asymptotics of string amplitudes!
- Information about analytic structure of topological string free energy
- Nontrivial check of conjectural dualities
- New conjectures about asymptotics of enumerative invariants

We consider:

- matrix models in double-scaling limit ↔ noncritical string theory
- matrix models off-criticality ↔ topological strings

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B-model on some local CYs $\overset{\text{large N dual}}{\longleftrightarrow}$ Matrix model

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- Identify analogue of spectral curve for the topological string theory in question →
- apply recursive matrix model formalism to generate topological string ampitudes
- no holomorphic ambiguity
- at large radius: mirror to topological vertex, but valid anywhere in moduli space

Instanton effects and Large Order behavior

A Quantum mechanics example

Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4}.$$

Take the perturbative expansion of the ground-state energy,

$$S_E(\lambda) = \sum_k E_k \lambda^k.$$

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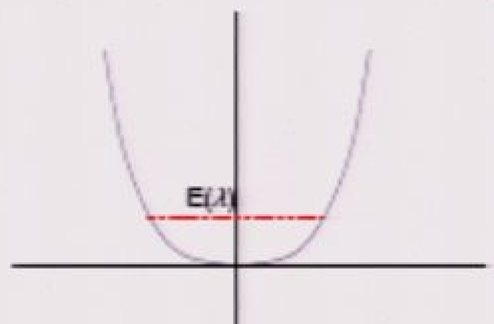

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→ We can deform the Cauchy representation to the dispersion relation

$$E_k = \frac{1}{2\pi i} \int_{-\infty}^{0} d\lambda' \frac{\mathrm{Disc}(E(\lambda'))}{\lambda'^{k+1}}$$

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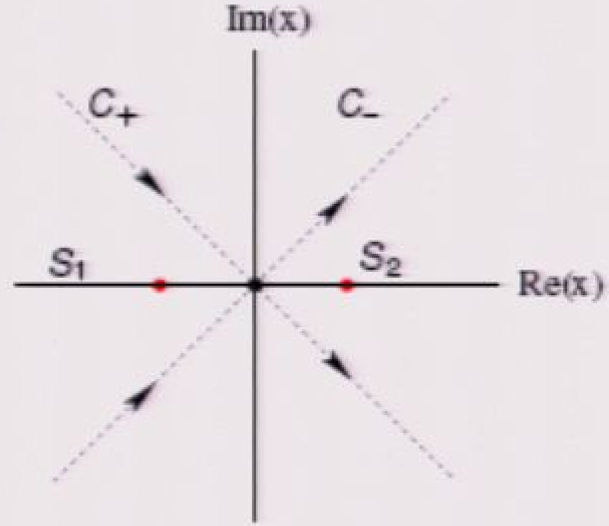

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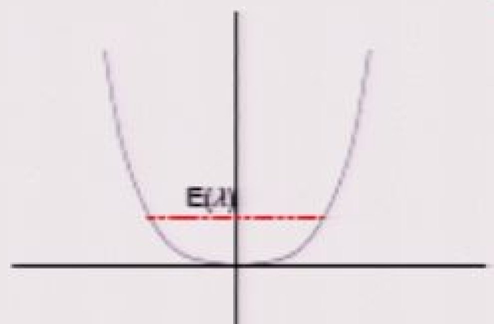

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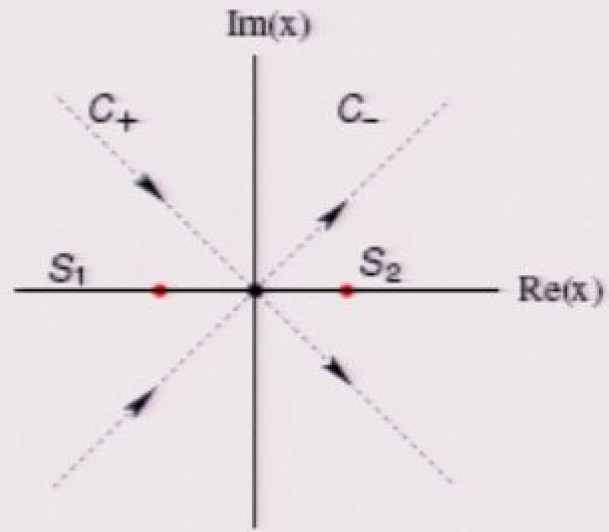

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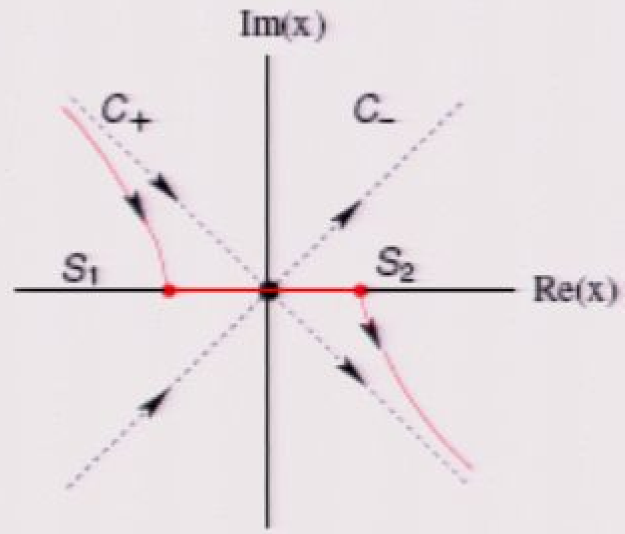

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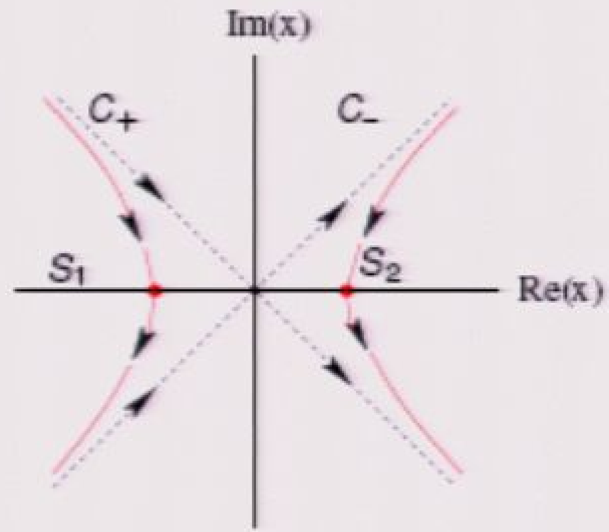

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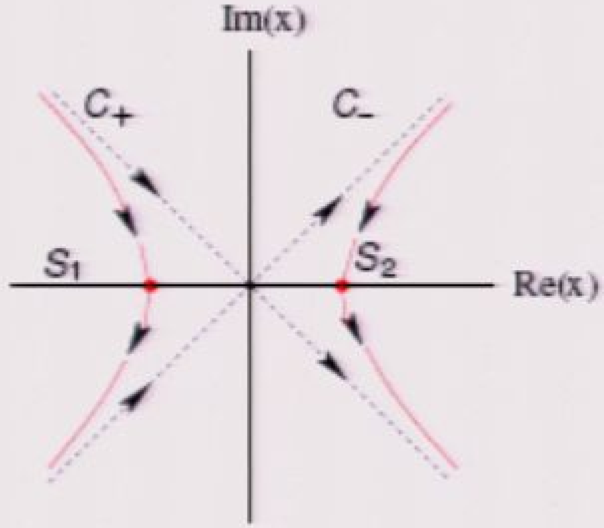
$$= i\mu_1 \lambda^{-b-1} e^{\frac{-\mathcal{A}_{inst}}{\lambda}} \left(1 + \lambda\mu_2 + O(\lambda^2)\right),$$

↑ 1-loop determinant



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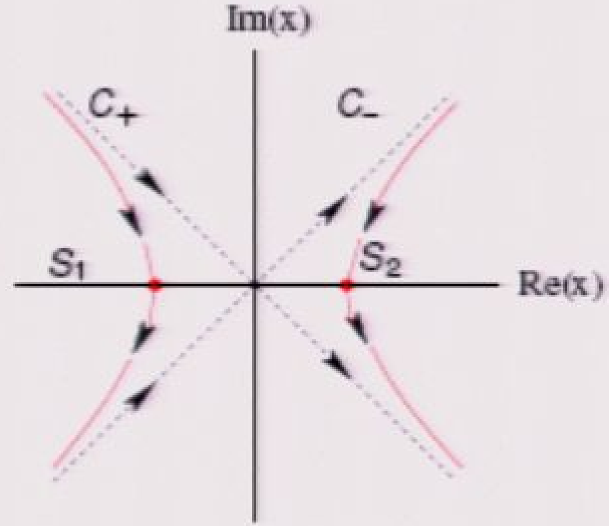
↑ saddle-point expansion



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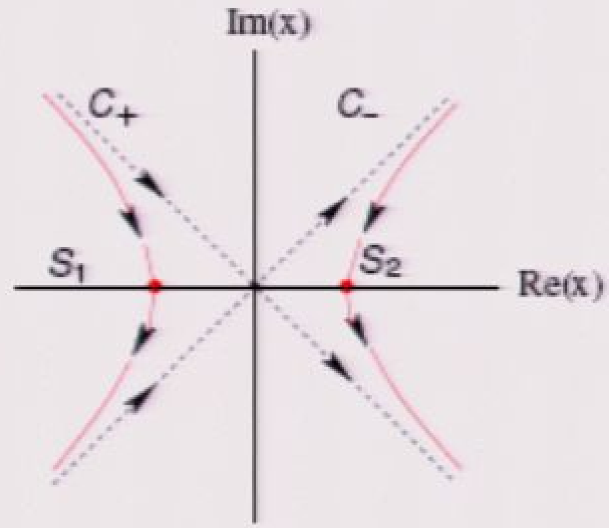

$$\mathcal{A}_{inst} = 2\int_0^{x_0}\sqrt{2V(x)}dx = -\frac{1}{3}$$ → action of tunneling-instanton

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Introduction and Motivation

Instantons & Large Order: The Anharmonic Oscillator

Matrix Models in $\frac{1}{N}$ Expansion

Examples

Conclusion and Outlook

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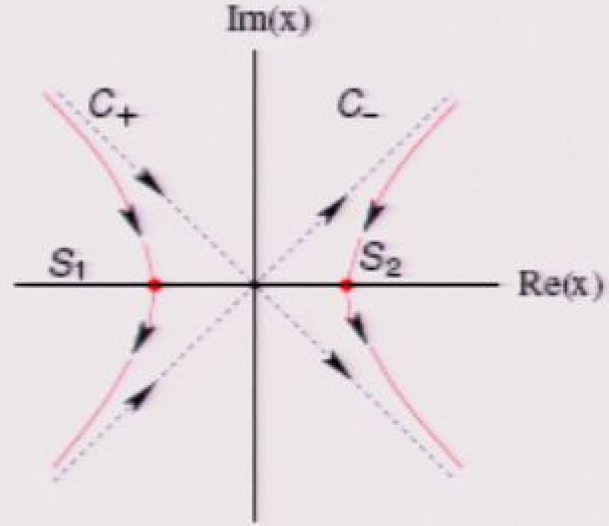

$$E_k \sim \frac{\mu_1}{2\pi}\mathcal{A}_{inst}^{-k-b}\Gamma(k+b)\left(1 + \frac{\mathcal{A}_{inst}}{(b+k-1)}\mu_2 + O\left(\frac{1}{k^2}\right)\right) \rightarrow$$

anharmonic oscillator: $E_k \sim (-1)^{k+1}\frac{\sqrt{6}}{\pi^{\frac{3}{2}}}3^k\Gamma(k+\frac{1}{2})$ [Bender Wu]

Matrix models in $1/N$ expansion

- Partition function

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int dM e^{-\frac{1}{g_s}\mathrm{Tr}V(M)} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dz_i}{2\pi} e^{N^2 V_{\mathrm{eff}}(z_i)}$$

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- effective potential $V_{\mathit{eff}}(z_i) = V(z_i) - 2\frac{t}{N}\sum_{i\neq j}\log|z_i - z_j| \rightarrow$ Coulomb repulsion $\rightarrow$ eigenvalues spread out over interval C
- The object we are interested in is the free energy;

$$F(t) = \sum_{g\geq 0} F_g(t) g_s^{2g-2}$$

where $t = g_s N$ is the 't Hooft parameter

- t fixed: expansion in $g_s \leftrightarrow$ expansion in $\frac{1}{N}$

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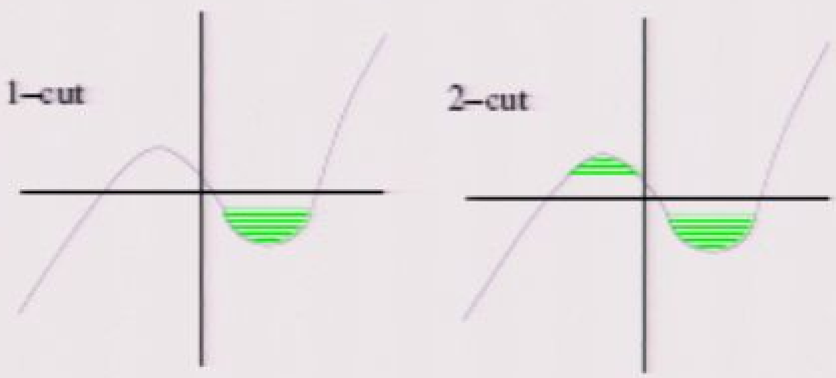

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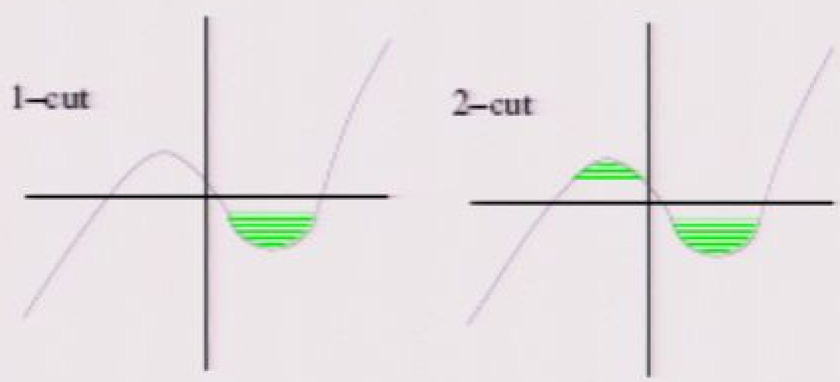

Here: Consider 1-cut case only

The planar solution

- When $N \to \infty$, the distribution of eigenvalues becomes continuous and one can write

$$V_{\text{eff}}(z) = V(z) - \frac{1}{\pi} \int \mathrm{Im}(y(z)) \log|z - z'| dz,$$

where $y(z)$ is the classical spectral curve of the matrix model

[Brézin Itzykson Parisi Zuber]

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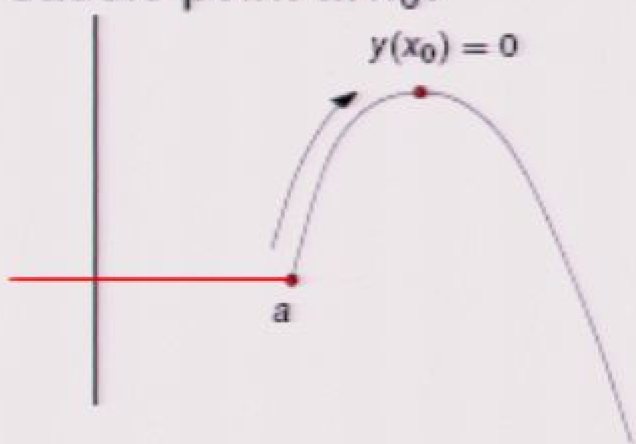
- The effective potential is constant along the cut and has a saddle point at x_0:



- Instanton configuration: an eigenvalue from the endpoint of the cut moves to the saddle of the effective potential barrier.

- The instanton action is

$$\mathcal{A}_{\text{inst}} = N \int_a^{x_0} y(z)dz$$

[David; Shenker]

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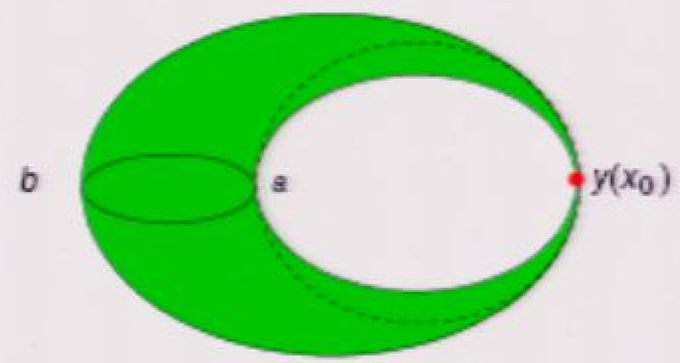
Geometrically: $\mathcal{A}_{\text{inst}}$ is a contour integral from endpoint of the cut to singularity of spectral curve



[Seiberg Shih]

Instanton analysis

We expect a relation instantons ↔ large-order analogous to the anharmonic oscillator:

$$F_g = \frac{1}{2\pi}\int_0^\infty ds \frac{\mathrm{Disc}(F(\sqrt{s}))}{s^{g+1}} = \mu_1 \frac{\mathcal{A}_{inst}^{-b-2g}}{\pi}\Gamma(2g+b)\left(1 + \frac{\mathcal{A}_{inst}}{2g+b-1}\mu_2 + O(\frac{1}{g^2}\right.$$

Matrix models in $1/N$ expansion

- Partition function

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int dM e^{-\frac{1}{g_s}\mathrm{Tr} V(M)} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dz_i}{2\pi} e^{N^2 V_{\mathrm{eff}}(z_i)}$$

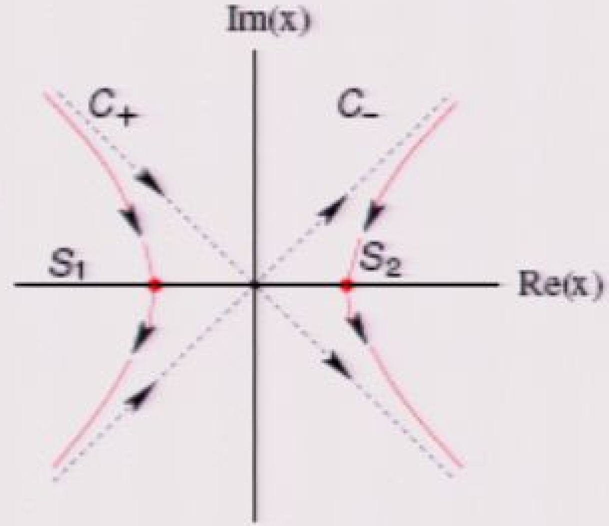
$$E_k = \frac{1}{2\pi i}\int_{-\infty}^{0} d\lambda \frac{\mathrm{Disc}(E(\lambda))}{\lambda^{k+1}}$$

- This result is rigorous and exact
- The perturbation coefficients are related to the lifetime of the state in the unstable potential with negative coupling ↔ instanton effect at $\lambda < 0$

One finds

$$\mathrm{Disc}(E(\lambda)) = \frac{Z^{1-inst}}{Z^{0-inst}}$$

$$= i\mu_1 \lambda^{-b-1} e^{\frac{-\mathcal{A}_{inst}}{\lambda}}\left(1 + \lambda\mu_2 + O(\lambda^2)\right),$$



$$E_k \sim \frac{\mu_1}{2\pi}\mathcal{A}_{inst}^{-k-b}\Gamma(k+b)\left(1 + \frac{\mathcal{A}_{inst}}{(b+k-1)}\mu_2 + O\left(\frac{1}{k^2}\right)\right) \to$$

anharmonic oscillator: $E_k \sim (-1)^{k+1}\frac{\sqrt{6}}{\pi^{\frac{3}{2}}}3^k\Gamma(k+\frac{1}{2})$ [Bender Wu]

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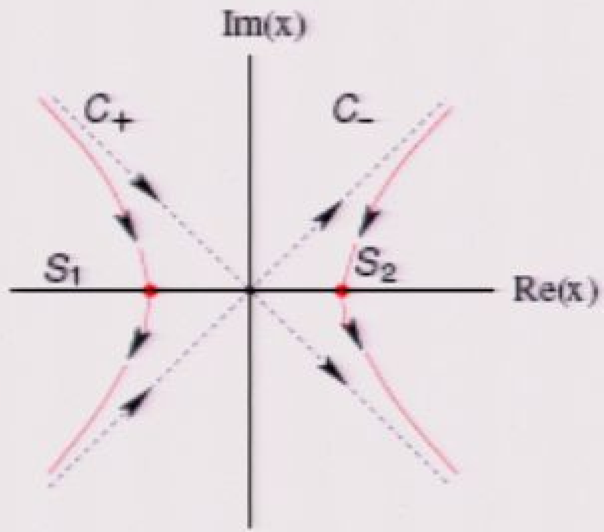
↑ saddle-point expansion



Nonperturbative effects in Matrix Models and Topological Strings

Marlene Weiss

- The instanton action is

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[David; Shenker]

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↑ 1-loop, leading

↑ 2-loop, subleading

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- $\langle \det(z - M)^2 \rangle$ can be expanded in terms of connected matrix correlation functions $W_{g,h}$ defined as

$$\langle \mathrm{Tr}\frac{1}{p_1 - M} \cdots \mathrm{Tr}\frac{1}{p_h - M} \rangle = \sum_{g=0}^{\infty} g_s^{2g-2+h} W_{g,h}(p_1, \cdots p_h)$$

- $W_{g,h}$ determined recursively from spectral curve by matrix model loop equations

[Ambjørn Chekhov Kristjansen Makeenko; Eynard Orantin]

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- The remaining ingredient is

$$\frac{Z_{N-1}^{(0)}}{Z_N^{(0)}} = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} (F_g(g_s(N-1)) - F_g(g_s N)) \right),$$

and we find in saddle-point analysis

$$\mathrm{Disc}(F) = \mu_1 g_s^{1/2} \exp\left(-\frac{\mathcal{A}_{inst}}{g_s}\right) (1 + g_s\mu_2 + \cdots)$$

For concreteness:

$$\mu_1 = \frac{(a-b)}{4}\sqrt{\frac{1}{2\pi y'(x_0)((x_0-a)(x_0-b))^{\frac{3}{2}}}}\,e^{-\frac{1}{g_s}\mathcal{A}_{inst}}$$

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- $a,\ b,\ x_0$ depend on 't Hooft parameter t

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- $a,\ b,\ x_0$ depend on 't Hooft parameter t
- μ_1 is proportional to $\mathrm{e}^{-N\mathcal{A}_{inst}/t}$ and therefore can not be seen by the perturbative $\frac{1}{N}$-expansion
- μ_1 has been computed before, but the result is not valid off criticality

[Hanada Hayakawa Ishibashi Kawai Kuroki Matsuo Tada]

- We have computed $\mathrm{Disc}(F)$ to two loops → μ_1, μ_2

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We now test our prediction for the asymptotics in the following examples:

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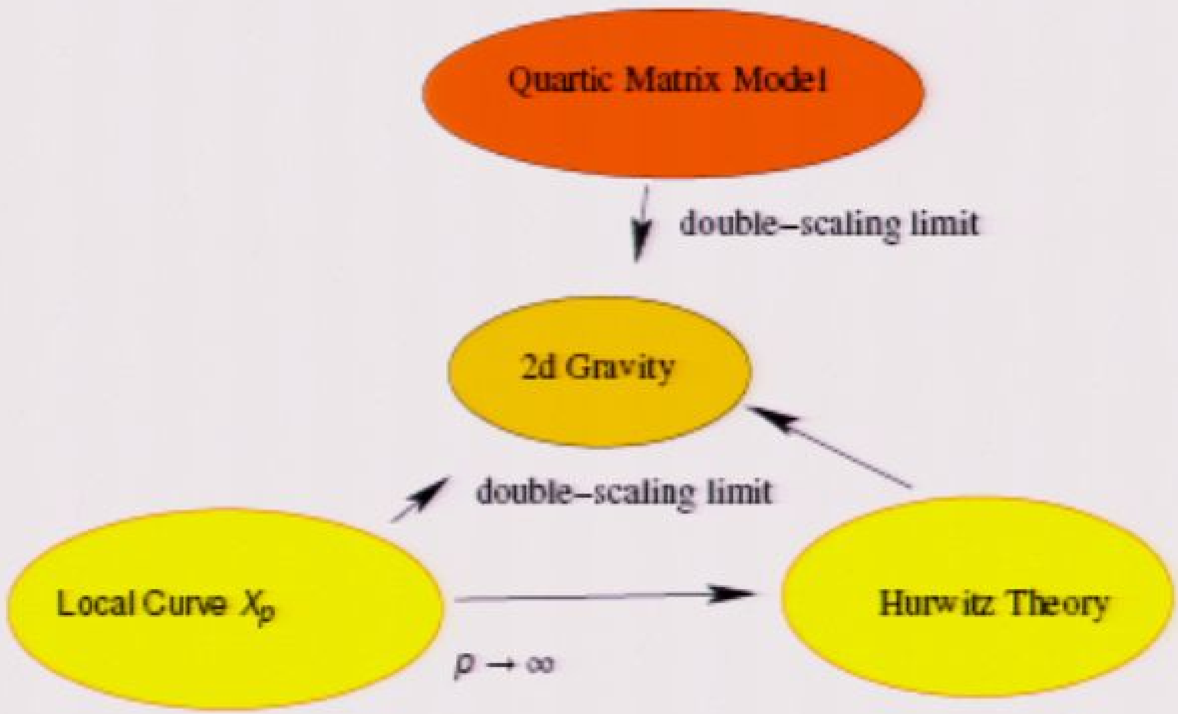

Numerical analysis:Richardson transformation

F_g are only available to limited genus, how to extract the asymptotics as $g \to \infty$? $\to$ Richardson transformation.

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Given a sequence $\{S_g\}$,

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the subleading corrections up to order $\frac{1}{g^n}$ can be removed defining

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If S_g truncates at $1/g^n$, this gives exactly s_0: for n=1;

$$S_g = s_0 + \frac{s_1}{g} \to A(g,1) = -(s_0 + \frac{s_1}{g}) + (s_0 + \frac{s_1}{g+1})(g+1) = s_0$$

- $A(g,n) = s_0 + O(\frac{1}{g^{n+1}}) \to$ accelerates convergence

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$$a(\lambda) = \frac{1}{24\lambda}\left(-1 + \sqrt{1 + 48\lambda}\right)$$

[Brézin Itzykson Parisi Zuber]

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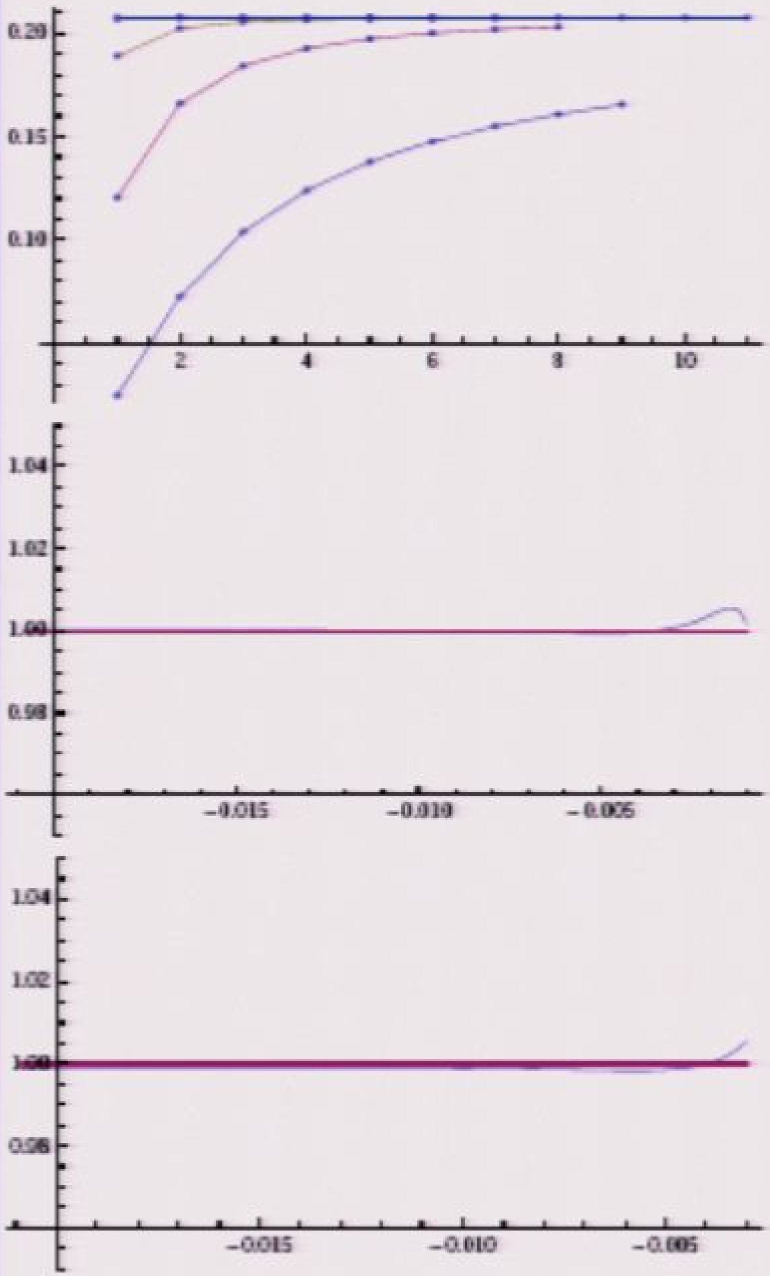
[Brézin Itzykson Parisi Zuber]

- Critical point at $\lambda = -\frac{1}{48}$
- The free energy in $\frac{1}{N}$-expansion can be computed by standard methods

[Bessis Itzykson Zuber]

We have computed $F_g(\lambda)$ up to genus 10:



The numerical asymptotics for the instanton action, along with the matrix prediction, at $\lambda = -0.1$

The leading asymptotics for F_g^{quart}, divided by the one-loop matrix prediction

The subleading asymptotics, divided by the two-loop prediction

2d gravity

- Taking $N \to \infty$ in a standard matrix model retains only planar surfaces unless one simultaneously takes $\lambda \to \lambda_c$ where higher-genus contributions are enhanced as $F_g \propto (\lambda - \lambda_c)^{(2-\gamma)(1-g)}$: double-scaling limit→ 2d gravity

  [Gross Migdal; Douglas Shenker]

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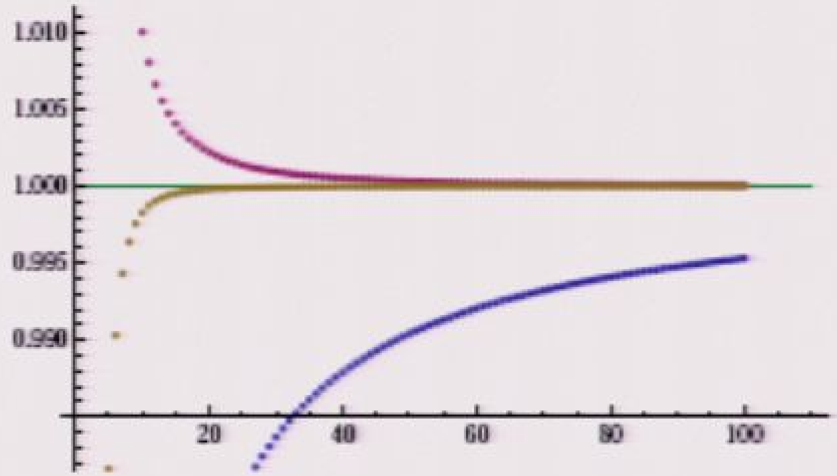
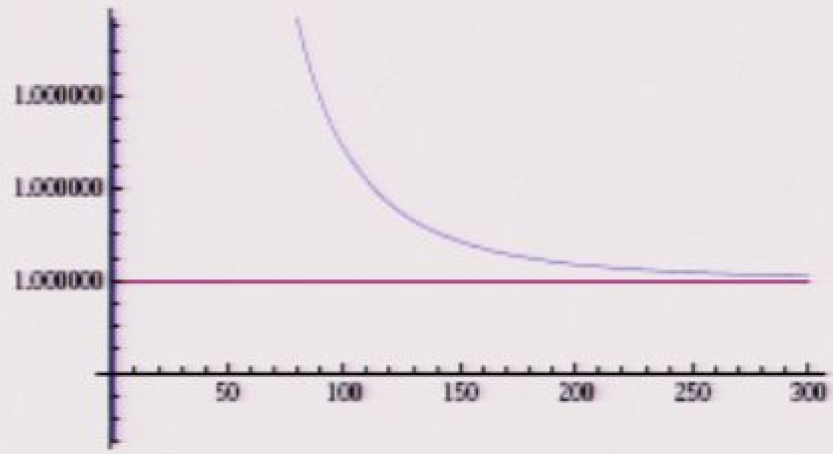
- The instanton action and 1-loop factor are

$$\mathcal{A}_{inst} = \frac{8\sqrt{3}}{5}, \; \mu_1 = \frac{1}{8\, 3^{3/4}\sqrt{\pi}}$$

[David]



The leading asymptotics, divided by the one-loop prediction



The subleading asymptotics, divided by the two-loop prediction

The local curve

Consider A-model topological strings on the local curve

$$X_p = \mathcal{O}(p) \oplus \mathcal{O}(2-p) \to \mathbb{P}^1, \ p \in \mathbb{Z}.$$

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- This is a toric Calabi-Yau threefold with one Kähler modulus

- The potential is unstable for all $p > 2$
- The free energy can be computed using the topological vertex or local Gromov-Witten theory

[Aganagic Klemm Mariño Vafa; Bryan Pandharipande]

- double-scaling limit → 2d gravity

The spectral curve corresponding to the matrix description of the mirror B-model is

$$y(z) = \frac{2}{z}\left(\left(\tanh^{-1}\left(\frac{\sqrt{(z-a)(z-b)}}{z-\frac{a+b}{2}}\right)\right) - p\tanh^{-1}\left(\frac{\sqrt{(z-a)(z-b)}}{z+\sqrt{ab}}\right)\right)$$

[Mariño]

- The endpoints of the cut a, b depend on the exponential of the Kähler parameter Q via the mirror map:

$$a = \frac{(1+\sqrt{\zeta})^2}{(1-\zeta)^p};\ b = \frac{(1-\sqrt{\zeta})^2}{(1-\zeta)^p}\ Q = (1-\zeta)^{p(p-2)}\zeta$$

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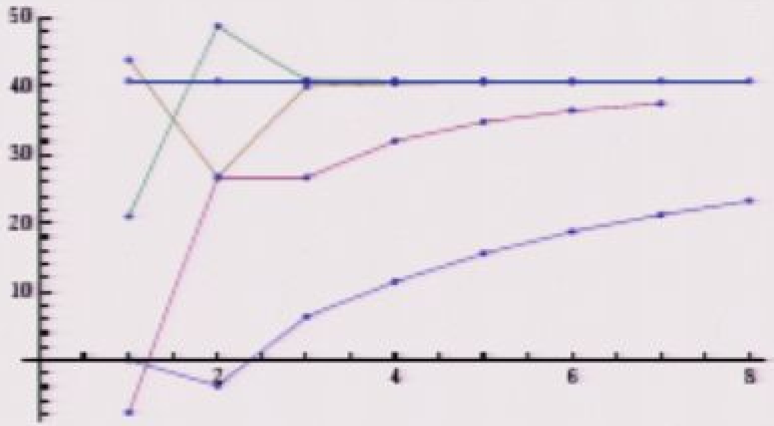
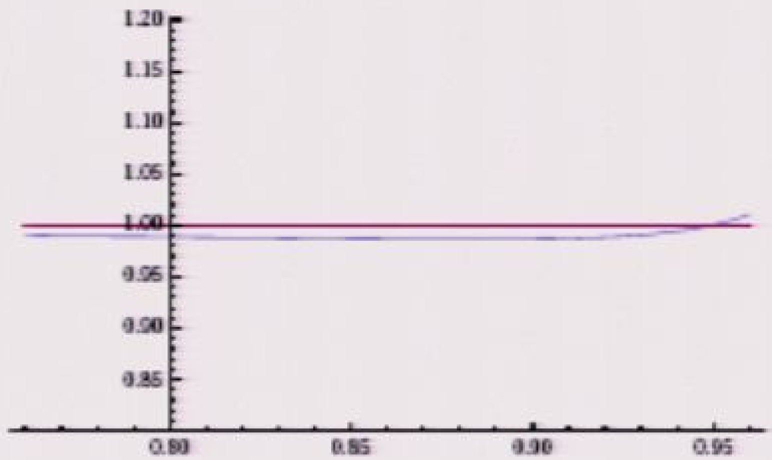
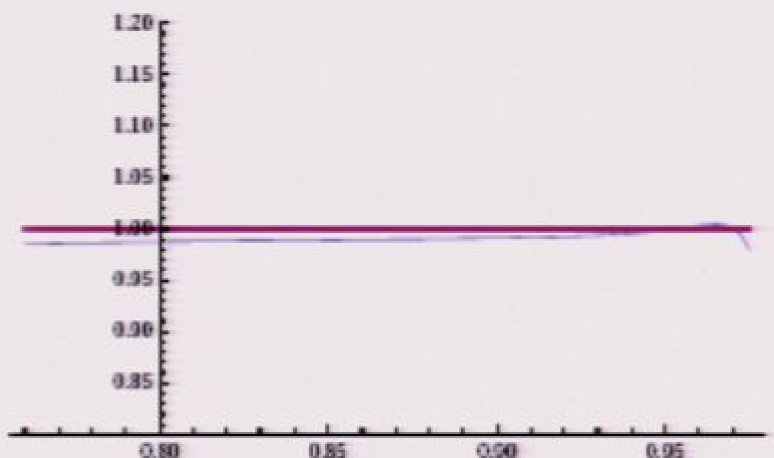
- The B-model matrix formalism provides a nonperturbative completion that is testable with the large-order behaviour of the perturbative amplitudes $F_g(Q)$
- Using the topological vertex, we computed F_g up to genus 9 (genus 7) for p=3 (p=4)



The numerical asymptotics for the instanton action, along with the matrix prediction, at $\zeta = .15, p = 3$

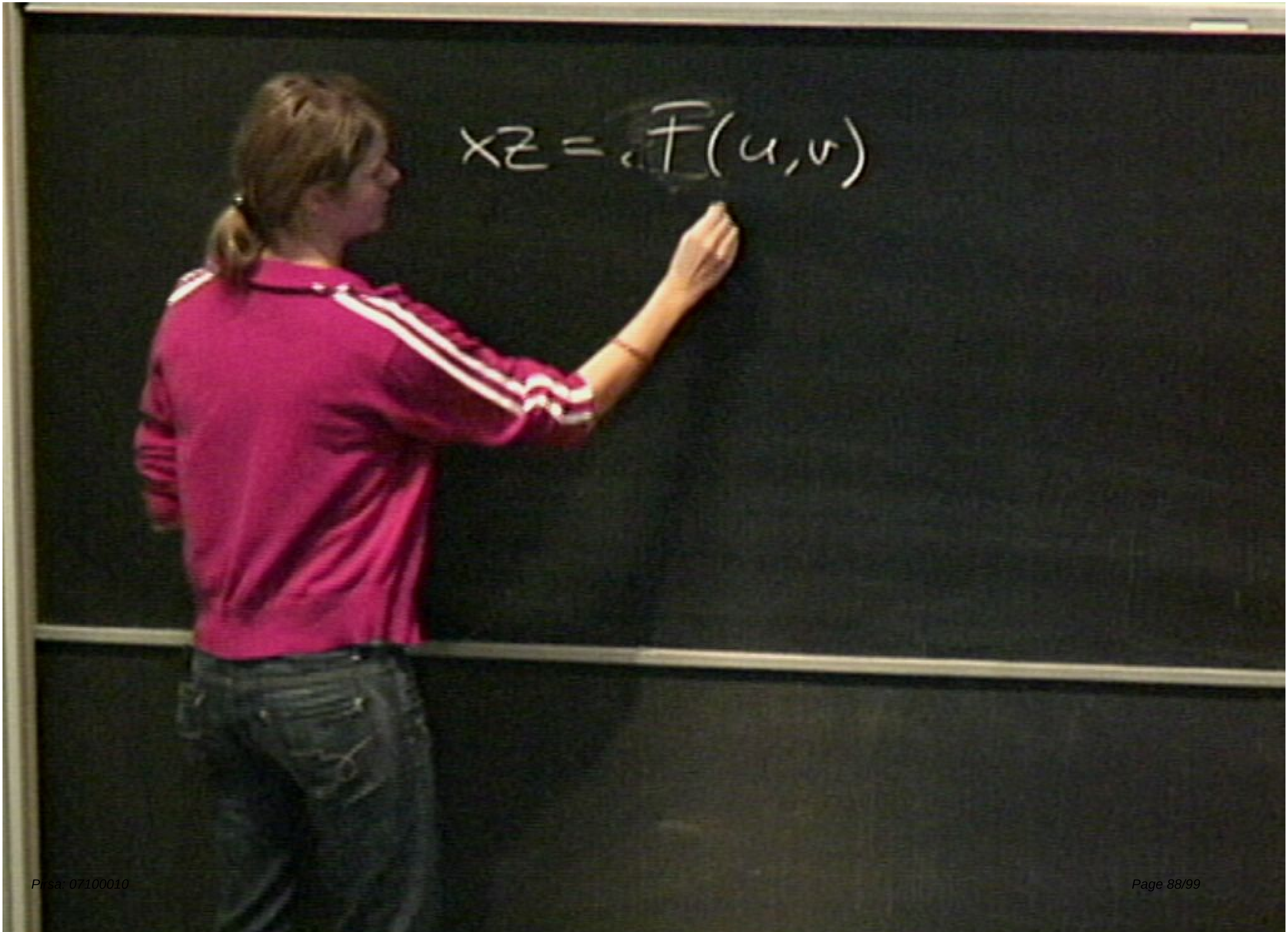


The leading asymptotics for $F_g^{p=3}$, divided by the one-loop prediction

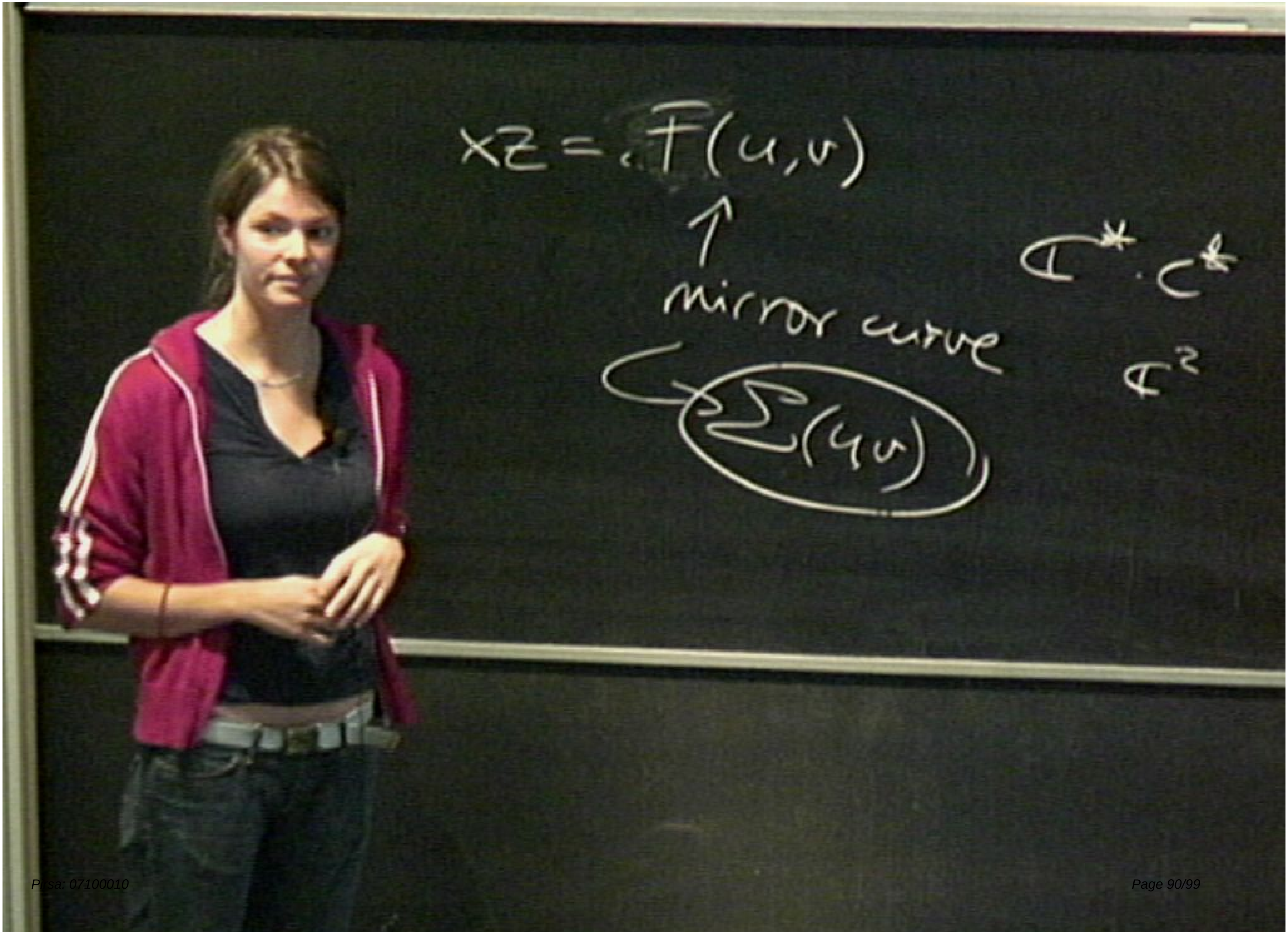


The subleading asymptotics, divided by the two-loop prediction

[redacted]


xz = F(u,v)


xZ = F(u,v)
mirror curve
Σ(u,v)


xZ = F(u,v)
mirror curve
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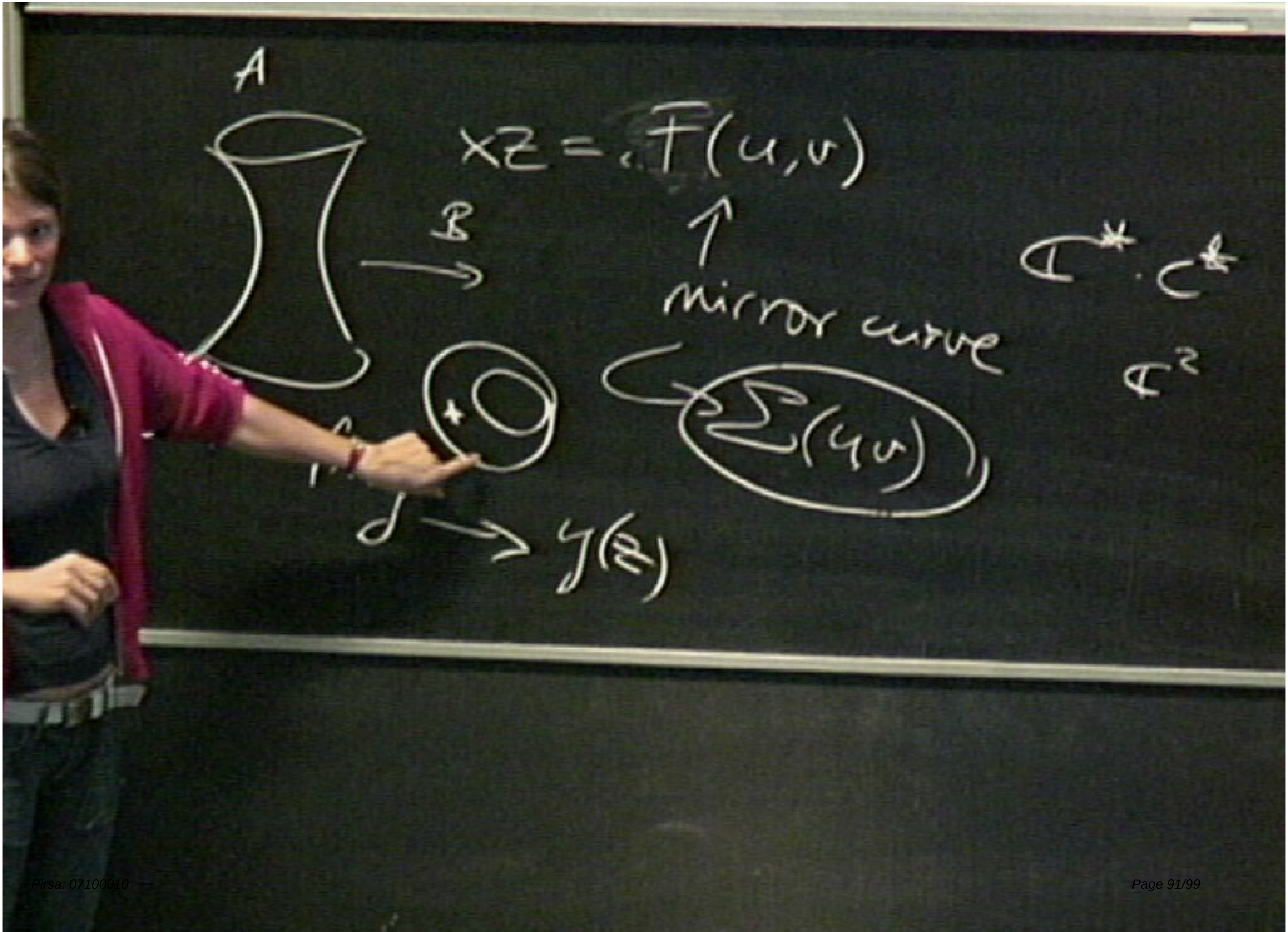
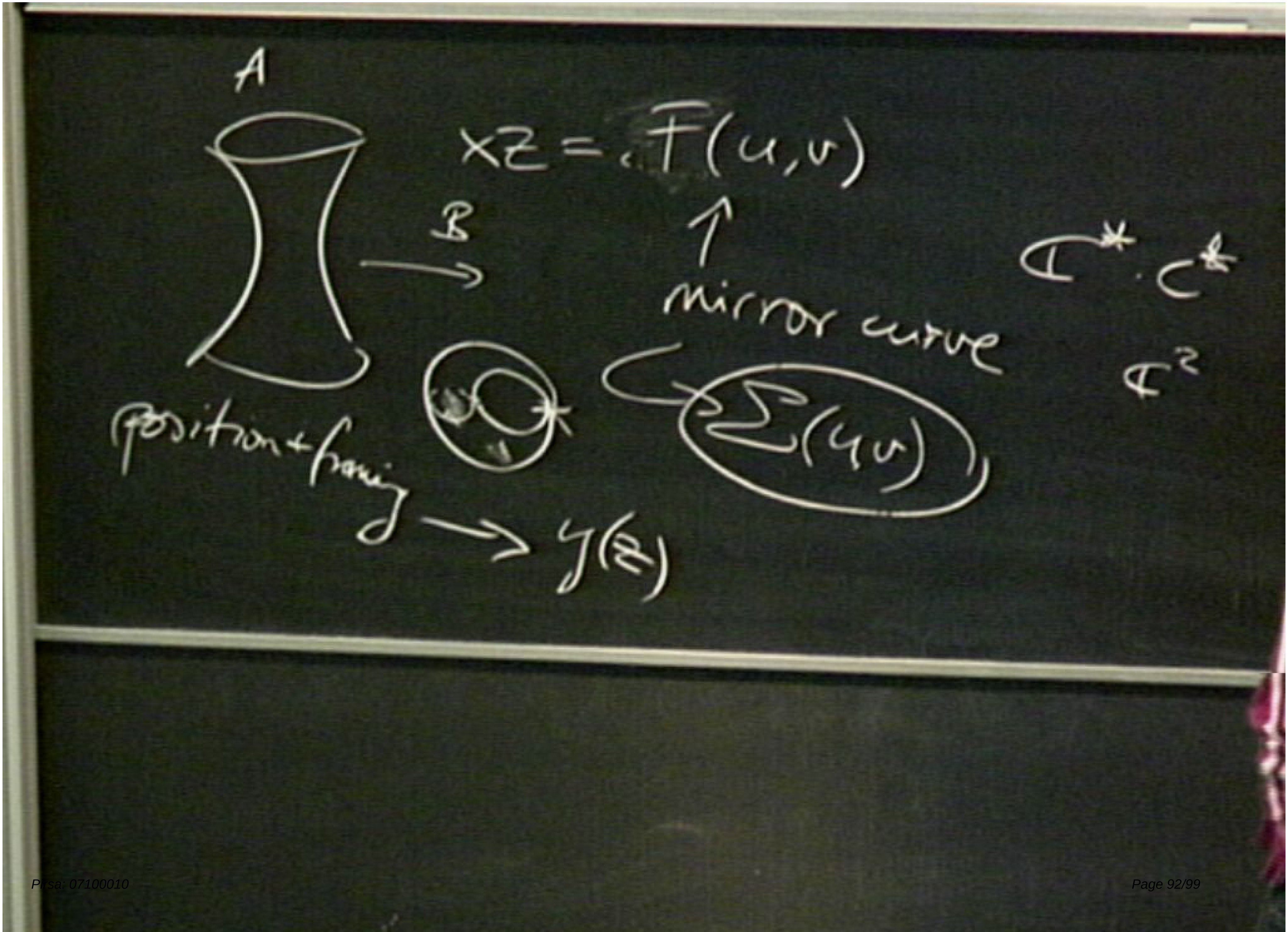
ℂ*·ℂ*
ℂ²



A
B
xz = F(u,v)
mirror curve
Σ(u,v)
y(z)


A
xZ = F(u,v)
B
mirror curve
position + framing
Σ(u,v)
y(z)
ℂ* · ℂ*
ℂ²

Hurwitz Theory

- Hurwitz theory counts branched covers of Riemann surfaces
- obtained as a special limit of the local curve X_p:

$$p \to \infty,\ g_s \to 0,\ Q \to 0;\ g^H = Npg_s,\ Q_H = \frac{(-1)^p}{(g_s N)^2} Q$$

-

$$F^H = \sum_{g \geq 0} N^{2-2g} \sum_{d \geq 0} Q_H^d H^{\mathbb{P}^1}_{g,d}(1^d)^\bullet \frac{g_H^{2g-2+2d}}{(2g-2+2d)!}$$

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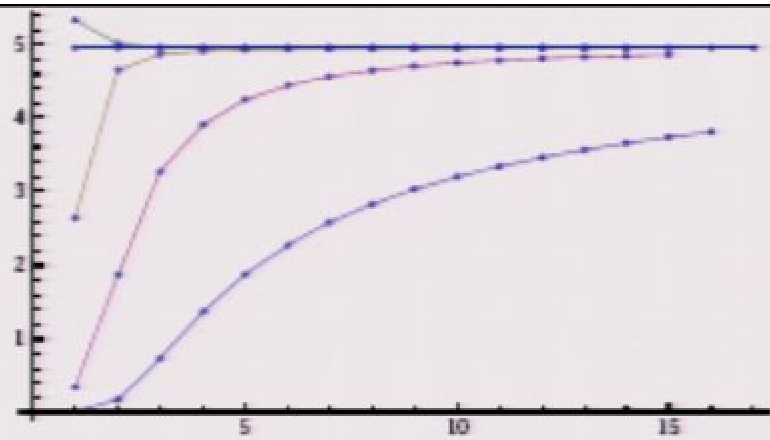
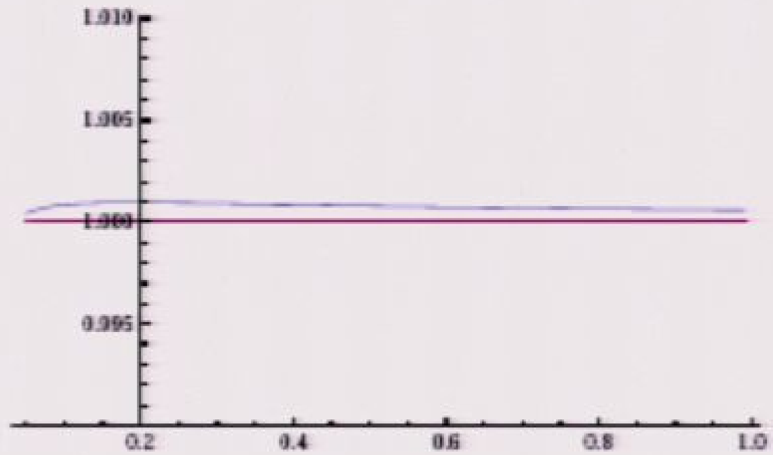
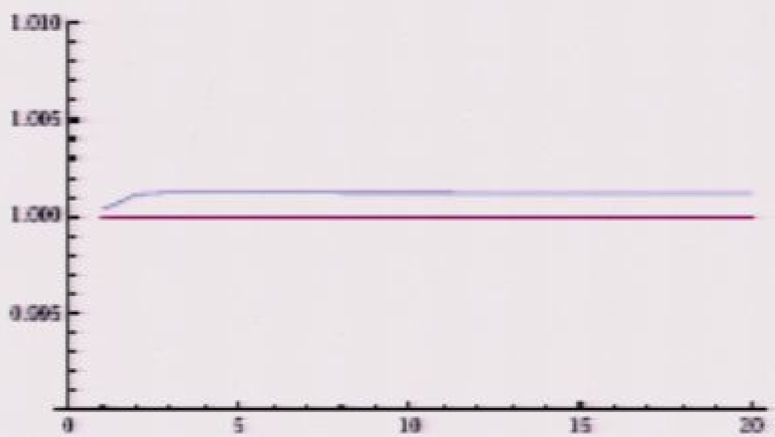


The numerical asymptot[ics] for the instanton action, along with the matrix prediction, at $\chi = 0.5$



The leading asymptotics for $F_g^H(\chi)$, divided by the one-loop matrix predicti[on]



The subleading asymptotics for $F_g^H(\chi)$, divided by the two-loop prediction

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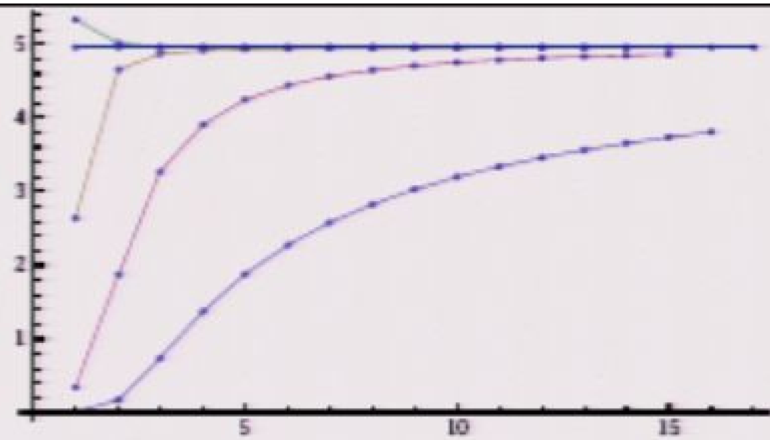

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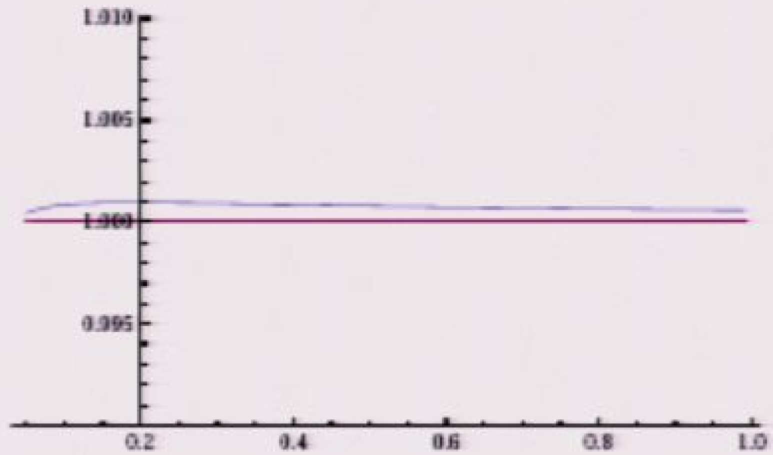

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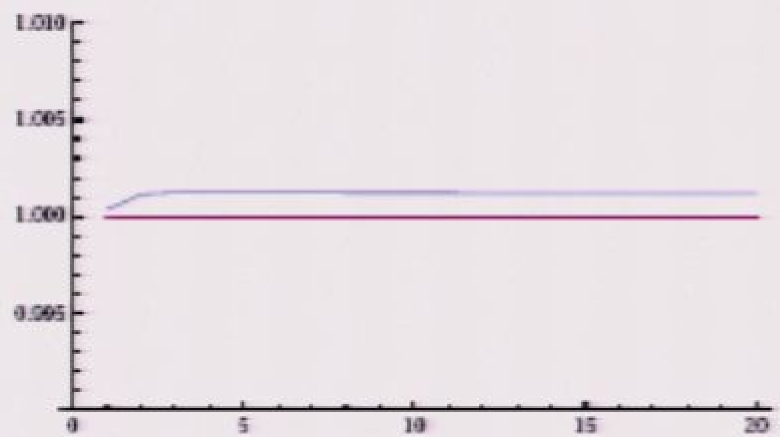

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Conclusion and Outlook

- We have computed nonperturbative effects for a generic matrix model
- Using the B-model matrix formalism, this defines a nonperturbative completion for topological strings on local geometries
- The results can be tested with the large-order behavior of the string perturbation series: agreement to very high precision

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